Peano's *Principles of Arithmetics*

Marcel Schütz (2025)

This is a translation of §1 of Giuseppe Peano's *Arithmetices Principia* (eng: *Principles of Arithmetics*) [1, 2] to ForTheL. It introduces Peano's language and famous axioms of arithmetics as well as a number of theorems about addition of natural numbers.

---

§1. Numbers and Addition.

—

Explanations.

Let a, b, c, d denote mathematical objects.

Signature. A *number* is a mathematical object.

Definition. N is the class of all numbers.

Signature. 1 is a mathematical object.

Signature. $a + 1$ is a mathematical object.

Axioms.

Axiom (1). $1 \mathrel{\epsilon} \mathrm{N}$.

Axiom (2). $a \mathrel{\epsilon} \mathrm{N} \;.\supset.\; a = a$.

Axiom (3). $a, b, c \mathrel{\epsilon} \mathrm{N} \;.\supset.\; (a = b \;.=.\; b = a)$.

Axiom (4). $a, b \mathrel{\epsilon} \mathrm{N} \;.\supset.\; ((a = b \;.\therefore\; b = c) \;.\supset.\; a = c)$.

Axiom (5). $((a = b \;.\therefore\; b \mathrel{\epsilon} \mathrm{N}) \;.\supset.\; a \mathrel{\epsilon} \mathrm{N})$.

Axiom (6). $a \epsilon \mathrm{N} \;.\supset.\; a+1 \epsilon \mathrm{N}$.

Axiom (7). $a, b \epsilon \mathrm{N} \;.\supset.\; (a = b \;.=.\; a+1 = b+1)$.

Axiom (8). $a \epsilon \mathrm{N} \;.\supset.\; a+1 \neq 1$.

Axiom (9). Let k be a class. Then $(1 \epsilon k \therefore \forall x \epsilon \mathrm{N} : x \epsilon k \;.\supset.\; x+1 \epsilon k) \;.\supset.\; \mathrm{N} \supset k$.

Definitions.

Definition (10.i). $2 := 1+1$.

Definition (10.ii). $3 := 2+1$.

Definition (10.iii). $4 := 3+1$.

Theorems.

Theorem (11). $2 \epsilon \mathrm{N}$.

Proof. (1) $1 \epsilon \mathrm{N}$ (by P1).

(2) $1 \epsilon \mathrm{N} \;.\supset.\; 1+1 \epsilon \mathrm{N}$ (by P6).

(3) $1+1 \epsilon \mathrm{N}$ (by 1, 2).

(4) $2 = 1+1$.

Therefore $2 \epsilon \mathrm{N}$. ■

Theorem (12). $3, 4 \epsilon \mathrm{N}$.

Theorem (13). $(a, b, c, d \epsilon \mathrm{N} \therefore a = b \therefore b = c \therefore c = d) \;.\supset.\; a = d$.

Theorem (14). $(a, b, c \epsilon \mathrm{N} \therefore a = b \therefore b = c \therefore a \neq c) \;.=.\; \Lambda$.

Theorem (15). $(a, b, c \epsilon \mathrm{N} \therefore a = b \therefore b \neq c) \;.\supset.\; a \neq c$.

Theorem (16). $(a, b \epsilon \mathrm{N} \therefore a+1 = b+1) \;.\supset.\; a = b$.

Theorem (17). $a, b \epsilon \mathrm{N} \;.\supset.\; (a \neq b \;.=.\; a+1 \neq b+1)$.

Definition.

Signature. $a + b$ is a mathematical object.

Axiom. $a \mathrel{\epsilon} \mathrm{N} \mathrel{.\supset.} a + 1 = a + 1$.

Axiom (18). $a, b \mathrel{\epsilon} \mathrm{N} \mathrel{.\supset.} a + (b + 1) = a + b + 1$.

Theorems.

Theorem (19). $a, b \mathrel{\epsilon} \mathrm{N} \mathrel{.\supset.} a + b \mathrel{\epsilon} \mathrm{N}$.

Proof. Let $a, b \mathrel{\epsilon} \mathrm{N}$. Define $T := \{b' \mid a + b' \mathrel{\epsilon} \mathrm{N}\}$.

We have $a \mathrel{\epsilon} \mathrm{N}$. Hence $a + 1 \mathrel{\epsilon} \mathrm{N}$ (by P6).

(1) Thus $1 \mathrel{\epsilon} T$.

(2) For all $b' \mathrel{\epsilon} \mathrm{N}$ if $b' \mathrel{\epsilon} T$ then $b' + 1 \mathrel{\epsilon} T$.

Proof. Let $b' \mathrel{\epsilon} \mathrm{N}$. Assume $b' \mathrel{\epsilon} T$. Then $a + b' \mathrel{\epsilon} \mathrm{N}$. Hence $a + b' + 1 \mathrel{\epsilon} \mathrm{N}$ (by P6). Thus $a + b' + 1 \mathrel{\epsilon} \mathrm{N}$ (by P18). Consequently $b' + 1 \mathrel{\epsilon} T$. □

(3) Therefore $\mathrm{N} \supset T$ (by 1, 2, P9).

We have $b \mathrel{\epsilon} \mathrm{N}$. Hence $b \mathrel{\epsilon} T$. Thus $a + b \mathrel{\epsilon} \mathrm{N}$. ■

Definition (20). $a + b + c := (a + b) + c$.

Theorem (21). $a, b, c \mathrel{\epsilon} \mathrm{N} \mathrel{.\supset.} a + b + c \mathrel{\epsilon} \mathrm{N}$.

References

[1] Giuseppe Peano. *Arithmetices Principia, Nova Methodo Exposita*. Libreria Bocca, 1889.

[2] Michael Nahas Vincent Verheyen. "Arithmetices Principia, Nova Methodo Exposita – or The Principles of Arithmetic, Presented by a New Method – both in the original Latin and in parallel English Translation with Modern Mathematical Notation". 2022. URL: https://github.com/mdnahas/Peano_Book/blob/master/Peano.pdf.

License

© Marcel Schütz (2025). This work is licensed under a Creative Commons "Attribution-NonCommercial-ShareAlike 4.0 International" (CC BY-NC-SA 4.0) license.