Kőnig's Theorem in ℕaproche

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Kőnig's Theorem is an important set-theoretical result about the arithmetic of cardinals. It was proved by Julius Kőnig in 1905 [1, p. 177–180]. The proof is reminiscent of Cantor's diagonal argument for proving that $\kappa < 2^\kappa$.

Sums and Products of cardinals

Let f_i stand for $f(i)$. Let D denote a set.

[synonym sequence/-s]

Definition. A sequence of cardinals on D is a function κ such that $\mathrm{dom}(\kappa) = D$ and κ_i is a cardinal for every element i of D.

Definition. Let κ be a sequence of cardinals on D.

$$\bigsqcup_{i \in D} \kappa_i := \{(n, i) \mid i \text{ is an element of } D \text{ and } n \text{ is an element of } \kappa_i\}.$$

Axiom. Let κ be a sequence of cardinals on D. Then $\bigsqcup_{i \in D} \kappa_i$ is a set.

Definition. Let κ be a sequence of cardinals on D.

$$\sum_{i \in D} \kappa_i := \left| \bigsqcup_{i \in D} \kappa_i \right|.$$

Definition. Let κ be a sequence of cardinals on D.

$$\bigtimes_{i \in ;D} \kappa_i := \left\{ f \,\middle|\, \begin{array}{l} f \text{ is a function and } \mathrm{dom}(f) = D \text{ and } f(i) \text{ is an element} \\ \text{of } \kappa_i \text{ for every element } i \text{ of } D \end{array} \right\}.$$

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$$\prod_{i \in D} \kappa_i := \left| \bigtimes_{i \in ;D} \kappa_i \right|.$$

Kőnig's Theorem requires some form of the axiom of choice. Currently choice is built into Naproche by the *choose* construct in function definitions. The axiom of choice is also required to show that products of non-empty factors are themselves non-empty:

Lemma (Choice). Let λ be a sequence of cardinals on D. Assume that λ_i has an element for every element i of D. Then $\bigtimes_{i \in; D} \lambda_i$ has an element.

Proof. Define $f(i) :=$ "choose an element v of λ_i in v" for $i \in D$. Then f is an element of $\bigtimes_{i \in; D} \lambda_i$. ■

Kőnig's theorem

Let D denote a set.

Theorem (Kőnig). Let κ, λ be sequences of cardinals on D. Assume that for every element i of D $\kappa_i < \lambda_i$. Then

$$\sum_{i \in D} \kappa_i < \prod_{i \in D} \lambda_i.$$

Proof by contradiction. Assume the contrary. Then

$$\prod_{i \in D} \lambda_i \leq \sum_{i \in D} \kappa_i.$$

Take a surjective map G from $\bigsqcup_{i \in D} \kappa_i$ to $\bigtimes_{i \in; D} \lambda_i$. Indeed $\bigtimes_{i \in; D} \lambda_i$ and $\sum_{i \in D} \kappa_i$ are nonempty sets. Take $\Lambda = \bigcup \mathrm{range}(\lambda)$. Then Λ is a set. Indeed $\mathrm{range}(\lambda)$ is a set.

$(n, i) \in \mathrm{dom}(G)$ for every $i \in D$ and every $n \in \kappa_i$. $G(n, i) \in \bigtimes_{i \in; D} \lambda_i$ for every $i \in D$ and every $n \in \kappa_i$. Hence for every $i \in D$ and every $n \in \kappa_i$ $G(n, i)$ is a map such that $i \in \mathrm{dom}(G(n, i))$.

Define $\Delta(i) := \{G(n, i)(i) {\in} \Lambda \mid n \in \kappa_i\}$ for $i \in D$.

For every element f of $\bigtimes_{i \in; D} \lambda_i$ and every element i of D we have $f(i) \in \Lambda$.

For every element i of D we have $|\Delta(i)| < \lambda_i$.

Proof by contradiction. Let i be an element of D. Define $F(n) := G(n, i)(i)$ for $n \in \kappa_i$. Then F is a map from κ_i to λ_i. We have $\Delta(i) = \{F(n) \mid n \in \kappa_i\}$. Thus $F[\kappa_i] = \Delta(i)$. Therefore $|\Delta(i)| = |F[\kappa_i]| \leq |\kappa_i| = \kappa_i < \lambda_i$. Indeed $|F[\kappa_i]| \leq |\kappa_i|$ (by cardinality of image). Indeed κ_i and λ_i are sets. □

Define $f(i) :=$ "choose an element v of $\lambda_i \setminus \Delta(i)$ in v" for $i \in D$. Indeed $\lambda_i \setminus \Delta(i)$ is nonempty for each $i \in D$. Then f is an element of $\bigtimes_{i \in; D} \lambda_i$. Take an element j of D and an element m of κ_j such that $G(m, j) = f$.

$G(m, j)(j)$ is an element of $\Delta(j)$ and $f(j)$ is not an element of $\Delta(j)$. Contradiction. ∎

References

[1] Gyula Kőnig. “Zum Kontinuumsproblem”. In: *Mathematische Annalen* 60 (1905).

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