Part I

Definition

Signature 1. Let n, m be natural numbers. $n \cdot m$ is a natural number. Let the *product of n and m* stand for $n \cdot m$.

Axiom 2. Let n be a natural number. Then $n \cdot 0 = 0$.

Axiom 3. Let n, m be natural numbers. Then $n \cdot (m+1) = (n \cdot m) + n$.

Part II

Computation Laws

1 Distributivity

Proposition 4. Let n, m, k be natural numbers. Then $n \cdot (m+k) = (n \cdot m) + (n \cdot k)$.

Proof. Define $\Phi := \{k' \in \mathbb{N} \mid n \cdot (m+k') = (n \cdot m) + (n \cdot k')\}$.

(1) 0 is an element of Φ. Indeed $n \cdot (m+0) = n \cdot m = (n \cdot m) + 0 = (n \cdot m) + (n \cdot 0)$.

(2) For all $k' \in \Phi$ we have $k' + 1 \in \Phi$.

Proof. Let $k' \in \Phi$. Then

$$\begin{aligned} & n \cdot (m + (k'+1)) \\ &= n \cdot ((m+k')+1) \\ &= (n \cdot (m+k')) + n \\ &= ((n \cdot m) + (n \cdot k')) + n \\ &= (n \cdot m) + ((n \cdot k') + n) \\ &= (n \cdot m) + (n \cdot (k'+1)). \end{aligned}$$

Hence $n \cdot (m + (k'+1)) = (n \cdot m) + (n \cdot (k'+1))$. Thus $k'+1 \in \Phi$. $\square$

Thus every natural number is contained in Φ. Therefore $n \cdot (m+k) = (n \cdot m) + (n \cdot k)$. ■

Proposition 5. Let n, m, k be natural numbers. Then $(n+m) \cdot k = (n \cdot k) + (m \cdot k)$.

Proof. Define $\Phi := \{k' \in \mathbb{N} \mid (n+m) \cdot k' = (n \cdot k') + (m \cdot k')\}$.

(1) 0 belongs to Φ. Indeed $((n+m) \cdot 0) = 0 = 0 + 0 = (n \cdot 0) + (m \cdot 0)$.

(2) For all $k' \in \Phi$ we have $k' + 1 \in \Phi$.

Proof. Let $k' \in \Phi$. Then

$$
\begin{aligned}
&(n+m) \cdot (k'+1) \\
&= ((n+m) \cdot k') + (n+m) \\
&= ((n \cdot k') + (m \cdot k')) + (n+m) \\
&= (((n \cdot k') + (m \cdot k')) + n) + m \\
&= ((n \cdot k') + ((m \cdot k') + n)) + m \\
&= ((n \cdot k') + (n + (m \cdot k'))) + m \\
&= (((n \cdot k') + n) + (m \cdot k')) + m \\
&= ((n \cdot k') + n) + ((m \cdot k') + m) \\
&= (n \cdot (k'+1)) + (m \cdot (k'+1)).
\end{aligned}
$$

Thus $(n+m) \cdot (k'+1) = (n \cdot (k'+1)) + (m \cdot (k'+1))$. □

Thus every natural number is an element of Φ. Therefore $(n+m) \cdot k = (n \cdot k) + (m \cdot k)$. ■

2 Multiplication with 1 and 2

Proposition 6. Let n be a natural number. Then $n \cdot 1 = n$.

Proof. $n \cdot 1 = n \cdot (0+1) = (n \cdot 0) + n = 0 + n = n$. ■

Corollary 7. Let n be a natural number. Then $n \cdot 2 = n + n$.

Proof. $n \cdot 2 = n \cdot (1+1) = (n \cdot 1) + n = n + n$. ■

3 Associativity

Proposition 8. Let n, m, k be natural numbers. Then $n \cdot (m \cdot k) = (n \cdot m) \cdot k$.

Proof. Define $\Phi := \{k' \in \mathbb{N} \mid n \cdot (m \cdot k') = (n \cdot m) \cdot k'\}$.

(1) 0 is contained in Φ. Indeed $n \cdot (m \cdot 0) = n \cdot 0 = 0 = (n \cdot m) \cdot 0$.

(2) For all $k' \in \Phi$ we have $k' + 1 \in \Phi$.

Proof. Let $k' \in \Phi$. Then

$$n \cdot (m \cdot (k'+1))$$

$$\begin{aligned} &= n \cdot ((m \cdot k') + m) \\ &= (n \cdot (m \cdot k')) + (n \cdot m) \\ &= ((n \cdot m) \cdot k') + (n \cdot m) \\ &= ((n \cdot m) \cdot k') + ((n \cdot m) \cdot 1) \\ &= (n \cdot m) \cdot (k'+1). \end{aligned}$$

□

Hence every natural number is contained in Φ (by induction). Thus $n \cdot (m \cdot k) = (n \cdot m) \cdot k$. ■

4 Commutativity

Proposition 9. Let n, m be natural numbers. Then

$$n \cdot m = m \cdot n.$$

Proof. Define $\Phi := \{m' \in \mathbb{N} \mid n \cdot m' = m' \cdot n\}$.

(1) 0 is contained in Φ.

Proof. Define $\Psi := \{n' \in \mathbb{N} \mid n' \cdot 0 = 0 \cdot n'\}$.

(1a) 0 is contained in Ψ.

(1b) For all $n' \in \Psi$ we have $n' + 1 \in \Psi$.

Proof. Let $n' \in \Psi$. Then

$$(n' + 1) \cdot 0 = 0 = n' \cdot 0 = 0 \cdot n' = (0 \cdot n') + 0 = 0 \cdot (n' + 1).$$

□

Hence every natural number is contained in Ψ (by induction). Thus $n \cdot 0 = 0 \cdot n$. □

(2) 1 belongs to Φ.

Proof. Define $\Theta := \{n' {\in} \mathbb{N} \mid n' \cdot 1 = 1 \cdot n'\}$.

(2a) 0 is contained in Θ.

(2b) For all $n' \in \Theta$ we have $n' + 1 \in \Theta$.

Proof. Let $n' \in \Theta$. Then

$$\begin{aligned} &(n' + 1) \cdot 1 \\ &= (n' \cdot 1) + 1 \\ &= (1 \cdot n') + 1 \\ &= 1 \cdot (n' + 1). \end{aligned}$$

□

Thus every natural number is contained in Θ (by induction). Therefore $n \cdot 1 = 1 \cdot n$. □

(3) For all $m' \in \Phi$ we have $m' + 1 \in \Phi$.

Proof. Let $m' \in \Phi$. Then

$$\begin{aligned} &n \cdot (m' + 1) \\ &= (n \cdot m') + (n \cdot 1) \\ &= (m' \cdot n) + (1 \cdot n) \\ &= (1 \cdot n) + (m' \cdot n) \\ &= (1 + m') \cdot n \\ &= (m' + 1) \cdot n. \end{aligned}$$

Indeed $((1 \cdot n) + (m' \cdot n)) = (1 + m') \cdot n$. □

Hence every natural number is contained in Φ (by induction). Thus $n \cdot m = m \cdot n$. ■

5 Non-Existence of Zero-Divisors

Proposition 10. Let n, m be natural numbers such that $n \cdot m = 0$. Then $n = 0$ or $m = 0$.

Proof. Suppose $n, m \neq 0$. Take natural numbers n', m' such that $n = (n' + 1)$ and $m = (m' + 1)$. Then

$$\begin{aligned} & 0 \\ &= n \cdot m \\ &= (n' + 1) \cdot (m' + 1) \\ &= ((n' + 1) \cdot m') + (n' + 1) \\ &= (((n' + 1) \cdot m') + n') + 1. \end{aligned}$$

Indeed $(n' + 1) \cdot (m' + 1) = ((n' + 1) \cdot m') + (n' + 1)$. Hence $0 = k + 1$ for some natural number k. Contradiction. ■

6 Cancellation

Proposition 11. Let n, m, k be natural numbers. Assume $k \neq 0$. If $n \cdot k = m \cdot k$ then $n = m$.

Proof. Define $\Phi := \{n' \in \mathbb{N} \mid$ for all $m' \in \mathbb{N}$ if $n' \cdot k = m' \cdot k$ and $k \neq 0$ then $n' = m'\}$.

(1) 0 is contained in Φ.

Proof. Let $m' \in \mathbb{N}$. Assume $0 \cdot k = m' \cdot k$ and $k \neq 0$. Then $m' \cdot k = 0$. Hence $m' = 0$ or $k = 0$. Thus $m' = 0$. □

(2) For all $n' \in \Phi$ we have $n' + 1 \in \Phi$.

Proof. Let $n' \in \Phi$.

Let us show that for all $m' \in \mathbb{N}$ if $(n' + 1) \cdot k = m' \cdot k$ and $k \neq 0$ then $n' + 1 = m'$. Let $m' \in \mathbb{N}$. Assume $(n' + 1) \cdot k = m' \cdot k$ and $k \neq 0$.

Case $m' = 0$. Then $(n' + 1) \cdot k = 0$. Hence $n' + 1 = 0$. Contradiction. □

Case $m' \neq 0$. Take a natural number l such that $m' = l + 1$. Then $(n' + 1) \cdot k = (l + 1) \cdot k$. Hence $(n' \cdot k) + k = (n' \cdot k) + (1 \cdot k) = (n' \cdot k) + k = (l + 1) \cdot k = (l \cdot k) + (1 \cdot k) = (l \cdot$ Thus $n' \cdot k = l \cdot k$ (by right-cancellability of addition). Indeed $n' \cdot k$ and $l \cdot k$ are natural numbers. Then we have $n' = l$. Indeed if $n' \cdot k = l \cdot k$ and $k \neq 0$ then $n' = l$. Therefore $n' + 1 = l + 1 = m'$. □

End.

Hence $n' + 1 \in \Phi$. $\square$

Thus every natural number is contained in Φ (by induction). Therefore if $n \cdot k = m \cdot k$ then $n = m$. $\blacksquare$

Corollary 12. Let n, m, k be natural numbers. Assume $k \neq 0$. If $k \cdot n = k \cdot m$ then $n = m$.

Proof. Assume $k \cdot n = k \cdot m$. We have $k \cdot n = n \cdot k$ and $k \cdot m = m \cdot k$. Hence $n \cdot k = m \cdot k$. Thus $n = m$ (by right-cancellability of multiplication). $\blacksquare$