Part I

Commutativity of Union and Intersection

Proposition 1. Let A, B be classes. Then $A \cup B = B \cup A$.

Proof. Let us show that $A \cup B \subset B \cup A$. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. Hence $x \in B$ or $x \in A$. Thus $x \in B \cup A$. End.

Let us show that $B \cup A \subset A \cup B$. Let $x \in B \cup A$. Then $x \in B$ or $x \in A$. Hence $x \in A$ or $x \in B$. Thus $x \in A \cup B$. End. ∎

Proposition 2. Let A, B be classes. Then $A \cap B = B \cap A$.

Proof. Let us show that $A \cap B \subset B \cap A$. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Hence $x \in B$ and $x \in A$. Thus $x \in B \cap A$. End.

Let us show that $B \cap A \subset A \cap B$. Let $x \in B \cap A$. Then $x \in B$ and $x \in A$. Hence $x \in A$ and $x \in B$. Thus $x \in A \cap B$. End. ∎

Part II

Associativity of Union and Intersection

Proposition 3. Let A, B, C be classes. Then $(A \cup B) \cup C = A \cup (B \cup C)$.

Proof. Let us show that $((A \cup B) \cup C) \subset A \cup (B \cup C)$. Let $x \in (A \cup B) \cup C$. Then $x \in A \cup B$ or $x \in C$. Hence $x \in A$ or $x \in B$ or $x \in C$. Thus $x \in A$ or $x \in (B \cup C)$. Therefore $x \in A \cup (B \cup C)$. End.

Let us show that $A \cup (B \cup C) \subset (A \cup B) \cup C$. Let $x \in A \cup (B \cup C)$. Then $x \in A$ or $x \in B \cup C$. Hence $x \in A$ or $x \in B$ or $x \in C$. Thus $x \in A \cup B$ or $x \in C$. Therefore $x \in (A \cup B) \cup C$. End. ∎

Proposition 4. Let A, B, C be classes. Then $(A \cap B) \cap C = A \cap (B \cap C)$.

Proof. Let us show that $((A \cap B) \cap C) \subset A \cap (B \cap C)$. Let $x \in (A \cap B) \cap C$. Then $x \in A \cap B$ and $x \in C$. Hence $x \in A$ and $x \in B$ and $x \in C$. Thus $x \in A$ and $x \in (B \cap C)$. Therefore$x \in A \cap (B \cap C)$. End.

Let us show that $A \cap (B \cap C) \subset (A \cap B) \cap C$. Let $x \in A \cap (B \cap C)$. Then $x \in A$ and $x \in B \cap C$. Hence $x \in A$ and $x \in B$ and $x \in C$. Thus $x \in A \cap B$ and $x \in C$. Therefore $x \in (A \cap B) \cap C$. End. ∎

Part III
Distributivity of Union and Intersection

Proposition 5. Let A, B, C be classes. Then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. Let us show that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Hence $x \in A$ and ($x \in B$ or $x \in C$). Thus ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$). Therefore $x \in A \cap B$ or $x \in A \cap C$. Hence $x \in (A \cap B) \cup (A \cap C)$. End.

Let us show that $((A \cap B) \cup (A \cap C)) \subset A \cap (B \cup C)$. Let $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$. Hence ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$). Thus $x \in A$ and ($x \in B$ or $x \in C$). Therefore $x \in A$ and $x \in B \cup C$. Hence$x \in A \cap (B \cup C)$. End. ∎

Proposition 6. Let A, B, C be classes. Then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof. Let us show that $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$. Let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. Hence $x \in A$ or ($x \in B$ and $x \in C$). Thus ($x \in A$ or $x \in B$) and ($x \in$ A or $x \in C$). Therefore $x \in A \cup B$ and $x \in A \cup C$. Hence $x \in (A \cup B) \cap (A \cup C)$. End.

Let us show that $((A \cup B) \cap (A \cup C)) \subset A \cup (B \cap C)$. Let $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. Hence ($x \in A$ or $x \in B$) and ($x \in A$ or

$x \in C$). Thus $x \in A$ or ($x \in B$ and $x \in C$). Therefore $x \in A$ or $x \in B \cap C$. Hence $x \in A \cup (B \cap C)$. End. ∎

Part IV

Idempocy Laws for Union and Intersection

Proposition 7. Let A be a class. Then $A \cup A = A$.

Proof. $A \cup A = \{x \,|\, x \in A \text{ or } x \in A\}$. Hence $A \cup A = \{x \,|\, x \in A\}$. Thus $A \cup A = A$. ∎

Proposition 8. Let A be a class. Then $A \cap A = A$.

Proof. $A \cap A = \{x \,|\, x \in A \text{ and } x \in A\}$. Hence $A \cap A = \{x \,|\, x \in A\}$. Thus $A \cap A = A$. ∎

Part V

Distributivity of Complement

Proposition 9. Let A, B, C be classes. Then $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

Proof. Let us show that $A \setminus (B \cap C) \subset (A \setminus B) \cup (A \setminus C)$. Let $x \in A \setminus (B \cap C)$. Then $x \in A$ and $x \notin B \cap C$. Hence it is wrong that ($x \in B$ and $x \in C$). Thus $x \notin B$ or $x \notin C$. Therefore $x \in A$ and ($x \notin B$ or $x \notin C$). Then ($x \in A$ and $x \notin B$) or ($x \in A$ and $x \notin C$). Hence $x \in A \setminus B$ or $x \in A \setminus C$. Thus $x \in (A \setminus B) \cup (A \setminus C)$. End.

Let us show that $((A \setminus B) \cup (A \setminus C)) \subset A \setminus (B \cap C)$. Let $x \in (A \setminus B) \cup (A \setminus C)$. Then $x \in A \setminus B$ or $x \in A \setminus C$. Hence ($x \in A$ and $x \notin B$) or ($x \in A$ and

$x \notin C$). Thus $x \in A$ and ($x \notin B$ or $x \notin C$). Therefore $x \in A$ and $\neg\,(x \in B \wedge x \in C)$. Then $x \in A$ and $x \notin B \cap C$. Hence $x \in A \setminus (B \cap C)$. End. ∎

Proposition 10. Let A, B, C be classes. Then $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Proof. Let us show that $A \setminus (B \cup C) \subset (A \setminus B) \cap (A \setminus C)$. Let $x \in A \setminus (B \cup C)$. Then $x \in A$ and $x \notin B \cup C$. Hence it is wrong that ($x \in B$ or $x \in C$). Thus $x \notin B$ and $x \notin C$. Therefore $x \in A$ and ($x \notin B$ and $x \notin C$). Then ($x \in A$ and $x \notin B$) and ($x \in A$ and $x \notin C$). Hence $x \in A \setminus B$ and $x \in A \setminus C$. Thus $x \in (A \setminus B) \cap (A \setminus C)$. End.

Let us show that $((A \setminus B) \cap (A \setminus C)) \subset A \setminus (B \cup C)$. Let $x \in (A \setminus B) \cap (A \setminus C)$. Then $x \in A \setminus B$ and $x \in A \setminus C$. Hence ($x \in A$ and $x \notin B$) and ($x \in A$ and $x \notin C$). Thus $x \in A$ and ($x \notin B$ and $x \notin C$). Therefore $x \in A$ and $\neg\,(x \in B \vee x \in C)$. Then $x \in A$ and $x \notin B \cup C$. Hence $x \in A \setminus (B \cup C)$. End. ∎

Part VI

Subclass Laws

Proposition 11. Let A, B be classes. Then $A \subset A \cup B$.

Proof. Let $x \in A$. Then $x \in A$ or $x \in B$. Hence $x \in A \cup B$. ∎

Proposition 12. Let A, B be classes. Then $A \cap B \subset A$.

Proof. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Hence $x \in A$. ∎

Proposition 13. Let A, B be classes. Then $A \subset B$ iff $A \cup B = B$.

Proof.

Case $A \subset B$.

Let us show that $A \cup B \subset B$. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$ then $x \in B$. Hence $x \in B$. End.

Let us show that $B \subset A \cup B$. Let $x \in B$. Then $x \in A$ or $x \in B$. Hence $x \in A \cup B$. End. $\square$

Case $A \cup B = B$. Let $x \in A$. Then $x \in A$ or $x \in B$. Hence $x \in A \cup B = B$. $\square$

$\blacksquare$

Proposition 14. Let A, B be classes. Then $A \subset B$ iff $A \cap B = A$.

Proof.

Case $A \subset B$.

Let us show that $A \cap B \subset A$. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Hence $x \in A$. End.

Let us show that $A \subset A \cap B$. Let $x \in A$. Then $x \in B$. Hence $x \in A$ and $x \in B$. Thus $x \in A \cap B$. End. $\square$

Case $A \cap B = A$. Let $x \in A$. Then $x \in A \cap B$. Hence $x \in A$ and $x \in B$. Thus $x \in B$. $\square$

$\blacksquare$

Part VII
Complement Laws

Proposition 15. Let A be a class. Then $A \setminus A = \emptyset$.

Proof. $A \setminus A$ has no elements. Indeed $A \setminus A = \{x \mid x \in A \text{ and } x \notin A\}$. Hence the thesis. $\blacksquare$

Proposition 16. Let A be a class. Then $A \setminus \emptyset = A$.

Proof. $A \setminus \emptyset = \{x \mid x \in A \text{ and } x \notin \emptyset\}$. No element is an element of $\emptyset$. Hence $A \setminus \emptyset = \{x \mid x \in A\}$. Then we have the thesis. ■

Proposition 17. Let A, B be classes. Then $A \setminus (A \setminus B) = A \cap B$.

Proof. Let us show that $A \setminus (A \setminus B) \subset A \cap B$. Let $x \in A \setminus (A \setminus B)$. Then $x \in A$ and $x \notin A \setminus B$. Hence $x \notin A$ or $x \in B$. Thus $x \in B$. Therefore $x \in A \cap B$. End.

Let us show that $A \cap B \subset A \setminus (A \setminus B)$. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Hence $x \notin A$ or $x \in B$. Thus $x \notin A \setminus B$. Therefore $x \in A \setminus (A \setminus B)$. End. ■

Proposition 18. Let A, B be classes. Then $B \subset A$ iff $A \setminus (A \setminus B) = B$.

Proof.

Case $B \subset A$. □

Case $A \setminus (A \setminus B) = B$. Then every element of B is an element of $A \setminus (A \setminus B)$. Thus every element of B is an element of A. Then we have the thesis. □

■

Proposition 19. Let A, B, C be classes. Then $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.

Proof. Let us show that $A \cap (B \setminus C) \subset (A \cap B) \setminus (A \cap C)$. Let $x \in A \cap (B \setminus C)$. Then $x \in A$ and $x \in B \setminus C$. Hence $x \in A$ and $x \in B$. Thus $x \in A \cap B$ and $x \notin C$. Therefore $x \notin A \cap C$. Then we have $x \in (A \cap B) \setminus (A \cap C)$. End.

Let us show that $((A \cap B) \setminus (A \cap C)) \subset A \cap (B \setminus C)$. Let $x \in (A \cap B) \setminus (A \cap C)$. Then $x \in A$ and $x \in B$. $x \notin A \cap C$. Hence $x \notin C$. Thus $x \in B \setminus C$. Therefore $x \in A \cap (B \setminus C)$. End. ■