Regularity of Successor Cardinals in ℕaproche

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This is a formalization of a theorem of Felix Hausdorff stating that successor cardinals are always regular.

In lack of a formalization in ℕaproche of a proof of *Hessenberg's Theorem* [2], i.e. the fact that $|X \times X| = |X|$ for every infinite set X, which is required by the subsequent proof of Hausdorff's Theorem, we postulate it as an axiom:

Axiom (Hessenberg's Theorem). $|X \times X| = |X|$ for every infinite set X.

The following result appears in [1, p. 443], where Hausdorff mentions that the proof of the regularity of successor cardinals is *"ganz einfach"* (*"very simple"*) and can be skipped.

Theorem (Hausdorff). Let κ be a cardinal. Then κ^+ is regular.

Proof by contradiction.

Case κ is finite. $\square$

Case κ is infinite. Assume the contrary. Take a cofinal subset x of κ^+ such that $|x| \neq \kappa^+$. Then $|x| \leq \kappa$. Take a surjective map f from κ onto x (by existence condition for surjections). Indeed x and κ are nonempty and $|\kappa| = \kappa$. Then $f(\xi) \in \kappa^+$ for all $\xi \in \kappa$.

Let us show that for all $z \in \kappa^+$ if z is nonempty then there exists a surjective map from κ onto z. Let $z \in \kappa^+$. Assume that z is nonempty. κ is nonempty. Hence the thesis (by existence condition for surjections). Indeed $|\kappa| \geq |z|$. End.

Define

$$g(z) := \begin{cases} \text{choose } h : \kappa \twoheadrightarrow z \text{ in } h & : z \text{ has an element} \\ \text{const}^0_\kappa & : z \text{ has no element} \end{cases}$$

for $z \in \kappa^+$.

Let us show that for all $\xi, \zeta \in \kappa$ $g(f(\xi))$ is a map such that $\zeta \in \mathrm{dom}(g(f(\xi)))$. Let $\xi, \zeta \in \kappa$. If $f(\xi)$ has an element then $g(f(\xi))$ is a surjective map from κ onto $f(\xi)$. If $f(\xi)$ has no element then $g(f(\xi)) = \mathrm{const}^0_\kappa$. Hence $\mathrm{dom}(g(f(\xi))) = \kappa$. Therefore $\zeta \in \mathrm{dom}(g(f(\xi)))$. End.

[prover vampire] For all objects ξ, ζ we have $\xi, \zeta \in \kappa$ iff $(\xi, \zeta) \in \kappa \times \kappa$. Define $h(\xi, \zeta) := g(f(\xi))(\zeta)$ for $(\xi, \zeta) \in \kappa \times \kappa$. [prover eprover]

Let us show that h is surjective onto κ^+.

Every element of κ^+ is an element of $h[\kappa \times \kappa]$.

Proof by contradiction. Let n be an element of κ^+. Take an element ξ of κ such that $n < f(\xi)$. Take an element ζ of κ such that $g(f(\xi))(\zeta) = n$. Indeed $g(f(\xi))$ is a surjective map from κ onto $f(\xi)$. Then $n = h(\xi, \zeta)$. $\square$

Every element of $h[\kappa \times \kappa]$ is an element of κ^+.

Proof by contradiction. Let n be an element of $h[\kappa \times \kappa]$. We can take elements a, b of κ such that $n = h(a, b)$. Indeed $h[\kappa \times \kappa] = \{h(a, b) \mid a, b \in \kappa\}$. Indeed $\kappa \times \kappa = \{(a, b) \mid a, b \in \kappa\}$. Then $n = g(f(a))(b)$. Indeed $(a, b) \in \kappa \times \kappa$ and $h(a, b) = g(f(a))(b)$. $f(a)$ is an element of κ^+. Every element of $f(a)$ is an element of κ^+.

Case $f(a)$ has an element. Then $g(f(a))$ is a surjective map from κ onto $f(a)$. Hence $n \in f(a) \in \kappa^+$. Thus $n \in \kappa^+$. $\square$

Case $f(a)$ has no element. Then $g(f(a)) = \mathrm{const}^0_\kappa$. Hence n is the empty set. Thus $n \in \kappa^+$. $\square$

$\square$

Hence $\mathrm{range}(h) = h[\kappa \times \kappa] = \kappa^+$. End.

Therefore $|\kappa^+| \leq |\kappa \times \kappa|$ (by existence condition for surjections). Indeed $\kappa \times \kappa$ and κ^+ are nonempty sets and h is a surjective map from $\kappa \times \kappa$ to κ^+. Consequently $\kappa^+ \leq \kappa$. Contradiction. $\square$

$\blacksquare$

References

[1] Felix Hausdorff. “Grundzüge einer Theorie der geordneten Mengen”. In: *Mathematische Annalen* 65 (1908), pp. 435–505.

[2] Gerhard Hessenberg. *Grundbegriffe der Mengenlehre*. Vandenhoeck und Ruprecht, Göttingen, 1906.

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