Part I

Definition

Signature 1. Let n, m be natural numbers. n^m is a natural number.

Axiom 2. Let n be a natural number. Then $n^0 = 1$.

Axiom 3. Let n, m be natural numbers. Then $n^{m+1} = n^m \cdot n$.

Part II

Computation Laws

1 Exponentiation with 0, 1 and 2

Proposition 4. Let n be a natural number. Assume $n \neq 0$. Then $0^n = 0$.

Proof. Take a natural number m such that $n = m + 1$. Then $0^n = 0^{m+1} = 0^m \cdot 0 = 0$. Indeed $0^{m+1} = 0^m \cdot 0$. ∎

Proposition 5. Let n be a natural number. Then $1^n = 1$.

Proof. Define $\Phi := \left\{ n' \in \mathbb{N} \,\middle|\, 1^{n'} = 1 \right\}$.

(1) Φ contains 0.

(2) For all $n' \in \Phi$ we have $n' + 1 \in \Phi$.

Proof. Let $n' \in \Phi$. Then $1^{n'+1} = 1^{n'} \cdot 1 = 1 \cdot 1 = 1$. □

Hence every natural number is contained in Φ (by induction). Thus $1^n = 1$. ∎

Proposition 6. Let n be a natural number. Then $n^1 = n$.

Proof. We have $n^1 = n^{0+1} = n^0 \cdot n = 1 \cdot n = n$. ∎

Proposition 7. Let n be a natural number. Then $n^2 = n \cdot n$.

Proof. We have $n^2 = n^{1+1} = n^1 \cdot n = n \cdot n$. ∎

2 Sums as Exponents

Proposition 8. Let n, m, k be natural numbers. Then $k^{n+m} = k^n \cdot k^m$.

Proof. Define $\Phi := \left\{ m' \in \mathbb{N} \,\middle|\, k^{n+m'} = k^n \cdot k^{m'} \right\}$.

(1) Φ contains 0. Indeed $k^{n+0} = k^n = k^n \cdot 1 = k^n \cdot k^0$.

(2) For all $m' \in \Phi$ we have $m' + 1 \in \Phi$.

Proof. Let $m' \in \Phi$. Then

$$\begin{aligned} & k^{n+(m'+1)} \\ &= k^{(n+m')+1} \\ &= k^{n+m'} \cdot k \\ &= (k^n \cdot k^{m'}) \cdot k \\ &= k^n \cdot (k^{m'} \cdot k) \\ &= k^n \cdot k^{m'+1}. \end{aligned}$$

□

Hence every natural number is contained in Φ (by induction). Thus $k^{n+m} = k^n \cdot k^m$. ∎

3 Products as Exponents

Proposition 9. Let n, m, k be natural numbers. Then $n^{m \cdot k} = n^{mk}$.

Proof. Define $\Phi := \left\{ k' \in \mathbb{N} \,\middle|\, n^{m \cdot k'} = n^{mk'} \right\}$.

(1) Φ contains 0. Indeed $n^{m0} = 1 = n^0 = n^{m \cdot 0}$.

(2) For all $k' \in \Phi$ we have $k' + 1 \in \Phi$.

Proof. Let $k' \in \Phi$. Then

$$\begin{aligned} & n^{mk'+1} \\ &= n^{mk'} \cdot n^m \\ &= n^{m \cdot k'} \cdot n^m \\ &= n^{(m \cdot k') + m} \\ &= n^{m \cdot (k'+1)}. \end{aligned}$$

□

Therefore every natural number is contained in Φ (by induction). Consequently $n^{m \cdot k} = n^{mk}$. ■

4 Products as Base

Proposition 10. Let n, m, k be natural numbers. Then $n \cdot m^k = n^k \cdot m^k$.

Proof. Define $\Phi := \left\{ k' \in \mathbb{N} \,\middle|\, n \cdot m^{k'} = n^{k'} \cdot m^{k'} \right\}$.

(1) Φ contains 0. Indeed $(n \cdot m^0) = 1 = 1 \cdot 1 = n^0 \cdot m^0$.

(2) For all $k' \in \Phi$ we have $k' + 1 \in \Phi$.

Proof. Let $k' \in \Phi$.

Let us show that $(n^{k'} \cdot m^{k'}) \cdot (n \cdot m) = (n^{k'} \cdot n) \cdot (m^{k'} \cdot m)$.

$$\begin{aligned} & (n^{k'} \cdot m^{k'}) \cdot (n \cdot m) \\ &= ((n^{k'} \cdot m^{k'}) \cdot n) \cdot m \\ &= (n^{k'} \cdot (m^{k'} \cdot n)) \cdot m \\ &= (n^{k'} \cdot (n \cdot m^{k'})) \cdot m \end{aligned}$$

$$= ((n^{k'} \cdot n) \cdot m^{k'}) \cdot m$$
$$= (n^{k'} \cdot n) \cdot (m^{k'} \cdot m).$$

End.

Hence

$$n \cdot m^{k'+1}$$
$$= n \cdot m^{k'} \cdot (n \cdot m)$$
$$= (n^{k'} \cdot m^{k'}) \cdot (n \cdot m)$$
$$= (n^{k'} \cdot n) \cdot (m^{k'} \cdot m)$$
$$= n^{k'+1} \cdot m^{k'+1}.$$

□

Therefore every natural number is contained in Φ (by induction). Consequently $n \cdot m^k = n^k \cdot m^k$. ■

5 Zeroes of Exponentiation

Proposition 11. Let n, m be natural numbers. Then $n^m = 0$ iff $n = 0$ and $m \neq 0$.

Proof.

Case $n^m = 0$. Define $\Phi := \left\{ m' \in \mathbb{N} \,\middle|\, \text{if } n^{m'} = 0 \text{ then } n = 0 \text{ and } m' \neq 0 \right\}$.

(1) Φ contains 0. Indeed if $n^0 = 0$ then we have a contradiction.

(2) For all $m' \in \Phi$ we have $m' + 1 \in \Phi$.

Proof. Let $m' \in \Phi$.

Let us show that if $n^{m'+1} = 0$ then $n = 0$ and $m' + 1 \neq 0$. Assume $n^{m'+1} = 0$. Then $0 = n^{m'+1} = n^{m'} \cdot n$. Hence $n^{m'} = 0$ or $n = 0$. We have $m' + 1 \neq 0$ and if $n^{m'} = 0$ then $n = 0$. Hence $n = 0$ and $m' + 1 \neq 0$. End. □

Thus every natural number is contained in Φ (by induction). Consequently $m \in \Phi$. Therefore $n = 0$ and $m \neq 0$. □

Case $n = 0$ *and* $m \neq 0$. Take a natural number k such that $m = k + 1$. Then $n^m = n^{k+1} = n^k \cdot n = 0^k \cdot 0 = 0$. □

■