Part I
Definitions

Definition 1. Let f be a map. An *inverse of f* is a map g from $\mathrm{range}(f)$ to $\mathrm{dom}(f)$ such that $f(a) = b \Longleftrightarrow g(b) = a$ for all $a \in \mathrm{dom}(f)$ and all $b \in \mathrm{dom}(g)$.

Definition 2. Let f be a map. f is *invertible* iff f has an inverse.

Lemma 3. Let f be a map and g, g' be inverses of f. Then $g = g'$.

Proof. We have $\mathrm{dom}(g) = \mathrm{range}(f) = \mathrm{dom}(g')$.

Let us show that $g(b) = g'(b)$ for all $b \in \mathrm{range}(f)$. Let $b \in \mathrm{range}(f)$. Take $a = g'(b)$. Then $g(b) = a$ iff $f(a) = b$. We have $f(a) = b$ iff $g'(b) = a$. Thus $g(b) = g'(b)$. End. ■

Definition 4. Let f be an invertible map. f^{-1} is the inverse of f. Let f is *involutory* stand for f is the inverse of f. Let f is *selfinverse* stand for f is the inverse of f.

Part II
Basic Properties

Proposition 5. Let A, B be classes and $f : A \twoheadrightarrow B$ and $g : B \twoheadrightarrow A$. Then g is the inverse of f iff $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Proof.

Case g is the inverse of f. We have $\mathrm{dom}(g \circ f) = \mathrm{dom}(f) = A = \mathrm{dom}(\mathrm{id}_A)$. For all $a \in A$ we have $(g \circ f)(a) = g(f(a)) = a$. Hence $g \circ f = \mathrm{id}_A$.

We have $\mathrm{dom}(f \circ g) = \mathrm{dom}(g) = B = \mathrm{dom}(\mathrm{id}_B)$. For all $b \in B$ we have $(f \circ g)(b) = f(g(b)) = b$. Hence $f \circ g = \mathrm{id}_B$. □

Case $g \circ f = id_A$ *and* $f \circ g = id_B$. Then $\mathrm{dom}(g) = B = \mathrm{range}(f)$ and $\mathrm{range}(g) = A = \mathrm{dom}(f)$. Let $a \in \mathrm{dom}(f)$ and $b \in \mathrm{dom}(g)$. If $f(a) = b$ then $g(b) = g(f(a)) = (g \circ f)(a) = \mathrm{id}_A(a) = a$. If $g(b) = a$ then $f(a) = f(g(b)) = (f \circ g)(b) = \mathrm{id}_B(b) = b$. Hence $f(a) = b$ iff $g(b) = a$. □

■

Proposition 6. Let A, B be classes and $f : A \twoheadrightarrow B$. Assume that f is invertible. Then f^{-1} is an invertible surjective map from B onto A such that $(f^{-1})^{-1} = f$.

Proof. f^{-1} is a map from B to A. Indeed $\mathrm{range}(f) = B$ and $\mathrm{dom}(f) = A$. f^{-1} is surjective onto A. Indeed for any $a \in A$ we have $f^{-1}(f(a)) = a$. f^{-1} is the inverse of f. Thus $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$. Therefore f is the inverse of f^{-1}. ■

Proposition 7. Let A, B be classes and $f : A \twoheadrightarrow B$. Assume that f is invertible. Then $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$.

Proof. f^{-1} is a surjective map from B onto A . f^{-1} is the inverse of f. ■

Proposition 8. Let A, B be classes and $f : A \twoheadrightarrow B$ and $a \in A$. Assume that f is invertible. Then $f^{-1}(f(a)) = a$.

Proof. We have $f^{-1}(f(a)) = (f^{-1} \circ f)(a) = \mathrm{id}_A(a) = a$. ■

Proposition 9. Let A, B be classes and $f : A \twoheadrightarrow B$ and $b \in B$. Assume that f is invertible. Then $f(f^{-1}(b)) = b$.

Proof. We have $f(f^{-1}(b)) = (f \circ f^{-1})(b) = \mathrm{id}_B(b) = b$. ■

Proposition 10. Let A, B, C be classes and $f : A \twoheadrightarrow B$ and $g : B \twoheadrightarrow C$. Assume that f and g are invertible. Then $g \circ f$ is invertible and $g \circ f^{-1} = f^{-1} \circ g^{-1}$.

Proof. f^{-1} is a surjective map from B onto A. g^{-1} is a surjective map from C onto B. Take $h = f^{-1} \circ g^{-1}$. Then h is a surjective map from C onto A (by surjectivity of composition of surjections). $g \circ f$ is a map from A to C.

Let us show that $((g \circ f) \circ h) = \mathrm{id}_C$. We have $f \circ (f^{-1} \circ g^{-1}) = (f \circ f^{-1}) \circ g^{-1}$. Indeed $f \circ (f^{-1} \circ g^{-1})$ and $(f \circ f^{-1}) \circ g^{-1}$ are maps of C. $f \circ h$ is a map from C to B. Hence

$$
\begin{aligned}
&(g \circ f) \circ h \\
&= g \circ (f \circ h) \\
&= g \circ (f \circ (f^{-1} \circ g^{-1})) \\
&= g \circ ((f \circ f^{-1}) \circ g^{-1}) \\
&= g \circ (\mathrm{id}_B \circ g^{-1}) \\
&= g \circ g^{-1} \\
&= \mathrm{id}_C.
\end{aligned}
$$

End.

Let us show that $h \circ (g \circ f) = \mathrm{id}_A$. We have $(f^{-1} \circ g^{-1}) \circ g = f^{-1} \circ (g^{-1} \circ g)$. $g \circ f$ is a map from A to C. Hence

$$
\begin{aligned}
&h \circ (g \circ f) \\
&= (h \circ g) \circ f \\
&= ((f^{-1} \circ g^{-1}) \circ g) \circ f \\
&= (f^{-1} \circ (g^{-1} \circ g)) \circ f \\
&= (f^{-1} \circ \mathrm{id}_B) \circ f \\
&= f^{-1} \circ f \\
&= \mathrm{id}_A.
\end{aligned}
$$

End.

Thus h is the inverse of $g \circ f$. Indeed $g \circ f$ is a surjective map from A onto C and h is a surjective map from C onto A. ∎

Proposition 11. Let A, B be classes and $f : A \twoheadrightarrow B$ and $X \subset A$. Assume that f is invertible. Then $f \restriction X$ is invertible and $f \restriction X^{-1} = f^{-1} \restriction (f[X])$.

Proof. $f \restriction X$ is a surjective map from X onto $f[X]$. Take $g = f^{-1} \restriction (f[X])$. Then g is a map of $f[X]$.

Let us show that $X \subset \mathrm{range}(g)$. Let $a \in X$. Then $f(a) \in f[X]$. Hence $g(f(a)) = f^{-1}(f(a)) = a$. Thus a is a value of g. End.

Let us show that $\mathrm{range}(g) \subset X$. Let $a \in \mathrm{range}(g)$. Take $b \in f[X]$ such that $a = g(b)$. Take $c \in X$ such that $b = f(c)$. Then $a = (f^{-1} \restriction (f[X]))(b) = f^{-1}(b) = f^{-1}(f(c)) = c$. Hence $a \in X$. End.

Hence $\mathrm{range}(g) = X$. Thus g is a surjective map onto X.

Let us show that $g((f \restriction X)(a)) = a$ for all $a \in X$. Let $a \in X$. Then $g((f \restriction X)(a)) = g(f(a)) = (f^{-1} \restriction (f[X]))(f$
End.

Let us show that $((f \restriction X)(g(b))) = b$ for all $b \in f[X]$. Let $b \in f[X]$. Take $a \in X$ such that $b = f(a)$. We have $g(b) = g(f(a)) = (f^{-1} \restriction (f[X]))(f(a)) = f^{-1}(f(a)) = a$. Hence $(f \restriction X)(g(b)) = (f \restriction X)(a) = f(a) = b$. End.

Thus $g \circ (f \restriction X) = \mathrm{id}_X$ and $(f \restriction X) \circ g = \mathrm{id}_{f[X]}$. Therefore g is the inverse of $f \restriction X$. ∎

Proposition 12. Let A, B be classes and $f : A \twoheadrightarrow B$ and $Y \subset B$. Assume that f is invertible. Then

$$f^{-1}[Y] = f^{-1}[Y].$$

Proof. We have $f^{-1}[Y] = \left\{ f^{-1}(b) \,\middle|\, b \in Y \right\}$ and $f^{-1}[Y] = \{ a{\in}A \mid f(a) \in Y \}$.

Let us show that $f^{-1}[Y] \subset f^{-1}[Y]$. Let $a \in f^{-1}[Y]$. Take $b \in Y$ such that $b = f(a)$. Then $f^{-1}(b) = f^{-1}(f(a)) = a$. Hence $a \in f^{-1}[Y]$. End.

Let us show that $f^{-1}[Y] \subset f^{-1}[Y]$. Let $a \in f^{-1}[Y]$. Take $b \in Y$ such that $a = f^{-1}(b)$. Then $f(a) = f(f^{-1}(b)) = b$. Hence $a \in f^{-1}[Y]$. End. ∎

Corollary 13. Let A, B be classes and $f : A \twoheadrightarrow B$ and $b \in B$. Assume that f is invertible. Then

$$f^{-1}[\{b\}] = \{f^{-1}(b)\}.$$

Proof. $f^{-1}[\{b\}] = f^{-1}[\{b\}]$. We have $f^{-1}[\{b\}] = \{f^{-1}(c) \mid c \in \{b\}\}$. Hence $f^{-1}[\{b\}] = \{f^{-1}(b)\}$. ■

Proposition 14. Let A, B be classes and $f : A \twoheadrightarrow B$. Then f is invertible iff f is injective.

Proof.

Case f is invertible. Let $a, b \in A$. Assume $f(a) = f(b)$. Then $a = f^{-1}(f(a)) = f^{-1}(f(b)) = b$. □

Case f is injective. Define $g(b) :=$ "choose $a \in A$ such that $f(a) = b$ in a" for $b \in B$. Then g is a map from B to A. For all $a \in A$ we have $a = g(f(a))$. Hence g is a surjective map from B onto A. For all $a \in A$ we have $g(f(a)) = a$. For all $b \in B$ we have $f(g(b)) = b$. Hence g is the inverse of f. □

■

Corollary 15. Let A, B be classes and $f : A \twoheadrightarrow B$. Assume that f is invertible. Then f^{-1} is a bijection between B and A.

Proof. f^{-1} is a surjective map from B onto A. f^{-1} is invertible. Hence f^{-1} is injective. Therefore f^{-1} is a bijection between B and A. ■

Part III
Involutions

Definition 16. Let A be a class. An *involution on A* is a selfinverse map f on A.

Proposition 17. Let A be a class. id_A is an involution on A.

Proof. We have $\mathrm{id}_A \circ \mathrm{id}_A = \mathrm{id}_A$. Hence id_A is selfinverse. ∎

Proposition 18. Let A be a class and f, g be involutions on A. Then $g \circ f$ is an involution on A iff $g \circ f = f \circ g$.

Proof.

Case $g \circ f$ is an involution on A. Then $g \circ f^{-1} = f^{-1} \circ g^{-1} = f \circ g$ (by invertibility of composition of invertible maps). Indeed f and g are invertible and surjective onto A. □

Case $g \circ f = f \circ g$. [prover vampire] $f \circ f$, $f \circ g$ and $f \circ g$ are maps on A. Hence

$$\begin{aligned}
&(g \circ f) \circ (g \circ f) \\
&= (g \circ f) \circ (f \circ g) \\
&= ((g \circ f) \circ f) \circ g \\
&= (g \circ (f \circ f)) \circ g \\
&= (g \circ \mathrm{id}_A) \circ g \\
&= g \circ g \\
&= \mathrm{id}_A.
\end{aligned}$$

Thus $g \circ f$ is selfinverse. □

∎

Corollary 19. Let A be a class and f be an involutions on A. Then $f \circ f$ is an involution on A.

Proposition 20. Let A be a class and f be an involution on A. Then f is a permutation of A.

Proof. f is an invertible map of A that surjects onto A. Hence f is a bijection between A and A. Thus f is a permutation of A. ∎