# The Cantor-Schröder-Bernstein Theoremin $\mathbb{N}$aproche

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This is a formalization of the *Cantor-Schröder-Bernstein Theorem*, i.e. of the fact that two sets are equinumerous iff they can be embedded into each other [1]. Its proof of is based on the Knaster-Tarski Fixed Point Theorem.

**Theorem 1 (Cantor-Schröder-Bernstein).** Let $x, y$ be sets. $x$ and $y$ are equinumerous iff there exists a injective map from $x$ to $y$ and there exists an injective map from $y$ to $x$.

*Proof.*

*Case $x$ and $y$ are equinumerous.* Take a bijection $f$ between $x$ and $y$. Then $f^{-1}$ is a bijection between $y$ and $x$. Hence $f$ is an injective map from $x$ to $y$ and $f^{-1}$ is an injective map from $y$ to $x$. $\square$

*Case there exists an injective map from $x$ to $y$ and there exists an injective map from $y$ to $x$.* Take an injective map $f$ from $x$ to $y$. Take an injective map $g$ from $y$ to $x$. We have $y \setminus f[a] \subset y$ for any $a \in \mathcal{P}(x)$.

(1) Define $h(a) := x \setminus g[y \setminus f[a]]$ for $a \in \mathcal{P}(x)$.

$h$ is a map from $\mathcal{P}(x)$ to $\mathcal{P}(x)$. Indeed $h(a)$ is a subset of $x$ for each subset $a$ of $x$.

Let us show that $h$ preserves subsets. Let $u, v$ be subsets of $x$. Assume $u \subset v$. Then $f[u] \subset f[v]$. Hence $y \setminus f[v] \subset y \setminus f[u]$. Thus $g[y \setminus f[v]] \subset g[y \setminus f[u]]$. Indeed $y \setminus f[v]$ and $y \setminus f[u]$ are subsets of $y$. Therefore $x \setminus g[y \setminus f[u]] \subset x \setminus g[y \setminus f[v]]$. Consequently $h[u] \subset h[v]$. End.

Hence we can take a fixed point $c$ of $h$ (by Knaster-Tarski).

(2) Define $F(u) := f(u)$ for $u \in c$.

We have $c = h(c)$ iff $x \setminus c = g[y \setminus f[c]]$. $g^{-1}$ is a bijection between $\mathrm{range}(g)$ and $y$. Thus $x \setminus c = g[y \setminus f[c]] \subset \mathrm{range}(g)$. Therefore $x \setminus c$ is a subset of $\mathrm{dom}(g^{-1})$.

(3) Define $G(u) := g^{-1}(u)$ for $u \in x \setminus c$.

$F$ is a bijection between $c$ and $\mathrm{range}(F)$. $G$ is a bijection between $x \setminus c$ and $\mathrm{range}(G)$.

Define

$$H(u) := \begin{cases} F(u) & : u \in c \\ G(u) & : u \notin c \end{cases}$$

for $u \in x$.

Let us show that $H$ is a map to $y$. $\mathrm{dom}(H)$ is a set and every value of $H$ is an object. Hence $H$ is a map.

Let us show that every value of $H$ lies in $y$. Let $v$ be a value of $H$. Take $u \in x$ such that $H(u) = v$. If $u \in c$ then $v = H(u) = F(u) = f(u) \in y$. If $u \notin c$ then $v = H(u) = G(u) = g^{-1}(u) \in y$. End. End.

(4) $H$ is surjective onto $y$. Indeed we can show that every element of $y$ is a value of $H$. Let $v \in y$.

*Case* $v \in f[c]$. Take $u \in c$ such that $f(u) = v$. Then $F(u) = v$. □

*Case* $v \notin f[c]$. Then $v \in y \setminus f[c]$. Hence $g(v) \in g[y \setminus f[c]]$. Thus $g(v) \in x \setminus h(c)$. We have $g(v) \in x \setminus c$. Therefore we can take $u \in x \setminus c$ such that $G(u) = v$. Then $v = H(u)$. □

End.

(5) $H$ is injective. Indeed we can show that for all $u, v \in x$ if $u \neq v$ then $H(u) \neq H(v)$. Let $u, v \in x$. Assume $u \neq v$.

*Case* $u, v \in c$. Then $H(u) = F(u)$ and $H(v) = F(v)$. We have $F(u) \neq F(v)$. Hence $H(u) \neq H(v)$. □

*Case* $u, v \notin c$. Then $H(u) = G(u)$ and $H(v) = G(v)$. We have $G(u) \neq G(v)$. Hence $H(u) \neq H(v)$. □

*Case* $u \in c$ *and* $v \notin c$. Then $H(u) = F(u)$ and $H(v) = G(v)$. Hence $v \in g[y \setminus F[c]]$. We have $G(v) = g^{-1}(v) \in y \setminus F[c]$. Indeed $v = g(v')$ for some $v' \in y \setminus F[c]$. Thus $G(v) \neq F(u)$. □

*Case* $u \notin c$ *and* $v \in c$. Then $H(u) = G(u)$ and $H(v) = F(v)$. Hence $u \in g[y \setminus F[c]]$. We have $G(u) = g^{-1}(u) \in y \setminus F[c]$. Indeed $u = g(u')$ for some $u' \in y \setminus F[c]$. Thus $G(u) \neq F(v)$. □

End.

Consequently $H$ is a bijection between $x$ and $y$ (by 4, 5). Therefore $x$ and $y$ are equinumerous. □

■

## References

[1] Bernd S. W. Schröder. “The fixed point property for ordered sets”. In: *Arabian Journal of Mathematics* 1 (2012), pp. 529–547.

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