Hilbert's Paradox in $\mathbb{N}$aproche

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Hilbert's Paradox, discovered around 1900 by David Hilbert, demonstates that there cannot exists a set that is closed under powersets and under unions of arbitrary subsets [1].

Theorem 1 (Hilbert's Paradox). There exists no system of sets that is closed under powersets and closed under unions.

Proof. Assume the contrary. Consider a system of sets S that is closed under powersets and closed under unions. We have $S \subset S$. Hence $\bigcup S \in S$. Thus $\mathcal{P}(\bigcup S) \in S$. Contradiction. ■

Using Hilbert's Paradox it can further be shown that there exists no universal set, i.e. that the class $\mathbb{V}$ of all sets is a proper class:

Corollary 2. $\mathbb{V}$ is not a set.

Proof. Assume the contrary. Then $\mathbb{V}$ is closed under powersets and closed under unions. Contradiction (by Hilbert's Paradox). ■

References

[1] Volker Peckhaus and Reinhard Kahl. "Hilbert's Paradox". In: *Historia Mathematica* 29 (2 2002), pp. 157–175. DOI: 10.1006/hmat.2002.2345.

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