Part I

Definitions and Immediate Consequences

Definition 1. Let n, m be natural numbers. $n < m$ iff there exists a nonzero natural number k such that $m = n + k$. Let n is *less than m* stand for $n < m$. Let $n > m$ stand for $m < n$. Let n is *greater than m* stand for $n > m$. Let $n \not< m$ stand for n is not less than m. Let $n \not> m$ stand for n is not greater than m.

Definition 2. Let n be a natural number. $\mathbb{N}_{<n} := \{k \in \mathbb{N} \mid k < n\}$.

Definition 3. Let n be a natural number. $[\mathbb{N}]_{>n} := \{k \in \mathbb{N} \mid k > n\}$.

Definition 4. Let n be a natural number. n is *positive* iff $n > 0$.

Definition 5. Let n, m be natural numbers. $n \leq m$ iff there exists a natural number k such that $m = n + k$. Let n is *less than or equal to m* stand for $n \leq m$. Let $n \geq m$ stand for $m \leq n$. Let n is *greater than or equal to m* stand for $n \geq m$. Let $n \not\leq m$ stand for n is not less than or equal to m. Let $n \not\geq m$ stand for n is not greater than or equal to m.

Definition 6. Let n be a natural number. $\mathbb{N}_{\leq n} := \{k \in \mathbb{N} \mid k \leq n\}$.

Definition 7. Let n be a natural number. $\mathbb{N}_{\geq n} := \{k \in \mathbb{N} \mid k \geq n\}$.

Proposition 8. Let n, m be natural numbers. $n \leq m$ iff $n < m$ or $n = m$.

Proof.

Case $n \leq m$. Take a natural number k such that $m = n + k$. If $k = 0$ then $n = m$. If $k \neq 0$ then $n < m$. □

Case $n < m$ or $n = m$. If $n < m$ then there is a positive natural number k such that $m = n + k$. If $n = m$ then $m = n + 0$. Thus if $n < m$ then there is a natural number k such that $m = n + k$. □

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Definition 9. Let n be a natural number. A *predecessor of n* is a natural number that is less than n.

Definition 10. Let n be a natural number. A *successor of n* is a natural number that is greater than n.

Proposition 11. Let n be a natural number. Then n is positive iff n is nonzero.

Proof.

Case n is positive. Take a positive natural number k such that $n = 0 + k = k$. Then we have $n \neq 0$. □

Case n is nonzero. Take a natural number k such that $n = k + 1$. Then $n = 0 + (k + 1)$. $k + 1$ is positive. Hence $0 < n$. □

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Part II
Basic Properties

Proposition 12. Let n be a natural number. Then

$$n \not< n.$$

Proof. Assume the contrary. Then we can take a positive natural number k such that $n = n + k$. Then we have $0 = k$. Contradiction. ■

Proposition 13. Let n, m be natural numbers. Then

$$n < m \Longrightarrow n \neq m.$$

Proof. Assume $n < m$. Take a positive natural number k such that $m = n + k$. If $n = m$ then $k = 0$. Hence $n \neq m$. ■

Proposition 14. Let n, m be natural numbers. If $n \leq m$ and $m \leq n$ then $n = m$.

Proof. Assume $n \leq m$ and $m \leq n$. Take natural numbers k, l such that $m = n + k$ and $n = m + l$. Then $m = n + k = (m + l) + k = m + (l + k)$. Hence $l + k = 0$. Thus $l = 0 = k$. Indeed if $l \neq 0$ or $k \neq 0$ then $l + k$ is the direct successor of some natural number. Therefore $m = n$. ■

Proposition 15. Let n, m, k be natural numbers. If $n < m < k$ then $n < k$.

Proof. Assume $n < m < k$. Take a positive natural number a such that $m = n + a$. Take a positive natural number b such that $k = m + b$. Then $k = m + b = (n + a) + b = n + (a + b)$. $a + b$ is positive. Hence $n < k$. ■

Proposition 16. Let n, m, k be natural numbers. If $n \leq m \leq k$ then $n \leq k$.

Proof. Assume $n \leq m \leq k$.

Case $n = m = k$. □

Case $n = m < k$. □

Case $n < m = k$. $\square$

Case $n < m < k$. $\square$

$\blacksquare$

Proposition 17. Let n, m, k be natural numbers. If $n \leq m < k$ then $n < k$.

Proof. Assume $n \leq m < k$. If $n = m$ then $n < k$. If $n < m$ then $n < k$. $\blacksquare$

Proposition 18. Let n, m, k be natural numbers. If $n < m \leq k$ then $n < k$.

Proof. Assume $n < m \leq k$. If $m = k$ then $n < k$. If $m < k$ then $n < k$. $\blacksquare$

Proposition 19. Let n, m be natural numbers. If $n < m$ then $n + 1 \leq m$.

Proof. Assume $n < m$. Take a positive natural number k such that $m = n + k$.

Case $k = 1$. Then $m = n + 1$. Hence $n + 1 \leq m$. $\square$

Case $k \neq 1$. Then we can take a natural number l such that $k = l + 1$. Then $m = n + (l + 1) = (n + l) + 1 = (n + 1) + l$. l is positive. Thus $n + 1 < m$. $\square$

$\blacksquare$

Proposition 20. Let n, m be natural numbers. Then $n < m$ or $n = m$ or $n > m$.

Proof. Define $\Phi := \{m' \in \mathbb{N} \mid n < m' \text{ or } n = m' \text{ or } n > m'\}$.

(1) Φ contains 0.

(2) For all $m' \in \Phi$ we have $m' + 1 \in \Phi$.

Proof. Let $m' \in \Phi$.

Case $n < m'$. $\square$

Case $n = m'$. $\square$

Case $n > m'$. Take a positive natural number k such that $n = m' + k$.

Case $k = 1$. $\square$

Case $k \neq 1$. Take a natural number l such that $n = (m' + 1) + l$. Hence $n > m' + 1$. Indeed l is positive. $\square$

$\square$

$\square$

Thus every natural number is contained in Φ (by induction). Therefore $n < m$ or $n = m$ or $n > m$. ■

Proposition 21. Let n, m be natural numbers. Then $n \not< m$ iff $n \geq m$.

Proof.

Case $n \not< m$. Then $n = m$ or $n > m$. Hence $n \geq m$. $\square$

Case $n \geq m$. Assume $n < m$. Then $n \leq m$. Hence $n = m$. Contradiction. $\square$

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Part III
Ordering and Successors

Proposition 22. Let n, m be natural numbers. If $n < m \leq n + 1$ then $m = n + 1$.

Proof. Assume $n < m \leq n + 1$. Take a positive natural number k such that $m = n + k$. Take a natural number l such that $n + 1 = m + l$. Then $n + 1 = m + l = (n + k) + l = n + (k + l)$. Hence $k + l = 1$.

We have $l = 0$.

Proof. Assume the contrary. Then $k, l > 0$.

Case $k, l = 1$. Then $k + l = 2 \neq 1$. Contradiction. □

Case $k = 1 and l \neq 1$. Then $l > 1$. Hence $k + l > 1 + l > 1$. Contradiction. □

Case $k \neq 1 and l = 1$. Then $k > 1$. Hence $k + l > k + 1 > 1$. Contradiction. □

Case $k, l \neq 1$. Take natural numbers a, b such that $k = a + 1$ and $l = b + 1$. Indeed $k, l \neq 0$. Hence $k = a + 1$ and $l = b + 1$. Thus $k, l > 1$. Indeed a, b are positive. □

□

Then we have $n + 1 = m + l = m + 0 = m$. ■

Proposition 23. Let n, m be natural numbers. If $n \leq m < n + 1$ then $n = m$.

Proof. Assume $n \leq m < n + 1$.

Case $n = m$. □

Case $n < m$. Then $n < m \leq n + 1$. Hence $n = m$. □

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Corollary 24. Let n be a natural number. There is no natural number m such that $n < m < n + 1$.

Proof. Assume the contrary. Take a natural number m such that $n < m < n + 1$. Then $n < m \leq n + 1$ and $n \leq m < n + 1$. Hence $m = n + 1$ and $m = n$. Hence $n = n + 1$. Contradiction. ■

Proposition 25. Let n be a natural number. Then $n + 1 \geq 1$.

Proof.

Case $n = 0$. $\square$

Case $n \neq 0$. Then $n > 0$. Hence $n + 1 > 0 + 1 = 1$. $\square$

$\blacksquare$

Part IV
Ordering and Addition

Proposition 26. Let n, m, k be natural numbers. Then $n < m$ iff $n + k < m + k$.

Proof.

Case $n < m$. Take a positive natural number l such that $m = n + l$. Then $m + k = (n + l) + k = (n + k) + l$. Hence $n + k < m + k$. $\square$

Case $n + k < m + k$. Take a positive natural number l such that $m + k = (n + k) + l$. $((n + k) + l) = n + (k + l) = n + (l + k) = (n + l) + k$. Hence $m + k = (n + l) + k$. Thus $m = n + l$ (by right-cancellability of addition). Therefore $n < m$. $\square$

$\blacksquare$

Corollary 27. Let n, m, k be natural numbers. Then $n < m$ iff $k + n < k + m$.

Proof. We have $k + n = n + k$ and $k + m = m + k$. Hence $k + n < k + m$ iff $n + k < m + k$. $\blacksquare$

Corollary 28. Let n, m, k be natural numbers. Then $n \leq m$ iff $k + n \leq k + m$.

Corollary 29. Let n, m, k be natural numbers. Then $n \leq m$ iff $n + k \leq m + k$.

Part V

Induction Revisited

Proposition 30. Let A be a nonempty subclass of $\mathbb{N}$. Then there exists a $m \in A$ such that $m \leq n$ for all $n \in A$.

Proof. Assume the contrary.

Let us show that for each $n \in A$ there exists a $m \in A$ such that $m < n$. Let $n \in A$. Assume that there exists no $m \in A$ such that $m < n$. Then $n \leq m$ for all $m \in A$. Contradiction. End.

(a) Define $\Phi := \{n{\in}\mathbb{N} \mid n \text{ is less than any element of } A\}$.

(1) Φ contains 0.

Proof. $0 \notin A$. Hence 0 is less than every element of A. Thus $0 \in \Phi$. □

(2) For all $n \in \Phi$ we have $n+1 \in \Phi$.

Proof. Let $n \in \Phi$. Then n is less than any element of A. Assume that Φ does not contain $n+1$. Then we can take an $m \in A$ such that $n+1 \not< m$ (by a). Then $n < m \leq n+1$. Hence $m = n+1$. Contradiction. □

Then Φ contains every natural number (by induction). Therefore every natural number is less than any element of A. Consequently A has no elements. Contradiction. ■

Theorem 31. Let A be a class. Assume for all $n \in \mathbb{N}$ if A contains all predecessors of n then A contains n. Then A contains every natural number.

Proof. Assume the contrary. Take a natural number n that is not contained in A. Then n is contained in $\mathbb{N} \setminus A$. Hence we can take a $m \in \mathbb{N} \setminus A$ such that $m \leq k$ for all $k \in \mathbb{N} \setminus A$. Then $\mathbb{N} \setminus A$ does not contain any predecessor of m. Therefore A contains all predecessors of m. Consequently A contains m. Contradiction. ■

Theorem 32. Let A be a class. Let k be a natural number such that $k \in A$. Assume that for all $n \in \mathbb{N}_{\geq k}$ if $n \in A$ then $n+1 \in A$. Then for all $n \in \mathbb{N}_{\geq k}$ we have $n \in A$.

Proof. Define $\Phi := \{n \in \mathbb{N} \mid \text{ if } n \geq k \text{ then } n \in A\}$.

(1) Φ contains 0. Indeed if $0 \geq k$ then $0 = k \in A$.

(2) For all $n \in \Phi$ we have $n + 1 \in \Phi$.

Proof. Let $n \in \Phi$.

Let us show that if $n + 1 \geq k$ then $n + 1 \in A$. Assume $n + 1 \geq k$.

Case $n < k$. Then $n + 1 = k$. Hence $n + 1 \in A$. $\square$

Case $n \geq k$. Then $n \in A$. Hence $n + 1 \in A$. $\square$

End.

Therefore $n + 1 \in \Phi$. $\square$

Thus Φ contains every natural number (by induction). Consequently for all $n \in \mathbb{N}_{\geq k}$ we have $n \in A$. $\blacksquare$