Proposition 1. Let x, y be sets and $f : x \hookrightarrow y$ and $a \subset x$. Then $|f[a]| = |a|$.

Proof. $f \restriction a$ is a bijection between a and $f[a]$. $f[a]$ is a set. Hence $|a| = |f[a]|$. ■

Proposition 2. Let κ be a cardinal and $x \subset \kappa$. Then $|x| \leq \kappa$.

Proof. Assume $|x| > \kappa$. Then $\kappa \subset |x|$. Take a bijection f between $|x|$ and x. Then $f \restriction \kappa$ is an injective map from κ to x. id_x is an injective map from x to κ. Hence x and κ are equinumerous (by Cantor-Schöder-Bernstein). Indeed x is a set. Thus $|x| = \kappa$. Contradiction. ■

Proposition 3. Let x, y be sets. Then there exists an injective map from x to y iff $|x| \leq |y|$.

Proof.

Case there exists an injective map from x to y. Consider an injective map f from x to y. Take a bijection g from $|x|$ to x and a bijection h from y to $|y|$. Then g is an injective map from $|x|$ to x and h is an injective map from y to $|y|$. Hence $h \circ f$ is an injective map from x to $|y|$. Thus $(h \circ f) \circ g$ is an injective map from $|x|$ to $|y|$. Therefore $|x| = ||x|| = |(h \circ f) \circ g[\,|x|\,]|$ (by cardinality of cardinal number, cardinality of image under injection). We have $(h \circ f) \circ g[\,|x|\,] \subset |y|$. Hence $|x| \leq |y|$. □

Case $|x| \leq |y|$. Take a bijection g from x to $|x|$ and a bijection h from $|y|$ to y. We have $|x| \subset |y|$. Hence g is an injective map from x to $|y|$. Take $f = h \circ g$. Then f is an injective map from x to y. Indeed f is injective. Indeed h is an injective map from $|y|$ to y. □

■

Corollary 4. Let x be a set and $y \subset x$. Then $|y| \leq |x|$.

Proof. Define $f(v) := v$ for $v \in y$. Then f is an injective map from y to x. Hence $|y| \leq |x|$. ■

Corollary 5. Let x, y be sets such that $|y| < |x|$. Then $x \setminus y$ is nonempty.

Proof. Assume the contrary. Then $x \subset y$. Hence $|x| \leq |y|$. Contradiction. ■

Proposition 6. Let x, y be nonempty sets. Then there exists a surjective map from x onto y iff $|x| \geq |y|$.

Proof.

Case there exists a surjective map from x onto y. Consider a surjective map f from x onto y. Define $g(v) :=$ "choose $u \in x$ such that $f(u) = v$ in u" for $v \in y$. Then g is an injective map from y to x. Indeed we can show that g is injective. Let $v, v' \in y$. Assume $g(v) = g(v')$. Take $u \in x$ such that $f(u) = v$ and $g(v) = u$. Take $u' \in x$ such that $f(u') = v'$ and $g(v') = u'$. Then $v = f(u) = f(g(v)) = f(g(v')) = f(u') = v'$. End. Hence $|x| \geq |y|$. □

Case $|x| \geq |y|$. Then we can take an injective map f from y to x. Then f^{-1} is a bijection between range(f) and y. Consider an element z of y. Define

$$g(u) := \begin{cases} f^{-1}(u) & : u \in \mathrm{range}(f) \\ z & : u \notin \mathrm{range}(f) \end{cases}$$

for $u \in x$. Then g is a surjective map from x onto y. Indeed we can show that every element of y is a value of g. Let $v \in y$. Then $f(v) \in \mathrm{range}(f)$. Hence $g(f(v)) = f^{-1}(f(v)) = v$. End. □

■

Proposition 7. Let x, y be nonempty sets. $|x| < |y|$ iff there exists an injective map from x to y and there exists no surjective map from x onto y.

Proof. There exists an injective map from x to y and there exists no surjective map from x onto y iff $|x| \leq |y|$ and $|x| \ngeq |y|$ (by existence condition for injections, existence condition for surjections). $|x| \leq |y|$ and $|x| \ngeq |y|$ iff $|x| \leq |y|$ and $|x| \neq |y|$. $|x| \leq |y|$ and $|x| \neq |y|$ iff $|x| < |y|$. ■

Proposition 8. Let x, y be sets and $f : x \to y$ and $a \subset x$. Then $|f[a]| \leq |a|$.

Proof.

Case a is empty. □

Case a is nonempty. $f \restriction a$ is a surjective map from a onto $f[a]$. $f[a]$ is nonempty. Hence $|f[a]| \leq |a|$ (by existence condition for surjections). Indeed a and $f[a]$ are sets. □

■