Part I
Definitions

Definition 1. Let A, B be classes. $A \triangle B := (A \cup B) \setminus (A \cap B)$. Let the *symmetric difference of A and B* stand for $A \triangle B$.

Proposition 2. Let A, B be classes. Then $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Proof. Let us show that $A \triangle B \subset (A \setminus B) \cup (B \setminus A)$. Let $u \in A \triangle B$. Then $u \in A \cup B$ and $u \notin A \cap B$. Hence ($u \in A$ or $u \in B$) and $\neg (u \in A \wedge u \in B)$. Thus ($u \in A$ or $u \in B$) and ($u \notin A$ or $u \notin B$). Therefore if $u \in A$ then $u \notin B$. If $u \in B$ then $u \notin A$. Then we have ($u \in A$ and $u \notin B$) or ($u \in B$ and $u \notin A$). Hence $u \in A \setminus B$ or $u \in B \setminus A$. Thus $u \in (A \setminus B) \cup (B \setminus A)$. End.

Let us show that $((A \setminus B) \cup (B \setminus A)) \subset A \triangle B$. Let $u \in (A \setminus B) \cup (B \setminus A)$. Then ($u \in A$ and $u \notin B$) or ($u \in B$ and $u \notin A$). If $u \in A$ and $u \notin B$ then $u \in A \cup B$ and $u \notin A \cap B$. If $u \in B$ and $u \notin A$ then $u \in A \cup B$ and $u \notin A \cap B$. Hence $u \in A \cup B$ and $u \notin A \cap B$. Thus $u \in (A \cup B) \setminus (A \cap B) = A \triangle B$. End. ∎

Part II
Computation Laws

1 Commutativity

Proposition 3. Let A, B be classes. Then $A \triangle B = B \triangle A$.

Proof. $A \triangle B = (A \cup B) \setminus (A \cap B) = (B \cup A) \setminus (B \cap A) = B \triangle A$. ∎

2 Associativity

Proposition 4. Let A, B, C be classes. Then $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

Proof. Take a class X such that $X = (((A \setminus B) \cup (B \setminus A)) \setminus C) \cup (C \setminus ((A \setminus B) \cup (B \setminus A)))$.

Take a class Y such that $Y = (A \setminus ((B \setminus C) \cup (C \setminus B))) \cup (((B \setminus C) \cup (C \setminus B)) \setminus A)$.

We have $A \triangle B = (A \setminus B) \cup (B \setminus A)$ and $B \triangle C = (B \setminus C) \cup (C \setminus B)$. Hence $(A \triangle B) \triangle C = X$ and $A \triangle (B \triangle C) = Y$.

Let us show that (I) $X \subset Y$. Let $x \in X$.

Case $x \in ((A \setminus B) \cup (B \setminus A)) \setminus C$. Then $x \notin C$.

Case $x \in A \setminus B$. Then $x \notin B \setminus C$ and $x \notin C \setminus B$. $x \in A$. Hence $x \in A \setminus ((B \setminus C) \cup (C \setminus B))$. Thus $x \in Y$. □

Case $x \in B \setminus A$. Then $x \in B \setminus C$. Hence $x \in (B \setminus C) \cup (C \setminus B)$. $x \notin A$. Thus $x \in ((B \setminus C) \cup (C \setminus B)) \setminus A$. Therefore $x \in Y$. □

□

Case $x \in C \setminus ((A \setminus B) \cup (B \setminus A))$. Then $x \in C$. $x \notin A \setminus B$ and $x \notin B \setminus A$. Hence $\neg (x \in A \setminus B \vee x \in B \setminus A)$. Thus $\neg ((x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A))$. Therefore ($x \notin A$ or $x \in B$) and ($x \notin B$ or $x \in A$).

Case $x \in A$. Then $x \in B$. Hence $x \notin (B \setminus C) \cup (C \setminus B)$. Thus $x \in A \setminus ((B \setminus C) \cup (C \setminus B))$. Therefore $x \in Y$. □

Case $x \notin A$. Then $x \notin B$. Hence $x \in C \setminus B$. Thus $x \in (B \setminus C) \cup (C \setminus B)$. Therefore $x \in ((B \setminus C) \cup (C \setminus B)) \setminus A$. Then we have $x \in Y$. □

□

End.

Let us show that (II) $Y \subset X$. Let $y \in Y$.

Case $y \in A \setminus ((B \setminus C) \cup (C \setminus B))$. Then $y \in A$. $y \notin B \setminus C$ and $y \notin C \setminus B$. Hence $\neg (y \in B \setminus C \vee y \in C \setminus B)$. Thus $\neg ((y \in B \wedge y \notin C) \vee (y \in C \wedge y \notin B))$. Therefore ($y \notin B$ or $y \in C$) and ($y \notin C$ or $y \in B$).

Case $y \in B$. Then $y \in C$. $y \notin A \setminus B$ and $y \notin B \setminus A$. Hence $y \notin (A \setminus B) \cup (B \setminus A)$. Thus $y \in C \setminus ((A \setminus B) \cup (B \setminus A))$. Therefore $y \in X$. □

Case $y \notin B$. Then $y \notin C$. $y \in A \setminus B$. Hence $y \in (A \setminus B) \cup (B \setminus A)$. Thus $y \in ((A \setminus B) \cup (B \setminus A)) \setminus C$. Therefore $y \in X$. □

□

Case $y \in ((B \setminus C) \cup (C \setminus B)) \setminus A$. Then $y \notin A$.

Case $y \in B \setminus C$. Then $y \in B \setminus A$. Hence $y \in (A \setminus B) \cup (B \setminus A)$. Thus $y \in ((A \setminus B) \cup (B \setminus A)) \setminus C$. Therefore $y \in X$. □

Case $y \in C \setminus B$. Then $y \in C$. $y \notin A \setminus B$ and $y \notin B \setminus A$. Hence $y \notin (A \setminus B) \cup (B \setminus A)$. Thus $y \in C \setminus ((A \setminus B) \cup (B \setminus A))$. Therefore $y \in X$. □

$\square$

End. $\blacksquare$

3 Distributivity

Proposition 5. Let A, B, C be classes. Then $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$.

Proof. $A \cap (B \triangle C) = A \cap ((B \setminus C) \cup (C \setminus B)) = (A \cap (B \setminus C)) \cup (A \cap (C \setminus B))$.

$A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$. $A \cap (C \setminus B) = (A \cap C) \setminus (A \cap B)$.

Hence $A \cap (B \triangle C) = ((A \cap B) \setminus (A \cap C)) \cup ((A \cap C) \setminus (A \cap B)) = (A \cap B) \triangle (A \cap C)$. $\blacksquare$

4 Miscellaneous Rules

Proposition 6. Let A, B be classes. Then $A \subset B \iff A \triangle B = B \setminus A$.

Proof.

Case $A \subset B$. Then $A \cup B = B$ and $A \cap B = A$. Hence the thesis. $\square$

Case $A \triangle B = B \setminus A$. Let $a \in A$. Then $a \notin B \setminus A$. Hence $a \notin A \triangle B$. Thus $a \notin A \cup B$ or $a \in A \cap B$. Indeed $A \triangle B = (A \cup B) \setminus (A \cap B)$. If $a \notin A \cup B$ then we have a contradiction. Therefore $a \in A \cap B$. Then we have the thesis. $\square$

$\blacksquare$

Proposition 7. Let A, B, C be classes. Then $A \triangle B = A \triangle C \iff B = C$.

Proof.

Case $A \triangle B = A \triangle C$.

Let us show that $B \subset C$. Let $b \in B$.

Case $b \in A$. Then $b \notin A \triangle B$. Hence $b \notin A \triangle C$. Therefore $b \in A \cap C$. Indeed $A \triangle C = (A \cup C) \setminus (A \cap C)$. Hence $b \in C$. $\square$

Case $b \notin A$. Then $b \in A \triangle B$. Indeed $b \in A \cup B$ and $b \notin A \cap B$. Hence $b \in A \triangle C$. Thus $b \in A \cup C$ and $b \notin A \cap C$. Therefore $b \in A$ or $b \in C$. Then we have the thesis. $\square$

End.

Let us show that $C \subset B$. Let $c \in C$.

Case $c \in A$. Then $c \notin A \triangle C$. Hence $c \notin A \triangle B$. Therefore $c \in A \cap B$. Indeed $c \notin A \cup B$ or $c \in A \cap B$. Hence $c \in B$. $\square$

Case $c \notin A$. Then $c \in A \triangle C$. Indeed $c \in A \cup C$ and $c \notin A \cap C$. Hence $c \in A \triangle B$. Thus $c \in A \cup B$ and $c \notin A \cap B$. Therefore $c \in A$ or $c \in B$. Then we have the thesis. $\square$

End. $\square$

$\blacksquare$

Proposition 8. Let A be a class. Then $A \triangle A = \emptyset$.

Proof. $A \triangle A = (A \cup A) \setminus (A \cap A) = A \setminus A = \emptyset$. $\blacksquare$

Proposition 9. Let A be a class. Then $A \triangle \emptyset = A$.

Proof. $A \triangle \emptyset = (A \cup \emptyset) \setminus (A \cap \emptyset) = A \setminus \emptyset = A$. $\blacksquare$

Proposition 10. Let A, B be classes. Then $A = B \iff A \triangle B = \emptyset$.

Proof.

Case $A = B$. Then $A \triangle B = (A \cup A) \setminus (A \cap A) = A \setminus A = \emptyset$. Hence the thesis. $\square$

Case $A \triangle B = \emptyset$. Then $(A \cup B) \setminus (A \cap B)$ is empty. Hence every element of $A \cup B$ is an element of $A \cap B$. Thus for all objects u if $u \in A$ or $u \in B$ then $u \in A$ and $u \in B$. Therefore every element of A is an element of B. Every element of B is an element of A. Then we have the thesis. $\square$