Definition 1. Let a be an object and f be a map. Let φ be a map from $\mathbb{N}$ to $\mathrm{dom}(f)$. φ is *recursively defined by a and f* iff $\varphi(0) = a$ and $\varphi(n+1) = f(\varphi(n))$ for every $n \in \mathbb{N}$.

Theorem 2 (Dedekind's Recursion Theorem: Existence). Let A be a set and $a \in A$ and $f : A \to A$. Then there exists a $\varphi : \mathbb{N} \to A$ that is recursively defined by a and f.

Proof. (a) Define

$$\Phi := \left\{ H \in \mathcal{P}(\mathbb{N} \times A) \,\middle|\, \begin{array}{l} (0, a) \in H \text{ and for all } n \in \mathbb{N} \text{ and all } x \in A \text{ if} \\ (n, x) \in H \text{ then } (n+1, f(x)) \in H \end{array} \right\}.$$

Let us show that $\bigcap \Phi \in \Phi$.

Proof. (0) $\mathbb{N} \times A \in \Phi$.

Proof. (i) $\mathbb{N} \times A \in \mathcal{P}(\mathbb{N} \times A)$.

(ii) $(0, a) \in \mathbb{N} \times A$.

(iii) for all $n \in \mathbb{N}$ and all $x \in A$ if $(n, x) \in \mathbb{N} \times A$ then $(n+1, f(x)) \in \mathbb{N} \times A$.

Proof. Let $n \in \mathbb{N}$ and $x \in A$. Assume $(n, x) \in \mathbb{N} \times A$. We have $n+1 \in \mathbb{N}$ and $f(x) \in A$. Hence $(n+1, f(x)) \in \mathbb{N} \times A$. $\square$ $\square$

(1) $\bigcap \Phi \in \mathcal{P}(\mathbb{N} \times A)$.

Proof. Any element of $\bigcap \Phi$ is contained in every element of Φ. Hence any element of $\bigcap \Phi$ is contained in $\mathbb{N} \times A$. Thus $\bigcap \Phi \subset \mathbb{N} \times A$. $\bigcap \Phi$ is a set. Hence $\bigcap \Phi$ is a subset of $\mathbb{N} \times A$. $\square$

(2) $(0, a) \in \bigcap \Phi$. Indeed $(0, a) \in \mathbb{N} \times A \in \Phi$.

(3) For all $n \in \mathbb{N}$ and all $x \in A$ if $(n, x) \in \bigcap \Phi$ then $(n+1, f(x)) \in \bigcap \Phi$.

Proof. Let $n \in \mathbb{N}$ and $x \in A$. Assume $(n, x) \in \bigcap \Phi$. Then (n, x) is contained in every element of Φ. Hence $(n+1, f(x))$ is contained in every element of Φ. Thus $(n+1, f(x)) \in \bigcap \Phi$. $\square$

Therefore $\bigcap \Phi \in \Phi$ (by a). $\square$

Let us show that for any $n \in \mathbb{N}$ there exists an $x \in A$ such that $(n, x) \in \bigcap \Phi$.

Proof. Define $\Psi := \{ n \in \mathbb{N} \mid \text{there exists an } x \in A \text{ such that } (n, x) \in \bigcap \Phi \}$.

(1) 0 is contained in Ψ. Indeed $(0, a) \in \bigcap \Phi$.

(2) For all $n \in \Psi$ we have $n+1 \in \Psi$.

Proof. Let $n \in \Psi$. Take an $x \in A$ such that $(n, x) \in \bigcap \Phi$. Then $(n+1, f(x)) \in \bigcap \Phi$. Hence $n+1 \in \Psi$. Indeed $f(x) \in A$. □

Therefore $n \in \Psi$ for every $n \in \mathbb{N}$ (by induction). □

Let us show that for all $n \in \mathbb{N}$ and all $x, y \in A$ if $(n, x), (n, y) \in \bigcap \Phi$ then $x = y$.

Proof. (b) Define $\Theta := \{n \in \mathbb{N} \mid \text{ for all } x, y \in A \text{ if } (n, x), (n, y) \in \bigcap \Phi \text{ then } x = y\}$.

(1) Θ contains 0.

Proof. Let us show that for all $x, y \in A$ if $(0, x), (0, y) \in \bigcap \Phi$ then $x = y$. Let $x, y \in A$. Assume $(0, x), (0, y) \in \bigcap \Phi$.

Let us show that $x, y = a$. Assume $x \neq a$ or $y \neq a$.

Case $x \neq a$. $(0, x), (0, a)$ are contained in every element of Φ. Then $(0, x), (0, a) \in \bigcap \Phi$. Take $H = (\bigcap \Phi) \setminus \{(0, x)\}$.

Let us show that $H \in \Phi$. (1) $H \in \mathcal{P}(\mathbb{N} \times A)$.

(2) $(0, a) \in H$.

(3) For all $n \in \mathbb{N}$ and all $b \in A$ if $(n, b) \in H$ then $(n+1, f(b)) \in H$.

Proof. Let $n \in \mathbb{N}$ and $b \in A$. Assume $(n, b) \in H$. Then $(n+1, f(b)) \in \bigcap \Phi$. [prover vampire] We have $(n+1, f(b)) \neq (0, x)$. Indeed $n+1, 0, f(b), x$ are objects and $n+1 \neq 0$. Hence $(n+1, f(b)) \in H$. □

[prover vampire] Thus $H \in \Phi$ (by a). [prover eprover] End.

Then $(0, x)$ is not contained in every member of Φ. Contradiction. □

Case $y \neq a$. $(0, y), (0, a)$ are contained in every element of Φ. Then $(0, y), (0, a) \in \bigcap \Phi$. Take $H = (\bigcap \Phi) \setminus \{(0, y)\}$.

Let us show that $H \in \Phi$. (1) $H \in \mathcal{P}(\mathbb{N} \times A)$.

(2) $(0, a) \in H$. Indeed $(0, a) \neq (0, y)$.

(3) For all $n \in \mathbb{N}$ and all $b \in A$ if $(n, b) \in H$ then $(n+1, f(b)) \in H$.

Proof. Let $n \in \mathbb{N}$ and $b \in A$. Assume $(n, b) \in H$. Then $(n+1, f(b)) \in \bigcap \Phi$. [prover vampire] We have $(n+1, f(b)) \neq (0, y)$. Indeed $n+1, 0, f(b), x$ are objects and $n+1 \neq 0$. Hence $(n+1, f(b)) \in H$. □

[prover vampire] Thus $H \in \Phi$ (by a). [prover eprover] End.

Then $(0, y)$ is not contained in every member of Φ. Contradiction. □

End. End. □

(2) For all $n \in \Theta$ we have $n+1 \in \Theta$.

Proof. Let $n \in \Theta$. Then for all $x, y \in A$ if $(n, x), (n, y) \in \bigcap \Phi$ then $x = y$. Consider a $b \in A$ such that $(n, b) \in \bigcap \Phi$. Then $(n+1, f(b)) \in \bigcap \Phi$.

Let us show that for all $x, y \in A$ if $(n+1, x), (n+1, y) \in \bigcap \Phi$ then $x = f(b) = y$. Let $x, y \in A$. Assume $(n+1, x), (n+1, y) \in \bigcap \Phi$. Suppose $x \neq f(b)$ or $y \neq f(b)$.

Case $x \neq f(b)$. Take $H = (\bigcap \Phi) \setminus \{(n+1, x)\}$.

(i) $H \in \mathcal{P}(\mathbb{N} \times A)$.

(ii) $(0, a) \in H$.

Proof. $(0, a) \in \bigcap \Phi$. [prover vampire] $(0, a) \notin \{(n+1, x)\}$. Indeed $(0, a) \neq (n+1, x)$. Indeed $n+1, 0, a, x$ are objects and $0 \neq n+1$. □

(iii) For all $m \in \mathbb{N}$ and all $z \in A$ if $(m, z) \in H$ then $(m+1, f(z)) \in H$.

Proof. Let $m \in \mathbb{N}$ and $z \in A$. Assume $(m, z) \in H$. Then $(m, z) \in \bigcap \Phi$. Hence $(m+1, f(z)) \in \bigcap \Phi$. $m+1, n+1, f(z), x$ are objects. [prover vampire] Thus if $(m+1, f(z)) = (n+1, x)$ then $m+1 = n+1$ and $f(z) = x$. Therefore if $(m+1, f(z)) = (n+1, x)$ then $m = n$ and $f(z) = x$ (by injectivity of successor function). Consequently $(m+1, f(z)) \neq (n+1, x)$. Thus $(m+1, f(z)) \notin \{(n+1, x)\}$. Therefore $(m+1, f(z)) \in H$. □

[prover vampire] Thus $H \in \Phi$ (by a, i, ii, iii). [prover eprover] Therefore every element of $\bigcap \Phi$ is contained in H. Consequently $(n+1, x) \in H$. Contradiction. □

Case $y \neq f(b)$. Take $H = (\bigcap \Phi) \setminus \{(n+1, y)\}$.

(i) $H \in \mathcal{P}(\mathbb{N} \times A)$.

(ii) $(0, a) \in H$.

Proof. $(0, a) \in \bigcap \Phi$. [prover vampire] $(0, a) \notin \{(n+1, y)\}$. Indeed $0, n+1, a, y$ are objects and $(0, a) \neq (n+1, y)$. Indeed $0 \neq n+1$. □

(iii) For all $m \in \mathbb{N}$ and all $z \in A$ if $(m, z) \in H$ then $(m+1, f(z)) \in H$.

Proof. Let $m \in \mathbb{N}$ and $z \in A$. Assume $(m, z) \in H$. Then $(m, z) \in \bigcap \Phi$. Hence $(m+1, f(z)) \in \bigcap \Phi$. $m+1, n+1, f(z), y$ are objects. [prover vampire] Thus if $(m+1, f(z)) = (n+1, y)$ then $m+1 = n+1$ and $f(z) = y$. Therefore if $(m+1, f(z)) = (n+1, y)$ then $m = n$ and $f(z) = y$ (by injectivity of successor function). Consequently $(m+1, f(z)) \neq (n+1, y)$. Thus $(m+1, f(z)) \notin \{(n+1, y)\}$. Therefore $(m+1, f(z)) \in H$. □

[prover vampire] Thus $H \in \Phi$ (by a, i, ii, iii). [prover eprover] Therefore every element of $\bigcap \Phi$ is contained in H. Consequently $(n+1, y) \in H$. Contradiction. □

Hence it is wrong that $x \neq f(b)$ or $y \neq f(b)$. Consequently $x = f(b) = y$. End.

Therefore $n+1 \in \Theta$ (by a). □

Consequently $n \in \Theta$ for every $n \in \mathbb{N}$ (by induction). □

Define $\varphi(n) :=$ "choose $x \in A$ such that $(n, x) \in \bigcap \Phi$ in x" for $n \in \mathbb{N}$.

(1) Then φ is a map from $\mathbb{N}$ to A and we have $\varphi(0) = a$.

(2) For all $n \in \mathbb{N}$ we have $\varphi(n+1) = f(\varphi(n))$.

Proof. Let $n \in \mathbb{N}$. Take $x \in A$ such that $\varphi(n) = x$. Then $(n, x) \in \bigcap \Phi$. Hence $(n+1, f(\varphi(n))) = (n+1, f(x)) \in \bigcap \Phi$. Thus $\varphi(n+1) = f(\varphi(n))$ (by a). $\square$ $\blacksquare$

Theorem 3 (Dedekind's Recursion Theorem: Uniqueness). Let A be a set and $a \in A$ and $f : A \to A$. Let $\varphi, \varphi' : \mathbb{N} \to A$. Assume that φ and φ' are recursively defined by a and f. Then $\varphi = \varphi'$.

Proof. Define $\Phi := \{n \in \mathbb{N} \mid \varphi(n) = \varphi'(n)\}$.

(1) Φ contains 0. Indeed $\varphi(0) = a = \varphi'(0)$.

(2) For all $n \in \Phi$ we have $n+1 \in \Phi$.

Proof. Let $n \in \Phi$. Then $\varphi(n) = \varphi'(n)$. Hence $\varphi(n+1) = f(\varphi(n)) = f(\varphi'(n)) = \varphi'(n+1)$. $\square$

Thus Φ contains every natural number (by induction). Consequently $\varphi(n) = \varphi'(n)$ for each $n \in \mathbb{N}$. $\blacksquare$