## Part I

# Finite Classes

**Definition 1.** Let $X$ be a class and $n$ be a natural number. $X$ *has $n$ elements* iff $X$ is equinumerous to $\{1,\ldots,n\}$.

**Definition 2.** Let $X$ be a class. $X$ is *finite* iff there exists a natural number $n$ such that $X$ has $n$ elements.

**Proposition 3.** Let $X, Y$ be classes. If $X$ is finite and $Y$ is equinumerous to $X$ then $Y$ is finite.

*Proof.* Assume that $X$ is finite and $Y$ is equinumerous to $X$. Take a natural number $n$ and a bijection $f$ between $\{1,\ldots,n\}$ and $X$ and a bijection $g$ between $X$ and $Y$. Then $g \circ f$ is a bijection between $\{1,\ldots,n\}$ and $Y$ (by bijectivity of composition of bijections). Indeed $X, Y$ are classes. Hence $Y$ is finite. ■

**Proposition 4.** Let $X$ be a class. $X$ has zero elements iff $X = \emptyset$.

*Proof.*

*Case $X$ has zero elements.* Then $X$ does not contain any element. □

*Case $X = \emptyset$.* Then $X$ does not contain any element. Hence $X = \{1,\ldots,0\}$. □

■

**Proposition 5.** Let $X$ be a class. $X$ has one element iff $X = \{a\}$ for some object $a$.

*Proof.*

*Case $X$ has one element.* Take a bijection $f$ between $\{1,\ldots,1\}$ and $X$. We have $\{1,\ldots,1\} = \{1\}$. Hence $X = \{f(1)\}$. $\square$

*Case $X = \{a\}$ for some object $a$.* Consider an object $a$ such that $X = \{a\}$. Define $f(x) := 1$ for $x \in \{a\}$. Then $f$ is a bijection between $\{a\}$ and $\{1,\ldots,1\}$. $\square$

$\blacksquare$

**Proposition 6.** Let $X$ be a class. $X$ has two elements iff $X = \{a,b\}$ for some distinct objects $a, b$.

*Proof.*

*Case $X$ has two elements.* Take a bijection $f$ between $\{1,\ldots,2\}$ and $X$. We have $\{1,\ldots,2\} = \{1,2\}$. Hence $X = \{f(1),f(2)\}$. We have $f(1) \neq f(2)$. $\square$

*Case $X = \{a,b\}$ for some distinct objects $a, b$.* Consider distinct objects $a, b$ such that $X = \{a,b\}$. Define

$$f(x) := \begin{cases} 1 & : x = a \\ 2 & : x = b \end{cases}$$

for $x \in \{a,b\}$. Then $f$ is a bijection between $\{a,b\}$ and $\{1,\ldots,2\}$. Indeed $f$ is injective and surjective onto $\{1,\ldots,2\}$. $\square$

$\blacksquare$

**Proposition 7.** Let $n$ be a natural number and $X$ be a class that has $n$ elements and $a$ be an object such that $a \notin X$. Then $X \cup \{a\}$ has $n+1$ elements.

*Proof.* Take a bijection $f$ between $X$ and $\{1,\ldots,n\}$. Define

$$g(x) := \begin{cases} f(x) & : x \in X \\ n+1 & : x = a \end{cases}$$

for $x \in X \cup \{a\}$.

(1) $g$ is a map from $X \cup \{a\}$ to $\{1, \ldots, n+1\}$. Indeed we can show that $g(x) \in \{1, \ldots, n+1\}$ for all $x \in X \cup \{a\}$. Let $x \in X \cup \{a\}$. If $x \in X$ then $g(x) \in \{1, \ldots, n\}$. If $x = a$ then $g(x) = n+1$. End.

(2) $g$ is injective.

*Proof.* Let $x, y \in \mathrm{dom}(g)$. Assume $x \neq y$.

*Case $x, y \in X$.* □

*Case $x \in X$ and $y = a$.* □

*Case $x = a$ and $y \in X$.* □

□

(3) $g$ is surjective onto $\{1, \ldots, n+1\}$. Indeed we can show that for every $k \in \{1, \ldots, n+1\}$ there exists an $x \in \mathrm{dom}(g)$ such that $k = g(x)$.

*Proof.* Let $k \in \{1, \ldots, n+1\}$.

*Case $k \leq n$.* Then $k \in \{1, \ldots, n\}$. Hence we can take a $x \in X$ such that $k = f(x)$. □

*Case $k = n+1$.* Then $k = g(a)$. □

□

Hence $g$ is a bijection between $X \cup \{a\}$ and $\{1, \ldots, n+1\}$. ■

# Part II
# Infinite Classes

**Definition 8.** Let $X$ be a class. $X$ is *infinite* iff $X$ is not finite.

**Proposition 9.** Let $X, Y$ be classes. If $X$ is infinite and $Y$ is equinumerous to $X$ then $Y$ is infinite.

*Proof.* Assume that $Y$ is equinumerous to $X$. If $Y$ is finite then $X$ is finite. Hence if $X$ is infinite then $Y$ is infinite. ■