Part I

Pairs

Definition 1. A *pair* is an object p such that $p = (a, b)$ for some objects a, b. Let an *ordered pair* stand for a pair.

Part II

Cartesian Products

Definition 2. Let A, B be classes. $A \times B := \{(a, b) \mid a \in A \text{ and } b \in B\}$. Let the *Cartesian product of A and B* stand for $A \times B$. Let the *direct product of A and B* stand for $A \times B$.

Proposition 3. Let A, B be classes and a, b be objects. Then $(a, b) \in A \times B$ iff $a \in A$ and $b \in B$.

Proof.

Case $(a, b) \in A \times B$. We can take $a' \in A$ and $b' \in B$ such that $(a, b) = (a', b')$. Then $a = a'$ and $b = b'$. Hence $a \in A$ and $b \in B$. □

Case $a \in A$ and $b \in B$. a and a are objects. Hence (a, b) is an object. Therefore $(a, b) \in A \times B$. □

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Proposition 4. Let A, B be classes. Then $A \times B$ is empty iff A is empty or B is empty.

Proof.

Case $A \times B$ is empty. Assume that A and B are nonempty. Then we can take an element a of A and an element b of B. Then $(a, b) \in A \times B$. Contradiction. □

Case A is empty or B is empty. Assume that $A \times B$ is nonempty. Then we can take an element c of $A \times B$. Then $c = (a, b)$ for some $a \in A$ and some $b \in B$. Hence A and B are nonempty. Contradiction. □

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Proposition 5. Let a, b be objects. Then $\{a\} \times \{b\} = \{(a, b)\}$.

Proof. Let us show that $\{a\} \times \{b\} \subset \{(a, b)\}$. Let $c \in \{a\} \times \{b\}$. Take $a' \in \{a\}$ and $b' \in \{b\}$ such that $c = (a', b')$. We have $a' = a$ and $b' = b$. Hence $c = (a, b)$. Thus $c \in \{(a, b)\}$. End.

Let us show that $\{(a, b)\} \subset \{a\} \times \{b\}$. Let $c \in \{(a, b)\}$. Then $c = (a, b)$. We have $a \in \{a\}$ and $b \in \{b\}$. Hence $c \in \{a\} \times \{b\}$. End. ■

Proposition 6. Let A, A', B, B' be nonempty classes. If $A \times B = A' \times B'$ then $A = A'$ and $B = B'$.

Proof. Assume $A \times B = A' \times B'$.

(1) $A \subset A'$ and $B \subset B'$.

Proof. Let $a \in A$ and $b \in B$. Then $(a, b) \in A \times B$. Hence $(a, b) \in A' \times B'$. Thus $a \in A'$ and $b \in B'$. □

(2) $A' \subset A$ and $B' \subset B$.

Proof. Let $a \in A'$ and $b \in B'$. Then $(a, b) \in A' \times B'$. Hence $(a, b) \in A \times B$. Thus $a \in A$ and $b \in B$. □ ■