The Ontology of $\mathbb{N}$aproche

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Abstract

This document provides an overview of the object-level ontology of $\mathbb{N}$aproche. Moreover, it introduces common (symbolic and verbal) macros for the notions provided by this ontology which extend the core language of ForTheL.

Contents

1 Introduction

This document describes the notions and axioms hard-coded in $\mathbb{N}$aproche. Furthermore, it presents some additional symbolic and verbal macros that extend the core language of ForTheL. These macros are provided by the library `libraries/meta` which contains a module `preliminaries.ftl` that exports all these macros. Every library module (excluding those in the library `meta`) is expected to import the module `preliminaries.ftl`. So by importing any library module to your formalization, you automatically have access to all macros described in this document.

2 Propositions

ForTheL is essentially a surface language for (unsorted) first-order logic. We therefore provide some symbolic notations for logical connectives and quantifiers that are supported by ForTheL:

Convention 1. Let $\top$ stand for truth.

Convention 2. Let $\bot$ stand for falsity.

Convention 3. Let $\neg\varphi$ stand for not φ.

Convention 4. Let $\varphi \wedge \psi$ stand for φ and ψ.

Convention 5. Let $\varphi \vee \psi$ stand for φ or ψ.

Convention 6. Let $\varphi \implies \psi$ stand for φ implies ψ.

Convention 7. Let $\varphi \iff \psi$ stand for φ iff ψ.

Convention 8. Let $\forall x\ R\ y : \varphi$ stand for for all $x\ R\ y$ we have φ.

Convention 9. Let $\exists x\ R\ y : \varphi$ stand for there exists an $x\ R\ y$ such that φ.

3 Notions

3.1 Objects

Objects are mathematical entities that are “small” enough to be contained in classes.

Signature 10. A *mathematical object* is a notion. Let an *object* stand for a mathematical object. Let an *element* stand for a mathematical object.

3.2 Ordered Pairs

Signature 11. Let a, b be objects. (a, b) is an object.

Convention 12. Let the *ordered pair of a and b* stand for (a, b). Let an *ordered pair* stand for an object x such that $x = (a, b)$ for some objects a, b.

3.3 Classes and Sets

Signature 13. A *class* is a notion. Let a *collection* stand for a class.

Sets are considered to be classes that are “small” enough to be contained in other classes:

Definition 14. A *set* is a class that is an object.

Signature 15. Let A be a class. A *element of A* is an object. Let $a \in A$ stand for a is an element of A. Let $a \notin A$ stand for a is not an element of A.

Convention 16. Let a *member of X* stand for an element of X. Let x *belongs to X* stand for x is an element of X. Let X *contains x* stand for x is an element of X. Let x is *contained in X* stand for x is an element of X. Let x *lies in X* stand for x is an element of X. Let x is *in X* stand for x is an element of X.

3.4 Comprehension Terms

In ForTheL we have three types of comprehension terms:

- "Enumeration classes", i.e. terms of the form $\{a_1, \dots, a_n\}$, that represent the class that contains precisely the objects $a_1, \dots, a_n$. Typical examples are $\{4,7,11\}$ (i.e. the class consisting of the numbers 4, 7 and 11) or $\{x, y+1, z-x\}$ (i.e. the class consisting of the elements x, $y+1$ and $z-x$ for some fixed objects x, y and z).

- "Comprehension classes", i.e. terms of the form $\{\tau \mid \varphi\}$, that represent the class of all objects of the form τ such that the components of τ satisfy the condition φ. Typical examples are $\{p \mid p \text{ is an odd prime number}\}$ (i.e the class of all odd prime numbers) or $\{(p,n) \mid n \in \mathbb{N} \text{ and } p \text{ is a prime divisor of } n\}$ (i.e. the class of all pairs (p,n) of natural numbers such that p is a prime divisor of n).

- "Separation classes", i.e. terms of the form $\{\tau \in A \mid \varphi\}$, that represent the subclass of a class A of all objects of the form τ such that the components of τ satisfy the condition φ. Typical examples are $\{p \in \mathbb{P} \mid p \text{ is odd}\}$ (i.e the class of all odd prime numbers) or $\{(p,n) \in \mathbb{N} \times \mathbb{N} \mid p \text{ is a prime divisor of } n\}$ (i.e. the class of all pairs (p,n) of natural numbers such that p is a prime divisor of n).

Signature 17. Let $a_1, \dots, a_n$ be objects. $\{a1, \dots, a_n\}$ is a class.

Signature 18. Let τ be a term and φ be a formula. $\{\tau \mid \varphi\}$ is a class.

Signature 19. Let τ be a term, A be a class and φ be a formula. $\{\tau \in A \mid \varphi\}$ is a class.

3.5 Maps and Functions

Signature 20. A *map* is a notion.

Similar to the notion of sets which are considered to be "small" classes, we consider *functions* to be "small" maps.

Definition 21. A *function* is a map that is an object.

Signature 22. Let f be a map. $\mathrm{dom}(f)$ is a class. Let the *domain of f* stand for $\mathrm{dom}(f)$.

Convention 23. Let f is *defined on X* stand for $\mathrm{dom}(f) = X$.

Signature 24. Let f be a map and $a \in \mathrm{dom}(f)$. $f(a)$ is an object.

Convention 25. Let the *value of f at x* stand for $f(x)$. Let $f(x, y)$ stand for $f((x, y))$.

3.6 λ-Terms

In low-level map definitions, ForTheL allows to define maps via λ-terms:

Signature 26. Let v be a variable, A be a class and τ be a term. $\lambda v \in A.\, \tau$ is a map.

4 Relations

Convention 27. Let $x = y$ stand for x is equal to y.

Convention 28. Let $x \neq y$ stand for x is not equal to y.

Convention 29. Let x and y *agree* stand for $x = y$. Let x *agrees with y* stand for $x = y$.

Convention 30. Let x and y are *distinct* stand for $x \neq y$.

ℕaproche has a hard-coded induction proof tactic (cf. section 5.4) that relies on an "induction ordering" $\prec$. An important special case of this tactic is the following induction principle: If, for all x, we can show that a predicate $\varphi[y]$ holds for every $y \prec x$, then $\varphi[x]$ holds for every x. Typical instances of this principle are the (strong) induction principle from natural number arithmetics (with $\prec$ being the $<$-relation) or the $\in$-induction principle from set theory (with $\prec$ being the $\in$-relation).

Convention 31. Let $x \prec y$ stand for x is inductively less than y.

5 Axioms

5.1 Extensionality Axioms

To provide equality conditions for ordered pairs, classes and maps, the following extensionality axioms are hard-coded in ℕaproche.

Axiom 32. Let a, a', b, b' be objects. If $(a, b) = (a', b')$ then $a = a'$ and $b = b'$.

Axiom 33. Let A, B be classes. If every element of A is an element of B and every element of B is an element of A then $A = B$.

Axiom 34. Let f, g be maps. If $\mathrm{dom}(f) = \mathrm{dom}(g)$ and $f(a) = g(a)$ for all $a \in \mathrm{dom}(f)$ then $g = g$.

5.2 Comprehension Term Axioms

The following axioms specify the elements of comprehension terms (cf. section 3.4).

Axiom 35. Let τ be a term, A be a class and φ be a formula. If A is a set then $\{\tau{\in}A \mid \varphi\}$ is a set.

Corollary 36. Let A be a set and B be a class. Assume that every element of B is an element of A. Then B is a set.

Proof. Define $C := \{x{\in}A \mid x \in B\}$. Then C is a set and $C = B$. ∎

Axiom 37. Let $a_1, \dots, a_n$ be objects. Then $x \in \{a_1, \dots, a_n\}$ iff $x = a_1$ or $\dots$ or $x = a_n$.

Axiom 38. Let $v_1, \dots, v_n$ be variables, $\tau[v_1, \dots, v_n]$ be a term and $\varphi[v_1, \dots, v_n]$ be a formula. Then $x \in \{\tau[v_1, \dots, v_n] \mid \varphi[v_1, \dots, v_n]\}$ iff x is an object and there exist $y_1, \dots, y_n$ such that $\varphi[y_1, \dots, y_n]$ and $x = \tau[y_1, \dots, y_n]$.

Corollary 39. Let v be a variable and $\varphi[v]$ be a formula. Then $x \in \{v \mid \varphi[v]\}$ iff x is an object and $\varphi[x]$.

5.3 λ-Term Axioms

The following axioms specify the domain and values of λ-terms:

Axiom 40. Let v be a variable, A be a class and $\tau[v]$ be a term. Then $\mathrm{dom}(\lambda v \in A.\, \tau) = A$.

Axiom 41. Let v be a variable, A be a class and $\tau[v]$ be a term. Then $(\lambda v \in A.\, \tau)(x) = \tau[x]$ for all $x \in A$.

5.4 Miscellaneous Axioms

Subclasses of sets that are formed via separation class terms (cf. 3.4) are again sets:

Axiom 42. Let $v_1, \dots, v_n$ be variables, $\tau[v_1, \dots, v_n]$ be a term, A be a class and $\varphi[v_1, \dots, v_n]$ be a formula. Then $x \in \{\tau[v_1, \dots, v_n]{\in}A \mid \varphi[v_1, \dots, v_n]\}$ iff $x \in A$ and there exist $y_1, \dots, y_n$ such that $\varphi[y_1, \dots, y_n]$ and $x = \tau[y_1, \dots, y_n]$.

Corollary 43. Let v be a variable, A be a class and $\varphi[v]$ be a formula. Then $x \in \{v{\in}A \mid \varphi[v]\}$ iff $x \in A$ and $\varphi[x]$.

As noted in section 3.5, functions are considered to be “small” maps. Whether a map is small or not is determined by the “size” of its domain:

Axiom 44. Let f be a map. f is a function iff $\mathrm{dom}(f)$ is a set.

To facilitate proofs by induction, Naproche provides a hard-coded induction proof tactic which corresponds to the following induction principle.

Axiom 45. Let $v_1, \dots, v_n$ be variables, $\tau[v_1, \dots, v_n]$ be a term and $\psi_1[v_1]$, $\psi_2[v_1, v_2]$, $\dots$, $\psi_n[v_1, \dots, v_n]$ and $\varphi[v_1, \dots, v_n]$ be formulas. Assume that the following holds for all $x_1, \dots, x_n$ with $\psi_i[x_1, \dots, x_i]$ for all $i \in \{1, \dots, n\}$: If $\tau[y_1, \dots, y_n] \prec \tau[x_1, \dots, x_n]$ for all $y_1, \dots, y_n$ with $\psi_i[y_1, \dots, y_i]$ for all $i \in \{1, \dots, n\}$ then $\varphi[x_1, \dots, x_n]$. Then $\varphi[x_1, \dots, x_n]$ for all $x_1, \dots, x_n$ with $\psi_i[x_1, \dots, x_i]$ for all $i \in \{1, \dots, n\}$.

Corollary 46. Let A be a class, v be a variable and $\varphi[v]$ be a formula. Assume that the following holds for all $x \in A$: If $\varphi[y]$ for for all $y \in A$ with $y \prec x$ then $\varphi[x]$. Then $\varphi[x]$ for all $x \in A$.

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