Definition 1. Let X be a system of sets. X is *closed under finite intersections* iff $\bigcap U \in X$ for every nonempty finite subclass U of X.

Proposition 2. Let X be a system of sets. X is closed under finite intersections iff $U \cap V \in X$ for every $U, V \in X$.

Proof.

Case X is closed under finite intersections. Let $U, V \in X$. Then $\{U,V\}$ is a nonempty finite subclass of X. Hence $U \cap V = \bigcap\{U,V\} \in X$. □

Case $U \cap V \in X$ for every $U, V \in X$. Define $\Phi := \{n{\in}\mathbb{N} \mid \bigcap U \in X$ for every nonempty subclass U of X th

(1) Φ contains 0.

(2) For every $n \in \Phi$ we have $n+1 \in \Phi$.

Proof. Let $n \in \Phi$. Then $\bigcap U \in X$ for every nonempty subclass U of X that has n elements.

Let us show that $\bigcap U \in X$ for every nonempty subclass U of X that has $n+1$ elements.

Case $n = 0$. □

Case $n \neq 0$. Let U be a nonempty subclass of X such that U has $n+1$ elements. Take a bijection f between $\{1,\ldots,n+1\}$ and U. We have $\{1,\ldots,n+1\} = \{1,\ldots,n\} \cup \{n+1\}$. Take $V = f[\{1,\ldots,n\}]$. We have $\{1,\ldots,n\} \subset \{1,\ldots,n+1\}$.

Let us show that $V \subset U$. Let $x \in V$. Take $k \in \{1,\ldots,n\}$ such that $x = f(k)$. Indeed we can show that there exists a $k \in \{1,\ldots,n\}$ such that $x = f(k)$. Assume the contrary. Then $x \neq f(k)$ for all $k \in \{1,\ldots,n\}$. Hence $x \notin f[\{1,\ldots,n\}] = V$. Contradiction. End. Hence $x \in U$. Indeed $x \in f[\{1,\ldots,n+1\}]$. End.

V is nonempty. Indeed $f(1) \in f[\{1,\ldots,n\}]$. Indeed $1 \in \{1,\ldots,n\}$. Hence $f \restriction \{1,\ldots,n\}$ is a bijection between $\{1,\ldots,n\}$ and V (by bijectivity of restriction of injection). Thus V has n elements. Consequently $\bigcap V \in X$.

Let us show that $U = V \cup \{f(n+1)\}$.

(1) $f[A \cup B] = f[A] \cup f[B]$ for all $A, B \subset \mathrm{dom}(f)$.

(2) $f[\{a\}] = \{f(a)\}$ for all $a \in \mathrm{dom}(f)$.

Hence $U = f[\mathrm{dom}(f)] = f[\{1,\ldots,n+1\}] = f[\{1,\ldots,n\} \cup \{n+1\}] = f[\{1,\ldots,n\}] \cup f[\{n+1\}] = f[\{1,\ldots$

Indeed $n+1 \in \mathrm{dom}(f)$ and $\{1,\ldots,n\}, \{1+n\} \subset \mathrm{dom}(f)$. End.

Let us show that $\bigcap(A \cup B) = (\bigcap A) \cap (\bigcap B)$ for any nonempty systems of sets A, B. Let A, B be nonempty systems of sets. $\bigcap(A \cup B) \subset (\bigcap A) \cap (\bigcap B)$. $((\bigcap A) \cap (\bigcap B)) \subset \bigcap(A \cup B)$. End.

Let us show that $f(n+1)$ and $f(k)$ are sets for every $k \in \{1, \ldots, n\}$. Let $k \in \{1, \ldots, n\}$. Then $f(k), f(n+1) \in U$. Hence $f(k)$ and $f(n+1)$ are sets. End.

Hence V and $\{f(n+1)\}$ are nonempty systems of sets. Thus $\bigcap U = \bigcap(V \cup \{f(n+1)\}) = (\bigcap V) \cap (\bigcap \{f($

$\square$

End. $\square$

Therefore Φ contains every natural number (by induction). Thus $\bigcap U \in X$ for every nonempty finite subclass U of X. Consequently X is closed under finite intersections. $\square$

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