Part I
The Language of Natural Number Arithmetic

Signature 1. A *natural number* is an object.

Definition 2. $\mathbb{N}$ is the class of natural numbers.

Signature 3. Let n, m be natural numbers. $n + m$ is a natural number. Let the *sum of n and m* stand for $n + m$.

Signature 4. 0 is a natural number. Let *zero* stand for 0. Let n is *nonzero* stand for $n \neq 0$.

Signature 5. 1 is a natural number. Let *one* stand for 1. Let the *direct successor of n* stand for $n + 1$.

Definition 6. $2 := 1 + 1$. Let *two* stand for 2.

Definition 7. $3 := 2 + 1$. Let *three* stand for 3.

Definition 8. $4 := 3 + 1$. Let *four* stand for 4.

Definition 9. $5 := 4 + 1$. Let *five* stand for 5.

Definition 10. $6 := 5 + 1$. Let *six* stand for 6.

Definition 11. $7 := 6+1$. Let *seven* stand for 7.

Definition 12. $8 := 7+1$. Let *eight* stand for 8.

Definition 13. $9 := 8+1$. Let *nine* stand for 9.

Part II
The Axioms of Natural Number Arithmetic

Axiom 14. Let n, m be natural numbers. If $n+1 = m+1$ then $n = m$.

Axiom 15. There exists no natural number n such that $n+1 = 0$.

Axiom 16 (Induction). Let Φ be a class. Assume $0 \in \Phi$ and for all natural numbers n if $n \in \Phi$ then $n+1 \in \Phi$. Then Φ contains every natural number.

Axiom 17. Then $1 = 0+1$.

Axiom 18. Let n be a natural number. Then $n+0 = n$.

Axiom 19. Let n, m be natural numbers. Then $n+(m+1) = (n+m)+1$.

Part III
Immediate Consequences of the Axioms

Proposition 20. Let n be a natural number. Then $n = 0$ or $n = m + 1$ for some natural number m.

Proof. Define $\Phi := \{n' \in \mathbb{N} \mid n' = 0 \text{ or } n' = m' + 1 \text{ for some natural number } m'\}$. $0 \in \Phi$ and for all $n' \in \Phi$ we have $n' + 1 \in \Phi$. Hence every natural number is contained in Φ. Thus $n = 0$ or $n = m + 1$ for some natural number m.
■

Proposition 21. Let n be a natural number. Then $n \neq n + 1$.

Proof. Define $\Phi := \{n' \in \mathbb{N} \mid n' \neq n' + 1\}$.

(1) 0 belongs to Φ.

(2) For all $n' \in \Phi$ we have $n' + 1 \in \Phi$.

Proof. Let $n' \in \Phi$. Then $n' \neq n' + 1$. If $n' + 1 = (n' + 1) + 1$ then $n' = n' + 1$. Thus it is wrong that $n' + 1 = (n' + 1) + 1$. Hence $n' + 1 \in \Phi$. □

Therefore every natural number is an element of Φ. Consequently $n \neq n + 1$.
■

Part IV
Computation Laws for Addition

1 Associativity

Proposition 22. Let n, m, k be natural numbers. Then $n + (m + k) = (n + m) + k$.

Proof. Define $\Phi := \{k' \in \mathbb{N} \mid n + (m + k') = (n + m) + k'\}$.

(1) 0 is contained in Φ. Indeed $n + (m + 0) = n + m = (n + m) + 0$.

(2) For all $k' \in \Phi$ we have $k' + 1 \in \Phi$.

Proof. Let $k' \in \Phi$. Then $n + (m + k') = (n + m) + k'$. Hence

$$n + (m + (k' + 1))$$

$$= n + ((m + k') + 1)$$
$$= (n + (m + k')) + 1$$
$$= ((n + m) + k') + 1$$
$$= (n + m) + (k' + 1).$$

Thus $k' + 1 \in \Phi$. □

Thus every natural number is an element of Φ. Therefore $n + (m + k) = (n + m) + k$. ■

2 Commutativity

Proposition 23. Let n, m be natural numbers. Then $n + m = m + n$.

Proof. Define $\Phi := \{m' \in \mathbb{N} \mid n + m' = m' + n\}$.

(1) 0 is an element of Φ.

Proof. Define $\Psi := \{n' \in \mathbb{N} \mid n' + 0 = 0 + n'\}$.

(1a) 0 belongs to Ψ.

(1b) For all $n' \in \Psi$ we have $n' + 1 \in \Psi$.

Proof. Let $n' \in \Psi$. Then $n' + 0 = 0 + n'$. Hence

$$(n' + 1) + 0$$
$$= n' + 1$$
$$= (n' + 0) + 1$$
$$= (0 + n') + 1$$
$$= 0 + (n' + 1).$$

□

Hence every natural number belongs to Ψ. Thus $n + 0 = 0 + n$. Therefore 0 is an element of Φ. □

Let us show that (2) $n + 1 = 1 + n$.

Proof. Define $\Theta := \{n' \in \mathbb{N} \mid n' + 1 = 1 + n'\}$.

(2a) 0 is an element of Θ.

(2b) For all $n' \in \Theta$ we have $n' + 1 \in \Theta$.

Proof. Let $n' \in \Theta$. Then $n'+1 = 1+n'$. Hence

$$(n'+1)+1$$

$$= (1+n')+1$$

$$= 1+(n'+1).$$

Thus $n'+1 \in \Theta$. $\square$

Thus every natural number belongs to Θ. Therefore $n+1 = 1+n$. $\square$

(3) For all $m' \in \Phi$ we have $m'+1 \in \Phi$.

Proof. Let $m' \in \Phi$. Then $n+m' = m'+n$. Hence

$$n+(m'+1)$$

$$= (n+m')+1$$

$$= (m'+n)+1$$

$$= m'+(n+1)$$

$$= m'+(1+n)$$

$$= (m'+1)+n.$$

Thus $m'+1 \in \Phi$. $\square$

Thus every natural number is an element of Φ. Therefore $n+m = m+n$. ■

3 Cancellation

Proposition 24. Let n, m, k be natural numbers. If $n+k = m+k$ then $n = m$.

Proof. Define $\Phi := \{k' \in \mathbb{N} \mid \text{ if } n+k' = m+k' \text{ then } n = m\}$.

(1) 0 is an element of Φ.

(2) For all $k' \in \Phi$ we have $k'+1 \in \Phi$.

Proof. Let $k' \in \Phi$. Suppose $n+(k'+1) = m+(k'+1)$. Then $(n+k')+1 = (m+k')+1$. Hence $n+k' = m+k'$. Thus $n = m$. $\square$

Therefore every natural number is an element of Φ. Consequently if $n+k = m+k$ then $n = m$. ■

Corollary 25. Let n, m, k be natural numbers. If $k+n = k+m$ then $n = m$.

Proof. Assume $k+n = k+m$. We have $k+n = n+k$ and $k+m = m+k$. Hence $n+k = m+k$. Thus $n = m$. ■

4 Zero Sums

Proposition 26. Let n, m be natural numbers. If $n+m = 0$ then $n = 0$ and $m = 0$.

Proof. Assume $n+m = 0$. Suppose $n \neq 0$ or $m \neq 0$. Then we can take a $k \in \mathbb{N}$ such that $n = k+1$ or $m = k+1$. Hence there exists a natural number l such that $n+m = l+(k+1) = (l+k)+1 \neq 0$. Contradiction. ■