Part I

Surjective Maps

Definition 1. Let f be a map and B be a class. f is *surjective onto* B iff $\mathrm{range}(f) = B$. Let f *surjects onto* B stand for f is surjective onto B. Let a *surjective map onto* B stand for a map that is surjective onto B.

Definition 2. Let A, B be classes. A *surjective map from* A *to* B is a map of A that is surjective onto B. Let a *surjective map from* A *onto* B stand for a surjective map from A to B. Let $f : A \twoheadrightarrow B$ stand for f is a surjective map from A onto B.

Proposition 3. Let B be a class and f be a map to B. f is surjective onto B iff every element of B is a value of f.

Proof.

Case f is surjective onto B. Then $B = \mathrm{range}(f)$. Let b be an element of B. Then $b \in \mathrm{range}(f)$. Hence b is a value of f. □

Case every element of B is a value of f. Let us show that $B \subset \mathrm{range}(f)$. Let $b \in B$. Then b is a value of f. Hence $b \in \mathrm{range}(f)$. End.

Let us show that $\mathrm{range}(f) \subset B$. Let $b \in \mathrm{range}(f)$. Then b is a value of f. Hence $b \in B$. End. □

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Part II

Injective Maps

Definition 4. Let f be a map. f is *injective* iff for all $a, a' \in \mathrm{dom}(f)$ if $f(a) = f(a')$ then $a = a'$. Let $f : A \hookrightarrow B$ stand for f is an injective map from A to B.

Part III
Bijective Maps

Definition 5. Let A, B be classes. A *bijection between A and B* is an injective map from A to B that is surjective onto B. Let a *bijection from A to B* stand for a bijection between A and B.

Proposition 6. Let A, B be classes and $f : A \hookrightarrow B$. Then f is a bijection between A and $\mathrm{range}(f)$.

Proof. f is injective and surjects onto $\mathrm{range}(f)$. Hence f is a bijection between A and $\mathrm{range}(f)$. ■

Definition 7. Let A be a class. A *permutation of A* is a bijection between A and A.

Part IV
Basic Properties

Proposition 8. Let A be a class. Then id_A is a permutation of A.

Proof. (1) id_A is a map on A.

(2) id_A is surjective onto A.

Proof. Let $a \in A$. Then $a = \mathrm{id}_A(a)$. Hence $a \in \mathrm{range}(\mathrm{id}_A)$. □

(3) id_A is injective.

Proof. Let $a, a' \in A$. Assume $\mathrm{id}_A(a) = \mathrm{id}_A(a')$. Then $a = a'$. □ ■

Proposition 9. Let A, B, C be classes and $f : A \twoheadrightarrow B$ and $g : B \twoheadrightarrow C$. Then $g \circ f$ is a surjective map from A onto C.

Proof. $g \circ f$ is a map of A.

Let us show that $g \circ f$ is surjective onto C. Let $c \in C$. Take $b \in B$ such that $c = g(b)$. Take $a \in A$ such that $b = f(a)$. Then $c = g(f(a)) = (g \circ f)(a)$. End. ■

Proposition 10. Let A, B, C be classes and $f : A \hookrightarrow B$ and $g : B \hookrightarrow C$. Then $g \circ f$ is an injective map from A to C.

Proof. $g \circ f$ is a map of A.

Let us show that $g \circ f$ is injective. Let $a, a' \in A$. Assume $(g \circ f)(a) = (g \circ f)(a')$. Then $g(f(a)) = g(f(a'))$. Hence $f(a) = f(a')$. Indeed $f(a), f(a') \in B$. Thus $a = a'$. End. ■

Corollary 11. Let A, B, C be classes. Let f be a bijection between A and B and g be a bijection between B and C. Then $g \circ f$ is a bijection between A and C.

Proof. $g \circ f$ is an injective map from A to C (by injectivity of composition of injections). $g \circ f$ is a surjective map from A onto C (by surjectivity of composition of surjections). ■

Proposition 12. Let A, B be classes and $f : A \hookrightarrow B$ and $X \subset A$. Then $f \restriction X$ is injective.

Proof. Let $a, a' \in X$. Assume $(f \restriction X)(a) = (f \restriction X)(a')$. Then $f(a) = f(a')$. Hence $a = a'$. ■

Proposition 13. Let A, B be classes and $f : A \hookrightarrow B$ and $X \subset A$. Then $f \restriction X$ is a bijection between X and $f[X]$.

Corollary 14. Let A, B be classes and $f : A \hookrightarrow B$. Then f is a bijection between A and $f[A]$.