Curry's Paradox in $\mathbb{N}$aproche

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Curry's Paradox is a paradox described by Haskell Curry in 1942 [1]. It allows the derivation of an arbitrary statement from a self-referential expression that presupposes its own validity.

Untyped λ-Calculus

Our formalization of Curry's Paradox in $\mathbb{N}$aproche is based on an untyped λ-calculus:

Signature. An *expression* is a notion.

Let E, E' denote expressions.

Signature. A *variable* is an expression.

Signature. x_0 is a variable.

Signature (Abstraction). Let x be a variable. $\lambda x. E$ is an expression.

Signature (Application). $(E)(E')$ is an expression.

Signature (Fixed-point combinator). `fix` is an expression such that

$$(\mathtt{fix})(E) = (E)((\mathtt{fix})(E))$$

for every expression E.

Propositional Logic

We extend this λ-calculus by a fragment of propositional logic:

Let E, E' denote expressions. Let x denote a variable.

Signature (Implication). $E \to E'$ is an expression.

Signature (Truth). E is *true* is a relation.

Axiom (β-reduction). $(\lambda x.\, x \to E')(E) = E \to E'$.

Axiom (Reflexivity). $E \to E$ is true.

Axiom (Modus Ponens). If E is true and $E \to E'$ is true then E' is true.

Axiom (Strengthening). If $E \to (E \to E')$ is true then $E \to E'$ is true.

Curry's paradox

Using the fixpoint combinator from above we can formulate a self-referential expression X of the form "If X is true then E is true" for any arbitrary expression E by defining $X = (\texttt{fix})(\lambda x_0.\, x_0 \to E)$. From the existence of such an expression X together with the above axioms we can then derive that any expression E is true.

Theorem (Curry's Paradox). Every expression is true.

Proof. Let E be an expression. Take $N = \lambda x_0.\, x_0 \to E$ and $X = (\texttt{fix})(N)$.

(1) Then $X = (N)(X) = X \to E$ (by β-reduction).

Hence $X \to (X \to E)$ is true (by 1, reflexivity).

(2) Thus $X \to E$ is true (by strengthening).

(3) Therefore X is true (by 1, 2).

Consequently E is true (by modus ponens, 2, 3). ■

References

[1] Haskell B. Curry. "The inconsistency of certain formal logics". In: *Journal of Symbolic Logic* 7.13 (1942), pp. 115–117.

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