Cantor's Paradoxes in $\mathbb{N}$aproche

Marcel Schütz

2025

Cantor's First Paradox, discovered by Georg Cantor in 1897, states that the collection $\mathbb{C}$ of all (infinite) cardinal numbers is a proper class [1, chapter 156].

Theorem 1 (Cantor's First Paradox). $\mathbb{C}$ is not a set.

Proof. Suppose that $\mathbb{C}$ is a set. Then $\bigcup \mathbb{C}$ is a set.

Let us show that $\bigcup \mathbb{C}$ contains every ordinal. Let α be an ordinal. Choose an infinite ordinal β such that $\beta \geq \alpha$. Choose a cardinal κ greater than β. Then $\alpha \in \kappa \in \mathbb{C}$. Hence $\alpha \in \bigcup \mathbb{C}$. End.

Therefore $\mathbb{O} \subset \bigcup \mathbb{C}$. Thus $\mathbb{O}$ is a set. Contradiction. ■

Cantor's Second Paradox denotes the observation that the collection $\mathbb{V}$ of all set is a proper class. It was shown by Georg Cantor in 1899 via his famous theorem stating that the cardinality of any set is strictly smaller than the cardinality of its powerset [1, chapter 163].

Theorem 2 (Cantor's Second Paradox). $\mathbb{V}$ is not a set.

Proof. Assume the contrary. Then $\mathcal{P}(\mathbb{V})$ is a subset of $\mathbb{V}$. Hence $|\mathcal{P}(\mathbb{V})| \leq |\mathbb{V}|$. Contradiction. Indeed $|x| < |\mathcal{P}(x)|$ for any set x. ■

References

[1] Georg Cantor. *Briefe*. Ed. by Herbert Meschkowski and Winfried Nilson. Springer-Verlag Berlin Heidelberg, 1991.

License