Zermelo's Well-ordering Theorem in $\mathbb{N}$aproche

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2025

This is a formalization of *Zermelo's Well-ordering Theorem*, i.e. of the assertion that under the assumption of the axiom of choice every set is equinumerous to some ordinal number, where an ordinal number is regarded as a transitive set whose elements are transitive sets as well. The proof of this theorem presented here is oriented on [1].

In the following, for any class A, we write $A^{<\infty}$ to denote the collection of all maps $f : \alpha \to A$ for some ordinal α. Moreover, for any map $G : A^{<\infty} \to A$ we say that a map $F : \mathbb{O} \to A$, where $\mathbb{O}$ denotes the class of all ordinals, is recursive regarding G if $F(\alpha) = G(F \restriction \alpha)$ for all $\alpha \in \mathbb{O}$.

Theorem 1 (Zermelo's Well-Ordering Theorem). Every set is equinumerous to some ordinal.

Proof. Let x be a set. Consider a choice map g for $\mathcal{P}(x) \setminus \{\emptyset\}$. Take $A = x \cup \{x\}$. Every $F \in A^{<\infty}$ is a map from some ordinal to A. For any $F \in A^{<\infty}$ if $x \setminus \mathrm{range}(F) \neq \emptyset$ then $x \setminus \mathrm{range}(F) \in \mathrm{dom}(g)$. Indeed $x \setminus \mathrm{range}(F)$ is a subset of x for any $F \in A^{<\infty}$. Define

$$G(F) := \begin{cases} g(x \setminus \mathrm{range}(F)) & : x \setminus \mathrm{range}(F) \neq \emptyset \\ x & : x \setminus \mathrm{range}(F) = \emptyset \end{cases}$$

for $F \in A^{<\infty}$. We can show that for any $F \in A^{<\infty}$ if $x \setminus \mathrm{range}(F) \neq \emptyset$ then $G(F) \in x \setminus \mathrm{range}(F)$. Let $F \in A^{<\infty}$. Assume $x \setminus \mathrm{range}(F) \neq \emptyset$. Then $G(F) \in x \setminus \mathrm{range}(F)$. End. G is a map from $A^{<\infty}$ to A. Indeed we can show that for any $F \in A^{<\infty}$ we have $G(F) \in A$. Let $F \in A^{<\infty}$. If $x \setminus \mathrm{range}(F) \neq \emptyset$ then $G(F) \in x \setminus \mathrm{range}(F)$. If $x \setminus \mathrm{range}(F) = \emptyset$ then $G(F) = x$. Hence $G(F) \in A$. End. Hence we can take a map F from $\mathbb{O}$ to A that is recursive regarding G. For any ordinal α we have $F \restriction \alpha \in A^{<\infty}$.

For any $\alpha \in \mathbb{O}$ we have

$$x \setminus F[\alpha] \neq \emptyset \Longrightarrow F(\alpha) \in x \setminus F[\alpha]$$

and

$$x \setminus F[\alpha] = \emptyset \Longrightarrow F(\alpha) = x.$$

Proof. Let $\alpha \in \mathbb{O}$. We have $F[\alpha] = \{F(\beta) \mid \beta \in \alpha\}$. Hence $F[\alpha] = \{G(F \restriction \beta) \mid \beta \in \alpha\}$. We have $\operatorname{range}(F \restriction \alpha) = \{F(\beta) \mid \beta \in \alpha\}$. Thus $\operatorname{range}(F \restriction \alpha) = F[\alpha]$.

Case $x \setminus F[\alpha] \neq \emptyset$. Then $x \setminus \operatorname{range}(F \restriction \alpha) \neq \emptyset$. Hence $F(\alpha) = G(F \restriction \alpha) \in x \setminus \operatorname{range}(F \restriction \alpha) = x \setminus F[\alpha]$. $\square$

Case $x \setminus F[\alpha] \neq \emptyset$. Then $x \setminus \operatorname{range}(F \restriction \alpha) = \emptyset$. Hence $F(\alpha) = G(F \restriction \alpha) = x$. $\square$

$\square$

(1) For any ordinals α, β such that $\alpha < \beta$ and $F(\beta) \neq x$ we have $F(\alpha)$, $F(\beta) \in x$ and $F(\alpha) \neq F(\beta)$.

Proof. Let $\alpha, \beta \in \mathbb{O}$. Assume $\alpha < \beta$ and $F(\beta) \neq x$. Then $x \setminus F[\beta] \neq \emptyset$. (a) Hence $F(\beta) \in x \setminus F[\beta]$. We have $F[\alpha] \subset F[\beta]$. Thus $x \setminus F[\alpha] \neq \emptyset$. (b) Therefore $F(\alpha) \in x \setminus F[\alpha]$. Consequently $F(\alpha), F(\beta) \in x$ (by a, b). We have $F(\alpha) \in F[\beta]$ and $F(\beta) \notin F[\beta]$. Thus $F(\alpha) \neq F(\beta)$. $\square$

(2) There exists an ordinal α such that $F(\alpha) = x$.

Proof. Assume the contrary. Then F is a map from $\mathbb{O}$ to x.

Let us show that F is injective. Let $\alpha, \beta \in \mathbb{O}$. Assume $\alpha \neq \beta$. Then $\alpha < \beta$ or $\beta < \alpha$. Hence $F(\alpha) \neq F(\beta)$ (by 1). Indeed $F(\alpha), F(\beta) \neq x$. End.

Thus F is an injective map from some proper class to some set. Contradiction. $\square$

Define $\Phi := \{\alpha \in \mathbb{O} \mid F(\alpha) = x\}$. Φ is nonempty. Let us show that we can take an $\alpha \in \Phi$ such that $\alpha \leq \beta$ for all $\beta \in \Phi$. Take an $\alpha \in \Phi$ such that for no $\beta \in \Phi$ we have $\beta < \alpha$ (by wellfoundedness of membership relation). Then $\alpha \leq \beta$ for all $\beta \in \Phi$. End. Take $f = F \restriction \alpha$. Then f is a map from α to x. Indeed for no $\beta \in \alpha$ we have $F(\beta) = x$.

(3) f is surjective onto x.

Proof. $x \setminus F[\alpha] = \emptyset$. Hence $\operatorname{range}(f) = f[\alpha] = F[\alpha] = x$. $\square$

(4) f is injective.

Proof. Let $\beta, \gamma \in \alpha$. Assume $\beta \neq \gamma$. We have $f(\beta), f(\gamma) \neq x$. Hence $f(\beta) \neq f(\gamma)$ (by 1). Indeed $\beta < \gamma$ or $\gamma < \beta$. $\square$

Therefore f is a bijection between α and x. Consequently x and α are equinumerous. ■

References

[1] Peter Koepke. "Set Theory". Lecture notes. 2018/19. URL: `http://www.math.uni-bonn.de/ag/logik/teaching/2018WS/set_theory/current_scriptum.pdf`.

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