Part I

Sub- and Superclasses

Definition 1. Let A be a class. A *subclass of A* is a class B such that every element of B is an element of A. Let $B \subset A$ stand for B is a subclass of A. Let a *superclass of B* stand for a class A such that $B \subset A$. Let $B \supset A$ stand for B is a superclass of A. Let a *proper subclass of A* stand for a subclass B of A such that $B \neq A$. Let $B \subsetneq A$ stand for B is a proper subclass of A. Let a *proper superclass of B* stand for a superclass A of B such that $A \neq B$. Let $B \supsetneq A$ stand for B is a proper superclass of A. Let *A includes B* stand for $B \subset A$. Let B is *included in A* stand for $B \subset A$.

Proposition 2. Let A be a class. Then $A \subset A$.

Proof. Every element of A is contained in A. Therefore $A \subset A$. ∎

Proposition 3. Let A, B, C be classes. If $A \subset B$ and $B \subset C$ then $A \subset C$.

Proof. Assume $A \subset B$ and $B \subset C$. Then every element of A is contained in B and every element of B is contained in C. Hence every element of A is contained in C. Thus $A \subset C$. ∎

Proposition 4. Let A, B be classes. If $A \subset B$ and $B \subset A$ then $A = B$.

Proof. Assume $A \subset B$ and $B \subset A$. Then every element of A is contained in B and every element of B is contained in A. Hence $A = B$. ∎

Part II

The Empty Class

Definition 5. Let A be a class. A is *empty* iff A has no elements. Let A is *nonempty* stand for A is not empty.

Definition 6. $\emptyset := \{x \mid x \neq x\}$.

Proposition 7. Let A be a class. A is empty iff $A = \emptyset$.

Proof. We can show that $\emptyset$ is empty. Indeed any element x of $\emptyset$ is nonequal to x. Hence if $A = \emptyset$ then A is empty. If A is empty then A and $\emptyset$ have no elements. Hence if A is empty then $A \subset \emptyset$ and $\emptyset \subset A$. Thus if A is empty then $A = \emptyset$. ■

Corollary 8. $\emptyset$ is empty.

Corollary 9. Let A be a class. Then $\emptyset \subset A$.

Proof. $\emptyset$ has no elements. Hence every element of $\emptyset$ is contained in A. ■

Part III
Unordered Pairs

Definition 10. Let a, b be objects. $\{a,b\} := \{x \mid x = a \text{ or } x = b\}$. Let the *unordered pair of a and b* stand for $\{a,b\}$.

Definition 11. An *unordered pair* is a class A such that $A = \{a,b\}$ for some distinct objects a, b.

Definition 12. Let a be an object. $\{a\} := \{x \mid x = a\}$. Let the *singleton class of a* stand for $\{a\}$.

Definition 13. A *singleton class* is a class A such that $A = \{a\}$ for some object a.

Proposition 14. Let a, a', b, b' be objects. Assume $\{a,b\} = \{a',b'\}$. Then ($a = a'$ and $b = b'$) or ($a = b'$ and $b = a'$).

Proof. We have $a = a'$ or $a = b'$. If $a = a'$ then $b = b'$. If $a = b'$ then $b = a'$. Hence ($a = a'$ and $b = b'$) or ($a = b'$ and $b = a'$). ■

Corollary 15. Let a, a' be objects. If $\{a\} = \{a'\}$ then $a = a'$.

Definition 16. Let A be a class. A *unique element of A* is an element a of A such that for each $x \in A$ we have $x = a$.

Proposition 17. Let A be a class. Then A has a unique element iff $A = \{a\}$ for some object a.

Part IV
Unions, Intersections, Complements

Definition 18. Let A, B be classes. $A \cup B := \{x \mid x \in A \text{ or } x \in B\}$. Let the *union of A and B* stand for $A \cup B$.

Definition 19. Let A, B be classes. $A \cap B := \{x \mid x \in A \text{ and } x \in B\}$. Let the *intersection of A and B* stand for $A \cap B$.

Definition 20. Let A, B be classes. $A \setminus B := \{x \mid x \in A \text{ and } x \notin B\}$. Let the *complement of B in A* stand for $A \setminus B$.

Part V

Disjoint Classes

Definition 21. Let A, B be classes. A and B are *disjoint* iff A and B have no common elements.

Proposition 22. Let A, B be classes. Then A and B are disjoint iff $A \cap B$ is empty.