Part I
Range

Definition 1. Let f be a map. A *value of f* is an object b such that $b = f(a)$ for some $a \in \mathrm{dom}(f)$.

Definition 2. Let f be a map. $\mathrm{range}(f) := \{f(a) \mid a \in \mathrm{dom}(f)\}$. Let the *range of f* stand for $\mathrm{range}(f)$.

Proposition 3. Let f be a map and b be an object. b is a value of f iff $b \in \mathrm{range}(f)$.

Proof.

Case b is a value of f. Take $a \in \mathrm{dom}(f)$ such that $b = f(a)$. b is an object. Hence $b \in \mathrm{range}(f)$. □

Case $b \in \mathrm{range}(f)$. Then b is an object such that $b = f(a)$ for some $a \in \mathrm{dom}(f)$. Hence b is a value of f. □

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Part II
Identity Map

Definition 4. Let A be a class. id_A is the map defined on A such that $\mathrm{id}_A(a) = a$ for all $a \in A$. Let the *identity map on A* stand for id_A.

Part III
Composition

Definition 5. Let f, g be maps. Assume $\mathrm{range}(f) \subset \mathrm{dom}(g)$. $g \circ f$ is the map defined on $\mathrm{dom}(f)$ such that $(g \circ f)(a) = g(f(a))$ for all $a \in \mathrm{dom}(f)$.

Let the *composition of g and f* stand for $g \circ f$.

Part IV
Restriction

Definition 6. Let f be a map and $X \subset \mathrm{dom}(f)$. $f \restriction X$ is the map defined on X such that $(f \restriction X)(a) = f(a)$ for all $a \in X$. Let the *restriction of f to X* stand for $f \restriction X$.

Proposition 7. Let A be a class and $X \subset A$. Then $\mathrm{id}_A \restriction X = \mathrm{id}_X$.

Part V
Image and Preimage

Definition 8. Let f be a map and A be a class. $f[A] := \{f(a) \mid a \in \mathrm{dom}(f) \cap A\}$. Let the *image of A under f* stand for $f[A]$. Let the *direct image of A under f* stand for $f[A]$.

Definition 9. Let f be a map and B be a class. $f^{-1}[B] := \{a \in \mathrm{dom}(f) \mid f(a) \in B\}$. Let the *preimage of B under f* stand for $f^{-1}[B]$. Let the *inverse image of B under f* stand for $f^{-1}[B]$.

Part VI
Maps Between Classes

Definition 10. Let A be a class. A *map of A* is a map f such that $\mathrm{dom}(f) = A$. Let a *function of A* stand for a map of A that is a function.

Definition 11. Let B be a class. A *map to B* is a map f such that $f(a) \in B$ for each $a \in \mathrm{dom}(f)$. Let a *function to B* stand for a map to B that is a function.

Definition 12. Let A, B be classes. A *map from A to B* is a map f such that $\mathrm{dom}(f) = A$ and $f(a) \in B$ for each $a \in A$. Let $f : A \to B$ stand for f is a map from A to B. Let a *function from A to B* stand for a map from A to B that is a function.

Definition 13. Let A be a class. A *map on A* is a map from A to A. Let a *function on A* stand for a map on A that is a function.

Proposition 14. Let A, B be classes and $f, g : A \to B$. Assume that $f(a) = g(a)$ for all $a \in A$. Then $f = g$.

Proposition 15. Let A, B be classes and f be a map of A. Assume that $f(a) \in B$ for all $a \in A$. Then f is a map from A to B iff $\mathrm{range}(f) \subset B$.

Proposition 16. Let A be a class. Then id_A is a map on A.

Proposition 17. Let A, B, C be classes and $f : A \to B$ and $g : B \to C$. Then $g \circ f : A \to C$.

Proposition 18. Let A, B be classes and $f : A \to B$ and $X \subset A$. Then $f \restriction X : X \to B$.

Proposition 19. Let A, B be classes and $f : A \to B$. Then $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$.

Proof. A is the domain of $f \circ \mathrm{id}_A$ and the domain of f and the domain of $\mathrm{id}_B \circ f$. We have $(f \circ \mathrm{id}_A)(a) = f(\mathrm{id}_A(a)) = f(a) = \mathrm{id}_B(f(a)) = (\mathrm{id}_B \circ f)(a)$ for all $a \in A$. Hence $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$. ∎

Proposition 20. Let A be a class and $X \subset A$. Then $\mathrm{id}_A \restriction X = \mathrm{id}_X$.

Proof. We have $\mathrm{dom}(\mathrm{id}_A \upharpoonright X) = X = \mathrm{dom}(\mathrm{id}_X)$. $(\mathrm{id}_A \upharpoonright X)(a) = \mathrm{id}_A(a) = a = \mathrm{id}_X(a)$ for all $a \in X$. Hence $\mathrm{id}_A \upharpoonright X = \mathrm{id}_X$. ∎

Proposition 21. Let A, B, C, D be classes and $f : A \to B$ and $g : B \to C$ and $h : C \to D$. Then $h \circ (g \circ f) = (h \circ g) \circ f$.

Proof. $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are maps from A to D.

Let us show that $(h \circ (g \circ f))(a) = ((h \circ g) \circ f)(a)$ for all $a \in A$. Let $a \in A$. Then $(h \circ (g \circ f))(a) = h((g \circ f)(a)) = h(g(f(a))) = (h \circ g)(f(a)) = ((h \circ g) \circ f)(a)$. End.

Hence $h \circ (g \circ f) = (h \circ g) \circ f$. ∎

Part VII
Classes of Functions

Definition 22. Let A, B be classes. $[A \to B]$ is the class of all functions from A to B.

Part VIII
Fixed Points

Definition 23. Let f be a map. A *fixed point of f* is an element x of $\mathrm{dom}(f)$ such that $f(x) = x$.