The Knaster-Tarski Fixed Point Theorem in $\mathbb{N}$aproche

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This is a formalization of the *Knaster-Tarski Fixed Point Theorem*, i.e. of the assertion that every subset-preserving map has a fixed point [1].

Theorem 1 (Knaster-Tarski). Let x be a set. Let h be a map from $\mathcal{P}(x)$ to $\mathcal{P}(x)$ that preserves subsets. Then h has a fixed point.

Proof. (1) Define $A := \{y \in \mathcal{P}(x) \mid y \subset h(y)\}$. Then A is a subset of $\mathcal{P}(x)$. We have $\bigcup A \in \mathcal{P}(x)$.

Let us show that (2) $\bigcup A \subset h(\bigcup A)$. Let $u \in \bigcup A$. Take $y \in A$ such that $u \in y$. Then $u \in h(y)$. We have $y \subset \bigcup A$. Hence $h(y) \subset h(\bigcup A)$. Indeed h is a subset preserving map between systems of sets and $y, \bigcup A \in \mathrm{dom}(h)$. Thus $h(y) \subset h(\bigcup A)$. Therefore $u \in h(\bigcup A)$. End.

Then $h(\bigcup A) \in A$ (by 1). Indeed $h(\bigcup A) \subset x$. (3) Hence $h(\bigcup A) \subset \bigcup A$. Indeed every element of $h(\bigcup A)$ is an element of some element of A.

Thus $h(\bigcup A) = \bigcup A$ (by 2, 3). ■

References

[1] Bernd S. W. Schröder. “The fixed point property for ordered sets”. In: *Arabian Journal of Mathematics* 1 (2012), pp. 529–547.

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