Definition 1. Let n, m be natural numbers. *$n \mid m$* iff there exists a natural number k such that $n \cdot k = m$. Let m is *divisible by n* stand for $n \mid m$. Let n *divides m* stand for $n \mid m$. Let *$n \nmid m$* stand for $\neg\, n \mid m$.

Lemma 2. Let n, m be natural numbers. n divides m iff there exists a natural number k such that $k \cdot n = m$.

Definition 3. Let n be a natural number. A *factor of n* is a natural number that divides n. Let a *divisor of n* stand for a factor of n.

Definition 4. Let n be a natural number. A *trivial divisor of n* is a divisor m of n such that $m = 1$ or $m = n$.

Definition 5. Let n be a natural number. A *nontrivial divisor of n* is a divisor m of n such that $m \neq 1$ and $m \neq n$.

Definition 6. Let n be a natural number. n is *composite* iff $n > 1$ and n has a nontrivial divisor.

Proposition 7. Let n be a natural number. Then $n \mid 0$.

Proof. We have $n \cdot 0 = 0$. Hence $n \mid 0$. ∎

Proposition 8. Let n be a natural number. If $0 \mid n$ then $n = 0$.

Proof. Assume $0 \mid n$. Consider a natural number k such that $0 \cdot k = n$. Then $n = 0$. ∎

Proposition 9. Let n be a natural number. Then $1 \mid n$.

Proof. We have $1 \cdot n = n$. Hence $1 \mid n$. ∎

Proposition 10. Let n be a natural number. Then $n \mid n$.

Proof. We have $n \cdot 1 = n$. Hence $n \mid n$. ∎

Proposition 11. Let n be a natural number. If $n \mid 1$ then $n = 1$.

Proof. Assume $n \mid 1$. Take a natural number k such that $n \cdot k = 1$. Suppose $n \neq 1$. Then $n < 1$ or $n > 1$.

Case $n < 1$. Then $n = 0$. Hence $0 = 0 \cdot k = n \cdot k = 1$. Contradiction. □

Case $n > 1$. We have $k \neq 0$. Indeed if $k = 0$ then $1 = n \cdot k = n \cdot 0 = 0$. Hence $k \geq 1$. Take a positive natural number l such that $n = 1 + l$. Then $1 < 1 + l = n = n \cdot 1 \leq n \cdot k$. Hence $1 < n$. Contradiction. □

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Proposition 12. Let n, m, k be natural numbers. If $n \mid m$ then $n \mid m \cdot k$.

Proof. Assume $n \mid m$. Take $l \in \mathbb{N}$ such that $n \cdot l = m$. Then $n \cdot (l \cdot k) = (n \cdot l) \cdot k = m \cdot k$. Hence $n \mid m \cdot k$. ∎

Corollary 13. Let n, m, k be natural numbers. If $n \mid m$ then $n \mid k \cdot m$.

Proposition 14. Let n, m, k be natural numbers. If $n \mid m \mid k$ then $n \mid k$.

Proof. Assume $n \mid m$ and $m \mid k$. Take natural numbers l, l' such that $n \cdot l = m$ and $m \cdot l' = k$. Then $n \cdot (l \cdot l') = (n \cdot l) \cdot l' = m \cdot l' = k$. Hence $n \mid k$. ∎

Proposition 15. Let n, m be natural numbers such that $n \neq 0$. If $n \mid m$ and $m \mid n$ then $n = m$.

Proof. Assume $n \mid m$ and $m \mid n$. Take natural numbers k, k' such that $n \cdot k = m$ and $m \cdot k' = n$. Then $n = m \cdot k' = (n \cdot k) \cdot k' = n \cdot (k \cdot k')$. Hence $k \cdot k' = 1$. Thus $k = 1 = k'$. Therefore $n = m$. ∎

Proposition 16. Let n, m, k be natural numbers. If $n \mid m$ then $k \cdot n \mid k \cdot m$.

Proof. Assume $n \mid m$. Take a natural number l such that $n \cdot l = m$. Then $(k \cdot n) \cdot l = k \cdot (n \cdot l) = k \cdot m$. Hence $k \cdot n \mid k \cdot m$. ∎

Proposition 17. Let n, m, k be natural numbers. Assume $k \neq 0$. If $k \cdot n \mid k \cdot m$ then $n \mid m$.

Proof. Assume $k \cdot n \mid k \cdot m$. Take a natural number l such that $(k \cdot n) \cdot l = k \cdot m$. Then $k \cdot (n \cdot l) = k \cdot m$. Hence $n \cdot l = m$ (by left-cancellability of multiplication). Thus $n \mid m$. ∎

Proposition 18. Let n, m, k be natural numbers. If $k \mid n$ and $k \mid m$ then $k \mid (n' \cdot n) + (m' \cdot m)$ for all natural numbers n', m'.

Proof. Assume $k \mid n$ and $k \mid m$. Let n', m' be natural numbers. Take natural numbers l, l' such that $k \cdot l = n$ and $k \cdot l' = m$. Then

$$k \cdot ((n' \cdot l) + (m' \cdot l'))$$

$$= (k \cdot (n' \cdot l)) + (k \cdot (m' \cdot l'))$$

$$
\begin{aligned}
&= ((k\cdot n')\cdot l) + ((k\cdot m')\cdot l') \\
&= (n'\cdot(k\cdot l)) + (m'\cdot(k\cdot l')) \\
&= (n'\cdot n) + (m'\cdot m).
\end{aligned}
$$

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Corollary 19. Let n, m, k be natural numbers. If $k \mid n$ and $k \mid m$ then $k \mid n+m$.

Proof. Assume $k \mid n$ and $k \mid m$. Take $n' = 1$ and $m' = 1$. Then $k \mid (n'\cdot n) + (m'\cdot m)$. $(n'\cdot n) + (m'\cdot m) = n+m$. Hence $k \mid n+m$. ■

Proposition 20. Let n, m, k be natural numbers. If $k \mid n$ and $k \mid n+m$ then $k \mid m$.

Proof. Assume $k \mid n$ and $k \mid n+m$.

Case $k = 0$. □

Case $k \neq 0$. Take a natural number l such that $n = k\cdot l$. Take a natural number l' such that $n+m = k\cdot l'$. Then $(k\cdot l)+m = k\cdot l'$. We have $l' \geq l$. Indeed if $l' < l$ then $n+m = k\cdot l' < k\cdot l = n$ (by preservation of ordering under left-multiplication). Hence we can take a natural number l'' such that $l' = l+l''$. Then $(k\cdot l)+m = k\cdot l' = k\cdot(l+l'') = (k\cdot l)+(k\cdot l'')$. Indeed $k\cdot(l+l'') = (k\cdot l)+(k\cdot l'')$ (by left-distributivity of multiplication over addition). Thus $m = (k\cdot l'')$ (by left-cancellability of addition). Indeed $k\cdot l$ and $k\cdot l''$ are natural numbers. Therefore $k \mid m$. □

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Proposition 21. Let n, m be natural numbers such that $n, m \neq 0$. If $m \mid n$ then $m \leq n$.

Proof. Assume $m \mid n$. Take a natural number k such that $m\cdot k = n$.

If $k = 0$ then $n = m \cdot k = m \cdot 0 = 0$. Thus $k \geq 1$. Assume $m > n$. Then $n = m \cdot k \geq m \cdot 1 = m > n$. Hence $n > n$. Contradiction. ∎