Proposition 1. Let n be a natural number. Assume $n \geq 2$. Then $n^n > n!$.

Proof. (a) Define $\Phi := \left\{ n' \in \mathbb{N}_{\geq 2} \,\middle|\, n'^{n'} > n'! \right\}$.

(1) Φ contains 2.

(2) For all $n' \in \Phi$ we have $n' + 1 \in \Phi$.

Proof. Let $n' \in \Phi$. Then $n' \geq 2$.

Take $X = n' + 1^{n'} \cdot (n' + 1)$ and $Y = n'^{n'} \cdot (n' + 1)$. Then $X > Y$.

Proof. We have $n' + 1 > n'$ and $n' \neq 0$. Thus $n' + 1^{n'} > n'^{n'}$. $n' + 1$ is nonzero. Hence $n' + 1^{n'} \cdot (n' + 1) > n'^{n'} \cdot (n' + 1)$. □

Take $Z = n'! \cdot (n' + 1)$. Then $Y > Z$. Indeed $n'^{n'} > n'!$ and $n' + 1 \neq 0$.

Hence $n' + 1^{n' + 1} = X > Y > Z = n' + 1!$. Thus $n' + 1^{n' + 1} > n' + 1!$. □

Therefore Φ contains every element of $\mathbb{N}_{\geq 2}$ (by induction). Consequently $n^n > n!$ (by a). ■

Proposition 2. Let n be a natural number. Assume $n \geq 4$. Then $n! > 2^n$.

Proof. (a) Define $\Phi := \left\{ n' \in \mathbb{N}_{\geq 4} \,\middle|\, n'! > 2^{n'} \right\}$.

(1) Φ contains 4.

Proof.

$$
\begin{aligned}
& 4! \\
&= 4 \cdot (3 \cdot 2) \\
&= 2 \cdot (2 \cdot (3 \cdot 2)) \\
&= 3 \cdot (2 \cdot (2 \cdot 2)) \\
&> 2 \cdot (2 \cdot (2 \cdot 2)) \\
&= 2^4.
\end{aligned}
$$

□

(2) For all $n' \in \Phi$ we have $n' + 1 \in \Phi$.

Proof. Let $n' \in \Phi$. Then $n' \geq 4$ and $n'! > 2^{n'}$.

Take $X = n'! \cdot (n' + 1)$ and $Y = 2^{n'} \cdot (n' + 1)$. Then $X > Y$.

Take $Z = 2^{n'} \cdot 2$. Then $Y > Z$ (by preservation of ordering under left-multiplication). Indeed $n' + 1 > 2$ and $2^{n'} \neq 0$.

Hence $n' + 1! = X > Y > Z = 2^{n' + 1}$. Thus $n' + 1! > 2^{n' + 1}$. $\square$

Therefore Φ contains every element of $\mathbb{N}_{\geq 4}$ (by induction). Consequently $n! > 2^n$ (by a). $\blacksquare$