“Little Gauß”’ Theorem in ℕaproche

Christian Schöner, Marcel Schütz

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This is a formalization of “Little Gauß”’ Theorem, i.e. of the assertion that

$$\sum_{k=0}^{n} k = \frac{k(k+1)}{2}$$

for all $n \in \mathbb{N}$. In ℕaproche we can define the sum $\sum_{n=0}^{k} n$ by the function $\mathcal{G} : \mathbb{N} \to \mathbb{N}$ with $\mathcal{G}(0) = 0$ and $\mathcal{G}(n+1) = \mathcal{G}(n) + (n+1)$:

Let n denote a natural number.

Signature. $\mathcal{G}(n)$ is a natural number.

Axiom. $\mathcal{G}(0) = 0$.

Axiom. $\mathcal{G}(n+1) = \mathcal{G}(n) + (n+1)$.

Then the theorem can be stated as follows.

Theorem (Little Gauß). For all natural numbers n we have

$$2 \cdot \mathcal{G}(n) = n \cdot (n+1).$$

Proof. Define $\Phi := \{n \in \mathbb{N} \mid 2 \cdot \mathcal{G}(n) = n \cdot (n+1)\}$.

(1) $0 \in \Phi$.

(2) For all $n \in \Phi$ we have $n+1 \in \Phi$.

Proof. Let $n \in \Phi$. Then $2 \cdot \mathcal{G}(n) = n \cdot (n+1)$. Hence

$$\begin{aligned}2 \cdot \mathcal{G}(n+1) &= 2 \cdot (\mathcal{G}(n) + (n+1)) \\ &= (2 \cdot \mathcal{G}(n)) + (2 \cdot (n+1)) \\ &= (n \cdot (n+1)) + (2 \cdot (n+1)) \\ &= (n+2) \cdot (n+1) \\ &= (n+1) \cdot (n+2) \\ &= (n+1) \cdot ((n+1)+1)\end{aligned}$$

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