Proposition 1. Let n, m, k be natural numbers. Assume $k \neq 0$. Then $n < m$ iff $n^k < m^k$.

Proof.

Case $n < m$. Define $\Phi := \left\{ k' \in \mathbb{N} \,\middle|\, \text{if } k' > 1 \text{ then } n^{k'} < m^{k'} \right\}$.

(1) Φ contains 0.

(2) Φ contains 1.

(3) Φ contains 2.

Proof.

Case $n = 0$ *or* $m = 0$. $\square$

Case $n, m \neq 0$. Then $n \cdot n < n \cdot m < m \cdot m$. Hence $n^2 = n \cdot n < n \cdot m < m \cdot m = m^2$. $\square$

$\square$

(4) For all $k' \in \Phi$ we have $k' + 1 \in \Phi$.

Proof. Let $k' \in \Phi$.

Let us show that if $k' + 1 > 1$ then $n^{k'+1} < m^{k'+1}$. Assume $k' + 1 > 1$. Then $n^{k'} < m^{k'}$. Indeed $k' \neq 0$ and $if k' = 1$ then $n^{k'} < m^{k'}$.

Case $k' \leq 1$. Then $k' = 0$ or $k' = 1$. Hence $k' + 1 = 1$ or $k' + 1 = 2$. Thus $k' + 1 \in \Phi$. Therefore $n^{k'+1} < m^{k'+1}$. $\square$

Case $k' > 1$.

Case $n = 0$. Then $m \neq 0$. Hence $n^{k'+1} = 0 < m^{k'} \cdot m = m^{k'+1}$. Thus $n^{k'+1} < m^{k'+1}$. $\square$

Case $n \neq 0$. Then $n^{k'} \cdot n < m^{k'} \cdot n < m^{k'} \cdot m$. Indeed $n^{k'} < m^{k'} \neq 0$. Take $A = n^{k'+1}$ and $B = m^{k'+1}$. Then $A = n^{k'+1} = n^{k'} \cdot n < m^{k'} \cdot n < m^{k'} \cdot m = m^{k'+1} = B$. Take $X = m^{k'} \cdot n$ and $Y = m^{k'} \cdot m$. Then $A < X < Y = B$. Hence $A < B$. Thus $n^{k'+1} < m^{k'+1}$. $\square$

$\square$

Hence $n^{k'+1} < m^{k'+1}$. Indeed $k' \leq 1$ or $k' > 1$. End.

Thus $k' + 1 \in \Phi$. $\square$

Therefore every natural number is contained in Φ (by induction). Consequently $n^k < m^k$. $\square$

Case $n^k < m^k$. Define $\Psi := \left\{ k' \in \mathbb{N} \,\middle|\, \text{if } n \geq m \text{ then } n^{k'} \geq m^{k'} \right\}$.

(1) Ψ contains 0.

(2) For all $k' \in \Psi$ we have $k'+1 \in \Psi$.

Proof. Let $k' \in \Psi$.

Let us show that if $n \geq m$ then $n^{k'+1} \geq m^{k'+1}$. Assume $n \geq m$. Then $n^{k'} \geq m^{k'}$. Hence $n^{k'} \cdot n \geq m^{k'} \cdot n \geq m^{k'} \cdot m$. Take $A = n^{k'+1}$ and $B = m^{k'+1}$. Thus $A = n^{k'+1} = n^{k'} \cdot n \geq m^{k'} \cdot n \geq m^{k'} \cdot m = m^{k'+1} = B$. Therefore $n^{k'+1} = A \geq B = m^{k'+1}$. End.

Hence $k'+1 \in \Psi$. □

Thus every natural number is contained in Ψ (by induction). Therefore if $n \geq m$ then $n^k \geq m^k$. Consequently $n < m$. □

■

Corollary 2. Let n, m, k be natural numbers. Assume $k \neq 0$. Then

$$n^k = m^k \implies n = m.$$

Proof. Assume $n^k = m^k$. Suppose $n \neq m$. Then $n < m$ or $m < n$. If $n < m$ then $n^k < m^k$. If $m < n$ then $m^k < n^k$. Thus $n^k \neq m^k$. Contradiction. ■

Corollary 3. Let n, m, k be natural numbers. Assume $k \neq 0$. Then

$$n^k \leq m^k \iff n \leq m.$$

Proof. If $n^k < m^k$ then $n < m$. If $n^k = m^k$ then $n = m$.

If $n < m$ then $n^k < m^k$. If $n = m$ then $n^k = m^k$. ■

Proposition 4. Let n, m, k be natural numbers. Assume $k > 1$. Then

$$n < m \iff k^n < k^m.$$

Proof.

Case $n < m$. Define $\Phi := \left\{ m' \in \mathbb{N} \;\middle|\; \text{if } n < m' \text{ then } k^n < k^{m'} \right\}$.

(1) Φ contains 0.

(2) For all $m' \in \Phi$ we have $m' + 1 \in \Phi$.

Proof. Let $m' \in \Phi$.

Let us show that if $n < m' + 1$ then $k^n < k^{m'+1}$. Assume $n < m' + 1$. Then $n \leq m'$. We have $k^{m'} \cdot 1 < k^{m'} \cdot k$. Indeed $k^{m'} \neq 0$.

Case $n = m'$. Take $A = k^n$ and $B = k^{m'+1}$. Then $A = k^n = k^{m'} < k^{m'} \cdot k = k^{m'+1} = B$. Hence $k^n = A < B = k^{m'+1}$. □

Case $n < m'$. Take $A = k^n$ and $B = k^{m'+1}$. Then $A = k^n < k^{m'} < k^{m'} \cdot k = k^{m'+1} = B$. Hence $k^n = A < B = k^{m'+1}$. □

End. □

Hence every natural number is contained in Φ (by induction). Thus $k^n < k^m$. □

Case $k^n < k^m$. Define $\Psi := \left\{ n' \in \mathbb{N} \,\middle|\, \text{if } n' \geq m \text{ then } k^{n'} \geq k^m \right\}$.

(1) 0 is contained in Ψ.

(2) For all $n' \in \Psi$ we have $n' + 1 \in \Psi$.

Proof. Let $n' \in \Psi$.

Let us show that if $n' + 1 \geq m$ then $k^{n'+1} \geq k^m$. Assume $n' + 1 \geq m$.

Case $n' + 1 = m$. □

Case $n' + 1 > m$. Then $n' \geq m$. Hence $k^{n'} \geq k^m$. We have $k^{n'} \cdot 1 \leq k^{n'} \cdot k$. Indeed $1 \leq k$ and $k^{n'} \neq 0$. Take $A = k^m$ and $B = k^{n'+1}$. Then $A = k^m \leq k^{n'} = k^{n'} \cdot 1 \leq k^{n'} \cdot k = k^{n'+1}$ Hence $k^m = A \leq B = k^{n'+1}$. □

End. □

Thus every natural number is contained in Ψ (by induction). Therefore if $n \geq m$ then $k^n \geq k^m$. Consequently $n < m$. □

■

Corollary 5. Let n, m, k be natural numbers. Assume $k > 1$. If $k^n = k^m$ then $n = m$.

Proof. Assume $k^n = k^m$. Suppose $n \neq m$. Then $n < m$ or $m < n$. If $n < m$ then $k^n < k^m$. If $m < n$ then $k^m < k^n$. Thus $k^n \neq k^m$. Contradiction. ■

Corollary 6. Let n, m, k be natural numbers. Assume $k > 1$. Then $n \leq m$ iff $k^n \leq k^m$.

Proof. If $n \leq m$ then $k^n \leq k^m$.

If $k^n = k^m$ then $n = m$. If $k^n < k^m$ then $n < m$. ■

Proposition 7. Let n be a natural number. Then $n + 1^2 = (n^2 + (2 \cdot n)) + 1$.

Proof. We have

$$
\begin{aligned}
& n + 1^2 \\
&= (n+1) \cdot (n+1) \\
&= ((n+1) \cdot n) + (n+1) \\
&= ((n \cdot n) + n) + (n+1) \\
&= (n^2 + n) + (n+1) \\
&= ((n^2 + n) + n) + 1 \\
&= (n^2 + (n+n)) + 1 \\
&= (n^2 + (2 \cdot n)) + 1.
\end{aligned}
$$

■

Proposition 8. Let n be a natural number. Assume $n \geq 3$. Then $n^2 > (2 \cdot n) + 1$.

Proof. (a) Define $\Phi := \left\{ n' \in \mathbb{N}_{\geq 3} \,\middle|\, {n'}^2 > (2 \cdot n') + 1 \right\}$.

(1) Φ contains 3. Indeed $3^2 > (2 \cdot 3) + 1$.

(2) For all $n' \in \Phi$ we have $n' + 1 \in \Phi$.

Proof. Let $n' \in \Phi$. Then $n' \geq 3$.

Take $V = ({n'}^2 + (2 \cdot n')) + 1$ and $W = (((2 \cdot n') + 1) + (2 \cdot n')) + 1$. Then $V > W$.

Proof. We have ${n'}^2 > (2 \cdot n') + 1$. Hence ${n'}^2 + (2 \cdot n') > ((2 \cdot n') + 1) + (2 \cdot n')$

(by preservation of ordering under right-addition). □

Take $X = ((2 \cdot n') + (2 \cdot n')) + 1$. Then $W > X$.

Proof. We have $((2 \cdot n') + 1) + (2 \cdot n') > (2 \cdot n') + (2 \cdot n')$. Indeed $(2 \cdot n') + 1 > 2 \cdot n'$.
□

Take $Y = (2 \cdot (n' + n')) + 1$ and $Z = (2 \cdot (n' + 1)) + 1$. Then $Y > Z$.

Proof. We have $n' + n' > n' + 1$ and $2 \neq 0$. Thus $2 \cdot (n' + n') > 2 \cdot (n' + 1)$ (by preservation of ordering under left-multiplication). Indeed $n' + n'$ and $n' + 1$ are natural numbers. □

Then $n' + 1^2 = V > W > X = Y > Z = (2 \cdot (n' + 1)) + 1$. Hence $n' + 1^2 > (2 \cdot (n' + 1)) + 1$.
□

Therefore Φ contains every element of $\mathbb{N}_{\geq 3}$ (by induction, 1, 2). Consequently $n^2 > (2 \cdot n) + 1$ (by a). ■

Proposition 9. Let n be a natural number. Assume $n \geq 5$. Then $2^n > n^2$.

Proof. (a) Define $\Phi := \left\{ n' \in \mathbb{N}_{\geq 5} \,\middle|\, 2^{n'} > n'^2 \right\}$.

(1) Φ contains 5. Indeed we can show that $2^5 > 5^2$. We have $2^5 = 2 \cdot (2 \cdot (2 \cdot (2 \cdot 2))) = 8 \cdot 4 = (5 + 3) \cdot 4 = (5 \cdot 4$
Indeed $((5 \cdot 4) + ((5 \cdot 2) + 2)) = (5 \cdot (4 + 2)) + 2$. End.

(2) For all $n' \in \Phi$ we have $n' + 1 \in \Phi$.

Proof. Let $n' \in \Phi$. Then $n' \geq 5$ and $2^{n'} > n'^2$.

Take $V = 2^{n'} \cdot 2$ and $W = n'^2 \cdot 2$. Then $V > W$. Indeed $2 \neq 0$.

Take $X = n'^2 + n'^2$. Then $W = X$.

Take $Y = n'^2 + ((2 \cdot n') + 1)$. Then $X > Y$. Indeed $n'^2 > (2 \cdot n') + 1$.

Take $Z = n' + 1^2$. Then $Y = Z$.

Then $2^{n' + 1} = V > W = X > Y = Z = n' + 1^2$. Hence $2^{n' + 1} > n' + 1^2$. □

Therefore Φ contains every element of $\mathbb{N}_{\geq 5}$ (by induction). Consequently $2^n > n^2$ (by a). ■