Proposition 1. Let n, m, k be natural numbers. Assume $k \neq 0$. Then $n < m$ iff $n \cdot k < m \cdot k$.

Proof.

Case $n \cdot k < m \cdot k$. Define $\Phi := \{n' \in \mathbb{N} \mid \text{ if } n' \cdot k < m \cdot k \text{ then } n' < m\}$.

(1) Φ contains 0.

(2) For all $n' \in \Phi$ we have $n' + 1 \in \Phi$.

Proof. Let $n' \in \Phi$.

Let us show that if $(n' + 1) \cdot k < m \cdot k$ then $n' + 1 < m$. Assume $(n' + 1) \cdot k < m \cdot k$. Then $(n' \cdot k) + k < m \cdot k$. Hence $n' \cdot k < m \cdot k$. Thus $n' < m$. Then $n' + 1 \leq m$. If $n' + 1 = m$ then $(n' + 1) \cdot k = m \cdot k$. Hence $n' + 1 < m$. End. □

Therefore every natural number is contained in Φ (by induction). Consequently $n < m$. □

Case $n < m$. Take a positive natural number l such that $m = n + l$. Then $m \cdot k = (n + l) \cdot k = (n \cdot k) + (l \cdot k)$. $l \cdot k$ is positive. Hence $n \cdot k < m \cdot k$. □

■

Corollary 2. Let n, m, k be natural numbers. Assume $k \neq 0$. Then $n < m$ iff $k \cdot n < k \cdot m$.

Proof. The thesis (by preservation of ordering under right-multiplication, commutativity of multiplication). ■

Proposition 3. Let n, m, k be natural numbers. If $n, m > k$ then $n \cdot m > k$.

Proof. Define $\Phi := \{n' \in \mathbb{N} \mid \text{ if } n', m > k \text{ then } n' \cdot m > k\}$.

(1) Φ contains 0.

(2) For all $n' \in \Phi$ we have $n' + 1 \in \Phi$.

Proof. Let $n' \in \Phi$.

Let us show that if $n' + 1, m > k$ then $(n' + 1) \cdot m > k$. Assume $n' + 1, m > k$. Then $(n' + 1) \cdot m = (n' \cdot m) + m$. If $n' = 0$ then $(n' \cdot m) + m = 0 + m = m > k$.

If $n' \neq 0$ then $(n' \cdot m) + m > m > k$. Indeed if $n' \neq 0$ then $n' \cdot m > 0$. Indeed $m > 0$. Hence $(n' + 1) \cdot m > k$. End. $\square$

Thus every natural number is contained in Φ (by induction). Therefore if $n, m > k$ then $n \cdot m > k$. ■

Corollary 4. Let n, m, k be natural numbers. If $n \leq m$ then $k \cdot n \leq k \cdot m$.

Corollary 5. Let n, m, k be natural numbers. Assume $k \neq 0$. If $k \cdot n \leq k \cdot m$ then $n \leq m$.

Proof. If $k \cdot n = k \cdot m$ then $n = m$ (by left-cancellability of multiplication). If $k \cdot n < k \cdot m$ then $n < m$ (by preservation of ordering under left-multiplication). ■

Corollary 6. Let n, m, k be natural numbers. If $n \leq m$ then $n \cdot k \leq m \cdot k$.

Corollary 7. Let n, m, k be natural numbers. Assume $k \neq 0$. If $n \cdot k \leq m \cdot k$ then $n \leq m$.

Proposition 8. Let n, m, k be natural numbers. Assume $m > 0$ and $k > 1$. Then $k \cdot m > m$.

Proof. Take a natural number l such that $k = l + 2$. Then

$$k \cdot m$$

$$= (l + 2) \cdot m$$

$$= (l \cdot m) + (2 \cdot m)$$

$$= (l \cdot m) + (m + m)$$

$$= ((l \cdot m) + m) + m$$

$$= ((l + 1) \cdot m) + m$$

$$\geq 1 + m$$

$$> m.$$

Indeed $((l+1)\cdot m)+m \geq 1+m$. ■