Definition 1. Let x be a set. $|x|$ is the ordinal κ such that κ is equinumerous to x and every ordinal that is equinumerous to x is greater than or equal to κ. Let the *cardinality of x* stand for $|x|$.

Definition 2. A *cardinal number* is an ordinal κ such that $\kappa = |x|$ for some set x. Let a *cardinal* stand for a cardinal number.

Proposition 3. Let κ be a cardinal. Then $|\kappa| = \kappa$.

Proof. κ is an ordinal that is equinumerous to κ. Hence $|\kappa| \leq \kappa$. Consider a set x such that $\kappa = |x|$. Then $|\kappa|$ is an ordinal that is equinumerous to x. Hence $\kappa \leq |\kappa|$. Thus $|\kappa| = \kappa$. ■

Proposition 4. Let x, y be sets. Then x and y are equinumerous iff $|x| = |y|$.

Proof.

Case x and y are equinumerous. Take a bijection f between x and y. Consider a bijection g between y and $|y|$. Then $g \circ f$ is a bijection between x and $|y|$ (by bijectivity of composition of bijections). Hence x and $|y|$ are equinumerous. Thus $|y| \geq |x|$.

f^{-1} is a bijection between y and x. Consider a bijection h between x and $|x|$. Then $h \circ f^{-1}$ is a bijection between y and $|x|$ (by bijectivity of composition of bijections). Hence y and $|x|$ are equinumerous. Thus $|x| \geq |y|$.

Therefore $|x| = |y|$. □

Case $|x| = |y|$. Consider a bijection f between x and $|x|$ and a bijection g between $|y|$ and y. Then $g \circ f$ is a bijection between x and y. Hence x and y are equinumerous. □

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