# Isabelle/HOLCF — Higher-Order Logic of Computable Functions

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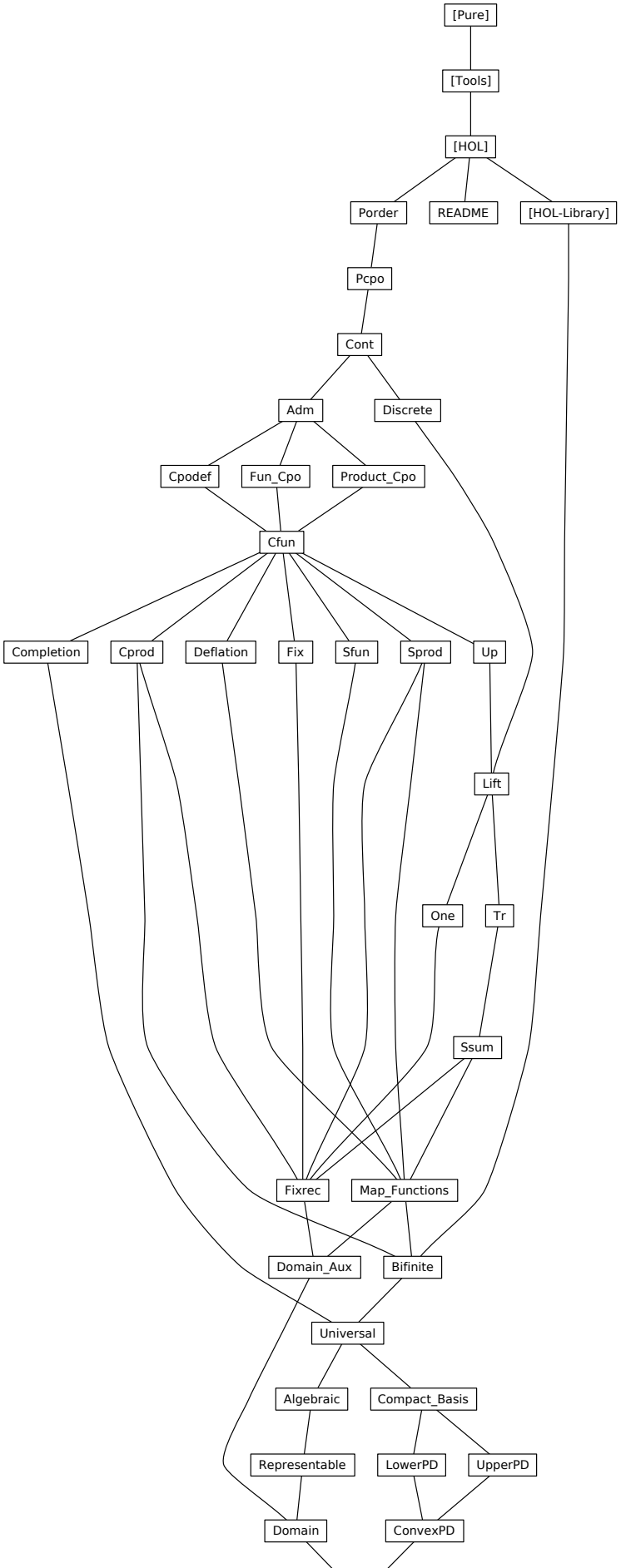
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# 1 Partial orders

**theory** *Porder*
**imports** *Main*
**begin**

**declare** [[*typedef-overloaded*]]

## 1.1 Type class for partial orders

**class** *below* =
**fixes** *below* :: $'a \Rightarrow 'a \Rightarrow bool$
**begin**

**notation** (*ASCII*)
*below* (**infix** $<<$ *50*)

**notation**
*below* (**infix** $\sqsubseteq$ *50*)

**abbreviation** *not-below* :: $'a \Rightarrow 'a \Rightarrow bool$ (**infix** $\not\sqsubseteq$ *50*)
**where** *not-below x y* $\equiv \neg$ *below x y*

**notation** (*ASCII*)
*not-below* (**infix** $^{\sim}<<$ *50*)

**lemma** *below-eq-trans*: $a \sqsubseteq b \Longrightarrow b = c \Longrightarrow a \sqsubseteq c$
**by** (*rule subst*)

**lemma** *eq-below-trans*: $a = b \Longrightarrow b \sqsubseteq c \Longrightarrow a \sqsubseteq c$
**by** (*rule ssubst*)

**end**

**class** *po* = *below* +
**assumes** *below-refl* [*iff*]: $x \sqsubseteq x$
**assumes** *below-trans*: $x \sqsubseteq y \Longrightarrow y \sqsubseteq z \Longrightarrow x \sqsubseteq z$
**assumes** *below-antisym*: $x \sqsubseteq y \Longrightarrow y \sqsubseteq x \Longrightarrow x = y$
**begin**

**lemma** *eq-imp-below*: $x = y \Longrightarrow x \sqsubseteq y$
**by** *simp*

**lemma** *box-below*: $a \sqsubseteq b \Longrightarrow c \sqsubseteq a \Longrightarrow b \sqsubseteq d \Longrightarrow c \sqsubseteq d$
**by** (*rule below-trans* [*OF below-trans*])

**lemma** *po-eq-conv*: $x = y \longleftrightarrow x \sqsubseteq y \wedge y \sqsubseteq x$
**by** (*fast intro*!: *below-antisym*)

**lemma** *rev-below-trans*: $y \sqsubseteq z \Longrightarrow x \sqsubseteq y \Longrightarrow x \sqsubseteq z$

**by** (*rule below-trans*)

**lemma** *not-below2not-eq*: $x \not\sqsubseteq y \Longrightarrow x \neq y$
**by** *auto*

**end**

**lemmas** *HOLCF-trans-rules* [*trans*] =
*below-trans*
*below-antisym*
*below-eq-trans*
*eq-below-trans*

**context** *po*
**begin**

## 1.2 Upper bounds

**definition** *is-ub* :: $'a\ set \Rightarrow 'a \Rightarrow bool$ (**infix** $<|$ *55*)
**where** $S <| x \longleftrightarrow (\forall y{\in}S.\ y \sqsubseteq x)$

**lemma** *is-ubI*: $(\bigwedge x.\ x \in S \Longrightarrow x \sqsubseteq u) \Longrightarrow S <| u$
**by** (*simp add*: *is-ub-def*)

**lemma** *is-ubD*: $[\![S <| u;\ x \in S]\!] \Longrightarrow x \sqsubseteq u$
**by** (*simp add*: *is-ub-def*)

**lemma** *ub-imageI*: $(\bigwedge x.\ x \in S \Longrightarrow f\ x \sqsubseteq u) \Longrightarrow (\lambda x.\ f\ x)\ `\ S <| u$
**unfolding** *is-ub-def* **by** *fast*

**lemma** *ub-imageD*: $[\![f\ `\ S <| u;\ x \in S]\!] \Longrightarrow f\ x \sqsubseteq u$
**unfolding** *is-ub-def* **by** *fast*

**lemma** *ub-rangeI*: $(\bigwedge i.\ S\ i \sqsubseteq x) \Longrightarrow range\ S <| x$
**unfolding** *is-ub-def* **by** *fast*

**lemma** *ub-rangeD*: $range\ S <| x \Longrightarrow S\ i \sqsubseteq x$
**unfolding** *is-ub-def* **by** *fast*

**lemma** *is-ub-empty* [*simp*]: $\{\} <| u$
**unfolding** *is-ub-def* **by** *fast*

**lemma** *is-ub-insert* [*simp*]: $(insert\ x\ A) <| y = (x \sqsubseteq y \wedge A <| y)$
**unfolding** *is-ub-def* **by** *fast*

**lemma** *is-ub-upward*: $[\![S <| x;\ x \sqsubseteq y]\!] \Longrightarrow S <| y$
**unfolding** *is-ub-def* **by** (*fast intro*: *below-trans*)

## 1.3 Least upper bounds

**definition** *is-lub* :: *'a set* ⇒ *'a* ⇒ *bool* (**infix** <<| *55*)
  **where** $S <<| x \longleftrightarrow S <| x \wedge (\forall u.\ S <| u \longrightarrow x \sqsubseteq u)$

**definition** *lub* :: *'a set* ⇒ *'a*
  **where** *lub S* = (*THE x*. *S* <<| *x*)

**end**

**syntax** (*ASCII*)
  *-BLub* :: [*pttrn*, *'a set*, *'b*] ⇒ *'b* ((*3LUB* -:-./ -) [*0*,*0*, *10*] *10*)

**syntax**
  *-BLub* :: [*pttrn*, *'a set*, *'b*] ⇒ *'b* ((*3*⨆-∈-./ -) [*0*,*0*, *10*] *10*)

**translations**
  *LUB x*:*A*. *t* ⇌ *CONST lub* ((λ*x*. *t*) ‘ *A*)

**context** *po*
**begin**

**abbreviation** *Lub* (**binder** ⨆ *10*)
  **where** ⨆*n*. *t n* ≡ *lub* (*range t*)

**notation** (*ASCII*)
  *Lub* (**binder** *LUB* *10*)

access to some definition as inference rule

**lemma** *is-lubD1*: $S <<| x \Longrightarrow S <| x$
  **unfolding** *is-lub-def* **by** *fast*

**lemma** *is-lubD2*: $[\![S <<| x;\ S <| u]\!] \Longrightarrow x \sqsubseteq u$
  **unfolding** *is-lub-def* **by** *fast*

**lemma** *is-lubI*: $[\![S <| x;\ \bigwedge u.\ S <| u \Longrightarrow x \sqsubseteq u]\!] \Longrightarrow S <<| x$
  **unfolding** *is-lub-def* **by** *fast*

**lemma** *is-lub-below-iff*: $S <<| x \Longrightarrow x \sqsubseteq u \longleftrightarrow S <| u$
  **unfolding** *is-lub-def is-ub-def* **by** (*metis below-trans*)

lubs are unique

**lemma** *is-lub-unique*: $S <<| x \Longrightarrow S <<| y \Longrightarrow x = y$
  **unfolding** *is-lub-def is-ub-def* **by** (*blast intro*: *below-antisym*)

technical lemmas about *lub* and (<<|)

**lemma** *is-lub-lub*: $M <<| x \Longrightarrow M <<| lub\ M$
  **unfolding** *lub-def* **by** (*rule theI* [*OF* - *is-lub-unique*])

**lemma** *lub-eqI*: $M <<| l \Longrightarrow lub\ M = l$
**by** (*rule is-lub-unique* [*OF is-lub-lub*])

**lemma** *is-lub-singleton* [*simp*]: $\{x\} <<| x$
**by** (*simp add*: *is-lub-def*)

**lemma** *lub-singleton* [*simp*]: $lub\ \{x\} = x$
**by** (*rule is-lub-singleton* [*THEN lub-eqI*])

**lemma** *is-lub-bin*: $x \sqsubseteq y \Longrightarrow \{x, y\} <<| y$
**by** (*simp add*: *is-lub-def*)

**lemma** *lub-bin*: $x \sqsubseteq y \Longrightarrow lub\ \{x, y\} = y$
**by** (*rule is-lub-bin* [*THEN lub-eqI*])

**lemma** *is-lub-maximal*: $S <| x \Longrightarrow x \in S \Longrightarrow S <<| x$
**by** (*erule is-lubI*, *erule* (*1*) *is-ubD*)

**lemma** *lub-maximal*: $S <| x \Longrightarrow x \in S \Longrightarrow lub\ S = x$
**by** (*rule is-lub-maximal* [*THEN lub-eqI*])

## 1.4 Countable chains

**definition** *chain* :: $(nat \Rightarrow 'a) \Rightarrow bool$
**where** — Here we use countable chains and I prefer to code them as functions!
$chain\ Y = (\forall i.\ Y\ i \sqsubseteq Y\ (Suc\ i))$

**lemma** *chainI*: $(\bigwedge i.\ Y\ i \sqsubseteq Y\ (Suc\ i)) \Longrightarrow chain\ Y$
**unfolding** *chain-def* **by** *fast*

**lemma** *chainE*: $chain\ Y \Longrightarrow Y\ i \sqsubseteq Y\ (Suc\ i)$
**unfolding** *chain-def* **by** *fast*

chains are monotone functions

**lemma** *chain-mono-less*: $chain\ Y \Longrightarrow i < j \Longrightarrow Y\ i \sqsubseteq Y\ j$
**by** (*erule less-Suc-induct*, *erule chainE*, *erule below-trans*)

**lemma** *chain-mono*: $chain\ Y \Longrightarrow i \leq j \Longrightarrow Y\ i \sqsubseteq Y\ j$
**by** (*cases* $i = j$) (*simp-all add*: *chain-mono-less*)

**lemma** *chain-shift*: $chain\ Y \Longrightarrow chain\ (\lambda i.\ Y\ (i + j))$
**by** (*rule chainI*, *simp*, *erule chainE*)

technical lemmas about (least) upper bounds of chains

**lemma** *is-lub-rangeD1*: $range\ S <<| x \Longrightarrow S\ i \sqsubseteq x$
**by** (*rule is-lubD1* [*THEN ub-rangeD*])

**lemma** *is-ub-range-shift*: $chain\ S \Longrightarrow range\ (\lambda i.\ S\ (i + j)) <| x = range\ S <| x$
**apply** (*rule iffI*)

**apply** (*rule ub-rangeI*)
**apply** (*rule-tac y=S (i + j)* **in** *below-trans*)
**apply** (*erule chain-mono*)
**apply** (*rule le-add1*)
**apply** (*erule ub-rangeD*)
**apply** (*rule ub-rangeI*)
**apply** (*erule ub-rangeD*)
**done**

**lemma** *is-lub-range-shift*: *chain S* $\Longrightarrow$ *range* ($\lambda i.$ *S* (*i* + *j*)) <<| *x* = *range S* <<| *x*
**by** (*simp add*: *is-lub-def is-ub-range-shift*)

the lub of a constant chain is the constant

**lemma** *chain-const* [*simp*]: *chain* ($\lambda i.$ *c*)
**by** (*simp add*: *chainI*)

**lemma** *is-lub-const*: *range* ($\lambda x.$ *c*) <<| *c*
**by** (*blast dest*: *ub-rangeD intro*: *is-lubI ub-rangeI*)

**lemma** *lub-const* [*simp*]: ($\bigsqcup i.$ *c*) = *c*
**by** (*rule is-lub-const* [*THEN lub-eqI*])

## 1.5 Finite chains

**definition** *max-in-chain* :: *nat* $\Rightarrow$ (*nat* $\Rightarrow$ *'a*) $\Rightarrow$ *bool*
**where** — finite chains, needed for monotony of continuous functions
*max-in-chain i C* $\longleftrightarrow$ ($\forall j.$ *i* $\leq$ *j* $\longrightarrow$ *C i* = *C j*)

**definition** *finite-chain* :: (*nat* $\Rightarrow$ *'a*) $\Rightarrow$ *bool*
**where** *finite-chain C* = (*chain C* $\wedge$ ($\exists i.$ *max-in-chain i C*))

results about finite chains

**lemma** *max-in-chainI*: ($\bigwedge j.$ *i* $\leq$ *j* $\Longrightarrow$ *Y i* = *Y j*) $\Longrightarrow$ *max-in-chain i Y*
**unfolding** *max-in-chain-def* **by** *fast*

**lemma** *max-in-chainD*: *max-in-chain i Y* $\Longrightarrow$ *i* $\leq$ *j* $\Longrightarrow$ *Y i* = *Y j*
**unfolding** *max-in-chain-def* **by** *fast*

**lemma** *finite-chainI*: *chain C* $\Longrightarrow$ *max-in-chain i C* $\Longrightarrow$ *finite-chain C*
**unfolding** *finite-chain-def* **by** *fast*

**lemma** *finite-chainE*: [[*finite-chain C*; $\bigwedge i.$ [[*chain C*; *max-in-chain i C*]] $\Longrightarrow$ *R*]] $\Longrightarrow$ *R*
**unfolding** *finite-chain-def* **by** *fast*

**lemma** *lub-finch1*: *chain C* $\Longrightarrow$ *max-in-chain i C* $\Longrightarrow$ *range C* <<| *C i*
**apply** (*rule is-lubI*)
**apply** (*rule ub-rangeI*, *rename-tac j*)

```
   apply (rule-tac x=i and y=j in linorder-le-cases)
    apply (drule (1) max-in-chainD, simp)
   apply (erule (1) chain-mono)
  apply (erule ub-rangeD)
  done

lemma lub-finch2: finite-chain C ⟹ range C <<| C (LEAST i. max-in-chain i
C)
  apply (erule finite-chainE)
  apply (erule LeastI2 [where Q=λi. range C <<| C i])
  apply (erule (1) lub-finch1)
  done

lemma finch-imp-finite-range: finite-chain Y ⟹ finite (range Y)
  apply (erule finite-chainE)
  apply (rule-tac B=Y ` {..i} in finite-subset)
   apply (rule subsetI)
   apply (erule rangeE, rename-tac j)
   apply (rule-tac x=i and y=j in linorder-le-cases)
    apply (subgoal-tac Y j = Y i, simp)
    apply (simp add: max-in-chain-def)
   apply simp
  apply simp
  done

lemma finite-range-has-max:
  fixes f :: nat ⇒ 'a
    and r :: 'a ⇒ 'a ⇒ bool
  assumes mono: ⋀i j. i ≤ j ⟹ r (f i) (f j)
  assumes finite-range: finite (range f)
  shows ∃k. ∀i. r (f i) (f k)
proof (intro exI allI)
  fix i :: nat
  let ?j = LEAST k. f k = f i
  let ?k = Max ((λx. LEAST k. f k = x) ` range f)
  have ?j ≤ ?k
  proof (rule Max-ge)
    show finite ((λx. LEAST k. f k = x) ` range f)
      using finite-range by (rule finite-imageI)
    show ?j ∈ (λx. LEAST k. f k = x) ` range f
      by (intro imageI rangeI)
  qed
  hence r (f ?j) (f ?k)
    by (rule mono)
  also have f ?j = f i
    by (rule LeastI, rule refl)
  finally show r (f i) (f ?k) .
qed
```

**lemma** *finite-range-imp-finch*: *chain* $Y \Longrightarrow$ *finite* (*range* $Y$) $\Longrightarrow$ *finite-chain* $Y$
**apply** (*subgoal-tac* $\exists k.\ \forall i.\ Y\ i \sqsubseteq Y\ k$)
**apply** (*erule exE*)
**apply** (*rule finite-chainI*, *assumption*)
**apply** (*rule max-in-chainI*)
**apply** (*rule below-antisym*)
**apply** (*erule* (*1*) *chain-mono*)
**apply** (*erule spec*)
**apply** (*rule finite-range-has-max*)
**apply** (*erule* (*1*) *chain-mono*)
**apply** *assumption*
**done**

**lemma** *bin-chain*: $x \sqsubseteq y \Longrightarrow$ *chain* ($\lambda i.$ *if i=0 then x else y*)
**by** (*rule chainI*) *simp*

**lemma** *bin-chainmax*: $x \sqsubseteq y \Longrightarrow$ *max-in-chain* (*Suc 0*) ($\lambda i.$ *if i=0 then x else y*)
**by** (*simp add*: *max-in-chain-def*)

**lemma** *is-lub-bin-chain*: $x \sqsubseteq y \Longrightarrow$ *range* ($\lambda i$::*nat. if i=0 then x else y*) <<| $y$
**apply** (*frule bin-chain*)
**apply** (*drule bin-chainmax*)
**apply** (*drule* (*1*) *lub-finch1*)
**apply** *simp*
**done**

the maximal element in a chain is its lub

**lemma** *lub-chain-maxelem*: $Y\ i = c \Longrightarrow \forall i.\ Y\ i \sqsubseteq c \Longrightarrow$ *lub* (*range* $Y$) $= c$
**by** (*blast dest*: *ub-rangeD intro*: *lub-eqI is-lubI ub-rangeI*)

**end**

**end**

# 2 Classes cpo and pcpo

**theory** *Pcpo*
**imports** *Porder*
**begin**

## 2.1 Complete partial orders

The class cpo of chain complete partial orders

**class** *cpo* = *po* +
**assumes** *cpo*: *chain* $S \Longrightarrow \exists x.$ *range* $S$ <<| $x$
**begin**

in cpo's everthing equal to THE lub has lub properties for every chain

**lemma** *cpo-lubI*: *chain S* $\Longrightarrow$ *range S* <<| $(\bigsqcup i.\ S\ i)$
**by** (*fast dest*: *cpo elim*: *is-lub-lub*)

**lemma** *thelubE*: $[\![$*chain S*; $(\bigsqcup i.\ S\ i) = l]\!] \Longrightarrow$ *range S* <<| *l*
**by** (*blast dest*: *cpo intro*: *is-lub-lub*)

Properties of the lub

**lemma** *is-ub-thelub*: *chain S* $\Longrightarrow$ *S x* $\sqsubseteq$ $(\bigsqcup i.\ S\ i)$
**by** (*blast dest*: *cpo intro*: *is-lub-lub* [*THEN is-lub-rangeD1*])

**lemma** *is-lub-thelub*: $[\![$*chain S*; *range S* <| *x*$]\!] \Longrightarrow (\bigsqcup i.\ S\ i) \sqsubseteq x$
**by** (*blast dest*: *cpo intro*: *is-lub-lub* [*THEN is-lubD2*])

**lemma** *lub-below-iff*: *chain S* $\Longrightarrow (\bigsqcup i.\ S\ i) \sqsubseteq x \longleftrightarrow (\forall i.\ S\ i \sqsubseteq x)$
**by** (*simp add*: *is-lub-below-iff* [*OF cpo-lubI*] *is-ub-def*)

**lemma** *lub-below*: $[\![$*chain S*; $\bigwedge i.\ S\ i \sqsubseteq x]\!] \Longrightarrow (\bigsqcup i.\ S\ i) \sqsubseteq x$
**by** (*simp add*: *lub-below-iff*)

**lemma** *below-lub*: $[\![$*chain S*; $x \sqsubseteq S\ i]\!] \Longrightarrow x \sqsubseteq (\bigsqcup i.\ S\ i)$
**by** (*erule below-trans*, *erule is-ub-thelub*)

**lemma** *lub-range-mono*: $[\![$*range X* $\subseteq$ *range Y*; *chain Y*; *chain X*$]\!] \Longrightarrow (\bigsqcup i.\ X\ i)$
$\sqsubseteq (\bigsqcup i.\ Y\ i)$
**apply** (*erule lub-below*)
**apply** (*subgoal-tac* $\exists j.\ X\ i = Y\ j$)
**apply** *clarsimp*
**apply** (*erule is-ub-thelub*)
**apply** *auto*
**done**

**lemma** *lub-range-shift*: *chain Y* $\Longrightarrow (\bigsqcup i.\ Y\ (i + j)) = (\bigsqcup i.\ Y\ i)$
**apply** (*rule below-antisym*)
**apply** (*rule lub-range-mono*)
**apply** *fast*
**apply** *assumption*
**apply** (*erule chain-shift*)
**apply** (*rule lub-below*)
**apply** *assumption*
**apply** (*rule-tac i=i* **in** *below-lub*)
**apply** (*erule chain-shift*)
**apply** (*erule chain-mono*)
**apply** (*rule le-add1*)
**done**

**lemma** *maxinch-is-thelub*: *chain Y* $\Longrightarrow$ *max-in-chain i Y* $= ((\bigsqcup i.\ Y\ i) = Y\ i)$
**apply** (*rule iffI*)
**apply** (*fast intro*!: *lub-eqI lub-finch1*)
**apply** (*unfold max-in-chain-def*)

**apply** (*safe intro*!: *below-antisym*)
**apply** (*fast elim*!: *chain-mono*)
**apply** (*drule sym*)
**apply** (*force elim*!: *is-ub-thelub*)
**done**

the $\sqsubseteq$ relation between two chains is preserved by their lubs

**lemma** *lub-mono*: $[\![chain\ X; chain\ Y; \bigwedge i.\ X\ i \sqsubseteq Y\ i]\!] \Longrightarrow (\bigsqcup i.\ X\ i) \sqsubseteq (\bigsqcup i.\ Y\ i)$
**by** (*fast elim*: *lub-below below-lub*)

the $=$ relation between two chains is preserved by their lubs

**lemma** *lub-eq*: $(\bigwedge i.\ X\ i = Y\ i) \Longrightarrow (\bigsqcup i.\ X\ i) = (\bigsqcup i.\ Y\ i)$
**by** *simp*

**lemma** *ch2ch-lub*:
**assumes** *1*: $\bigwedge j.\ chain\ (\lambda i.\ Y\ i\ j)$
**assumes** *2*: $\bigwedge i.\ chain\ (\lambda j.\ Y\ i\ j)$
**shows** $chain\ (\lambda i.\ \bigsqcup j.\ Y\ i\ j)$
**apply** (*rule chainI*)
**apply** (*rule lub-mono* [*OF 2 2*])
**apply** (*rule chainE* [*OF 1*])
**done**

**lemma** *diag-lub*:
**assumes** *1*: $\bigwedge j.\ chain\ (\lambda i.\ Y\ i\ j)$
**assumes** *2*: $\bigwedge i.\ chain\ (\lambda j.\ Y\ i\ j)$
**shows** $(\bigsqcup i.\ \bigsqcup j.\ Y\ i\ j) = (\bigsqcup i.\ Y\ i\ i)$
**proof** (*rule below-antisym*)
**have** *3*: $chain\ (\lambda i.\ Y\ i\ i)$
**apply** (*rule chainI*)
**apply** (*rule below-trans*)
**apply** (*rule chainE* [*OF 1*])
**apply** (*rule chainE* [*OF 2*])
**done**
**have** *4*: $chain\ (\lambda i.\ \bigsqcup j.\ Y\ i\ j)$
**by** (*rule ch2ch-lub* [*OF 1 2*])
**show** $(\bigsqcup i.\ \bigsqcup j.\ Y\ i\ j) \sqsubseteq (\bigsqcup i.\ Y\ i\ i)$
**apply** (*rule lub-below* [*OF 4*])
**apply** (*rule lub-below* [*OF 2*])
**apply** (*rule below-lub* [*OF 3*])
**apply** (*rule below-trans*)
**apply** (*rule chain-mono* [*OF 1 max.cobounded1*])
**apply** (*rule chain-mono* [*OF 2 max.cobounded2*])
**done**
**show** $(\bigsqcup i.\ Y\ i\ i) \sqsubseteq (\bigsqcup i.\ \bigsqcup j.\ Y\ i\ j)$
**apply** (*rule lub-mono* [*OF 3 4*])
**apply** (*rule is-ub-thelub* [*OF 2*])
**done**
**qed**

**lemma** *ex-lub*:
**assumes** *1*: $\bigwedge j.\ chain\ (\lambda i.\ Y\ i\ j)$
**assumes** *2*: $\bigwedge i.\ chain\ (\lambda j.\ Y\ i\ j)$
**shows** $(\bigsqcup i.\ \bigsqcup j.\ Y\ i\ j) = (\bigsqcup j.\ \bigsqcup i.\ Y\ i\ j)$
**by** (*simp add*: *diag-lub 1 2*)

**end**

## 2.2 Pointed cpos

The class pcpo of pointed cpos

**class** *pcpo* = *cpo* +
**assumes** *least*: $\exists\, x.\ \forall\, y.\ x \sqsubseteq y$
**begin**

**definition** *bottom* :: *'a* ($\bot$)
**where** *bottom* = (*THE x*. $\forall\, y.\ x \sqsubseteq y$)

**lemma** *minimal* [*iff*]: $\bot \sqsubseteq x$
**unfolding** *bottom-def*
**apply** (*rule the1I2*)
**apply** (*rule ex-ex1I*)
**apply** (*rule least*)
**apply** (*blast intro*: *below-antisym*)
**apply** *simp*
**done**

**end**

Old "UU" syntax:

**syntax** *UU* :: *logic*
**translations** *UU* $\rightharpoonup$ *CONST bottom*

Simproc to rewrite $\bot = x$ to $x = \bot$.

**setup** ‹*Reorient-Proc.add* (*fn* ***Const-*** ‹*bottom* -› => *true* | - => *false*)›
**simproc-setup** *reorient-bottom* ($\bot = x$) = ‹*K Reorient-Proc.proc*›

useful lemmas about $\bot$

**lemma** *below-bottom-iff* [*simp*]: $x \sqsubseteq \bot \longleftrightarrow x = \bot$
**by** (*simp add*: *po-eq-conv*)

**lemma** *eq-bottom-iff*: $x = \bot \longleftrightarrow x \sqsubseteq \bot$
**by** *simp*

**lemma** *bottomI*: $x \sqsubseteq \bot \Longrightarrow x = \bot$
**by** (*subst eq-bottom-iff*)

**lemma** *lub-eq-bottom-iff*: *chain* $Y \Longrightarrow (\bigsqcup i.\ Y\ i) = \bot \longleftrightarrow (\forall i.\ Y\ i = \bot)$
**by** (*simp only*: *eq-bottom-iff lub-below-iff*)

## 2.3 Chain-finite and flat cpos

further useful classes for HOLCF domains

**class** *chfin* = *po* +
**assumes** *chfin*: *chain* $Y \Longrightarrow \exists n.$ *max-in-chain* $n\ Y$
**begin**

**subclass** *cpo*
**apply** *standard*
**apply** (*frule chfin*)
**apply** (*blast intro*: *lub-finch1*)
**done**

**lemma** *chfin2finch*: *chain* $Y \Longrightarrow$ *finite-chain* $Y$
**by** (*simp add*: *chfin finite-chain-def*)

**end**

**class** *flat* = *pcpo* +
**assumes** *ax-flat*: $x \sqsubseteq y \Longrightarrow x = \bot \vee x = y$
**begin**

**subclass** *chfin*
**proof**
**fix** $Y$
**assume** $*$: *chain* $Y$
**show** $\exists n.$ *max-in-chain* $n\ Y$
**apply** (*unfold max-in-chain-def*)
**apply** (*cases* $\forall i.\ Y\ i = \bot$)
**apply** *simp*
**apply** *simp*
**apply** (*erule exE*)
**apply** (*rule-tac* $x{=}i$ **in** *exI*)
**apply** *clarify*
**using** $*$ **apply** (*blast dest*: *chain-mono ax-flat*)
**done**
**qed**

**lemma** *flat-below-iff*: $x \sqsubseteq y \longleftrightarrow x = \bot \vee x = y$
**by** (*safe dest*!: *ax-flat*)

**lemma** *flat-eq*: $a \neq \bot \Longrightarrow a \sqsubseteq b = (a = b)$
**by** (*safe dest*!: *ax-flat*)

**end**

### 2.4 Discrete cpos

**class** *discrete-cpo* = *below* +
  **assumes** *discrete-cpo* [*simp*]: $x \sqsubseteq y \longleftrightarrow x = y$
**begin**

**subclass** *po*
  **by** *standard simp-all*

In a discrete cpo, every chain is constant

**lemma** *discrete-chain-const*:
  **assumes** *S*: *chain S*
  **shows** $\exists x.\ S = (\lambda i.\ x)$
**proof** (*intro exI ext*)
  **fix** *i* :: *nat*
  **from** *S le0* **have** *S 0* $\sqsubseteq$ *S i* **by** (*rule chain-mono*)
  **then have** *S 0* = *S i* **by** *simp*
  **then show** *S i* = *S 0* **by** (*rule sym*)
**qed**

**subclass** *chfin*
**proof**
  **fix** *S* :: *nat* ⇒ *'a*
  **assume** *S*: *chain S*
  **then have** $\exists x.\ S = (\lambda i.\ x)$
    **by** (*rule discrete-chain-const*)
  **then have** *max-in-chain 0 S*
    **by** (*auto simp*: *max-in-chain-def*)
  **then show** $\exists i.$ *max-in-chain i S* **..**
**qed**

**end**

**end**

## 3 Continuity and monotonicity

**theory** *Cont*
  **imports** *Pcpo*
**begin**

Now we change the default class! Form now on all untyped type variables are of default class po

**default-sort** *po*

### 3.1 Definitions

**definition** *monofun* :: (*'a* ⇒ *'b*) ⇒ *bool* — monotonicity
  **where** *monofun f* $\longleftrightarrow$ $(\forall x\ y.\ x \sqsubseteq y \longrightarrow f\ x \sqsubseteq f\ y)$

**definition** *cont* :: ('*a*::*cpo* ⇒ '*b*::*cpo*) ⇒ *bool*
**where** *cont f* = (∀ *Y*. *chain Y* ⟶ *range* (λ*i*. *f* (*Y i*)) <<| *f* (⨆ *i*. *Y i*))

**lemma** *contI*: (⋀*Y*. *chain Y* ⟹ *range* (λ*i*. *f* (*Y i*)) <<| *f* (⨆ *i*. *Y i*)) ⟹ *cont f*
**by** (*simp add*: *cont-def*)

**lemma** *contE*: *cont f* ⟹ *chain Y* ⟹ *range* (λ*i*. *f* (*Y i*)) <<| *f* (⨆ *i*. *Y i*)
**by** (*simp add*: *cont-def*)

**lemma** *monofunI*: (⋀*x y*. *x* ⊑ *y* ⟹ *f x* ⊑ *f y*) ⟹ *monofun f*
**by** (*simp add*: *monofun-def*)

**lemma** *monofunE*: *monofun f* ⟹ *x* ⊑ *y* ⟹ *f x* ⊑ *f y*
**by** (*simp add*: *monofun-def*)

## 3.2 Equivalence of alternate definition

monotone functions map chains to chains

**lemma** *ch2ch-monofun*: *monofun f* ⟹ *chain Y* ⟹ *chain* (λ*i*. *f* (*Y i*))
**apply** (*rule chainI*)
**apply** (*erule monofunE*)
**apply** (*erule chainE*)
**done**

monotone functions map upper bound to upper bounds

**lemma** *ub2ub-monofun*: *monofun f* ⟹ *range Y* <| *u* ⟹ *range* (λ*i*. *f* (*Y i*)) <| *f u*
**apply** (*rule ub-rangeI*)
**apply** (*erule monofunE*)
**apply** (*erule ub-rangeD*)
**done**

a lemma about binary chains

**lemma** *binchain-cont*: *cont f* ⟹ *x* ⊑ *y* ⟹ *range* (λ*i*::*nat*. *f* (*if i* = *0 then x else y*)) <<| *f y*
**apply** (*subgoal-tac f* (⨆ *i*::*nat*. *if i* = *0 then x else y*) = *f y*)
**apply** (*erule subst*)
**apply** (*erule contE*)
**apply** (*erule bin-chain*)
**apply** (*rule-tac f*=*f* **in** *arg-cong*)
**apply** (*erule is-lub-bin-chain* [*THEN lub-eqI*])
**done**

continuity implies monotonicity

**lemma** *cont2mono*: *cont f* ⟹ *monofun f*
**apply** (*rule monofunI*)
**apply** (*drule* (*1*) *binchain-cont*)

**apply** (*drule-tac i=0* **in** *is-lub-rangeD1*)
**apply** *simp*
**done**

**lemmas** *cont2monofunE* = *cont2mono* [*THEN monofunE*]

**lemmas** *ch2ch-cont* = *cont2mono* [*THEN ch2ch-monofun*]

continuity implies preservation of lubs

**lemma** *cont2contlubE*: *cont f* ⟹ *chain Y* ⟹ *f* (⨆ *i*. *Y i*) = (⨆ *i*. *f* (*Y i*))
**apply** (*rule lub-eqI* [*symmetric*])
**apply** (*erule* (*1*) *contE*)
**done**

**lemma** *contI2*:
**fixes** *f* :: *'a*::*cpo* ⇒ *'b*::*cpo*
**assumes** *mono*: *monofun f*
**assumes** *below*: ⋀*Y*. ⟦*chain Y*; *chain* (λ*i*. *f* (*Y i*))⟧ ⟹ *f* (⨆ *i*. *Y i*) ⊑ (⨆ *i*. *f* (*Y i*))
**shows** *cont f*
**proof** (*rule contI*)
**fix** *Y* :: *nat* ⇒ *'a*
**assume** *Y*: *chain Y*
**with** *mono* **have** *fY*: *chain* (λ*i*. *f* (*Y i*))
**by** (*rule ch2ch-monofun*)
**have** (⨆ *i*. *f* (*Y i*)) = *f* (⨆ *i*. *Y i*)
**apply** (*rule below-antisym*)
**apply** (*rule lub-below* [*OF fY*])
**apply** (*rule monofunE* [*OF mono*])
**apply** (*rule is-ub-thelub* [*OF Y*])
**apply** (*rule below* [*OF Y fY*])
**done**
**with** *fY* **show** *range* (λ*i*. *f* (*Y i*)) <<| *f* (⨆ *i*. *Y i*)
**by** (*rule thelubE*)
**qed**

## 3.3 Collection of continuity rules

**named-theorems** *cont2cont continuity intro rule*

## 3.4 Continuity of basic functions

The identity function is continuous

**lemma** *cont-id* [*simp*, *cont2cont*]: *cont* (λ*x*. *x*)
**apply** (*rule contI*)
**apply** (*erule cpo-lubI*)
**done**

constant functions are continuous

**lemma** *cont-const* [*simp*, *cont2cont*]: *cont* (λ*x*. *c*)
**using** *is-lub-const* **by** (*rule contI*)

application of functions is continuous

**lemma** *cont-apply*:
**fixes** *f* :: *'a*::*cpo* ⇒ *'b*::*cpo* ⇒ *'c*::*cpo* **and** *t* :: *'a* ⇒ *'b*
**assumes** *1*: *cont* (λ*x*. *t x*)
**assumes** *2*: ⋀*x*. *cont* (λ*y*. *f x y*)
**assumes** *3*: ⋀*y*. *cont* (λ*x*. *f x y*)
**shows** *cont* (λ*x*. (*f x*) (*t x*))
**proof** (*rule contI2* [*OF monofunI*])
**fix** *x y* :: *'a*
**assume** *x* ⊑ *y*
**then show** *f x* (*t x*) ⊑ *f y* (*t y*)
**by** (*auto intro*: *cont2monofunE* [*OF 1*]
*cont2monofunE* [*OF 2*]
*cont2monofunE* [*OF 3*]
*below-trans*)
**next**
**fix** *Y* :: *nat* ⇒ *'a*
**assume** *chain Y*
**then show** *f* (⨆ *i*. *Y i*) (*t* (⨆ *i*. *Y i*)) ⊑ (⨆ *i*. *f* (*Y i*) (*t* (*Y i*)))
**by** (*simp only*: *cont2contlubE* [*OF 1*] *ch2ch-cont* [*OF 1*]
*cont2contlubE* [*OF 2*] *ch2ch-cont* [*OF 2*]
*cont2contlubE* [*OF 3*] *ch2ch-cont* [*OF 3*]
*diag-lub below-refl*)
**qed**

**lemma** *cont-compose*: *cont c* ⟹ *cont* (λ*x*. *f x*) ⟹ *cont* (λ*x*. *c* (*f x*))
**by** (*rule cont-apply* [*OF - - cont-const*])

Least upper bounds preserve continuity

**lemma** *cont2cont-lub* [*simp*]:
**assumes** *chain*: ⋀*x*. *chain* (λ*i*. *F i x*)
**and** *cont*: ⋀*i*. *cont* (λ*x*. *F i x*)
**shows** *cont* (λ*x*. ⨆ *i*. *F i x*)
**apply** (*rule contI2*)
**apply** (*simp add*: *monofunI cont2monofunE* [*OF cont*] *lub-mono chain*)
**apply** (*simp add*: *cont2contlubE* [*OF cont*])
**apply** (*simp add*: *diag-lub ch2ch-cont* [*OF cont*] *chain*)
**done**

if-then-else is continuous

**lemma** *cont-if* [*simp*, *cont2cont*]: *cont f* ⟹ *cont g* ⟹ *cont* (λ*x*. *if b then f x else g x*)
**by** (*induct b*) *simp-all*

### 3.5 Finite chains and flat pcpos

Monotone functions map finite chains to finite chains.

**lemma** *monofun-finch2finch*: *monofun f* $\Longrightarrow$ *finite-chain Y* $\Longrightarrow$ *finite-chain* ($\lambda n.$ *f* (*Y n*))
**by** (*force simp add*: *finite-chain-def ch2ch-monofun max-in-chain-def*)

The same holds for continuous functions.

**lemma** *cont-finch2finch*: *cont f* $\Longrightarrow$ *finite-chain Y* $\Longrightarrow$ *finite-chain* ($\lambda n.$ *f* (*Y n*))
**by** (*rule cont2mono* [*THEN monofun-finch2finch*])

All monotone functions with chain-finite domain are continuous.

**lemma** *chfindom-monofun2cont*: *monofun f* $\Longrightarrow$ *cont f*
**for** *f* :: $'a$::*chfin* $\Rightarrow$ $'b$::*cpo*
**apply** (*erule contI2*)
**apply** (*frule chfin2finch*)
**apply** (*clarsimp simp add*: *finite-chain-def*)
**apply** (*subgoal-tac max-in-chain i* ($\lambda i.$ *f* (*Y i*)))
**apply** (*simp add*: *maxinch-is-thelub ch2ch-monofun*)
**apply** (*force simp add*: *max-in-chain-def*)
**done**

All strict functions with flat domain are continuous.

**lemma** *flatdom-strict2mono*: *f* $\bot$ = $\bot$ $\Longrightarrow$ *monofun f*
**for** *f* :: $'a$::*flat* $\Rightarrow$ $'b$::*pcpo*
**apply** (*rule monofunI*)
**apply** (*drule ax-flat*)
**apply** *auto*
**done**

**lemma** *flatdom-strict2cont*: *f* $\bot$ = $\bot$ $\Longrightarrow$ *cont f*
**for** *f* :: $'a$::*flat* $\Rightarrow$ $'b$::*pcpo*
**by** (*rule flatdom-strict2mono* [*THEN chfindom-monofun2cont*])

All functions with discrete domain are continuous.

**lemma** *cont-discrete-cpo* [*simp*, *cont2cont*]: *cont f*
**for** *f* :: $'a$::*discrete-cpo* $\Rightarrow$ $'b$::*cpo*
**apply** (*rule contI*)
**apply** (*drule discrete-chain-const*, *clarify*)
**apply** *simp*
**done**

**end**

# 4 Admissibility and compactness

**theory** *Adm*

**imports** *Cont*
**begin**

**default-sort** *cpo*

## 4.1 Definitions

**definition** *adm* :: ('*a*::*cpo* ⇒ *bool*) ⇒ *bool*
**where** *adm P* ⟷ (∀ *Y*. *chain Y* ⟶ (∀ *i*. *P* (*Y i*)) ⟶ *P* (⨆ *i*. *Y i*))

**lemma** *admI*: (⋀ *Y*. ⟦*chain Y*; ∀ *i*. *P* (*Y i*)⟧ ⟹ *P* (⨆ *i*. *Y i*)) ⟹ *adm P*
**unfolding** *adm-def* **by** *fast*

**lemma** *admD*: *adm P* ⟹ *chain Y* ⟹ (⋀*i*. *P* (*Y i*)) ⟹ *P* (⨆ *i*. *Y i*)
**unfolding** *adm-def* **by** *fast*

**lemma** *admD2*: *adm* (λ*x*. ¬ *P x*) ⟹ *chain Y* ⟹ *P* (⨆ *i*. *Y i*) ⟹ ∃ *i*. *P* (*Y i*)
**unfolding** *adm-def* **by** *fast*

**lemma** *triv-admI*: ∀ *x*. *P x* ⟹ *adm P*
**by** (*rule admI*) (*erule spec*)

## 4.2 Admissibility on chain-finite types

For chain-finite (easy) types every formula is admissible.

**lemma** *adm-chfin* [*simp*]: *adm P*
**for** *P* :: '*a*::*chfin* ⇒ *bool*
**by** (*rule admI*, *frule chfin*, *auto simp add*: *maxinch-is-thelub*)

## 4.3 Admissibility of special formulae and propagation

**lemma** *adm-const* [*simp*]: *adm* (λ*x*. *t*)
**by** (*rule admI*, *simp*)

**lemma** *adm-conj* [*simp*]: *adm* (λ*x*. *P x*) ⟹ *adm* (λ*x*. *Q x*) ⟹ *adm* (λ*x*. *P x* ∧ *Q x*)
**by** (*fast intro*: *admI elim*: *admD*)

**lemma** *adm-all* [*simp*]: (⋀*y*. *adm* (λ*x*. *P x y*)) ⟹ *adm* (λ*x*. ∀ *y*. *P x y*)
**by** (*fast intro*: *admI elim*: *admD*)

**lemma** *adm-ball* [*simp*]: (⋀*y*. *y* ∈ *A* ⟹ *adm* (λ*x*. *P x y*)) ⟹ *adm* (λ*x*. ∀ *y*∈*A*. *P x y*)
**by** (*fast intro*: *admI elim*: *admD*)

Admissibility for disjunction is hard to prove. It requires 2 lemmas.

**lemma** *adm-disj-lemma1*:
**assumes** *adm*: *adm P*
**assumes** *chain*: *chain Y*

**assumes** $P$: $\forall i.\ \exists j{\geq}i.\ P\ (Y\ j)$
**shows** $P\ (\bigsqcup i.\ Y\ i)$
**proof** −
**define** $f$ **where** $f\ i = (LEAST\ j.\ i \leq j \wedge P\ (Y\ j))$ **for** $i$
**have** *chain'*: *chain* $(\lambda i.\ Y\ (f\ i))$
**unfolding** *f-def*
**apply** (*rule chainI*)
**apply** (*rule chain-mono* [*OF chain*])
**apply** (*rule Least-le*)
**apply** (*rule LeastI2-ex*)
**apply** (*simp-all add*: *P*)
**done**
**have** *f1*: $\bigwedge i.\ i \leq f\ i$ **and** *f2*: $\bigwedge i.\ P\ (Y\ (f\ i))$
**using** *LeastI-ex* [*OF P* [*rule-format*]] **by** (*simp-all add*: *f-def*)
**have** *lub-eq*: $(\bigsqcup i.\ Y\ i) = (\bigsqcup i.\ Y\ (f\ i))$
**apply** (*rule below-antisym*)
**apply** (*rule lub-mono* [*OF chain chain'*])
**apply** (*rule chain-mono* [*OF chain f1*])
**apply** (*rule lub-range-mono* [*OF - chain chain'*])
**apply** *clarsimp*
**done**
**show** $P\ (\bigsqcup i.\ Y\ i)$
**unfolding** *lub-eq* **using** *adm chain' f2* **by** (*rule admD*)
**qed**

**lemma** *adm-disj-lemma2*: $\forall n{::}nat.\ P\ n \vee Q\ n \Longrightarrow (\forall i.\ \exists j{\geq}i.\ P\ j) \vee (\forall i.\ \exists j{\geq}i.\ Q\ j)$
**apply** (*erule contrapos-pp*)
**apply** (*clarsimp*, *rename-tac a b*)
**apply** (*rule-tac x=max a b* **in** *exI*)
**apply** *simp*
**done**

**lemma** *adm-disj* [*simp*]: *adm* $(\lambda x.\ P\ x) \Longrightarrow$ *adm* $(\lambda x.\ Q\ x) \Longrightarrow$ *adm* $(\lambda x.\ P\ x \vee Q\ x)$
**apply** (*rule admI*)
**apply** (*erule adm-disj-lemma2* [*THEN disjE*])
**apply** (*erule* (*2*) *adm-disj-lemma1* [*THEN disjI1*])
**apply** (*erule* (*2*) *adm-disj-lemma1* [*THEN disjI2*])
**done**

**lemma** *adm-imp* [*simp*]: *adm* $(\lambda x.\ \neg\ P\ x) \Longrightarrow$ *adm* $(\lambda x.\ Q\ x) \Longrightarrow$ *adm* $(\lambda x.\ P\ x \longrightarrow Q\ x)$
**by** (*subst imp-conv-disj*) (*rule adm-disj*)

**lemma** *adm-iff* [*simp*]: *adm* $(\lambda x.\ P\ x \longrightarrow Q\ x) \Longrightarrow$ *adm* $(\lambda x.\ Q\ x \longrightarrow P\ x) \Longrightarrow$ *adm* $(\lambda x.\ P\ x \longleftrightarrow Q\ x)$
**by** (*subst iff-conv-conj-imp*) (*rule adm-conj*)

admissibility and continuity

**lemma** *adm-below* [*simp*]: *cont* $(\lambda x.\ u\ x) \Longrightarrow$ *cont* $(\lambda x.\ v\ x) \Longrightarrow$ *adm* $(\lambda x.\ u\ x \sqsubseteq v\ x)$
**by** (*simp add*: *adm-def cont2contlubE lub-mono ch2ch-cont*)

**lemma** *adm-eq* [*simp*]: *cont* $(\lambda x.\ u\ x) \Longrightarrow$ *cont* $(\lambda x.\ v\ x) \Longrightarrow$ *adm* $(\lambda x.\ u\ x = v\ x)$
**by** (*simp add*: *po-eq-conv*)

**lemma** *adm-subst*: *cont* $(\lambda x.\ t\ x) \Longrightarrow$ *adm* $P \Longrightarrow$ *adm* $(\lambda x.\ P\ (t\ x))$
**by** (*simp add*: *adm-def cont2contlubE ch2ch-cont*)

**lemma** *adm-not-below* [*simp*]: *cont* $(\lambda x.\ t\ x) \Longrightarrow$ *adm* $(\lambda x.\ t\ x \not\sqsubseteq u)$
**by** (*rule admI*) (*simp add*: *cont2contlubE ch2ch-cont lub-below-iff*)

## 4.4 Compactness

**definition** *compact* :: $'a$::*cpo* $\Rightarrow$ *bool*
**where** *compact k* = *adm* $(\lambda x.\ k \not\sqsubseteq x)$

**lemma** *compactI*: *adm* $(\lambda x.\ k \not\sqsubseteq x) \Longrightarrow$ *compact k*
**unfolding** *compact-def* **.**

**lemma** *compactD*: *compact k* $\Longrightarrow$ *adm* $(\lambda x.\ k \not\sqsubseteq x)$
**unfolding** *compact-def* **.**

**lemma** *compactI2*: $(\bigwedge Y.\ [\![chain\ Y; x \sqsubseteq (\bigsqcup i.\ Y\ i)]\!] \Longrightarrow \exists i.\ x \sqsubseteq Y\ i) \Longrightarrow$ *compact x*
**unfolding** *compact-def adm-def* **by** *fast*

**lemma** *compactD2*: *compact x* $\Longrightarrow$ *chain Y* $\Longrightarrow x \sqsubseteq (\bigsqcup i.\ Y\ i) \Longrightarrow \exists i.\ x \sqsubseteq Y\ i$
**unfolding** *compact-def adm-def* **by** *fast*

**lemma** *compact-below-lub-iff*: *compact x* $\Longrightarrow$ *chain Y* $\Longrightarrow x \sqsubseteq (\bigsqcup i.\ Y\ i) \longleftrightarrow (\exists i.\ x \sqsubseteq Y\ i)$
**by** (*fast intro*: *compactD2 elim*: *below-lub*)

**lemma** *compact-chfin* [*simp*]: *compact x*
**for** *x* :: $'a$::*chfin*
**by** (*rule compactI* [*OF adm-chfin*])

**lemma** *compact-imp-max-in-chain*: *chain Y* $\Longrightarrow$ *compact* $(\bigsqcup i.\ Y\ i) \Longrightarrow \exists i.$ *max-in-chain i Y*
**apply** (*drule* (*1*) *compactD2*, *simp*)
**apply** (*erule exE*, *rule-tac x*=*i* **in** *exI*)
**apply** (*rule max-in-chainI*)
**apply** (*rule below-antisym*)
**apply** (*erule* (*1*) *chain-mono*)
**apply** (*erule* (*1*) *below-trans* [*OF is-ub-thelub*])
**done**

admissibility and compactness

**lemma** *adm-compact-not-below* [*simp*]:
*compact k* $\Longrightarrow$ *cont* ($\lambda x.\ t\ x$) $\Longrightarrow$ *adm* ($\lambda x.\ k \not\sqsubseteq t\ x$)
**unfolding** *compact-def* **by** (*rule adm-subst*)

**lemma** *adm-neq-compact* [*simp*]: *compact k* $\Longrightarrow$ *cont* ($\lambda x.\ t\ x$) $\Longrightarrow$ *adm* ($\lambda x.\ t\ x \neq k$)
**by** (*simp add*: *po-eq-conv*)

**lemma** *adm-compact-neq* [*simp*]: *compact k* $\Longrightarrow$ *cont* ($\lambda x.\ t\ x$) $\Longrightarrow$ *adm* ($\lambda x.\ k \neq t\ x$)
**by** (*simp add*: *po-eq-conv*)

**lemma** *compact-bottom* [*simp*, *intro*]: *compact* $\bot$
**by** (*rule compactI*) *simp*

Any upward-closed predicate is admissible.

**lemma** *adm-upward*:
**assumes** *P*: $\bigwedge x\ y.\ [\![P\ x;\ x \sqsubseteq y]\!] \Longrightarrow P\ y$
**shows** *adm P*
**by** (*rule admI*, *drule spec*, *erule P*, *erule is-ub-thelub*)

**lemmas** *adm-lemmas* =
*adm-const adm-conj adm-all adm-ball adm-disj adm-imp adm-iff*
*adm-below adm-eq adm-not-below*
*adm-compact-not-below adm-compact-neq adm-neq-compact*

**end**

# 5 Subtypes of pcpos

**theory** *Cpodef*
**imports** *Adm*
**keywords** *pcpodef cpodef* :: *thy-goal-defn*
**begin**

## 5.1 Proving a subtype is a partial order

A subtype of a partial order is itself a partial order, if the ordering is defined in the standard way.

**setup** ‹*Sign.add-const-constraint* (***const-name*** ‹*Porder.below*›, *NONE*)›

**theorem** *typedef-po*:
**fixes** *Abs* :: *'a*::*po* $\Rightarrow$ *'b*::*type*
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: ($\sqsubseteq$) $\equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
**shows** *OFCLASS*(*'b*, *po-class*)
**apply** (*intro-classes*, *unfold below*)
**apply** (*rule below-refl*)

**apply** (*erule* (*1*) *below-trans*)
**apply** (*rule type-definition.Rep-inject* [*OF type*, *THEN iffD1*])
**apply** (*erule* (*1*) *below-antisym*)
**done**

**setup** ‹*Sign.add-const-constraint* (***const-name*** ‹*Porder.below*›, *SOME* ***typ*** ‹'*a*::*below* ⇒ '*a*::*below* ⇒ *bool*›)›

## 5.2 Proving a subtype is finite

**lemma** *typedef-finite-UNIV*:
**fixes** *Abs* :: '*a*::*type* ⇒ '*b*::*type*
**assumes** *type*: *type-definition Rep Abs A*
**shows** *finite A* ⟹ *finite* (*UNIV* :: '*b set*)
**proof** −
**assume** *finite A*
**then have** *finite* (*Abs* ‘ *A*)
**by** (*rule finite-imageI*)
**then show** *finite* (*UNIV* :: '*b set*)
**by** (*simp only*: *type-definition.Abs-image* [*OF type*])
**qed**

## 5.3 Proving a subtype is chain-finite

**lemma** *ch2ch-Rep*:
**assumes** *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**shows** *chain S* ⟹ *chain* (λ*i*. *Rep* (*S i*))
**unfolding** *chain-def below* **.**

**theorem** *typedef-chfin*:
**fixes** *Abs* :: '*a*::*chfin* ⇒ '*b*::*po*
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**shows** *OFCLASS*('*b*, *chfin-class*)
**apply** *intro-classes*
**apply** (*drule ch2ch-Rep* [*OF below*])
**apply** (*drule chfin*)
**apply** (*unfold max-in-chain-def*)
**apply** (*simp add*: *type-definition.Rep-inject* [*OF type*])
**done**

## 5.4 Proving a subtype is complete

A subtype of a cpo is itself a cpo if the ordering is defined in the standard way, and the defining subset is closed with respect to limits of chains. A set is closed if and only if membership in the set is an admissible predicate.

**lemma** *typedef-is-lubI*:
**assumes** *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**shows** *range* (λ*i*. *Rep* (*S i*)) <<| *Rep x* ⟹ *range S* <<| *x*

**by** (*simp add*: *is-lub-def is-ub-def below*)

**lemma** *Abs-inverse-lub-Rep*:
**fixes** *Abs* :: *'a*::*cpo* ⇒ *'b*::*po*
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**and** *adm*: *adm* (λ*x*. *x* ∈ *A*)
**shows** *chain S* ⟹ *Rep* (*Abs* (⨆ *i*. *Rep* (*S i*))) = (⨆ *i*. *Rep* (*S i*))
**apply** (*rule type-definition.Abs-inverse* [*OF type*])
**apply** (*erule admD* [*OF adm ch2ch-Rep* [*OF below*]])
**apply** (*rule type-definition.Rep* [*OF type*])
**done**

**theorem** *typedef-is-lub*:
**fixes** *Abs* :: *'a*::*cpo* ⇒ *'b*::*po*
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**and** *adm*: *adm* (λ*x*. *x* ∈ *A*)
**assumes** *S*: *chain S*
**shows** *range S* <<| *Abs* (⨆ *i*. *Rep* (*S i*))
**proof** −
**from** *S* **have** *chain* (λ*i*. *Rep* (*S i*))
**by** (*rule ch2ch-Rep* [*OF below*])
**then have** *range* (λ*i*. *Rep* (*S i*)) <<| (⨆ *i*. *Rep* (*S i*))
**by** (*rule cpo-lubI*)
**then have** *range* (λ*i*. *Rep* (*S i*)) <<| *Rep* (*Abs* (⨆ *i*. *Rep* (*S i*)))
**by** (*simp only*: *Abs-inverse-lub-Rep* [*OF type below adm S*])
**then show** *range S* <<| *Abs* (⨆ *i*. *Rep* (*S i*))
**by** (*rule typedef-is-lubI* [*OF below*])
**qed**

**lemmas** *typedef-lub* = *typedef-is-lub* [*THEN lub-eqI*]

**theorem** *typedef-cpo*:
**fixes** *Abs* :: *'a*::*cpo* ⇒ *'b*::*po*
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**and** *adm*: *adm* (λ*x*. *x* ∈ *A*)
**shows** *OFCLASS*(*'b*, *cpo-class*)
**proof**
**fix** *S* :: *nat* ⇒ *'b*
**assume** *chain S*
**then have** *range S* <<| *Abs* (⨆ *i*. *Rep* (*S i*))
**by** (*rule typedef-is-lub* [*OF type below adm*])
**then show** ∃ *x*. *range S* <<| *x* **..**
**qed**

### 5.4.1 Continuity of *Rep* and *Abs*

For any sub-cpo, the *Rep* function is continuous.

**theorem** *typedef-cont-Rep*:
**fixes** *Abs* :: *'a*::*cpo* ⇒ *'b*::*cpo*
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**and** *adm*: *adm* (λ*x*. *x* ∈ *A*)
**shows** *cont* (λ*x*. *f x*) ⟹ *cont* (λ*x*. *Rep* (*f x*))
**apply** (*erule cont-apply* [*OF - - cont-const*])
**apply** (*rule contI*)
**apply** (*simp only*: *typedef-lub* [*OF type below adm*])
**apply** (*simp only*: *Abs-inverse-lub-Rep* [*OF type below adm*])
**apply** (*rule cpo-lubI*)
**apply** (*erule ch2ch-Rep* [*OF below*])
**done**

For a sub-cpo, we can make the *Abs* function continuous only if we restrict its domain to the defining subset by composing it with another continuous function.

**theorem** *typedef-cont-Abs*:
**fixes** *Abs* :: *'a*::*cpo* ⇒ *'b*::*cpo*
**fixes** *f* :: *'c*::*cpo* ⇒ *'a*::*cpo*
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**and** *adm*: *adm* (λ*x*. *x* ∈ *A*)
**and** *f-in-A*: ⋀*x*. *f x* ∈ *A*
**shows** *cont f* ⟹ *cont* (λ*x*. *Abs* (*f x*))
**unfolding** *cont-def is-lub-def is-ub-def ball-simps below*
**by** (*simp add*: *type-definition.Abs-inverse* [*OF type f-in-A*])

## 5.5 Proving subtype elements are compact

**theorem** *typedef-compact*:
**fixes** *Abs* :: *'a*::*cpo* ⇒ *'b*::*cpo*
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**and** *adm*: *adm* (λ*x*. *x* ∈ *A*)
**shows** *compact* (*Rep k*) ⟹ *compact k*
**proof** (*unfold compact-def*)
**have** *cont-Rep*: *cont Rep*
**by** (*rule typedef-cont-Rep* [*OF type below adm cont-id*])
**assume** *adm* (λ*x*. *Rep k* ⋢ *x*)
**with** *cont-Rep* **have** *adm* (λ*x*. *Rep k* ⋢ *Rep x*) **by** (*rule adm-subst*)
**then show** *adm* (λ*x*. *k* ⋢ *x*) **by** (*unfold below*)
**qed**

## 5.6 Proving a subtype is pointed

A subtype of a cpo has a least element if and only if the defining subset has a least element.

**theorem** *typedef-pcpo-generic*:
**fixes** *Abs* :: *'a*::*cpo* ⇒ *'b*::*cpo*
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**and** *z-in-A*: *z* ∈ *A*
**and** *z-least*: ⋀*x*. *x* ∈ *A* ⟹ *z* ⊑ *x*
**shows** *OFCLASS*(*'b*, *pcpo-class*)
**apply** (*intro-classes*)
**apply** (*rule-tac x*=*Abs z* **in** *exI*, *rule allI*)
**apply** (*unfold below*)
**apply** (*subst type-definition.Abs-inverse* [*OF type z-in-A*])
**apply** (*rule z-least* [*OF type-definition.Rep* [*OF type*]])
**done**

As a special case, a subtype of a pcpo has a least element if the defining subset contains ⊥.

**theorem** *typedef-pcpo*:
**fixes** *Abs* :: *'a*::*pcpo* ⇒ *'b*::*cpo*
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**and** *bottom-in-A*: ⊥ ∈ *A*
**shows** *OFCLASS*(*'b*, *pcpo-class*)
**by** (*rule typedef-pcpo-generic* [*OF type below bottom-in-A*], *rule minimal*)

### 5.6.1 Strictness of *Rep* and *Abs*

For a sub-pcpo where ⊥ is a member of the defining subset, *Rep* and *Abs* are both strict.

**theorem** *typedef-Abs-strict*:
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**and** *bottom-in-A*: ⊥ ∈ *A*
**shows** *Abs* ⊥ = ⊥
**apply** (*rule bottomI*, *unfold below*)
**apply** (*simp add*: *type-definition.Abs-inverse* [*OF type bottom-in-A*])
**done**

**theorem** *typedef-Rep-strict*:
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**and** *bottom-in-A*: ⊥ ∈ *A*
**shows** *Rep* ⊥ = ⊥
**apply** (*rule typedef-Abs-strict* [*OF type below bottom-in-A*, *THEN subst*])
**apply** (*rule type-definition.Abs-inverse* [*OF type bottom-in-A*])

**done**

**theorem** *typedef-Abs-bottom-iff*:
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
**and** *bottom-in-A*: $\bot \in A$
**shows** $x \in A \Longrightarrow (Abs\ x = \bot) = (x = \bot)$
**apply** (*rule typedef-Abs-strict* [*OF type below bottom-in-A*, *THEN subst*])
**apply** (*simp add*: *type-definition.Abs-inject* [*OF type*] *bottom-in-A*)
**done**

**theorem** *typedef-Rep-bottom-iff*:
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
**and** *bottom-in-A*: $\bot \in A$
**shows** $(Rep\ x = \bot) = (x = \bot)$
**apply** (*rule typedef-Rep-strict* [*OF type below bottom-in-A*, *THEN subst*])
**apply** (*simp add*: *type-definition.Rep-inject* [*OF type*])
**done**

## 5.7 Proving a subtype is flat

**theorem** *typedef-flat*:
**fixes** *Abs* :: $'a$::*flat* $\Rightarrow$ $'b$::*pcpo*
**assumes** *type*: *type-definition Rep Abs A*
**and** *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
**and** *bottom-in-A*: $\bot \in A$
**shows** *OFCLASS*($'b$, *flat-class*)
**apply** (*intro-classes*)
**apply** (*unfold below*)
**apply** (*simp add*: *type-definition.Rep-inject* [*OF type*, *symmetric*])
**apply** (*simp add*: *typedef-Rep-strict* [*OF type below bottom-in-A*])
**apply** (*simp add*: *ax-flat*)
**done**

## 5.8 HOLCF type definition package

**ML-file** ‹*Tools/cpodef.ML*›

**end**

# 6 Class instances for the full function space

**theory** *Fun-Cpo*
**imports** *Adm*
**begin**

## 6.1 Full function space is a partial order

**instantiation** *fun* :: (*type*, *below*) *below*
**begin**

**definition** *below-fun-def*: $(\sqsubseteq) \equiv (\lambda f\ g.\ \forall x.\ f\ x \sqsubseteq g\ x)$

**instance ..**
**end**

**instance** *fun* :: (*type*, *po*) *po*
**proof**
  **fix** $f :: 'a \Rightarrow 'b$
  **show** $f \sqsubseteq f$
    **by** (*simp add*: *below-fun-def*)
**next**
  **fix** $f\ g :: 'a \Rightarrow 'b$
  **assume** $f \sqsubseteq g$ **and** $g \sqsubseteq f$ **then show** $f = g$
    **by** (*simp add*: *below-fun-def fun-eq-iff below-antisym*)
**next**
  **fix** $f\ g\ h :: 'a \Rightarrow 'b$
  **assume** $f \sqsubseteq g$ **and** $g \sqsubseteq h$ **then show** $f \sqsubseteq h$
    **unfolding** *below-fun-def* **by** (*fast elim*: *below-trans*)
**qed**

**lemma** *fun-below-iff*: $f \sqsubseteq g \longleftrightarrow (\forall x.\ f\ x \sqsubseteq g\ x)$
  **by** (*simp add*: *below-fun-def*)

**lemma** *fun-belowI*: $(\bigwedge x.\ f\ x \sqsubseteq g\ x) \Longrightarrow f \sqsubseteq g$
  **by** (*simp add*: *below-fun-def*)

**lemma** *fun-belowD*: $f \sqsubseteq g \Longrightarrow f\ x \sqsubseteq g\ x$
  **by** (*simp add*: *below-fun-def*)

## 6.2 Full function space is chain complete

Properties of chains of functions.

**lemma** *fun-chain-iff*: *chain* $S \longleftrightarrow (\forall x.\ chain\ (\lambda i.\ S\ i\ x))$
  **by** (*auto simp*: *chain-def fun-below-iff*)

**lemma** *ch2ch-fun*: *chain* $S \Longrightarrow chain\ (\lambda i.\ S\ i\ x)$
  **by** (*simp add*: *chain-def below-fun-def*)

**lemma** *ch2ch-lambda*: $(\bigwedge x.\ chain\ (\lambda i.\ S\ i\ x)) \Longrightarrow chain\ S$
  **by** (*simp add*: *chain-def below-fun-def*)

Type $'a \Rightarrow 'b$ is chain complete

**lemma** *is-lub-lambda*: $(\bigwedge x.\ range\ (\lambda i.\ Y\ i\ x) <<|\ f\ x) \Longrightarrow range\ Y <<|\ f$
  **by** (*simp add*: *is-lub-def is-ub-def below-fun-def*)

**lemma** *is-lub-fun*: *chain S* $\Longrightarrow$ *range S* <<| $(\lambda x.\ \bigsqcup i.\ S\ i\ x)$
**for** *S* :: *nat* $\Rightarrow$ *'a*::*type* $\Rightarrow$ *'b*::*cpo*
**apply** (*rule is-lub-lambda*)
**apply** (*rule cpo-lubI*)
**apply** (*erule ch2ch-fun*)
**done**

**lemma** *lub-fun*: *chain S* $\Longrightarrow$ $(\bigsqcup i.\ S\ i) = (\lambda x.\ \bigsqcup i.\ S\ i\ x)$
**for** *S* :: *nat* $\Rightarrow$ *'a*::*type* $\Rightarrow$ *'b*::*cpo*
**by** (*rule is-lub-fun* [*THEN lub-eqI*])

**instance** *fun* :: (*type*, *cpo*) *cpo*
**by** *intro-classes* (*rule exI*, *erule is-lub-fun*)

**instance** *fun* :: (*type*, *discrete-cpo*) *discrete-cpo*
**proof**
**fix** *f g* :: *'a* $\Rightarrow$ *'b*
**show** *f* $\sqsubseteq$ *g* $\longleftrightarrow$ *f* = *g*
**by** (*simp add*: *fun-below-iff fun-eq-iff*)
**qed**

## 6.3 Full function space is pointed

**lemma** *minimal-fun*: $(\lambda x.\ \bot) \sqsubseteq f$
**by** (*simp add*: *below-fun-def*)

**instance** *fun* :: (*type*, *pcpo*) *pcpo*
**by** *standard* (*fast intro*: *minimal-fun*)

**lemma** *inst-fun-pcpo*: $\bot = (\lambda x.\ \bot)$
**by** (*rule minimal-fun* [*THEN bottomI*, *symmetric*])

**lemma** *app-strict* [*simp*]: $\bot\ x = \bot$
**by** (*simp add*: *inst-fun-pcpo*)

**lemma** *lambda-strict*: $(\lambda x.\ \bot) = \bot$
**by** (*rule bottomI*, *rule minimal-fun*)

## 6.4 Propagation of monotonicity and continuity

The lub of a chain of monotone functions is monotone.

**lemma** *adm-monofun*: *adm monofun*
**by** (*rule admI*) (*simp add*: *lub-fun fun-chain-iff monofun-def lub-mono*)

The lub of a chain of continuous functions is continuous.

**lemma** *adm-cont*: *adm cont*
**by** (*rule admI*) (*simp add*: *lub-fun fun-chain-iff*)

Function application preserves monotonicity and continuity.

**lemma** *mono2mono-fun*: *monofun f* $\Longrightarrow$ *monofun* ($\lambda x.\ f\ x\ y$)
  **by** (*simp add*: *monofun-def fun-below-iff*)

**lemma** *cont2cont-fun*: *cont f* $\Longrightarrow$ *cont* ($\lambda x.\ f\ x\ y$)
  **apply** (*rule contI2*)
   **apply** (*erule cont2mono* [*THEN mono2mono-fun*])
  **apply** (*simp add*: *cont2contlubE lub-fun ch2ch-cont*)
  **done**

**lemma** *cont-fun*: *cont* ($\lambda f.\ f\ x$)
  **using** *cont-id* **by** (*rule cont2cont-fun*)

Lambda abstraction preserves monotonicity and continuity. (Note ($\lambda x.\ \lambda y.\ f\ x\ y$) = *f*.)

**lemma** *mono2mono-lambda*: ($\bigwedge y.$ *monofun* ($\lambda x.\ f\ x\ y$)) $\Longrightarrow$ *monofun f*
  **by** (*simp add*: *monofun-def fun-below-iff*)

**lemma** *cont2cont-lambda* [*simp*]:
  **assumes** *f*: $\bigwedge y.$ *cont* ($\lambda x.\ f\ x\ y$)
  **shows** *cont f*
  **by** (*rule contI*, *rule is-lub-lambda*, *rule contE* [*OF f*])

What D.A.Schmidt calls continuity of abstraction; never used here

**lemma** *contlub-lambda*: ($\bigwedge x.$ *chain* ($\lambda i.\ S\ i\ x$)) $\Longrightarrow$ ($\lambda x.\ \bigsqcup i.\ S\ i\ x$) = ($\bigsqcup i.$ ($\lambda x.\ S\ i\ x$))
  **for** *S* :: *nat* $\Rightarrow$ *'a*::*type* $\Rightarrow$ *'b*::*cpo*
  **by** (*simp add*: *lub-fun ch2ch-lambda*)

**end**

# 7 The cpo of cartesian products

**theory** *Product-Cpo*
  **imports** *Adm*
**begin**

**default-sort** *cpo*

## 7.1 Unit type is a pcpo

**instantiation** *unit* :: *discrete-cpo*
**begin**

**definition** *below-unit-def* [*simp*]: $x \sqsubseteq (y::unit) \longleftrightarrow$ *True*

**instance**
  **by** *standard simp*

**end**

**instance** *unit* :: *pcpo*
**by** *standard simp*

## 7.2 Product type is a partial order

**instantiation** *prod* :: (*below*, *below*) *below*
**begin**

**definition** *below-prod-def*: $(\sqsubseteq) \equiv \lambda p1\ p2.\ (fst\ p1 \sqsubseteq fst\ p2 \wedge snd\ p1 \sqsubseteq snd\ p2)$

**instance ..**

**end**

**instance** *prod* :: (*po*, *po*) *po*
**proof**
**fix** $x :: 'a \times 'b$
**show** $x \sqsubseteq x$
**by** (*simp add*: *below-prod-def*)
**next**
**fix** $x\ y :: 'a \times 'b$
**assume** $x \sqsubseteq y\ y \sqsubseteq x$
**then show** $x = y$
**unfolding** *below-prod-def prod-eq-iff*
**by** (*fast intro*: *below-antisym*)
**next**
**fix** $x\ y\ z :: 'a \times 'b$
**assume** $x \sqsubseteq y\ y \sqsubseteq z$
**then show** $x \sqsubseteq z$
**unfolding** *below-prod-def*
**by** (*fast intro*: *below-trans*)
**qed**

## 7.3 Monotonicity of *Pair*, *fst*, *snd*

**lemma** *prod-belowI*: $fst\ p \sqsubseteq fst\ q \Longrightarrow snd\ p \sqsubseteq snd\ q \Longrightarrow p \sqsubseteq q$
**by** (*simp add*: *below-prod-def*)

**lemma** *Pair-below-iff* [*simp*]: $(a, b) \sqsubseteq (c, d) \longleftrightarrow a \sqsubseteq c \wedge b \sqsubseteq d$
**by** (*simp add*: *below-prod-def*)

Pair (-,-) is monotone in both arguments

**lemma** *monofun-pair1*: *monofun* $(\lambda x.\ (x, y))$
**by** (*simp add*: *monofun-def*)

**lemma** *monofun-pair2*: *monofun* $(\lambda y.\ (x, y))$

**by** (*simp add*: *monofun-def*)

**lemma** *monofun-pair*: *x1* $\sqsubseteq$ *x2* $\Longrightarrow$ *y1* $\sqsubseteq$ *y2* $\Longrightarrow$ (*x1*, *y1*) $\sqsubseteq$ (*x2*, *y2*)
**by** *simp*

**lemma** *ch2ch-Pair* [*simp*]: *chain X* $\Longrightarrow$ *chain Y* $\Longrightarrow$ *chain* ($\lambda i.$ (*X i*, *Y i*))
**by** (*rule chainI*, *simp add*: *chainE*)

*fst* and *snd* are monotone

**lemma** *fst-monofun*: *x* $\sqsubseteq$ *y* $\Longrightarrow$ *fst x* $\sqsubseteq$ *fst y*
**by** (*simp add*: *below-prod-def*)

**lemma** *snd-monofun*: *x* $\sqsubseteq$ *y* $\Longrightarrow$ *snd x* $\sqsubseteq$ *snd y*
**by** (*simp add*: *below-prod-def*)

**lemma** *monofun-fst*: *monofun fst*
**by** (*simp add*: *monofun-def below-prod-def*)

**lemma** *monofun-snd*: *monofun snd*
**by** (*simp add*: *monofun-def below-prod-def*)

**lemmas** *ch2ch-fst* [*simp*] = *ch2ch-monofun* [*OF monofun-fst*]

**lemmas** *ch2ch-snd* [*simp*] = *ch2ch-monofun* [*OF monofun-snd*]

**lemma** *prod-chain-cases*:
**assumes** *chain*: *chain Y*
**obtains** *A B*
**where** *chain A* **and** *chain B* **and** *Y* = ($\lambda i.$ (*A i*, *B i*))
**proof**
**from** *chain* **show** *chain* ($\lambda i.$ *fst* (*Y i*))
**by** (*rule ch2ch-fst*)
**from** *chain* **show** *chain* ($\lambda i.$ *snd* (*Y i*))
**by** (*rule ch2ch-snd*)
**show** *Y* = ($\lambda i.$ (*fst* (*Y i*), *snd* (*Y i*)))
**by** *simp*
**qed**

## 7.4 Product type is a cpo

**lemma** *is-lub-Pair*: *range A* <<| *x* $\Longrightarrow$ *range B* <<| *y* $\Longrightarrow$ *range* ($\lambda i.$ (*A i*, *B i*)) <<| (*x*, *y*)
**by** (*simp add*: *is-lub-def is-ub-def below-prod-def*)

**lemma** *lub-Pair*: *chain A* $\Longrightarrow$ *chain B* $\Longrightarrow$ ($\bigsqcup i.$ (*A i*, *B i*)) = ($\bigsqcup i.$ *A i*, $\bigsqcup i.$ *B i*)
**for** *A* :: *nat* $\Rightarrow$ *'a*::*cpo* **and** *B* :: *nat* $\Rightarrow$ *'b*::*cpo*
**by** (*fast intro*: *lub-eqI is-lub-Pair elim*: *thelubE*)

**lemma** *is-lub-prod*:

**fixes** $S :: nat \Rightarrow ('a::cpo \times 'b::cpo)$
**assumes** *chain S*
**shows** *range S* <<| $(\bigsqcup i.\ fst\ (S\ i), \bigsqcup i.\ snd\ (S\ i))$
**using** *assms* **by** (*auto elim*: *prod-chain-cases simp*: *is-lub-Pair cpo-lubI*)

**lemma** *lub-prod*: *chain S* $\Longrightarrow$ $(\bigsqcup i.\ S\ i) = (\bigsqcup i.\ fst\ (S\ i), \bigsqcup i.\ snd\ (S\ i))$
**for** $S :: nat \Rightarrow 'a::cpo \times 'b::cpo$
**by** (*rule is-lub-prod* [*THEN lub-eqI*])

**instance** *prod* :: (*cpo*, *cpo*) *cpo*
**proof**
**fix** $S :: nat \Rightarrow ('a \times 'b)$
**assume** *chain S*
**then have** *range S* <<| $(\bigsqcup i.\ fst\ (S\ i), \bigsqcup i.\ snd\ (S\ i))$
**by** (*rule is-lub-prod*)
**then show** $\exists x.$ *range S* <<| *x* **..**
**qed**

**instance** *prod* :: (*discrete-cpo*, *discrete-cpo*) *discrete-cpo*
**proof**
**fix** $x\ y :: 'a \times 'b$
**show** $x \sqsubseteq y \longleftrightarrow x = y$
**by** (*simp add*: *below-prod-def prod-eq-iff*)
**qed**

## 7.5 Product type is pointed

**lemma** *minimal-prod*: $(\bot, \bot) \sqsubseteq p$
**by** (*simp add*: *below-prod-def*)

**instance** *prod* :: (*pcpo*, *pcpo*) *pcpo*
**by** *intro-classes* (*fast intro*: *minimal-prod*)

**lemma** *inst-prod-pcpo*: $\bot = (\bot, \bot)$
**by** (*rule minimal-prod* [*THEN bottomI*, *symmetric*])

**lemma** *Pair-bottom-iff* [*simp*]: $(x, y) = \bot \longleftrightarrow x = \bot \wedge y = \bot$
**by** (*simp add*: *inst-prod-pcpo*)

**lemma** *fst-strict* [*simp*]: *fst* $\bot = \bot$
**unfolding** *inst-prod-pcpo* **by** (*rule fst-conv*)

**lemma** *snd-strict* [*simp*]: *snd* $\bot = \bot$
**unfolding** *inst-prod-pcpo* **by** (*rule snd-conv*)

**lemma** *Pair-strict* [*simp*]: $(\bot, \bot) = \bot$
**by** *simp*

**lemma** *split-strict* [*simp*]: *case-prod f* $\bot = f\ \bot\ \bot$

**by** (*simp add*: *split-def*)

## 7.6 Continuity of *Pair*, *fst*, *snd*

**lemma** *cont-pair1*: *cont* (λ*x*. (*x*, *y*))
**apply** (*rule contI*)
**apply** (*rule is-lub-Pair*)
**apply** (*erule cpo-lubI*)
**apply** (*rule is-lub-const*)
**done**

**lemma** *cont-pair2*: *cont* (λ*y*. (*x*, *y*))
**apply** (*rule contI*)
**apply** (*rule is-lub-Pair*)
**apply** (*rule is-lub-const*)
**apply** (*erule cpo-lubI*)
**done**

**lemma** *cont-fst*: *cont fst*
**apply** (*rule contI*)
**apply** (*simp add*: *lub-prod*)
**apply** (*erule cpo-lubI* [*OF ch2ch-fst*])
**done**

**lemma** *cont-snd*: *cont snd*
**apply** (*rule contI*)
**apply** (*simp add*: *lub-prod*)
**apply** (*erule cpo-lubI* [*OF ch2ch-snd*])
**done**

**lemma** *cont2cont-Pair* [*simp*, *cont2cont*]:
**assumes** *f*: *cont* (λ*x*. *f x*)
**assumes** *g*: *cont* (λ*x*. *g x*)
**shows** *cont* (λ*x*. (*f x*, *g x*))
**apply** (*rule cont-apply* [*OF f cont-pair1*])
**apply** (*rule cont-apply* [*OF g cont-pair2*])
**apply** (*rule cont-const*)
**done**

**lemmas** *cont2cont-fst* [*simp*, *cont2cont*] = *cont-compose* [*OF cont-fst*]

**lemmas** *cont2cont-snd* [*simp*, *cont2cont*] = *cont-compose* [*OF cont-snd*]

**lemma** *cont2cont-case-prod*:
**assumes** *f1*: ⋀*a b*. *cont* (λ*x*. *f x a b*)
**assumes** *f2*: ⋀*x b*. *cont* (λ*a*. *f x a b*)
**assumes** *f3*: ⋀*x a*. *cont* (λ*b*. *f x a b*)
**assumes** *g*: *cont* (λ*x*. *g x*)
**shows** *cont* (λ*x*. *case g x of* (*a*, *b*) ⇒ *f x a b*)

**unfolding** *split-def*
**apply** (*rule cont-apply* [*OF g*])
**apply** (*rule cont-apply* [*OF cont-fst f2*])
**apply** (*rule cont-apply* [*OF cont-snd f3*])
**apply** (*rule cont-const*)
**apply** (*rule f1*)
**done**

**lemma** *prod-contI*:
**assumes** *f1*: $\bigwedge y.$ *cont* ($\lambda x.$ *f* (*x*, *y*))
**assumes** *f2*: $\bigwedge x.$ *cont* ($\lambda y.$ *f* (*x*, *y*))
**shows** *cont f*
**proof** −
**have** *cont* ($\lambda(x, y).$ *f* (*x*, *y*))
**by** (*intro cont2cont-case-prod f1 f2 cont2cont*)
**then show** *cont f*
**by** (*simp only*: *case-prod-eta*)
**qed**

**lemma** *prod-cont-iff*: *cont f* $\longleftrightarrow$ ($\forall y.$ *cont* ($\lambda x.$ *f* (*x*, *y*))) $\wedge$ ($\forall x.$ *cont* ($\lambda y.$ *f* (*x*, *y*)))
**apply** *safe*
**apply** (*erule cont-compose* [*OF - cont-pair1*])
**apply** (*erule cont-compose* [*OF - cont-pair2*])
**apply** (*simp only*: *prod-contI*)
**done**

**lemma** *cont2cont-case-prod'* [*simp*, *cont2cont*]:
**assumes** *f*: *cont* ($\lambda p.$ *f* (*fst p*) (*fst* (*snd p*)) (*snd* (*snd p*)))
**assumes** *g*: *cont* ($\lambda x.$ *g x*)
**shows** *cont* ($\lambda x.$ *case-prod* (*f x*) (*g x*))
**using** *assms* **by** (*simp add*: *cont2cont-case-prod prod-cont-iff*)

The simple version (due to Joachim Breitner) is needed if either element type of the pair is not a cpo.

**lemma** *cont2cont-split-simple* [*simp*, *cont2cont*]:
**assumes** $\bigwedge a\ b.$ *cont* ($\lambda x.$ *f x a b*)
**shows** *cont* ($\lambda x.$ *case p of* (*a*, *b*) $\Rightarrow$ *f x a b*)
**using** *assms* **by** (*cases p*) *auto*

Admissibility of predicates on product types.

**lemma** *adm-case-prod* [*simp*]:
**assumes** *adm* ($\lambda x.$ *P x* (*fst* (*f x*)) (*snd* (*f x*)))
**shows** *adm* ($\lambda x.$ *case f x of* (*a*, *b*) $\Rightarrow$ *P x a b*)
**unfolding** *case-prod-beta* **using** *assms* **.**

## 7.7 Compactness and chain-finiteness

**lemma** *fst-below-iff*: *fst x* $\sqsubseteq$ *y* $\longleftrightarrow$ *x* $\sqsubseteq$ (*y*, *snd x*)

**for** $x :: 'a \times 'b$
**by** (*simp add*: *below-prod-def*)

**lemma** *snd-below-iff*: $snd\ x \sqsubseteq y \longleftrightarrow x \sqsubseteq (fst\ x,\ y)$
**for** $x :: 'a \times 'b$
**by** (*simp add*: *below-prod-def*)

**lemma** *compact-fst*: $compact\ x \Longrightarrow compact\ (fst\ x)$
**by** (*rule compactI*) (*simp add*: *fst-below-iff*)

**lemma** *compact-snd*: $compact\ x \Longrightarrow compact\ (snd\ x)$
**by** (*rule compactI*) (*simp add*: *snd-below-iff*)

**lemma** *compact-Pair*: $compact\ x \Longrightarrow compact\ y \Longrightarrow compact\ (x,\ y)$
**by** (*rule compactI*) (*simp add*: *below-prod-def*)

**lemma** *compact-Pair-iff* [*simp*]: $compact\ (x,\ y) \longleftrightarrow compact\ x \wedge compact\ y$
**apply** (*safe intro*!: *compact-Pair*)
**apply** (*drule compact-fst*, *simp*)
**apply** (*drule compact-snd*, *simp*)
**done**

**instance** *prod* :: (*chfin*, *chfin*) *chfin*
**apply** *intro-classes*
**apply** (*erule compact-imp-max-in-chain*)
**apply** (*case-tac* $\bigsqcup i.\ Y\ i$, *simp*)
**done**

**end**

# 8 The type of continuous functions

**theory** *Cfun*
**imports** *Cpodef Fun-Cpo Product-Cpo*
**begin**

**default-sort** *cpo*

## 8.1 Definition of continuous function type

**definition** $cfun = \{f :: 'a \Rightarrow 'b.\ cont\ f\}$

**cpodef** ('*a*, '*b*) *cfun* ((- →/ -) [*1*, *0*] *0*) = *cfun* :: ('*a* ⇒ '*b*) *set*
**by** (*auto simp*: *cfun-def intro*: *cont-const adm-cont*)

**type-notation** (*ASCII*)
*cfun* (**infixr** *–>* *0*)

**notation** (*ASCII*)

*Rep-cfun* ((-$/-) [*999*,*1000*] *999*)

**notation**
*Rep-cfun* ((-·/-) [*999*,*1000*] *999*)

## 8.2 Syntax for continuous lambda abstraction

**syntax** *-cabs* :: [*logic*, *logic*] ⇒ *logic*

**parse-translation** ‹
(∗ *rewrite* (*-cabs x t*) => (*Abs-cfun* (%*x. t*)) ∗)
[*Syntax-Trans.mk-binder-tr* (***syntax-const*** ‹*-cabs*›, ***const-syntax*** ‹*Abs-cfun*›)]
›

**print-translation** ‹
[(***const-syntax*** ‹*Abs-cfun*›, *fn - => fn* [*Abs abs*] =>
*let val* (*x*, *t*) = *Syntax-Trans.atomic-abs-tr′ abs*
*in Syntax.const* ***syntax-const*** ‹*-cabs*› $ *x* $ *t end*)]
› — To avoid eta-contraction of body

Syntax for nested abstractions

**syntax** (*ASCII*)
*-Lambda* :: [*cargs*, *logic*] ⇒ *logic* ((*3LAM -./ -*) [*1000*, *10*] *10*)

**syntax**
*-Lambda* :: [*cargs*, *logic*] ⇒ *logic* ((*3*Λ *-./ -*) [*1000*, *10*] *10*)

**parse-ast-translation** ‹
(∗ *rewrite* (*LAM x y z. t*) => (*-cabs x* (*-cabs y* (*-cabs z t*))) ∗)
(∗ *cf. Syntax.lambda-ast-tr from src/Pure/Syntax/syn-trans.ML* ∗)
*let*
*fun Lambda-ast-tr* [*pats*, *body*] =
*Ast.fold-ast-p* ***syntax-const*** ‹*-cabs*›
(*Ast.unfold-ast* ***syntax-const*** ‹*-cargs*› (*Ast.strip-positions pats*), *body*)
| *Lambda-ast-tr asts* = *raise Ast.AST* (*Lambda-ast-tr*, *asts*);
*in* [(***syntax-const*** ‹*-Lambda*›, *K Lambda-ast-tr*)] *end*
›

**print-ast-translation** ‹
(∗ *rewrite* (*-cabs x* (*-cabs y* (*-cabs z t*))) => (*LAM x y z. t*) ∗)
(∗ *cf. Syntax.abs-ast-tr′ from src/Pure/Syntax/syn-trans.ML* ∗)
*let*
*fun cabs-ast-tr′ asts* =
(*case Ast.unfold-ast-p* ***syntax-const*** ‹*-cabs*›
(*Ast.Appl* (*Ast.Constant* ***syntax-const*** ‹*-cabs*› :: *asts*)) *of*
([], -) => *raise Ast.AST* (*cabs-ast-tr′*, *asts*)
| (*xs*, *body*) => *Ast.Appl*
[*Ast.Constant* ***syntax-const*** ‹*-Lambda*›,
*Ast.fold-ast* ***syntax-const*** ‹*-cargs*› *xs*, *body*]);

*in* [(***syntax-const*** ‹*-cabs*›, *K cabs-ast-tr'*)] *end*
›

Dummy patterns for continuous abstraction

**translations**
Λ -. *t* ⇀ *CONST Abs-cfun* (λ-. *t*)

## 8.3 Continuous function space is pointed

**lemma** *bottom-cfun*: ⊥ ∈ *cfun*
**by** (*simp add*: *cfun-def inst-fun-pcpo*)

**instance** *cfun* :: (*cpo*, *discrete-cpo*) *discrete-cpo*
**by** *intro-classes* (*simp add*: *below-cfun-def Rep-cfun-inject*)

**instance** *cfun* :: (*cpo*, *pcpo*) *pcpo*
**by** (*rule typedef-pcpo* [*OF type-definition-cfun below-cfun-def bottom-cfun*])

**lemmas** *Rep-cfun-strict* =
*typedef-Rep-strict* [*OF type-definition-cfun below-cfun-def bottom-cfun*]

**lemmas** *Abs-cfun-strict* =
*typedef-Abs-strict* [*OF type-definition-cfun below-cfun-def bottom-cfun*]

function application is strict in its first argument

**lemma** *Rep-cfun-strict1* [*simp*]: ⊥·*x* = ⊥
**by** (*simp add*: *Rep-cfun-strict*)

**lemma** *LAM-strict* [*simp*]: (Λ *x*. ⊥) = ⊥
**by** (*simp add*: *inst-fun-pcpo* [*symmetric*] *Abs-cfun-strict*)

for compatibility with old HOLCF-Version

**lemma** *inst-cfun-pcpo*: ⊥ = (Λ *x*. ⊥)
**by** *simp*

## 8.4 Basic properties of continuous functions

Beta-equality for continuous functions

**lemma** *Abs-cfun-inverse2*: *cont f* ⟹ *Rep-cfun* (*Abs-cfun f*) = *f*
**by** (*simp add*: *Abs-cfun-inverse cfun-def*)

**lemma** *beta-cfun*: *cont f* ⟹ (Λ *x*. *f x*)·*u* = *f u*
**by** (*simp add*: *Abs-cfun-inverse2*)

### 8.4.1 Beta-reduction simproc

Given the term (Λ *x*. *f x*)·*y*, the procedure tries to construct the theorem (Λ *x*. *f x*)·*y* ≡ *f y*. If this theorem cannot be completely solved by the cont2cont rules, then the procedure returns the ordinary conditional *beta-cfun* rule.

The simproc does not solve any more goals that would be solved by using *beta-cfun* as a simp rule. The advantage of the simproc is that it can avoid deeply-nested calls to the simplifier that would otherwise be caused by large continuity side conditions.

Update: The simproc now uses rule *Abs-cfun-inverse2* instead of *beta-cfun*, to avoid problems with eta-contraction.

**simproc-setup** *beta-cfun-proc* (*Rep-cfun* (*Abs-cfun f*)) = ‹
*K* (*fn ctxt => fn ct =>*
*let*
*val f* = *#2* (*Thm.dest-comb* (*#2* (*Thm.dest-comb ct*)));
*val* [*T*, *U*] = *Thm.dest-ctyp* (*Thm.ctyp-of-cterm f*);
*val tr* = *Thm.instantiate'* [*SOME T*, *SOME U*] [*SOME f*] (*mk-meta-eq* @{*thm Abs-cfun-inverse2*});
*val rules* = *Named-Theorems.get ctxt* ***named-theorems*** ‹*cont2cont*›;
*val tac* = *SOLVED'* (*REPEAT-ALL-NEW* (*match-tac ctxt* (*rev rules*)));
*in SOME* (*perhaps* (*SINGLE* (*tac 1*)) *tr*) *end*)
›

Eta-equality for continuous functions

**lemma** *eta-cfun*: $(\Lambda\ x.\ f{\cdot}x) = f$
**by** (*rule Rep-cfun-inverse*)

Extensionality for continuous functions

**lemma** *cfun-eq-iff*: $f = g \longleftrightarrow (\forall\, x.\ f{\cdot}x = g{\cdot}x)$
**by** (*simp add*: *Rep-cfun-inject* [*symmetric*] *fun-eq-iff*)

**lemma** *cfun-eqI*: $(\bigwedge x.\ f{\cdot}x = g{\cdot}x) \Longrightarrow f = g$
**by** (*simp add*: *cfun-eq-iff*)

Extensionality wrt. ordering for continuous functions

**lemma** *cfun-below-iff*: $f \sqsubseteq g \longleftrightarrow (\forall\, x.\ f{\cdot}x \sqsubseteq g{\cdot}x)$
**by** (*simp add*: *below-cfun-def fun-below-iff*)

**lemma** *cfun-belowI*: $(\bigwedge x.\ f{\cdot}x \sqsubseteq g{\cdot}x) \Longrightarrow f \sqsubseteq g$
**by** (*simp add*: *cfun-below-iff*)

Congruence for continuous function application

**lemma** *cfun-cong*: $f = g \Longrightarrow x = y \Longrightarrow f{\cdot}x = g{\cdot}y$
**by** *simp*

**lemma** *cfun-fun-cong*: $f = g \Longrightarrow f{\cdot}x = g{\cdot}x$
**by** *simp*

**lemma** *cfun-arg-cong*: $x = y \Longrightarrow f{\cdot}x = f{\cdot}y$
**by** *simp*

## 8.5 Continuity of application

**lemma** *cont-Rep-cfun1*: *cont* $(\lambda f.\ f{\cdot}x)$
**by** (*rule cont-Rep-cfun* [*OF cont-id*, *THEN cont2cont-fun*])

**lemma** *cont-Rep-cfun2*: *cont* $(\lambda x.\ f{\cdot}x)$
**using** *Rep-cfun* [**where** $x = f$] **by** (*simp add*: *cfun-def*)

**lemmas** *monofun-Rep-cfun* = *cont-Rep-cfun* [*THEN cont2mono*]

**lemmas** *monofun-Rep-cfun1* = *cont-Rep-cfun1* [*THEN cont2mono*]
**lemmas** *monofun-Rep-cfun2* = *cont-Rep-cfun2* [*THEN cont2mono*]

contlub, cont properties of *Rep-cfun* in each argument

**lemma** *contlub-cfun-arg*: *chain* $Y \Longrightarrow f{\cdot}(\bigsqcup i.\ Y\ i) = (\bigsqcup i.\ f{\cdot}(Y\ i))$
**by** (*rule cont-Rep-cfun2* [*THEN cont2contlubE*])

**lemma** *contlub-cfun-fun*: *chain* $F \Longrightarrow (\bigsqcup i.\ F\ i){\cdot}x = (\bigsqcup i.\ F\ i{\cdot}x)$
**by** (*rule cont-Rep-cfun1* [*THEN cont2contlubE*])

monotonicity of application

**lemma** *monofun-cfun-fun*: $f \sqsubseteq g \Longrightarrow f{\cdot}x \sqsubseteq g{\cdot}x$
**by** (*simp add*: *cfun-below-iff*)

**lemma** *monofun-cfun-arg*: $x \sqsubseteq y \Longrightarrow f{\cdot}x \sqsubseteq f{\cdot}y$
**by** (*rule monofun-Rep-cfun2* [*THEN monofunE*])

**lemma** *monofun-cfun*: $f \sqsubseteq g \Longrightarrow x \sqsubseteq y \Longrightarrow f{\cdot}x \sqsubseteq g{\cdot}y$
**by** (*rule below-trans* [*OF monofun-cfun-fun monofun-cfun-arg*])

ch2ch - rules for the type $'a \to 'b$

**lemma** *chain-monofun*: *chain* $Y \Longrightarrow$ *chain* $(\lambda i.\ f{\cdot}(Y\ i))$
**by** (*erule monofun-Rep-cfun2* [*THEN ch2ch-monofun*])

**lemma** *ch2ch-Rep-cfunR*: *chain* $Y \Longrightarrow$ *chain* $(\lambda i.\ f{\cdot}(Y\ i))$
**by** (*rule monofun-Rep-cfun2* [*THEN ch2ch-monofun*])

**lemma** *ch2ch-Rep-cfunL*: *chain* $F \Longrightarrow$ *chain* $(\lambda i.\ (F\ i){\cdot}x)$
**by** (*rule monofun-Rep-cfun1* [*THEN ch2ch-monofun*])

**lemma** *ch2ch-Rep-cfun* [*simp*]: *chain* $F \Longrightarrow$ *chain* $Y \Longrightarrow$ *chain* $(\lambda i.\ (F\ i){\cdot}(Y\ i))$
**by** (*simp add*: *chain-def monofun-cfun*)

**lemma** *ch2ch-LAM* [*simp*]:
$(\bigwedge x.$ *chain* $(\lambda i.\ S\ i\ x)) \Longrightarrow (\bigwedge i.$ *cont* $(\lambda x.\ S\ i\ x)) \Longrightarrow$ *chain* $(\lambda i.\ \Lambda\ x.\ S\ i\ x)$
**by** (*simp add*: *chain-def cfun-below-iff*)

contlub, cont properties of *Rep-cfun* in both arguments

**lemma** *lub-APP*: *chain* $F \Longrightarrow$ *chain* $Y \Longrightarrow (\bigsqcup i.\ F\ i{\cdot}(Y\ i)) = (\bigsqcup i.\ F\ i){\cdot}(\bigsqcup i.\ Y\ i)$

**by** (*simp add*: *contlub-cfun-fun contlub-cfun-arg diag-lub*)

**lemma** *lub-LAM*:
**assumes** $\bigwedge x.\ chain\ (\lambda i.\ F\ i\ x)$
**and** $\bigwedge i.\ cont\ (\lambda x.\ F\ i\ x)$
**shows** $(\bigsqcup i.\ \Lambda\ x.\ F\ i\ x) = (\Lambda\ x.\ \bigsqcup i.\ F\ i\ x)$
**using** *assms* **by** (*simp add*: *lub-cfun lub-fun ch2ch-lambda*)

**lemmas** *lub-distribs* = *lub-APP lub-LAM*

strictness

**lemma** *strictI*: $f{\cdot}x = \bot \Longrightarrow f{\cdot}\bot = \bot$
**apply** (*rule bottomI*)
**apply** (*erule subst*)
**apply** (*rule minimal* [*THEN monofun-cfun-arg*])
**done**

type $'a \to 'b$ is chain complete

**lemma** *lub-cfun*: *chain F* $\Longrightarrow (\bigsqcup i.\ F\ i) = (\Lambda\ x.\ \bigsqcup i.\ F\ i{\cdot}x)$
**by** (*simp add*: *lub-cfun lub-fun ch2ch-lambda*)

## 8.6 Continuity simplification procedure

cont2cont lemma for *Rep-cfun*

**lemma** *cont2cont-APP* [*simp*, *cont2cont*]:
**assumes** *f*: *cont* $(\lambda x.\ f\ x)$
**assumes** *t*: *cont* $(\lambda x.\ t\ x)$
**shows** *cont* $(\lambda x.\ (f\ x){\cdot}(t\ x))$
**proof** −
**from** *cont-Rep-cfun1 f* **have** *cont* $(\lambda x.\ (f\ x){\cdot}y)$ **for** *y*
**by** (*rule cont-compose*)
**with** *t cont-Rep-cfun2* **show** *cont* $(\lambda x.\ (f\ x){\cdot}(t\ x))$
**by** (*rule cont-apply*)
**qed**

Two specific lemmas for the combination of LCF and HOL terms. These lemmas are needed in theories that use types like $'a \to 'b \Rightarrow 'c$.

**lemma** *cont-APP-app* [*simp*]: *cont f* $\Longrightarrow$ *cont g* $\Longrightarrow$ *cont* $(\lambda x.\ ((f\ x){\cdot}(g\ x))\ s)$
**by** (*rule cont2cont-APP* [*THEN cont2cont-fun*])

**lemma** *cont-APP-app-app* [*simp*]: *cont f* $\Longrightarrow$ *cont g* $\Longrightarrow$ *cont* $(\lambda x.\ ((f\ x){\cdot}(g\ x))\ s\ t)$
**by** (*rule cont-APP-app* [*THEN cont2cont-fun*])

cont2mono Lemma for $\lambda x.\ \Lambda\ y.\ c1\ x\ y$

**lemma** *cont2mono-LAM*:
$[\![\bigwedge x.\ cont\ (\lambda y.\ f\ x\ y);\ \bigwedge y.\ monofun\ (\lambda x.\ f\ x\ y)]\!]$
$\Longrightarrow monofun\ (\lambda x.\ \Lambda\ y.\ f\ x\ y)$

**by** (*simp add*: *monofun-def cfun-below-iff*)

cont2cont Lemma for $\lambda x.\ \Lambda\ y.\ f\ x\ y$

Not suitable as a cont2cont rule, because on nested lambdas it causes exponential blow-up in the number of subgoals.

**lemma** *cont2cont-LAM*:
**assumes** *f1*: $\bigwedge x.\ cont\ (\lambda y.\ f\ x\ y)$
**assumes** *f2*: $\bigwedge y.\ cont\ (\lambda x.\ f\ x\ y)$
**shows** *cont* $(\lambda x.\ \Lambda\ y.\ f\ x\ y)$
**proof** (*rule cont-Abs-cfun*)
**from** *f1* **show** $f\ x \in cfun$ **for** $x$
**by** (*simp add*: *cfun-def*)
**from** *f2* **show** *cont f*
**by** (*rule cont2cont-lambda*)
**qed**

This version does work as a cont2cont rule, since it has only a single subgoal.

**lemma** *cont2cont-LAM'* [*simp*, *cont2cont*]:
**fixes** $f :: 'a{::}cpo \Rightarrow 'b{::}cpo \Rightarrow 'c{::}cpo$
**assumes** *f*: *cont* $(\lambda p.\ f\ (fst\ p)\ (snd\ p))$
**shows** *cont* $(\lambda x.\ \Lambda\ y.\ f\ x\ y)$
**using** *assms* **by** (*simp add*: *cont2cont-LAM prod-cont-iff*)

**lemma** *cont2cont-LAM-discrete* [*simp*, *cont2cont*]:
$(\bigwedge y{::}'a{::}discrete\text{-}cpo.\ cont\ (\lambda x.\ f\ x\ y)) \Longrightarrow cont\ (\lambda x.\ \Lambda\ y.\ f\ x\ y)$
**by** (*simp add*: *cont2cont-LAM*)

## 8.7 Miscellaneous

Monotonicity of *Abs-cfun*

**lemma** *monofun-LAM*: $cont\ f \Longrightarrow cont\ g \Longrightarrow (\bigwedge x.\ f\ x \sqsubseteq g\ x) \Longrightarrow (\Lambda\ x.\ f\ x) \sqsubseteq (\Lambda\ x.\ g\ x)$
**by** (*simp add*: *cfun-below-iff*)

some lemmata for functions with flat/chfin domain/range types

**lemma** *chfin-Rep-cfunR*: $chain\ Y \Longrightarrow \forall s.\ \exists n.\ (LUB\ i.\ Y\ i)\cdot s = Y\ n\cdot s$
**for** $Y :: nat \Rightarrow 'a{::}cpo \to 'b{::}chfin$
**apply** (*rule allI*)
**apply** (*subst contlub-cfun-fun*)
**apply** *assumption*
**apply** (*fast intro*!: *lub-eqI chfin lub-finch2 chfin2finch ch2ch-Rep-cfunL*)
**done**

**lemma** *adm-chfindom*: $adm\ (\lambda(u{::}'a{::}cpo \to 'b{::}chfin).\ P(u\cdot s))$
**by** (*rule adm-subst*, *simp*, *rule adm-chfin*)

## 8.8 Continuous injection-retraction pairs

Continuous retractions are strict.

**lemma** *retraction-strict*: $\forall x.\ f{\cdot}(g{\cdot}x) = x \Longrightarrow f{\cdot}\bot = \bot$
**apply** (*rule bottomI*)
**apply** (*drule-tac x=⊥* **in** *spec*)
**apply** (*erule subst*)
**apply** (*rule monofun-cfun-arg*)
**apply** (*rule minimal*)
**done**

**lemma** *injection-eq*: $\forall x.\ f{\cdot}(g{\cdot}x) = x \Longrightarrow (g{\cdot}x = g{\cdot}y) = (x = y)$
**apply** (*rule iffI*)
**apply** (*drule-tac f=f* **in** *cfun-arg-cong*)
**apply** *simp*
**apply** *simp*
**done**

**lemma** *injection-below*: $\forall x.\ f{\cdot}(g{\cdot}x) = x \Longrightarrow (g{\cdot}x \sqsubseteq g{\cdot}y) = (x \sqsubseteq y)$
**apply** (*rule iffI*)
**apply** (*drule-tac f=f* **in** *monofun-cfun-arg*)
**apply** *simp*
**apply** (*erule monofun-cfun-arg*)
**done**

**lemma** *injection-defined-rev*: $\forall x.\ f{\cdot}(g{\cdot}x) = x \Longrightarrow g{\cdot}z = \bot \Longrightarrow z = \bot$
**apply** (*drule-tac f=f* **in** *cfun-arg-cong*)
**apply** (*simp add*: *retraction-strict*)
**done**

**lemma** *injection-defined*: $\forall x.\ f{\cdot}(g{\cdot}x) = x \Longrightarrow z \neq \bot \Longrightarrow g{\cdot}z \neq \bot$
**by** (*erule contrapos-nn*, *rule injection-defined-rev*)

a result about functions with flat codomain

**lemma** *flat-eqI*: $x \sqsubseteq y \Longrightarrow x \neq \bot \Longrightarrow x = y$
**for** $x\ y :: 'a{::}flat$
**by** (*drule ax-flat*) *simp*

**lemma** *flat-codom*: $f{\cdot}x = c \Longrightarrow f{\cdot}\bot = \bot \vee (\forall z.\ f{\cdot}z = c)$
**for** $c :: 'b{::}flat$
**apply** (*cases* $f{\cdot}x = \bot$)
**apply** (*rule disjI1*)
**apply** (*rule bottomI*)
**apply** (*erule-tac t=⊥* **in** *subst*)
**apply** (*rule minimal* [*THEN monofun-cfun-arg*])
**apply** *clarify*
**apply** (*rule-tac a* $= f{\cdot}\bot$ **in** *refl* [*THEN box-equals*])
**apply** (*erule minimal* [*THEN monofun-cfun-arg*, *THEN flat-eqI*])
**apply** (*erule minimal* [*THEN monofun-cfun-arg*, *THEN flat-eqI*])

**done**

## 8.9 Identity and composition

**definition** *ID* :: $'a \to 'a$
**where** $ID = (\Lambda\ x.\ x)$

**definition** *cfcomp* :: $('b \to 'c) \to ('a \to 'b) \to 'a \to 'c$
**where** *oo-def*: $cfcomp = (\Lambda\ f\ g\ x.\ f{\cdot}(g{\cdot}x))$

**abbreviation** *cfcomp-syn* :: $['b \to 'c,\ 'a \to 'b] \Rightarrow 'a \to 'c$ (**infixr** *oo 100*)
**where** $f\ oo\ g == cfcomp{\cdot}f{\cdot}g$

**lemma** *ID1* [*simp*]: $ID{\cdot}x = x$
**by** (*simp add*: *ID-def*)

**lemma** *cfcomp1*: $(f\ oo\ g) = (\Lambda\ x.\ f{\cdot}(g{\cdot}x))$
**by** (*simp add*: *oo-def*)

**lemma** *cfcomp2* [*simp*]: $(f\ oo\ g){\cdot}x = f{\cdot}(g{\cdot}x)$
**by** (*simp add*: *cfcomp1*)

**lemma** *cfcomp-LAM*: $cont\ g \Longrightarrow f\ oo\ (\Lambda\ x.\ g\ x) = (\Lambda\ x.\ f{\cdot}(g\ x))$
**by** (*simp add*: *cfcomp1*)

**lemma** *cfcomp-strict* [*simp*]: $\bot\ oo\ f = \bot$
**by** (*simp add*: *cfun-eq-iff*)

Show that interpretation of (pcpo, -→-) is a category.

- The class of objects is interpretation of syntactical class pcpo.
- The class of arrows between objects $'a$ and $'b$ is interpret. of $'a \to 'b$.
- The identity arrow is interpretation of *ID*.
- The composition of f and g is interpretation of *oo*.

**lemma** *ID2* [*simp*]: $f\ oo\ ID = f$
**by** (*rule cfun-eqI*, *simp*)

**lemma** *ID3* [*simp*]: $ID\ oo\ f = f$
**by** (*rule cfun-eqI*) *simp*

**lemma** *assoc-oo*: $f\ oo\ (g\ oo\ h) = (f\ oo\ g)\ oo\ h$
**by** (*rule cfun-eqI*) *simp*

## 8.10 Strictified functions

**default-sort** *pcpo*

**definition** *seq* :: $'a \to 'b \to 'b$
  **where** *seq* = $(\Lambda\ x.\ \mathit{if}\ x = \bot\ \mathit{then}\ \bot\ \mathit{else}\ \mathit{ID})$

**lemma** *cont2cont-if-bottom* [*cont2cont*, *simp*]:
  **assumes** *f*: *cont* $(\lambda x.\ f\ x)$
    **and** *g*: *cont* $(\lambda x.\ g\ x)$
  **shows** *cont* $(\lambda x.\ \mathit{if}\ f\ x = \bot\ \mathit{then}\ \bot\ \mathit{else}\ g\ x)$
**proof** (*rule cont-apply* [*OF f*])
  **show** *cont* $(\lambda y.\ \mathit{if}\ y = \bot\ \mathit{then}\ \bot\ \mathit{else}\ g\ x)$ **for** *x*
    **unfolding** *cont-def is-lub-def is-ub-def ball-simps*
    **by** (*simp add*: *lub-eq-bottom-iff*)
  **show** *cont* $(\lambda x.\ \mathit{if}\ y = \bot\ \mathit{then}\ \bot\ \mathit{else}\ g\ x)$ **for** *y*
    **by** (*simp add*: *g*)
**qed**

**lemma** *seq-conv-if*: $\mathit{seq}{\cdot}x = (\mathit{if}\ x = \bot\ \mathit{then}\ \bot\ \mathit{else}\ \mathit{ID})$
  **by** (*simp add*: *seq-def*)

**lemma** *seq-simps* [*simp*]:
  $\mathit{seq}{\cdot}\bot = \bot$
  $\mathit{seq}{\cdot}x{\cdot}\bot = \bot$
  $x \neq \bot \Longrightarrow \mathit{seq}{\cdot}x = \mathit{ID}$
  **by** (*simp-all add*: *seq-conv-if*)

**definition** *strictify* :: $('a \to 'b) \to 'a \to 'b$
  **where** *strictify* = $(\Lambda\ f\ x.\ \mathit{seq}{\cdot}x{\cdot}(f{\cdot}x))$

**lemma** *strictify-conv-if*: $\mathit{strictify}{\cdot}f{\cdot}x = (\mathit{if}\ x = \bot\ \mathit{then}\ \bot\ \mathit{else}\ f{\cdot}x)$
  **by** (*simp add*: *strictify-def*)

**lemma** *strictify1* [*simp*]: $\mathit{strictify}{\cdot}f{\cdot}\bot = \bot$
  **by** (*simp add*: *strictify-conv-if*)

**lemma** *strictify2* [*simp*]: $x \neq \bot \Longrightarrow \mathit{strictify}{\cdot}f{\cdot}x = f{\cdot}x$
  **by** (*simp add*: *strictify-conv-if*)

## 8.11 Continuity of let-bindings

**lemma** *cont2cont-Let*:
  **assumes** *f*: *cont* $(\lambda x.\ f\ x)$
  **assumes** *g1*: $\bigwedge y.$ *cont* $(\lambda x.\ g\ x\ y)$
  **assumes** *g2*: $\bigwedge x.$ *cont* $(\lambda y.\ g\ x\ y)$
  **shows** *cont* $(\lambda x.\ \mathit{let}\ y = f\ x\ \mathit{in}\ g\ x\ y)$
  **unfolding** *Let-def* **using** *f g2 g1* **by** (*rule cont-apply*)

**lemma** *cont2cont-Let'* [*simp*, *cont2cont*]:
  **assumes** *f*: *cont* $(\lambda x.\ f\ x)$
  **assumes** *g*: *cont* $(\lambda p.\ g\ (\mathit{fst}\ p)\ (\mathit{snd}\ p))$

**shows** *cont* ($\lambda x.$ *let* $y = f\ x$ *in* $g\ x\ y$)
**using** $f$
**proof** (*rule cont2cont-Let*)
**from** $g$ **show** *cont* ($\lambda y.\ g\ x\ y$) **for** $x$
**by** (*simp add*: *prod-cont-iff*)
**from** $g$ **show** *cont* ($\lambda x.\ g\ x\ y$) **for** $y$
**by** (*simp add*: *prod-cont-iff*)
**qed**

The simple version (suggested by Joachim Breitner) is needed if the type of the defined term is not a cpo.

**lemma** *cont2cont-Let-simple* [*simp*, *cont2cont*]:
**assumes** $\bigwedge y.$ *cont* ($\lambda x.\ g\ x\ y$)
**shows** *cont* ($\lambda x.$ *let* $y = t$ *in* $g\ x\ y$)
**unfolding** *Let-def* **using** *assms* **.**

**end**

# 9 Continuous deflations and ep-pairs

**theory** *Deflation*
**imports** *Cfun*
**begin**

**default-sort** *cpo*

## 9.1 Continuous deflations

**locale** *deflation* =
**fixes** $d :: 'a \to 'a$
**assumes** *idem*: $\bigwedge x.\ d{\cdot}(d{\cdot}x) = d{\cdot}x$
**assumes** *below*: $\bigwedge x.\ d{\cdot}x \sqsubseteq x$
**begin**

**lemma** *below-ID*: $d \sqsubseteq ID$
**by** (*rule cfun-belowI*) (*simp add*: *below*)

The set of fixed points is the same as the range.

**lemma** *fixes-eq-range*: $\{x.\ d{\cdot}x = x\} = range\ (\lambda x.\ d{\cdot}x)$
**by** (*auto simp add*: *eq-sym-conv idem*)

**lemma** *range-eq-fixes*: $range\ (\lambda x.\ d{\cdot}x) = \{x.\ d{\cdot}x = x\}$
**by** (*auto simp add*: *eq-sym-conv idem*)

The pointwise ordering on deflation functions coincides with the subset ordering of their sets of fixed-points.

**lemma** *belowI*:
**assumes** $f$: $\bigwedge x.\ d{\cdot}x = x \Longrightarrow f{\cdot}x = x$

**shows** $d \sqsubseteq f$
**proof** (*rule cfun-belowI*)
**fix** $x$
**from** *below* **have** $f{\cdot}(d{\cdot}x) \sqsubseteq f{\cdot}x$
**by** (*rule monofun-cfun-arg*)
**also from** *idem* **have** $f{\cdot}(d{\cdot}x) = d{\cdot}x$
**by** (*rule f*)
**finally show** $d{\cdot}x \sqsubseteq f{\cdot}x$ **.**
**qed**

**lemma** *belowD*: $[\![f \sqsubseteq d; f{\cdot}x = x]\!] \Longrightarrow d{\cdot}x = x$
**proof** (*rule below-antisym*)
**from** *below* **show** $d{\cdot}x \sqsubseteq x$ **.**
**assume** $f \sqsubseteq d$
**then have** $f{\cdot}x \sqsubseteq d{\cdot}x$ **by** (*rule monofun-cfun-fun*)
**also assume** $f{\cdot}x = x$
**finally show** $x \sqsubseteq d{\cdot}x$ **.**
**qed**

**end**

**lemma** *deflation-strict*: *deflation* $d \Longrightarrow d{\cdot}\bot = \bot$
**by** (*rule deflation.below* [*THEN bottomI*])

**lemma** *adm-deflation*: *adm* $(\lambda d.$ *deflation* $d)$
**by** (*simp add*: *deflation-def*)

**lemma** *deflation-ID*: *deflation ID*
**by** (*simp add*: *deflation.intro*)

**lemma** *deflation-bottom*: *deflation* $\bot$
**by** (*simp add*: *deflation.intro*)

**lemma** *deflation-below-iff*: *deflation* $p \Longrightarrow$ *deflation* $q \Longrightarrow p \sqsubseteq q \longleftrightarrow (\forall x.\ p{\cdot}x = x \longrightarrow q{\cdot}x = x)$
**apply** *safe*
**apply** (*simp add*: *deflation.belowD*)
**apply** (*simp add*: *deflation.belowI*)
**done**

The composition of two deflations is equal to the lesser of the two (if they are comparable).

**lemma** *deflation-below-comp1*:
**assumes** *deflation f*
**assumes** *deflation g*
**shows** $f \sqsubseteq g \Longrightarrow f{\cdot}(g{\cdot}x) = f{\cdot}x$
**proof** (*rule below-antisym*)
**interpret** *g*: *deflation g* **by** *fact*
**from** *g.below* **show** $f{\cdot}(g{\cdot}x) \sqsubseteq f{\cdot}x$ **by** (*rule monofun-cfun-arg*)

**next**
**interpret** $f$: *deflation f* **by** *fact*
**assume** $f \sqsubseteq g$
**then have** $f{\cdot}x \sqsubseteq g{\cdot}x$ **by** (*rule monofun-cfun-fun*)
**then have** $f{\cdot}(f{\cdot}x) \sqsubseteq f{\cdot}(g{\cdot}x)$ **by** (*rule monofun-cfun-arg*)
**also have** $f{\cdot}(f{\cdot}x) = f{\cdot}x$ **by** (*rule f.idem*)
**finally show** $f{\cdot}x \sqsubseteq f{\cdot}(g{\cdot}x)$ **.**
**qed**

**lemma** *deflation-below-comp2*: *deflation f* $\Longrightarrow$ *deflation g* $\Longrightarrow$ $f \sqsubseteq g \Longrightarrow g{\cdot}(f{\cdot}x) = f{\cdot}x$
**by** (*simp only*: *deflation.belowD deflation.idem*)

## 9.2 Deflations with finite range

**lemma** *finite-range-imp-finite-fixes*:
**assumes** *finite* (*range f*)
**shows** *finite* $\{x.\ f\ x = x\}$
**proof** −
**have** $\{x.\ f\ x = x\} \subseteq$ *range f*
**by** (*clarify*, *erule subst*, *rule rangeI*)
**from** *this assms* **show** *finite* $\{x.\ f\ x = x\}$
**by** (*rule finite-subset*)
**qed**

**locale** *finite-deflation* = *deflation* +
**assumes** *finite-fixes*: *finite* $\{x.\ d{\cdot}x = x\}$
**begin**

**lemma** *finite-range*: *finite* (*range* $(\lambda x.\ d{\cdot}x)$)
**by** (*simp add*: *range-eq-fixes finite-fixes*)

**lemma** *finite-image*: *finite* $((\lambda x.\ d{\cdot}x)\ `\ A)$
**by** (*rule finite-subset* [*OF image-mono* [*OF subset-UNIV*] *finite-range*])

**lemma** *compact*: *compact* $(d{\cdot}x)$
**proof** (*rule compactI2*)
**fix** $Y :: nat \Rightarrow 'a$
**assume** *Y*: *chain Y*
**have** *finite-chain* $(\lambda i.\ d{\cdot}(Y\ i))$
**proof** (*rule finite-range-imp-finch*)
**from** *Y* **show** *chain* $(\lambda i.\ d{\cdot}(Y\ i))$ **by** *simp*
**have** *range* $(\lambda i.\ d{\cdot}(Y\ i)) \subseteq$ *range* $(\lambda x.\ d{\cdot}x)$ **by** *auto*
**then show** *finite* (*range* $(\lambda i.\ d{\cdot}(Y\ i))$)
**using** *finite-range* **by** (*rule finite-subset*)
**qed**
**then have** $\exists j.\ (\bigsqcup i.\ d{\cdot}(Y\ i)) = d{\cdot}(Y\ j)$
**by** (*simp add*: *finite-chain-def maxinch-is-thelub Y*)
**then obtain** *j* **where** *j*: $(\bigsqcup i.\ d{\cdot}(Y\ i)) = d{\cdot}(Y\ j)$ **..**

**assume** $d{\cdot}x \sqsubseteq (\bigsqcup i.\ Y\ i)$
**then have** $d{\cdot}(d{\cdot}x) \sqsubseteq d{\cdot}(\bigsqcup i.\ Y\ i)$
**by** (*rule monofun-cfun-arg*)
**then have** $d{\cdot}x \sqsubseteq (\bigsqcup i.\ d{\cdot}(Y\ i))$
**by** (*simp add*: *contlub-cfun-arg Y idem*)
**with** $j$ **have** $d{\cdot}x \sqsubseteq d{\cdot}(Y\ j)$ **by** *simp*
**then have** $d{\cdot}x \sqsubseteq Y\ j$
**using** *below* **by** (*rule below-trans*)
**then show** $\exists j.\ d{\cdot}x \sqsubseteq Y\ j$ **..**
**qed**

**end**

**lemma** *finite-deflation-intro*: *deflation* $d \Longrightarrow$ *finite* $\{x.\ d{\cdot}x = x\} \Longrightarrow$ *finite-deflation* $d$
**by** (*intro finite-deflation.intro finite-deflation-axioms.intro*)

**lemma** *finite-deflation-imp-deflation*: *finite-deflation* $d \Longrightarrow$ *deflation* $d$
**by** (*simp add*: *finite-deflation-def*)

**lemma** *finite-deflation-bottom*: *finite-deflation* $\bot$
**by** *standard simp-all*

## 9.3 Continuous embedding-projection pairs

**locale** *ep-pair* =
**fixes** $e :: 'a \to 'b$ **and** $p :: 'b \to 'a$
**assumes** *e-inverse* [*simp*]: $\bigwedge x.\ p{\cdot}(e{\cdot}x) = x$
**and** *e-p-below*: $\bigwedge y.\ e{\cdot}(p{\cdot}y) \sqsubseteq y$
**begin**

**lemma** *e-below-iff* [*simp*]: $e{\cdot}x \sqsubseteq e{\cdot}y \longleftrightarrow x \sqsubseteq y$
**proof**
**assume** $e{\cdot}x \sqsubseteq e{\cdot}y$
**then have** $p{\cdot}(e{\cdot}x) \sqsubseteq p{\cdot}(e{\cdot}y)$ **by** (*rule monofun-cfun-arg*)
**then show** $x \sqsubseteq y$ **by** *simp*
**next**
**assume** $x \sqsubseteq y$
**then show** $e{\cdot}x \sqsubseteq e{\cdot}y$ **by** (*rule monofun-cfun-arg*)
**qed**

**lemma** *e-eq-iff* [*simp*]: $e{\cdot}x = e{\cdot}y \longleftrightarrow x = y$
**unfolding** *po-eq-conv e-below-iff* **..**

**lemma** *p-eq-iff*: $e{\cdot}(p{\cdot}x) = x \Longrightarrow e{\cdot}(p{\cdot}y) = y \Longrightarrow p{\cdot}x = p{\cdot}y \longleftrightarrow x = y$
**by** (*safe*, *erule subst*, *erule subst*, *simp*)

**lemma** *p-inverse*: $(\exists x.\ y = e{\cdot}x) \longleftrightarrow e{\cdot}(p{\cdot}y) = y$

**by** (*auto*, *rule exI*, *erule sym*)

**lemma** *e-below-iff-below-p*: $e{\cdot}x \sqsubseteq y \longleftrightarrow x \sqsubseteq p{\cdot}y$
**proof**
**assume** $e{\cdot}x \sqsubseteq y$
**then have** $p{\cdot}(e{\cdot}x) \sqsubseteq p{\cdot}y$ **by** (*rule monofun-cfun-arg*)
**then show** $x \sqsubseteq p{\cdot}y$ **by** *simp*
**next**
**assume** $x \sqsubseteq p{\cdot}y$
**then have** $e{\cdot}x \sqsubseteq e{\cdot}(p{\cdot}y)$ **by** (*rule monofun-cfun-arg*)
**then show** $e{\cdot}x \sqsubseteq y$ **using** *e-p-below* **by** (*rule below-trans*)
**qed**

**lemma** *compact-e-rev*: *compact* $(e{\cdot}x) \Longrightarrow$ *compact* $x$
**proof** −
**assume** *compact* $(e{\cdot}x)$
**then have** *adm* $(\lambda y.\ e{\cdot}x \not\sqsubseteq y)$ **by** (*rule compactD*)
**then have** *adm* $(\lambda y.\ e{\cdot}x \not\sqsubseteq e{\cdot}y)$ **by** (*rule adm-subst* [*OF cont-Rep-cfun2*])
**then have** *adm* $(\lambda y.\ x \not\sqsubseteq y)$ **by** *simp*
**then show** *compact* $x$ **by** (*rule compactI*)
**qed**

**lemma** *compact-e*:
**assumes** *compact* $x$
**shows** *compact* $(e{\cdot}x)$
**proof** −
**from** *assms* **have** *adm* $(\lambda y.\ x \not\sqsubseteq y)$ **by** (*rule compactD*)
**then have** *adm* $(\lambda y.\ x \not\sqsubseteq p{\cdot}y)$ **by** (*rule adm-subst* [*OF cont-Rep-cfun2*])
**then have** *adm* $(\lambda y.\ e{\cdot}x \not\sqsubseteq y)$ **by** (*simp add*: *e-below-iff-below-p*)
**then show** *compact* $(e{\cdot}x)$ **by** (*rule compactI*)
**qed**

**lemma** *compact-e-iff*: *compact* $(e{\cdot}x) \longleftrightarrow$ *compact* $x$
**by** (*rule iffI* [*OF compact-e-rev compact-e*])

Deflations from ep-pairs

**lemma** *deflation-e-p*: *deflation* $(e \mathbin{oo} p)$
**by** (*simp add*: *deflation.intro e-p-below*)

**lemma** *deflation-e-d-p*:
**assumes** *deflation* $d$
**shows** *deflation* $(e \mathbin{oo} d \mathbin{oo} p)$
**proof**
**interpret** *deflation* $d$ **by** *fact*
**fix** $x :: 'b$
**show** $(e \mathbin{oo} d \mathbin{oo} p){\cdot}((e \mathbin{oo} d \mathbin{oo} p){\cdot}x) = (e \mathbin{oo} d \mathbin{oo} p){\cdot}x$
**by** (*simp add*: *idem*)
**show** $(e \mathbin{oo} d \mathbin{oo} p){\cdot}x \sqsubseteq x$
**by** (*simp add*: *e-below-iff-below-p below*)

**qed**

**lemma** *finite-deflation-e-d-p*:
**assumes** *finite-deflation d*
**shows** *finite-deflation* (*e oo d oo p*)
**proof**
**interpret** *finite-deflation d* **by** *fact*
**fix** *x* :: *'b*
**show** (*e oo d oo p*)·((*e oo d oo p*)·*x*) = (*e oo d oo p*)·*x*
**by** (*simp add*: *idem*)
**show** (*e oo d oo p*)·*x* ⊑ *x*
**by** (*simp add*: *e-below-iff-below-p below*)
**have** *finite* ((λ*x*. *e*·*x*) ' (λ*x*. *d*·*x*) ' *range* (λ*x*. *p*·*x*))
**by** (*simp add*: *finite-image*)
**then have** *finite* (*range* (λ*x*. (*e oo d oo p*)·*x*))
**by** (*simp add*: *image-image*)
**then show** *finite* {*x*. (*e oo d oo p*)·*x* = *x*}
**by** (*rule finite-range-imp-finite-fixes*)
**qed**

**lemma** *deflation-p-d-e*:
**assumes** *deflation d*
**assumes** *d*: ⋀*x*. *d*·*x* ⊑ *e*·(*p*·*x*)
**shows** *deflation* (*p oo d oo e*)
**proof** −
**interpret** *d*: *deflation d* **by** *fact*
**have** *p-d-e-below*: (*p oo d oo e*)·*x* ⊑ *x* **for** *x*
**proof** −
**have** *d*·(*e*·*x*) ⊑ *e*·*x*
**by** (*rule d.below*)
**then have** *p*·(*d*·(*e*·*x*)) ⊑ *p*·(*e*·*x*)
**by** (*rule monofun-cfun-arg*)
**then show** *?thesis* **by** *simp*
**qed**
**show** *?thesis*
**proof**
**show** (*p oo d oo e*)·*x* ⊑ *x* **for** *x*
**by** (*rule p-d-e-below*)
**show** (*p oo d oo e*)·((*p oo d oo e*)·*x*) = (*p oo d oo e*)·*x* **for** *x*
**proof** (*rule below-antisym*)
**show** (*p oo d oo e*)·((*p oo d oo e*)·*x*) ⊑ (*p oo d oo e*)·*x*
**by** (*rule p-d-e-below*)
**have** *p*·(*d*·(*d*·(*d*·(*e*·*x*)))) ⊑ *p*·(*d*·(*e*·(*p*·(*d*·(*e*·*x*)))))
**by** (*intro monofun-cfun-arg d*)
**then have** *p*·(*d*·(*e*·*x*)) ⊑ *p*·(*d*·(*e*·(*p*·(*d*·(*e*·*x*)))))
**by** (*simp only*: *d.idem*)
**then show** (*p oo d oo e*)·*x* ⊑ (*p oo d oo e*)·((*p oo d oo e*)·*x*)
**by** *simp*
**qed**

```
  qed
qed

lemma finite-deflation-p-d-e:
  assumes finite-deflation d
  assumes d: ⋀x. d·x ⊑ e·(p·x)
  shows finite-deflation (p oo d oo e)
proof −
  interpret d: finite-deflation d by fact
  show ?thesis
  proof (rule finite-deflation-intro)
    have deflation d ..
    then show deflation (p oo d oo e)
      using d by (rule deflation-p-d-e)
  next
    have finite ((λx. d·x) ‘ range (λx. e·x))
      by (rule d.finite-image)
    then have finite ((λx. p·x) ‘ (λx. d·x) ‘ range (λx. e·x))
      by (rule finite-imageI)
    then have finite (range (λx. (p oo d oo e)·x))
      by (simp add: image-image)
    then show finite {x. (p oo d oo e)·x = x}
      by (rule finite-range-imp-finite-fixes)
  qed
qed

end
```

## 9.4 Uniqueness of ep-pairs

```
lemma ep-pair-unique-e-lemma:
  assumes 1: ep-pair e1 p
    and 2: ep-pair e2 p
  shows e1 ⊑ e2
proof (rule cfun-belowI)
  fix x
  have e1·(p·(e2·x)) ⊑ e2·x
    by (rule ep-pair.e-p-below [OF 1])
  then show e1·x ⊑ e2·x
    by (simp only: ep-pair.e-inverse [OF 2])
qed

lemma ep-pair-unique-e: ep-pair e1 p ⟹ ep-pair e2 p ⟹ e1 = e2
  by (fast intro: below-antisym elim: ep-pair-unique-e-lemma)

lemma ep-pair-unique-p-lemma:
  assumes 1: ep-pair e p1
    and 2: ep-pair e p2
  shows p1 ⊑ p2
```

**proof** (*rule cfun-belowI*)
**fix** $x$
**have** $e{\cdot}(p1{\cdot}x) \sqsubseteq x$
**by** (*rule ep-pair.e-p-below* [*OF 1*])
**then have** $p2{\cdot}(e{\cdot}(p1{\cdot}x)) \sqsubseteq p2{\cdot}x$
**by** (*rule monofun-cfun-arg*)
**then show** $p1{\cdot}x \sqsubseteq p2{\cdot}x$
**by** (*simp only*: *ep-pair.e-inverse* [*OF 2*])
**qed**

**lemma** *ep-pair-unique-p*: *ep-pair e p1* $\Longrightarrow$ *ep-pair e p2* $\Longrightarrow$ $p1 = p2$
**by** (*fast intro*: *below-antisym elim*: *ep-pair-unique-p-lemma*)

## 9.5 Composing ep-pairs

**lemma** *ep-pair-ID-ID*: *ep-pair ID ID*
**by** *standard simp-all*

**lemma** *ep-pair-comp*:
**assumes** *ep-pair e1 p1* **and** *ep-pair e2 p2*
**shows** *ep-pair* (*e2 oo e1*) (*p1 oo p2*)
**proof**
**interpret** *ep1*: *ep-pair e1 p1* **by** *fact*
**interpret** *ep2*: *ep-pair e2 p2* **by** *fact*
**fix** $x$ $y$
**show** $(p1 \mathbin{oo} p2){\cdot}((e2 \mathbin{oo} e1){\cdot}x) = x$
**by** *simp*
**have** $e1{\cdot}(p1{\cdot}(p2{\cdot}y)) \sqsubseteq p2{\cdot}y$
**by** (*rule ep1.e-p-below*)
**then have** $e2{\cdot}(e1{\cdot}(p1{\cdot}(p2{\cdot}y))) \sqsubseteq e2{\cdot}(p2{\cdot}y)$
**by** (*rule monofun-cfun-arg*)
**also have** $e2{\cdot}(p2{\cdot}y) \sqsubseteq y$
**by** (*rule ep2.e-p-below*)
**finally show** $(e2 \mathbin{oo} e1){\cdot}((p1 \mathbin{oo} p2){\cdot}y) \sqsubseteq y$
**by** *simp*
**qed**

**locale** *pcpo-ep-pair* = *ep-pair e p*
**for** $e$ :: $'a$::*pcpo* $\to$ $'b$::*pcpo*
**and** $p$ :: $'b$::*pcpo* $\to$ $'a$::*pcpo*
**begin**

**lemma** *e-strict* [*simp*]: $e{\cdot}\bot = \bot$
**proof** −
**have** $\bot \sqsubseteq p{\cdot}\bot$ **by** (*rule minimal*)
**then have** $e{\cdot}\bot \sqsubseteq e{\cdot}(p{\cdot}\bot)$ **by** (*rule monofun-cfun-arg*)
**also have** $e{\cdot}(p{\cdot}\bot) \sqsubseteq \bot$ **by** (*rule e-p-below*)
**finally show** $e{\cdot}\bot = \bot$ **by** *simp*
**qed**

**lemma** *e-bottom-iff* [*simp*]: $e{\cdot}x = \bot \longleftrightarrow x = \bot$
**by** (*rule e-eq-iff* [**where** $y{=}\bot$, *unfolded e-strict*])

**lemma** *e-defined*: $x \neq \bot \Longrightarrow e{\cdot}x \neq \bot$
**by** *simp*

**lemma** *p-strict* [*simp*]: $p{\cdot}\bot = \bot$
**by** (*rule e-inverse* [**where** $x{=}\bot$, *unfolded e-strict*])

**lemmas** *stricts* = *e-strict p-strict*

**end**

**end**

# 10 The type of strict products

**theory** *Sprod*
**imports** *Cfun*
**begin**

**default-sort** *pcpo*

## 10.1 Definition of strict product type

**definition** *sprod* = $\{p{::}'a \times 'b.\ p = \bot \vee (fst\ p \neq \bot \wedge snd\ p \neq \bot)\}$

**pcpodef** ('*a*, '*b*) *sprod* ((- ⊗/ -) [*21*,*20*] *20*) = *sprod* :: ('*a* × '*b*) *set*
**by** (*simp-all add*: *sprod-def*)

**instance** *sprod* :: ({*chfin*,*pcpo*}, {*chfin*,*pcpo*}) *chfin*
**by** (*rule typedef-chfin* [*OF type-definition-sprod below-sprod-def*])

**type-notation** (*ASCII*)
*sprod* (**infixr** ** *20*)

## 10.2 Definitions of constants

**definition** *sfst* :: ('*a* ** '*b*) → '*a*
**where** *sfst* = (Λ *p*. *fst* (*Rep-sprod p*))

**definition** *ssnd* :: ('*a* ** '*b*) → '*b*
**where** *ssnd* = (Λ *p*. *snd* (*Rep-sprod p*))

**definition** *spair* :: '*a* → '*b* → ('*a* ** '*b*)
**where** *spair* = (Λ *a b*. *Abs-sprod* (*seq*·*b*·*a*, *seq*·*a*·*b*))

**definition** *ssplit* :: ('*a* → '*b* → '*c*) → ('*a* ** '*b*) → '*c*

**where** *ssplit* = (Λ *f p*. *seq*·*p*·(*f*·(*sfst*·*p*)·(*ssnd*·*p*)))

**syntax** -*stuple* :: [*logic*, *args*] ⇒ *logic* ((*1* '(:-,/ -:')))
**translations**
(:*x*, *y*, *z*:) ⇌ (:*x*, (:*y*, *z*:):)
(:*x*, *y*:) ⇌ *CONST spair*·*x*·*y*

**translations**
Λ(*CONST spair*·*x*·*y*). *t* ⇌ *CONST ssplit*·(Λ *x y*. *t*)

## 10.3 Case analysis

**lemma** *spair-sprod*: (*seq*·*b*·*a*, *seq*·*a*·*b*) ∈ *sprod*
**by** (*simp add*: *sprod-def seq-conv-if*)

**lemma** *Rep-sprod-spair*: *Rep-sprod* (:*a*, *b*:) = (*seq*·*b*·*a*, *seq*·*a*·*b*)
**by** (*simp add*: *spair-def cont-Abs-sprod Abs-sprod-inverse spair-sprod*)

**lemmas** *Rep-sprod-simps* =
*Rep-sprod-inject* [*symmetric*] *below-sprod-def*
*prod-eq-iff below-prod-def*
*Rep-sprod-strict Rep-sprod-spair*

**lemma** *sprodE* [*case-names bottom spair*, *cases type*: *sprod*]:
**obtains** *p* = ⊥ | *x y* **where** *p* = (:*x*, *y*:) **and** *x* ≠ ⊥ **and** *y* ≠ ⊥
**using** *Rep-sprod* [*of p*] **by** (*auto simp add*: *sprod-def Rep-sprod-simps*)

**lemma** *sprod-induct* [*case-names bottom spair*, *induct type*: *sprod*]:
⟦*P* ⊥; ⋀*x y*. ⟦*x* ≠ ⊥; *y* ≠ ⊥⟧ ⟹ *P* (:*x*, *y*:)⟧ ⟹ *P x*
**by** (*cases x*) *simp-all*

## 10.4 Properties of *spair*

**lemma** *spair-strict1* [*simp*]: (:⊥, *y*:) = ⊥
**by** (*simp add*: *Rep-sprod-simps*)

**lemma** *spair-strict2* [*simp*]: (:*x*, ⊥:) = ⊥
**by** (*simp add*: *Rep-sprod-simps*)

**lemma** *spair-bottom-iff* [*simp*]: (:*x*, *y*:) = ⊥ ⟷ *x* = ⊥ ∨ *y* = ⊥
**by** (*simp add*: *Rep-sprod-simps seq-conv-if*)

**lemma** *spair-below-iff*: (:*a*, *b*:) ⊑ (:*c*, *d*:) ⟷ *a* = ⊥ ∨ *b* = ⊥ ∨ (*a* ⊑ *c* ∧ *b* ⊑ *d*)
**by** (*simp add*: *Rep-sprod-simps seq-conv-if*)

**lemma** *spair-eq-iff*: (:*a*, *b*:) = (:*c*, *d*:) ⟷ *a* = *c* ∧ *b* = *d* ∨ (*a* = ⊥ ∨ *b* = ⊥) ∧ (*c* = ⊥ ∨ *d* = ⊥)
**by** (*simp add*: *Rep-sprod-simps seq-conv-if*)

**lemma** *spair-strict*: *x* = ⊥ ∨ *y* = ⊥ ⟹ (:*x*, *y*:) = ⊥

**by** *simp*

**lemma** *spair-strict-rev*: $(:x,\ y:) \neq \bot \Longrightarrow x \neq \bot \wedge y \neq \bot$
**by** *simp*

**lemma** *spair-defined*: $[\![x \neq \bot;\ y \neq \bot]\!] \Longrightarrow (:x,\ y:) \neq \bot$
**by** *simp*

**lemma** *spair-defined-rev*: $(:x,\ y:) = \bot \Longrightarrow x = \bot \vee y = \bot$
**by** *simp*

**lemma** *spair-below*: $x \neq \bot \Longrightarrow y \neq \bot \Longrightarrow (:x,\ y:) \sqsubseteq (:a,\ b:) \longleftrightarrow x \sqsubseteq a \wedge y \sqsubseteq b$
**by** (*simp add*: *spair-below-iff*)

**lemma** *spair-eq*: $x \neq \bot \Longrightarrow y \neq \bot \Longrightarrow (:x,\ y:) = (:a,\ b:) \longleftrightarrow x = a \wedge y = b$
**by** (*simp add*: *spair-eq-iff*)

**lemma** *spair-inject*: $x \neq \bot \Longrightarrow y \neq \bot \Longrightarrow (:x,\ y:) = (:a,\ b:) \Longrightarrow x = a \wedge y = b$
**by** (*rule spair-eq* [*THEN iffD1*])

**lemma** *inst-sprod-pcpo2*: $\bot = (:\bot,\ \bot:)$
**by** *simp*

**lemma** *sprodE2*: $(\bigwedge x\ y.\ p = (:x,\ y:) \Longrightarrow Q) \Longrightarrow Q$
**by** (*cases p*) (*simp only*: *inst-sprod-pcpo2*, *simp*)

## 10.5 Properties of *sfst* and *ssnd*

**lemma** *sfst-strict* [*simp*]: $sfst{\cdot}\bot = \bot$
**by** (*simp add*: *sfst-def cont-Rep-sprod Rep-sprod-strict*)

**lemma** *ssnd-strict* [*simp*]: $ssnd{\cdot}\bot = \bot$
**by** (*simp add*: *ssnd-def cont-Rep-sprod Rep-sprod-strict*)

**lemma** *sfst-spair* [*simp*]: $y \neq \bot \Longrightarrow sfst{\cdot}(:x,\ y:) = x$
**by** (*simp add*: *sfst-def cont-Rep-sprod Rep-sprod-spair*)

**lemma** *ssnd-spair* [*simp*]: $x \neq \bot \Longrightarrow ssnd{\cdot}(:x,\ y:) = y$
**by** (*simp add*: *ssnd-def cont-Rep-sprod Rep-sprod-spair*)

**lemma** *sfst-bottom-iff* [*simp*]: $sfst{\cdot}p = \bot \longleftrightarrow p = \bot$
**by** (*cases p*) *simp-all*

**lemma** *ssnd-bottom-iff* [*simp*]: $ssnd{\cdot}p = \bot \longleftrightarrow p = \bot$
**by** (*cases p*) *simp-all*

**lemma** *sfst-defined*: $p \neq \bot \Longrightarrow sfst{\cdot}p \neq \bot$
**by** *simp*

**lemma** *ssnd-defined*: $p \neq \bot \Longrightarrow ssnd{\cdot}p \neq \bot$
**by** *simp*

**lemma** *spair-sfst-ssnd*: $(:sfst{\cdot}p,\ ssnd{\cdot}p:) = p$
**by** (*cases p*) *simp-all*

**lemma** *below-sprod*: $x \sqsubseteq y \longleftrightarrow sfst{\cdot}x \sqsubseteq sfst{\cdot}y \wedge ssnd{\cdot}x \sqsubseteq ssnd{\cdot}y$
**by** (*simp add*: *Rep-sprod-simps sfst-def ssnd-def cont-Rep-sprod*)

**lemma** *eq-sprod*: $x = y \longleftrightarrow sfst{\cdot}x = sfst{\cdot}y \wedge ssnd{\cdot}x = ssnd{\cdot}y$
**by** (*auto simp add*: *po-eq-conv below-sprod*)

**lemma** *sfst-below-iff*: $sfst{\cdot}x \sqsubseteq y \longleftrightarrow x \sqsubseteq (:y,\ ssnd{\cdot}x:)$
**by** (*cases* $x = \bot$, *simp*, *cases* $y = \bot$, *simp*, *simp add*: *below-sprod*)

**lemma** *ssnd-below-iff*: $ssnd{\cdot}x \sqsubseteq y \longleftrightarrow x \sqsubseteq (:sfst{\cdot}x,\ y:)$
**by** (*cases* $x = \bot$, *simp*, *cases* $y = \bot$, *simp*, *simp add*: *below-sprod*)

## 10.6 Compactness

**lemma** *compact-sfst*: $compact\ x \Longrightarrow compact\ (sfst{\cdot}x)$
**by** (*rule compactI*) (*simp add*: *sfst-below-iff*)

**lemma** *compact-ssnd*: $compact\ x \Longrightarrow compact\ (ssnd{\cdot}x)$
**by** (*rule compactI*) (*simp add*: *ssnd-below-iff*)

**lemma** *compact-spair*: $compact\ x \Longrightarrow compact\ y \Longrightarrow compact\ (:x,\ y:)$
**by** (*rule compact-sprod*) (*simp add*: *Rep-sprod-spair seq-conv-if*)

**lemma** *compact-spair-iff*: $compact\ (:x,\ y:) \longleftrightarrow x = \bot \vee y = \bot \vee (compact\ x \wedge compact\ y)$
**apply** (*safe elim*!: *compact-spair*)
**apply** (*drule compact-sfst*, *simp*)
**apply** (*drule compact-ssnd*, *simp*)
**apply** *simp*
**apply** *simp*
**done**

## 10.7 Properties of *ssplit*

**lemma** *ssplit1* [*simp*]: $ssplit{\cdot}f{\cdot}\bot = \bot$
**by** (*simp add*: *ssplit-def*)

**lemma** *ssplit2* [*simp*]: $x \neq \bot \Longrightarrow y \neq \bot \Longrightarrow ssplit{\cdot}f{\cdot}(:x,\ y:) = f{\cdot}x{\cdot}y$
**by** (*simp add*: *ssplit-def*)

**lemma** *ssplit3* [*simp*]: $ssplit{\cdot}spair{\cdot}z = z$
**by** (*cases z*) *simp-all*

### 10.8 Strict product preserves flatness

**instance** *sprod* :: (*flat*, *flat*) *flat*
**proof**
  **fix** *x y* :: *'a* ⊗ *'b*
  **assume** *x* ⊑ *y*
  **then show** *x* = ⊥ ∨ *x* = *y*
    **apply** (*induct x*, *simp*)
    **apply** (*induct y*, *simp*)
    **apply** (*simp add*: *spair-below-iff flat-below-iff*)
    **done**
**qed**

**end**

## 11 Discrete cpo types

**theory** *Discrete*
  **imports** *Cont*
**begin**

**datatype** *'a discr* = *Discr 'a* :: *type*

### 11.1 Discrete cpo class instance

**instantiation** *discr* :: (*type*) *discrete-cpo*
**begin**

**definition** ((⊑) :: *'a discr* ⇒ *'a discr* ⇒ *bool*) = (=)

**instance**
  **by** *standard* (*simp add*: *below-discr-def*)

**end**

### 11.2 *undiscr*

**definition** *undiscr* :: (*'a*::*type*)*discr* ⇒ *'a*
  **where** *undiscr x* = (*case x of Discr y* ⇒ *y*)

**lemma** *undiscr-Discr* [*simp*]: *undiscr* (*Discr x*) = *x*
  **by** (*simp add*: *undiscr-def*)

**lemma** *Discr-undiscr* [*simp*]: *Discr* (*undiscr y*) = *y*
  **by** (*induct y*) *simp*

**end**

# 12 The type of lifted values

**theory** *Up*
**imports** *Cfun*
**begin**

**default-sort** *cpo*

## 12.1 Definition of new type for lifting

**datatype** $'a\ u\ \ ((\text{-}_\perp)\ [1000]\ 999) = Ibottom \mid Iup\ 'a$

**primrec** *Ifup* :: $('a \to 'b{::}pcpo) \Rightarrow 'a\ u \Rightarrow 'b$
**where**
*Ifup f Ibottom* $= \perp$
| *Ifup f* (*Iup x*) $= f{\cdot}x$

## 12.2 Ordering on lifted cpo

**instantiation** $u :: (cpo)\ below$
**begin**

**definition** *below-up-def*:
$(\sqsubseteq) \equiv$
$(\lambda x\ y.$
(*case x of*
*Ibottom* $\Rightarrow$ *True*
| *Iup a* $\Rightarrow$ (*case y of Ibottom* $\Rightarrow$ *False* | *Iup b* $\Rightarrow$ $a \sqsubseteq b$)))

**instance** ..

**end**

**lemma** *minimal-up* [*iff*]: *Ibottom* $\sqsubseteq z$
**by** (*simp add*: *below-up-def*)

**lemma** *not-Iup-below* [*iff*]: *Iup x* $\not\sqsubseteq$ *Ibottom*
**by** (*simp add*: *below-up-def*)

**lemma** *Iup-below* [*iff*]: (*Iup x* $\sqsubseteq$ *Iup y*) $= (x \sqsubseteq y)$
**by** (*simp add*: *below-up-def*)

## 12.3 Lifted cpo is a partial order

**instance** $u :: (cpo)\ po$
**proof**
**fix** $x :: 'a\ u$
**show** $x \sqsubseteq x$
**by** (*simp add*: *below-up-def split*: *u.split*)
**next**

**fix** $x\ y :: 'a\ u$
**assume** $x \sqsubseteq y\ y \sqsubseteq x$
**then show** $x = y$
**by** (*auto simp*: *below-up-def split*: *u.split-asm intro*: *below-antisym*)
**next**
**fix** $x\ y\ z :: 'a\ u$
**assume** $x \sqsubseteq y\ y \sqsubseteq z$
**then show** $x \sqsubseteq z$
**by** (*auto simp*: *below-up-def split*: *u.split-asm intro*: *below-trans*)
**qed**

## 12.4 Lifted cpo is a cpo

**lemma** *is-lub-Iup*: *range S <<| x ⟹ range (λi. Iup (S i)) <<| Iup x*
**by** (*auto simp*: *is-lub-def is-ub-def ball-simps below-up-def split*: *u.split*)

**lemma** *up-chain-lemma*:
**assumes** *Y*: *chain Y*
**obtains** $\forall i.\ Y\ i = Ibottom$
| *A k* **where** $\forall i.\ Iup\ (A\ i) = Y\ (i + k)$ **and** *chain A* **and** *range Y <<| Iup* $(\bigsqcup i.\ A\ i)$
**proof** (*cases* $\exists k.\ Y\ k \neq Ibottom$)
**case** *True*
**then obtain** *k* **where** *k*: $Y\ k \neq Ibottom$ **..**
**define** *A* **where** $A\ i = (THE\ a.\ Iup\ a = Y\ (i + k))$ **for** *i*
**have** *Iup-A*: $\forall i.\ Iup\ (A\ i) = Y\ (i + k)$
**proof**
**fix** *i* :: *nat*
**from** *Y le-add2* **have** $Y\ k \sqsubseteq Y\ (i + k)$ **by** (*rule chain-mono*)
**with** *k* **have** $Y\ (i + k) \neq Ibottom$ **by** (*cases Y k*) *auto*
**then show** $Iup\ (A\ i) = Y\ (i + k)$
**by** (*cases Y* $(i + k)$, *simp-all add*: *A-def*)
**qed**
**from** *Y* **have** *chain-A*: *chain A*
**by** (*simp add*: *chain-def Iup-below* [*symmetric*] *Iup-A*)
**then have** *range A <<|* $(\bigsqcup i.\ A\ i)$
**by** (*rule cpo-lubI*)
**then have** *range* $(\lambda i.\ Iup\ (A\ i))$ *<<| Iup* $(\bigsqcup i.\ A\ i)$
**by** (*rule is-lub-Iup*)
**then have** *range* $(\lambda i.\ Y\ (i + k))$ *<<| Iup* $(\bigsqcup i.\ A\ i)$
**by** (*simp only*: *Iup-A*)
**then have** *range* $(\lambda i.\ Y\ i)$ *<<| Iup* $(\bigsqcup i.\ A\ i)$
**by** (*simp only*: *is-lub-range-shift* [*OF Y*])
**with** *Iup-A chain-A* **show** *?thesis* **..**
**next**
**case** *False*
**then have** $\forall i.\ Y\ i = Ibottom$ **by** *simp*
**then show** *?thesis* **..**
**qed**

**instance** $u :: (cpo)\ cpo$
**proof**
  **fix** $S :: nat \Rightarrow 'a\ u$
  **assume** $S$: *chain* $S$
  **then show** $\exists x.\ range\ (\lambda i.\ S\ i) <<|\ x$
  **proof** (*rule up-chain-lemma*)
    **assume** $\forall i.\ S\ i = Ibottom$
    **then have** $range\ (\lambda i.\ S\ i) <<|\ Ibottom$
      **by** (*simp add*: *is-lub-const*)
    **then show** *?thesis* **..**
  **next**
    **fix** $A :: nat \Rightarrow 'a$
    **assume** $range\ S <<|\ Iup\ (\bigsqcup i.\ A\ i)$
    **then show** *?thesis* **..**
  **qed**
**qed**

## 12.5 Lifted cpo is pointed

**instance** $u :: (cpo)\ pcpo$
  **by** *intro-classes fast*

for compatibility with old HOLCF-Version

**lemma** *inst-up-pcpo*: $\bot = Ibottom$
  **by** (*rule minimal-up* [*THEN bottomI*, *symmetric*])

## 12.6 Continuity of *Iup* and *Ifup*

continuity for *Iup*

**lemma** *cont-Iup*: *cont Iup*
  **apply** (*rule contI*)
  **apply** (*rule is-lub-Iup*)
  **apply** (*erule cpo-lubI*)
  **done**

continuity for *Ifup*

**lemma** *cont-Ifup1*: *cont* $(\lambda f.\ Ifup\ f\ x)$
  **by** (*induct x*) *simp-all*

**lemma** *monofun-Ifup2*: *monofun* $(\lambda x.\ Ifup\ f\ x)$
  **apply** (*rule monofunI*)
  **apply** (*case-tac x*, *simp*)
  **apply** (*case-tac y*, *simp*)
  **apply** (*simp add*: *monofun-cfun-arg*)
  **done**

**lemma** *cont-Ifup2*: *cont* $(\lambda x.\ Ifup\ f\ x)$

**proof** (*rule contI2*)
 **fix** *Y*
 **assume** *Y*: *chain Y* **and** *Y'*: *chain* (λ*i*. *Ifup f* (*Y i*))
 **from** *Y* **show** *Ifup f* (⨆ *i*. *Y i*) ⊑ (⨆ *i*. *Ifup f* (*Y i*))
 **proof** (*rule up-chain-lemma*)
  **fix** *A* **and** *k*
  **assume** *A*: ∀ *i*. *Iup* (*A i*) = *Y* (*i* + *k*)
  **assume** *chain A* **and** *range Y* <<| *Iup* (⨆ *i*. *A i*)
  **then have** *Ifup f* (⨆ *i*. *Y i*) = (⨆ *i*. *Ifup f* (*Iup* (*A i*)))
   **by** (*simp add*: *lub-eqI contlub-cfun-arg*)
  **also have** ... = (⨆ *i*. *Ifup f* (*Y* (*i* + *k*)))
   **by** (*simp add*: *A*)
  **also have** ... = (⨆ *i*. *Ifup f* (*Y i*))
   **using** *Y'* **by** (*rule lub-range-shift*)
  **finally show** *?thesis* **by** *simp*
 **qed** *simp*
**qed** (*rule monofun-Ifup2*)

## 12.7 Continuous versions of constants

**definition** *up* :: *'a* → *'a u*
 **where** *up* = (Λ *x*. *Iup x*)

**definition** *fup* :: (*'a* → *'b*::*pcpo*) → *'a u* → *'b*
 **where** *fup* = (Λ *f p*. *Ifup f p*)

**translations**
 *case l of XCONST up*·*x* ⇒ *t* ⇌ *CONST fup*·(Λ *x*. *t*)·*l*
 *case l of* (*XCONST up* :: *'a*)·*x* ⇒ *t* ⇀ *CONST fup*·(Λ *x*. *t*)·*l*
 Λ(*XCONST up*·*x*). *t* ⇌ *CONST fup*·(Λ *x*. *t*)

continuous versions of lemmas for $'a_\perp$

**lemma** *Exh-Up*: *z* = ⊥ ∨ (∃ *x*. *z* = *up*·*x*)
 **by** (*induct z*) (*simp add*: *inst-up-pcpo*, *simp add*: *up-def cont-Iup*)

**lemma** *up-eq* [*simp*]: (*up*·*x* = *up*·*y*) = (*x* = *y*)
 **by** (*simp add*: *up-def cont-Iup*)

**lemma** *up-inject*: *up*·*x* = *up*·*y* ⟹ *x* = *y*
 **by** *simp*

**lemma** *up-defined* [*simp*]: *up*·*x* ≠ ⊥
 **by** (*simp add*: *up-def cont-Iup inst-up-pcpo*)

**lemma** *not-up-less-UU*: *up*·*x* ⋢ ⊥
 **by** *simp*

**lemma** *up-below* [*simp*]: *up*·*x* ⊑ *up*·*y* ⟷ *x* ⊑ *y*
 **by** (*simp add*: *up-def cont-Iup*)

**lemma** *upE* [*case-names bottom up*, *cases type*: *u*]: ⟦*p* = ⊥ ⟹ *Q*; ⋀*x*. *p* = *up*·*x* ⟹ *Q*⟧ ⟹ *Q*
**by** (*cases p*) (*simp add*: *inst-up-pcpo*, *simp add*: *up-def cont-Iup*)

**lemma** *up-induct* [*case-names bottom up*, *induct type*: *u*]: *P* ⊥ ⟹ (⋀*x*. *P* (*up*·*x*)) ⟹ *P x*
**by** (*cases x*) *simp-all*

lifting preserves chain-finiteness

**lemma** *up-chain-cases*:
**assumes** *Y*: *chain Y*
**obtains** ∀ *i*. *Y i* = ⊥
| *A k* **where** ∀ *i*. *up*·(*A i*) = *Y* (*i* + *k*) **and** *chain A* **and** (⨆ *i*. *Y i*) = *up*·(⨆ *i*. *A i*)
**by** (*rule up-chain-lemma* [*OF Y*]) (*simp-all add*: *inst-up-pcpo up-def cont-Iup lub-eqI*)

**lemma** *compact-up*: *compact x* ⟹ *compact* (*up*·*x*)
**apply** (*rule compactI2*)
**apply** (*erule up-chain-cases*)
**apply** *simp*
**apply** (*drule* (*1*) *compactD2*, *simp*)
**apply** (*erule exE*)
**apply** (*drule-tac f*=*up* **and** *x*=*x* **in** *monofun-cfun-arg*)
**apply** (*simp*, *erule exI*)
**done**

**lemma** *compact-upD*: *compact* (*up*·*x*) ⟹ *compact x*
**unfolding** *compact-def*
**by** (*drule adm-subst* [*OF cont-Rep-cfun2* [**where** *f*=*up*]], *simp*)

**lemma** *compact-up-iff* [*simp*]: *compact* (*up*·*x*) = *compact x*
**by** (*safe elim*!: *compact-up compact-upD*)

**instance** *u* :: (*chfin*) *chfin*
**apply** *intro-classes*
**apply** (*erule compact-imp-max-in-chain*)
**apply** (*rule-tac p*=⨆ *i*. *Y i* **in** *upE*, *simp-all*)
**done**

properties of fup

**lemma** *fup1* [*simp*]: *fup*·*f*·⊥ = ⊥
**by** (*simp add*: *fup-def cont-Ifup1 cont-Ifup2 inst-up-pcpo cont2cont-LAM*)

**lemma** *fup2* [*simp*]: *fup*·*f*·(*up*·*x*) = *f*·*x*
**by** (*simp add*: *up-def fup-def cont-Iup cont-Ifup1 cont-Ifup2 cont2cont-LAM*)

**lemma** *fup3* [*simp*]: *fup*·*up*·*x* = *x*

**by** (*cases x*) *simp-all*

**end**

# 13 Lifting types of class type to flat pcpo's

**theory** *Lift*
**imports** *Discrete Up*
**begin**

**default-sort** *type*

**pcpodef** *'a lift* = *UNIV* :: *'a discr u set*
**by** *simp-all*

**lemmas** *inst-lift-pcpo* = *Abs-lift-strict* [*symmetric*]

**definition**
*Def* :: *'a* ⇒ *'a lift* **where**
*Def x* = *Abs-lift* (*up*·(*Discr x*))

## 13.1 Lift as a datatype

**lemma** *lift-induct*: ⟦*P* ⊥; ⋀*x*. *P* (*Def x*)⟧ ⟹ *P y*
**apply** (*induct y*)
**apply** (*rule-tac p*=*y* **in** *upE*)
**apply** (*simp add*: *Abs-lift-strict*)
**apply** (*case-tac x*)
**apply** (*simp add*: *Def-def*)
**done**

**old-rep-datatype** ⊥::*'a lift Def*
**by** (*erule lift-induct*) (*simp-all add*: *Def-def Abs-lift-inject inst-lift-pcpo*)

⊥ and *Def*

**lemma** *not-Undef-is-Def*: (*x* ≠ ⊥) = (∃ *y*. *x* = *Def y*)
**by** (*cases x*) *simp-all*

**lemma** *lift-definedE*: ⟦*x* ≠ ⊥; ⋀*a*. *x* = *Def a* ⟹ *R*⟧ ⟹ *R*
**by** (*cases x*) *simp-all*

For *x* ≠ ⊥ in assumptions *defined* replaces *x* by *Def a* in conclusion.

**method-setup** *defined* = ‹
*Scan.succeed* (*fn ctxt* => *SIMPLE-METHOD'*
(*eresolve-tac ctxt* @{*thms lift-definedE*} *THEN' asm-simp-tac ctxt*))
›

**lemma** *DefE*: *Def x* = ⊥ ⟹ *R*

**by** *simp*

**lemma** *DefE2*: $[\![x = Def\ s;\ x = \bot]\!] \Longrightarrow R$
**by** *simp*

**lemma** *Def-below-Def*: $Def\ x \sqsubseteq Def\ y \longleftrightarrow x = y$
**by** (*simp add*: *below-lift-def Def-def Abs-lift-inverse*)

**lemma** *Def-below-iff* [*simp*]: $Def\ x \sqsubseteq y \longleftrightarrow Def\ x = y$
**by** (*induct y*, *simp*, *simp add*: *Def-below-Def*)

## 13.2 Lift is flat

**instance** *lift* :: (*type*) *flat*
**proof**
**fix** *x y* :: *'a lift*
**assume** $x \sqsubseteq y$ **thus** $x = \bot \vee x = y$
**by** (*induct x*) *auto*
**qed**

## 13.3 Continuity of *case-lift*

**lemma** *case-lift-eq*: $case\text{-}lift\ \bot\ f\ x = fup{\cdot}(\Lambda\ y.\ f\ (undiscr\ y)){\cdot}(Rep\text{-}lift\ x)$
**apply** (*induct x*, *unfold lift.case*)
**apply** (*simp add*: *Rep-lift-strict*)
**apply** (*simp add*: *Def-def Abs-lift-inverse*)
**done**

**lemma** *cont2cont-case-lift* [*simp*]:
$[\![\bigwedge y.\ cont\ (\lambda x.\ f\ x\ y);\ cont\ g]\!] \Longrightarrow cont\ (\lambda x.\ case\text{-}lift\ \bot\ (f\ x)\ (g\ x))$
**unfolding** *case-lift-eq* **by** (*simp add*: *cont-Rep-lift*)

## 13.4 Further operations

**definition**
$flift1 :: ('a \Rightarrow 'b{::}pcpo) \Rightarrow ('a\ lift \to 'b)$ (**binder** *FLIFT* 10) **where**
$flift1 = (\lambda f.\ (\Lambda\ x.\ case\text{-}lift\ \bot\ f\ x))$

**translations**
$\Lambda(XCONST\ Def\ x).\ t => CONST\ flift1\ (\lambda x.\ t)$
$\Lambda(CONST\ Def\ x).\ FLIFT\ y.\ t <= FLIFT\ x\ y.\ t$
$\Lambda(CONST\ Def\ x).\ t <= FLIFT\ x.\ t$

**definition**
$flift2 :: ('a \Rightarrow 'b) \Rightarrow ('a\ lift \to 'b\ lift)$ **where**
$flift2\ f = (FLIFT\ x.\ Def\ (f\ x))$

**lemma** *flift1-Def* [*simp*]: $flift1\ f{\cdot}(Def\ x) = (f\ x)$
**by** (*simp add*: *flift1-def*)

**lemma** *flift2-Def* [*simp*]: *flift2* $f{\cdot}(Def\ x) = Def\ (f\ x)$
**by** (*simp add*: *flift2-def*)

**lemma** *flift1-strict* [*simp*]: *flift1* $f{\cdot}\bot = \bot$
**by** (*simp add*: *flift1-def*)

**lemma** *flift2-strict* [*simp*]: *flift2* $f{\cdot}\bot = \bot$
**by** (*simp add*: *flift2-def*)

**lemma** *flift2-defined* [*simp*]: $x \neq \bot \Longrightarrow (\mathit{flift2}\ f){\cdot}x \neq \bot$
**by** (*erule lift-definedE*, *simp*)

**lemma** *flift2-bottom-iff* [*simp*]: $(\mathit{flift2}\ f{\cdot}x = \bot) = (x = \bot)$
**by** (*cases x*, *simp-all*)

**lemma** *FLIFT-mono*:
$(\bigwedge x.\ f\ x \sqsubseteq g\ x) \Longrightarrow (FLIFT\ x.\ f\ x) \sqsubseteq (FLIFT\ x.\ g\ x)$
**by** (*rule cfun-belowI*, *case-tac x*, *simp-all*)

**lemma** *cont2cont-flift1* [*simp*, *cont2cont*]:
$[\![\bigwedge y.\ cont\ (\lambda x.\ f\ x\ y)]\!] \Longrightarrow cont\ (\lambda x.\ FLIFT\ y.\ f\ x\ y)$
**by** (*simp add*: *flift1-def cont2cont-LAM*)

**end**

# 14 The type of lifted booleans

**theory** *Tr*
**imports** *Lift*
**begin**

## 14.1 Type definition and constructors

**type-synonym** *tr* = *bool lift*

**translations**
(*type*) *tr* $\leftharpoonup$ (*type*) *bool lift*

**definition** *TT* :: *tr*
**where** *TT* = *Def True*

**definition** *FF* :: *tr*
**where** *FF* = *Def False*

Exhaustion and Elimination for type *tr*

**lemma** *Exh-tr*: $t = \bot \vee t = TT \vee t = FF$
**by** (*induct t*) (*auto simp*: *FF-def TT-def*)

**lemma** *trE* [*case-names bottom TT FF*, *cases type*: *tr*]:

$[\![p = \bot \Longrightarrow Q;\ p = TT \Longrightarrow Q;\ p = FF \Longrightarrow Q]\!] \Longrightarrow Q$
**by** (*induct p*) (*auto simp*: *FF-def TT-def*)

**lemma** *tr-induct* [*case-names bottom TT FF*, *induct type*: *tr*]:
$P\ \bot \Longrightarrow P\ TT \Longrightarrow P\ FF \Longrightarrow P\ x$
**by** (*cases x*) *simp-all*

distinctness for type *tr*

**lemma** *dist-below-tr* [*simp*]:
$TT \not\sqsubseteq \bot\ FF \not\sqsubseteq \bot\ TT \not\sqsubseteq FF\ FF \not\sqsubseteq TT$
**by** (*simp-all add*: *TT-def FF-def*)

**lemma** *dist-eq-tr* [*simp*]: $TT \neq \bot\ FF \neq \bot\ TT \neq FF \perp \neq TT \perp \neq FF\ FF \neq TT$
**by** (*simp-all add*: *TT-def FF-def*)

**lemma** *TT-below-iff* [*simp*]: $TT \sqsubseteq x \longleftrightarrow x = TT$
**by** (*induct x*) *simp-all*

**lemma** *FF-below-iff* [*simp*]: $FF \sqsubseteq x \longleftrightarrow x = FF$
**by** (*induct x*) *simp-all*

**lemma** *not-below-TT-iff* [*simp*]: $x \not\sqsubseteq TT \longleftrightarrow x = FF$
**by** (*induct x*) *simp-all*

**lemma** *not-below-FF-iff* [*simp*]: $x \not\sqsubseteq FF \longleftrightarrow x = TT$
**by** (*induct x*) *simp-all*

## 14.2 Case analysis

**default-sort** *pcpo*

**definition** *tr-case* :: $'a \to 'a \to tr \to 'a$
**where** *tr-case* = $(\Lambda\ t\ e\ (Def\ b).$ *if b then t else e*)

**abbreviation** *cifte-syn* :: $[tr, 'c, 'c] \Rightarrow 'c$ ((*If* (-)/ *then* (-)/ *else* (-)) [*0*, *0*, *60*] *60*)
**where** *If b then e1 else e2* $\equiv$ *tr-case·e1·e2·b*

**translations**
$\Lambda\ (XCONST\ TT).\ t \rightleftharpoons CONST\ tr\text{-}case{\cdot}t{\cdot}\bot$
$\Lambda\ (XCONST\ FF).\ t \rightleftharpoons CONST\ tr\text{-}case{\cdot}\bot{\cdot}t$

**lemma** *ifte-thms* [*simp*]:
*If* $\bot$ *then e1 else e2* $= \bot$
*If FF then e1 else e2* $= e2$
*If TT then e1 else e2* $= e1$
**by** (*simp-all add*: *tr-case-def TT-def FF-def*)

## 14.3 Boolean connectives

**definition** *trand* :: $tr \to tr \to tr$
  **where** *andalso-def*: $trand = (\Lambda\ x\ y.\ If\ x\ then\ y\ else\ FF)$

**abbreviation** *andalso-syn* :: $tr \Rightarrow tr \Rightarrow tr$ (- *andalso* - [*36*,*35*] *35*)
  **where** $x\ andalso\ y \equiv trand{\cdot}x{\cdot}y$

**definition** *tror* :: $tr \to tr \to tr$
  **where** *orelse-def*: $tror = (\Lambda\ x\ y.\ If\ x\ then\ TT\ else\ y)$

**abbreviation** *orelse-syn* :: $tr \Rightarrow tr \Rightarrow tr$ (- *orelse* - [*31*,*30*] *30*)
  **where** $x\ orelse\ y \equiv tror{\cdot}x{\cdot}y$

**definition** *neg* :: $tr \to tr$
  **where** $neg = flift2\ Not$

**definition** *If2* :: $tr \Rightarrow 'c \Rightarrow 'c \Rightarrow 'c$
  **where** $If2\ Q\ x\ y = (If\ Q\ then\ x\ else\ y)$

tactic for tr-thms with case split

**lemmas** *tr-defs* = *andalso-def orelse-def neg-def tr-case-def TT-def FF-def*

lemmas about andalso, orelse, neg and if

**lemma** *andalso-thms* [*simp*]:
  $(TT\ andalso\ y) = y$
  $(FF\ andalso\ y) = FF$
  $(\bot\ andalso\ y) = \bot$
  $(y\ andalso\ TT) = y$
  $(y\ andalso\ y) = y$
    **apply** (*unfold andalso-def*, *simp-all*)
   **apply** (*cases y*, *simp-all*)
  **apply** (*cases y*, *simp-all*)
  **done**

**lemma** *orelse-thms* [*simp*]:
  $(TT\ orelse\ y) = TT$
  $(FF\ orelse\ y) = y$
  $(\bot\ orelse\ y) = \bot$
  $(y\ orelse\ FF) = y$
  $(y\ orelse\ y) = y$
    **apply** (*unfold orelse-def*, *simp-all*)
   **apply** (*cases y*, *simp-all*)
  **apply** (*cases y*, *simp-all*)
  **done**

**lemma** *neg-thms* [*simp*]:
  $neg{\cdot}TT = FF$
  $neg{\cdot}FF = TT$
  $neg{\cdot}\bot = \bot$

**by** (*simp-all add*: *neg-def TT-def FF-def*)

split-tac for If via If2 because the constant has to be a constant

**lemma** *split-If2*: $P\ (\mathit{If2}\ Q\ x\ y) \longleftrightarrow ((Q = \bot \longrightarrow P\ \bot) \wedge (Q = \mathit{TT} \longrightarrow P\ x) \wedge (Q = \mathit{FF} \longrightarrow P\ y))$
**by** (*cases Q*) (*simp-all add*: *If2-def*)

**ML** ‹
*fun split-If-tac ctxt =*
*simp-tac* (*put-simpset HOL-basic-ss ctxt addsimps* [@{*thm If2-def*} *RS sym*])
*THEN'* (*split-tac ctxt* [@{*thm split-If2*}])
›

## 14.4 Rewriting of HOLCF operations to HOL functions

**lemma** *andalso-or*: $t \neq \bot \Longrightarrow (t\ \mathit{andalso}\ s) = \mathit{FF} \longleftrightarrow t = \mathit{FF} \vee s = \mathit{FF}$
**by** (*cases t*) *simp-all*

**lemma** *andalso-and*: $t \neq \bot \Longrightarrow ((t\ \mathit{andalso}\ s) \neq \mathit{FF}) \longleftrightarrow t \neq \mathit{FF} \wedge s \neq \mathit{FF}$
**by** (*cases t*) *simp-all*

**lemma** *Def-bool1* [*simp*]: $\mathit{Def}\ x \neq \mathit{FF} \longleftrightarrow x$
**by** (*simp add*: *FF-def*)

**lemma** *Def-bool2* [*simp*]: $\mathit{Def}\ x = \mathit{FF} \longleftrightarrow \neg\ x$
**by** (*simp add*: *FF-def*)

**lemma** *Def-bool3* [*simp*]: $\mathit{Def}\ x = \mathit{TT} \longleftrightarrow x$
**by** (*simp add*: *TT-def*)

**lemma** *Def-bool4* [*simp*]: $\mathit{Def}\ x \neq \mathit{TT} \longleftrightarrow \neg\ x$
**by** (*simp add*: *TT-def*)

**lemma** *If-and-if*: $(\mathit{If}\ \mathit{Def}\ P\ \mathit{then}\ A\ \mathit{else}\ B) = (\mathit{if}\ P\ \mathit{then}\ A\ \mathit{else}\ B)$
**by** (*cases Def P*) (*auto simp add*: *TT-def*[*symmetric*] *FF-def*[*symmetric*])

## 14.5 Compactness

**lemma** *compact-TT*: *compact TT*
**by** (*rule compact-chfin*)

**lemma** *compact-FF*: *compact FF*
**by** (*rule compact-chfin*)

**end**

# 15 The type of strict sums

**theory** *Ssum*
**imports** *Tr*
**begin**

**default-sort** *pcpo*

## 15.1 Definition of strict sum type

**definition** *ssum* =
{$p$ :: $tr \times ('a \times 'b)$. $p = \bot \vee$
($fst\ p = TT \wedge fst\ (snd\ p) \neq \bot \wedge snd\ (snd\ p) = \bot$) $\vee$
($fst\ p = FF \wedge fst\ (snd\ p) = \bot \wedge snd\ (snd\ p) \neq \bot$)}

**pcpodef** ('$a$, '$b$) *ssum* ((- $\oplus$/ -) [*21*, *20*] *20*) = *ssum* :: ($tr \times 'a \times 'b$) *set*
**by** (*simp-all add*: *ssum-def*)

**instance** *ssum* :: ({*chfin*,*pcpo*}, {*chfin*,*pcpo*}) *chfin*
**by** (*rule typedef-chfin* [*OF type-definition-ssum below-ssum-def*])

**type-notation** (*ASCII*)
*ssum* (**infixr** ++ *10*)

## 15.2 Definitions of constructors

**definition** *sinl* :: $'a \to ('a ++ 'b)$
**where** *sinl* = ($\Lambda$ *a*. *Abs-ssum* (*seq*·*a*·*TT*, *a*, $\bot$))

**definition** *sinr* :: $'b \to ('a ++ 'b)$
**where** *sinr* = ($\Lambda$ *b*. *Abs-ssum* (*seq*·*b*·*FF*, $\bot$, *b*))

**lemma** *sinl-ssum*: (*seq*·*a*·*TT*, *a*, $\bot$) $\in$ *ssum*
**by** (*simp add*: *ssum-def seq-conv-if*)

**lemma** *sinr-ssum*: (*seq*·*b*·*FF*, $\bot$, *b*) $\in$ *ssum*
**by** (*simp add*: *ssum-def seq-conv-if*)

**lemma** *Rep-ssum-sinl*: *Rep-ssum* (*sinl*·*a*) = (*seq*·*a*·*TT*, *a*, $\bot$)
**by** (*simp add*: *sinl-def cont-Abs-ssum Abs-ssum-inverse sinl-ssum*)

**lemma** *Rep-ssum-sinr*: *Rep-ssum* (*sinr*·*b*) = (*seq*·*b*·*FF*, $\bot$, *b*)
**by** (*simp add*: *sinr-def cont-Abs-ssum Abs-ssum-inverse sinr-ssum*)

**lemmas** *Rep-ssum-simps* =
*Rep-ssum-inject* [*symmetric*] *below-ssum-def*
*prod-eq-iff below-prod-def*
*Rep-ssum-strict Rep-ssum-sinl Rep-ssum-sinr*

## 15.3 Properties of *sinl* and *sinr*

Ordering

**lemma** *sinl-below* [*simp*]: $sinl{\cdot}x \sqsubseteq sinl{\cdot}y \longleftrightarrow x \sqsubseteq y$
**by** (*simp add*: *Rep-ssum-simps seq-conv-if*)

**lemma** *sinr-below* [*simp*]: $sinr{\cdot}x \sqsubseteq sinr{\cdot}y \longleftrightarrow x \sqsubseteq y$
**by** (*simp add*: *Rep-ssum-simps seq-conv-if*)

**lemma** *sinl-below-sinr* [*simp*]: $sinl{\cdot}x \sqsubseteq sinr{\cdot}y \longleftrightarrow x = \bot$
**by** (*simp add*: *Rep-ssum-simps seq-conv-if*)

**lemma** *sinr-below-sinl* [*simp*]: $sinr{\cdot}x \sqsubseteq sinl{\cdot}y \longleftrightarrow x = \bot$
**by** (*simp add*: *Rep-ssum-simps seq-conv-if*)

Equality

**lemma** *sinl-eq* [*simp*]: $sinl{\cdot}x = sinl{\cdot}y \longleftrightarrow x = y$
**by** (*simp add*: *po-eq-conv*)

**lemma** *sinr-eq* [*simp*]: $sinr{\cdot}x = sinr{\cdot}y \longleftrightarrow x = y$
**by** (*simp add*: *po-eq-conv*)

**lemma** *sinl-eq-sinr* [*simp*]: $sinl{\cdot}x = sinr{\cdot}y \longleftrightarrow x = \bot \wedge y = \bot$
**by** (*subst po-eq-conv*) *simp*

**lemma** *sinr-eq-sinl* [*simp*]: $sinr{\cdot}x = sinl{\cdot}y \longleftrightarrow x = \bot \wedge y = \bot$
**by** (*subst po-eq-conv*) *simp*

**lemma** *sinl-inject*: $sinl{\cdot}x = sinl{\cdot}y \Longrightarrow x = y$
**by** (*rule sinl-eq* [*THEN iffD1*])

**lemma** *sinr-inject*: $sinr{\cdot}x = sinr{\cdot}y \Longrightarrow x = y$
**by** (*rule sinr-eq* [*THEN iffD1*])

Strictness

**lemma** *sinl-strict* [*simp*]: $sinl{\cdot}\bot = \bot$
**by** (*simp add*: *Rep-ssum-simps*)

**lemma** *sinr-strict* [*simp*]: $sinr{\cdot}\bot = \bot$
**by** (*simp add*: *Rep-ssum-simps*)

**lemma** *sinl-bottom-iff* [*simp*]: $sinl{\cdot}x = \bot \longleftrightarrow x = \bot$
**using** *sinl-eq* [*of x* $\bot$] **by** *simp*

**lemma** *sinr-bottom-iff* [*simp*]: $sinr{\cdot}x = \bot \longleftrightarrow x = \bot$
**using** *sinr-eq* [*of x* $\bot$] **by** *simp*

**lemma** *sinl-defined*: $x \neq \bot \Longrightarrow sinl{\cdot}x \neq \bot$
**by** *simp*

**lemma** *sinr-defined*: $x \neq \bot \Longrightarrow sinr{\cdot}x \neq \bot$
**by** *simp*

Compactness

**lemma** *compact-sinl*: *compact* $x \Longrightarrow$ *compact* $(sinl{\cdot}x)$
**by** (*rule compact-ssum*) (*simp add*: *Rep-ssum-sinl*)

**lemma** *compact-sinr*: *compact* $x \Longrightarrow$ *compact* $(sinr{\cdot}x)$
**by** (*rule compact-ssum*) (*simp add*: *Rep-ssum-sinr*)

**lemma** *compact-sinlD*: *compact* $(sinl{\cdot}x) \Longrightarrow$ *compact* $x$
**unfolding** *compact-def*
**by** (*drule adm-subst* [*OF cont-Rep-cfun2* [**where** *f*=*sinl*]], *simp*)

**lemma** *compact-sinrD*: *compact* $(sinr{\cdot}x) \Longrightarrow$ *compact* $x$
**unfolding** *compact-def*
**by** (*drule adm-subst* [*OF cont-Rep-cfun2* [**where** *f*=*sinr*]], *simp*)

**lemma** *compact-sinl-iff* [*simp*]: *compact* $(sinl{\cdot}x) =$ *compact* $x$
**by** (*safe elim*!: *compact-sinl compact-sinlD*)

**lemma** *compact-sinr-iff* [*simp*]: *compact* $(sinr{\cdot}x) =$ *compact* $x$
**by** (*safe elim*!: *compact-sinr compact-sinrD*)

## 15.4 Case analysis

**lemma** *ssumE* [*case-names bottom sinl sinr*, *cases type*: *ssum*]:
**obtains** $p = \bot$
| $x$ **where** $p = sinl{\cdot}x$ **and** $x \neq \bot$
| $y$ **where** $p = sinr{\cdot}y$ **and** $y \neq \bot$
**using** *Rep-ssum* [*of p*] **by** (*auto simp add*: *ssum-def Rep-ssum-simps*)

**lemma** *ssum-induct* [*case-names bottom sinl sinr*, *induct type*: *ssum*]:
$[\![P\ \bot;$
$\bigwedge x.\ x \neq \bot \Longrightarrow P\ (sinl{\cdot}x);$
$\bigwedge y.\ y \neq \bot \Longrightarrow P\ (sinr{\cdot}y)]\!] \Longrightarrow P\ x$
**by** (*cases x*) *simp-all*

**lemma** *ssumE2* [*case-names sinl sinr*]:
$[\![\bigwedge x.\ p = sinl{\cdot}x \Longrightarrow Q;\ \bigwedge y.\ p = sinr{\cdot}y \Longrightarrow Q]\!] \Longrightarrow Q$
**by** (*cases p*, *simp only*: *sinl-strict* [*symmetric*], *simp*, *simp*)

**lemma** *below-sinlD*: $p \sqsubseteq sinl{\cdot}x \Longrightarrow \exists\, y.\ p = sinl{\cdot}y \wedge y \sqsubseteq x$
**by** (*cases p*, *rule-tac* $x{=}\bot$ **in** *exI*, *simp-all*)

**lemma** *below-sinrD*: $p \sqsubseteq sinr{\cdot}x \Longrightarrow \exists\, y.\ p = sinr{\cdot}y \wedge y \sqsubseteq x$
**by** (*cases p*, *rule-tac* $x{=}\bot$ **in** *exI*, *simp-all*)

### 15.5 Case analysis combinator

**definition** *sscase* :: $('a \to 'c) \to ('b \to 'c) \to ('a \mathbin{++} 'b) \to 'c$
**where** *sscase* = $(\Lambda\ f\ g\ s.\ (\lambda(t,\ x,\ y).\ \mathit{If}\ t\ \mathit{then}\ f{\cdot}x\ \mathit{else}\ g{\cdot}y)\ (\mathit{Rep\text{-}ssum}\ s))$

**translations**
*case s of XCONST sinl·x ⇒ t1 | XCONST sinr·y ⇒ t2* ⇌ *CONST sscase·(Λ x. t1)·(Λ y. t2)·s*
*case s of (XCONST sinl :: 'a)·x ⇒ t1 | XCONST sinr·y ⇒ t2* ↽ *CONST sscase·(Λ x. t1)·(Λ y. t2)·s*

**translations**
$\Lambda(\mathit{XCONST\ sinl}{\cdot}x).\ t \rightleftharpoons \mathit{CONST\ sscase}{\cdot}(\Lambda\ x.\ t){\cdot}\bot$
$\Lambda(\mathit{XCONST\ sinr}{\cdot}y).\ t \rightleftharpoons \mathit{CONST\ sscase}{\cdot}\bot{\cdot}(\Lambda\ y.\ t)$

**lemma** *beta-sscase*: $\mathit{sscase}{\cdot}f{\cdot}g{\cdot}s = (\lambda(t,\ x,\ y).\ \mathit{If}\ t\ \mathit{then}\ f{\cdot}x\ \mathit{else}\ g{\cdot}y)\ (\mathit{Rep\text{-}ssum}\ s)$
**by** (*simp add*: *sscase-def cont-Rep-ssum*)

**lemma** *sscase1* [*simp*]: $\mathit{sscase}{\cdot}f{\cdot}g{\cdot}\bot = \bot$
**by** (*simp add*: *beta-sscase Rep-ssum-strict*)

**lemma** *sscase2* [*simp*]: $x \neq \bot \Longrightarrow \mathit{sscase}{\cdot}f{\cdot}g{\cdot}(\mathit{sinl}{\cdot}x) = f{\cdot}x$
**by** (*simp add*: *beta-sscase Rep-ssum-sinl*)

**lemma** *sscase3* [*simp*]: $y \neq \bot \Longrightarrow \mathit{sscase}{\cdot}f{\cdot}g{\cdot}(\mathit{sinr}{\cdot}y) = g{\cdot}y$
**by** (*simp add*: *beta-sscase Rep-ssum-sinr*)

**lemma** *sscase4* [*simp*]: $\mathit{sscase}{\cdot}\mathit{sinl}{\cdot}\mathit{sinr}{\cdot}z = z$
**by** (*cases z*) *simp-all*

### 15.6 Strict sum preserves flatness

**instance** *ssum* :: (*flat*, *flat*) *flat*
**apply** (*intro-classes*, *clarify*)
**apply** (*case-tac x*, *simp*)
**apply** (*case-tac y*, *simp-all add*: *flat-below-iff*)
**apply** (*case-tac y*, *simp-all add*: *flat-below-iff*)
**done**

**end**

## 16 The Strict Function Type

**theory** *Sfun*
**imports** *Cfun*
**begin**

**pcpodef** ('*a*, '*b*) *sfun* (**infixr** →! *0*) = $\{f :: 'a \to 'b.\ f{\cdot}\bot = \bot\}$
**by** *simp-all*

**type-notation** (*ASCII*)
  *sfun* (**infixr** *−>!* *0*)

TODO: Define nice syntax for abstraction, application.

**definition** *sfun-abs* :: $('a \to 'b) \to ('a \to\!! 'b)$
  **where** *sfun-abs* = $(\Lambda\ f.$ *Abs-sfun* (*strictify*·*f*))

**definition** *sfun-rep* :: $('a \to\!! 'b) \to 'a \to 'b$
  **where** *sfun-rep* = $(\Lambda\ f.$ *Rep-sfun f*)

**lemma** *sfun-rep-beta*: *sfun-rep*·*f* = *Rep-sfun f*
  **by** (*simp add*: *sfun-rep-def cont-Rep-sfun*)

**lemma** *sfun-rep-strict1* [*simp*]: *sfun-rep*·$\bot$ = $\bot$
  **unfolding** *sfun-rep-beta* **by** (*rule Rep-sfun-strict*)

**lemma** *sfun-rep-strict2* [*simp*]: *sfun-rep*·*f*·$\bot$ = $\bot$
  **unfolding** *sfun-rep-beta* **by** (*rule Rep-sfun* [*simplified*])

**lemma** *strictify-cancel*: *f*·$\bot$ = $\bot \Longrightarrow$ *strictify*·*f* = *f*
  **by** (*simp add*: *cfun-eq-iff strictify-conv-if*)

**lemma** *sfun-abs-sfun-rep* [*simp*]: *sfun-abs*·(*sfun-rep*·*f*) = *f*
  **unfolding** *sfun-abs-def sfun-rep-def*
  **apply** (*simp add*: *cont-Abs-sfun cont-Rep-sfun*)
  **apply** (*simp add*: *Rep-sfun-inject* [*symmetric*] *Abs-sfun-inverse*)
  **apply** (*simp add*: *cfun-eq-iff strictify-conv-if*)
  **apply** (*simp add*: *Rep-sfun* [*simplified*])
  **done**

**lemma** *sfun-rep-sfun-abs* [*simp*]: *sfun-rep*·(*sfun-abs*·*f*) = *strictify*·*f*
  **unfolding** *sfun-abs-def sfun-rep-def*
  **apply** (*simp add*: *cont-Abs-sfun cont-Rep-sfun*)
  **apply** (*simp add*: *Abs-sfun-inverse*)
  **done**

**lemma** *sfun-eq-iff*: *f* = *g* $\longleftrightarrow$ *sfun-rep*·*f* = *sfun-rep*·*g*
  **by** (*simp add*: *sfun-rep-def cont-Rep-sfun Rep-sfun-inject*)

**lemma** *sfun-below-iff*: *f* $\sqsubseteq$ *g* $\longleftrightarrow$ *sfun-rep*·*f* $\sqsubseteq$ *sfun-rep*·*g*
  **by** (*simp add*: *sfun-rep-def cont-Rep-sfun below-sfun-def*)

**end**

# 17 Map functions for various types

**theory** *Map-Functions*
  **imports** *Deflation Sprod Ssum Sfun Up*

**begin**

## 17.1 Map operator for continuous function space

**default-sort** *cpo*

**definition** *cfun-map* :: $('b \to 'a) \to ('c \to 'd) \to ('a \to 'c) \to ('b \to 'd)$
  **where** *cfun-map* = $(\Lambda\ a\ b\ f\ x.\ b{\cdot}(f{\cdot}(a{\cdot}x)))$

**lemma** *cfun-map-beta* [*simp*]: *cfun-map*·*a*·*b*·*f*·*x* = *b*·(*f*·(*a*·*x*))
  **by** (*simp add*: *cfun-map-def*)

**lemma** *cfun-map-ID*: *cfun-map*·*ID*·*ID* = *ID*
  **by** (*simp add*: *cfun-eq-iff*)

**lemma** *cfun-map-map*: *cfun-map*·*f1*·*g1*·(*cfun-map*·*f2*·*g2*·*p*) = *cfun-map*·(Λ *x*. *f2*·(*f1*·*x*))·(Λ *x*. *g1*·(*g2*·*x*))·*p*
  **by** (*rule cfun-eqI*) *simp*

**lemma** *ep-pair-cfun-map*:
  **assumes** *ep-pair e1 p1* **and** *ep-pair e2 p2*
  **shows** *ep-pair* (*cfun-map*·*p1*·*e2*) (*cfun-map*·*e1*·*p2*)
**proof**
  **interpret** *e1p1*: *ep-pair e1 p1* **by** *fact*
  **interpret** *e2p2*: *ep-pair e2 p2* **by** *fact*
  **show** *cfun-map*·*e1*·*p2*·(*cfun-map*·*p1*·*e2*·*f*) = *f* **for** *f*
    **by** (*simp add*: *cfun-eq-iff*)
  **show** *cfun-map*·*p1*·*e2*·(*cfun-map*·*e1*·*p2*·*g*) ⊑ *g* **for** *g*
    **apply** (*rule cfun-belowI*, *simp*)
    **apply** (*rule below-trans* [*OF e2p2.e-p-below*])
    **apply** (*rule monofun-cfun-arg*)
    **apply** (*rule e1p1.e-p-below*)
    **done**
**qed**

**lemma** *deflation-cfun-map*:
  **assumes** *deflation d1* **and** *deflation d2*
  **shows** *deflation* (*cfun-map*·*d1*·*d2*)
**proof**
  **interpret** *d1*: *deflation d1* **by** *fact*
  **interpret** *d2*: *deflation d2* **by** *fact*
  **fix** *f*
  **show** *cfun-map*·*d1*·*d2*·(*cfun-map*·*d1*·*d2*·*f*) = *cfun-map*·*d1*·*d2*·*f*
    **by** (*simp add*: *cfun-eq-iff d1.idem d2.idem*)
  **show** *cfun-map*·*d1*·*d2*·*f* ⊑ *f*
    **apply** (*rule cfun-belowI*, *simp*)
    **apply** (*rule below-trans* [*OF d2.below*])
    **apply** (*rule monofun-cfun-arg*)
    **apply** (*rule d1.below*)

**done**
**qed**

**lemma** *finite-range-cfun-map*:
**assumes** *a*: *finite* (*range* (λ*x*. *a*·*x*))
**assumes** *b*: *finite* (*range* (λ*y*. *b*·*y*))
**shows** *finite* (*range* (λ*f*. *cfun-map*·*a*·*b*·*f*)) (**is** *finite* (*range* *?h*))
**proof** (*rule finite-imageD*)
**let** *?f* = λ*g*. *range* (λ*x*. (*a*·*x*, *g*·*x*))
**show** *finite* (*?f* ‘ *range* *?h*)
**proof** (*rule finite-subset*)
**let** *?B* = *Pow* (*range* (λ*x*. *a*·*x*) × *range* (λ*y*. *b*·*y*))
**show** *?f* ‘ *range* *?h* ⊆ *?B*
**by** *clarsimp*
**show** *finite* *?B*
**by** (*simp add*: *a b*)
**qed**
**show** *inj-on* *?f* (*range* *?h*)
**proof** (*rule inj-onI*, *rule cfun-eqI*, *clarsimp*)
**fix** *x f g*
**assume** *range* (λ*x*. (*a*·*x*, *b*·(*f*·(*a*·*x*)))) = *range* (λ*x*. (*a*·*x*, *b*·(*g*·(*a*·*x*))))
**then have** *range* (λ*x*. (*a*·*x*, *b*·(*f*·(*a*·*x*)))) ⊆ *range* (λ*x*. (*a*·*x*, *b*·(*g*·(*a*·*x*))))
**by** (*rule equalityD1*)
**then have** (*a*·*x*, *b*·(*f*·(*a*·*x*))) ∈ *range* (λ*x*. (*a*·*x*, *b*·(*g*·(*a*·*x*))))
**by** (*simp add*: *subset-eq*)
**then obtain** *y* **where** (*a*·*x*, *b*·(*f*·(*a*·*x*))) = (*a*·*y*, *b*·(*g*·(*a*·*y*)))
**by** (*rule rangeE*)
**then show** *b*·(*f*·(*a*·*x*)) = *b*·(*g*·(*a*·*x*))
**by** *clarsimp*
**qed**
**qed**

**lemma** *finite-deflation-cfun-map*:
**assumes** *finite-deflation d1* **and** *finite-deflation d2*
**shows** *finite-deflation* (*cfun-map*·*d1*·*d2*)
**proof** (*rule finite-deflation-intro*)
**interpret** *d1*: *finite-deflation d1* **by** *fact*
**interpret** *d2*: *finite-deflation d2* **by** *fact*
**from** *d1.deflation-axioms d2.deflation-axioms* **show** *deflation* (*cfun-map*·*d1*·*d2*)
**by** (*rule deflation-cfun-map*)
**have** *finite* (*range* (λ*f*. *cfun-map*·*d1*·*d2*·*f*))
**using** *d1.finite-range d2.finite-range*
**by** (*rule finite-range-cfun-map*)
**then show** *finite* {*f*. *cfun-map*·*d1*·*d2*·*f* = *f*}
**by** (*rule finite-range-imp-finite-fixes*)
**qed**

Finite deflations are compact elements of the function space

**lemma** *finite-deflation-imp-compact*: *finite-deflation d* ⟹ *compact d*

**apply** (*frule finite-deflation-imp-deflation*)
**apply** (*subgoal-tac compact* (*cfun-map·d·d·d*))
**apply** (*simp add*: *cfun-map-def deflation.idem eta-cfun*)
**apply** (*rule finite-deflation.compact*)
**apply** (*simp only*: *finite-deflation-cfun-map*)
**done**

## 17.2 Map operator for product type

**definition** *prod-map* :: $('a \to 'b) \to ('c \to 'd) \to 'a \times 'c \to 'b \times 'd$
**where** *prod-map* = $(\Lambda\ f\ g\ p.\ (f{\cdot}(fst\ p),\ g{\cdot}(snd\ p)))$

**lemma** *prod-map-Pair* [*simp*]: *prod-map·f·g·*$(x,\ y)$ = $(f{\cdot}x,\ g{\cdot}y)$
**by** (*simp add*: *prod-map-def*)

**lemma** *prod-map-ID*: *prod-map·ID·ID* = *ID*
**by** (*auto simp*: *cfun-eq-iff*)

**lemma** *prod-map-map*: *prod-map·f1·g1·*(*prod-map·f2·g2·p*) = *prod-map·*$(\Lambda\ x.\ f1{\cdot}(f2{\cdot}x)){\cdot}(\Lambda\ x.\ g1{\cdot}(g2{\cdot}x)){\cdot}p$
**by** (*induct p*) *simp*

**lemma** *ep-pair-prod-map*:
**assumes** *ep-pair e1 p1* **and** *ep-pair e2 p2*
**shows** *ep-pair* (*prod-map·e1·e2*) (*prod-map·p1·p2*)
**proof**
**interpret** *e1p1*: *ep-pair e1 p1* **by** *fact*
**interpret** *e2p2*: *ep-pair e2 p2* **by** *fact*
**show** *prod-map·p1·p2·*(*prod-map·e1·e2·x*) = *x* **for** *x*
**by** (*induct x*) *simp*
**show** *prod-map·e1·e2·*(*prod-map·p1·p2·y*) $\sqsubseteq$ *y* **for** *y*
**by** (*induct y*) (*simp add*: *e1p1.e-p-below e2p2.e-p-below*)
**qed**

**lemma** *deflation-prod-map*:
**assumes** *deflation d1* **and** *deflation d2*
**shows** *deflation* (*prod-map·d1·d2*)
**proof**
**interpret** *d1*: *deflation d1* **by** *fact*
**interpret** *d2*: *deflation d2* **by** *fact*
**fix** *x*
**show** *prod-map·d1·d2·*(*prod-map·d1·d2·x*) = *prod-map·d1·d2·x*
**by** (*induct x*) (*simp add*: *d1.idem d2.idem*)
**show** *prod-map·d1·d2·x* $\sqsubseteq$ *x*
**by** (*induct x*) (*simp add*: *d1.below d2.below*)
**qed**

**lemma** *finite-deflation-prod-map*:
**assumes** *finite-deflation d1* **and** *finite-deflation d2*

**shows** *finite-deflation* (*prod-map*·*d1*·*d2*)
**proof** (*rule finite-deflation-intro*)
**interpret** *d1*: *finite-deflation d1* **by** *fact*
**interpret** *d2*: *finite-deflation d2* **by** *fact*
**from** *d1.deflation-axioms d2.deflation-axioms* **show** *deflation* (*prod-map*·*d1*·*d2*)
**by** (*rule deflation-prod-map*)
**have** {*p*. *prod-map*·*d1*·*d2*·*p* = *p*} ⊆ {*x*. *d1*·*x* = *x*} × {*y*. *d2*·*y* = *y*}
**by** *auto*
**then show** *finite* {*p*. *prod-map*·*d1*·*d2*·*p* = *p*}
**by** (*rule finite-subset*, *simp add*: *d1.finite-fixes d2.finite-fixes*)
**qed**

## 17.3 Map function for lifted cpo

**definition** *u-map* :: ('*a* → '*b*) → '*a u* → '*b u*
**where** *u-map* = (Λ *f*. *fup*·(*up oo f*))

**lemma** *u-map-strict* [*simp*]: *u-map*·*f*·⊥ = ⊥
**by** (*simp add*: *u-map-def*)

**lemma** *u-map-up* [*simp*]: *u-map*·*f*·(*up*·*x*) = *up*·(*f*·*x*)
**by** (*simp add*: *u-map-def*)

**lemma** *u-map-ID*: *u-map*·*ID* = *ID*
**by** (*simp add*: *u-map-def cfun-eq-iff eta-cfun*)

**lemma** *u-map-map*: *u-map*·*f*·(*u-map*·*g*·*p*) = *u-map*·(Λ *x*. *f*·(*g*·*x*))·*p*
**by** (*induct p*) *simp-all*

**lemma** *u-map-oo*: *u-map*·(*f oo g*) = *u-map*·*f oo u-map*·*g*
**by** (*simp add*: *cfcomp1 u-map-map eta-cfun*)

**lemma** *ep-pair-u-map*: *ep-pair e p* ⟹ *ep-pair* (*u-map*·*e*) (*u-map*·*p*)
**apply** *standard*
**subgoal for** *x* **by** (*cases x*) (*simp-all add*: *ep-pair.e-inverse*)
**subgoal for** *y* **by** (*cases y*) (*simp-all add*: *ep-pair.e-p-below*)
**done**

**lemma** *deflation-u-map*: *deflation d* ⟹ *deflation* (*u-map*·*d*)
**apply** *standard*
**subgoal for** *x* **by** (*cases x*) (*simp-all add*: *deflation.idem*)
**subgoal for** *x* **by** (*cases x*) (*simp-all add*: *deflation.below*)
**done**

**lemma** *finite-deflation-u-map*:
**assumes** *finite-deflation d*
**shows** *finite-deflation* (*u-map*·*d*)
**proof** (*rule finite-deflation-intro*)
**interpret** *d*: *finite-deflation d* **by** *fact*

**from** *d.deflation-axioms* **show** *deflation* (*u-map*·*d*)
**by** (*rule deflation-u-map*)
**have** {*x*. *u-map*·*d*·*x* = *x*} ⊆ *insert* ⊥ ((λ*x*. *up*·*x*) ‘ {*x*. *d*·*x* = *x*})
**by** (*rule subsetI*, *case-tac x*, *simp-all*)
**then show** *finite* {*x*. *u-map*·*d*·*x* = *x*}
**by** (*rule finite-subset*) (*simp add*: *d.finite-fixes*)
**qed**

## 17.4 Map function for strict products

**default-sort** *pcpo*

**definition** *sprod-map* :: ('*a* → '*b*) → ('*c* → '*d*) → '*a* ⊗ '*c* → '*b* ⊗ '*d*
**where** *sprod-map* = (Λ *f g*. *ssplit*·(Λ *x y*. (:*f*·*x*, *g*·*y*:)))

**lemma** *sprod-map-strict* [*simp*]: *sprod-map*·*a*·*b*·⊥ = ⊥
**by** (*simp add*: *sprod-map-def*)

**lemma** *sprod-map-spair* [*simp*]: *x* ≠ ⊥ ⟹ *y* ≠ ⊥ ⟹ *sprod-map*·*f*·*g*·(:*x*, *y*:) = (:*f*·*x*, *g*·*y*:)
**by** (*simp add*: *sprod-map-def*)

**lemma** *sprod-map-spair'*: *f*·⊥ = ⊥ ⟹ *g*·⊥ = ⊥ ⟹ *sprod-map*·*f*·*g*·(:*x*, *y*:) = (:*f*·*x*, *g*·*y*:)
**by** (*cases x* = ⊥ ∨ *y* = ⊥) *auto*

**lemma** *sprod-map-ID*: *sprod-map*·*ID*·*ID* = *ID*
**by** (*simp add*: *sprod-map-def cfun-eq-iff eta-cfun*)

**lemma** *sprod-map-map*:
⟦*f1*·⊥ = ⊥; *g1*·⊥ = ⊥⟧ ⟹
*sprod-map*·*f1*·*g1*·(*sprod-map*·*f2*·*g2*·*p*) =
*sprod-map*·(Λ *x*. *f1*·(*f2*·*x*))·(Λ *x*. *g1*·(*g2*·*x*))·*p*
**proof** (*induct p*)
**case** *bottom*
**then show** *?case* **by** *simp*
**next**
**case** (*spair x y*)
**then show** *?case*
**apply** (*cases f2*·*x* = ⊥, *simp*)
**apply** (*cases g2*·*y* = ⊥, *simp*)
**apply** *simp*
**done**
**qed**

**lemma** *ep-pair-sprod-map*:
**assumes** *ep-pair e1 p1* **and** *ep-pair e2 p2*
**shows** *ep-pair* (*sprod-map*·*e1*·*e2*) (*sprod-map*·*p1*·*p2*)
**proof**

```
  interpret e1p1: pcpo-ep-pair e1 p1 unfolding pcpo-ep-pair-def by fact
  interpret e2p2: pcpo-ep-pair e2 p2 unfolding pcpo-ep-pair-def by fact
  show sprod-map·p1·p2·(sprod-map·e1·e2·x) = x for x
    by (induct x) simp-all
  show sprod-map·e1·e2·(sprod-map·p1·p2·y) ⊑ y for y
  proof (induct y)
    case bottom
    then show ?case by simp
  next
    case (spair x y)
    then show ?case
      apply simp
      apply (cases p1·x = ⊥, simp, cases p2·y = ⊥, simp)
      apply (simp add: monofun-cfun e1p1.e-p-below e2p2.e-p-below)
      done
  qed
qed

lemma deflation-sprod-map:
  assumes deflation d1 and deflation d2
  shows deflation (sprod-map·d1·d2)
proof
  interpret d1: deflation d1 by fact
  interpret d2: deflation d2 by fact
  fix x
  show sprod-map·d1·d2·(sprod-map·d1·d2·x) = sprod-map·d1·d2·x
  proof (induct x)
    case bottom
    then show ?case by simp
  next
    case (spair x y)
    then show ?case
      apply (cases d1·x = ⊥, simp, cases d2·y = ⊥, simp)
      apply (simp add: d1.idem d2.idem)
      done
  qed
  show sprod-map·d1·d2·x ⊑ x
  proof (induct x)
    case bottom
    then show ?case by simp
  next
    case spair
    then show ?case by (simp add: monofun-cfun d1.below d2.below)
  qed
qed

lemma finite-deflation-sprod-map:
  assumes finite-deflation d1 and finite-deflation d2
  shows finite-deflation (sprod-map·d1·d2)
```

**proof** (*rule finite-deflation-intro*)
**interpret** *d1*: *finite-deflation d1* **by** *fact*
**interpret** *d2*: *finite-deflation d2* **by** *fact*
**from** *d1.deflation-axioms d2.deflation-axioms* **show** *deflation* (*sprod-map·d1·d2*)
**by** (*rule deflation-sprod-map*)
**have** {*x*. *sprod-map·d1·d2·x* = *x*} ⊆
*insert* ⊥ ((λ(*x*, *y*). (:*x*, *y*:)) ' ({*x*. *d1·x* = *x*} × {*y*. *d2·y* = *y*}))
**by** (*rule subsetI*, *case-tac x*, *auto simp add*: *spair-eq-iff*)
**then show** *finite* {*x*. *sprod-map·d1·d2·x* = *x*}
**by** (*rule finite-subset*) (*simp add*: *d1.finite-fixes d2.finite-fixes*)
**qed**

## 17.5 Map function for strict sums

**definition** *ssum-map* :: ('*a* → '*b*) → ('*c* → '*d*) → '*a* ⊕ '*c* → '*b* ⊕ '*d*
**where** *ssum-map* = (Λ *f g*. *sscase·*(*sinl oo f*)*·*(*sinr oo g*))

**lemma** *ssum-map-strict* [*simp*]: *ssum-map·f·g·*⊥ = ⊥
**by** (*simp add*: *ssum-map-def*)

**lemma** *ssum-map-sinl* [*simp*]: *x* ≠ ⊥ ⟹ *ssum-map·f·g·*(*sinl·x*) = *sinl·*(*f·x*)
**by** (*simp add*: *ssum-map-def*)

**lemma** *ssum-map-sinr* [*simp*]: *x* ≠ ⊥ ⟹ *ssum-map·f·g·*(*sinr·x*) = *sinr·*(*g·x*)
**by** (*simp add*: *ssum-map-def*)

**lemma** *ssum-map-sinl'*: *f·*⊥ = ⊥ ⟹ *ssum-map·f·g·*(*sinl·x*) = *sinl·*(*f·x*)
**by** (*cases x* = ⊥) *simp-all*

**lemma** *ssum-map-sinr'*: *g·*⊥ = ⊥ ⟹ *ssum-map·f·g·*(*sinr·x*) = *sinr·*(*g·x*)
**by** (*cases x* = ⊥) *simp-all*

**lemma** *ssum-map-ID*: *ssum-map·ID·ID* = *ID*
**by** (*simp add*: *ssum-map-def cfun-eq-iff eta-cfun*)

**lemma** *ssum-map-map*:
⟦*f1·*⊥ = ⊥; *g1·*⊥ = ⊥⟧ ⟹
*ssum-map·f1·g1·*(*ssum-map·f2·g2·p*) =
*ssum-map·*(Λ *x*. *f1·*(*f2·x*))*·*(Λ *x*. *g1·*(*g2·x*))*·p*
**proof** (*induct p*)
**case** *bottom*
**then show** *?case* **by** *simp*
**next**
**case** (*sinl x*)
**then show** *?case* **by** (*cases f2·x* = ⊥) *simp-all*
**next**
**case** (*sinr y*)
**then show** *?case* **by** (*cases g2·y* = ⊥) *simp-all*
**qed**

**lemma** *ep-pair-ssum-map*:
**assumes** *ep-pair e1 p1* **and** *ep-pair e2 p2*
**shows** *ep-pair* (*ssum-map*·*e1*·*e2*) (*ssum-map*·*p1*·*p2*)
**proof**
**interpret** *e1p1*: *pcpo-ep-pair e1 p1* **unfolding** *pcpo-ep-pair-def* **by** *fact*
**interpret** *e2p2*: *pcpo-ep-pair e2 p2* **unfolding** *pcpo-ep-pair-def* **by** *fact*
**show** *ssum-map*·*p1*·*p2*·(*ssum-map*·*e1*·*e2*·*x*) = *x* **for** *x*
**by** (*induct x*) *simp-all*
**show** *ssum-map*·*e1*·*e2*·(*ssum-map*·*p1*·*p2*·*y*) ⊑ *y* **for** *y*
**proof** (*induct y*)
**case** *bottom*
**then show** *?case* **by** *simp*
**next**
**case** (*sinl x*)
**then show** *?case* **by** (*cases p1*·*x* = ⊥) (*simp-all add*: *e1p1.e-p-below*)
**next**
**case** (*sinr y*)
**then show** *?case* **by** (*cases p2*·*y* = ⊥) (*simp-all add*: *e2p2.e-p-below*)
**qed**
**qed**

**lemma** *deflation-ssum-map*:
**assumes** *deflation d1* **and** *deflation d2*
**shows** *deflation* (*ssum-map*·*d1*·*d2*)
**proof**
**interpret** *d1*: *deflation d1* **by** *fact*
**interpret** *d2*: *deflation d2* **by** *fact*
**fix** *x*
**show** *ssum-map*·*d1*·*d2*·(*ssum-map*·*d1*·*d2*·*x*) = *ssum-map*·*d1*·*d2*·*x*
**proof** (*induct x*)
**case** *bottom*
**then show** *?case* **by** *simp*
**next**
**case** (*sinl x*)
**then show** *?case* **by** (*cases d1*·*x* = ⊥) (*simp-all add*: *d1.idem*)
**next**
**case** (*sinr y*)
**then show** *?case* **by** (*cases d2*·*y* = ⊥) (*simp-all add*: *d2.idem*)
**qed**
**show** *ssum-map*·*d1*·*d2*·*x* ⊑ *x*
**proof** (*induct x*)
**case** *bottom*
**then show** *?case* **by** *simp*
**next**
**case** (*sinl x*)
**then show** *?case* **by** (*cases d1*·*x* = ⊥) (*simp-all add*: *d1.below*)
**next**
**case** (*sinr y*)

```
    then show ?case by (cases d2·y = ⊥) (simp-all add: d2.below)
  qed
qed

lemma finite-deflation-ssum-map:
  assumes finite-deflation d1 and finite-deflation d2
  shows finite-deflation (ssum-map·d1·d2)
proof (rule finite-deflation-intro)
  interpret d1: finite-deflation d1 by fact
  interpret d2: finite-deflation d2 by fact
  from d1.deflation-axioms d2.deflation-axioms show deflation (ssum-map·d1·d2)
    by (rule deflation-ssum-map)
  have {x. ssum-map·d1·d2·x = x} ⊆
        (λx. sinl·x) ` {x. d1·x = x} ∪
        (λx. sinr·x) ` {x. d2·x = x} ∪ {⊥}
    by (rule subsetI, case-tac x, simp-all)
  then show finite {x. ssum-map·d1·d2·x = x}
    by (rule finite-subset, simp add: d1.finite-fixes d2.finite-fixes)
qed
```

## 17.6 Map operator for strict function space

```
definition sfun-map :: ('b → 'a) → ('c → 'd) → ('a →! 'c) → ('b →! 'd)
  where sfun-map = (Λ a b. sfun-abs oo cfun-map·a·b oo sfun-rep)

lemma sfun-map-ID: sfun-map·ID·ID = ID
  by (simp add: sfun-map-def cfun-map-ID cfun-eq-iff)

lemma sfun-map-map:
  assumes f2·⊥ = ⊥ and g2·⊥ = ⊥
  shows sfun-map·f1·g1·(sfun-map·f2·g2·p) =
    sfun-map·(Λ x. f2·(f1·x))·(Λ x. g1·(g2·x))·p
  by (simp add: sfun-map-def cfun-eq-iff strictify-cancel assms cfun-map-map)

lemma ep-pair-sfun-map:
  assumes 1: ep-pair e1 p1
  assumes 2: ep-pair e2 p2
  shows ep-pair (sfun-map·p1·e2) (sfun-map·e1·p2)
proof
  interpret e1p1: pcpo-ep-pair e1 p1
    unfolding pcpo-ep-pair-def by fact
  interpret e2p2: pcpo-ep-pair e2 p2
    unfolding pcpo-ep-pair-def by fact
  show sfun-map·e1·p2·(sfun-map·p1·e2·f) = f for f
    unfolding sfun-map-def
    apply (simp add: sfun-eq-iff strictify-cancel)
    apply (rule ep-pair.e-inverse)
    apply (rule ep-pair-cfun-map [OF 1 2])
    done
```

```
  show sfun-map·p1·e2·(sfun-map·e1·p2·g) ⊑ g for g
    unfolding sfun-map-def
    apply (simp add: sfun-below-iff strictify-cancel)
    apply (rule ep-pair.e-p-below)
    apply (rule ep-pair-cfun-map [OF 1 2])
    done
qed

lemma deflation-sfun-map:
  assumes 1: deflation d1
  assumes 2: deflation d2
  shows deflation (sfun-map·d1·d2)
  apply (simp add: sfun-map-def)
  apply (rule deflation.intro)
   apply simp
   apply (subst strictify-cancel)
    apply (simp add: cfun-map-def deflation-strict 1 2)
   apply (simp add: cfun-map-def deflation.idem 1 2)
  apply (simp add: sfun-below-iff)
  apply (subst strictify-cancel)
   apply (simp add: cfun-map-def deflation-strict 1 2)
  apply (rule deflation.below)
  apply (rule deflation-cfun-map [OF 1 2])
  done

lemma finite-deflation-sfun-map:
  assumes finite-deflation d1
    and finite-deflation d2
  shows finite-deflation (sfun-map·d1·d2)
proof (intro finite-deflation-intro)
  interpret d1: finite-deflation d1 by fact
  interpret d2: finite-deflation d2 by fact
  from d1.deflation-axioms d2.deflation-axioms show deflation (sfun-map·d1·d2)
    by (rule deflation-sfun-map)
  from assms have finite-deflation (cfun-map·d1·d2)
    by (rule finite-deflation-cfun-map)
  then have finite {f. cfun-map·d1·d2·f = f}
    by (rule finite-deflation.finite-fixes)
  moreover have inj (λf. sfun-rep·f)
    by (rule inj-onI) (simp add: sfun-eq-iff)
  ultimately have finite ((λf. sfun-rep·f) -` {f. cfun-map·d1·d2·f = f})
    by (rule finite-vimageI)
  with ‹deflation d1› ‹deflation d2› show finite {f. sfun-map·d1·d2·f = f}
    by (simp add: sfun-map-def sfun-eq-iff strictify-cancel deflation-strict)
qed

end
```

# 18 The cpo of cartesian products

**theory** *Cprod*
**imports** *Cfun*
**begin**

**default-sort** *cpo*

## 18.1 Continuous case function for unit type

**definition** *unit-when* :: $'a \to unit \to 'a$
**where** *unit-when* = $(\Lambda\ a\ \text{-}.\ a)$

**translations**
$\Lambda().\ t \rightleftharpoons CONST\ unit\text{-}when{\cdot}t$

**lemma** *unit-when* [*simp*]: $unit\text{-}when{\cdot}a{\cdot}u = a$
**by** (*simp add*: *unit-when-def*)

## 18.2 Continuous version of split function

**definition** *csplit* :: $('a \to 'b \to 'c) \to ('a \times 'b) \to 'c$
**where** *csplit* = $(\Lambda\ f\ p.\ f{\cdot}(fst\ p){\cdot}(snd\ p))$

**translations**
$\Lambda(CONST\ Pair\ x\ y).\ t \rightleftharpoons CONST\ csplit{\cdot}(\Lambda\ x\ y.\ t)$

**abbreviation** *cfst* :: $'a \times 'b \to 'a$
**where** *cfst* $\equiv$ *Abs-cfun fst*

**abbreviation** *csnd* :: $'a \times 'b \to 'b$
**where** *csnd* $\equiv$ *Abs-cfun snd*

## 18.3 Convert all lemmas to the continuous versions

**lemma** *csplit1* [*simp*]: $csplit{\cdot}f{\cdot}\bot = f{\cdot}\bot{\cdot}\bot$
**by** (*simp add*: *csplit-def*)

**lemma** *csplit-Pair* [*simp*]: $csplit{\cdot}f{\cdot}(x,\ y) = f{\cdot}x{\cdot}y$
**by** (*simp add*: *csplit-def*)

**end**

# 19 Profinite and bifinite cpos

**theory** *Bifinite*
**imports** *Map-Functions Cprod Sprod Sfun Up HOL−Library.Countable*
**begin**

**default-sort** *cpo*

## 19.1 Chains of finite deflations

**locale** *approx-chain* =
  **fixes** *approx* :: *nat* ⇒ *'a* → *'a*
  **assumes** *chain-approx* [*simp*]: *chain* (λ*i*. *approx i*)
  **assumes** *lub-approx* [*simp*]: (⨆ *i*. *approx i*) = *ID*
  **assumes** *finite-deflation-approx* [*simp*]: ⋀*i*. *finite-deflation* (*approx i*)
**begin**

**lemma** *deflation-approx*: *deflation* (*approx i*)
**using** *finite-deflation-approx* **by** (*rule finite-deflation-imp-deflation*)

**lemma** *approx-idem*: *approx i*·(*approx i*·*x*) = *approx i*·*x*
**using** *deflation-approx* **by** (*rule deflation.idem*)

**lemma** *approx-below*: *approx i*·*x* ⊑ *x*
**using** *deflation-approx* **by** (*rule deflation.below*)

**lemma** *finite-range-approx*: *finite* (*range* (λ*x*. *approx i*·*x*))
**apply** (*rule finite-deflation.finite-range*)
**apply** (*rule finite-deflation-approx*)
**done**

**lemma** *compact-approx* [*simp*]: *compact* (*approx n*·*x*)
**apply** (*rule finite-deflation.compact*)
**apply** (*rule finite-deflation-approx*)
**done**

**lemma** *compact-eq-approx*: *compact x* ⟹ ∃ *i*. *approx i*·*x* = *x*
**by** (*rule admD2*, *simp-all*)

**end**

## 19.2 Omega-profinite and bifinite domains

**class** *bifinite* = *pcpo* +
  **assumes** *bifinite*: ∃ (*a*::*nat* ⇒ *'a* → *'a*). *approx-chain a*

**class** *profinite* = *cpo* +
  **assumes** *profinite*: ∃ (*a*::*nat* ⇒ *'a*⊥ → *'a*⊥). *approx-chain a*

## 19.3 Building approx chains

**lemma** *approx-chain-iso*:
  **assumes** *a*: *approx-chain a*
  **assumes** [*simp*]: ⋀*x*. *f*·(*g*·*x*) = *x*
  **assumes** [*simp*]: ⋀*y*. *g*·(*f*·*y*) = *y*
  **shows** *approx-chain* (λ*i*. *f oo a i oo g*)
**proof** −
  **have** *1*: *f oo g* = *ID* **by** (*simp add*: *cfun-eqI*)

**have** *2*: *ep-pair f g* **by** (*simp add*: *ep-pair-def*)
**from** *1 2* **show** *?thesis*
**using** *a* **unfolding** *approx-chain-def*
**by** (*simp add*: *lub-APP ep-pair.finite-deflation-e-d-p*)
**qed**

**lemma** *approx-chain-u-map*:
**assumes** *approx-chain a*
**shows** *approx-chain* (λ*i*. *u-map*·(*a i*))
**using** *assms* **unfolding** *approx-chain-def*
**by** (*simp add*: *lub-APP u-map-ID finite-deflation-u-map*)

**lemma** *approx-chain-sfun-map*:
**assumes** *approx-chain a* **and** *approx-chain b*
**shows** *approx-chain* (λ*i*. *sfun-map*·(*a i*)·(*b i*))
**using** *assms* **unfolding** *approx-chain-def*
**by** (*simp add*: *lub-APP sfun-map-ID finite-deflation-sfun-map*)

**lemma** *approx-chain-sprod-map*:
**assumes** *approx-chain a* **and** *approx-chain b*
**shows** *approx-chain* (λ*i*. *sprod-map*·(*a i*)·(*b i*))
**using** *assms* **unfolding** *approx-chain-def*
**by** (*simp add*: *lub-APP sprod-map-ID finite-deflation-sprod-map*)

**lemma** *approx-chain-ssum-map*:
**assumes** *approx-chain a* **and** *approx-chain b*
**shows** *approx-chain* (λ*i*. *ssum-map*·(*a i*)·(*b i*))
**using** *assms* **unfolding** *approx-chain-def*
**by** (*simp add*: *lub-APP ssum-map-ID finite-deflation-ssum-map*)

**lemma** *approx-chain-cfun-map*:
**assumes** *approx-chain a* **and** *approx-chain b*
**shows** *approx-chain* (λ*i*. *cfun-map*·(*a i*)·(*b i*))
**using** *assms* **unfolding** *approx-chain-def*
**by** (*simp add*: *lub-APP cfun-map-ID finite-deflation-cfun-map*)

**lemma** *approx-chain-prod-map*:
**assumes** *approx-chain a* **and** *approx-chain b*
**shows** *approx-chain* (λ*i*. *prod-map*·(*a i*)·(*b i*))
**using** *assms* **unfolding** *approx-chain-def*
**by** (*simp add*: *lub-APP prod-map-ID finite-deflation-prod-map*)

Approx chains for countable discrete types.

**definition** *discr-approx* :: *nat* ⇒ *'a*::*countable discr u* → *'a discr u*
**where** *discr-approx* = (λ*i*. Λ(*up*·*x*). *if to-nat* (*undiscr x*) < *i then up*·*x else* ⊥)

**lemma** *chain-discr-approx* [*simp*]: *chain discr-approx*
**unfolding** *discr-approx-def*
**by** (*rule chainI*, *simp add*: *monofun-cfun monofun-LAM*)

**lemma** *lub-discr-approx* [*simp*]: $(\bigsqcup i.\ \textit{discr-approx}\ i) = \textit{ID}$
**apply** (*rule cfun-eqI*)
**apply** (*simp add*: *contlub-cfun-fun*)
**apply** (*simp add*: *discr-approx-def*)
**subgoal for** *x*
**apply** (*cases x*)
**apply** *simp*
**apply** (*rule lub-eqI*)
**apply** (*rule is-lubI*)
**apply** (*rule ub-rangeI*, *simp*)
**apply** (*drule ub-rangeD*)
**apply** (*erule rev-below-trans*)
**apply** *simp*
**apply** (*rule lessI*)
**done**
**done**

**lemma** *inj-on-undiscr* [*simp*]: *inj-on undiscr A*
**using** *Discr-undiscr* **by** (*rule inj-on-inverseI*)

**lemma** *finite-deflation-discr-approx*: *finite-deflation* (*discr-approx i*)
**proof**
**fix** $x :: 'a\ \textit{discr}\ u$
**show** $\textit{discr-approx}\ i{\cdot}x \sqsubseteq x$
**unfolding** *discr-approx-def*
**by** (*cases x*, *simp*, *simp*)
**show** $\textit{discr-approx}\ i{\cdot}(\textit{discr-approx}\ i{\cdot}x) = \textit{discr-approx}\ i{\cdot}x$
**unfolding** *discr-approx-def*
**by** (*cases x*, *simp*, *simp*)
**show** *finite* $\{x{::}'a\ \textit{discr}\ u.\ \textit{discr-approx}\ i{\cdot}x = x\}$
**proof** (*rule finite-subset*)
**let** *?S* = *insert* $(\bot{::}'a\ \textit{discr}\ u)$ $((\lambda x.\ \textit{up}{\cdot}x)\ `\ \textit{undiscr} -`\ \textit{to-nat} -`\ \{..{<}i\})$
**show** $\{x{::}'a\ \textit{discr}\ u.\ \textit{discr-approx}\ i{\cdot}x = x\} \subseteq \textit{?S}$
**unfolding** *discr-approx-def*
**by** (*rule subsetI*, *case-tac x*, *simp*, *simp split*: *if-split-asm*)
**show** *finite ?S*
**by** (*simp add*: *finite-vimageI*)
**qed**
**qed**

**lemma** *discr-approx*: *approx-chain discr-approx*
**using** *chain-discr-approx lub-discr-approx finite-deflation-discr-approx*
**by** (*rule approx-chain.intro*)

## 19.4 Class instance proofs

**instance** *bifinite* $\subseteq$ *profinite*
**proof**

**show** $\exists (a{::}nat \Rightarrow {'a}_\bot \to {'a}_\bot).\ approx\text{-}chain\ a$
**using** *bifinite* [**where** $'a='a$]
**by** (*fast intro*!: *approx-chain-u-map*)
**qed**

**instance** $u$ :: (*profinite*) *bifinite*
**by** *standard* (*rule profinite*)

Types $'a \to 'b$ and ${'a}_\bot \to! 'b$ are isomorphic.

**definition** *encode-cfun* = $(\Lambda\ f.\ sfun\text{-}abs{\cdot}(fup{\cdot}f))$

**definition** *decode-cfun* = $(\Lambda\ g\ x.\ sfun\text{-}rep{\cdot}g{\cdot}(up{\cdot}x))$

**lemma** *decode-encode-cfun* [*simp*]: $decode\text{-}cfun{\cdot}(encode\text{-}cfun{\cdot}x) = x$
**unfolding** *encode-cfun-def decode-cfun-def*
**by** (*simp add*: *eta-cfun*)

**lemma** *encode-decode-cfun* [*simp*]: $encode\text{-}cfun{\cdot}(decode\text{-}cfun{\cdot}y) = y$
**unfolding** *encode-cfun-def decode-cfun-def*
**apply** (*simp add*: *sfun-eq-iff strictify-cancel*)
**apply** (*rule cfun-eqI*, *case-tac x*, *simp-all*)
**done**

**instance** *cfun* :: (*profinite*, *bifinite*) *bifinite*
**proof**
**obtain** $a :: nat \Rightarrow {'a}_\bot \to {'a}_\bot$ **where** *a*: *approx-chain a*
**using** *profinite* **..**
**obtain** $b :: nat \Rightarrow 'b \to 'b$ **where** *b*: *approx-chain b*
**using** *bifinite* **..**
**have** *approx-chain* $(\lambda i.\ decode\text{-}cfun\ oo\ sfun\text{-}map{\cdot}(a\ i){\cdot}(b\ i)\ oo\ encode\text{-}cfun)$
**using** *a b* **by** (*simp add*: *approx-chain-iso approx-chain-sfun-map*)
**thus** $\exists (a{::}nat \Rightarrow ('a \to 'b) \to ('a \to 'b)).\ approx\text{-}chain\ a$
**by** − (*rule exI*)
**qed**

Types $('a \times 'b)_\bot$ and ${'a}_\bot \otimes {'b}_\bot$ are isomorphic.

**definition** *encode-prod-u* = $(\Lambda(up{\cdot}(x,\ y)).\ (:up{\cdot}x,\ up{\cdot}y:))$

**definition** *decode-prod-u* = $(\Lambda(:up{\cdot}x,\ up{\cdot}y:).\ up{\cdot}(x,\ y))$

**lemma** *decode-encode-prod-u* [*simp*]: $decode\text{-}prod\text{-}u{\cdot}(encode\text{-}prod\text{-}u{\cdot}x) = x$
**unfolding** *encode-prod-u-def decode-prod-u-def*
**apply** (*cases x*)
**apply** *simp*
**subgoal for** *y* **by** (*cases y*) *simp*
**done**

**lemma** *encode-decode-prod-u* [*simp*]: $encode\text{-}prod\text{-}u{\cdot}(decode\text{-}prod\text{-}u{\cdot}y) = y$
**unfolding** *encode-prod-u-def decode-prod-u-def*

```
  apply (cases y)
   apply simp
  subgoal for a b
    apply (cases a, simp)
    apply (cases b, simp, simp)
    done
  done

instance prod :: (profinite, profinite) profinite
proof
  obtain a :: nat ⇒ 'a⊥ → 'a⊥ where a: approx-chain a
    using profinite ..
  obtain b :: nat ⇒ 'b⊥ → 'b⊥ where b: approx-chain b
    using profinite ..
  have approx-chain (λi. decode-prod-u oo sprod-map·(a i)·(b i) oo encode-prod-u)
    using a b by (simp add: approx-chain-iso approx-chain-sprod-map)
  thus ∃(a::nat ⇒ ('a × 'b)⊥ → ('a × 'b)⊥). approx-chain a
    by − (rule exI)
qed

instance prod :: (bifinite, bifinite) bifinite
proof
  show ∃(a::nat ⇒ ('a × 'b) → ('a × 'b)). approx-chain a
    using bifinite [where 'a='a] and bifinite [where 'a='b]
    by (fast intro!: approx-chain-prod-map)
qed

instance sfun :: (bifinite, bifinite) bifinite
proof
  show ∃(a::nat ⇒ ('a →! 'b) → ('a →! 'b)). approx-chain a
    using bifinite [where 'a='a] and bifinite [where 'a='b]
    by (fast intro!: approx-chain-sfun-map)
qed

instance sprod :: (bifinite, bifinite) bifinite
proof
  show ∃(a::nat ⇒ ('a ⊗ 'b) → ('a ⊗ 'b)). approx-chain a
    using bifinite [where 'a='a] and bifinite [where 'a='b]
    by (fast intro!: approx-chain-sprod-map)
qed

instance ssum :: (bifinite, bifinite) bifinite
proof
  show ∃(a::nat ⇒ ('a ⊕ 'b) → ('a ⊕ 'b)). approx-chain a
    using bifinite [where 'a='a] and bifinite [where 'a='b]
    by (fast intro!: approx-chain-ssum-map)
qed

lemma approx-chain-unit: approx-chain (⊥ :: nat ⇒ unit → unit)
```

**by** (*simp add*: *approx-chain-def cfun-eq-iff finite-deflation-bottom*)

**instance** *unit* :: *bifinite*
  **by** *standard* (*fast intro*!: *approx-chain-unit*)

**instance** *discr* :: (*countable*) *profinite*
  **by** *standard* (*fast intro*!: *discr-approx*)

**instance** *lift* :: (*countable*) *bifinite*
**proof**
  **note** [*simp*] = *cont-Abs-lift cont-Rep-lift Rep-lift-inverse Abs-lift-inverse*
  **obtain** *a* :: *nat* ⇒ ('*a discr*)$_\perp$ → ('*a discr*)$_\perp$ **where** *a*: *approx-chain a*
    **using** *profinite* **..**
  **hence** *approx-chain* (λ*i*. (Λ *y*. *Abs-lift y*) *oo a i oo* (Λ *x*. *Rep-lift x*))
    **by** (*rule approx-chain-iso*) *simp-all*
  **thus** ∃ (*a*::*nat* ⇒ '*a lift* → '*a lift*). *approx-chain a*
    **by** − (*rule exI*)
**qed**

**end**

# 20 Defining algebraic domains by ideal completion

**theory** *Completion*
**imports** *Cfun*
**begin**

## 20.1 Ideals over a preorder

**locale** *preorder* =
  **fixes** *r* :: '*a*::*type* ⇒ '*a* ⇒ *bool* (**infix** ≼ *50*)
  **assumes** *r-refl*: *x* ≼ *x*
  **assumes** *r-trans*: ⟦*x* ≼ *y*; *y* ≼ *z*⟧ ⟹ *x* ≼ *z*
**begin**

**definition**
  *ideal* :: '*a set* ⇒ *bool* **where**
  *ideal A* = ((∃ *x*. *x* ∈ *A*) ∧ (∀ *x*∈*A*. ∀ *y*∈*A*. ∃ *z*∈*A*. *x* ≼ *z* ∧ *y* ≼ *z*) ∧
    (∀ *x y*. *x* ≼ *y* ⟶ *y* ∈ *A* ⟶ *x* ∈ *A*))

**lemma** *idealI*:
  **assumes** ∃ *x*. *x* ∈ *A*
  **assumes** ⋀*x y*. ⟦*x* ∈ *A*; *y* ∈ *A*⟧ ⟹ ∃ *z*∈*A*. *x* ≼ *z* ∧ *y* ≼ *z*
  **assumes** ⋀*x y*. ⟦*x* ≼ *y*; *y* ∈ *A*⟧ ⟹ *x* ∈ *A*
  **shows** *ideal A*
**unfolding** *ideal-def* **using** *assms* **by** *fast*

**lemma** *idealD1*:
  *ideal A* ⟹ ∃ *x*. *x* ∈ *A*

**unfolding** *ideal-def* **by** *fast*

**lemma** *idealD2*:
$[\![ideal\ A;\ x \in A;\ y \in A]\!] \Longrightarrow \exists\, z{\in}A.\ x \preceq z \wedge y \preceq z$
**unfolding** *ideal-def* **by** *fast*

**lemma** *idealD3*:
$[\![ideal\ A;\ x \preceq y;\ y \in A]\!] \Longrightarrow x \in A$
**unfolding** *ideal-def* **by** *fast*

**lemma** *ideal-principal*: *ideal* $\{x.\ x \preceq z\}$
**apply** (*rule idealI*)
**apply** (*rule exI* [**where** $x = z$])
**apply** (*fast intro*: *r-refl*)
**apply** (*rule bexI* [**where** $x = z$], *fast*)
**apply** (*fast intro*: *r-refl*)
**apply** (*fast intro*: *r-trans*)
**done**

**lemma** *ex-ideal*: $\exists\, A.\ A \in \{A.\ ideal\ A\}$
**by** (*fast intro*: *ideal-principal*)

The set of ideals is a cpo

**lemma** *ideal-UN*:
**fixes** *A* :: *nat* ⇒ *'a set*
**assumes** *ideal-A*: $\bigwedge i.\ ideal\ (A\ i)$
**assumes** *chain-A*: $\bigwedge i\ j.\ i \leq j \Longrightarrow A\ i \subseteq A\ j$
**shows** *ideal* $(\bigcup i.\ A\ i)$
**apply** (*rule idealI*)
**using** *idealD1* [*OF ideal-A*] **apply** *fast*
**apply** (*clarify*)
**subgoal for** *i j*
**apply** (*drule subsetD* [*OF chain-A* [*OF max.cobounded1*]])
**apply** (*drule subsetD* [*OF chain-A* [*OF max.cobounded2*]])
**apply** (*drule* (*1*) *idealD2* [*OF ideal-A*])
**apply** *blast*
**done**
**apply** *clarify*
**apply** (*drule* (*1*) *idealD3* [*OF ideal-A*])
**apply** *fast*
**done**

**lemma** *typedef-ideal-po*:
**fixes** *Abs* :: *'a set* ⇒ *'b*::*below*
**assumes** *type*: *type-definition Rep Abs* $\{S.\ ideal\ S\}$
**assumes** *below*: $\bigwedge x\ y.\ x \sqsubseteq y \longleftrightarrow Rep\ x \subseteq Rep\ y$
**shows** *OFCLASS*(*'b*, *po-class*)
**apply** (*intro-classes*, *unfold below*)
**apply** (*rule subset-refl*)

**apply** (*erule* (*1*) *subset-trans*)
**apply** (*rule type-definition.Rep-inject* [*OF type*, *THEN iffD1*])
**apply** (*erule* (*1*) *subset-antisym*)
**done**

**lemma**
**fixes** *Abs* :: *'a set* ⇒ *'b*::*po*
**assumes** *type*: *type-definition Rep Abs* {*S*. *ideal S*}
**assumes** *below*: ⋀*x y*. *x* ⊑ *y* ⟷ *Rep x* ⊆ *Rep y*
**assumes** *S*: *chain S*
**shows** *typedef-ideal-lub*: *range S* <<| *Abs* (⋃ *i*. *Rep* (*S i*))
**and** *typedef-ideal-rep-lub*: *Rep* (⨆ *i*. *S i*) = (⋃ *i*. *Rep* (*S i*))
**proof** −
**have** *1*: *ideal* (⋃ *i*. *Rep* (*S i*))
**apply** (*rule ideal-UN*)
**apply** (*rule type-definition.Rep* [*OF type*, *unfolded mem-Collect-eq*])
**apply** (*subst below* [*symmetric*])
**apply** (*erule chain-mono* [*OF S*])
**done**
**hence** *2*: *Rep* (*Abs* (⋃ *i*. *Rep* (*S i*))) = (⋃ *i*. *Rep* (*S i*))
**by** (*simp add*: *type-definition.Abs-inverse* [*OF type*])
**show** *3*: *range S* <<| *Abs* (⋃ *i*. *Rep* (*S i*))
**apply** (*rule is-lubI*)
**apply** (*rule is-ubI*)
**apply** (*simp add*: *below 2*, *fast*)
**apply** (*simp add*: *below 2 is-ub-def*, *fast*)
**done**
**hence** *4*: (⨆ *i*. *S i*) = *Abs* (⋃ *i*. *Rep* (*S i*))
**by** (*rule lub-eqI*)
**show** *5*: *Rep* (⨆ *i*. *S i*) = (⋃ *i*. *Rep* (*S i*))
**by** (*simp add*: *4 2*)
**qed**

**lemma** *typedef-ideal-cpo*:
**fixes** *Abs* :: *'a set* ⇒ *'b*::*po*
**assumes** *type*: *type-definition Rep Abs* {*S*. *ideal S*}
**assumes** *below*: ⋀*x y*. *x* ⊑ *y* ⟷ *Rep x* ⊆ *Rep y*
**shows** *OFCLASS*(*'b*, *cpo-class*)
**by** *standard* (*rule exI*, *erule typedef-ideal-lub* [*OF type below*])

**end**

**interpretation** *below*: *preorder below* :: *'a*::*po* ⇒ *'a* ⇒ *bool*
**apply** *unfold-locales*
**apply** (*rule below-refl*)
**apply** (*erule* (*1*) *below-trans*)
**done**

## 20.2 Lemmas about least upper bounds

**lemma** *is-ub-thelub-ex*: $[\![\exists\, u.\ S <<|\ u;\ x \in S]\!] \Longrightarrow x \sqsubseteq \mathit{lub}\ S$
**apply** (*erule exE*, *drule is-lub-lub*)
**apply** (*drule is-lubD1*)
**apply** (*erule* (*1*) *is-ubD*)
**done**

**lemma** *is-lub-thelub-ex*: $[\![\exists\, u.\ S <<|\ u;\ S <|\ x]\!] \Longrightarrow \mathit{lub}\ S \sqsubseteq x$
**by** (*erule exE*, *drule is-lub-lub*, *erule is-lubD2*)

## 20.3 Locale for ideal completion

**hide-const** (**open**) *Filter.principal*

**locale** *ideal-completion* = *preorder* +
  **fixes** *principal* :: $'a$::*type* $\Rightarrow$ $'b$::*cpo*
  **fixes** *rep* :: $'b$::*cpo* $\Rightarrow$ $'a$::*type set*
  **assumes** *ideal-rep*: $\bigwedge x.$ *ideal* (*rep x*)
  **assumes** *rep-lub*: $\bigwedge Y.$ *chain Y* $\Longrightarrow$ *rep* $(\bigsqcup i.\ Y\ i) = (\bigcup i.\ \mathit{rep}\ (Y\ i))$
  **assumes** *rep-principal*: $\bigwedge a.$ *rep* (*principal a*) $= \{b.\ b \preceq a\}$
  **assumes** *belowI*: $\bigwedge x\ y.$ *rep x* $\subseteq$ *rep y* $\Longrightarrow x \sqsubseteq y$
  **assumes** *countable*: $\exists f$::$'a \Rightarrow$ *nat*. *inj f*
**begin**

**lemma** *rep-mono*: $x \sqsubseteq y \Longrightarrow \mathit{rep}\ x \subseteq \mathit{rep}\ y$
**apply** (*frule bin-chain*)
**apply** (*drule rep-lub*)
**apply** (*simp only*: *lub-eqI* [*OF is-lub-bin-chain*])
**apply** (*rule subsetI*, *rule UN-I* [**where** *a=0*], *simp-all*)
**done**

**lemma** *below-def*: $x \sqsubseteq y \longleftrightarrow \mathit{rep}\ x \subseteq \mathit{rep}\ y$
**by** (*rule iffI* [*OF rep-mono belowI*])

**lemma** *principal-below-iff-mem-rep*: *principal a* $\sqsubseteq x \longleftrightarrow a \in \mathit{rep}\ x$
**unfolding** *below-def rep-principal*
**by** (*auto intro*: *r-refl elim*: *idealD3* [*OF ideal-rep*])

**lemma** *principal-below-iff* [*simp*]: *principal a* $\sqsubseteq$ *principal b* $\longleftrightarrow a \preceq b$
**by** (*simp add*: *principal-below-iff-mem-rep rep-principal*)

**lemma** *principal-eq-iff*: *principal a* = *principal b* $\longleftrightarrow a \preceq b \wedge b \preceq a$
**unfolding** *po-eq-conv* [**where** $'a='b$] *principal-below-iff* **..**

**lemma** *eq-iff*: $x = y \longleftrightarrow \mathit{rep}\ x = \mathit{rep}\ y$
**unfolding** *po-eq-conv below-def* **by** *auto*

**lemma** *principal-mono*: $a \preceq b \Longrightarrow$ *principal a* $\sqsubseteq$ *principal b*
**by** (*simp only*: *principal-below-iff*)

**lemma** *ch2ch-principal* [*simp*]:
$\forall i.\ Y\ i \preceq Y\ (Suc\ i) \Longrightarrow chain\ (\lambda i.\ principal\ (Y\ i))$
**by** (*simp add*: *chainI principal-mono*)

### 20.3.1 Principal ideals approximate all elements

**lemma** *compact-principal* [*simp*]: *compact* (*principal a*)
**by** (*rule compactI2*, *simp add*: *principal-below-iff-mem-rep rep-lub*)

Construct a chain whose lub is the same as a given ideal

**lemma** *obtain-principal-chain*:
**obtains** $Y$ **where** $\forall i.\ Y\ i \preceq Y\ (Suc\ i)$ **and** $x = (\bigsqcup i.\ principal\ (Y\ i))$
**proof** −
**obtain** *count* :: $'a \Rightarrow nat$ **where** *inj*: *inj count*
**using** *countable* **..**
**define** *enum* **where** *enum i* = (*THE a*. *count a* = *i*) **for** *i*
**have** *enum-count* [*simp*]: $\bigwedge x.$ *enum* (*count x*) = *x*
**unfolding** *enum-def* **by** (*simp add*: *inj-eq* [*OF inj*])
**define** *a* **where** *a* = (*LEAST i*. *enum i* ∈ *rep x*)
**define** *b* **where** *b i* = (*LEAST j*. *enum j* ∈ *rep x* ∧ ¬ *enum j* ≼ *enum i*) **for** *i*
**define** *c* **where** *c i j* = (*LEAST k*. *enum k* ∈ *rep x* ∧ *enum i* ≼ *enum k* ∧ *enum j* ≼ *enum k*) **for** *i j*
**define** *P* **where** *P i* ⟷ (∃ *j*. *enum j* ∈ *rep x* ∧ ¬ *enum j* ≼ *enum i*) **for** *i*
**define** *X* **where** *X* = *rec-nat a* (λ*n i*. *if P i then c i* (*b i*) *else i*)
**have** *X-0*: *X 0* = *a* **unfolding** *X-def* **by** *simp*
**have** *X-Suc*: $\bigwedge n.$ *X* (*Suc n*) = (*if P* (*X n*) *then c* (*X n*) (*b* (*X n*)) *else X n*)
**unfolding** *X-def* **by** *simp*
**have** *a-mem*: *enum a* ∈ *rep x*
**unfolding** *a-def*
**apply** (*rule LeastI-ex*)
**apply** (*insert ideal-rep* [*of x*])
**apply** (*drule idealD1*)
**apply** (*clarify*)
**subgoal for** *a* **by** (*rule exI* [**where** *x*=*count a*]) *simp*
**done**
**have** *b*: $\bigwedge i.$ *P i* ⟹ *enum i* ∈ *rep x*
⟹ *enum* (*b i*) ∈ *rep x* ∧ ¬ *enum* (*b i*) ≼ *enum i*
**unfolding** *P-def b-def* **by** (*erule LeastI2-ex*, *simp*)
**have** *c*: $\bigwedge i\ j.$ *enum i* ∈ *rep x* ⟹ *enum j* ∈ *rep x*
⟹ *enum* (*c i j*) ∈ *rep x* ∧ *enum i* ≼ *enum* (*c i j*) ∧ *enum j* ≼ *enum* (*c i j*)
**unfolding** *c-def*
**apply** (*drule* (*1*) *idealD2* [*OF ideal-rep*], *clarify*)
**subgoal for** ... *z* **by** (*rule LeastI2* [**where** *a*=*count z*], *simp*, *simp*)
**done**
**have** *X-mem*: *enum* (*X n*) ∈ *rep x* **for** *n*
**proof** (*induct n*)
**case** *0*
**then show** *?case* **by** (*simp add*: *X-0 a-mem*)

**next**
**case** (*Suc n*)
**with** *b c* **show** *?case* **by** (*auto simp*: *X-Suc*)
**qed**
**have** *X-chain*: $\bigwedge n.\ enum\ (X\ n) \preceq enum\ (X\ (Suc\ n))$
**apply** (*clarsimp simp add*: *X-Suc r-refl*)
**apply** (*simp add*: *b c X-mem*)
**done**
**have** *less-b*: $\bigwedge n\ i.\ n < b\ i \Longrightarrow enum\ n \in rep\ x \Longrightarrow enum\ n \preceq enum\ i$
**unfolding** *b-def* **by** (*drule not-less-Least*, *simp*)
**have** *X-covers*: $\forall k{\leq}n.\ enum\ k \in rep\ x \longrightarrow enum\ k \preceq enum\ (X\ n)$ **for** *n*
**proof** (*induct n*)
**case** *0*
**then show** *?case*
**apply** (*clarsimp simp add*: *X-0 a-def*)
**apply** (*drule Least-le* [**where** *k=0*], *simp add*: *r-refl*)
**done**
**next**
**case** (*Suc n*)
**then show** *?case*
**apply** *clarsimp*
**apply** (*erule le-SucE*)
**apply** (*rule r-trans* [*OF - X-chain*], *simp*)
**apply** (*cases P* (*X n*), *simp add*: *X-Suc*)
**apply** (*rule linorder-cases* [**where** *x=b* (*X n*) **and** *y=Suc n*])
**apply** (*simp only*: *less-Suc-eq-le*)
**apply** (*drule spec*, *drule* (*1*) *mp*, *simp add*: *b X-mem*)
**apply** (*simp add*: *c X-mem*)
**apply** (*drule* (*1*) *less-b*)
**apply** (*erule r-trans*)
**apply** (*simp add*: *b c X-mem*)
**apply** (*simp add*: *X-Suc*)
**apply** (*simp add*: *P-def*)
**done**
**qed**
**have** *1*: $\forall i.\ enum\ (X\ i) \preceq enum\ (X\ (Suc\ i))$
**by** (*simp add*: *X-chain*)
**have** $x = (\bigsqcup n.\ principal\ (enum\ (X\ n)))$
**apply** (*simp add*: *eq-iff rep-lub 1 rep-principal*)
**apply** *auto*
**subgoal for** *a*
**apply** (*subgoal-tac* $\exists i.\ a = enum\ i$, *erule exE*)
**apply** (*rule-tac x=i* **in** *exI*, *simp add*: *X-covers*)
**apply** (*rule-tac x=count a* **in** *exI*, *simp*)
**done**
**subgoal**
**apply** (*erule idealD3* [*OF ideal-rep*])
**apply** (*rule X-mem*)
**done**

```
    done
  with 1 show ?thesis ..
qed

lemma principal-induct:
  assumes adm: adm P
  assumes P: ⋀a. P (principal a)
  shows P x
apply (rule obtain-principal-chain [of x])
apply (simp add: admD [OF adm] P)
done

lemma compact-imp-principal: compact x ⟹ ∃ a. x = principal a
apply (rule obtain-principal-chain [of x])
apply (drule adm-compact-neq [OF - cont-id])
apply (subgoal-tac chain (λi. principal (Y i)))
apply (drule (2) admD2, fast, simp)
done
```

## 20.4 Defining functions in terms of basis elements

```
definition
  extension :: ('a::type ⇒ 'c::cpo) ⇒ 'b → 'c where
  extension = (λf. (Λ x. lub (f ‘ rep x)))

lemma extension-lemma:
  fixes f :: 'a::type ⇒ 'c::cpo
  assumes f-mono: ⋀a b. a ⪯ b ⟹ f a ⊑ f b
  shows ∃ u. f ‘ rep x <<| u
proof −
  obtain Y where Y: ∀ i. Y i ⪯ Y (Suc i)
  and x: x = (⨆ i. principal (Y i))
    by (rule obtain-principal-chain [of x])
  have chain: chain (λi. f (Y i))
    by (rule chainI, simp add: f-mono Y)
  have rep-x: rep x = (⋃ n. {a. a ⪯ Y n})
    by (simp add: x rep-lub Y rep-principal)
  have f ‘ rep x <<| (⨆ n. f (Y n))
    apply (rule is-lubI)
     apply (rule ub-imageI)
    subgoal for a
      apply (clarsimp simp add: rep-x)
      apply (drule f-mono)
      apply (erule below-lub [OF chain])
      done
    apply (rule lub-below [OF chain])
    subgoal for ... n
      apply (drule ub-imageD [where x=Y n])
       apply (simp add: rep-x, fast intro: r-refl)
```

```
      apply assumption
      done
    done
  then show ?thesis ..
qed

lemma extension-beta:
  fixes f :: 'a::type ⇒ 'c::cpo
  assumes f-mono: ⋀a b. a ⊑ b ⟹ f a ⊑ f b
  shows extension f·x = lub (f ‘ rep x)
unfolding extension-def
proof (rule beta-cfun)
  have lub: ⋀x. ∃ u. f ‘ rep x <<| u
    using f-mono by (rule extension-lemma)
  show cont: cont (λx. lub (f ‘ rep x))
    apply (rule contI2)
     apply (rule monofunI)
     apply (rule is-lub-thelub-ex [OF lub ub-imageI])
     apply (rule is-ub-thelub-ex [OF lub imageI])
     apply (erule (1) subsetD [OF rep-mono])
    apply (rule is-lub-thelub-ex [OF lub ub-imageI])
    apply (simp add: rep-lub, clarify)
    apply (erule rev-below-trans [OF is-ub-thelub])
    apply (erule is-ub-thelub-ex [OF lub imageI])
    done
qed

lemma extension-principal:
  fixes f :: 'a::type ⇒ 'c::cpo
  assumes f-mono: ⋀a b. a ⊑ b ⟹ f a ⊑ f b
  shows extension f·(principal a) = f a
apply (subst extension-beta, erule f-mono)
apply (subst rep-principal)
apply (rule lub-eqI)
apply (rule is-lub-maximal)
apply (rule ub-imageI)
apply (simp add: f-mono)
apply (rule imageI)
apply (simp add: r-refl)
done

lemma extension-mono:
  assumes f-mono: ⋀a b. a ⊑ b ⟹ f a ⊑ f b
  assumes g-mono: ⋀a b. a ⊑ b ⟹ g a ⊑ g b
  assumes below: ⋀a. f a ⊑ g a
  shows extension f ⊑ extension g
  apply (rule cfun-belowI)
  apply (simp only: extension-beta f-mono g-mono)
  apply (rule is-lub-thelub-ex)
```

```
  apply (rule extension-lemma, erule f-mono)
 apply (rule ub-imageI)
 subgoal for x a
   apply (rule below-trans [OF below])
   apply (rule is-ub-thelub-ex)
    apply (rule extension-lemma, erule g-mono)
   apply (erule imageI)
   done
 done

lemma cont-extension:
 assumes f-mono: ⋀a b x. a ⊑ b ⟹ f x a ⊑ f x b
 assumes f-cont: ⋀a. cont (λx. f x a)
 shows cont (λx. extension (λa. f x a))
apply (rule contI2)
 apply (rule monofunI)
 apply (rule extension-mono, erule f-mono, erule f-mono)
 apply (erule cont2monofunE [OF f-cont])
apply (rule cfun-belowI)
apply (rule principal-induct, simp)
apply (simp only: contlub-cfun-fun)
apply (simp only: extension-principal f-mono)
apply (simp add: cont2contlubE [OF f-cont])
done

end

lemma (in preorder) typedef-ideal-completion:
 fixes Abs :: 'a set ⇒ 'b::cpo
 assumes type: type-definition Rep Abs {S. ideal S}
 assumes below: ⋀x y. x ⊑ y ⟷ Rep x ⊆ Rep y
 assumes principal: ⋀a. principal a = Abs {b. b ⊑ a}
 assumes countable: ∃f::'a ⇒ nat. inj f
 shows ideal-completion r principal Rep
proof
 interpret type-definition Rep Abs {S. ideal S} by fact
 fix a b :: 'a and x y :: 'b and Y :: nat ⇒ 'b
 show ideal (Rep x)
   using Rep [of x] by simp
 show chain Y ⟹ Rep (⨆ i. Y i) = (⋃ i. Rep (Y i))
   using type below by (rule typedef-ideal-rep-lub)
 show Rep (principal a) = {b. b ⊑ a}
   by (simp add: principal Abs-inverse ideal-principal)
 show Rep x ⊆ Rep y ⟹ x ⊑ y
   by (simp only: below)
 show ∃f::'a ⇒ nat. inj f
   by (rule countable)
qed
```

**end**

# 21 A universal bifinite domain

**theory** *Universal*
**imports** *Bifinite Completion HOL−Library.Nat-Bijection*
**begin**

**no-notation** *binomial* (**infixl** *choose 65*)

## 21.1 Basis for universal domain

### 21.1.1 Basis datatype

**type-synonym** *ubasis* = *nat*

**definition**
*node* :: *nat* ⇒ *ubasis* ⇒ *ubasis set* ⇒ *ubasis*
**where**
*node i a S* = *Suc* (*prod-encode* (*i*, *prod-encode* (*a*, *set-encode S*)))

**lemma** *node-not-0* [*simp*]: *node i a S* ≠ *0*
**unfolding** *node-def* **by** *simp*

**lemma** *node-gt-0* [*simp*]: *0* < *node i a S*
**unfolding** *node-def* **by** *simp*

**lemma** *node-inject* [*simp*]:
⟦*finite S*; *finite T*⟧
⟹ *node i a S* = *node j b T* ⟷ *i* = *j* ∧ *a* = *b* ∧ *S* = *T*
**unfolding** *node-def* **by** (*simp add*: *prod-encode-eq set-encode-eq*)

**lemma** *node-gt0*: *i* < *node i a S*
**unfolding** *node-def less-Suc-eq-le*
**by** (*rule le-prod-encode-1*)

**lemma** *node-gt1*: *a* < *node i a S*
**unfolding** *node-def less-Suc-eq-le*
**by** (*rule order-trans* [*OF le-prod-encode-1 le-prod-encode-2*])

**lemma** *nat-less-power2*: *n* < *2*^*n*
**by** (*fact less-exp*)

**lemma** *node-gt2*: ⟦*finite S*; *b* ∈ *S*⟧ ⟹ *b* < *node i a S*
**unfolding** *node-def less-Suc-eq-le set-encode-def*
**apply** (*rule order-trans* [*OF - le-prod-encode-2*])
**apply** (*rule order-trans* [*OF - le-prod-encode-2*])
**apply** (*rule order-trans* [**where** *y*=*sum* ((^) *2*) {*b*}])
**apply** (*simp add*: *nat-less-power2* [*THEN order-less-imp-le*])

**apply** (*erule sum-mono2*, *simp*, *simp*)
**done**

**lemma** *eq-prod-encode-pairI*:
  ⟦*fst* (*prod-decode x*) = *a*; *snd* (*prod-decode x*) = *b*⟧ ⟹ *x* = *prod-encode* (*a*, *b*)
**by** (*erule subst*, *erule subst*, *simp*)

**lemma** *node-cases*:
  **assumes** *1*: *x* = *0* ⟹ *P*
  **assumes** *2*: ⋀*i a S*. ⟦*finite S*; *x* = *node i a S*⟧ ⟹ *P*
  **shows** *P*
 **apply** (*cases x*)
  **apply** (*erule 1*)
 **apply** (*rule 2*)
  **apply** (*rule finite-set-decode*)
 **apply** (*simp add*: *node-def*)
 **apply** (*rule eq-prod-encode-pairI* [*OF refl*])
 **apply** (*rule eq-prod-encode-pairI* [*OF refl refl*])
**done**

**lemma** *node-induct*:
  **assumes** *1*: *P 0*
  **assumes** *2*: ⋀*i a S*. ⟦*P a*; *finite S*; ∀ *b*∈*S*. *P b*⟧ ⟹ *P* (*node i a S*)
  **shows** *P x*
 **apply** (*induct x rule*: *nat-less-induct*)
 **apply** (*case-tac n rule*: *node-cases*)
  **apply** (*simp add*: *1*)
 **apply** (*simp add*: *2 node-gt1 node-gt2*)
**done**

### 21.1.2 Basis ordering

**inductive**
  *ubasis-le* :: *nat* ⇒ *nat* ⇒ *bool*
**where**
  *ubasis-le-refl*: *ubasis-le a a*
| *ubasis-le-trans*:
    ⟦*ubasis-le a b*; *ubasis-le b c*⟧ ⟹ *ubasis-le a c*
| *ubasis-le-lower*:
    *finite S* ⟹ *ubasis-le a* (*node i a S*)
| *ubasis-le-upper*:
    ⟦*finite S*; *b* ∈ *S*; *ubasis-le a b*⟧ ⟹ *ubasis-le* (*node i a S*) *b*

**lemma** *ubasis-le-minimal*: *ubasis-le 0 x*
**apply** (*induct x rule*: *node-induct*)
**apply** (*rule ubasis-le-refl*)
**apply** (*erule ubasis-le-trans*)
**apply** (*erule ubasis-le-lower*)
**done**

**interpretation** *udom*: *preorder ubasis-le*
**apply** *standard*
**apply** (*rule ubasis-le-refl*)
**apply** (*erule* (*1*) *ubasis-le-trans*)
**done**

### 21.1.3 Generic take function

**function**
  *ubasis-until* :: (*ubasis* ⇒ *bool*) ⇒ *ubasis* ⇒ *ubasis*
**where**
  *ubasis-until P 0 = 0*
| *finite S* ⟹ *ubasis-until P* (*node i a S*) =
    (*if P* (*node i a S*) *then node i a S else ubasis-until P a*)
   **apply** *clarify*
   **apply** (*rule-tac x*=*b* **in** *node-cases*)
    **apply** *simp-all*
**done**

**termination** *ubasis-until*
**apply** (*relation measure snd*)
**apply** (*rule wf-measure*)
**apply** (*simp add*: *node-gt1*)
**done**

**lemma** *ubasis-until*: *P 0* ⟹ *P* (*ubasis-until P x*)
**by** (*induct x rule*: *node-induct*) *simp-all*

**lemma** *ubasis-until'*: *0 < ubasis-until P x* ⟹ *P* (*ubasis-until P x*)
**by** (*induct x rule*: *node-induct*) *auto*

**lemma** *ubasis-until-same*: *P x* ⟹ *ubasis-until P x = x*
**by** (*induct x rule*: *node-induct*) *simp-all*

**lemma** *ubasis-until-idem*:
  *P 0* ⟹ *ubasis-until P* (*ubasis-until P x*) = *ubasis-until P x*
**by** (*rule ubasis-until-same* [*OF ubasis-until*])

**lemma** *ubasis-until-0*:
  ∀ *x*. *x* ≠ *0* ⟶ ¬ *P x* ⟹ *ubasis-until P x = 0*
**by** (*induct x rule*: *node-induct*) *simp-all*

**lemma** *ubasis-until-less*: *ubasis-le* (*ubasis-until P x*) *x*
**apply** (*induct x rule*: *node-induct*)
**apply** (*simp add*: *ubasis-le-refl*)
**apply** (*simp add*: *ubasis-le-refl*)
**apply** (*rule impI*)
**apply** (*erule ubasis-le-trans*)

**apply** (*erule ubasis-le-lower*)
**done**

**lemma** *ubasis-until-chain*:
  **assumes** *PQ*: $\bigwedge x.\ P\ x \Longrightarrow Q\ x$
  **shows** *ubasis-le* (*ubasis-until P x*) (*ubasis-until Q x*)
**apply** (*induct x rule*: *node-induct*)
**apply** (*simp add*: *ubasis-le-refl*)
**apply** (*simp add*: *ubasis-le-refl*)
**apply** (*simp add*: *PQ*)
**apply** *clarify*
**apply** (*rule ubasis-le-trans*)
**apply** (*rule ubasis-until-less*)
**apply** (*erule ubasis-le-lower*)
**done**

**lemma** *ubasis-until-mono*:
  **assumes** $\bigwedge i\ a\ S\ b.\ [\![\mathit{finite}\ S;\ P\ (\mathit{node}\ i\ a\ S);\ b \in S;\ \mathit{ubasis\text{-}le}\ a\ b]\!] \Longrightarrow P\ b$
  **shows** *ubasis-le a b* $\Longrightarrow$ *ubasis-le* (*ubasis-until P a*) (*ubasis-until P b*)
**proof** (*induct set*: *ubasis-le*)
  **case** (*ubasis-le-refl a*) **show** *?case* **by** (*rule ubasis-le.ubasis-le-refl*)
**next**
  **case** (*ubasis-le-trans a b c*) **thus** *?case* **by** − (*rule ubasis-le.ubasis-le-trans*)
**next**
  **case** (*ubasis-le-lower S a i*) **thus** *?case*
    **apply** (*clarsimp simp add*: *ubasis-le-refl*)
    **apply** (*rule ubasis-le-trans* [*OF ubasis-until-less*])
    **apply** (*erule ubasis-le.ubasis-le-lower*)
    **done**
**next**
  **case** (*ubasis-le-upper S b a i*) **thus** *?case*
    **apply** *clarsimp*
    **apply** (*subst ubasis-until-same*)
     **apply** (*erule* (*3*) *assms*)
    **apply** (*erule* (*2*) *ubasis-le.ubasis-le-upper*)
    **done**
**qed**

**lemma** *finite-range-ubasis-until*:
  *finite* $\{x.\ P\ x\}$ $\Longrightarrow$ *finite* (*range* (*ubasis-until P*))
**apply** (*rule finite-subset* [**where** *B*=*insert 0* $\{x.\ P\ x\}$])
**apply** (*clarsimp simp add*: *ubasis-until′*)
**apply** *simp*
**done**

## 21.2 Defining the universal domain by ideal completion

**typedef** *udom* = $\{S.\ \mathit{udom.ideal}\ S\}$
**by** (*rule udom.ex-ideal*)

**instantiation** *udom* :: *below*
**begin**

**definition**
*x* ⊑ *y* ⟷ *Rep-udom x* ⊆ *Rep-udom y*

**instance ..**
**end**

**instance** *udom* :: *po*
**using** *type-definition-udom below-udom-def*
**by** (*rule udom.typedef-ideal-po*)

**instance** *udom* :: *cpo*
**using** *type-definition-udom below-udom-def*
**by** (*rule udom.typedef-ideal-cpo*)

**definition**
*udom-principal* :: *nat* ⇒ *udom* **where**
*udom-principal t* = *Abs-udom* {*u*. *ubasis-le u t*}

**lemma** *ubasis-countable*: ∃*f*::*ubasis* ⇒ *nat*. *inj f*
**by** (*rule exI*, *rule inj-on-id*)

**interpretation** *udom*:
*ideal-completion ubasis-le udom-principal Rep-udom*
**using** *type-definition-udom below-udom-def*
**using** *udom-principal-def ubasis-countable*
**by** (*rule udom.typedef-ideal-completion*)

Universal domain is pointed

**lemma** *udom-minimal*: *udom-principal 0* ⊑ *x*
**apply** (*induct x rule*: *udom.principal-induct*)
**apply** (*simp*, *simp add*: *ubasis-le-minimal*)
**done**

**instance** *udom* :: *pcpo*
**by** *intro-classes* (*fast intro*: *udom-minimal*)

**lemma** *inst-udom-pcpo*: ⊥ = *udom-principal 0*
**by** (*rule udom-minimal* [*THEN bottomI*, *symmetric*])

## 21.3 Compact bases of domains

**typedef** *'a compact-basis* = {*x*::*'a*::*pcpo*. *compact x*}
**by** *auto*

**lemma** *Rep-compact-basis'* [*simp*]: *compact* (*Rep-compact-basis a*)

**by** (*rule Rep-compact-basis* [*unfolded mem-Collect-eq*])

**lemma** *Abs-compact-basis-inverse'* [*simp*]:
*compact x* $\Longrightarrow$ *Rep-compact-basis* (*Abs-compact-basis x*) = *x*
**by** (*rule Abs-compact-basis-inverse* [*unfolded mem-Collect-eq*])

**instantiation** *compact-basis* :: (*pcpo*) *below*
**begin**

**definition**
*compact-le-def*:
($\sqsubseteq$) $\equiv$ ($\lambda x\ y.$ *Rep-compact-basis x* $\sqsubseteq$ *Rep-compact-basis y*)

**instance ..**
**end**

**instance** *compact-basis* :: (*pcpo*) *po*
**using** *type-definition-compact-basis compact-le-def*
**by** (*rule typedef-po*)

**definition**
*approximants* :: *'a* $\Rightarrow$ *'a compact-basis set* **where**
*approximants* = ($\lambda x.$ {*a. Rep-compact-basis a* $\sqsubseteq$ *x*})

**definition**
*compact-bot* :: *'a*::*pcpo compact-basis* **where**
*compact-bot* = *Abs-compact-basis* $\bot$

**lemma** *Rep-compact-bot* [*simp*]: *Rep-compact-basis compact-bot* = $\bot$
**unfolding** *compact-bot-def* **by** *simp*

**lemma** *compact-bot-minimal* [*simp*]: *compact-bot* $\sqsubseteq$ *a*
**unfolding** *compact-le-def Rep-compact-bot* **by** *simp*

## 21.4 Universality of *udom*

We use a locale to parameterize the construction over a chain of approx functions on the type to be embedded.

**locale** *bifinite-approx-chain* =
*approx-chain approx* **for** *approx* :: *nat* $\Rightarrow$ *'a*::*bifinite* $\rightarrow$ *'a*
**begin**

### 21.4.1 Choosing a maximal element from a finite set

**lemma** *finite-has-maximal*:
**fixes** *A* :: *'a compact-basis set*
**shows** $[\![$*finite A*; *A* $\neq$ {}$]\!] \Longrightarrow \exists\, x{\in}A.\ \forall\, y{\in}A.\ x \sqsubseteq y \longrightarrow x = y$
**proof** (*induct rule*: *finite-ne-induct*)
**case** (*singleton x*)

```
    show ?case by simp
next
  case (insert a A)
  from ‹∃ x∈A. ∀ y∈A. x ⊑ y ⟶ x = y›
  obtain x where x: x ∈ A
          and x-eq: ⋀y. ⟦y ∈ A; x ⊑ y⟧ ⟹ x = y by fast
  show ?case
  proof (intro bexI ballI impI)
    fix y
    assume y ∈ insert a A and (if x ⊑ a then a else x) ⊑ y
    thus (if x ⊑ a then a else x) = y
      apply auto
      apply (frule (1) below-trans)
      apply (frule (1) x-eq)
      apply (rule below-antisym, assumption)
      apply simp
      apply (erule (1) x-eq)
      done
  next
    show (if x ⊑ a then a else x) ∈ insert a A
      by (simp add: x)
  qed
qed

definition
  choose :: 'a compact-basis set ⇒ 'a compact-basis
where
  choose A = (SOME x. x ∈ {x∈A. ∀ y∈A. x ⊑ y ⟶ x = y})

lemma choose-lemma:
  ⟦finite A; A ≠ {}⟧ ⟹ choose A ∈ {x∈A. ∀ y∈A. x ⊑ y ⟶ x = y}
unfolding choose-def
apply (rule someI-ex)
apply (frule (1) finite-has-maximal, fast)
done

lemma maximal-choose:
  ⟦finite A; y ∈ A; choose A ⊑ y⟧ ⟹ choose A = y
apply (cases A = {}, simp)
apply (frule (1) choose-lemma, simp)
done

lemma choose-in: ⟦finite A; A ≠ {}⟧ ⟹ choose A ∈ A
by (frule (1) choose-lemma, simp)

function
  choose-pos :: 'a compact-basis set ⇒ 'a compact-basis ⇒ nat
where
  choose-pos A x =
```

```
    (if finite A ∧ x ∈ A ∧ x ≠ choose A
      then Suc (choose-pos (A − {choose A}) x) else 0)
by auto

termination choose-pos
apply (relation measure (card ∘ fst), simp)
apply clarsimp
apply (rule card-Diff1-less)
apply assumption
apply (erule choose-in)
apply clarsimp
done

declare choose-pos.simps [simp del]

lemma choose-pos-choose: finite A ⟹ choose-pos A (choose A) = 0
by (simp add: choose-pos.simps)

lemma inj-on-choose-pos [OF refl]:
  ⟦card A = n; finite A⟧ ⟹ inj-on (choose-pos A) A
 apply (induct n arbitrary: A)
  apply simp
 apply (case-tac A = {}, simp)
 apply (frule (1) choose-in)
 apply (rule inj-onI)
 apply (drule-tac x=A − {choose A} in meta-spec, simp)
 apply (simp add: choose-pos.simps)
 apply (simp split: if-split-asm)
 apply (erule (1) inj-onD, simp, simp)
done

lemma choose-pos-bounded [OF refl]:
  ⟦card A = n; finite A; x ∈ A⟧ ⟹ choose-pos A x < n
apply (induct n arbitrary: A)
apply simp
 apply (case-tac A = {}, simp)
 apply (frule (1) choose-in)
apply (subst choose-pos.simps)
apply simp
done

lemma choose-pos-lessD:
  ⟦choose-pos A x < choose-pos A y; finite A; x ∈ A; y ∈ A⟧ ⟹ x ⊑ y
 apply (induct A x arbitrary: y rule: choose-pos.induct)
 apply simp
 apply (case-tac x = choose A)
  apply simp
  apply (rule notI)
  apply (frule (2) maximal-choose)
```

**apply** *simp*
**apply** (*case-tac y = choose A*)
**apply** (*simp add*: *choose-pos-choose*)
**apply** (*drule-tac x=y* **in** *meta-spec*)
**apply** *simp*
**apply** (*erule meta-mp*)
**apply** (*simp add*: *choose-pos.simps*)
**done**

### 21.4.2 Compact basis take function

**primrec**
*cb-take* :: *nat* ⇒ *'a compact-basis* ⇒ *'a compact-basis* **where**
*cb-take 0* = (λ*x*. *compact-bot*)
| *cb-take* (*Suc n*) = (λ*a*. *Abs-compact-basis* (*approx n*·(*Rep-compact-basis a*)))

**declare** *cb-take.simps* [*simp del*]

**lemma** *cb-take-zero* [*simp*]: *cb-take 0 a* = *compact-bot*
**by** (*simp only*: *cb-take.simps*)

**lemma** *Rep-cb-take*:
*Rep-compact-basis* (*cb-take* (*Suc n*) *a*) = *approx n*·(*Rep-compact-basis a*)
**by** (*simp add*: *cb-take.simps*(*2*))

**lemmas** *approx-Rep-compact-basis* = *Rep-cb-take* [*symmetric*]

**lemma** *cb-take-covers*: ∃ *n*. *cb-take n x* = *x*
**apply** (*subgoal-tac* ∃ *n*. *cb-take* (*Suc n*) *x* = *x*, *fast*)
**apply** (*simp add*: *Rep-compact-basis-inject* [*symmetric*])
**apply** (*simp add*: *Rep-cb-take*)
**apply** (*rule compact-eq-approx*)
**apply** (*rule Rep-compact-basis'*)
**done**

**lemma** *cb-take-less*: *cb-take n x* ⊑ *x*
**unfolding** *compact-le-def*
**by** (*cases n*, *simp*, *simp add*: *Rep-cb-take approx-below*)

**lemma** *cb-take-idem*: *cb-take n* (*cb-take n x*) = *cb-take n x*
**unfolding** *Rep-compact-basis-inject* [*symmetric*]
**by** (*cases n*, *simp*, *simp add*: *Rep-cb-take approx-idem*)

**lemma** *cb-take-mono*: *x* ⊑ *y* ⟹ *cb-take n x* ⊑ *cb-take n y*
**unfolding** *compact-le-def*
**by** (*cases n*, *simp*, *simp add*: *Rep-cb-take monofun-cfun-arg*)

**lemma** *cb-take-chain-le*: *m* ≤ *n* ⟹ *cb-take m x* ⊑ *cb-take n x*
**unfolding** *compact-le-def*

**apply** (*cases m*, *simp*, *cases n*, *simp*)
**apply** (*simp add*: *Rep-cb-take*, *rule chain-mono*, *simp*, *simp*)
**done**

**lemma** *finite-range-cb-take*: *finite* (*range* (*cb-take n*))
**apply** (*cases n*)
**apply** (*subgoal-tac range* (*cb-take 0*) = {*compact-bot*}, *simp*, *force*)
**apply** (*rule finite-imageD* [**where** *f*=*Rep-compact-basis*])
**apply** (*rule finite-subset* [**where** *B*=*range* (λ*x*. *approx* (*n* − *1*)·*x*)])
**apply** (*clarsimp simp add*: *Rep-cb-take*)
**apply** (*rule finite-range-approx*)
**apply** (*rule inj-onI*, *simp add*: *Rep-compact-basis-inject*)
**done**

### 21.4.3 Rank of basis elements

**definition**
*rank* :: *'a compact-basis* ⇒ *nat*
**where**
*rank x* = (*LEAST n*. *cb-take n x* = *x*)

**lemma** *compact-approx-rank*: *cb-take* (*rank x*) *x* = *x*
**unfolding** *rank-def*
**apply** (*rule LeastI-ex*)
**apply** (*rule cb-take-covers*)
**done**

**lemma** *rank-leD*: *rank x* ≤ *n* ⟹ *cb-take n x* = *x*
**apply** (*rule below-antisym* [*OF cb-take-less*])
**apply** (*subst compact-approx-rank* [*symmetric*])
**apply** (*erule cb-take-chain-le*)
**done**

**lemma** *rank-leI*: *cb-take n x* = *x* ⟹ *rank x* ≤ *n*
**unfolding** *rank-def* **by** (*rule Least-le*)

**lemma** *rank-le-iff*: *rank x* ≤ *n* ⟷ *cb-take n x* = *x*
**by** (*rule iffI* [*OF rank-leD rank-leI*])

**lemma** *rank-compact-bot* [*simp*]: *rank compact-bot* = *0*
**using** *rank-leI* [*of 0 compact-bot*] **by** *simp*

**lemma** *rank-eq-0-iff* [*simp*]: *rank x* = *0* ⟷ *x* = *compact-bot*
**using** *rank-le-iff* [*of x 0*] **by** *auto*

**definition**
*rank-le* :: *'a compact-basis* ⇒ *'a compact-basis set*
**where**
*rank-le x* = {*y*. *rank y* ≤ *rank x*}

**definition**
*rank-lt* :: *'a compact-basis* ⇒ *'a compact-basis set*
**where**
*rank-lt x* = {*y*. *rank y* < *rank x*}

**definition**
*rank-eq* :: *'a compact-basis* ⇒ *'a compact-basis set*
**where**
*rank-eq x* = {*y*. *rank y* = *rank x*}

**lemma** *rank-eq-cong*: *rank x* = *rank y* ⟹ *rank-eq x* = *rank-eq y*
**unfolding** *rank-eq-def* **by** *simp*

**lemma** *rank-lt-cong*: *rank x* = *rank y* ⟹ *rank-lt x* = *rank-lt y*
**unfolding** *rank-lt-def* **by** *simp*

**lemma** *rank-eq-subset*: *rank-eq x* ⊆ *rank-le x*
**unfolding** *rank-eq-def rank-le-def* **by** *auto*

**lemma** *rank-lt-subset*: *rank-lt x* ⊆ *rank-le x*
**unfolding** *rank-lt-def rank-le-def* **by** *auto*

**lemma** *finite-rank-le*: *finite* (*rank-le x*)
**unfolding** *rank-le-def*
**apply** (*rule finite-subset* [**where** *B*=*range* (*cb-take* (*rank x*))])
**apply** *clarify*
**apply** (*rule range-eqI*)
**apply** (*erule rank-leD* [*symmetric*])
**apply** (*rule finite-range-cb-take*)
**done**

**lemma** *finite-rank-eq*: *finite* (*rank-eq x*)
**by** (*rule finite-subset* [*OF rank-eq-subset finite-rank-le*])

**lemma** *finite-rank-lt*: *finite* (*rank-lt x*)
**by** (*rule finite-subset* [*OF rank-lt-subset finite-rank-le*])

**lemma** *rank-lt-Int-rank-eq*: *rank-lt x* ∩ *rank-eq x* = {}
**unfolding** *rank-lt-def rank-eq-def rank-le-def* **by** *auto*

**lemma** *rank-lt-Un-rank-eq*: *rank-lt x* ∪ *rank-eq x* = *rank-le x*
**unfolding** *rank-lt-def rank-eq-def rank-le-def* **by** *auto*

### 21.4.4 Sequencing basis elements

**definition**
*place* :: *'a compact-basis* ⇒ *nat*
**where**

*place x = card (rank-lt x) + choose-pos (rank-eq x) x*

**lemma** *place-bounded*: *place x < card (rank-le x)*
**unfolding** *place-def*
**apply** (*rule ord-less-eq-trans*)
**apply** (*rule add-strict-left-mono*)
**apply** (*rule choose-pos-bounded*)
**apply** (*rule finite-rank-eq*)
**apply** (*simp add*: *rank-eq-def*)
**apply** (*subst card-Un-disjoint* [*symmetric*])
**apply** (*rule finite-rank-lt*)
**apply** (*rule finite-rank-eq*)
**apply** (*rule rank-lt-Int-rank-eq*)
**apply** (*simp add*: *rank-lt-Un-rank-eq*)
**done**

**lemma** *place-ge*: *card (rank-lt x) ≤ place x*
**unfolding** *place-def* **by** *simp*

**lemma** *place-rank-mono*:
**fixes** *x y* :: *'a compact-basis*
**shows** *rank x < rank y ⟹ place x < place y*
**apply** (*rule less-le-trans* [*OF place-bounded*])
**apply** (*rule order-trans* [*OF - place-ge*])
**apply** (*rule card-mono*)
**apply** (*rule finite-rank-lt*)
**apply** (*simp add*: *rank-le-def rank-lt-def subset-eq*)
**done**

**lemma** *place-eqD*: *place x = place y ⟹ x = y*
**apply** (*rule linorder-cases* [**where** *x=rank x* **and** *y=rank y*])
**apply** (*drule place-rank-mono*, *simp*)
**apply** (*simp add*: *place-def*)
**apply** (*rule inj-on-choose-pos* [**where** *A=rank-eq x*, *THEN inj-onD*])
**apply** (*rule finite-rank-eq*)
**apply** (*simp cong*: *rank-lt-cong rank-eq-cong*)
**apply** (*simp add*: *rank-eq-def*)
**apply** (*simp add*: *rank-eq-def*)
**apply** (*drule place-rank-mono*, *simp*)
**done**

**lemma** *inj-place*: *inj place*
**by** (*rule inj-onI*, *erule place-eqD*)

### 21.4.5 Embedding and projection on basis elements

**definition**
*sub* :: *'a compact-basis ⇒ 'a compact-basis*
**where**

*sub x = (case rank x of 0 ⇒ compact-bot | Suc k ⇒ cb-take k x)*

**lemma** *rank-sub-less*: *x ≠ compact-bot ⟹ rank (sub x) < rank x*
**unfolding** *sub-def*
**apply** (*cases rank x*, *simp*)
**apply** (*simp add*: *less-Suc-eq-le*)
**apply** (*rule rank-leI*)
**apply** (*rule cb-take-idem*)
**done**

**lemma** *place-sub-less*: *x ≠ compact-bot ⟹ place (sub x) < place x*
**apply** (*rule place-rank-mono*)
**apply** (*erule rank-sub-less*)
**done**

**lemma** *sub-below*: *sub x ⊑ x*
**unfolding** *sub-def* **by** (*cases rank x*, *simp-all add*: *cb-take-less*)

**lemma** *rank-less-imp-below-sub*: *⟦x ⊑ y; rank x < rank y⟧ ⟹ x ⊑ sub y*
**unfolding** *sub-def*
**apply** (*cases rank y*, *simp*)
**apply** (*simp add*: *less-Suc-eq-le*)
**apply** (*subgoal-tac cb-take nat x ⊑ cb-take nat y*)
**apply** (*simp add*: *rank-leD*)
**apply** (*erule cb-take-mono*)
**done**

**function** *basis-emb* :: *'a compact-basis ⇒ ubasis*
**where** *basis-emb x = (if x = compact-bot then 0 else*
*node (place x) (basis-emb (sub x))*
*(basis-emb ‘ {y. place y < place x ∧ x ⊑ y}))*
**by** *simp-all*

**termination** *basis-emb*
**by** (*relation measure place*) (*simp-all add*: *place-sub-less*)

**declare** *basis-emb.simps* [*simp del*]

**lemma** *basis-emb-compact-bot* [*simp*]:
*basis-emb compact-bot = 0*
**using** *basis-emb.simps* [*of compact-bot*] **by** *simp*

**lemma** *basis-emb-rec*:
*basis-emb x = node (place x) (basis-emb (sub x)) (basis-emb ‘ {y. place y < place x ∧ x ⊑ y})*
**if** *x ≠ compact-bot*
**using** *that basis-emb.simps* [*of x*] **by** *simp*

**lemma** *basis-emb-eq-0-iff* [*simp*]:

```
  basis-emb x = 0 ⟷ x = compact-bot
  by (cases x = compact-bot) (simp-all add: basis-emb-rec)

lemma fin1: finite {y. place y < place x ∧ x ⊑ y}
apply (subst Collect-conj-eq)
apply (rule finite-Int)
apply (rule disjI1)
apply (subgoal-tac finite (place −` {n. n < place x}), simp)
apply (rule finite-vimageI [OF - inj-place])
apply (simp add: lessThan-def [symmetric])
done

lemma fin2: finite (basis-emb ` {y. place y < place x ∧ x ⊑ y})
by (rule finite-imageI [OF fin1])

lemma rank-place-mono:
  ⟦place x < place y; x ⊑ y⟧ ⟹ rank x < rank y
apply (rule linorder-cases, assumption)
apply (simp add: place-def cong: rank-lt-cong rank-eq-cong)
apply (drule choose-pos-lessD)
apply (rule finite-rank-eq)
apply (simp add: rank-eq-def)
apply (simp add: rank-eq-def)
apply simp
apply (drule place-rank-mono, simp)
done

lemma basis-emb-mono:
  x ⊑ y ⟹ ubasis-le (basis-emb x) (basis-emb y)
proof (induct max (place x) (place y) arbitrary: x y rule: less-induct)
  case less
  show ?case proof (rule linorder-cases)
    assume place x < place y
    then have rank x < rank y
      using ‹x ⊑ y› by (rule rank-place-mono)
    with ‹place x < place y› show ?case
      apply (case-tac y = compact-bot, simp)
      apply (simp add: basis-emb.simps [of y])
      apply (rule ubasis-le-trans [OF - ubasis-le-lower [OF fin2]])
      apply (rule less)
       apply (simp add: less-max-iff-disj)
       apply (erule place-sub-less)
      apply (erule rank-less-imp-below-sub [OF ‹x ⊑ y›])
      done
  next
    assume place x = place y
    hence x = y by (rule place-eqD)
    thus ?case by (simp add: ubasis-le-refl)
  next
```

```
    assume place x > place y
    with ‹x ⊑ y› show ?case
     apply (case-tac x = compact-bot, simp add: ubasis-le-minimal)
     apply (simp add: basis-emb.simps [of x])
     apply (rule ubasis-le-upper [OF fin2], simp)
     apply (rule less)
      apply (simp add: less-max-iff-disj)
      apply (erule place-sub-less)
     apply (erule rev-below-trans)
     apply (rule sub-below)
     done
  qed
qed

lemma inj-basis-emb: inj basis-emb
proof (rule injI)
  fix x y
  assume basis-emb x = basis-emb y
  then show x = y
    by (cases x = compact-bot ∨ y = compact-bot) (auto simp add: basis-emb-rec
fin2 place-eqD)
qed

definition
  basis-prj :: ubasis ⇒ 'a compact-basis
where
  basis-prj x = inv basis-emb
    (ubasis-until (λx. x ∈ range (basis-emb :: 'a compact-basis ⇒ ubasis)) x)

lemma basis-prj-basis-emb: ⋀x. basis-prj (basis-emb x) = x
unfolding basis-prj-def
 apply (subst ubasis-until-same)
  apply (rule rangeI)
 apply (rule inv-f-f)
 apply (rule inj-basis-emb)
done

lemma basis-prj-node:
  ⟦finite S; node i a S ∉ range (basis-emb :: 'a compact-basis ⇒ nat)⟧
    ⟹ basis-prj (node i a S) = (basis-prj a :: 'a compact-basis)
unfolding basis-prj-def by simp

lemma basis-prj-0: basis-prj 0 = compact-bot
apply (subst basis-emb-compact-bot [symmetric])
apply (rule basis-prj-basis-emb)
done

lemma node-eq-basis-emb-iff:
  finite S ⟹ node i a S = basis-emb x ⟷
```

```
    x ≠ compact-bot ∧ i = place x ∧ a = basis-emb (sub x) ∧
        S = basis-emb ‘ {y. place y < place x ∧ x ⊑ y}
apply (cases x = compact-bot, simp)
apply (simp add: basis-emb.simps [of x])
apply (simp add: fin2)
done

lemma basis-prj-mono: ubasis-le a b ⟹ basis-prj a ⊑ basis-prj b
proof (induct a b rule: ubasis-le.induct)
  case (ubasis-le-refl a) show ?case by (rule below-refl)
next
  case (ubasis-le-trans a b c) thus ?case by − (rule below-trans)
next
  case (ubasis-le-lower S a i) thus ?case
    apply (cases node i a S ∈ range (basis-emb :: 'a compact-basis ⇒ nat))
     apply (erule rangeE, rename-tac x)
     apply (simp add: basis-prj-basis-emb)
     apply (simp add: node-eq-basis-emb-iff)
     apply (simp add: basis-prj-basis-emb)
     apply (rule sub-below)
    apply (simp add: basis-prj-node)
    done
next
  case (ubasis-le-upper S b a i) thus ?case
    apply (cases node i a S ∈ range (basis-emb :: 'a compact-basis ⇒ nat))
     apply (erule rangeE, rename-tac x)
     apply (simp add: basis-prj-basis-emb)
     apply (clarsimp simp add: node-eq-basis-emb-iff)
     apply (simp add: basis-prj-basis-emb)
    apply (simp add: basis-prj-node)
    done
qed

lemma basis-emb-prj-less: ubasis-le (basis-emb (basis-prj x)) x
unfolding basis-prj-def
 apply (subst f-inv-into-f [where f=basis-emb])
  apply (rule ubasis-until)
  apply (rule range-eqI [where x=compact-bot])
  apply simp
 apply (rule ubasis-until-less)
done

lemma ideal-completion:
  ideal-completion below Rep-compact-basis (approximants :: 'a ⇒ -)
proof
  fix w :: 'a
  show below.ideal (approximants w)
  proof (rule below.idealI)
    have Abs-compact-basis (approx 0·w) ∈ approximants w
```

```
    by (simp add: approximants-def approx-below)
   thus ∃ x. x ∈ approximants w ..
  next
   fix x y :: 'a compact-basis
   assume x: x ∈ approximants w and y: y ∈ approximants w
   obtain i where i: approx i·(Rep-compact-basis x) = Rep-compact-basis x
    using compact-eq-approx Rep-compact-basis' by fast
   obtain j where j: approx j·(Rep-compact-basis y) = Rep-compact-basis y
    using compact-eq-approx Rep-compact-basis' by fast
   let ?z = Abs-compact-basis (approx (max i j)·w)
   have ?z ∈ approximants w
    by (simp add: approximants-def approx-below)
   moreover from x y have x ⊑ ?z ∧ y ⊑ ?z
    by (simp add: approximants-def compact-le-def)
     (metis i j monofun-cfun chain-mono chain-approx max.cobounded1 max.cobounded2)
   ultimately show ∃ z ∈ approximants w. x ⊑ z ∧ y ⊑ z ..
  next
   fix x y :: 'a compact-basis
   assume x ⊑ y y ∈ approximants w thus x ∈ approximants w
    unfolding approximants-def compact-le-def
    by (auto elim: below-trans)
  qed
next
  fix Y :: nat ⇒ 'a
  assume chain Y
  thus approximants (⨆ i. Y i) = (⋃ i. approximants (Y i))
   unfolding approximants-def
   by (auto simp add: compact-below-lub-iff)
next
  fix a :: 'a compact-basis
  show approximants (Rep-compact-basis a) = {b. b ⊑ a}
   unfolding approximants-def compact-le-def ..
next
  fix x y :: 'a
  assume approximants x ⊆ approximants y
  hence ∀ z. compact z ⟶ z ⊑ x ⟶ z ⊑ y
   by (simp add: approximants-def subset-eq)
     (metis Abs-compact-basis-inverse')
  hence (⨆ i. approx i·x) ⊑ y
   by (simp add: lub-below approx-below)
  thus x ⊑ y
   by (simp add: lub-distribs)
next
  show ∃ f::'a compact-basis ⇒ nat. inj f
   by (rule exI, rule inj-place)
qed

end
```

**interpretation** *compact-basis*:
*ideal-completion below Rep-compact-basis*
*approximants* :: *'a*::*bifinite* ⇒ *'a compact-basis set*
**proof** −
**obtain** *a* :: *nat* ⇒ *'a* → *'a* **where** *approx-chain a*
**using** *bifinite* **..**
**hence** *bifinite-approx-chain a*
**unfolding** *bifinite-approx-chain-def* **.**
**thus** *ideal-completion below Rep-compact-basis* (*approximants* :: *'a* ⇒ -)
**by** (*rule bifinite-approx-chain.ideal-completion*)
**qed**

### 21.4.6 EP-pair from any bifinite domain into *udom*

**context** *bifinite-approx-chain* **begin**

**definition**
*udom-emb* :: *'a* → *udom*
**where**
*udom-emb* = *compact-basis.extension* (λ*x*. *udom-principal* (*basis-emb x*))

**definition**
*udom-prj* :: *udom* → *'a*
**where**
*udom-prj* = *udom.extension* (λ*x*. *Rep-compact-basis* (*basis-prj x*))

**lemma** *udom-emb-principal*:
*udom-emb*·(*Rep-compact-basis x*) = *udom-principal* (*basis-emb x*)
**unfolding** *udom-emb-def*
**apply** (*rule compact-basis.extension-principal*)
**apply** (*rule udom.principal-mono*)
**apply** (*erule basis-emb-mono*)
**done**

**lemma** *udom-prj-principal*:
*udom-prj*·(*udom-principal x*) = *Rep-compact-basis* (*basis-prj x*)
**unfolding** *udom-prj-def*
**apply** (*rule udom.extension-principal*)
**apply** (*rule compact-basis.principal-mono*)
**apply** (*erule basis-prj-mono*)
**done**

**lemma** *ep-pair-udom*: *ep-pair udom-emb udom-prj*
**apply** *standard*
**apply** (*rule compact-basis.principal-induct*, *simp*)
**apply** (*simp add*: *udom-emb-principal udom-prj-principal*)
**apply** (*simp add*: *basis-prj-basis-emb*)
**apply** (*rule udom.principal-induct*, *simp*)
**apply** (*simp add*: *udom-emb-principal udom-prj-principal*)

**apply** (*rule basis-emb-prj-less*)
**done**

**end**

**abbreviation** *udom-emb* ≡ *bifinite-approx-chain.udom-emb*
**abbreviation** *udom-prj* ≡ *bifinite-approx-chain.udom-prj*

**lemmas** *ep-pair-udom* =
*bifinite-approx-chain.ep-pair-udom* [*unfolded bifinite-approx-chain-def*]

## 21.5 Chain of approx functions for type *udom*

**definition**
*udom-approx* :: *nat* ⇒ *udom* → *udom*
**where**
*udom-approx i* =
*udom.extension* (λ*x*. *udom-principal* (*ubasis-until* (λ*y*. *y* ≤ *i*) *x*))

**lemma** *udom-approx-mono*:
*ubasis-le a b* ⟹
*udom-principal* (*ubasis-until* (λ*y*. *y* ≤ *i*) *a*) ⊑
*udom-principal* (*ubasis-until* (λ*y*. *y* ≤ *i*) *b*)
**apply** (*rule udom.principal-mono*)
**apply** (*rule ubasis-until-mono*)
**apply** (*frule* (*2*) *order-less-le-trans* [*OF node-gt2*])
**apply** (*erule order-less-imp-le*)
**apply** *assumption*
**done**

**lemma** *adm-mem-finite*: ⟦*cont f*; *finite S*⟧ ⟹ *adm* (λ*x*. *f x* ∈ *S*)
**by** (*erule adm-subst*, *induct set*: *finite*, *simp-all*)

**lemma** *udom-approx-principal*:
*udom-approx i*·(*udom-principal x*) =
*udom-principal* (*ubasis-until* (λ*y*. *y* ≤ *i*) *x*)
**unfolding** *udom-approx-def*
**apply** (*rule udom.extension-principal*)
**apply** (*erule udom-approx-mono*)
**done**

**lemma** *finite-deflation-udom-approx*: *finite-deflation* (*udom-approx i*)
**proof**
**fix** *x* **show** *udom-approx i*·(*udom-approx i*·*x*) = *udom-approx i*·*x*
**by** (*induct x rule*: *udom.principal-induct*, *simp*)
(*simp add*: *udom-approx-principal ubasis-until-idem*)
**next**
**fix** *x* **show** *udom-approx i*·*x* ⊑ *x*
**by** (*induct x rule*: *udom.principal-induct*, *simp*)

```
      (simp add: udom-approx-principal ubasis-until-less)
next
  have *: finite (range (λx. udom-principal (ubasis-until (λy. y ≤ i) x)))
    apply (subst range-composition [where f=udom-principal])
    apply (simp add: finite-range-ubasis-until)
    done
  show finite {x. udom-approx i·x = x}
    apply (rule finite-range-imp-finite-fixes)
    apply (rule rev-finite-subset [OF *])
    apply (clarsimp, rename-tac x)
    apply (induct-tac x rule: udom.principal-induct)
    apply (simp add: adm-mem-finite *)
    apply (simp add: udom-approx-principal)
    done
qed

interpretation udom-approx: finite-deflation udom-approx i
by (rule finite-deflation-udom-approx)

lemma chain-udom-approx [simp]: chain (λi. udom-approx i)
unfolding udom-approx-def
apply (rule chainI)
apply (rule udom.extension-mono)
apply (erule udom-approx-mono)
apply (erule udom-approx-mono)
apply (rule udom.principal-mono)
apply (rule ubasis-until-chain, simp)
done

lemma lub-udom-approx [simp]: (⨆ i. udom-approx i) = ID
apply (rule cfun-eqI, simp add: contlub-cfun-fun)
apply (rule below-antisym)
apply (rule lub-below)
apply (simp)
apply (rule udom-approx.below)
apply (rule-tac x=x in udom.principal-induct)
apply (simp add: lub-distribs)
apply (rule-tac i=a in below-lub)
apply simp
apply (simp add: udom-approx-principal)
apply (simp add: ubasis-until-same ubasis-le-refl)
done

lemma udom-approx [simp]: approx-chain udom-approx
proof
  show chain (λi. udom-approx i)
    by (rule chain-udom-approx)
  show (⨆ i. udom-approx i) = ID
    by (rule lub-udom-approx)
```

**qed**

**instance** *udom* :: *bifinite*
**by** *standard* (*fast intro*: *udom-approx*)

**hide-const** (**open**) *node*

**notation** *binomial* (**infixl** *choose 65*)

**end**

# 22 Algebraic deflations

**theory** *Algebraic*
**imports** *Universal Map-Functions*
**begin**

**default-sort** *bifinite*

## 22.1 Type constructor for finite deflations

**typedef** $'a$ *fin-defl* = $\{d::'a \to 'a.$ *finite-deflation* $d\}$
**by** (*fast intro*: *finite-deflation-bottom*)

**instantiation** *fin-defl* :: (*bifinite*) *below*
**begin**

**definition** *below-fin-defl-def*:
*below* $\equiv \lambda x\ y.$ *Rep-fin-defl* $x \sqsubseteq$ *Rep-fin-defl* $y$

**instance ..**
**end**

**instance** *fin-defl* :: (*bifinite*) *po*
**using** *type-definition-fin-defl below-fin-defl-def*
**by** (*rule typedef-po*)

**lemma** *finite-deflation-Rep-fin-defl*: *finite-deflation* (*Rep-fin-defl* $d$)
**using** *Rep-fin-defl* **by** *simp*

**lemma** *deflation-Rep-fin-defl*: *deflation* (*Rep-fin-defl* $d$)
**using** *finite-deflation-Rep-fin-defl*
**by** (*rule finite-deflation-imp-deflation*)

**interpretation** *Rep-fin-defl*: *finite-deflation Rep-fin-defl d*
**by** (*rule finite-deflation-Rep-fin-defl*)

**lemma** *fin-defl-belowI*:
$(\bigwedge x.$ *Rep-fin-defl* $a{\cdot}x = x \Longrightarrow$ *Rep-fin-defl* $b{\cdot}x = x) \Longrightarrow a \sqsubseteq b$

**unfolding** *below-fin-defl-def*
**by** (*rule Rep-fin-defl.belowI*)

**lemma** *fin-defl-belowD*:
  ⟦$a \sqsubseteq b$; *Rep-fin-defl* $a{\cdot}x = x$⟧ ⟹ *Rep-fin-defl* $b{\cdot}x = x$
**unfolding** *below-fin-defl-def*
**by** (*rule Rep-fin-defl.belowD*)

**lemma** *fin-defl-eqI*:
  ($\bigwedge x$. *Rep-fin-defl* $a{\cdot}x = x \longleftrightarrow$ *Rep-fin-defl* $b{\cdot}x = x$) ⟹ $a = b$
**apply** (*rule below-antisym*)
**apply** (*rule fin-defl-belowI*, *simp*)
**apply** (*rule fin-defl-belowI*, *simp*)
**done**

**lemma** *Rep-fin-defl-mono*: $a \sqsubseteq b$ ⟹ *Rep-fin-defl* $a \sqsubseteq$ *Rep-fin-defl* $b$
**unfolding** *below-fin-defl-def* **.**

**lemma** *Abs-fin-defl-mono*:
  ⟦*finite-deflation* $a$; *finite-deflation* $b$; $a \sqsubseteq b$⟧
    ⟹ *Abs-fin-defl* $a \sqsubseteq$ *Abs-fin-defl* $b$
**unfolding** *below-fin-defl-def*
**by** (*simp add*: *Abs-fin-defl-inverse*)

**lemma** (**in** *finite-deflation*) *compact-belowI*:
  **assumes** $\bigwedge x$. *compact* $x$ ⟹ $d{\cdot}x = x$ ⟹ $f{\cdot}x = x$ **shows** $d \sqsubseteq f$
**by** (*rule belowI*, *rule assms*, *erule subst*, *rule compact*)

**lemma** *compact-Rep-fin-defl* [*simp*]: *compact* (*Rep-fin-defl* $a$)
**using** *finite-deflation-Rep-fin-defl*
**by** (*rule finite-deflation-imp-compact*)

## 22.2 Defining algebraic deflations by ideal completion

**typedef** $'a$ *defl* = {$S$::$'a$ *fin-defl set*. *below.ideal* $S$}
**by** (*rule below.ex-ideal*)

**instantiation** *defl* :: (*bifinite*) *below*
**begin**

**definition**
  $x \sqsubseteq y \longleftrightarrow$ *Rep-defl* $x \subseteq$ *Rep-defl* $y$

**instance ..**
**end**

**instance** *defl* :: (*bifinite*) *po*
**using** *type-definition-defl below-defl-def*
**by** (*rule below.typedef-ideal-po*)

**instance** *defl* :: (*bifinite*) *cpo*
**using** *type-definition-defl below-defl-def*
**by** (*rule below.typedef-ideal-cpo*)

**definition**
*defl-principal* :: *'a fin-defl* ⇒ *'a defl* **where**
*defl-principal t* = *Abs-defl* {*u*. *u* ⊑ *t*}

**lemma** *fin-defl-countable*: ∃*f*::*'a fin-defl* ⇒ *nat*. *inj f*
**proof** −
**obtain** *f* :: *'a compact-basis* ⇒ *nat* **where** *inj-f*: *inj f*
**using** *compact-basis.countable* **..**
**have** ∗: ⋀*d*. *finite* (*f* ‘ *Rep-compact-basis* −‘ {*x*. *Rep-fin-defl d*·*x* = *x*})
**apply** (*rule finite-imageI*)
**apply** (*rule finite-vimageI*)
**apply** (*rule Rep-fin-defl.finite-fixes*)
**apply** (*simp add*: *inj-on-def Rep-compact-basis-inject*)
**done**
**have** *range-eq*: *range Rep-compact-basis* = {*x*. *compact x*}
**using** *type-definition-compact-basis* **by** (*rule type-definition.Rep-range*)
**have** *inj* (λ*d*. *set-encode*
(*f* ‘ *Rep-compact-basis* −‘ {*x*. *Rep-fin-defl d*·*x* = *x*}))
**apply** (*rule inj-onI*)
**apply** (*simp only*: *set-encode-eq* ∗)
**apply** (*simp only*: *inj-image-eq-iff inj-f*)
**apply** (*drule-tac f*=*image Rep-compact-basis* **in** *arg-cong*)
**apply** (*simp del*: *vimage-Collect-eq add*: *range-eq set-eq-iff*)
**apply** (*rule Rep-fin-defl-inject* [*THEN iffD1*])
**apply** (*rule below-antisym*)
**apply** (*rule Rep-fin-defl.compact-belowI*, *rename-tac z*)
**apply** (*drule-tac x*=*z* **in** *spec*, *simp*)
**apply** (*rule Rep-fin-defl.compact-belowI*, *rename-tac z*)
**apply** (*drule-tac x*=*z* **in** *spec*, *simp*)
**done**
**thus** *?thesis* **by** − (*rule exI*)
**qed**

**interpretation** *defl*: *ideal-completion below defl-principal Rep-defl*
**using** *type-definition-defl below-defl-def*
**using** *defl-principal-def fin-defl-countable*
**by** (*rule below.typedef-ideal-completion*)

Algebraic deflations are pointed

**lemma** *defl-minimal*: *defl-principal* (*Abs-fin-defl* ⊥) ⊑ *x*
**apply** (*induct x rule*: *defl.principal-induct*, *simp*)
**apply** (*rule defl.principal-mono*)
**apply** (*simp add*: *below-fin-defl-def*)
**apply** (*simp add*: *Abs-fin-defl-inverse finite-deflation-bottom*)

**done**

**instance** *defl* :: (*bifinite*) *pcpo*
**by** *intro-classes* (*fast intro*: *defl-minimal*)

**lemma** *inst-defl-pcpo*: $\bot = \textit{defl-principal}\ (\textit{Abs-fin-defl}\ \bot)$
**by** (*rule defl-minimal* [*THEN bottomI*, *symmetric*])

## 22.3 Applying algebraic deflations

**definition**
  *cast* :: *'a defl* $\to$ *'a* $\to$ *'a*
**where**
  *cast* = *defl.extension Rep-fin-defl*

**lemma** *cast-defl-principal*:
  $\textit{cast}\cdot(\textit{defl-principal}\ a) = \textit{Rep-fin-defl}\ a$
**unfolding** *cast-def*
**apply** (*rule defl.extension-principal*)
**apply** (*simp only*: *below-fin-defl-def*)
**done**

**lemma** *deflation-cast*: *deflation* $(\textit{cast}\cdot d)$
**apply** (*induct d rule*: *defl.principal-induct*)
**apply** (*rule adm-subst* [*OF - adm-deflation*], *simp*)
**apply** (*simp add*: *cast-defl-principal*)
**apply** (*rule finite-deflation-imp-deflation*)
**apply** (*rule finite-deflation-Rep-fin-defl*)
**done**

**lemma** *finite-deflation-cast*:
  *compact d* $\Longrightarrow$ *finite-deflation* $(\textit{cast}\cdot d)$
**apply** (*drule defl.compact-imp-principal*, *clarify*)
**apply** (*simp add*: *cast-defl-principal*)
**apply** (*rule finite-deflation-Rep-fin-defl*)
**done**

**interpretation** *cast*: *deflation cast*$\cdot d$
**by** (*rule deflation-cast*)

**declare** *cast.idem* [*simp*]

**lemma** *compact-cast* [*simp*]: *compact d* $\Longrightarrow$ *compact* $(\textit{cast}\cdot d)$
**apply** (*rule finite-deflation-imp-compact*)
**apply** (*erule finite-deflation-cast*)
**done**

**lemma** *cast-below-cast*: $\textit{cast}\cdot A \sqsubseteq \textit{cast}\cdot B \longleftrightarrow A \sqsubseteq B$
**apply** (*induct A rule*: *defl.principal-induct*, *simp*)

**apply** (*induct B rule*: *defl.principal-induct*, *simp*)
**apply** (*simp add*: *cast-defl-principal below-fin-defl-def*)
**done**

**lemma** *compact-cast-iff*: *compact* (*cast*·*d*) ⟷ *compact d*
**apply** (*rule iffI*)
**apply** (*simp only*: *compact-def cast-below-cast* [*symmetric*])
**apply** (*erule adm-subst* [*OF cont-Rep-cfun2*])
**apply** (*erule compact-cast*)
**done**

**lemma** *cast-below-imp-below*: *cast*·*A* ⊑ *cast*·*B* ⟹ *A* ⊑ *B*
**by** (*simp only*: *cast-below-cast*)

**lemma** *cast-eq-imp-eq*: *cast*·*A* = *cast*·*B* ⟹ *A* = *B*
**by** (*simp add*: *below-antisym cast-below-imp-below*)

**lemma** *cast-strict1* [*simp*]: *cast*·⊥ = ⊥
**apply** (*subst inst-defl-pcpo*)
**apply** (*subst cast-defl-principal*)
**apply** (*rule Abs-fin-defl-inverse*)
**apply** (*simp add*: *finite-deflation-bottom*)
**done**

**lemma** *cast-strict2* [*simp*]: *cast*·*A*·⊥ = ⊥
**by** (*rule cast.below* [*THEN bottomI*])

## 22.4 Deflation combinators

**definition**
*defl-fun1 e p f* =
*defl.extension* (λ*a*.
*defl-principal* (*Abs-fin-defl*
(*e oo f*·(*Rep-fin-defl a*) *oo p*)))

**definition**
*defl-fun2 e p f* =
*defl.extension* (λ*a*.
*defl.extension* (λ*b*.
*defl-principal* (*Abs-fin-defl*
(*e oo f*·(*Rep-fin-defl a*)·(*Rep-fin-defl b*) *oo p*))))

**lemma** *cast-defl-fun1*:
**assumes** *ep*: *ep-pair e p*
**assumes** *f*: ⋀*a*. *finite-deflation a* ⟹ *finite-deflation* (*f*·*a*)
**shows** *cast*·(*defl-fun1 e p f*·*A*) = *e oo f*·(*cast*·*A*) *oo p*
**proof** −
**have** *1*: ⋀*a*. *finite-deflation* (*e oo f*·(*Rep-fin-defl a*) *oo p*)
**apply** (*rule ep-pair.finite-deflation-e-d-p* [*OF ep*])

```
    apply (rule f, rule finite-deflation-Rep-fin-defl)
    done
  show ?thesis
    by (induct A rule: defl.principal-induct, simp)
       (simp only: defl-fun1-def
                   defl.extension-principal
                   defl.extension-mono
                   defl.principal-mono
                   Abs-fin-defl-mono [OF 1 1]
                   monofun-cfun below-refl
                   Rep-fin-defl-mono
                   cast-defl-principal
                   Abs-fin-defl-inverse [unfolded mem-Collect-eq, OF 1])
qed

lemma cast-defl-fun2:
  assumes ep: ep-pair e p
  assumes f: ⋀a b. finite-deflation a ⟹ finite-deflation b ⟹
              finite-deflation (f·a·b)
  shows cast·(defl-fun2 e p f·A·B) = e oo f·(cast·A)·(cast·B) oo p
proof −
  have 1: ⋀a b. finite-deflation
     (e oo f·(Rep-fin-defl a)·(Rep-fin-defl b) oo p)
    apply (rule ep-pair.finite-deflation-e-d-p [OF ep])
    apply (rule f, (rule finite-deflation-Rep-fin-defl)+)
    done
  show ?thesis
    apply (induct A rule: defl.principal-induct, simp)
    apply (induct B rule: defl.principal-induct, simp)
    by (simp only: defl-fun2-def
                   defl.extension-principal
                   defl.extension-mono
                   defl.principal-mono
                   Abs-fin-defl-mono [OF 1 1]
                   monofun-cfun below-refl
                   Rep-fin-defl-mono
                   cast-defl-principal
                   Abs-fin-defl-inverse [unfolded mem-Collect-eq, OF 1])
qed

end
```

# 23 Representable domains

```
theory Representable
imports Algebraic Map-Functions HOL−Library.Countable
begin

default-sort cpo
```

## 23.1 Class of representable domains

We define a "domain" as a pcpo that is isomorphic to some algebraic deflation over the universal domain; this is equivalent to being omega-bifinite.

A predomain is a cpo that, when lifted, becomes a domain. Predomains are represented by deflations over a lifted universal domain type.

**class** *predomain-syn* = *cpo* +
  **fixes** *liftemb* :: $'a_\perp \to udom_\perp$
  **fixes** *liftprj* :: $udom_\perp \to 'a_\perp$
  **fixes** *liftdefl* :: *'a itself* ⇒ *udom u defl*

**class** *predomain* = *predomain-syn* +
  **assumes** *predomain-ep*: *ep-pair liftemb liftprj*
  **assumes** *cast-liftdefl*: *cast*·(*liftdefl TYPE*('*a*)) = *liftemb oo liftprj*

**syntax** -*LIFTDEFL* :: *type* ⇒ *logic* (($1$*LIFTDEFL*/($1$'(-'))))
**translations** *LIFTDEFL*('*t*) ⇌ *CONST liftdefl TYPE*('*t*)

**definition** *liftdefl-of* :: *udom defl* → *udom u defl*
  **where** *liftdefl-of* = *defl-fun1 ID ID u-map*

**lemma** *cast-liftdefl-of*: *cast*·(*liftdefl-of*·*t*) = *u-map*·(*cast*·*t*)
**by** (*simp add*: *liftdefl-of-def cast-defl-fun1 ep-pair-def finite-deflation-u-map*)

**class** *domain* = *predomain-syn* + *pcpo* +
  **fixes** *emb* :: *'a* → *udom*
  **fixes** *prj* :: *udom* → *'a*
  **fixes** *defl* :: *'a itself* ⇒ *udom defl*
  **assumes** *ep-pair-emb-prj*: *ep-pair emb prj*
  **assumes** *cast-DEFL*: *cast*·(*defl TYPE*('*a*)) = *emb oo prj*
  **assumes** *liftemb-eq*: *liftemb* = *u-map*·*emb*
  **assumes** *liftprj-eq*: *liftprj* = *u-map*·*prj*
  **assumes** *liftdefl-eq*: *liftdefl TYPE*('*a*) = *liftdefl-of*·(*defl TYPE*('*a*))

**syntax** -*DEFL* :: *type* ⇒ *logic* (($1$*DEFL*/($1$'(-'))))
**translations** *DEFL*('*t*) ⇌ *CONST defl TYPE*('*t*)

**instance** *domain* ⊆ *predomain*
**proof**
  **show** *ep-pair liftemb* (*liftprj*::$udom_\perp \to 'a_\perp$)
    **unfolding** *liftemb-eq liftprj-eq*
    **by** (*intro ep-pair-u-map ep-pair-emb-prj*)
  **show** *cast*·*LIFTDEFL*('*a*) = *liftemb oo* (*liftprj*::$udom_\perp \to 'a_\perp$)
    **unfolding** *liftemb-eq liftprj-eq liftdefl-eq*
    **by** (*simp add*: *cast-liftdefl-of cast-DEFL u-map-oo*)
**qed**

Constants *liftemb* and *liftprj* imply class predomain.

**setup** ‹
*fold Sign.add-const-constraint*
[(***const-name*** ‹*liftemb*›, *SOME* ***typ*** ‹'*a*::*predomain u* → *udom u*›),
(***const-name*** ‹*liftprj*›, *SOME* ***typ*** ‹*udom u* → '*a*::*predomain u*›),
(***const-name*** ‹*liftdefl*›, *SOME* ***typ*** ‹'*a*::*predomain itself* ⇒ *udom u defl*›)]
›

**interpretation** *predomain*: *pcpo-ep-pair liftemb liftprj*
**unfolding** *pcpo-ep-pair-def* **by** (*rule predomain-ep*)

**interpretation** *domain*: *pcpo-ep-pair emb prj*
**unfolding** *pcpo-ep-pair-def* **by** (*rule ep-pair-emb-prj*)

**lemmas** *emb-inverse* = *domain.e-inverse*
**lemmas** *emb-prj-below* = *domain.e-p-below*
**lemmas** *emb-eq-iff* = *domain.e-eq-iff*
**lemmas** *emb-strict* = *domain.e-strict*
**lemmas** *prj-strict* = *domain.p-strict*

## 23.2 Domains are bifinite

**lemma** *approx-chain-ep-cast*:
**assumes** *ep*: *ep-pair* (*e*::'*a*::*pcpo* → '*b*::*bifinite*) (*p*::'*b* → '*a*)
**assumes** *cast-t*: *cast*·*t* = *e oo p*
**shows** ∃(*a*::*nat* ⇒ '*a*::*pcpo* → '*a*). *approx-chain a*
**proof** −
**interpret** *ep-pair e p* **by** *fact*
**obtain** *Y* **where** *Y*: ∀ *i*. *Y i* ⊑ *Y* (*Suc i*)
**and** *t*: *t* = (⨆ *i*. *defl-principal* (*Y i*))
**by** (*rule defl.obtain-principal-chain*)
**define** *approx* **where** *approx i* = (*p oo cast*·(*defl-principal* (*Y i*)) *oo e*) **for** *i*
**have** *approx-chain approx*
**proof** (*rule approx-chain.intro*)
**show** *chain* (λ*i*. *approx i*)
**unfolding** *approx-def* **by** (*simp add*: *Y*)
**show** (⨆ *i*. *approx i*) = *ID*
**unfolding** *approx-def*
**by** (*simp add*: *lub-distribs Y t* [*symmetric*] *cast-t cfun-eq-iff*)
**show** ⋀*i*. *finite-deflation* (*approx i*)
**unfolding** *approx-def*
**apply** (*rule finite-deflation-p-d-e*)
**apply** (*rule finite-deflation-cast*)
**apply** (*rule defl.compact-principal*)
**apply** (*rule below-trans* [*OF monofun-cfun-fun*])
**apply** (*rule is-ub-thelub*, *simp add*: *Y*)
**apply** (*simp add*: *lub-distribs Y t* [*symmetric*] *cast-t*)
**done**
**qed**
**thus** ∃(*a*::*nat* ⇒ '*a* → '*a*). *approx-chain a* **by** − (*rule exI*)

**qed**

**instance** *domain* $\subseteq$ *bifinite*
**by** *standard* (*rule approx-chain-ep-cast* [*OF ep-pair-emb-prj cast-DEFL*])

**instance** *predomain* $\subseteq$ *profinite*
**by** *standard* (*rule approx-chain-ep-cast* [*OF predomain-ep cast-liftdefl*])

## 23.3 Universal domain ep-pairs

**definition** *u-emb* = *udom-emb* ($\lambda i$. *u-map*·(*udom-approx i*))
**definition** *u-prj* = *udom-prj* ($\lambda i$. *u-map*·(*udom-approx i*))

**definition** *prod-emb* = *udom-emb* ($\lambda i$. *prod-map*·(*udom-approx i*)·(*udom-approx i*))
**definition** *prod-prj* = *udom-prj* ($\lambda i$. *prod-map*·(*udom-approx i*)·(*udom-approx i*))

**definition** *sprod-emb* = *udom-emb* ($\lambda i$. *sprod-map*·(*udom-approx i*)·(*udom-approx i*))
**definition** *sprod-prj* = *udom-prj* ($\lambda i$. *sprod-map*·(*udom-approx i*)·(*udom-approx i*))

**definition** *ssum-emb* = *udom-emb* ($\lambda i$. *ssum-map*·(*udom-approx i*)·(*udom-approx i*))
**definition** *ssum-prj* = *udom-prj* ($\lambda i$. *ssum-map*·(*udom-approx i*)·(*udom-approx i*))

**definition** *sfun-emb* = *udom-emb* ($\lambda i$. *sfun-map*·(*udom-approx i*)·(*udom-approx i*))
**definition** *sfun-prj* = *udom-prj* ($\lambda i$. *sfun-map*·(*udom-approx i*)·(*udom-approx i*))

**lemma** *ep-pair-u*: *ep-pair u-emb u-prj*
  **unfolding** *u-emb-def u-prj-def*
  **by** (*simp add*: *ep-pair-udom approx-chain-u-map*)

**lemma** *ep-pair-prod*: *ep-pair prod-emb prod-prj*
  **unfolding** *prod-emb-def prod-prj-def*
  **by** (*simp add*: *ep-pair-udom approx-chain-prod-map*)

**lemma** *ep-pair-sprod*: *ep-pair sprod-emb sprod-prj*
  **unfolding** *sprod-emb-def sprod-prj-def*
  **by** (*simp add*: *ep-pair-udom approx-chain-sprod-map*)

**lemma** *ep-pair-ssum*: *ep-pair ssum-emb ssum-prj*
  **unfolding** *ssum-emb-def ssum-prj-def*
  **by** (*simp add*: *ep-pair-udom approx-chain-ssum-map*)

**lemma** *ep-pair-sfun*: *ep-pair sfun-emb sfun-prj*
  **unfolding** *sfun-emb-def sfun-prj-def*
  **by** (*simp add*: *ep-pair-udom approx-chain-sfun-map*)

## 23.4 Type combinators

**definition** *u-defl* :: *udom defl* → *udom defl*
**where** *u-defl* = *defl-fun1 u-emb u-prj u-map*

**definition** *prod-defl* :: *udom defl* → *udom defl* → *udom defl*
**where** *prod-defl* = *defl-fun2 prod-emb prod-prj prod-map*

**definition** *sprod-defl* :: *udom defl* → *udom defl* → *udom defl*
**where** *sprod-defl* = *defl-fun2 sprod-emb sprod-prj sprod-map*

**definition** *ssum-defl* :: *udom defl* → *udom defl* → *udom defl*
**where** *ssum-defl* = *defl-fun2 ssum-emb ssum-prj ssum-map*

**definition** *sfun-defl* :: *udom defl* → *udom defl* → *udom defl*
**where** *sfun-defl* = *defl-fun2 sfun-emb sfun-prj sfun-map*

**lemma** *cast-u-defl*:
*cast*·(*u-defl*·*A*) = *u-emb oo u-map*·(*cast*·*A*) *oo u-prj*
**using** *ep-pair-u finite-deflation-u-map*
**unfolding** *u-defl-def* **by** (*rule cast-defl-fun1*)

**lemma** *cast-prod-defl*:
*cast*·(*prod-defl*·*A*·*B*) =
*prod-emb oo prod-map*·(*cast*·*A*)·(*cast*·*B*) *oo prod-prj*
**using** *ep-pair-prod finite-deflation-prod-map*
**unfolding** *prod-defl-def* **by** (*rule cast-defl-fun2*)

**lemma** *cast-sprod-defl*:
*cast*·(*sprod-defl*·*A*·*B*) =
*sprod-emb oo sprod-map*·(*cast*·*A*)·(*cast*·*B*) *oo sprod-prj*
**using** *ep-pair-sprod finite-deflation-sprod-map*
**unfolding** *sprod-defl-def* **by** (*rule cast-defl-fun2*)

**lemma** *cast-ssum-defl*:
*cast*·(*ssum-defl*·*A*·*B*) =
*ssum-emb oo ssum-map*·(*cast*·*A*)·(*cast*·*B*) *oo ssum-prj*
**using** *ep-pair-ssum finite-deflation-ssum-map*
**unfolding** *ssum-defl-def* **by** (*rule cast-defl-fun2*)

**lemma** *cast-sfun-defl*:
*cast*·(*sfun-defl*·*A*·*B*) =
*sfun-emb oo sfun-map*·(*cast*·*A*)·(*cast*·*B*) *oo sfun-prj*
**using** *ep-pair-sfun finite-deflation-sfun-map*
**unfolding** *sfun-defl-def* **by** (*rule cast-defl-fun2*)

Special deflation combinator for unpointed types.

**definition** *u-liftdefl* :: *udom u defl* → *udom defl*
**where** *u-liftdefl* = *defl-fun1 u-emb u-prj ID*

**lemma** *cast-u-liftdefl*:
*cast*·(*u-liftdefl*·*A*) = *u-emb oo cast*·*A oo u-prj*
**unfolding** *u-liftdefl-def* **by** (*simp add*: *cast-defl-fun1 ep-pair-u*)

**lemma** *u-liftdefl-liftdefl-of*:
*u-liftdefl*·(*liftdefl-of*·*A*) = *u-defl*·*A*
**by** (*rule cast-eq-imp-eq*)
(*simp add*: *cast-u-liftdefl cast-liftdefl-of cast-u-defl*)

## 23.5 Class instance proofs

### 23.5.1 Universal domain

**instantiation** *udom* :: *domain*
**begin**

**definition** [*simp*]:
*emb* = (*ID* :: *udom* → *udom*)

**definition** [*simp*]:
*prj* = (*ID* :: *udom* → *udom*)

**definition**
*defl* (*t*::*udom itself*) = (⨆ *i*. *defl-principal* (*Abs-fin-defl* (*udom-approx i*)))

**definition**
(*liftemb* :: *udom u* → *udom u*) = *u-map*·*emb*

**definition**
(*liftprj* :: *udom u* → *udom u*) = *u-map*·*prj*

**definition**
*liftdefl* (*t*::*udom itself*) = *liftdefl-of*·*DEFL*(*udom*)

**instance proof**
**show** *ep-pair emb* (*prj* :: *udom* → *udom*)
**by** (*simp add*: *ep-pair.intro*)
**show** *cast*·*DEFL*(*udom*) = *emb oo* (*prj* :: *udom* → *udom*)
**unfolding** *defl-udom-def*
**apply** (*subst contlub-cfun-arg*)
**apply** (*rule chainI*)
**apply** (*rule defl.principal-mono*)
**apply** (*simp add*: *below-fin-defl-def*)
**apply** (*simp add*: *Abs-fin-defl-inverse finite-deflation-udom-approx*)
**apply** (*rule chainE*)
**apply** (*rule chain-udom-approx*)
**apply** (*subst cast-defl-principal*)
**apply** (*simp add*: *Abs-fin-defl-inverse finite-deflation-udom-approx*)
**done**
**qed** (*fact liftemb-udom-def liftprj-udom-def liftdefl-udom-def*)+

**end**

### 23.5.2 Lifted cpo

**instantiation** $u :: (predomain)\ domain$
**begin**

**definition**
$emb = u\text{-}emb\ oo\ liftemb$

**definition**
$prj = liftprj\ oo\ u\text{-}prj$

**definition**
$defl\ (t{::}'a\ u\ itself) = u\text{-}liftdefl{\cdot}LIFTDEFL('a)$

**definition**
$(liftemb :: 'a\ u\ u \to udom\ u) = u\text{-}map{\cdot}emb$

**definition**
$(liftprj :: udom\ u \to 'a\ u\ u) = u\text{-}map{\cdot}prj$

**definition**
$liftdefl\ (t{::}'a\ u\ itself) = liftdefl\text{-}of{\cdot}DEFL('a\ u)$

**instance proof**
**show** $ep\text{-}pair\ emb\ (prj :: udom \to 'a\ u)$
**unfolding** $emb\text{-}u\text{-}def\ prj\text{-}u\text{-}def$
**by** $(intro\ ep\text{-}pair\text{-}comp\ ep\text{-}pair\text{-}u\ predomain\text{-}ep)$
**show** $cast{\cdot}DEFL('a\ u) = emb\ oo\ (prj :: udom \to 'a\ u)$
**unfolding** $emb\text{-}u\text{-}def\ prj\text{-}u\text{-}def\ defl\text{-}u\text{-}def$
**by** $(simp\ add{:}\ cast\text{-}u\text{-}liftdefl\ cast\text{-}liftdefl\ assoc\text{-}oo)$
**qed** $(fact\ liftemb\text{-}u\text{-}def\ liftprj\text{-}u\text{-}def\ liftdefl\text{-}u\text{-}def)+$

**end**

**lemma** $DEFL\text{-}u{:}\ DEFL('a{::}predomain\ u) = u\text{-}liftdefl{\cdot}LIFTDEFL('a)$
**by** $(rule\ defl\text{-}u\text{-}def)$

### 23.5.3 Strict function space

**instantiation** $sfun :: (domain,\ domain)\ domain$
**begin**

**definition**
$emb = sfun\text{-}emb\ oo\ sfun\text{-}map{\cdot}prj{\cdot}emb$

**definition**
$prj = sfun\text{-}map{\cdot}emb{\cdot}prj\ oo\ sfun\text{-}prj$

**definition**
*defl* (*t*::('*a* →! '*b*) *itself*) = *sfun-defl*·*DEFL*('*a*)·*DEFL*('*b*)

**definition**
(*liftemb* :: ('*a* →! '*b*) *u* → *udom u*) = *u-map*·*emb*

**definition**
(*liftprj* :: *udom u* → ('*a* →! '*b*) *u*) = *u-map*·*prj*

**definition**
*liftdefl* (*t*::('*a* →! '*b*) *itself*) = *liftdefl-of*·*DEFL*('*a* →! '*b*)

**instance proof**
**show** *ep-pair emb* (*prj* :: *udom* → '*a* →! '*b*)
**unfolding** *emb-sfun-def prj-sfun-def*
**by** (*intro ep-pair-comp ep-pair-sfun ep-pair-sfun-map ep-pair-emb-prj*)
**show** *cast*·*DEFL*('*a* →! '*b*) = *emb oo* (*prj* :: *udom* → '*a* →! '*b*)
**unfolding** *emb-sfun-def prj-sfun-def defl-sfun-def cast-sfun-defl*
**by** (*simp add*: *cast-DEFL oo-def sfun-eq-iff sfun-map-map*)
**qed** (*fact liftemb-sfun-def liftprj-sfun-def liftdefl-sfun-def*)+

**end**

**lemma** *DEFL-sfun*:
*DEFL*('*a*::*domain* →! '*b*::*domain*) = *sfun-defl*·*DEFL*('*a*)·*DEFL*('*b*)
**by** (*rule defl-sfun-def*)

### 23.5.4 Continuous function space

**instantiation** *cfun* :: (*predomain*, *domain*) *domain*
**begin**

**definition**
*emb* = *emb oo encode-cfun*

**definition**
*prj* = *decode-cfun oo prj*

**definition**
*defl* (*t*::('*a* → '*b*) *itself*) = *DEFL*('*a u* →! '*b*)

**definition**
(*liftemb* :: ('*a* → '*b*) *u* → *udom u*) = *u-map*·*emb*

**definition**
(*liftprj* :: *udom u* → ('*a* → '*b*) *u*) = *u-map*·*prj*

**definition**

*liftdefl* (*t*::('*a* → '*b*) *itself*) = *liftdefl-of*·*DEFL*('*a* → '*b*)

**instance proof**
**have** *ep-pair encode-cfun decode-cfun*
**by** (*rule ep-pair.intro*, *simp-all*)
**thus** *ep-pair emb* (*prj* :: *udom* → '*a* → '*b*)
**unfolding** *emb-cfun-def prj-cfun-def*
**using** *ep-pair-emb-prj* **by** (*rule ep-pair-comp*)
**show** *cast*·*DEFL*('*a* → '*b*) = *emb oo* (*prj* :: *udom* → '*a* → '*b*)
**unfolding** *emb-cfun-def prj-cfun-def defl-cfun-def*
**by** (*simp add*: *cast-DEFL cfcomp1*)
**qed** (*fact liftemb-cfun-def liftprj-cfun-def liftdefl-cfun-def*)+

**end**

**lemma** *DEFL-cfun*:
*DEFL*('*a*::*predomain* → '*b*::*domain*) = *DEFL*('*a u* →! '*b*)
**by** (*rule defl-cfun-def*)

### 23.5.5 Strict product

**instantiation** *sprod* :: (*domain*, *domain*) *domain*
**begin**

**definition**
*emb* = *sprod-emb oo sprod-map*·*emb*·*emb*

**definition**
*prj* = *sprod-map*·*prj*·*prj oo sprod-prj*

**definition**
*defl* (*t*::('*a* ⊗ '*b*) *itself*) = *sprod-defl*·*DEFL*('*a*)·*DEFL*('*b*)

**definition**
(*liftemb* :: ('*a* ⊗ '*b*) *u* → *udom u*) = *u-map*·*emb*

**definition**
(*liftprj* :: *udom u* → ('*a* ⊗ '*b*) *u*) = *u-map*·*prj*

**definition**
*liftdefl* (*t*::('*a* ⊗ '*b*) *itself*) = *liftdefl-of*·*DEFL*('*a* ⊗ '*b*)

**instance proof**
**show** *ep-pair emb* (*prj* :: *udom* → '*a* ⊗ '*b*)
**unfolding** *emb-sprod-def prj-sprod-def*
**by** (*intro ep-pair-comp ep-pair-sprod ep-pair-sprod-map ep-pair-emb-prj*)
**show** *cast*·*DEFL*('*a* ⊗ '*b*) = *emb oo* (*prj* :: *udom* → '*a* ⊗ '*b*)
**unfolding** *emb-sprod-def prj-sprod-def defl-sprod-def cast-sprod-defl*
**by** (*simp add*: *cast-DEFL oo-def cfun-eq-iff sprod-map-map*)

**qed** (*fact liftemb-sprod-def liftprj-sprod-def liftdefl-sprod-def*)+

**end**

**lemma** *DEFL-sprod*:
*DEFL*('*a*::*domain* ⊗ '*b*::*domain*) = *sprod-defl*·*DEFL*('*a*)·*DEFL*('*b*)
**by** (*rule defl-sprod-def*)

### 23.5.6 Cartesian product

**definition** *prod-liftdefl* :: *udom u defl* → *udom u defl* → *udom u defl*
**where** *prod-liftdefl* = *defl-fun2* (*u-map*·*prod-emb oo decode-prod-u*)
(*encode-prod-u oo u-map*·*prod-prj*) *sprod-map*

**lemma** *cast-prod-liftdefl*:
*cast*·(*prod-liftdefl*·*a*·*b*) =
(*u-map*·*prod-emb oo decode-prod-u*) *oo sprod-map*·(*cast*·*a*)·(*cast*·*b*) *oo*
(*encode-prod-u oo u-map*·*prod-prj*)
**unfolding** *prod-liftdefl-def*
**apply** (*rule cast-defl-fun2*)
**apply** (*intro ep-pair-comp ep-pair-u-map ep-pair-prod*)
**apply** (*simp add*: *ep-pair.intro*)
**apply** (*erule* (*1*) *finite-deflation-sprod-map*)
**done**

**instantiation** *prod* :: (*predomain*, *predomain*) *predomain*
**begin**

**definition**
*liftemb* = (*u-map*·*prod-emb oo decode-prod-u*) *oo*
(*sprod-map*·*liftemb*·*liftemb oo encode-prod-u*)

**definition**
*liftprj* = (*decode-prod-u oo sprod-map*·*liftprj*·*liftprj*) *oo*
(*encode-prod-u oo u-map*·*prod-prj*)

**definition**
*liftdefl* (*t*::('*a* × '*b*) *itself*) = *prod-liftdefl*·*LIFTDEFL*('*a*)·*LIFTDEFL*('*b*)

**instance proof**
**show** *ep-pair liftemb* (*liftprj* :: *udom u* → ('*a* × '*b*) *u*)
**unfolding** *liftemb-prod-def liftprj-prod-def*
**by** (*intro ep-pair-comp ep-pair-sprod-map ep-pair-u-map*
*ep-pair-prod predomain-ep*, *simp-all add*: *ep-pair.intro*)
**show** *cast*·*LIFTDEFL*('*a* × '*b*) = *liftemb oo* (*liftprj* :: *udom u* → ('*a* × '*b*) *u*)
**unfolding** *liftemb-prod-def liftprj-prod-def liftdefl-prod-def*
**by** (*simp add*: *cast-prod-liftdefl cast-liftdefl cfcomp1 sprod-map-map*)
**qed**

**end**

**instantiation** *prod* :: (*domain*, *domain*) *domain*
**begin**

**definition**
*emb* = *prod-emb oo prod-map·emb·emb*

**definition**
*prj* = *prod-map·prj·prj oo prod-prj*

**definition**
*defl* (*t*::(*'a* × *'b*) *itself*) = *prod-defl·DEFL*(*'a*)·*DEFL*(*'b*)

**instance proof**
**show** *1*: *ep-pair emb* (*prj* :: *udom* → *'a* × *'b*)
**unfolding** *emb-prod-def prj-prod-def*
**by** (*intro ep-pair-comp ep-pair-prod ep-pair-prod-map ep-pair-emb-prj*)
**show** *2*: *cast·DEFL*(*'a* × *'b*) = *emb oo* (*prj* :: *udom* → *'a* × *'b*)
**unfolding** *emb-prod-def prj-prod-def defl-prod-def cast-prod-defl*
**by** (*simp add*: *cast-DEFL oo-def cfun-eq-iff prod-map-map*)
**show** *3*: *liftemb* = *u-map*·(*emb* :: *'a* × *'b* → *udom*)
**unfolding** *emb-prod-def liftemb-prod-def liftemb-eq*
**unfolding** *encode-prod-u-def decode-prod-u-def*
**by** (*rule cfun-eqI*, *case-tac x*, *simp*, *clarsimp*)
**show** *4*: *liftprj* = *u-map*·(*prj* :: *udom* → *'a* × *'b*)
**unfolding** *prj-prod-def liftprj-prod-def liftprj-eq*
**unfolding** *encode-prod-u-def decode-prod-u-def*
**apply** (*rule cfun-eqI*, *case-tac x*, *simp*)
**apply** (*rename-tac y*, *case-tac prod-prj·y*, *simp*)
**done**
**show** *5*: *LIFTDEFL*(*'a* × *'b*) = *liftdefl-of·DEFL*(*'a* × *'b*)
**by** (*rule cast-eq-imp-eq*)
(*simp add*: *cast-liftdefl cast-liftdefl-of cast-DEFL 2 3 4 u-map-oo*)
**qed**

**end**

**lemma** *DEFL-prod*:
*DEFL*(*'a*::*domain* × *'b*::*domain*) = *prod-defl·DEFL*(*'a*)·*DEFL*(*'b*)
**by** (*rule defl-prod-def*)

**lemma** *LIFTDEFL-prod*:
*LIFTDEFL*(*'a*::*predomain* × *'b*::*predomain*) =
*prod-liftdefl·LIFTDEFL*(*'a*)·*LIFTDEFL*(*'b*)
**by** (*rule liftdefl-prod-def*)

### 23.5.7 Unit type

**instantiation** *unit* :: *domain*
**begin**

**definition**
*emb* = (⊥ :: *unit* → *udom*)

**definition**
*prj* = (⊥ :: *udom* → *unit*)

**definition**
*defl* (*t*::*unit itself*) = ⊥

**definition**
(*liftemb* :: *unit u* → *udom u*) = *u-map*·*emb*

**definition**
(*liftprj* :: *udom u* → *unit u*) = *u-map*·*prj*

**definition**
*liftdefl* (*t*::*unit itself*) = *liftdefl-of*·*DEFL*(*unit*)

**instance proof**
**show** *ep-pair emb* (*prj* :: *udom* → *unit*)
**unfolding** *emb-unit-def prj-unit-def*
**by** (*simp add*: *ep-pair.intro*)
**show** *cast*·*DEFL*(*unit*) = *emb oo* (*prj* :: *udom* → *unit*)
**unfolding** *emb-unit-def prj-unit-def defl-unit-def* **by** *simp*
**qed** (*fact liftemb-unit-def liftprj-unit-def liftdefl-unit-def*)+

**end**

### 23.5.8 Discrete cpo

**instantiation** *discr* :: (*countable*) *predomain*
**begin**

**definition**
(*liftemb* :: *'a discr u* → *udom u*) = *strictify*·*up oo udom-emb discr-approx*

**definition**
(*liftprj* :: *udom u* → *'a discr u*) = *udom-prj discr-approx oo fup*·*ID*

**definition**
*liftdefl* (*t*::*'a discr itself*) =
(⨆ *i*. *defl-principal* (*Abs-fin-defl* (*liftemb oo discr-approx i oo* (*liftprj*::*udom u* → *'a discr u*))))

**instance proof**

```
  show 1: ep-pair liftemb (liftprj :: udom u → 'a discr u)
    unfolding liftemb-discr-def liftprj-discr-def
    apply (intro ep-pair-comp ep-pair-udom [OF discr-approx])
    apply (rule ep-pair.intro)
    apply (simp add: strictify-conv-if)
    apply (case-tac y, simp, simp add: strictify-conv-if)
    done
  show cast·LIFTDEFL('a discr) = liftemb oo (liftprj :: udom u → 'a discr u)
    unfolding liftdefl-discr-def
    apply (subst contlub-cfun-arg)
    apply (rule chainI)
    apply (rule defl.principal-mono)
    apply (simp add: below-fin-defl-def)
    apply (simp add: Abs-fin-defl-inverse
        ep-pair.finite-deflation-e-d-p [OF 1]
        approx-chain.finite-deflation-approx [OF discr-approx])
    apply (intro monofun-cfun below-refl)
    apply (rule chainE)
    apply (rule chain-discr-approx)
    apply (subst cast-defl-principal)
    apply (simp add: Abs-fin-defl-inverse
        ep-pair.finite-deflation-e-d-p [OF 1]
        approx-chain.finite-deflation-approx [OF discr-approx])
    apply (simp add: lub-distribs)
    done
qed

end
```

### 23.5.9 Strict sum

```
instantiation ssum :: (domain, domain) domain
begin

definition
  emb = ssum-emb oo ssum-map·emb·emb

definition
  prj = ssum-map·prj·prj oo ssum-prj

definition
  defl (t::('a ⊕ 'b) itself) = ssum-defl·DEFL('a)·DEFL('b)

definition
  (liftemb :: ('a ⊕ 'b) u → udom u) = u-map·emb

definition
  (liftprj :: udom u → ('a ⊕ 'b) u) = u-map·prj
```

**definition**
*liftdefl* (*t*::('*a* ⊕ '*b*) *itself*) = *liftdefl-of*·*DEFL*('*a* ⊕ '*b*)

**instance proof**
**show** *ep-pair emb* (*prj* :: *udom* → '*a* ⊕ '*b*)
**unfolding** *emb-ssum-def prj-ssum-def*
**by** (*intro ep-pair-comp ep-pair-ssum ep-pair-ssum-map ep-pair-emb-prj*)
**show** *cast*·*DEFL*('*a* ⊕ '*b*) = *emb oo* (*prj* :: *udom* → '*a* ⊕ '*b*)
**unfolding** *emb-ssum-def prj-ssum-def defl-ssum-def cast-ssum-defl*
**by** (*simp add*: *cast-DEFL oo-def cfun-eq-iff ssum-map-map*)
**qed** (*fact liftemb-ssum-def liftprj-ssum-def liftdefl-ssum-def*)+

**end**

**lemma** *DEFL-ssum*:
*DEFL*('*a*::*domain* ⊕ '*b*::*domain*) = *ssum-defl*·*DEFL*('*a*)·*DEFL*('*b*)
**by** (*rule defl-ssum-def*)

### 23.5.10 Lifted HOL type

**instantiation** *lift* :: (*countable*) *domain*
**begin**

**definition**
*emb* = *emb oo* (Λ *x*. *Rep-lift x*)

**definition**
*prj* = (Λ *y*. *Abs-lift y*) *oo prj*

**definition**
*defl* (*t*::'*a lift itself*) = *DEFL*('*a discr u*)

**definition**
(*liftemb* :: '*a lift u* → *udom u*) = *u-map*·*emb*

**definition**
(*liftprj* :: *udom u* → '*a lift u*) = *u-map*·*prj*

**definition**
*liftdefl* (*t*::'*a lift itself*) = *liftdefl-of*·*DEFL*('*a lift*)

**instance proof**
**note** [*simp*] = *cont-Rep-lift cont-Abs-lift Rep-lift-inverse Abs-lift-inverse*
**have** *ep-pair* (Λ(*x*::'*a lift*). *Rep-lift x*) (Λ *y*. *Abs-lift y*)
**by** (*simp add*: *ep-pair-def*)
**thus** *ep-pair emb* (*prj* :: *udom* → '*a lift*)
**unfolding** *emb-lift-def prj-lift-def*
**using** *ep-pair-emb-prj* **by** (*rule ep-pair-comp*)
**show** *cast*·*DEFL*('*a lift*) = *emb oo* (*prj* :: *udom* → '*a lift*)

**unfolding** *emb-lift-def prj-lift-def defl-lift-def cast-DEFL*
**by** (*simp add*: *cfcomp1*)
**qed** (*fact liftemb-lift-def liftprj-lift-def liftdefl-lift-def*)+

**end**

**end**

# 24 The unit domain

**theory** *One*
**imports** *Lift*
**begin**

**type-synonym** *one* = *unit lift*

**translations**
(*type*) *one* ↽ (*type*) *unit lift*

**definition** *ONE* :: *one*
**where** *ONE* ≡ *Def* ()

Exhaustion and Elimination for type *one*

**lemma** *Exh-one*: $t = \bot \vee t = ONE$
**by** (*induct t*) (*simp-all add*: *ONE-def*)

**lemma** *oneE* [*case-names bottom ONE*]: $[\![p = \bot \Longrightarrow Q; p = ONE \Longrightarrow Q]\!] \Longrightarrow Q$
**by** (*induct p*) (*simp-all add*: *ONE-def*)

**lemma** *one-induct* [*case-names bottom ONE*]: $P\ \bot \Longrightarrow P\ ONE \Longrightarrow P\ x$
**by** (*cases x rule*: *oneE*) *simp-all*

**lemma** *dist-below-one* [*simp*]: $ONE \not\sqsubseteq \bot$
**by** (*simp add*: *ONE-def*)

**lemma** *below-ONE* [*simp*]: $x \sqsubseteq ONE$
**by** (*induct x rule*: *one-induct*) *simp-all*

**lemma** *ONE-below-iff* [*simp*]: $ONE \sqsubseteq x \longleftrightarrow x = ONE$
**by** (*induct x rule*: *one-induct*) *simp-all*

**lemma** *ONE-defined* [*simp*]: $ONE \neq \bot$
**by** (*simp add*: *ONE-def*)

**lemma** *one-neq-iffs* [*simp*]:
$x \neq ONE \longleftrightarrow x = \bot$
$ONE \neq x \longleftrightarrow x = \bot$
$x \neq \bot \longleftrightarrow x = ONE$
$\bot \neq x \longleftrightarrow x = ONE$

**by** (*induct x rule*: *one-induct*) *simp-all*

**lemma** *compact-ONE*: *compact ONE*
**by** (*rule compact-chfin*)

Case analysis function for type *one*

**definition** *one-case* :: *'a::pcpo* → *one* → *'a*
**where** *one-case* = (Λ *a x*. *seq*·*x*·*a*)

**translations**
*case x of XCONST ONE* ⇒ *t* ⇌ *CONST one-case*·*t*·*x*
*case x of XCONST ONE* :: *'a* ⇒ *t* ⇀ *CONST one-case*·*t*·*x*
Λ (*XCONST ONE*). *t* ⇌ *CONST one-case*·*t*

**lemma** *one-case1* [*simp*]: (*case* ⊥ *of ONE* ⇒ *t*) = ⊥
**by** (*simp add*: *one-case-def*)

**lemma** *one-case2* [*simp*]: (*case ONE of ONE* ⇒ *t*) = *t*
**by** (*simp add*: *one-case-def*)

**lemma** *one-case3* [*simp*]: (*case x of ONE* ⇒ *ONE*) = *x*
**by** (*induct x rule*: *one-induct*) *simp-all*

**end**

# 25 Fixed point operator and admissibility

**theory** *Fix*
**imports** *Cfun*
**begin**

**default-sort** *pcpo*

## 25.1 Iteration

**primrec** *iterate* :: *nat* ⇒ (*'a::cpo* → *'a*) → (*'a* → *'a*)
**where**
*iterate 0* = (Λ *F x*. *x*)
| *iterate* (*Suc n*) = (Λ *F x*. *F*·(*iterate n*·*F*·*x*))

Derive inductive properties of iterate from primitive recursion

**lemma** *iterate-0* [*simp*]: *iterate 0*·*F*·*x* = *x*
**by** *simp*

**lemma** *iterate-Suc* [*simp*]: *iterate* (*Suc n*)·*F*·*x* = *F*·(*iterate n*·*F*·*x*)
**by** *simp*

**declare** *iterate.simps* [*simp del*]

**lemma** *iterate-Suc2*: *iterate* (*Suc n*)·*F*·*x* = *iterate n*·*F*·(*F*·*x*)
**by** (*induct n*) *simp-all*

**lemma** *iterate-iterate*: *iterate m*·*F*·(*iterate n*·*F*·*x*) = *iterate* (*m* + *n*)·*F*·*x*
**by** (*induct m*) *simp-all*

The sequence of function iterations is a chain.

**lemma** *chain-iterate* [*simp*]: *chain* (λ*i*. *iterate i*·*F*·⊥)
**by** (*rule chainI*, *unfold iterate-Suc2*, *rule monofun-cfun-arg*, *rule minimal*)

## 25.2 Least fixed point operator

**definition** *fix* :: ('*a* → '*a*) → '*a*
**where** *fix* = (Λ *F*. ⨆ *i*. *iterate i*·*F*·⊥)

Binder syntax for *fix*

**abbreviation** *fix-syn* :: ('*a* ⇒ '*a*) ⇒ '*a* (**binder** μ 10)
**where** *fix-syn* (λ*x*. *f x*) ≡ *fix*·(Λ *x*. *f x*)

**notation** (*ASCII*)
*fix-syn* (**binder** *FIX* 10)

Properties of *fix*

direct connection between *fix* and iteration

**lemma** *fix-def2*: *fix*·*F* = (⨆ *i*. *iterate i*·*F*·⊥)
**by** (*simp add*: *fix-def*)

**lemma** *iterate-below-fix*: *iterate n*·*f*·⊥ ⊑ *fix*·*f*
**unfolding** *fix-def2*
**using** *chain-iterate* **by** (*rule is-ub-thelub*)

Kleene's fixed point theorems for continuous functions in pointed omega cpo's

**lemma** *fix-eq*: *fix*·*F* = *F*·(*fix*·*F*)
**apply** (*simp add*: *fix-def2*)
**apply** (*subst lub-range-shift* [*of - 1*, *symmetric*])
**apply** (*rule chain-iterate*)
**apply** (*subst contlub-cfun-arg*)
**apply** (*rule chain-iterate*)
**apply** *simp*
**done**

**lemma** *fix-least-below*: *F*·*x* ⊑ *x* ⟹ *fix*·*F* ⊑ *x*
**apply** (*simp add*: *fix-def2*)
**apply** (*rule lub-below*)
**apply** (*rule chain-iterate*)
**apply** (*induct-tac i*)

**apply** *simp*
**apply** *simp*
**apply** (*erule rev-below-trans*)
**apply** (*erule monofun-cfun-arg*)
**done**

**lemma** *fix-least*: $F{\cdot}x = x \Longrightarrow \mathit{fix}{\cdot}F \sqsubseteq x$
**by** (*rule fix-least-below*) *simp*

**lemma** *fix-eqI*:
**assumes** *fixed*: $F{\cdot}x = x$
**and** *least*: $\bigwedge z.\ F{\cdot}z = z \Longrightarrow x \sqsubseteq z$
**shows** $\mathit{fix}{\cdot}F = x$
**apply** (*rule below-antisym*)
**apply** (*rule fix-least* [*OF fixed*])
**apply** (*rule least* [*OF fix-eq* [*symmetric*]])
**done**

**lemma** *fix-eq2*: $f \equiv \mathit{fix}{\cdot}F \Longrightarrow f = F{\cdot}f$
**by** (*simp add*: *fix-eq* [*symmetric*])

**lemma** *fix-eq3*: $f \equiv \mathit{fix}{\cdot}F \Longrightarrow f{\cdot}x = F{\cdot}f{\cdot}x$
**by** (*erule fix-eq2* [*THEN cfun-fun-cong*])

**lemma** *fix-eq4*: $f = \mathit{fix}{\cdot}F \Longrightarrow f = F{\cdot}f$
**by** (*erule ssubst*) (*rule fix-eq*)

**lemma** *fix-eq5*: $f = \mathit{fix}{\cdot}F \Longrightarrow f{\cdot}x = F{\cdot}f{\cdot}x$
**by** (*erule fix-eq4* [*THEN cfun-fun-cong*])

strictness of *fix*

**lemma** *fix-bottom-iff*: $\mathit{fix}{\cdot}F = \bot \longleftrightarrow F{\cdot}\bot = \bot$
**apply** (*rule iffI*)
**apply** (*erule subst*)
**apply** (*rule fix-eq* [*symmetric*])
**apply** (*erule fix-least* [*THEN bottomI*])
**done**

**lemma** *fix-strict*: $F{\cdot}\bot = \bot \Longrightarrow \mathit{fix}{\cdot}F = \bot$
**by** (*simp add*: *fix-bottom-iff*)

**lemma** *fix-defined*: $F{\cdot}\bot \neq \bot \Longrightarrow \mathit{fix}{\cdot}F \neq \bot$
**by** (*simp add*: *fix-bottom-iff*)

*fix* applied to identity and constant functions

**lemma** *fix-id*: $(\mu\ x.\ x) = \bot$
**by** (*simp add*: *fix-strict*)

**lemma** *fix-const*: $(\mu\ x.\ c) = c$

**by** (*subst fix-eq*) *simp*

## 25.3 Fixed point induction

**lemma** *fix-ind*: *adm P* $\Longrightarrow$ *P* $\bot$ $\Longrightarrow$ ($\bigwedge$*x*. *P x* $\Longrightarrow$ *P* (*F*·*x*)) $\Longrightarrow$ *P* (*fix*·*F*)
**unfolding** *fix-def2*
**apply** (*erule admD*)
**apply** (*rule chain-iterate*)
**apply** (*rule nat-induct*, *simp-all*)
**done**

**lemma** *cont-fix-ind*: *cont F* $\Longrightarrow$ *adm P* $\Longrightarrow$ *P* $\bot$ $\Longrightarrow$ ($\bigwedge$*x*. *P x* $\Longrightarrow$ *P* (*F x*)) $\Longrightarrow$ *P* (*fix*·(*Abs-cfun F*))
**by** (*simp add*: *fix-ind*)

**lemma** *def-fix-ind*: $[\![f \equiv fix{\cdot}F;\ adm\ P;\ P\ \bot;\ \bigwedge x.\ P\ x \Longrightarrow P\ (F{\cdot}x)]\!] \Longrightarrow P\ f$
**by** (*simp add*: *fix-ind*)

**lemma** *fix-ind2*:
**assumes** *adm*: *adm P*
**assumes** *0*: *P* $\bot$ **and** *1*: *P* (*F*·$\bot$)
**assumes** *step*: $\bigwedge x.\ [\![P\ x;\ P\ (F{\cdot}x)]\!] \Longrightarrow P\ (F{\cdot}(F{\cdot}x))$
**shows** *P* (*fix*·*F*)
**unfolding** *fix-def2*
**apply** (*rule admD* [*OF adm chain-iterate*])
**apply** (*rule nat-less-induct*)
**apply** (*case-tac n*)
**apply** (*simp add*: *0*)
**apply** (*case-tac nat*)
**apply** (*simp add*: *1*)
**apply** (*frule-tac x*=*nat* **in** *spec*)
**apply** (*simp add*: *step*)
**done**

**lemma** *parallel-fix-ind*:
**assumes** *adm*: *adm* ($\lambda$*x*. *P* (*fst x*) (*snd x*))
**assumes** *base*: *P* $\bot$ $\bot$
**assumes** *step*: $\bigwedge x\ y.\ P\ x\ y \Longrightarrow P\ (F{\cdot}x)\ (G{\cdot}y)$
**shows** *P* (*fix*·*F*) (*fix*·*G*)
**proof** −
**from** *adm* **have** *adm'*: *adm* (*case-prod P*)
**unfolding** *split-def* **.**
**have** *P* (*iterate i*·*F*·$\bot$) (*iterate i*·*G*·$\bot$) **for** *i*
**by** (*induct i*) (*simp add*: *base*, *simp add*: *step*)
**then have** $\bigwedge i$. *case-prod P* (*iterate i*·*F*·$\bot$, *iterate i*·*G*·$\bot$)
**by** *simp*
**then have** *case-prod P* ($\bigsqcup i$. (*iterate i*·*F*·$\bot$, *iterate i*·*G*·$\bot$))
**by** − (*rule admD* [*OF adm'*], *simp*, *assumption*)
**then have** *case-prod P* ($\bigsqcup i$. *iterate i*·*F*·$\bot$, $\bigsqcup i$. *iterate i*·*G*·$\bot$)

**by** (*simp add*: *lub-Pair*)
**then have** *P* (⨆ *i*. *iterate i*·*F*·⊥) (⨆ *i*. *iterate i*·*G*·⊥)
**by** *simp*
**then show** *P* (*fix*·*F*) (*fix*·*G*)
**by** (*simp add*: *fix-def2*)
**qed**

**lemma** *cont-parallel-fix-ind*:
**assumes** *cont F* **and** *cont G*
**assumes** *adm* (λ*x*. *P* (*fst x*) (*snd x*))
**assumes** *P* ⊥ ⊥
**assumes** ⋀*x y*. *P x y* ⟹ *P* (*F x*) (*G y*)
**shows** *P* (*fix*·(*Abs-cfun F*)) (*fix*·(*Abs-cfun G*))
**by** (*rule parallel-fix-ind*) (*simp-all add*: *assms*)

## 25.4 Fixed-points on product types

Bekic's Theorem: Simultaneous fixed points over pairs can be written in terms of separate fixed points.

**lemma** *fix-cprod*:
*fix*·(*F*::'*a* × '*b* → '*a* × '*b*) =
(μ *x*. *fst* (*F*·(*x*, μ *y*. *snd* (*F*·(*x*, *y*)))),
μ *y*. *snd* (*F*·(μ *x*. *fst* (*F*·(*x*, μ *y*. *snd* (*F*·(*x*, *y*)))), *y*)))
(**is** *fix*·*F* = (*?x*, *?y*))
**proof** (*rule fix-eqI*)
**have** ∗: *fst* (*F*·(*?x*, *?y*)) = *?x*
**by** (*rule trans* [*symmetric*, *OF fix-eq*], *simp*)
**have** *snd* (*F*·(*?x*, *?y*)) = *?y*
**by** (*rule trans* [*symmetric*, *OF fix-eq*], *simp*)
**with** ∗ **show** *F*·(*?x*, *?y*) = (*?x*, *?y*)
**by** (*simp add*: *prod-eq-iff*)
**next**
**fix** *z*
**assume** *F-z*: *F*·*z* = *z*
**obtain** *x y* **where** *z*: *z* = (*x*, *y*) **by** (*rule prod.exhaust*)
**from** *F-z z* **have** *F-x*: *fst* (*F*·(*x*, *y*)) = *x* **by** *simp*
**from** *F-z z* **have** *F-y*: *snd* (*F*·(*x*, *y*)) = *y* **by** *simp*
**let** *?y1* = μ *y*. *snd* (*F*·(*x*, *y*))
**have** *?y1* ⊑ *y*
**by** (*rule fix-least*) (*simp add*: *F-y*)
**then have** *fst* (*F*·(*x*, *?y1*)) ⊑ *fst* (*F*·(*x*, *y*))
**by** (*simp add*: *fst-monofun monofun-cfun*)
**with** *F-x* **have** *fst* (*F*·(*x*, *?y1*)) ⊑ *x*
**by** *simp*
**then have** ∗: *?x* ⊑ *x*
**by** (*simp add*: *fix-least-below*)
**then have** *snd* (*F*·(*?x*, *y*)) ⊑ *snd* (*F*·(*x*, *y*))
**by** (*simp add*: *snd-monofun monofun-cfun*)
**with** *F-y* **have** *snd* (*F*·(*?x*, *y*)) ⊑ *y*

**by** *simp*
**then have** *?y* $\sqsubseteq$ *y*
**by** (*simp add*: *fix-least-below*)
**with** *z* $*$ **show** (*?x*, *?y*) $\sqsubseteq$ *z*
**by** *simp*
**qed**

**end**

# 26 Package for defining recursive functions in HOLCF

**theory** *Fixrec*
**imports** *Cprod Sprod Ssum Up One Tr Fix*
**keywords** *fixrec* :: *thy-defn*
**begin**

## 26.1 Pattern-match monad

**default-sort** *cpo*

**pcpodef** *'a match* = *UNIV*::(*one* ++ *'a u*) *set*
**by** *simp-all*

**definition**
*fail* :: *'a match* **where**
*fail* = *Abs-match* (*sinl*·*ONE*)

**definition**
*succeed* :: *'a* $\rightarrow$ *'a match* **where**
*succeed* = ($\Lambda$ *x*. *Abs-match* (*sinr*·(*up*·*x*)))

**lemma** *matchE* [*case-names bottom fail succeed*, *cases type*: *match*]:
⟦*p* = $\bot$ $\Longrightarrow$ *Q*; *p* = *fail* $\Longrightarrow$ *Q*; $\bigwedge$*x*. *p* = *succeed*·*x* $\Longrightarrow$ *Q*⟧ $\Longrightarrow$ *Q*
**unfolding** *fail-def succeed-def*
**apply** (*cases p*, *rename-tac r*)
**apply** (*rule-tac p*=*r* **in** *ssumE*, *simp add*: *Abs-match-strict*)
**apply** (*rule-tac p*=*x* **in** *oneE*, *simp*, *simp*)
**apply** (*rule-tac p*=*y* **in** *upE*, *simp*, *simp add*: *cont-Abs-match*)
**done**

**lemma** *succeed-defined* [*simp*]: *succeed*·*x* $\neq$ $\bot$
**by** (*simp add*: *succeed-def cont-Abs-match Abs-match-bottom-iff*)

**lemma** *fail-defined* [*simp*]: *fail* $\neq$ $\bot$
**by** (*simp add*: *fail-def Abs-match-bottom-iff*)

**lemma** *succeed-eq* [*simp*]: (*succeed*·*x* = *succeed*·*y*) = (*x* = *y*)
**by** (*simp add*: *succeed-def cont-Abs-match Abs-match-inject*)

**lemma** *succeed-neq-fail* [*simp*]:
*succeed·x* ≠ *fail fail* ≠ *succeed·x*
**by** (*simp-all add*: *succeed-def fail-def cont-Abs-match Abs-match-inject*)

### 26.1.1 Run operator

**definition**
*run* :: *'a match* → *'a*::*pcpo* **where**
*run* = (Λ *m*. *sscase·*⊥*·*(*fup·ID*)*·*(*Rep-match m*))

rewrite rules for run

**lemma** *run-strict* [*simp*]: *run·*⊥ = ⊥
**unfolding** *run-def*
**by** (*simp add*: *cont-Rep-match Rep-match-strict*)

**lemma** *run-fail* [*simp*]: *run·fail* = ⊥
**unfolding** *run-def fail-def*
**by** (*simp add*: *cont-Rep-match Abs-match-inverse*)

**lemma** *run-succeed* [*simp*]: *run·*(*succeed·x*) = *x*
**unfolding** *run-def succeed-def*
**by** (*simp add*: *cont-Rep-match cont-Abs-match Abs-match-inverse*)

### 26.1.2 Monad plus operator

**definition**
*mplus* :: *'a match* → *'a match* → *'a match* **where**
*mplus* = (Λ *m1 m2*. *sscase·*(Λ -. *m2*)*·*(Λ -. *m1*)*·*(*Rep-match m1*))

**abbreviation**
*mplus-syn* :: [*'a match*, *'a match*] ⇒ *'a match* (**infixr** +++ *65*) **where**
*m1* +++ *m2* == *mplus·m1·m2*

rewrite rules for mplus

**lemma** *mplus-strict* [*simp*]: ⊥ +++ *m* = ⊥
**unfolding** *mplus-def*
**by** (*simp add*: *cont-Rep-match Rep-match-strict*)

**lemma** *mplus-fail* [*simp*]: *fail* +++ *m* = *m*
**unfolding** *mplus-def fail-def*
**by** (*simp add*: *cont-Rep-match Abs-match-inverse*)

**lemma** *mplus-succeed* [*simp*]: *succeed·x* +++ *m* = *succeed·x*
**unfolding** *mplus-def succeed-def*
**by** (*simp add*: *cont-Rep-match cont-Abs-match Abs-match-inverse*)

**lemma** *mplus-fail2* [*simp*]: *m* +++ *fail* = *m*
**by** (*cases m*, *simp-all*)

**lemma** *mplus-assoc*: $(x +\!\!+\!\!+ y) +\!\!+\!\!+ z = x +\!\!+\!\!+ (y +\!\!+\!\!+ z)$
**by** (*cases x*, *simp-all*)

## 26.2 Match functions for built-in types

**default-sort** *pcpo*

**definition**
*match-bottom* :: $'a \to 'c\ match \to 'c\ match$
**where**
*match-bottom* = $(\Lambda\ x\ k.\ seq{\cdot}x{\cdot}fail)$

**definition**
*match-Pair* :: $'a{::}cpo \times 'b{::}cpo \to ('a \to 'b \to 'c\ match) \to 'c\ match$
**where**
*match-Pair* = $(\Lambda\ x\ k.\ csplit{\cdot}k{\cdot}x)$

**definition**
*match-spair* :: $'a \otimes 'b \to ('a \to 'b \to 'c\ match) \to 'c\ match$
**where**
*match-spair* = $(\Lambda\ x\ k.\ ssplit{\cdot}k{\cdot}x)$

**definition**
*match-sinl* :: $'a \oplus 'b \to ('a \to 'c\ match) \to 'c\ match$
**where**
*match-sinl* = $(\Lambda\ x\ k.\ sscase{\cdot}k{\cdot}(\Lambda\ b.\ fail){\cdot}x)$

**definition**
*match-sinr* :: $'a \oplus 'b \to ('b \to 'c\ match) \to 'c\ match$
**where**
*match-sinr* = $(\Lambda\ x\ k.\ sscase{\cdot}(\Lambda\ a.\ fail){\cdot}k{\cdot}x)$

**definition**
*match-up* :: $'a{::}cpo\ u \to ('a \to 'c\ match) \to 'c\ match$
**where**
*match-up* = $(\Lambda\ x\ k.\ fup{\cdot}k{\cdot}x)$

**definition**
*match-ONE* :: *one* $\to 'c\ match \to 'c\ match$
**where**
*match-ONE* = $(\Lambda\ ONE\ k.\ k)$

**definition**
*match-TT* :: *tr* $\to 'c\ match \to 'c\ match$
**where**
*match-TT* = $(\Lambda\ x\ k.\ If\ x\ then\ k\ else\ fail)$

**definition**
*match-FF* :: *tr* $\to 'c\ match \to 'c\ match$

**where**
*match-FF* = (Λ *x k*. *If x then fail else k*)

**lemma** *match-bottom-simps* [*simp*]:
*match-bottom*·*x*·*k* = (*if x* = ⊥ *then* ⊥ *else fail*)
**by** (*simp add*: *match-bottom-def*)

**lemma** *match-Pair-simps* [*simp*]:
*match-Pair*·(*x*, *y*)·*k* = *k*·*x*·*y*
**by** (*simp-all add*: *match-Pair-def*)

**lemma** *match-spair-simps* [*simp*]:
⟦*x* ≠ ⊥; *y* ≠ ⊥⟧ ⟹ *match-spair*·(:*x*, *y*:)·*k* = *k*·*x*·*y*
*match-spair*·⊥·*k* = ⊥
**by** (*simp-all add*: *match-spair-def*)

**lemma** *match-sinl-simps* [*simp*]:
*x* ≠ ⊥ ⟹ *match-sinl*·(*sinl*·*x*)·*k* = *k*·*x*
*y* ≠ ⊥ ⟹ *match-sinl*·(*sinr*·*y*)·*k* = *fail*
*match-sinl*·⊥·*k* = ⊥
**by** (*simp-all add*: *match-sinl-def*)

**lemma** *match-sinr-simps* [*simp*]:
*x* ≠ ⊥ ⟹ *match-sinr*·(*sinl*·*x*)·*k* = *fail*
*y* ≠ ⊥ ⟹ *match-sinr*·(*sinr*·*y*)·*k* = *k*·*y*
*match-sinr*·⊥·*k* = ⊥
**by** (*simp-all add*: *match-sinr-def*)

**lemma** *match-up-simps* [*simp*]:
*match-up*·(*up*·*x*)·*k* = *k*·*x*
*match-up*·⊥·*k* = ⊥
**by** (*simp-all add*: *match-up-def*)

**lemma** *match-ONE-simps* [*simp*]:
*match-ONE*·*ONE*·*k* = *k*
*match-ONE*·⊥·*k* = ⊥
**by** (*simp-all add*: *match-ONE-def*)

**lemma** *match-TT-simps* [*simp*]:
*match-TT*·*TT*·*k* = *k*
*match-TT*·*FF*·*k* = *fail*
*match-TT*·⊥·*k* = ⊥
**by** (*simp-all add*: *match-TT-def*)

**lemma** *match-FF-simps* [*simp*]:
*match-FF*·*FF*·*k* = *k*
*match-FF*·*TT*·*k* = *fail*
*match-FF*·⊥·*k* = ⊥
**by** (*simp-all add*: *match-FF-def*)

## 26.3 Mutual recursion

The following rules are used to prove unfolding theorems from fixed-point definitions of mutually recursive functions.

**lemma** *Pair-equalI*: $[\![x \equiv \mathit{fst}\ p;\ y \equiv \mathit{snd}\ p]\!] \Longrightarrow (x,\ y) \equiv p$
**by** *simp*

**lemma** *Pair-eqD1*: $(x,\ y) = (x',\ y') \Longrightarrow x = x'$
**by** *simp*

**lemma** *Pair-eqD2*: $(x,\ y) = (x',\ y') \Longrightarrow y = y'$
**by** *simp*

**lemma** *def-cont-fix-eq*:
$[\![f \equiv \mathit{fix}{\cdot}(\mathit{Abs\text{-}cfun}\ F);\ \mathit{cont}\ F]\!] \Longrightarrow f = F\ f$
**by** (*simp*, *subst fix-eq*, *simp*)

**lemma** *def-cont-fix-ind*:
$[\![f \equiv \mathit{fix}{\cdot}(\mathit{Abs\text{-}cfun}\ F);\ \mathit{cont}\ F;\ \mathit{adm}\ P;\ P\ \bot;\ \bigwedge x.\ P\ x \Longrightarrow P\ (F\ x)]\!] \Longrightarrow P\ f$
**by** (*simp add*: *fix-ind*)

lemma for proving rewrite rules

**lemma** *ssubst-lhs*: $[\![t = s;\ P\ s = Q]\!] \Longrightarrow P\ t = Q$
**by** *simp*

## 26.4 Initializing the fixrec package

**ML-file** ‹*Tools/holcf-library.ML*›
**ML-file** ‹*Tools/fixrec.ML*›

**method-setup** *fixrec-simp* = ‹
*Scan.succeed* (*SIMPLE-METHOD'* *o* *Fixrec.fixrec-simp-tac*)
› *pattern prover for fixrec constants*

**setup** ‹
*Fixrec.add-matchers*
[ (***const-name*** ‹*up*›, ***const-name*** ‹*match-up*›),
(***const-name*** ‹*sinl*›, ***const-name*** ‹*match-sinl*›),
(***const-name*** ‹*sinr*›, ***const-name*** ‹*match-sinr*›),
(***const-name*** ‹*spair*›, ***const-name*** ‹*match-spair*›),
(***const-name*** ‹*Pair*›, ***const-name*** ‹*match-Pair*›),
(***const-name*** ‹*ONE*›, ***const-name*** ‹*match-ONE*›),
(***const-name*** ‹*TT*›, ***const-name*** ‹*match-TT*›),
(***const-name*** ‹*FF*›, ***const-name*** ‹*match-FF*›),
(***const-name*** ‹*bottom*›, ***const-name*** ‹*match-bottom*›) ]
›

**hide-const** (**open**) *succeed fail run*

**end**

# 27 Domain package support

**theory** *Domain-Aux*
**imports** *Map-Functions Fixrec*
**begin**

## 27.1 Continuous isomorphisms

A locale for continuous isomorphisms

**locale** *iso* =
  **fixes** *abs* :: $'a \to 'b$
  **fixes** *rep* :: $'b \to 'a$
  **assumes** *abs-iso* [*simp*]: $rep{\cdot}(abs{\cdot}x) = x$
  **assumes** *rep-iso* [*simp*]: $abs{\cdot}(rep{\cdot}y) = y$
**begin**

**lemma** *swap*: *iso rep abs*
  **by** (*rule iso.intro* [*OF rep-iso abs-iso*])

**lemma** *abs-below*: $(abs{\cdot}x \sqsubseteq abs{\cdot}y) = (x \sqsubseteq y)$
**proof**
  **assume** $abs{\cdot}x \sqsubseteq abs{\cdot}y$
  **then have** $rep{\cdot}(abs{\cdot}x) \sqsubseteq rep{\cdot}(abs{\cdot}y)$ **by** (*rule monofun-cfun-arg*)
  **then show** $x \sqsubseteq y$ **by** *simp*
**next**
  **assume** $x \sqsubseteq y$
  **then show** $abs{\cdot}x \sqsubseteq abs{\cdot}y$ **by** (*rule monofun-cfun-arg*)
**qed**

**lemma** *rep-below*: $(rep{\cdot}x \sqsubseteq rep{\cdot}y) = (x \sqsubseteq y)$
  **by** (*rule iso.abs-below* [*OF swap*])

**lemma** *abs-eq*: $(abs{\cdot}x = abs{\cdot}y) = (x = y)$
  **by** (*simp add*: *po-eq-conv abs-below*)

**lemma** *rep-eq*: $(rep{\cdot}x = rep{\cdot}y) = (x = y)$
  **by** (*rule iso.abs-eq* [*OF swap*])

**lemma** *abs-strict*: $abs{\cdot}\bot = \bot$
**proof** −
  **have** $\bot \sqsubseteq rep{\cdot}\bot$ **..**
  **then have** $abs{\cdot}\bot \sqsubseteq abs{\cdot}(rep{\cdot}\bot)$ **by** (*rule monofun-cfun-arg*)
  **then have** $abs{\cdot}\bot \sqsubseteq \bot$ **by** *simp*
  **then show** *?thesis* **by** (*rule bottomI*)
**qed**

**lemma** *rep-strict*: $rep{\cdot}\bot = \bot$
**by** (*rule iso.abs-strict* [*OF swap*])

**lemma** *abs-defin'*: $abs{\cdot}x = \bot \Longrightarrow x = \bot$
**proof** −
**have** $x = rep{\cdot}(abs{\cdot}x)$ **by** *simp*
**also assume** $abs{\cdot}x = \bot$
**also note** *rep-strict*
**finally show** $x = \bot$ **.**
**qed**

**lemma** *rep-defin'*: $rep{\cdot}z = \bot \Longrightarrow z = \bot$
**by** (*rule iso.abs-defin'* [*OF swap*])

**lemma** *abs-defined*: $z \neq \bot \Longrightarrow abs{\cdot}z \neq \bot$
**by** (*erule contrapos-nn*, *erule abs-defin'*)

**lemma** *rep-defined*: $z \neq \bot \Longrightarrow rep{\cdot}z \neq \bot$
**by** (*rule iso.abs-defined* [*OF iso.swap*]) (*rule iso-axioms*)

**lemma** *abs-bottom-iff*: $(abs{\cdot}x = \bot) = (x = \bot)$
**by** (*auto elim*: *abs-defin' intro*: *abs-strict*)

**lemma** *rep-bottom-iff*: $(rep{\cdot}x = \bot) = (x = \bot)$
**by** (*rule iso.abs-bottom-iff* [*OF iso.swap*]) (*rule iso-axioms*)

**lemma** *casedist-rule*: $rep{\cdot}x = \bot \vee P \Longrightarrow x = \bot \vee P$
**by** (*simp add*: *rep-bottom-iff*)

**lemma** *compact-abs-rev*: *compact* $(abs{\cdot}x) \Longrightarrow$ *compact* $x$
**proof** (*unfold compact-def*)
**assume** *adm* $(\lambda y.\ abs{\cdot}x \not\sqsubseteq y)$
**with** *cont-Rep-cfun2*
**have** *adm* $(\lambda y.\ abs{\cdot}x \not\sqsubseteq abs{\cdot}y)$ **by** (*rule adm-subst*)
**then show** *adm* $(\lambda y.\ x \not\sqsubseteq y)$ **using** *abs-below* **by** *simp*
**qed**

**lemma** *compact-rep-rev*: *compact* $(rep{\cdot}x) \Longrightarrow$ *compact* $x$
**by** (*rule iso.compact-abs-rev* [*OF iso.swap*]) (*rule iso-axioms*)

**lemma** *compact-abs*: *compact* $x \Longrightarrow$ *compact* $(abs{\cdot}x)$
**by** (*rule compact-rep-rev*) *simp*

**lemma** *compact-rep*: *compact* $x \Longrightarrow$ *compact* $(rep{\cdot}x)$
**by** (*rule iso.compact-abs* [*OF iso.swap*]) (*rule iso-axioms*)

**lemma** *iso-swap*: $(x = abs{\cdot}y) = (rep{\cdot}x = y)$
**proof**
**assume** $x = abs{\cdot}y$

**then have** $rep{\cdot}x = rep{\cdot}(abs{\cdot}y)$ **by** *simp*
**then show** $rep{\cdot}x = y$ **by** *simp*
**next**
**assume** $rep{\cdot}x = y$
**then have** $abs{\cdot}(rep{\cdot}x) = abs{\cdot}y$ **by** *simp*
**then show** $x = abs{\cdot}y$ **by** *simp*
**qed**

**end**

## 27.2 Proofs about take functions

This section contains lemmas that are used in a module that supports the domain isomorphism package; the module contains proofs related to take functions and the finiteness predicate.

**lemma** *deflation-abs-rep*:
**fixes** *abs* **and** *rep* **and** *d*
**assumes** *abs-iso*: $\bigwedge x.\ rep{\cdot}(abs{\cdot}x) = x$
**assumes** *rep-iso*: $\bigwedge y.\ abs{\cdot}(rep{\cdot}y) = y$
**shows** *deflation d* $\Longrightarrow$ *deflation* (*abs oo d oo rep*)
**by** (*rule ep-pair.deflation-e-d-p*) (*simp add*: *ep-pair.intro assms*)

**lemma** *deflation-chain-min*:
**assumes** *chain*: *chain d*
**assumes** *defl*: $\bigwedge n.$ *deflation* (*d n*)
**shows** $d\ m{\cdot}(d\ n{\cdot}x) = d\ (min\ m\ n){\cdot}x$
**proof** (*rule linorder-le-cases*)
**assume** $m \le n$
**with** *chain* **have** $d\ m \sqsubseteq d\ n$ **by** (*rule chain-mono*)
**then have** $d\ m{\cdot}(d\ n{\cdot}x) = d\ m{\cdot}x$
**by** (*rule deflation-below-comp1* [*OF defl defl*])
**moreover from** ‹$m \le n$› **have** $min\ m\ n = m$ **by** *simp*
**ultimately show** *?thesis* **by** *simp*
**next**
**assume** $n \le m$
**with** *chain* **have** $d\ n \sqsubseteq d\ m$ **by** (*rule chain-mono*)
**then have** $d\ m{\cdot}(d\ n{\cdot}x) = d\ n{\cdot}x$
**by** (*rule deflation-below-comp2* [*OF defl defl*])
**moreover from** ‹$n \le m$› **have** $min\ m\ n = n$ **by** *simp*
**ultimately show** *?thesis* **by** *simp*
**qed**

**lemma** *lub-ID-take-lemma*:
**assumes** *chain t* **and** $(\bigsqcup n.\ t\ n) = ID$
**assumes** $\bigwedge n.\ t\ n{\cdot}x = t\ n{\cdot}y$ **shows** $x = y$
**proof** −
**have** $(\bigsqcup n.\ t\ n{\cdot}x) = (\bigsqcup n.\ t\ n{\cdot}y)$
**using** *assms*(*3*) **by** *simp*
**then have** $(\bigsqcup n.\ t\ n){\cdot}x = (\bigsqcup n.\ t\ n){\cdot}y$

**using** *assms*(*1*) **by** (*simp add*: *lub-distribs*)
**then show** $x = y$
**using** *assms*(*2*) **by** *simp*
**qed**

**lemma** *lub-ID-reach*:
**assumes** *chain t* **and** $(\bigsqcup n.\ t\ n) = ID$
**shows** $(\bigsqcup n.\ t\ n{\cdot}x) = x$
**using** *assms* **by** (*simp add*: *lub-distribs*)

**lemma** *lub-ID-take-induct*:
**assumes** *chain t* **and** $(\bigsqcup n.\ t\ n) = ID$
**assumes** *adm P* **and** $\bigwedge n.\ P\ (t\ n{\cdot}x)$ **shows** $P\ x$
**proof** −
**from** ‹*chain t*› **have** *chain* $(\lambda n.\ t\ n{\cdot}x)$ **by** *simp*
**from** ‹*adm P*› *this* ‹$\bigwedge n.\ P\ (t\ n{\cdot}x)$› **have** $P\ (\bigsqcup n.\ t\ n{\cdot}x)$ **by** (*rule admD*)
**with** ‹*chain t*› ‹$(\bigsqcup n.\ t\ n) = ID$› **show** $P\ x$ **by** (*simp add*: *lub-distribs*)
**qed**

## 27.3 Finiteness

Let a "decisive" function be a deflation that maps every input to either itself or bottom. Then if a domain's take functions are all decisive, then all values in the domain are finite.

**definition**
*decisive* :: $('a{::}pcpo \to 'a) \Rightarrow bool$
**where**
*decisive d* $\longleftrightarrow (\forall x.\ d{\cdot}x = x \vee d{\cdot}x = \bot)$

**lemma** *decisiveI*: $(\bigwedge x.\ d{\cdot}x = x \vee d{\cdot}x = \bot) \Longrightarrow$ *decisive d*
**unfolding** *decisive-def* **by** *simp*

**lemma** *decisive-cases*:
**assumes** *decisive d* **obtains** $d{\cdot}x = x \mid d{\cdot}x = \bot$
**using** *assms* **unfolding** *decisive-def* **by** *auto*

**lemma** *decisive-bottom*: *decisive* $\bot$
**unfolding** *decisive-def* **by** *simp*

**lemma** *decisive-ID*: *decisive ID*
**unfolding** *decisive-def* **by** *simp*

**lemma** *decisive-ssum-map*:
**assumes** *f*: *decisive f*
**assumes** *g*: *decisive g*
**shows** *decisive* (*ssum-map*·*f*·*g*)
**apply** (*rule decisiveI*)
**subgoal for** *s*
**apply** (*cases s*, *simp-all*)

```
    apply (rule-tac x=x in decisive-cases [OF f], simp-all)
   apply (rule-tac x=y in decisive-cases [OF g], simp-all)
   done
 done

lemma decisive-sprod-map:
 assumes f: decisive f
 assumes g: decisive g
 shows decisive (sprod-map·f·g)
 apply (rule decisiveI)
 subgoal for s
  apply (cases s, simp)
  subgoal for x y
   apply (rule decisive-cases [OF f, where x = x], simp-all)
   apply (rule decisive-cases [OF g, where x = y], simp-all)
   done
  done
 done

lemma decisive-abs-rep:
 fixes abs rep
 assumes iso: iso abs rep
 assumes d: decisive d
 shows decisive (abs oo d oo rep)
 apply (rule decisiveI)
 subgoal for s
  apply (rule decisive-cases [OF d, where x=rep·s])
   apply (simp add: iso.rep-iso [OF iso])
  apply (simp add: iso.abs-strict [OF iso])
  done
 done

lemma lub-ID-finite:
 assumes chain: chain d
 assumes lub: (⨆ n. d n) = ID
 assumes decisive: ⋀n. decisive (d n)
 shows ∃ n. d n·x = x
proof −
 have 1: chain (λn. d n·x) using chain by simp
 have 2: (⨆ n. d n·x) = x using chain lub by (rule lub-ID-reach)
 have ∀ n. d n·x = x ∨ d n·x = ⊥
  using decisive unfolding decisive-def by simp
 hence range (λn. d n·x) ⊆ {x, ⊥}
  by auto
 hence finite (range (λn. d n·x))
  by (rule finite-subset, simp)
 with 1 have finite-chain (λn. d n·x)
  by (rule finite-range-imp-finch)
 then have ∃ n. (⨆ n. d n·x) = d n·x
```

**unfolding** *finite-chain-def* **by** (*auto simp add*: *maxinch-is-thelub*)
**with** *2* **show** $\exists\, n.\ d\ n{\cdot}x = x$ **by** (*auto elim*: *sym*)
**qed**

**lemma** *lub-ID-finite-take-induct*:
**assumes** *chain d* **and** $(\bigsqcup n.\ d\ n) = ID$ **and** $\bigwedge n.\ decisive\ (d\ n)$
**shows** $(\bigwedge n.\ P\ (d\ n{\cdot}x)) \Longrightarrow P\ x$
**using** *lub-ID-finite* [*OF assms*] **by** *metis*

## 27.4 Proofs about constructor functions

Lemmas for proving nchotomy rule:

**lemma** *ex-one-bottom-iff*:
$(\exists\, x.\ P\ x \wedge x \neq \bot) = P\ ONE$
**by** *simp*

**lemma** *ex-up-bottom-iff*:
$(\exists\, x.\ P\ x \wedge x \neq \bot) = (\exists\, x.\ P\ (up{\cdot}x))$
**by** (*safe*, *case-tac x*, *auto*)

**lemma** *ex-sprod-bottom-iff*:
$(\exists\, y.\ P\ y \wedge y \neq \bot) =$
$(\exists\, x\ y.\ (P\ (:x,\ y:) \wedge x \neq \bot) \wedge y \neq \bot)$
**by** (*safe*, *case-tac y*, *auto*)

**lemma** *ex-sprod-up-bottom-iff*:
$(\exists\, y.\ P\ y \wedge y \neq \bot) =$
$(\exists\, x\ y.\ P\ (:up{\cdot}x,\ y:) \wedge y \neq \bot)$
**by** (*safe*, *case-tac y*, *simp*, *case-tac x*, *auto*)

**lemma** *ex-ssum-bottom-iff*:
$(\exists\, x.\ P\ x \wedge x \neq \bot) =$
$((\exists\, x.\ P\ (sinl{\cdot}x) \wedge x \neq \bot) \vee$
$(\exists\, x.\ P\ (sinr{\cdot}x) \wedge x \neq \bot))$
**by** (*safe*, *case-tac x*, *auto*)

**lemma** *exh-start*: $p = \bot \vee (\exists\, x.\ p = x \wedge x \neq \bot)$
**by** *auto*

**lemmas** *ex-bottom-iffs* =
*ex-ssum-bottom-iff*
*ex-sprod-up-bottom-iff*
*ex-sprod-bottom-iff*
*ex-up-bottom-iff*
*ex-one-bottom-iff*

Rules for turning nchotomy into exhaust:

**lemma** *exh-casedist0*: $[\![R;\ R \Longrightarrow P]\!] \Longrightarrow P$
**by** *auto*

**lemma** *exh-casedist1*: $((P \lor Q \Longrightarrow R) \Longrightarrow S) \equiv (\llbracket P \Longrightarrow R;\ Q \Longrightarrow R \rrbracket \Longrightarrow S)$
**by** *rule auto*

**lemma** *exh-casedist2*: $(\exists\, x.\ P\ x \Longrightarrow Q) \equiv (\bigwedge x.\ P\ x \Longrightarrow Q)$
**by** *rule auto*

**lemma** *exh-casedist3*: $(P \land Q \Longrightarrow R) \equiv (P \Longrightarrow Q \Longrightarrow R)$
**by** *rule auto*

**lemmas** *exh-casedists* = *exh-casedist1 exh-casedist2 exh-casedist3*

Rules for proving constructor properties

**lemmas** *con-strict-rules* =
*sinl-strict sinr-strict spair-strict1 spair-strict2*

**lemmas** *con-bottom-iff-rules* =
*sinl-bottom-iff sinr-bottom-iff spair-bottom-iff up-defined ONE-defined*

**lemmas** *con-below-iff-rules* =
*sinl-below sinr-below sinl-below-sinr sinr-below-sinl con-bottom-iff-rules*

**lemmas** *con-eq-iff-rules* =
*sinl-eq sinr-eq sinl-eq-sinr sinr-eq-sinl con-bottom-iff-rules*

**lemmas** *sel-strict-rules* =
*cfcomp2 sscase1 sfst-strict ssnd-strict fup1*

**lemma** *sel-app-extra-rules*:
$sscase{\cdot}ID{\cdot}\bot{\cdot}(sinr{\cdot}x) = \bot$
$sscase{\cdot}ID{\cdot}\bot{\cdot}(sinl{\cdot}x) = x$
$sscase{\cdot}\bot{\cdot}ID{\cdot}(sinl{\cdot}x) = \bot$
$sscase{\cdot}\bot{\cdot}ID{\cdot}(sinr{\cdot}x) = x$
$fup{\cdot}ID{\cdot}(up{\cdot}x) = x$
**by** (*cases* $x = \bot$, *simp*, *simp*)+

**lemmas** *sel-app-rules* =
*sel-strict-rules sel-app-extra-rules*
*ssnd-spair sfst-spair up-defined spair-defined*

**lemmas** *sel-bottom-iff-rules* =
*cfcomp2 sfst-bottom-iff ssnd-bottom-iff*

**lemmas** *take-con-rules* =
*ssum-map-sinl' ssum-map-sinr' sprod-map-spair' u-map-up*
*deflation-strict deflation-ID ID1 cfcomp2*

### 27.5 ML setup

**named-theorems** *domain-deflation theorems like deflation a ==> deflation (foo-map$a)*
**and** *domain-map-ID theorems like foo-map$ID = ID*

**ML-file** ‹*Tools/Domain/domain-take-proofs.ML*›
**ML-file** ‹*Tools/cont-consts.ML*›
**ML-file** ‹*Tools/cont-proc.ML*›
**ML-file** ‹*Tools/Domain/domain-constructors.ML*›
**ML-file** ‹*Tools/Domain/domain-induction.ML*›

**end**

## 28 Domain package

**theory** *Domain*
**imports** *Representable Domain-Aux*
**keywords**
*lazy unsafe* **and**
*domaindef domain* :: *thy-defn* **and**
*domain-isomorphism* :: *thy-decl*
**begin**

**default-sort** *domain*

### 28.1 Representations of types

**lemma** *emb-prj*: *emb·((prj·x)::'a) = cast·DEFL('a)·x*
**by** (*simp add*: *cast-DEFL*)

**lemma** *emb-prj-emb*:
**fixes** *x* :: *'a*
**assumes** *DEFL('a)* ⊑ *DEFL('b)*
**shows** *emb·(prj·(emb·x) :: 'b) = emb·x*
**unfolding** *emb-prj*
**apply** (*rule cast.belowD*)
**apply** (*rule monofun-cfun-arg* [*OF assms*])
**apply** (*simp add*: *cast-DEFL*)
**done**

**lemma** *prj-emb-prj*:
**assumes** *DEFL('a)* ⊑ *DEFL('b)*
**shows** *prj·(emb·(prj·x :: 'b)) = (prj·x :: 'a)*
**apply** (*rule emb-eq-iff* [*THEN iffD1*])
**apply** (*simp only*: *emb-prj*)
**apply** (*rule deflation-below-comp1*)
**apply** (*rule deflation-cast*)
**apply** (*rule deflation-cast*)
**apply** (*rule monofun-cfun-arg* [*OF assms*])

**done**

Isomorphism lemmas used internally by the domain package:

**lemma** *domain-abs-iso*:
**fixes** *abs* **and** *rep*
**assumes** *DEFL*: $DEFL('b) = DEFL('a)$
**assumes** *abs-def*: $(abs :: 'a \to 'b) \equiv prj\ oo\ emb$
**assumes** *rep-def*: $(rep :: 'b \to 'a) \equiv prj\ oo\ emb$
**shows** $rep{\cdot}(abs{\cdot}x) = x$
**unfolding** *abs-def rep-def*
**by** (*simp add*: *emb-prj-emb DEFL*)

**lemma** *domain-rep-iso*:
**fixes** *abs* **and** *rep*
**assumes** *DEFL*: $DEFL('b) = DEFL('a)$
**assumes** *abs-def*: $(abs :: 'a \to 'b) \equiv prj\ oo\ emb$
**assumes** *rep-def*: $(rep :: 'b \to 'a) \equiv prj\ oo\ emb$
**shows** $abs{\cdot}(rep{\cdot}x) = x$
**unfolding** *abs-def rep-def*
**by** (*simp add*: *emb-prj-emb DEFL*)

## 28.2 Deflations as sets

**definition** *defl-set* :: $'a$::*bifinite defl* $\Rightarrow$ $'a$ *set*
**where** $defl\text{-}set\ A = \{x.\ cast{\cdot}A{\cdot}x = x\}$

**lemma** *adm-defl-set*: $adm\ (\lambda x.\ x \in defl\text{-}set\ A)$
**unfolding** *defl-set-def* **by** *simp*

**lemma** *defl-set-bottom*: $\bot \in defl\text{-}set\ A$
**unfolding** *defl-set-def* **by** *simp*

**lemma** *defl-set-cast* [*simp*]: $cast{\cdot}A{\cdot}x \in defl\text{-}set\ A$
**unfolding** *defl-set-def* **by** *simp*

**lemma** *defl-set-subset-iff*: $defl\text{-}set\ A \subseteq defl\text{-}set\ B \longleftrightarrow A \sqsubseteq B$
**apply** (*simp add*: *defl-set-def subset-eq cast-below-cast* [*symmetric*])
**apply** (*auto simp add*: *cast.belowI cast.belowD*)
**done**

## 28.3 Proving a subtype is representable

Temporarily relax type constraints.

**setup** ‹
*fold Sign.add-const-constraint*
[ (***const-name*** ‹*defl*›, *SOME* ***typ*** ‹$'a$::*pcpo itself* $\Rightarrow$ *udom defl*›)
, (***const-name*** ‹*emb*›, *SOME* ***typ*** ‹$'a$::*pcpo* $\to$ *udom*›)
, (***const-name*** ‹*prj*›, *SOME* ***typ*** ‹*udom* $\to$ $'a$::*pcpo*›)
, (***const-name*** ‹*liftdefl*›, *SOME* ***typ*** ‹$'a$::*pcpo itself* $\Rightarrow$ *udom u defl*›)

, (***const-name*** ‹*liftemb*›, *SOME* ***typ*** ‹'*a*::*pcpo u* → *udom u*›)
, (***const-name*** ‹*liftprj*›, *SOME* ***typ*** ‹*udom u* → '*a*::*pcpo u*›) ]
›

**lemma** *typedef-domain-class*:
**fixes** *Rep* :: '*a*::*pcpo* ⇒ *udom*
**fixes** *Abs* :: *udom* ⇒ '*a*::*pcpo*
**fixes** *t* :: *udom defl*
**assumes** *type*: *type-definition Rep Abs* (*defl-set t*)
**assumes** *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
**assumes** *emb*: *emb* ≡ (Λ *x*. *Rep x*)
**assumes** *prj*: *prj* ≡ (Λ *x*. *Abs* (*cast*·*t*·*x*))
**assumes** *defl*: *defl* ≡ (λ *a*::'*a itself*. *t*)
**assumes** *liftemb*: (*liftemb* :: '*a u* → *udom u*) ≡ *u-map*·*emb*
**assumes** *liftprj*: (*liftprj* :: *udom u* → '*a u*) ≡ *u-map*·*prj*
**assumes** *liftdefl*: (*liftdefl* :: '*a itself* ⇒ -) ≡ (λ*t*. *liftdefl-of*·*DEFL*('*a*))
**shows** *OFCLASS*('*a*, *domain-class*)
**proof**
**have** *emb-beta*: ⋀*x*. *emb*·*x* = *Rep x*
**unfolding** *emb*
**apply** (*rule beta-cfun*)
**apply** (*rule typedef-cont-Rep* [*OF type below adm-defl-set cont-id*])
**done**
**have** *prj-beta*: ⋀*y*. *prj*·*y* = *Abs* (*cast*·*t*·*y*)
**unfolding** *prj*
**apply** (*rule beta-cfun*)
**apply** (*rule typedef-cont-Abs* [*OF type below adm-defl-set*])
**apply** *simp-all*
**done**
**have** *prj-emb*: ⋀*x*::'*a*. *prj*·(*emb*·*x*) = *x*
**using** *type-definition.Rep* [*OF type*]
**unfolding** *prj-beta emb-beta defl-set-def*
**by** (*simp add*: *type-definition.Rep-inverse* [*OF type*])
**have** *emb-prj*: ⋀*y*. *emb*·(*prj*·*y* :: '*a*) = *cast*·*t*·*y*
**unfolding** *prj-beta emb-beta*
**by** (*simp add*: *type-definition.Abs-inverse* [*OF type*])
**show** *ep-pair* (*emb* :: '*a* → *udom*) *prj*
**apply** *standard*
**apply** (*simp add*: *prj-emb*)
**apply** (*simp add*: *emb-prj cast.below*)
**done**
**show** *cast*·*DEFL*('*a*) = *emb oo* (*prj* :: *udom* → '*a*)
**by** (*rule cfun-eqI*, *simp add*: *defl emb-prj*)
**qed** (*simp-all only*: *liftemb liftprj liftdefl*)

**lemma** *typedef-DEFL*:
**assumes** *defl* ≡ (λ*a*::'*a*::*pcpo itself*. *t*)
**shows** *DEFL*('*a*::*pcpo*) = *t*
**unfolding** *assms* **..**

Restore original typing constraints.

**setup** ‹
*fold Sign.add-const-constraint*
[(***const-name*** ‹*defl*›, *SOME* ***typ*** ‹'*a*::*domain itself* ⇒ *udom defl*›),
(***const-name*** ‹*emb*›, *SOME* ***typ*** ‹'*a*::*domain* → *udom*›),
(***const-name*** ‹*prj*›, *SOME* ***typ*** ‹*udom* → '*a*::*domain*›),
(***const-name*** ‹*liftdefl*›, *SOME* ***typ*** ‹'*a*::*predomain itself* ⇒ *udom u defl*›),
(***const-name*** ‹*liftemb*›, *SOME* ***typ*** ‹'*a*::*predomain u* → *udom u*›),
(***const-name*** ‹*liftprj*›, *SOME* ***typ*** ‹*udom u* → '*a*::*predomain u*›)]
›

**ML-file** ‹*Tools/domaindef.ML*›

## 28.4 Isomorphic deflations

**definition** *isodefl* :: ('*a* → '*a*) ⇒ *udom defl* ⇒ *bool*
**where** *isodefl d t* ⟷ *cast*·*t* = *emb oo d oo prj*

**definition** *isodefl'* :: ('*a*::*predomain* → '*a*) ⇒ *udom u defl* ⇒ *bool*
**where** *isodefl' d t* ⟷ *cast*·*t* = *liftemb oo u-map*·*d oo liftprj*

**lemma** *isodeflI*: (⋀*x*. *cast*·*t*·*x* = *emb*·(*d*·(*prj*·*x*))) ⟹ *isodefl d t*
**unfolding** *isodefl-def* **by** (*simp add*: *cfun-eqI*)

**lemma** *cast-isodefl*: *isodefl d t* ⟹ *cast*·*t* = (Λ *x*. *emb*·(*d*·(*prj*·*x*)))
**unfolding** *isodefl-def* **by** (*simp add*: *cfun-eqI*)

**lemma** *isodefl-strict*: *isodefl d t* ⟹ *d*·⊥ = ⊥
**unfolding** *isodefl-def*
**by** (*drule cfun-fun-cong* [**where** *x*=⊥], *simp*)

**lemma** *isodefl-imp-deflation*:
**fixes** *d* :: '*a* → '*a*
**assumes** *isodefl d t* **shows** *deflation d*
**proof**
**note** *assms* [*unfolded isodefl-def*, *simp*]
**fix** *x* :: '*a*
**show** *d*·(*d*·*x*) = *d*·*x*
**using** *cast.idem* [*of t emb*·*x*] **by** *simp*
**show** *d*·*x* ⊑ *x*
**using** *cast.below* [*of t emb*·*x*] **by** *simp*
**qed**

**lemma** *isodefl-ID-DEFL*: *isodefl* (*ID* :: '*a* → '*a*) *DEFL*('*a*)
**unfolding** *isodefl-def* **by** (*simp add*: *cast-DEFL*)

**lemma** *isodefl-LIFTDEFL*:
*isodefl'* (*ID* :: '*a* → '*a*) *LIFTDEFL*('*a*::*predomain*)
**unfolding** *isodefl'-def* **by** (*simp add*: *cast-liftdefl u-map-ID*)

**lemma** *isodefl-DEFL-imp-ID*: *isodefl* (*d* :: *'a* → *'a*) *DEFL*(*'a*) ⟹ *d* = *ID*
**unfolding** *isodefl-def*
**apply** (*simp add*: *cast-DEFL*)
**apply** (*simp add*: *cfun-eq-iff*)
**apply** (*rule allI*)
**apply** (*drule-tac x*=*emb*·*x* **in** *spec*)
**apply** *simp*
**done**

**lemma** *isodefl-bottom*: *isodefl* ⊥ ⊥
**unfolding** *isodefl-def* **by** (*simp add*: *cfun-eq-iff*)

**lemma** *adm-isodefl*:
*cont f* ⟹ *cont g* ⟹ *adm* (λ*x*. *isodefl* (*f x*) (*g x*))
**unfolding** *isodefl-def* **by** *simp*

**lemma** *isodefl-lub*:
**assumes** *chain d* **and** *chain t*
**assumes** ⋀*i*. *isodefl* (*d i*) (*t i*)
**shows** *isodefl* (⨆ *i*. *d i*) (⨆ *i*. *t i*)
**using** *assms* **unfolding** *isodefl-def*
**by** (*simp add*: *contlub-cfun-arg contlub-cfun-fun*)

**lemma** *isodefl-fix*:
**assumes** ⋀*d t*. *isodefl d t* ⟹ *isodefl* (*f*·*d*) (*g*·*t*)
**shows** *isodefl* (*fix*·*f*) (*fix*·*g*)
**unfolding** *fix-def2*
**apply** (*rule isodefl-lub*, *simp*, *simp*)
**apply** (*induct-tac i*)
**apply** (*simp add*: *isodefl-bottom*)
**apply** (*simp add*: *assms*)
**done**

**lemma** *isodefl-abs-rep*:
**fixes** *abs* **and** *rep* **and** *d*
**assumes** *DEFL*: *DEFL*(*'b*) = *DEFL*(*'a*)
**assumes** *abs-def*: (*abs* :: *'a* → *'b*) ≡ *prj oo emb*
**assumes** *rep-def*: (*rep* :: *'b* → *'a*) ≡ *prj oo emb*
**shows** *isodefl d t* ⟹ *isodefl* (*abs oo d oo rep*) *t*
**unfolding** *isodefl-def*
**by** (*simp add*: *cfun-eq-iff assms prj-emb-prj emb-prj-emb*)

**lemma** *isodefl'-liftdefl-of*: *isodefl d t* ⟹ *isodefl' d* (*liftdefl-of*·*t*)
**unfolding** *isodefl-def isodefl'-def*
**by** (*simp add*: *cast-liftdefl-of u-map-oo liftemb-eq liftprj-eq*)

**lemma** *isodefl-sfun*:
*isodefl d1 t1* ⟹ *isodefl d2 t2* ⟹

*isodefl* (*sfun-map*·*d1*·*d2*) (*sfun-defl*·*t1*·*t2*)
**apply** (*rule isodeflI*)
**apply** (*simp add*: *cast-sfun-defl cast-isodefl*)
**apply** (*simp add*: *emb-sfun-def prj-sfun-def*)
**apply** (*simp add*: *sfun-map-map isodefl-strict*)
**done**

**lemma** *isodefl-ssum*:
*isodefl d1 t1* ⟹ *isodefl d2 t2* ⟹
*isodefl* (*ssum-map*·*d1*·*d2*) (*ssum-defl*·*t1*·*t2*)
**apply** (*rule isodeflI*)
**apply** (*simp add*: *cast-ssum-defl cast-isodefl*)
**apply** (*simp add*: *emb-ssum-def prj-ssum-def*)
**apply** (*simp add*: *ssum-map-map isodefl-strict*)
**done**

**lemma** *isodefl-sprod*:
*isodefl d1 t1* ⟹ *isodefl d2 t2* ⟹
*isodefl* (*sprod-map*·*d1*·*d2*) (*sprod-defl*·*t1*·*t2*)
**apply** (*rule isodeflI*)
**apply** (*simp add*: *cast-sprod-defl cast-isodefl*)
**apply** (*simp add*: *emb-sprod-def prj-sprod-def*)
**apply** (*simp add*: *sprod-map-map isodefl-strict*)
**done**

**lemma** *isodefl-prod*:
*isodefl d1 t1* ⟹ *isodefl d2 t2* ⟹
*isodefl* (*prod-map*·*d1*·*d2*) (*prod-defl*·*t1*·*t2*)
**apply** (*rule isodeflI*)
**apply** (*simp add*: *cast-prod-defl cast-isodefl*)
**apply** (*simp add*: *emb-prod-def prj-prod-def*)
**apply** (*simp add*: *prod-map-map cfcomp1*)
**done**

**lemma** *isodefl-u*:
*isodefl d t* ⟹ *isodefl* (*u-map*·*d*) (*u-defl*·*t*)
**apply** (*rule isodeflI*)
**apply** (*simp add*: *cast-u-defl cast-isodefl*)
**apply** (*simp add*: *emb-u-def prj-u-def liftemb-eq liftprj-eq u-map-map*)
**done**

**lemma** *isodefl-u-liftdefl*:
*isodefl′ d t* ⟹ *isodefl* (*u-map*·*d*) (*u-liftdefl*·*t*)
**apply** (*rule isodeflI*)
**apply** (*simp add*: *cast-u-liftdefl isodefl′-def*)
**apply** (*simp add*: *emb-u-def prj-u-def liftemb-eq liftprj-eq*)
**done**

**lemma** *encode-prod-u-map*:

*encode-prod-u·(u-map·(prod-map·f·g)·(decode-prod-u·x))*
*= sprod-map·(u-map·f)·(u-map·g)·x*
**unfolding** *encode-prod-u-def decode-prod-u-def*
**apply** (*case-tac x*, *simp*, *rename-tac a b*)
**apply** (*case-tac a*, *simp*, *case-tac b*, *simp*, *simp*)
**done**

**lemma** *isodefl-prod-u*:
**assumes** *isodefl' d1 t1* **and** *isodefl' d2 t2*
**shows** *isodefl' (prod-map·d1·d2) (prod-liftdefl·t1·t2)*
**using** *assms* **unfolding** *isodefl'-def*
**unfolding** *liftemb-prod-def liftprj-prod-def*
**by** (*simp add*: *cast-prod-liftdefl cfcomp1 encode-prod-u-map sprod-map-map*)

**lemma** *encode-cfun-map*:
*encode-cfun·(cfun-map·f·g·(decode-cfun·x))*
*= sfun-map·(u-map·f)·g·x*
**unfolding** *encode-cfun-def decode-cfun-def*
**apply** (*simp add*: *sfun-eq-iff cfun-map-def sfun-map-def*)
**apply** (*rule cfun-eqI*, *rename-tac y*, *case-tac y*, *simp-all*)
**done**

**lemma** *isodefl-cfun*:
**assumes** *isodefl (u-map·d1) t1* **and** *isodefl d2 t2*
**shows** *isodefl (cfun-map·d1·d2) (sfun-defl·t1·t2)*
**using** *isodefl-sfun* [*OF assms*] **unfolding** *isodefl-def*
**by** (*simp add*: *emb-cfun-def prj-cfun-def cfcomp1 encode-cfun-map*)

## 28.5 Setting up the domain package

**named-theorems** *domain-defl-simps theorems like DEFL('a t) = t-defl$DEFL('a)*
**and** *domain-isodefl theorems like isodefl d t ==> isodefl (foo-map$d) (foo-defl$t)*

**ML-file** ‹*Tools/Domain/domain-isomorphism.ML*›
**ML-file** ‹*Tools/Domain/domain-axioms.ML*›
**ML-file** ‹*Tools/Domain/domain.ML*›

**lemmas** [*domain-defl-simps*] =
*DEFL-cfun DEFL-sfun DEFL-ssum DEFL-sprod DEFL-prod DEFL-u*
*liftdefl-eq LIFTDEFL-prod u-liftdefl-liftdefl-of*

**lemmas** [*domain-map-ID*] =
*cfun-map-ID sfun-map-ID ssum-map-ID sprod-map-ID prod-map-ID u-map-ID*

**lemmas** [*domain-isodefl*] =
*isodefl-u isodefl-sfun isodefl-ssum isodefl-sprod*
*isodefl-cfun isodefl-prod isodefl-prod-u isodefl'-liftdefl-of*
*isodefl-u-liftdefl*

**lemmas** [*domain-deflation*] =
*deflation-cfun-map deflation-sfun-map deflation-ssum-map*
*deflation-sprod-map deflation-prod-map deflation-u-map*

**setup** ‹
*fold Domain-Take-Proofs.add-rec-type*
[(***type-name*** ‹*cfun*›, [*true*, *true*]),
(***type-name*** ‹*sfun*›, [*true*, *true*]),
(***type-name*** ‹*ssum*›, [*true*, *true*]),
(***type-name*** ‹*sprod*›, [*true*, *true*]),
(***type-name*** ‹*prod*›, [*true*, *true*]),
(***type-name*** ‹*u*›, [*true*])]
›

**end**

# 29 A compact basis for powerdomains

**theory** *Compact-Basis*
**imports** *Universal*
**begin**

**default-sort** *bifinite*

## 29.1 A compact basis for powerdomains

**definition** *pd-basis* = {*S*::'*a compact-basis set*. *finite S* ∧ *S* ≠ {}}

**typedef** '*a pd-basis* = *pd-basis* :: '*a compact-basis set set*
**unfolding** *pd-basis-def*
**apply** (*rule-tac x*={-} **in** *exI*)
**apply** *simp*
**done**

**lemma** *finite-Rep-pd-basis* [*simp*]: *finite* (*Rep-pd-basis u*)
**by** (*insert Rep-pd-basis* [*of u*, *unfolded pd-basis-def*]) *simp*

**lemma** *Rep-pd-basis-nonempty* [*simp*]: *Rep-pd-basis u* ≠ {}
**by** (*insert Rep-pd-basis* [*of u*, *unfolded pd-basis-def*]) *simp*

The powerdomain basis type is countable.

**lemma** *pd-basis-countable*: ∃*f*::'*a pd-basis* ⇒ *nat*. *inj f*
**proof** −
**obtain** *g* :: '*a compact-basis* ⇒ *nat* **where** *inj g*
**using** *compact-basis.countable* **..**
**hence** *image-g-eq*: ⋀*A B*. *g* ‘ *A* = *g* ‘ *B* ⟷ *A* = *B*
**by** (*rule inj-image-eq-iff*)
**have** *inj* (λ*t*. *set-encode* (*g* ‘ *Rep-pd-basis t*))
**by** (*simp add*: *inj-on-def set-encode-eq image-g-eq Rep-pd-basis-inject*)

**thus** *?thesis* **by** − (*rule exI*)

**qed**

## 29.2 Unit and plus constructors

**definition**
*PDUnit* :: *'a compact-basis* ⇒ *'a pd-basis* **where**
*PDUnit* = (λ*x*. *Abs-pd-basis* {*x*})

**definition**
*PDPlus* :: *'a pd-basis* ⇒ *'a pd-basis* ⇒ *'a pd-basis* **where**
*PDPlus t u* = *Abs-pd-basis* (*Rep-pd-basis t* ∪ *Rep-pd-basis u*)

**lemma** *Rep-PDUnit*:
*Rep-pd-basis* (*PDUnit x*) = {*x*}
**unfolding** *PDUnit-def* **by** (*rule Abs-pd-basis-inverse*) (*simp add*: *pd-basis-def*)

**lemma** *Rep-PDPlus*:
*Rep-pd-basis* (*PDPlus u v*) = *Rep-pd-basis u* ∪ *Rep-pd-basis v*
**unfolding** *PDPlus-def* **by** (*rule Abs-pd-basis-inverse*) (*simp add*: *pd-basis-def*)

**lemma** *PDUnit-inject* [*simp*]: (*PDUnit a* = *PDUnit b*) = (*a* = *b*)
**unfolding** *Rep-pd-basis-inject* [*symmetric*] *Rep-PDUnit* **by** *simp*

**lemma** *PDPlus-assoc*: *PDPlus* (*PDPlus t u*) *v* = *PDPlus t* (*PDPlus u v*)
**unfolding** *Rep-pd-basis-inject* [*symmetric*] *Rep-PDPlus* **by** (*rule Un-assoc*)

**lemma** *PDPlus-commute*: *PDPlus t u* = *PDPlus u t*
**unfolding** *Rep-pd-basis-inject* [*symmetric*] *Rep-PDPlus* **by** (*rule Un-commute*)

**lemma** *PDPlus-absorb*: *PDPlus t t* = *t*
**unfolding** *Rep-pd-basis-inject* [*symmetric*] *Rep-PDPlus* **by** (*rule Un-absorb*)

**lemma** *pd-basis-induct1*:
**assumes** *PDUnit*: ⋀*a*. *P* (*PDUnit a*)
**assumes** *PDPlus*: ⋀*a t*. *P t* ⟹ *P* (*PDPlus* (*PDUnit a*) *t*)
**shows** *P x*
**apply** (*induct x*, *unfold pd-basis-def*, *clarify*)
**apply** (*erule* (*1*) *finite-ne-induct*)
**apply** (*cut-tac a*=*x* **in** *PDUnit*)
**apply** (*simp add*: *PDUnit-def*)
**apply** (*drule-tac a*=*x* **in** *PDPlus*)
**apply** (*simp add*: *PDUnit-def PDPlus-def*
*Abs-pd-basis-inverse* [*unfolded pd-basis-def*])
**done**

**lemma** *pd-basis-induct*:
**assumes** *PDUnit*: ⋀*a*. *P* (*PDUnit a*)

**assumes** *PDPlus*: $\bigwedge t\ u.\ [\![P\ t;\ P\ u]\!] \Longrightarrow P\ (PDPlus\ t\ u)$
**shows** $P\ x$
**apply** (*induct x rule*: *pd-basis-induct1*)
**apply** (*rule PDUnit*, *erule PDPlus* [*OF PDUnit*])
**done**

### 29.3 Fold operator

**definition**
*fold-pd* ::
$('a\ compact\text{-}basis \Rightarrow 'b{::}type) \Rightarrow ('b \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'a\ pd\text{-}basis \Rightarrow 'b$
**where** *fold-pd g f t* = *semilattice-set.F f* (*g* ‘ *Rep-pd-basis t*)

**lemma** *fold-pd-PDUnit*:
**assumes** *semilattice f*
**shows** *fold-pd g f* (*PDUnit x*) = *g x*
**proof** −
**from** *assms* **interpret** *semilattice-set f* **by** (*rule semilattice-set.intro*)
**show** *?thesis* **by** (*simp add*: *fold-pd-def Rep-PDUnit*)
**qed**

**lemma** *fold-pd-PDPlus*:
**assumes** *semilattice f*
**shows** *fold-pd g f* (*PDPlus t u*) = *f* (*fold-pd g f t*) (*fold-pd g f u*)
**proof** −
**from** *assms* **interpret** *semilattice-set f* **by** (*rule semilattice-set.intro*)
**show** *?thesis* **by** (*simp add*: *image-Un fold-pd-def Rep-PDPlus union*)
**qed**

**end**

## 30 Upper powerdomain

**theory** *UpperPD*
**imports** *Compact-Basis*
**begin**

### 30.1 Basis preorder

**definition**
*upper-le* :: *'a pd-basis* ⇒ *'a pd-basis* ⇒ *bool* (**infix** $\leq\sharp$ *50*) **where**
*upper-le* = $(\lambda u\ v.\ \forall y{\in}Rep\text{-}pd\text{-}basis\ v.\ \exists x{\in}Rep\text{-}pd\text{-}basis\ u.\ x \sqsubseteq y)$

**lemma** *upper-le-refl* [*simp*]: $t \leq\sharp t$
**unfolding** *upper-le-def* **by** *fast*

**lemma** *upper-le-trans*: $[\![t \leq\sharp u;\ u \leq\sharp v]\!] \Longrightarrow t \leq\sharp v$
**unfolding** *upper-le-def*
**apply** (*rule ballI*)

**apply** (*drule* (*1*) *bspec*, *erule bexE*)
**apply** (*drule* (*1*) *bspec*, *erule bexE*)
**apply** (*erule rev-bexI*)
**apply** (*erule* (*1*) *below-trans*)
**done**

**interpretation** *upper-le*: *preorder upper-le*
**by** (*rule preorder.intro*, *rule upper-le-refl*, *rule upper-le-trans*)

**lemma** *upper-le-minimal* [*simp*]: *PDUnit compact-bot* $\leq\sharp$ *t*
**unfolding** *upper-le-def Rep-PDUnit* **by** *simp*

**lemma** *PDUnit-upper-mono*: $x \sqsubseteq y \Longrightarrow$ *PDUnit x* $\leq\sharp$ *PDUnit y*
**unfolding** *upper-le-def Rep-PDUnit* **by** *simp*

**lemma** *PDPlus-upper-mono*: $[\![s \leq\sharp t;\ u \leq\sharp v]\!] \Longrightarrow$ *PDPlus s u* $\leq\sharp$ *PDPlus t v*
**unfolding** *upper-le-def Rep-PDPlus* **by** *fast*

**lemma** *PDPlus-upper-le*: *PDPlus t u* $\leq\sharp$ *t*
**unfolding** *upper-le-def Rep-PDPlus* **by** *fast*

**lemma** *upper-le-PDUnit-PDUnit-iff* [*simp*]:
(*PDUnit a* $\leq\sharp$ *PDUnit b*) = ($a \sqsubseteq b$)
**unfolding** *upper-le-def Rep-PDUnit* **by** *fast*

**lemma** *upper-le-PDPlus-PDUnit-iff*:
(*PDPlus t u* $\leq\sharp$ *PDUnit a*) = (*t* $\leq\sharp$ *PDUnit a* $\vee$ *u* $\leq\sharp$ *PDUnit a*)
**unfolding** *upper-le-def Rep-PDPlus Rep-PDUnit* **by** *fast*

**lemma** *upper-le-PDPlus-iff*: (*t* $\leq\sharp$ *PDPlus u v*) = (*t* $\leq\sharp$ *u* $\wedge$ *t* $\leq\sharp$ *v*)
**unfolding** *upper-le-def Rep-PDPlus* **by** *fast*

**lemma** *upper-le-induct* [*induct set*: *upper-le*]:
**assumes** *le*: *t* $\leq\sharp$ *u*
**assumes** *1*: $\bigwedge a\ b.\ a \sqsubseteq b \Longrightarrow P$ (*PDUnit a*) (*PDUnit b*)
**assumes** *2*: $\bigwedge t\ u\ a.\ P\ t$ (*PDUnit a*) $\Longrightarrow P$ (*PDPlus t u*) (*PDUnit a*)
**assumes** *3*: $\bigwedge t\ u\ v.\ [\![P\ t\ u;\ P\ t\ v]\!] \Longrightarrow P\ t$ (*PDPlus u v*)
**shows** *P t u*
**using** *le* **apply** (*induct u arbitrary*: *t rule*: *pd-basis-induct*)
**apply** (*erule rev-mp*)
**apply** (*induct-tac t rule*: *pd-basis-induct*)
**apply** (*simp add*: *1*)
**apply** (*simp add*: *upper-le-PDPlus-PDUnit-iff*)
**apply** (*simp add*: *2*)
**apply** (*subst PDPlus-commute*)
**apply** (*simp add*: *2*)
**apply** (*simp add*: *upper-le-PDPlus-iff 3*)
**done**

## 30.2 Type definition

**typedef** *'a upper-pd* (*('(-')♯*) =
{*S*::*'a pd-basis set*. *upper-le.ideal S*}
**by** (*rule upper-le.ex-ideal*)

**instantiation** *upper-pd* :: (*bifinite*) *below*
**begin**

**definition**
*x* ⊑ *y* ⟷ *Rep-upper-pd x* ⊆ *Rep-upper-pd y*

**instance ..**
**end**

**instance** *upper-pd* :: (*bifinite*) *po*
**using** *type-definition-upper-pd below-upper-pd-def*
**by** (*rule upper-le.typedef-ideal-po*)

**instance** *upper-pd* :: (*bifinite*) *cpo*
**using** *type-definition-upper-pd below-upper-pd-def*
**by** (*rule upper-le.typedef-ideal-cpo*)

**definition**
*upper-principal* :: *'a pd-basis* ⇒ *'a upper-pd* **where**
*upper-principal t* = *Abs-upper-pd* {*u*. *u* ≤♯ *t*}

**interpretation** *upper-pd*:
*ideal-completion upper-le upper-principal Rep-upper-pd*
**using** *type-definition-upper-pd below-upper-pd-def*
**using** *upper-principal-def pd-basis-countable*
**by** (*rule upper-le.typedef-ideal-completion*)

Upper powerdomain is pointed

**lemma** *upper-pd-minimal*: *upper-principal* (*PDUnit compact-bot*) ⊑ *ys*
**by** (*induct ys rule*: *upper-pd.principal-induct*, *simp*, *simp*)

**instance** *upper-pd* :: (*bifinite*) *pcpo*
**by** *intro-classes* (*fast intro*: *upper-pd-minimal*)

**lemma** *inst-upper-pd-pcpo*: ⊥ = *upper-principal* (*PDUnit compact-bot*)
**by** (*rule upper-pd-minimal* [*THEN bottomI*, *symmetric*])

## 30.3 Monadic unit and plus

**definition**
*upper-unit* :: *'a* → *'a upper-pd* **where**
*upper-unit* = *compact-basis.extension* (λ*a*. *upper-principal* (*PDUnit a*))

**definition**

*upper-plus* :: *'a upper-pd* → *'a upper-pd* → *'a upper-pd* **where**
*upper-plus* = *upper-pd.extension* (λ*t*. *upper-pd.extension* (λ*u*.
*upper-principal* (*PDPlus t u*)))

**abbreviation**
*upper-add* :: *'a upper-pd* ⇒ *'a upper-pd* ⇒ *'a upper-pd*
(**infixl** ∪♯ *65*) **where**
*xs* ∪♯ *ys* == *upper-plus*·*xs*·*ys*

**syntax**
-*upper-pd* :: *args* ⇒ *logic* ({-}♯)

**translations**
{*x*,*xs*}♯ == {*x*}♯ ∪♯ {*xs*}♯
{*x*}♯ == *CONST upper-unit*·*x*

**lemma** *upper-unit-Rep-compact-basis* [*simp*]:
{*Rep-compact-basis a*}♯ = *upper-principal* (*PDUnit a*)
**unfolding** *upper-unit-def*
**by** (*simp add*: *compact-basis.extension-principal PDUnit-upper-mono*)

**lemma** *upper-plus-principal* [*simp*]:
*upper-principal t* ∪♯ *upper-principal u* = *upper-principal* (*PDPlus t u*)
**unfolding** *upper-plus-def*
**by** (*simp add*: *upper-pd.extension-principal*
*upper-pd.extension-mono PDPlus-upper-mono*)

**interpretation** *upper-add*: *semilattice upper-add* **proof**
**fix** *xs ys zs* :: *'a upper-pd*
**show** (*xs* ∪♯ *ys*) ∪♯ *zs* = *xs* ∪♯ (*ys* ∪♯ *zs*)
**apply** (*induct xs rule*: *upper-pd.principal-induct*, *simp*)
**apply** (*induct ys rule*: *upper-pd.principal-induct*, *simp*)
**apply** (*induct zs rule*: *upper-pd.principal-induct*, *simp*)
**apply** (*simp add*: *PDPlus-assoc*)
**done**
**show** *xs* ∪♯ *ys* = *ys* ∪♯ *xs*
**apply** (*induct xs rule*: *upper-pd.principal-induct*, *simp*)
**apply** (*induct ys rule*: *upper-pd.principal-induct*, *simp*)
**apply** (*simp add*: *PDPlus-commute*)
**done**
**show** *xs* ∪♯ *xs* = *xs*
**apply** (*induct xs rule*: *upper-pd.principal-induct*, *simp*)
**apply** (*simp add*: *PDPlus-absorb*)
**done**
**qed**

**lemmas** *upper-plus-assoc* = *upper-add.assoc*
**lemmas** *upper-plus-commute* = *upper-add.commute*
**lemmas** *upper-plus-absorb* = *upper-add.idem*

**lemmas** *upper-plus-left-commute* = *upper-add.left-commute*
**lemmas** *upper-plus-left-absorb* = *upper-add.left-idem*

Useful for *simp add*: *upper-plus-ac*

**lemmas** *upper-plus-ac* =
*upper-plus-assoc upper-plus-commute upper-plus-left-commute*

Useful for *simp only*: *upper-plus-aci*

**lemmas** *upper-plus-aci* =
*upper-plus-ac upper-plus-absorb upper-plus-left-absorb*

**lemma** *upper-plus-below1*: $xs \cup\sharp\ ys \sqsubseteq xs$
**apply** (*induct xs rule*: *upper-pd.principal-induct*, *simp*)
**apply** (*induct ys rule*: *upper-pd.principal-induct*, *simp*)
**apply** (*simp add*: *PDPlus-upper-le*)
**done**

**lemma** *upper-plus-below2*: $xs \cup\sharp\ ys \sqsubseteq ys$
**by** (*subst upper-plus-commute*, *rule upper-plus-below1*)

**lemma** *upper-plus-greatest*: $[\![xs \sqsubseteq ys;\ xs \sqsubseteq zs]\!] \Longrightarrow xs \sqsubseteq ys \cup\sharp\ zs$
**apply** (*subst upper-plus-absorb* [*of xs*, *symmetric*])
**apply** (*erule* (*1*) *monofun-cfun* [*OF monofun-cfun-arg*])
**done**

**lemma** *upper-below-plus-iff* [*simp*]:
$xs \sqsubseteq ys \cup\sharp\ zs \longleftrightarrow xs \sqsubseteq ys \wedge xs \sqsubseteq zs$
**apply** *safe*
**apply** (*erule below-trans* [*OF - upper-plus-below1*])
**apply** (*erule below-trans* [*OF - upper-plus-below2*])
**apply** (*erule* (*1*) *upper-plus-greatest*)
**done**

**lemma** *upper-plus-below-unit-iff* [*simp*]:
$xs \cup\sharp\ ys \sqsubseteq \{z\}\sharp \longleftrightarrow xs \sqsubseteq \{z\}\sharp \vee ys \sqsubseteq \{z\}\sharp$
**apply** (*induct xs rule*: *upper-pd.principal-induct*, *simp*)
**apply** (*induct ys rule*: *upper-pd.principal-induct*, *simp*)
**apply** (*induct z rule*: *compact-basis.principal-induct*, *simp*)
**apply** (*simp add*: *upper-le-PDPlus-PDUnit-iff*)
**done**

**lemma** *upper-unit-below-iff* [*simp*]: $\{x\}\sharp \sqsubseteq \{y\}\sharp \longleftrightarrow x \sqsubseteq y$
**apply** (*induct x rule*: *compact-basis.principal-induct*, *simp*)
**apply** (*induct y rule*: *compact-basis.principal-induct*, *simp*)
**apply** *simp*
**done**

**lemmas** *upper-pd-below-simps* =
*upper-unit-below-iff*

*upper-below-plus-iff*
*upper-plus-below-unit-iff*

**lemma** *upper-unit-eq-iff* [*simp*]: $\{x\}\sharp = \{y\}\sharp \longleftrightarrow x = y$
**unfolding** *po-eq-conv* **by** *simp*

**lemma** *upper-unit-strict* [*simp*]: $\{\bot\}\sharp = \bot$
**using** *upper-unit-Rep-compact-basis* [*of compact-bot*]
**by** (*simp add*: *inst-upper-pd-pcpo*)

**lemma** *upper-plus-strict1* [*simp*]: $\bot \cup\sharp\ ys = \bot$
**by** (*rule bottomI*, *rule upper-plus-below1*)

**lemma** *upper-plus-strict2* [*simp*]: $xs \cup\sharp\ \bot = \bot$
**by** (*rule bottomI*, *rule upper-plus-below2*)

**lemma** *upper-unit-bottom-iff* [*simp*]: $\{x\}\sharp = \bot \longleftrightarrow x = \bot$
**unfolding** *upper-unit-strict* [*symmetric*] **by** (*rule upper-unit-eq-iff*)

**lemma** *upper-plus-bottom-iff* [*simp*]:
$xs \cup\sharp\ ys = \bot \longleftrightarrow xs = \bot \vee ys = \bot$
**apply** (*induct xs rule*: *upper-pd.principal-induct*, *simp*)
**apply** (*induct ys rule*: *upper-pd.principal-induct*, *simp*)
**apply** (*simp add*: *inst-upper-pd-pcpo upper-pd.principal-eq-iff*
*upper-le-PDPlus-PDUnit-iff*)
**done**

**lemma** *compact-upper-unit*: *compact* $x \Longrightarrow$ *compact* $\{x\}\sharp$
**by** (*auto dest*!: *compact-basis.compact-imp-principal*)

**lemma** *compact-upper-unit-iff* [*simp*]: *compact* $\{x\}\sharp \longleftrightarrow$ *compact* $x$
**apply** (*safe elim*!: *compact-upper-unit*)
**apply** (*simp only*: *compact-def upper-unit-below-iff* [*symmetric*])
**apply** (*erule adm-subst* [*OF cont-Rep-cfun2*])
**done**

**lemma** *compact-upper-plus* [*simp*]:
$[\![$*compact xs*; *compact ys*$]\!] \Longrightarrow$ *compact* $(xs \cup\sharp\ ys)$
**by** (*auto dest*!: *upper-pd.compact-imp-principal*)

## 30.4 Induction rules

**lemma** *upper-pd-induct1*:
**assumes** *P*: *adm P*
**assumes** *unit*: $\bigwedge x.\ P\ \{x\}\sharp$
**assumes** *insert*: $\bigwedge x\ ys.\ [\![P\ \{x\}\sharp; P\ ys]\!] \Longrightarrow P\ (\{x\}\sharp \cup\sharp\ ys)$
**shows** *P* (*xs*::'*a upper-pd*)
**apply** (*induct xs rule*: *upper-pd.principal-induct*, *rule P*)
**apply** (*induct-tac a rule*: *pd-basis-induct1*)

**apply** (*simp only*: *upper-unit-Rep-compact-basis* [*symmetric*])
**apply** (*rule unit*)
**apply** (*simp only*: *upper-unit-Rep-compact-basis* [*symmetric*]
*upper-plus-principal* [*symmetric*])
**apply** (*erule insert* [*OF unit*])
**done**

**lemma** *upper-pd-induct*
[*case-names adm upper-unit upper-plus*, *induct type*: *upper-pd*]:
**assumes** *P*: *adm P*
**assumes** *unit*: $\bigwedge x.\ P\ \{x\}\sharp$
**assumes** *plus*: $\bigwedge xs\ ys.\ [\![P\ xs;\ P\ ys]\!] \Longrightarrow P\ (xs \cup\sharp\ ys)$
**shows** *P* (*xs*::'*a upper-pd*)
**apply** (*induct xs rule*: *upper-pd.principal-induct*, *rule P*)
**apply** (*induct-tac a rule*: *pd-basis-induct*)
**apply** (*simp only*: *upper-unit-Rep-compact-basis* [*symmetric*] *unit*)
**apply** (*simp only*: *upper-plus-principal* [*symmetric*] *plus*)
**done**

## 30.5 Monadic bind

**definition**
*upper-bind-basis* ::
'*a pd-basis* ⇒ ('*a* → '*b upper-pd*) → '*b upper-pd* **where**
*upper-bind-basis* = *fold-pd*
(λ*a*. Λ *f*. *f*·(*Rep-compact-basis a*))
(λ*x y*. Λ *f*. *x*·*f* ∪♯ *y*·*f*)

**lemma** *ACI-upper-bind*:
*semilattice* (λ*x y*. Λ *f*. *x*·*f* ∪♯ *y*·*f*)
**apply** *unfold-locales*
**apply** (*simp add*: *upper-plus-assoc*)
**apply** (*simp add*: *upper-plus-commute*)
**apply** (*simp add*: *eta-cfun*)
**done**

**lemma** *upper-bind-basis-simps* [*simp*]:
*upper-bind-basis* (*PDUnit a*) =
(Λ *f*. *f*·(*Rep-compact-basis a*))
*upper-bind-basis* (*PDPlus t u*) =
(Λ *f*. *upper-bind-basis t*·*f* ∪♯ *upper-bind-basis u*·*f*)
**unfolding** *upper-bind-basis-def*
**apply** −
**apply** (*rule fold-pd-PDUnit* [*OF ACI-upper-bind*])
**apply** (*rule fold-pd-PDPlus* [*OF ACI-upper-bind*])
**done**

**lemma** *upper-bind-basis-mono*:
*t* ≤♯ *u* ⟹ *upper-bind-basis t* ⊑ *upper-bind-basis u*

**unfolding** *cfun-below-iff*
**apply** (*erule upper-le-induct*, *safe*)
**apply** (*simp add*: *monofun-cfun*)
**apply** (*simp add*: *below-trans* [*OF upper-plus-below1*])
**apply** *simp*
**done**

**definition**
*upper-bind* :: *'a upper-pd* → (*'a* → *'b upper-pd*) → *'b upper-pd* **where**
*upper-bind* = *upper-pd.extension upper-bind-basis*

**syntax**
*-upper-bind* :: [*logic*, *logic*, *logic*] ⇒ *logic*
((*3*⋃♯*-*∈*-./ -) [*0*, *0*, *10*] *10*)

**translations**
⋃♯*x*∈*xs*. *e* == *CONST upper-bind*·*xs*·(Λ *x*. *e*)

**lemma** *upper-bind-principal* [*simp*]:
*upper-bind*·(*upper-principal t*) = *upper-bind-basis t*
**unfolding** *upper-bind-def*
**apply** (*rule upper-pd.extension-principal*)
**apply** (*erule upper-bind-basis-mono*)
**done**

**lemma** *upper-bind-unit* [*simp*]:
*upper-bind*·{*x*}♯·*f* = *f*·*x*
**by** (*induct x rule*: *compact-basis.principal-induct*, *simp*, *simp*)

**lemma** *upper-bind-plus* [*simp*]:
*upper-bind*·(*xs* ∪♯ *ys*)·*f* = *upper-bind*·*xs*·*f* ∪♯ *upper-bind*·*ys*·*f*
**by** (*induct xs rule*: *upper-pd.principal-induct*, *simp*,
*induct ys rule*: *upper-pd.principal-induct*, *simp*, *simp*)

**lemma** *upper-bind-strict* [*simp*]: *upper-bind*·⊥·*f* = *f*·⊥
**unfolding** *upper-unit-strict* [*symmetric*] **by** (*rule upper-bind-unit*)

**lemma** *upper-bind-bind*:
*upper-bind*·(*upper-bind*·*xs*·*f*)·*g* = *upper-bind*·*xs*·(Λ *x*. *upper-bind*·(*f*·*x*)·*g*)
**by** (*induct xs*, *simp-all*)

## 30.6 Map

**definition**
*upper-map* :: (*'a* → *'b*) → *'a upper-pd* → *'b upper-pd* **where**
*upper-map* = (Λ *f xs*. *upper-bind*·*xs*·(Λ *x*. {*f*·*x*}♯))

**lemma** *upper-map-unit* [*simp*]:
*upper-map*·*f*·{*x*}♯ = {*f*·*x*}♯

**unfolding** *upper-map-def* **by** *simp*

**lemma** *upper-map-plus* [*simp*]:
*upper-map·f·(xs ∪♯ ys) = upper-map·f·xs ∪♯ upper-map·f·ys*
**unfolding** *upper-map-def* **by** *simp*

**lemma** *upper-map-bottom* [*simp*]: *upper-map·f·⊥ = {f·⊥}♯*
**unfolding** *upper-map-def* **by** *simp*

**lemma** *upper-map-ident*: *upper-map·(Λ x. x)·xs = xs*
**by** (*induct xs rule*: *upper-pd-induct*, *simp-all*)

**lemma** *upper-map-ID*: *upper-map·ID = ID*
**by** (*simp add*: *cfun-eq-iff ID-def upper-map-ident*)

**lemma** *upper-map-map*:
*upper-map·f·(upper-map·g·xs) = upper-map·(Λ x. f·(g·x))·xs*
**by** (*induct xs rule*: *upper-pd-induct*, *simp-all*)

**lemma** *upper-bind-map*:
*upper-bind·(upper-map·f·xs)·g = upper-bind·xs·(Λ x. g·(f·x))*
**by** (*simp add*: *upper-map-def upper-bind-bind*)

**lemma** *upper-map-bind*:
*upper-map·f·(upper-bind·xs·g) = upper-bind·xs·(Λ x. upper-map·f·(g·x))*
**by** (*simp add*: *upper-map-def upper-bind-bind*)

**lemma** *ep-pair-upper-map*: *ep-pair e p ⟹ ep-pair (upper-map·e) (upper-map·p)*
**apply** *standard*
**apply** (*induct-tac x rule*: *upper-pd-induct*, *simp-all add*: *ep-pair.e-inverse*)
**apply** (*induct-tac y rule*: *upper-pd-induct*)
**apply** (*simp-all add*: *ep-pair.e-p-below monofun-cfun del*: *upper-below-plus-iff*)
**done**

**lemma** *deflation-upper-map*: *deflation d ⟹ deflation (upper-map·d)*
**apply** *standard*
**apply** (*induct-tac x rule*: *upper-pd-induct*, *simp-all add*: *deflation.idem*)
**apply** (*induct-tac x rule*: *upper-pd-induct*)
**apply** (*simp-all add*: *deflation.below monofun-cfun del*: *upper-below-plus-iff*)
**done**

**lemma** *finite-deflation-upper-map*:
**assumes** *finite-deflation d* **shows** *finite-deflation (upper-map·d)*
**proof** (*rule finite-deflation-intro*)
**interpret** *d*: *finite-deflation d* **by** *fact*
**from** *d.deflation-axioms* **show** *deflation (upper-map·d)*
**by** (*rule deflation-upper-map*)
**have** *finite* (*range* (λ*x*. *d·x*)) **by** (*rule d.finite-range*)

**hence** *finite* (*Rep-compact-basis* −‘ *range* (λ*x*. *d*·*x*))
**by** (*rule finite-vimageI*, *simp add*: *inj-on-def Rep-compact-basis-inject*)
**hence** *finite* (*Pow* (*Rep-compact-basis* −‘ *range* (λ*x*. *d*·*x*))) **by** *simp*
**hence** *finite* (*Rep-pd-basis* −‘ (*Pow* (*Rep-compact-basis* −‘ *range* (λ*x*. *d*·*x*))))
**by** (*rule finite-vimageI*, *simp add*: *inj-on-def Rep-pd-basis-inject*)
**hence** ∗: *finite* (*upper-principal* ‘ *Rep-pd-basis* −‘ (*Pow* (*Rep-compact-basis* −‘ *range* (λ*x*. *d*·*x*)))) **by** *simp*
**hence** *finite* (*range* (λ*xs*. *upper-map*·*d*·*xs*))
**apply** (*rule rev-finite-subset*)
**apply** *clarsimp*
**apply** (*induct-tac xs rule*: *upper-pd.principal-induct*)
**apply** (*simp add*: *adm-mem-finite* ∗)
**apply** (*rename-tac t*, *induct-tac t rule*: *pd-basis-induct*)
**apply** (*simp only*: *upper-unit-Rep-compact-basis* [*symmetric*] *upper-map-unit*)
**apply** *simp*
**apply** (*subgoal-tac* ∃ *b*. *d*·(*Rep-compact-basis a*) = *Rep-compact-basis b*)
**apply** *clarsimp*
**apply** (*rule imageI*)
**apply** (*rule vimageI2*)
**apply** (*simp add*: *Rep-PDUnit*)
**apply** (*rule range-eqI*)
**apply** (*erule sym*)
**apply** (*rule exI*)
**apply** (*rule Abs-compact-basis-inverse* [*symmetric*])
**apply** (*simp add*: *d.compact*)
**apply** (*simp only*: *upper-plus-principal* [*symmetric*] *upper-map-plus*)
**apply** *clarsimp*
**apply** (*rule imageI*)
**apply** (*rule vimageI2*)
**apply** (*simp add*: *Rep-PDPlus*)
**done**
**thus** *finite* {*xs*. *upper-map*·*d*·*xs* = *xs*}
**by** (*rule finite-range-imp-finite-fixes*)
**qed**

## 30.7 Upper powerdomain is bifinite

**lemma** *approx-chain-upper-map*:
**assumes** *approx-chain a*
**shows** *approx-chain* (λ*i*. *upper-map*·(*a i*))
**using** *assms* **unfolding** *approx-chain-def*
**by** (*simp add*: *lub-APP upper-map-ID finite-deflation-upper-map*)

**instance** *upper-pd* :: (*bifinite*) *bifinite*
**proof**
**show** ∃(*a*::*nat* ⇒ ′*a upper-pd* → ′*a upper-pd*). *approx-chain a*
**using** *bifinite* [**where** ′*a*=′*a*]
**by** (*fast intro*!: *approx-chain-upper-map*)
**qed**

### 30.8 Join

**definition**
*upper-join* :: *'a upper-pd upper-pd* → *'a upper-pd* **where**
*upper-join* = (Λ *xss*. *upper-bind*·*xss*·(Λ *xs*. *xs*))

**lemma** *upper-join-unit* [*simp*]:
*upper-join*·{*xs*}♯ = *xs*
**unfolding** *upper-join-def* **by** *simp*

**lemma** *upper-join-plus* [*simp*]:
*upper-join*·(*xss* ∪♯ *yss*) = *upper-join*·*xss* ∪♯ *upper-join*·*yss*
**unfolding** *upper-join-def* **by** *simp*

**lemma** *upper-join-bottom* [*simp*]: *upper-join*·⊥ = ⊥
**unfolding** *upper-join-def* **by** *simp*

**lemma** *upper-join-map-unit*:
*upper-join*·(*upper-map*·*upper-unit*·*xs*) = *xs*
**by** (*induct xs rule*: *upper-pd-induct*, *simp-all*)

**lemma** *upper-join-map-join*:
*upper-join*·(*upper-map*·*upper-join*·*xsss*) = *upper-join*·(*upper-join*·*xsss*)
**by** (*induct xsss rule*: *upper-pd-induct*, *simp-all*)

**lemma** *upper-join-map-map*:
*upper-join*·(*upper-map*·(*upper-map*·*f*)·*xss*) =
*upper-map*·*f*·(*upper-join*·*xss*)
**by** (*induct xss rule*: *upper-pd-induct*, *simp-all*)

**end**

## 31 Lower powerdomain

**theory** *LowerPD*
**imports** *Compact-Basis*
**begin**

### 31.1 Basis preorder

**definition**
*lower-le* :: *'a pd-basis* ⇒ *'a pd-basis* ⇒ *bool* (**infix** ≤♭ *50*) **where**
*lower-le* = (λ*u v*. ∀ *x*∈*Rep-pd-basis u*. ∃ *y*∈*Rep-pd-basis v*. *x* ⊑ *y*)

**lemma** *lower-le-refl* [*simp*]: *t* ≤♭ *t*
**unfolding** *lower-le-def* **by** *fast*

**lemma** *lower-le-trans*: ⟦*t* ≤♭ *u*; *u* ≤♭ *v*⟧ ⟹ *t* ≤♭ *v*
**unfolding** *lower-le-def*

**apply** (*rule ballI*)
**apply** (*drule* (*1*) *bspec*, *erule bexE*)
**apply** (*drule* (*1*) *bspec*, *erule bexE*)
**apply** (*erule rev-bexI*)
**apply** (*erule* (*1*) *below-trans*)
**done**

**interpretation** *lower-le*: *preorder lower-le*
**by** (*rule preorder.intro*, *rule lower-le-refl*, *rule lower-le-trans*)

**lemma** *lower-le-minimal* [*simp*]: *PDUnit compact-bot* $\leq\flat$ *t*
**unfolding** *lower-le-def Rep-PDUnit*
**by** (*simp*, *rule Rep-pd-basis-nonempty* [*folded ex-in-conv*])

**lemma** *PDUnit-lower-mono*: $x \sqsubseteq y \Longrightarrow$ *PDUnit x* $\leq\flat$ *PDUnit y*
**unfolding** *lower-le-def Rep-PDUnit* **by** *fast*

**lemma** *PDPlus-lower-mono*: $[\![s \leq\flat t;\ u \leq\flat v]\!] \Longrightarrow$ *PDPlus s u* $\leq\flat$ *PDPlus t v*
**unfolding** *lower-le-def Rep-PDPlus* **by** *fast*

**lemma** *PDPlus-lower-le*: *t* $\leq\flat$ *PDPlus t u*
**unfolding** *lower-le-def Rep-PDPlus* **by** *fast*

**lemma** *lower-le-PDUnit-PDUnit-iff* [*simp*]:
(*PDUnit a* $\leq\flat$ *PDUnit b*) = ($a \sqsubseteq b$)
**unfolding** *lower-le-def Rep-PDUnit* **by** *fast*

**lemma** *lower-le-PDUnit-PDPlus-iff*:
(*PDUnit a* $\leq\flat$ *PDPlus t u*) = (*PDUnit a* $\leq\flat$ *t* $\vee$ *PDUnit a* $\leq\flat$ *u*)
**unfolding** *lower-le-def Rep-PDPlus Rep-PDUnit* **by** *fast*

**lemma** *lower-le-PDPlus-iff*: (*PDPlus t u* $\leq\flat$ *v*) = (*t* $\leq\flat$ *v* $\wedge$ *u* $\leq\flat$ *v*)
**unfolding** *lower-le-def Rep-PDPlus* **by** *fast*

**lemma** *lower-le-induct* [*induct set*: *lower-le*]:
**assumes** *le*: *t* $\leq\flat$ *u*
**assumes** *1*: $\bigwedge a\ b.\ a \sqsubseteq b \Longrightarrow P$ (*PDUnit a*) (*PDUnit b*)
**assumes** *2*: $\bigwedge t\ u\ a.\ P$ (*PDUnit a*) *t* $\Longrightarrow P$ (*PDUnit a*) (*PDPlus t u*)
**assumes** *3*: $\bigwedge t\ u\ v.\ [\![P\ t\ v;\ P\ u\ v]\!] \Longrightarrow P$ (*PDPlus t u*) *v*
**shows** *P t u*
**using** *le*
**apply** (*induct t arbitrary*: *u rule*: *pd-basis-induct*)
**apply** (*erule rev-mp*)
**apply** (*induct-tac u rule*: *pd-basis-induct*)
**apply** (*simp add*: *1*)
**apply** (*simp add*: *lower-le-PDUnit-PDPlus-iff*)
**apply** (*simp add*: *2*)
**apply** (*subst PDPlus-commute*)
**apply** (*simp add*: *2*)

**apply** (*simp add*: *lower-le-PDPlus-iff 3*)
**done**

## 31.2 Type definition

**typedef** *'a lower-pd* (*('(-')⊵)*) =
{*S*::*'a pd-basis set*. *lower-le.ideal S*}
**by** (*rule lower-le.ex-ideal*)

**instantiation** *lower-pd* :: (*bifinite*) *below*
**begin**

**definition**
*x* ⊑ *y* ⟷ *Rep-lower-pd x* ⊆ *Rep-lower-pd y*

**instance ..**
**end**

**instance** *lower-pd* :: (*bifinite*) *po*
**using** *type-definition-lower-pd below-lower-pd-def*
**by** (*rule lower-le.typedef-ideal-po*)

**instance** *lower-pd* :: (*bifinite*) *cpo*
**using** *type-definition-lower-pd below-lower-pd-def*
**by** (*rule lower-le.typedef-ideal-cpo*)

**definition**
*lower-principal* :: *'a pd-basis* ⇒ *'a lower-pd* **where**
*lower-principal t* = *Abs-lower-pd* {*u*. *u* ≤⊵ *t*}

**interpretation** *lower-pd*:
*ideal-completion lower-le lower-principal Rep-lower-pd*
**using** *type-definition-lower-pd below-lower-pd-def*
**using** *lower-principal-def pd-basis-countable*
**by** (*rule lower-le.typedef-ideal-completion*)

Lower powerdomain is pointed

**lemma** *lower-pd-minimal*: *lower-principal* (*PDUnit compact-bot*) ⊑ *ys*
**by** (*induct ys rule*: *lower-pd.principal-induct*, *simp*, *simp*)

**instance** *lower-pd* :: (*bifinite*) *pcpo*
**by** *intro-classes* (*fast intro*: *lower-pd-minimal*)

**lemma** *inst-lower-pd-pcpo*: ⊥ = *lower-principal* (*PDUnit compact-bot*)
**by** (*rule lower-pd-minimal* [*THEN bottomI*, *symmetric*])

## 31.3 Monadic unit and plus

**definition**

*lower-unit* :: *'a* → *'a lower-pd* **where**
*lower-unit* = *compact-basis.extension* (λ*a*. *lower-principal* (*PDUnit a*))

**definition**
*lower-plus* :: *'a lower-pd* → *'a lower-pd* → *'a lower-pd* **where**
*lower-plus* = *lower-pd.extension* (λ*t*. *lower-pd.extension* (λ*u*.
*lower-principal* (*PDPlus t u*)))

**abbreviation**
*lower-add* :: *'a lower-pd* ⇒ *'a lower-pd* ⇒ *'a lower-pd*
(**infixl** ∪♭ *65*) **where**
*xs* ∪♭ *ys* == *lower-plus*·*xs*·*ys*

**syntax**
*-lower-pd* :: *args* ⇒ *logic* ({-}♭)

**translations**
{*x*,*xs*}♭ == {*x*}♭ ∪♭ {*xs*}♭
{*x*}♭ == *CONST lower-unit*·*x*

**lemma** *lower-unit-Rep-compact-basis* [*simp*]:
{*Rep-compact-basis a*}♭ = *lower-principal* (*PDUnit a*)
**unfolding** *lower-unit-def*
**by** (*simp add*: *compact-basis.extension-principal PDUnit-lower-mono*)

**lemma** *lower-plus-principal* [*simp*]:
*lower-principal t* ∪♭ *lower-principal u* = *lower-principal* (*PDPlus t u*)
**unfolding** *lower-plus-def*
**by** (*simp add*: *lower-pd.extension-principal*
*lower-pd.extension-mono PDPlus-lower-mono*)

**interpretation** *lower-add*: *semilattice lower-add* **proof**
**fix** *xs ys zs* :: *'a lower-pd*
**show** (*xs* ∪♭ *ys*) ∪♭ *zs* = *xs* ∪♭ (*ys* ∪♭ *zs*)
**apply** (*induct xs rule*: *lower-pd.principal-induct*, *simp*)
**apply** (*induct ys rule*: *lower-pd.principal-induct*, *simp*)
**apply** (*induct zs rule*: *lower-pd.principal-induct*, *simp*)
**apply** (*simp add*: *PDPlus-assoc*)
**done**
**show** *xs* ∪♭ *ys* = *ys* ∪♭ *xs*
**apply** (*induct xs rule*: *lower-pd.principal-induct*, *simp*)
**apply** (*induct ys rule*: *lower-pd.principal-induct*, *simp*)
**apply** (*simp add*: *PDPlus-commute*)
**done**
**show** *xs* ∪♭ *xs* = *xs*
**apply** (*induct xs rule*: *lower-pd.principal-induct*, *simp*)
**apply** (*simp add*: *PDPlus-absorb*)
**done**
**qed**

**lemmas** *lower-plus-assoc* = *lower-add.assoc*
**lemmas** *lower-plus-commute* = *lower-add.commute*
**lemmas** *lower-plus-absorb* = *lower-add.idem*
**lemmas** *lower-plus-left-commute* = *lower-add.left-commute*
**lemmas** *lower-plus-left-absorb* = *lower-add.left-idem*

Useful for *simp add*: *lower-plus-ac*

**lemmas** *lower-plus-ac* =
*lower-plus-assoc lower-plus-commute lower-plus-left-commute*

Useful for *simp only*: *lower-plus-aci*

**lemmas** *lower-plus-aci* =
*lower-plus-ac lower-plus-absorb lower-plus-left-absorb*

**lemma** *lower-plus-below1*: $xs \sqsubseteq xs \mathbin{\cup\!\flat} ys$
**apply** (*induct xs rule*: *lower-pd.principal-induct*, *simp*)
**apply** (*induct ys rule*: *lower-pd.principal-induct*, *simp*)
**apply** (*simp add*: *PDPlus-lower-le*)
**done**

**lemma** *lower-plus-below2*: $ys \sqsubseteq xs \mathbin{\cup\!\flat} ys$
**by** (*subst lower-plus-commute*, *rule lower-plus-below1*)

**lemma** *lower-plus-least*: $[\![xs \sqsubseteq zs; ys \sqsubseteq zs]\!] \Longrightarrow xs \mathbin{\cup\!\flat} ys \sqsubseteq zs$
**apply** (*subst lower-plus-absorb* [*of zs*, *symmetric*])
**apply** (*erule* (*1*) *monofun-cfun* [*OF monofun-cfun-arg*])
**done**

**lemma** *lower-plus-below-iff* [*simp*]:
$xs \mathbin{\cup\!\flat} ys \sqsubseteq zs \longleftrightarrow xs \sqsubseteq zs \wedge ys \sqsubseteq zs$
**apply** *safe*
**apply** (*erule below-trans* [*OF lower-plus-below1*])
**apply** (*erule below-trans* [*OF lower-plus-below2*])
**apply** (*erule* (*1*) *lower-plus-least*)
**done**

**lemma** *lower-unit-below-plus-iff* [*simp*]:
$\{x\}\flat \sqsubseteq ys \mathbin{\cup\!\flat} zs \longleftrightarrow \{x\}\flat \sqsubseteq ys \vee \{x\}\flat \sqsubseteq zs$
**apply** (*induct x rule*: *compact-basis.principal-induct*, *simp*)
**apply** (*induct ys rule*: *lower-pd.principal-induct*, *simp*)
**apply** (*induct zs rule*: *lower-pd.principal-induct*, *simp*)
**apply** (*simp add*: *lower-le-PDUnit-PDPlus-iff*)
**done**

**lemma** *lower-unit-below-iff* [*simp*]: $\{x\}\flat \sqsubseteq \{y\}\flat \longleftrightarrow x \sqsubseteq y$
**apply** (*induct x rule*: *compact-basis.principal-induct*, *simp*)
**apply** (*induct y rule*: *compact-basis.principal-induct*, *simp*)
**apply** *simp*

**done**

**lemmas** *lower-pd-below-simps* =
*lower-unit-below-iff*
*lower-plus-below-iff*
*lower-unit-below-plus-iff*

**lemma** *lower-unit-eq-iff* [*simp*]: $\{x\}\flat = \{y\}\flat \longleftrightarrow x = y$
**by** (*simp add*: *po-eq-conv*)

**lemma** *lower-unit-strict* [*simp*]: $\{\bot\}\flat = \bot$
**using** *lower-unit-Rep-compact-basis* [*of compact-bot*]
**by** (*simp add*: *inst-lower-pd-pcpo*)

**lemma** *lower-unit-bottom-iff* [*simp*]: $\{x\}\flat = \bot \longleftrightarrow x = \bot$
**unfolding** *lower-unit-strict* [*symmetric*] **by** (*rule lower-unit-eq-iff*)

**lemma** *lower-plus-bottom-iff* [*simp*]:
$xs \cup\flat\ ys = \bot \longleftrightarrow xs = \bot \wedge ys = \bot$
**apply** *safe*
**apply** (*rule bottomI*, *erule subst*, *rule lower-plus-below1*)
**apply** (*rule bottomI*, *erule subst*, *rule lower-plus-below2*)
**apply** (*rule lower-plus-absorb*)
**done**

**lemma** *lower-plus-strict1* [*simp*]: $\bot \cup\flat\ ys = ys$
**apply** (*rule below-antisym* [*OF - lower-plus-below2*])
**apply** (*simp add*: *lower-plus-least*)
**done**

**lemma** *lower-plus-strict2* [*simp*]: $xs \cup\flat \bot = xs$
**apply** (*rule below-antisym* [*OF - lower-plus-below1*])
**apply** (*simp add*: *lower-plus-least*)
**done**

**lemma** *compact-lower-unit*: *compact* $x \Longrightarrow$ *compact* $\{x\}\flat$
**by** (*auto dest*!: *compact-basis.compact-imp-principal*)

**lemma** *compact-lower-unit-iff* [*simp*]: *compact* $\{x\}\flat \longleftrightarrow$ *compact* $x$
**apply** (*safe elim*!: *compact-lower-unit*)
**apply** (*simp only*: *compact-def lower-unit-below-iff* [*symmetric*])
**apply** (*erule adm-subst* [*OF cont-Rep-cfun2*])
**done**

**lemma** *compact-lower-plus* [*simp*]:
$[\![$*compact xs*; *compact ys*$]\!] \Longrightarrow$ *compact* $(xs \cup\flat\ ys)$
**by** (*auto dest*!: *lower-pd.compact-imp-principal*)

## 31.4 Induction rules

**lemma** *lower-pd-induct1*:
  **assumes** *P*: *adm P*
  **assumes** *unit*: $\bigwedge x.\ P\ \{x\}\flat$
  **assumes** *insert*:
    $\bigwedge x\ ys.\ [\![P\ \{x\}\flat;\ P\ ys]\!] \Longrightarrow P\ (\{x\}\flat \cup\flat\ ys)$
  **shows** *P* (*xs*::'*a lower-pd*)
**apply** (*induct xs rule*: *lower-pd.principal-induct*, *rule P*)
**apply** (*induct-tac a rule*: *pd-basis-induct1*)
**apply** (*simp only*: *lower-unit-Rep-compact-basis* [*symmetric*])
**apply** (*rule unit*)
**apply** (*simp only*: *lower-unit-Rep-compact-basis* [*symmetric*]
            *lower-plus-principal* [*symmetric*])
**apply** (*erule insert* [*OF unit*])
**done**

**lemma** *lower-pd-induct*
  [*case-names adm lower-unit lower-plus*, *induct type*: *lower-pd*]:
  **assumes** *P*: *adm P*
  **assumes** *unit*: $\bigwedge x.\ P\ \{x\}\flat$
  **assumes** *plus*: $\bigwedge xs\ ys.\ [\![P\ xs;\ P\ ys]\!] \Longrightarrow P\ (xs \cup\flat\ ys)$
  **shows** *P* (*xs*::'*a lower-pd*)
**apply** (*induct xs rule*: *lower-pd.principal-induct*, *rule P*)
**apply** (*induct-tac a rule*: *pd-basis-induct*)
**apply** (*simp only*: *lower-unit-Rep-compact-basis* [*symmetric*] *unit*)
**apply** (*simp only*: *lower-plus-principal* [*symmetric*] *plus*)
**done**

## 31.5 Monadic bind

**definition**
  *lower-bind-basis* ::
  '*a pd-basis* ⇒ ('*a* → '*b lower-pd*) → '*b lower-pd* **where**
  *lower-bind-basis* = *fold-pd*
    (λ*a*. Λ *f*. *f*·(*Rep-compact-basis a*))
    (λ*x y*. Λ *f*. *x*·*f* ∪♭ *y*·*f*)

**lemma** *ACI-lower-bind*:
  *semilattice* (λ*x y*. Λ *f*. *x*·*f* ∪♭ *y*·*f*)
**apply** *unfold-locales*
**apply** (*simp add*: *lower-plus-assoc*)
**apply** (*simp add*: *lower-plus-commute*)
**apply** (*simp add*: *eta-cfun*)
**done**

**lemma** *lower-bind-basis-simps* [*simp*]:
  *lower-bind-basis* (*PDUnit a*) =
    (Λ *f*. *f*·(*Rep-compact-basis a*))
  *lower-bind-basis* (*PDPlus t u*) =

(Λ *f*. *lower-bind-basis t*·*f* ∪♭ *lower-bind-basis u*·*f*)
**unfolding** *lower-bind-basis-def*
**apply** −
**apply** (*rule fold-pd-PDUnit* [*OF ACI-lower-bind*])
**apply** (*rule fold-pd-PDPlus* [*OF ACI-lower-bind*])
**done**

**lemma** *lower-bind-basis-mono*:
*t* ≤♭ *u* ⟹ *lower-bind-basis t* ⊑ *lower-bind-basis u*
**unfolding** *cfun-below-iff*
**apply** (*erule lower-le-induct*, *safe*)
**apply** (*simp add*: *monofun-cfun*)
**apply** (*simp add*: *rev-below-trans* [*OF lower-plus-below1*])
**apply** *simp*
**done**

**definition**
*lower-bind* :: *'a lower-pd* → (*'a* → *'b lower-pd*) → *'b lower-pd* **where**
*lower-bind* = *lower-pd.extension lower-bind-basis*

**syntax**
*-lower-bind* :: [*logic*, *logic*, *logic*] ⇒ *logic*
(*3*⋃♭*-∈-./ -*) [*0*, *0*, *10*] *10*)

**translations**
⋃♭*x*∈*xs*. *e* == *CONST lower-bind*·*xs*·(Λ *x*. *e*)

**lemma** *lower-bind-principal* [*simp*]:
*lower-bind*·(*lower-principal t*) = *lower-bind-basis t*
**unfolding** *lower-bind-def*
**apply** (*rule lower-pd.extension-principal*)
**apply** (*erule lower-bind-basis-mono*)
**done**

**lemma** *lower-bind-unit* [*simp*]:
*lower-bind*·{*x*}♭·*f* = *f*·*x*
**by** (*induct x rule*: *compact-basis.principal-induct*, *simp*, *simp*)

**lemma** *lower-bind-plus* [*simp*]:
*lower-bind*·(*xs* ∪♭ *ys*)·*f* = *lower-bind*·*xs*·*f* ∪♭ *lower-bind*·*ys*·*f*
**by** (*induct xs rule*: *lower-pd.principal-induct*, *simp*,
*induct ys rule*: *lower-pd.principal-induct*, *simp*, *simp*)

**lemma** *lower-bind-strict* [*simp*]: *lower-bind*·⊥·*f* = *f*·⊥
**unfolding** *lower-unit-strict* [*symmetric*] **by** (*rule lower-bind-unit*)

**lemma** *lower-bind-bind*:
*lower-bind*·(*lower-bind*·*xs*·*f*)·*g* = *lower-bind*·*xs*·(Λ *x*. *lower-bind*·(*f*·*x*)·*g*)
**by** (*induct xs*, *simp-all*)

## 31.6 Map

**definition**
*lower-map* :: $('a \to 'b) \to 'a$ *lower-pd* $\to 'b$ *lower-pd* **where**
*lower-map* = $(\Lambda\ f\ xs.\ \textit{lower-bind}{\cdot}xs{\cdot}(\Lambda\ x.\ \{f{\cdot}x\}\flat))$

**lemma** *lower-map-unit* [*simp*]:
*lower-map*·*f*·$\{x\}\flat$ = $\{f{\cdot}x\}\flat$
**unfolding** *lower-map-def* **by** *simp*

**lemma** *lower-map-plus* [*simp*]:
*lower-map*·*f*·(*xs* $\cup\flat$ *ys*) = *lower-map*·*f*·*xs* $\cup\flat$ *lower-map*·*f*·*ys*
**unfolding** *lower-map-def* **by** *simp*

**lemma** *lower-map-bottom* [*simp*]: *lower-map*·*f*·$\bot$ = $\{f{\cdot}\bot\}\flat$
**unfolding** *lower-map-def* **by** *simp*

**lemma** *lower-map-ident*: *lower-map*·$(\Lambda\ x.\ x)$·*xs* = *xs*
**by** (*induct xs rule*: *lower-pd-induct*, *simp-all*)

**lemma** *lower-map-ID*: *lower-map*·*ID* = *ID*
**by** (*simp add*: *cfun-eq-iff ID-def lower-map-ident*)

**lemma** *lower-map-map*:
*lower-map*·*f*·(*lower-map*·*g*·*xs*) = *lower-map*·$(\Lambda\ x.\ f{\cdot}(g{\cdot}x))$·*xs*
**by** (*induct xs rule*: *lower-pd-induct*, *simp-all*)

**lemma** *lower-bind-map*:
*lower-bind*·(*lower-map*·*f*·*xs*)·*g* = *lower-bind*·*xs*·$(\Lambda\ x.\ g{\cdot}(f{\cdot}x))$
**by** (*simp add*: *lower-map-def lower-bind-bind*)

**lemma** *lower-map-bind*:
*lower-map*·*f*·(*lower-bind*·*xs*·*g*) = *lower-bind*·*xs*·$(\Lambda\ x.$ *lower-map*·*f*·$(g{\cdot}x))$
**by** (*simp add*: *lower-map-def lower-bind-bind*)

**lemma** *ep-pair-lower-map*: *ep-pair e p* $\Longrightarrow$ *ep-pair* (*lower-map*·*e*) (*lower-map*·*p*)
**apply** *standard*
**apply** (*induct-tac x rule*: *lower-pd-induct*, *simp-all add*: *ep-pair.e-inverse*)
**apply** (*induct-tac y rule*: *lower-pd-induct*)
**apply** (*simp-all add*: *ep-pair.e-p-below monofun-cfun del*: *lower-plus-below-iff*)
**done**

**lemma** *deflation-lower-map*: *deflation d* $\Longrightarrow$ *deflation* (*lower-map*·*d*)
**apply** *standard*
**apply** (*induct-tac x rule*: *lower-pd-induct*, *simp-all add*: *deflation.idem*)
**apply** (*induct-tac x rule*: *lower-pd-induct*)
**apply** (*simp-all add*: *deflation.below monofun-cfun del*: *lower-plus-below-iff*)
**done**

**lemma** *finite-deflation-lower-map*:
**assumes** *finite-deflation d* **shows** *finite-deflation* (*lower-map*·*d*)
**proof** (*rule finite-deflation-intro*)
**interpret** *d*: *finite-deflation d* **by** *fact*
**from** *d.deflation-axioms* **show** *deflation* (*lower-map*·*d*)
**by** (*rule deflation-lower-map*)
**have** *finite* (*range* (λ*x*. *d*·*x*)) **by** (*rule d.finite-range*)
**hence** *finite* (*Rep-compact-basis* −‘ *range* (λ*x*. *d*·*x*))
**by** (*rule finite-vimageI*, *simp add*: *inj-on-def Rep-compact-basis-inject*)
**hence** *finite* (*Pow* (*Rep-compact-basis* −‘ *range* (λ*x*. *d*·*x*))) **by** *simp*
**hence** *finite* (*Rep-pd-basis* −‘ (*Pow* (*Rep-compact-basis* −‘ *range* (λ*x*. *d*·*x*))))
**by** (*rule finite-vimageI*, *simp add*: *inj-on-def Rep-pd-basis-inject*)
**hence** ∗: *finite* (*lower-principal* ‘ *Rep-pd-basis* −‘ (*Pow* (*Rep-compact-basis* −‘ *range* (λ*x*. *d*·*x*)))) **by** *simp*
**hence** *finite* (*range* (λ*xs*. *lower-map*·*d*·*xs*))
**apply** (*rule rev-finite-subset*)
**apply** *clarsimp*
**apply** (*induct-tac xs rule*: *lower-pd.principal-induct*)
**apply** (*simp add*: *adm-mem-finite* ∗)
**apply** (*rename-tac t*, *induct-tac t rule*: *pd-basis-induct*)
**apply** (*simp only*: *lower-unit-Rep-compact-basis* [*symmetric*] *lower-map-unit*)
**apply** *simp*
**apply** (*subgoal-tac* ∃ *b*. *d*·(*Rep-compact-basis a*) = *Rep-compact-basis b*)
**apply** *clarsimp*
**apply** (*rule imageI*)
**apply** (*rule vimageI2*)
**apply** (*simp add*: *Rep-PDUnit*)
**apply** (*rule range-eqI*)
**apply** (*erule sym*)
**apply** (*rule exI*)
**apply** (*rule Abs-compact-basis-inverse* [*symmetric*])
**apply** (*simp add*: *d.compact*)
**apply** (*simp only*: *lower-plus-principal* [*symmetric*] *lower-map-plus*)
**apply** *clarsimp*
**apply** (*rule imageI*)
**apply** (*rule vimageI2*)
**apply** (*simp add*: *Rep-PDPlus*)
**done**
**thus** *finite* {*xs*. *lower-map*·*d*·*xs* = *xs*}
**by** (*rule finite-range-imp-finite-fixes*)
**qed**

## 31.7 Lower powerdomain is bifinite

**lemma** *approx-chain-lower-map*:
**assumes** *approx-chain a*
**shows** *approx-chain* (λ*i*. *lower-map*·(*a i*))
**using** *assms* **unfolding** *approx-chain-def*
**by** (*simp add*: *lub-APP lower-map-ID finite-deflation-lower-map*)

**instance** *lower-pd* :: (*bifinite*) *bifinite*
**proof**
**show** $\exists (a{::}nat \Rightarrow 'a\ lower\text{-}pd \to 'a\ lower\text{-}pd).\ approx\text{-}chain\ a$
**using** *bifinite* [**where** $'a='a$]
**by** (*fast intro*!: *approx-chain-lower-map*)
**qed**

### 31.8 Join

**definition**
*lower-join* :: $'a\ lower\text{-}pd\ lower\text{-}pd \to 'a\ lower\text{-}pd$ **where**
*lower-join* = $(\Lambda\ xss.\ lower\text{-}bind{\cdot}xss{\cdot}(\Lambda\ xs.\ xs))$

**lemma** *lower-join-unit* [*simp*]:
*lower-join*·$\{xs\}\flat$ = *xs*
**unfolding** *lower-join-def* **by** *simp*

**lemma** *lower-join-plus* [*simp*]:
*lower-join*·(*xss* $\cup\flat$ *yss*) = *lower-join*·*xss* $\cup\flat$ *lower-join*·*yss*
**unfolding** *lower-join-def* **by** *simp*

**lemma** *lower-join-bottom* [*simp*]: *lower-join*·$\bot$ = $\bot$
**unfolding** *lower-join-def* **by** *simp*

**lemma** *lower-join-map-unit*:
*lower-join*·(*lower-map*·*lower-unit*·*xs*) = *xs*
**by** (*induct xs rule*: *lower-pd-induct*, *simp-all*)

**lemma** *lower-join-map-join*:
*lower-join*·(*lower-map*·*lower-join*·*xsss*) = *lower-join*·(*lower-join*·*xsss*)
**by** (*induct xsss rule*: *lower-pd-induct*, *simp-all*)

**lemma** *lower-join-map-map*:
*lower-join*·(*lower-map*·(*lower-map*·*f*)·*xss*) =
*lower-map*·*f*·(*lower-join*·*xss*)
**by** (*induct xss rule*: *lower-pd-induct*, *simp-all*)

**end**

## 32 Convex powerdomain

**theory** *ConvexPD*
**imports** *UpperPD LowerPD*
**begin**

### 32.1 Basis preorder

**definition**

*convex-le* :: *'a pd-basis* ⇒ *'a pd-basis* ⇒ *bool* (**infix** ≤♮ *50*) **where**
*convex-le* = (λ*u v*. *u* ≤♯ *v* ∧ *u* ≤♭ *v*)

**lemma** *convex-le-refl* [*simp*]: *t* ≤♮ *t*
**unfolding** *convex-le-def* **by** (*fast intro*: *upper-le-refl lower-le-refl*)

**lemma** *convex-le-trans*: ⟦*t* ≤♮ *u*; *u* ≤♮ *v*⟧ ⟹ *t* ≤♮ *v*
**unfolding** *convex-le-def* **by** (*fast intro*: *upper-le-trans lower-le-trans*)

**interpretation** *convex-le*: *preorder convex-le*
**by** (*rule preorder.intro*, *rule convex-le-refl*, *rule convex-le-trans*)

**lemma** *upper-le-minimal* [*simp*]: *PDUnit compact-bot* ≤♮ *t*
**unfolding** *convex-le-def Rep-PDUnit* **by** *simp*

**lemma** *PDUnit-convex-mono*: *x* ⊑ *y* ⟹ *PDUnit x* ≤♮ *PDUnit y*
**unfolding** *convex-le-def* **by** (*fast intro*: *PDUnit-upper-mono PDUnit-lower-mono*)

**lemma** *PDPlus-convex-mono*: ⟦*s* ≤♮ *t*; *u* ≤♮ *v*⟧ ⟹ *PDPlus s u* ≤♮ *PDPlus t v*
**unfolding** *convex-le-def* **by** (*fast intro*: *PDPlus-upper-mono PDPlus-lower-mono*)

**lemma** *convex-le-PDUnit-PDUnit-iff* [*simp*]:
(*PDUnit a* ≤♮ *PDUnit b*) = (*a* ⊑ *b*)
**unfolding** *convex-le-def upper-le-def lower-le-def Rep-PDUnit* **by** *fast*

**lemma** *convex-le-PDUnit-lemma1*:
(*PDUnit a* ≤♮ *t*) = (∀ *b*∈*Rep-pd-basis t*. *a* ⊑ *b*)
**unfolding** *convex-le-def upper-le-def lower-le-def Rep-PDUnit*
**using** *Rep-pd-basis-nonempty* [*of t*, *folded ex-in-conv*] **by** *fast*

**lemma** *convex-le-PDUnit-PDPlus-iff* [*simp*]:
(*PDUnit a* ≤♮ *PDPlus t u*) = (*PDUnit a* ≤♮ *t* ∧ *PDUnit a* ≤♮ *u*)
**unfolding** *convex-le-PDUnit-lemma1 Rep-PDPlus* **by** *fast*

**lemma** *convex-le-PDUnit-lemma2*:
(*t* ≤♮ *PDUnit b*) = (∀ *a*∈*Rep-pd-basis t*. *a* ⊑ *b*)
**unfolding** *convex-le-def upper-le-def lower-le-def Rep-PDUnit*
**using** *Rep-pd-basis-nonempty* [*of t*, *folded ex-in-conv*] **by** *fast*

**lemma** *convex-le-PDPlus-PDUnit-iff* [*simp*]:
(*PDPlus t u* ≤♮ *PDUnit a*) = (*t* ≤♮ *PDUnit a* ∧ *u* ≤♮ *PDUnit a*)
**unfolding** *convex-le-PDUnit-lemma2 Rep-PDPlus* **by** *fast*

**lemma** *convex-le-PDPlus-lemma*:
**assumes** *z*: *PDPlus t u* ≤♮ *z*
**shows** ∃ *v w*. *z* = *PDPlus v w* ∧ *t* ≤♮ *v* ∧ *u* ≤♮ *w*
**proof** (*intro exI conjI*)
**let** *?A* = {*b*∈*Rep-pd-basis z*. ∃ *a*∈*Rep-pd-basis t*. *a* ⊑ *b*}
**let** *?B* = {*b*∈*Rep-pd-basis z*. ∃ *a*∈*Rep-pd-basis u*. *a* ⊑ *b*}

**let** *?v = Abs-pd-basis ?A*
**let** *?w = Abs-pd-basis ?B*
**have** *Rep-v*: *Rep-pd-basis ?v = ?A*
**apply** (*rule Abs-pd-basis-inverse*)
**apply** (*rule Rep-pd-basis-nonempty* [*of t*, *folded ex-in-conv*, *THEN exE*])
**apply** (*cut-tac z*, *simp only*: *convex-le-def lower-le-def*, *clarify*)
**apply** (*drule-tac x=x* **in** *bspec*, *simp add*: *Rep-PDPlus*, *erule bexE*)
**apply** (*simp add*: *pd-basis-def*)
**apply** *fast*
**done**
**have** *Rep-w*: *Rep-pd-basis ?w = ?B*
**apply** (*rule Abs-pd-basis-inverse*)
**apply** (*rule Rep-pd-basis-nonempty* [*of u*, *folded ex-in-conv*, *THEN exE*])
**apply** (*cut-tac z*, *simp only*: *convex-le-def lower-le-def*, *clarify*)
**apply** (*drule-tac x=x* **in** *bspec*, *simp add*: *Rep-PDPlus*, *erule bexE*)
**apply** (*simp add*: *pd-basis-def*)
**apply** *fast*
**done**
**show** *z = PDPlus ?v ?w*
**apply** (*insert z*)
**apply** (*simp add*: *convex-le-def*, *erule conjE*)
**apply** (*simp add*: *Rep-pd-basis-inject* [*symmetric*] *Rep-PDPlus*)
**apply** (*simp add*: *Rep-v Rep-w*)
**apply** (*rule equalityI*)
**apply** (*rule subsetI*)
**apply** (*simp only*: *upper-le-def*)
**apply** (*drule* (*1*) *bspec*, *erule bexE*)
**apply** (*simp add*: *Rep-PDPlus*)
**apply** *fast*
**apply** *fast*
**done**
**show** *t* $\leq\natural$ *?v u* $\leq\natural$ *?w*
**apply** (*insert z*)
**apply** (*simp-all add*: *convex-le-def upper-le-def lower-le-def Rep-PDPlus Rep-v Rep-w*)
**apply** *fast+*
**done**
**qed**

**lemma** *convex-le-induct* [*induct set*: *convex-le*]:
**assumes** *le*: *t* $\leq\natural$ *u*
**assumes** *2*: $\bigwedge t\ u\ v.\ [\![P\ t\ u;\ P\ u\ v]\!] \Longrightarrow P\ t\ v$
**assumes** *3*: $\bigwedge a\ b.\ a \sqsubseteq b \Longrightarrow P\ (PDUnit\ a)\ (PDUnit\ b)$
**assumes** *4*: $\bigwedge t\ u\ v\ w.\ [\![P\ t\ v;\ P\ u\ w]\!] \Longrightarrow P\ (PDPlus\ t\ u)\ (PDPlus\ v\ w)$
**shows** *P t u*
**using** *le* **apply** (*induct t arbitrary*: *u rule*: *pd-basis-induct*)
**apply** (*erule rev-mp*)
**apply** (*induct-tac u rule*: *pd-basis-induct1*)
**apply** (*simp add*: *3*)

**apply** (*simp*, *clarify*, *rename-tac a b t*)
**apply** (*subgoal-tac P* (*PDPlus* (*PDUnit a*) (*PDUnit a*)) (*PDPlus* (*PDUnit b*) *t*))
**apply** (*simp add*: *PDPlus-absorb*)
**apply** (*erule* (*1*) *4* [*OF 3*])
**apply** (*drule convex-le-PDPlus-lemma*, *clarify*)
**apply** (*simp add*: *4*)
**done**

## 32.2 Type definition

**typedef** *'a convex-pd* (*('(-')♮*) =
{*S*::*'a pd-basis set*. *convex-le.ideal S*}
**by** (*rule convex-le.ex-ideal*)

**instantiation** *convex-pd* :: (*bifinite*) *below*
**begin**

**definition**
*x* ⊑ *y* ⟷ *Rep-convex-pd x* ⊆ *Rep-convex-pd y*

**instance ..**
**end**

**instance** *convex-pd* :: (*bifinite*) *po*
**using** *type-definition-convex-pd below-convex-pd-def*
**by** (*rule convex-le.typedef-ideal-po*)

**instance** *convex-pd* :: (*bifinite*) *cpo*
**using** *type-definition-convex-pd below-convex-pd-def*
**by** (*rule convex-le.typedef-ideal-cpo*)

**definition**
*convex-principal* :: *'a pd-basis* ⇒ *'a convex-pd* **where**
*convex-principal t* = *Abs-convex-pd* {*u*. *u* ≤♮ *t*}

**interpretation** *convex-pd*:
*ideal-completion convex-le convex-principal Rep-convex-pd*
**using** *type-definition-convex-pd below-convex-pd-def*
**using** *convex-principal-def pd-basis-countable*
**by** (*rule convex-le.typedef-ideal-completion*)

Convex powerdomain is pointed

**lemma** *convex-pd-minimal*: *convex-principal* (*PDUnit compact-bot*) ⊑ *ys*
**by** (*induct ys rule*: *convex-pd.principal-induct*, *simp*, *simp*)

**instance** *convex-pd* :: (*bifinite*) *pcpo*
**by** *intro-classes* (*fast intro*: *convex-pd-minimal*)

**lemma** *inst-convex-pd-pcpo*: ⊥ = *convex-principal* (*PDUnit compact-bot*)

**by** (*rule convex-pd-minimal* [*THEN bottomI*, *symmetric*])

## 32.3 Monadic unit and plus

**definition**
*convex-unit* :: *'a* → *'a convex-pd* **where**
*convex-unit* = *compact-basis.extension* (λ*a*. *convex-principal* (*PDUnit a*))

**definition**
*convex-plus* :: *'a convex-pd* → *'a convex-pd* → *'a convex-pd* **where**
*convex-plus* = *convex-pd.extension* (λ*t*. *convex-pd.extension* (λ*u*.
*convex-principal* (*PDPlus t u*)))

**abbreviation**
*convex-add* :: *'a convex-pd* ⇒ *'a convex-pd* ⇒ *'a convex-pd*
(**infixl** ∪♮ *65*) **where**
*xs* ∪♮ *ys* == *convex-plus*·*xs*·*ys*

**syntax**
-*convex-pd* :: *args* ⇒ *logic* ({-}♮)

**translations**
{*x*,*xs*}♮ == {*x*}♮ ∪♮ {*xs*}♮
{*x*}♮ == *CONST convex-unit*·*x*

**lemma** *convex-unit-Rep-compact-basis* [*simp*]:
{*Rep-compact-basis a*}♮ = *convex-principal* (*PDUnit a*)
**unfolding** *convex-unit-def*
**by** (*simp add*: *compact-basis.extension-principal PDUnit-convex-mono*)

**lemma** *convex-plus-principal* [*simp*]:
*convex-principal t* ∪♮ *convex-principal u* = *convex-principal* (*PDPlus t u*)
**unfolding** *convex-plus-def*
**by** (*simp add*: *convex-pd.extension-principal*
*convex-pd.extension-mono PDPlus-convex-mono*)

**interpretation** *convex-add*: *semilattice convex-add* **proof**
**fix** *xs ys zs* :: *'a convex-pd*
**show** (*xs* ∪♮ *ys*) ∪♮ *zs* = *xs* ∪♮ (*ys* ∪♮ *zs*)
**apply** (*induct xs rule*: *convex-pd.principal-induct*, *simp*)
**apply** (*induct ys rule*: *convex-pd.principal-induct*, *simp*)
**apply** (*induct zs rule*: *convex-pd.principal-induct*, *simp*)
**apply** (*simp add*: *PDPlus-assoc*)
**done**
**show** *xs* ∪♮ *ys* = *ys* ∪♮ *xs*
**apply** (*induct xs rule*: *convex-pd.principal-induct*, *simp*)
**apply** (*induct ys rule*: *convex-pd.principal-induct*, *simp*)
**apply** (*simp add*: *PDPlus-commute*)
**done**

**show** $xs \cup\natural\ xs = xs$
**apply** (*induct xs rule*: *convex-pd.principal-induct*, *simp*)
**apply** (*simp add*: *PDPlus-absorb*)
**done**
**qed**

**lemmas** *convex-plus-assoc* = *convex-add.assoc*
**lemmas** *convex-plus-commute* = *convex-add.commute*
**lemmas** *convex-plus-absorb* = *convex-add.idem*
**lemmas** *convex-plus-left-commute* = *convex-add.left-commute*
**lemmas** *convex-plus-left-absorb* = *convex-add.left-idem*

Useful for *simp add*: *convex-plus-ac*

**lemmas** *convex-plus-ac* =
*convex-plus-assoc convex-plus-commute convex-plus-left-commute*

Useful for *simp only*: *convex-plus-aci*

**lemmas** *convex-plus-aci* =
*convex-plus-ac convex-plus-absorb convex-plus-left-absorb*

**lemma** *convex-unit-below-plus-iff* [*simp*]:
$\{x\}\natural \sqsubseteq ys \cup\natural\ zs \longleftrightarrow \{x\}\natural \sqsubseteq ys \wedge \{x\}\natural \sqsubseteq zs$
**apply** (*induct x rule*: *compact-basis.principal-induct*, *simp*)
**apply** (*induct ys rule*: *convex-pd.principal-induct*, *simp*)
**apply** (*induct zs rule*: *convex-pd.principal-induct*, *simp*)
**apply** *simp*
**done**

**lemma** *convex-plus-below-unit-iff* [*simp*]:
$xs \cup\natural\ ys \sqsubseteq \{z\}\natural \longleftrightarrow xs \sqsubseteq \{z\}\natural \wedge ys \sqsubseteq \{z\}\natural$
**apply** (*induct xs rule*: *convex-pd.principal-induct*, *simp*)
**apply** (*induct ys rule*: *convex-pd.principal-induct*, *simp*)
**apply** (*induct z rule*: *compact-basis.principal-induct*, *simp*)
**apply** *simp*
**done**

**lemma** *convex-unit-below-iff* [*simp*]: $\{x\}\natural \sqsubseteq \{y\}\natural \longleftrightarrow x \sqsubseteq y$
**apply** (*induct x rule*: *compact-basis.principal-induct*, *simp*)
**apply** (*induct y rule*: *compact-basis.principal-induct*, *simp*)
**apply** *simp*
**done**

**lemma** *convex-unit-eq-iff* [*simp*]: $\{x\}\natural = \{y\}\natural \longleftrightarrow x = y$
**unfolding** *po-eq-conv* **by** *simp*

**lemma** *convex-unit-strict* [*simp*]: $\{\bot\}\natural = \bot$
**using** *convex-unit-Rep-compact-basis* [*of compact-bot*]
**by** (*simp add*: *inst-convex-pd-pcpo*)

**lemma** *convex-unit-bottom-iff* [*simp*]: $\{x\}\natural = \bot \longleftrightarrow x = \bot$
**unfolding** *convex-unit-strict* [*symmetric*] **by** (*rule convex-unit-eq-iff*)

**lemma** *compact-convex-unit*: *compact x* $\Longrightarrow$ *compact* $\{x\}\natural$
**by** (*auto dest*!: *compact-basis.compact-imp-principal*)

**lemma** *compact-convex-unit-iff* [*simp*]: *compact* $\{x\}\natural \longleftrightarrow$ *compact x*
**apply** (*safe elim*!: *compact-convex-unit*)
**apply** (*simp only*: *compact-def convex-unit-below-iff* [*symmetric*])
**apply** (*erule adm-subst* [*OF cont-Rep-cfun2*])
**done**

**lemma** *compact-convex-plus* [*simp*]:
$[\![$*compact xs*; *compact ys*$]\!] \Longrightarrow$ *compact* (*xs* $\cup\natural$ *ys*)
**by** (*auto dest*!: *convex-pd.compact-imp-principal*)

## 32.4 Induction rules

**lemma** *convex-pd-induct1*:
**assumes** *P*: *adm P*
**assumes** *unit*: $\bigwedge x.$ *P* $\{x\}\natural$
**assumes** *insert*: $\bigwedge x$ *ys*. $[\![P\ \{x\}\natural;\ P\ ys]\!] \Longrightarrow P\ (\{x\}\natural \cup\natural\ ys)$
**shows** *P* (*xs*::*'a convex-pd*)
**apply** (*induct xs rule*: *convex-pd.principal-induct*, *rule P*)
**apply** (*induct-tac a rule*: *pd-basis-induct1*)
**apply** (*simp only*: *convex-unit-Rep-compact-basis* [*symmetric*])
**apply** (*rule unit*)
**apply** (*simp only*: *convex-unit-Rep-compact-basis* [*symmetric*]
*convex-plus-principal* [*symmetric*])
**apply** (*erule insert* [*OF unit*])
**done**

**lemma** *convex-pd-induct*
[*case-names adm convex-unit convex-plus*, *induct type*: *convex-pd*]:
**assumes** *P*: *adm P*
**assumes** *unit*: $\bigwedge x.$ *P* $\{x\}\natural$
**assumes** *plus*: $\bigwedge$*xs ys*. $[\![P\ xs;\ P\ ys]\!] \Longrightarrow P\ (xs \cup\natural\ ys)$
**shows** *P* (*xs*::*'a convex-pd*)
**apply** (*induct xs rule*: *convex-pd.principal-induct*, *rule P*)
**apply** (*induct-tac a rule*: *pd-basis-induct*)
**apply** (*simp only*: *convex-unit-Rep-compact-basis* [*symmetric*] *unit*)
**apply** (*simp only*: *convex-plus-principal* [*symmetric*] *plus*)
**done**

## 32.5 Monadic bind

**definition**
*convex-bind-basis* ::
*'a pd-basis* $\Rightarrow$ (*'a* $\rightarrow$ *'b convex-pd*) $\rightarrow$ *'b convex-pd* **where**
*convex-bind-basis* = *fold-pd*

(λ*a*. Λ *f*. *f*·(*Rep-compact-basis a*))
(λ*x y*. Λ *f*. *x*·*f* ∪♮ *y*·*f*)

**lemma** *ACI-convex-bind*:
*semilattice* (λ*x y*. Λ *f*. *x*·*f* ∪♮ *y*·*f*)
**apply** *unfold-locales*
**apply** (*simp add*: *convex-plus-assoc*)
**apply** (*simp add*: *convex-plus-commute*)
**apply** (*simp add*: *eta-cfun*)
**done**

**lemma** *convex-bind-basis-simps* [*simp*]:
*convex-bind-basis* (*PDUnit a*) =
(Λ *f*. *f*·(*Rep-compact-basis a*))
*convex-bind-basis* (*PDPlus t u*) =
(Λ *f*. *convex-bind-basis t*·*f* ∪♮ *convex-bind-basis u*·*f*)
**unfolding** *convex-bind-basis-def*
**apply** −
**apply** (*rule fold-pd-PDUnit* [*OF ACI-convex-bind*])
**apply** (*rule fold-pd-PDPlus* [*OF ACI-convex-bind*])
**done**

**lemma** *convex-bind-basis-mono*:
*t* ≤♮ *u* ⟹ *convex-bind-basis t* ⊑ *convex-bind-basis u*
**apply** (*erule convex-le-induct*)
**apply** (*erule* (*1*) *below-trans*)
**apply** (*simp add*: *monofun-LAM monofun-cfun*)
**apply** (*simp add*: *monofun-LAM monofun-cfun*)
**done**

**definition**
*convex-bind* :: *'a convex-pd* → (*'a* → *'b convex-pd*) → *'b convex-pd* **where**
*convex-bind* = *convex-pd.extension convex-bind-basis*

**syntax**
*-convex-bind* :: [*logic*, *logic*, *logic*] ⇒ *logic*
((*3*⋃♮*-*∈*-./ -*) [*0*, *0*, *10*] *10*)

**translations**
⋃♮*x*∈*xs*. *e* == *CONST convex-bind*·*xs*·(Λ *x*. *e*)

**lemma** *convex-bind-principal* [*simp*]:
*convex-bind*·(*convex-principal t*) = *convex-bind-basis t*
**unfolding** *convex-bind-def*
**apply** (*rule convex-pd.extension-principal*)
**apply** (*erule convex-bind-basis-mono*)
**done**

**lemma** *convex-bind-unit* [*simp*]:

*convex-bind*·{*x*}♮·*f* = *f*·*x*
**by** (*induct x rule*: *compact-basis.principal-induct*, *simp*, *simp*)

**lemma** *convex-bind-plus* [*simp*]:
*convex-bind*·(*xs* ∪♮ *ys*)·*f* = *convex-bind*·*xs*·*f* ∪♮ *convex-bind*·*ys*·*f*
**by** (*induct xs rule*: *convex-pd.principal-induct*, *simp*,
*induct ys rule*: *convex-pd.principal-induct*, *simp*, *simp*)

**lemma** *convex-bind-strict* [*simp*]: *convex-bind*·⊥·*f* = *f*·⊥
**unfolding** *convex-unit-strict* [*symmetric*] **by** (*rule convex-bind-unit*)

**lemma** *convex-bind-bind*:
*convex-bind*·(*convex-bind*·*xs*·*f*)·*g* =
*convex-bind*·*xs*·(Λ *x*. *convex-bind*·(*f*·*x*)·*g*)
**by** (*induct xs*, *simp-all*)

## 32.6 Map

**definition**
*convex-map* :: ('*a* → '*b*) → '*a convex-pd* → '*b convex-pd* **where**
*convex-map* = (Λ *f xs*. *convex-bind*·*xs*·(Λ *x*. {*f*·*x*}♮))

**lemma** *convex-map-unit* [*simp*]:
*convex-map*·*f*·{*x*}♮ = {*f*·*x*}♮
**unfolding** *convex-map-def* **by** *simp*

**lemma** *convex-map-plus* [*simp*]:
*convex-map*·*f*·(*xs* ∪♮ *ys*) = *convex-map*·*f*·*xs* ∪♮ *convex-map*·*f*·*ys*
**unfolding** *convex-map-def* **by** *simp*

**lemma** *convex-map-bottom* [*simp*]: *convex-map*·*f*·⊥ = {*f*·⊥}♮
**unfolding** *convex-map-def* **by** *simp*

**lemma** *convex-map-ident*: *convex-map*·(Λ *x*. *x*)·*xs* = *xs*
**by** (*induct xs rule*: *convex-pd-induct*, *simp-all*)

**lemma** *convex-map-ID*: *convex-map*·*ID* = *ID*
**by** (*simp add*: *cfun-eq-iff ID-def convex-map-ident*)

**lemma** *convex-map-map*:
*convex-map*·*f*·(*convex-map*·*g*·*xs*) = *convex-map*·(Λ *x*. *f*·(*g*·*x*))·*xs*
**by** (*induct xs rule*: *convex-pd-induct*, *simp-all*)

**lemma** *convex-bind-map*:
*convex-bind*·(*convex-map*·*f*·*xs*)·*g* = *convex-bind*·*xs*·(Λ *x*. *g*·(*f*·*x*))
**by** (*simp add*: *convex-map-def convex-bind-bind*)

**lemma** *convex-map-bind*:
*convex-map*·*f*·(*convex-bind*·*xs*·*g*) = *convex-bind*·*xs*·(Λ *x*. *convex-map*·*f*·(*g*·*x*))

**by** (*simp add*: *convex-map-def convex-bind-bind*)

**lemma** *ep-pair-convex-map*: *ep-pair e p* ⟹ *ep-pair* (*convex-map·e*) (*convex-map·p*)
**apply** *standard*
**apply** (*induct-tac x rule*: *convex-pd-induct*, *simp-all add*: *ep-pair.e-inverse*)
**apply** (*induct-tac y rule*: *convex-pd-induct*)
**apply** (*simp-all add*: *ep-pair.e-p-below monofun-cfun*)
**done**

**lemma** *deflation-convex-map*: *deflation d* ⟹ *deflation* (*convex-map·d*)
**apply** *standard*
**apply** (*induct-tac x rule*: *convex-pd-induct*, *simp-all add*: *deflation.idem*)
**apply** (*induct-tac x rule*: *convex-pd-induct*)
**apply** (*simp-all add*: *deflation.below monofun-cfun*)
**done**

**lemma** *finite-deflation-convex-map*:
**assumes** *finite-deflation d* **shows** *finite-deflation* (*convex-map·d*)
**proof** (*rule finite-deflation-intro*)
**interpret** *d*: *finite-deflation d* **by** *fact*
**from** *d.deflation-axioms* **show** *deflation* (*convex-map·d*)
**by** (*rule deflation-convex-map*)
**have** *finite* (*range* (λ*x*. *d·x*)) **by** (*rule d.finite-range*)
**hence** *finite* (*Rep-compact-basis* −‘ *range* (λ*x*. *d·x*))
**by** (*rule finite-vimageI*, *simp add*: *inj-on-def Rep-compact-basis-inject*)
**hence** *finite* (*Pow* (*Rep-compact-basis* −‘ *range* (λ*x*. *d·x*))) **by** *simp*
**hence** *finite* (*Rep-pd-basis* −‘ (*Pow* (*Rep-compact-basis* −‘ *range* (λ*x*. *d·x*))))
**by** (*rule finite-vimageI*, *simp add*: *inj-on-def Rep-pd-basis-inject*)
**hence** ∗: *finite* (*convex-principal* ‘ *Rep-pd-basis* −‘ (*Pow* (*Rep-compact-basis* −‘ *range* (λ*x*. *d·x*)))) **by** *simp*
**hence** *finite* (*range* (λ*xs*. *convex-map·d·xs*))
**apply** (*rule rev-finite-subset*)
**apply** *clarsimp*
**apply** (*induct-tac xs rule*: *convex-pd.principal-induct*)
**apply** (*simp add*: *adm-mem-finite* ∗)
**apply** (*rename-tac t*, *induct-tac t rule*: *pd-basis-induct*)
**apply** (*simp only*: *convex-unit-Rep-compact-basis* [*symmetric*] *convex-map-unit*)
**apply** *simp*
**apply** (*subgoal-tac* ∃ *b*. *d·*(*Rep-compact-basis a*) = *Rep-compact-basis b*)
**apply** *clarsimp*
**apply** (*rule imageI*)
**apply** (*rule vimageI2*)
**apply** (*simp add*: *Rep-PDUnit*)
**apply** (*rule range-eqI*)
**apply** (*erule sym*)
**apply** (*rule exI*)
**apply** (*rule Abs-compact-basis-inverse* [*symmetric*])
**apply** (*simp add*: *d.compact*)

**apply** (*simp only*: *convex-plus-principal* [*symmetric*] *convex-map-plus*)
**apply** *clarsimp*
**apply** (*rule imageI*)
**apply** (*rule vimageI2*)
**apply** (*simp add*: *Rep-PDPlus*)
**done**
**thus** *finite* {*xs*. *convex-map*·*d*·*xs* = *xs*}
**by** (*rule finite-range-imp-finite-fixes*)
**qed**

## 32.7 Convex powerdomain is bifinite

**lemma** *approx-chain-convex-map*:
**assumes** *approx-chain a*
**shows** *approx-chain* (λ*i*. *convex-map*·(*a i*))
**using** *assms* **unfolding** *approx-chain-def*
**by** (*simp add*: *lub-APP convex-map-ID finite-deflation-convex-map*)

**instance** *convex-pd* :: (*bifinite*) *bifinite*
**proof**
**show** ∃(*a*::*nat* ⇒ 'a *convex-pd* → 'a *convex-pd*). *approx-chain a*
**using** *bifinite* [**where** 'a='a]
**by** (*fast intro*!: *approx-chain-convex-map*)
**qed**

## 32.8 Join

**definition**
*convex-join* :: 'a *convex-pd convex-pd* → 'a *convex-pd* **where**
*convex-join* = (Λ *xss*. *convex-bind*·*xss*·(Λ *xs*. *xs*))

**lemma** *convex-join-unit* [*simp*]:
*convex-join*·{*xs*}♮ = *xs*
**unfolding** *convex-join-def* **by** *simp*

**lemma** *convex-join-plus* [*simp*]:
*convex-join*·(*xss* ∪♮ *yss*) = *convex-join*·*xss* ∪♮ *convex-join*·*yss*
**unfolding** *convex-join-def* **by** *simp*

**lemma** *convex-join-bottom* [*simp*]: *convex-join*·⊥ = ⊥
**unfolding** *convex-join-def* **by** *simp*

**lemma** *convex-join-map-unit*:
*convex-join*·(*convex-map*·*convex-unit*·*xs*) = *xs*
**by** (*induct xs rule*: *convex-pd-induct*, *simp-all*)

**lemma** *convex-join-map-join*:
*convex-join*·(*convex-map*·*convex-join*·*xsss*) = *convex-join*·(*convex-join*·*xsss*)
**by** (*induct xsss rule*: *convex-pd-induct*, *simp-all*)

**lemma** *convex-join-map-map*:
*convex-join*·(*convex-map*·(*convex-map*·*f*)·*xss*) =
*convex-map*·*f*·(*convex-join*·*xss*)
**by** (*induct xss rule*: *convex-pd-induct*, *simp-all*)

## 32.9 Conversions to other powerdomains

Convex to upper

**lemma** *convex-le-imp-upper-le*: $t \leq\natural u \Longrightarrow t \leq\sharp u$
**unfolding** *convex-le-def* **by** *simp*

**definition**
*convex-to-upper* :: *'a convex-pd* → *'a upper-pd* **where**
*convex-to-upper* = *convex-pd.extension upper-principal*

**lemma** *convex-to-upper-principal* [*simp*]:
*convex-to-upper*·(*convex-principal t*) = *upper-principal t*
**unfolding** *convex-to-upper-def*
**apply** (*rule convex-pd.extension-principal*)
**apply** (*rule upper-pd.principal-mono*)
**apply** (*erule convex-le-imp-upper-le*)
**done**

**lemma** *convex-to-upper-unit* [*simp*]:
*convex-to-upper*·{*x*}$\natural$ = {*x*}$\sharp$
**by** (*induct x rule*: *compact-basis.principal-induct*, *simp*, *simp*)

**lemma** *convex-to-upper-plus* [*simp*]:
*convex-to-upper*·(*xs* ∪$\natural$ *ys*) = *convex-to-upper*·*xs* ∪$\sharp$ *convex-to-upper*·*ys*
**by** (*induct xs rule*: *convex-pd.principal-induct*, *simp*,
*induct ys rule*: *convex-pd.principal-induct*, *simp*, *simp*)

**lemma** *convex-to-upper-bind* [*simp*]:
*convex-to-upper*·(*convex-bind*·*xs*·*f*) =
*upper-bind*·(*convex-to-upper*·*xs*)·(*convex-to-upper oo f*)
**by** (*induct xs rule*: *convex-pd-induct*, *simp*, *simp*, *simp*)

**lemma** *convex-to-upper-map* [*simp*]:
*convex-to-upper*·(*convex-map*·*f*·*xs*) = *upper-map*·*f*·(*convex-to-upper*·*xs*)
**by** (*simp add*: *convex-map-def upper-map-def cfcomp-LAM*)

**lemma** *convex-to-upper-join* [*simp*]:
*convex-to-upper*·(*convex-join*·*xss*) =
*upper-bind*·(*convex-to-upper*·*xss*)·*convex-to-upper*
**by** (*simp add*: *convex-join-def upper-join-def cfcomp-LAM eta-cfun*)

Convex to lower

**lemma** *convex-le-imp-lower-le*: $t \leq\natural u \Longrightarrow t \leq\flat u$
**unfolding** *convex-le-def* **by** *simp*

**definition**
*convex-to-lower* :: *'a convex-pd* → *'a lower-pd* **where**
*convex-to-lower* = *convex-pd.extension lower-principal*

**lemma** *convex-to-lower-principal* [*simp*]:
*convex-to-lower*·(*convex-principal t*) = *lower-principal t*
**unfolding** *convex-to-lower-def*
**apply** (*rule convex-pd.extension-principal*)
**apply** (*rule lower-pd.principal-mono*)
**apply** (*erule convex-le-imp-lower-le*)
**done**

**lemma** *convex-to-lower-unit* [*simp*]:
*convex-to-lower*·{*x*}♮ = {*x*}♭
**by** (*induct x rule*: *compact-basis.principal-induct*, *simp*, *simp*)

**lemma** *convex-to-lower-plus* [*simp*]:
*convex-to-lower*·(*xs* ∪♮ *ys*) = *convex-to-lower*·*xs* ∪♭ *convex-to-lower*·*ys*
**by** (*induct xs rule*: *convex-pd.principal-induct*, *simp*,
*induct ys rule*: *convex-pd.principal-induct*, *simp*, *simp*)

**lemma** *convex-to-lower-bind* [*simp*]:
*convex-to-lower*·(*convex-bind*·*xs*·*f*) =
*lower-bind*·(*convex-to-lower*·*xs*)·(*convex-to-lower oo f*)
**by** (*induct xs rule*: *convex-pd-induct*, *simp*, *simp*, *simp*)

**lemma** *convex-to-lower-map* [*simp*]:
*convex-to-lower*·(*convex-map*·*f*·*xs*) = *lower-map*·*f*·(*convex-to-lower*·*xs*)
**by** (*simp add*: *convex-map-def lower-map-def cfcomp-LAM*)

**lemma** *convex-to-lower-join* [*simp*]:
*convex-to-lower*·(*convex-join*·*xss*) =
*lower-bind*·(*convex-to-lower*·*xss*)·*convex-to-lower*
**by** (*simp add*: *convex-join-def lower-join-def cfcomp-LAM eta-cfun*)

Ordering property

**lemma** *convex-pd-below-iff*:
(*xs* ⊑ *ys*) =
(*convex-to-upper*·*xs* ⊑ *convex-to-upper*·*ys* ∧
*convex-to-lower*·*xs* ⊑ *convex-to-lower*·*ys*)
**apply** (*induct xs rule*: *convex-pd.principal-induct*, *simp*)
**apply** (*induct ys rule*: *convex-pd.principal-induct*, *simp*)
**apply** (*simp add*: *convex-le-def*)
**done**

**lemmas** *convex-plus-below-plus-iff* =
*convex-pd-below-iff* [**where** *xs*=*xs* ∪♮ *ys* **and** *ys*=*zs* ∪♮ *ws*]
**for** *xs ys zs ws*

**lemmas** *convex-pd-below-simps* =
*convex-unit-below-plus-iff*
*convex-plus-below-unit-iff*
*convex-plus-below-plus-iff*
*convex-unit-below-iff*
*convex-to-upper-unit*
*convex-to-upper-plus*
*convex-to-lower-unit*
*convex-to-lower-plus*
*upper-pd-below-simps*
*lower-pd-below-simps*

**end**

# 33 Powerdomains

**theory** *Powerdomains*
**imports** *ConvexPD Domain*
**begin**

## 33.1 Universal domain embeddings

**definition** *upper-emb* = *udom-emb* (λ*i*. *upper-map*·(*udom-approx i*))
**definition** *upper-prj* = *udom-prj* (λ*i*. *upper-map*·(*udom-approx i*))

**definition** *lower-emb* = *udom-emb* (λ*i*. *lower-map*·(*udom-approx i*))
**definition** *lower-prj* = *udom-prj* (λ*i*. *lower-map*·(*udom-approx i*))

**definition** *convex-emb* = *udom-emb* (λ*i*. *convex-map*·(*udom-approx i*))
**definition** *convex-prj* = *udom-prj* (λ*i*. *convex-map*·(*udom-approx i*))

**lemma** *ep-pair-upper*: *ep-pair upper-emb upper-prj*
**unfolding** *upper-emb-def upper-prj-def*
**by** (*simp add*: *ep-pair-udom approx-chain-upper-map*)

**lemma** *ep-pair-lower*: *ep-pair lower-emb lower-prj*
**unfolding** *lower-emb-def lower-prj-def*
**by** (*simp add*: *ep-pair-udom approx-chain-lower-map*)

**lemma** *ep-pair-convex*: *ep-pair convex-emb convex-prj*
**unfolding** *convex-emb-def convex-prj-def*
**by** (*simp add*: *ep-pair-udom approx-chain-convex-map*)

## 33.2 Deflation combinators

**definition** *upper-defl* :: *udom defl* → *udom defl*
**where** *upper-defl* = *defl-fun1 upper-emb upper-prj upper-map*

**definition** *lower-defl* :: *udom defl* → *udom defl*
**where** *lower-defl* = *defl-fun1 lower-emb lower-prj lower-map*

**definition** *convex-defl* :: *udom defl* → *udom defl*
**where** *convex-defl* = *defl-fun1 convex-emb convex-prj convex-map*

**lemma** *cast-upper-defl*:
*cast*·(*upper-defl*·*A*) = *upper-emb oo upper-map*·(*cast*·*A*) *oo upper-prj*
**using** *ep-pair-upper finite-deflation-upper-map*
**unfolding** *upper-defl-def* **by** (*rule cast-defl-fun1*)

**lemma** *cast-lower-defl*:
*cast*·(*lower-defl*·*A*) = *lower-emb oo lower-map*·(*cast*·*A*) *oo lower-prj*
**using** *ep-pair-lower finite-deflation-lower-map*
**unfolding** *lower-defl-def* **by** (*rule cast-defl-fun1*)

**lemma** *cast-convex-defl*:
*cast*·(*convex-defl*·*A*) = *convex-emb oo convex-map*·(*cast*·*A*) *oo convex-prj*
**using** *ep-pair-convex finite-deflation-convex-map*
**unfolding** *convex-defl-def* **by** (*rule cast-defl-fun1*)

## 33.3 Domain class instances

**instantiation** *upper-pd* :: (*domain*) *domain*
**begin**

**definition**
*emb* = *upper-emb oo upper-map*·*emb*

**definition**
*prj* = *upper-map*·*prj oo upper-prj*

**definition**
*defl* (*t*::*'a upper-pd itself*) = *upper-defl*·*DEFL*(*'a*)

**definition**
(*liftemb* :: *'a upper-pd u* → *udom u*) = *u-map*·*emb*

**definition**
(*liftprj* :: *udom u* → *'a upper-pd u*) = *u-map*·*prj*

**definition**
*liftdefl* (*t*::*'a upper-pd itself*) = *liftdefl-of*·*DEFL*(*'a upper-pd*)

**instance proof**
**show** *ep-pair emb* (*prj* :: *udom* → *'a upper-pd*)
**unfolding** *emb-upper-pd-def prj-upper-pd-def*
**by** (*intro ep-pair-comp ep-pair-upper ep-pair-upper-map ep-pair-emb-prj*)
**next**

**show** *cast·DEFL('a upper-pd) = emb oo (prj :: udom → 'a upper-pd)*
**unfolding** *emb-upper-pd-def prj-upper-pd-def defl-upper-pd-def cast-upper-defl*
**by** (*simp add: cast-DEFL oo-def cfun-eq-iff upper-map-map*)
**qed** (*fact liftemb-upper-pd-def liftprj-upper-pd-def liftdefl-upper-pd-def*)+

**end**

**instantiation** *lower-pd* :: (*domain*) *domain*
**begin**

**definition**
*emb = lower-emb oo lower-map·emb*

**definition**
*prj = lower-map·prj oo lower-prj*

**definition**
*defl* (*t*::*'a lower-pd itself*) = *lower-defl·DEFL('a)*

**definition**
(*liftemb* :: *'a lower-pd u → udom u*) = *u-map·emb*

**definition**
(*liftprj* :: *udom u → 'a lower-pd u*) = *u-map·prj*

**definition**
*liftdefl* (*t*::*'a lower-pd itself*) = *liftdefl-of·DEFL('a lower-pd)*

**instance proof**
**show** *ep-pair emb* (*prj* :: *udom → 'a lower-pd*)
**unfolding** *emb-lower-pd-def prj-lower-pd-def*
**by** (*intro ep-pair-comp ep-pair-lower ep-pair-lower-map ep-pair-emb-prj*)
**next**
**show** *cast·DEFL('a lower-pd) = emb oo* (*prj* :: *udom → 'a lower-pd*)
**unfolding** *emb-lower-pd-def prj-lower-pd-def defl-lower-pd-def cast-lower-defl*
**by** (*simp add: cast-DEFL oo-def cfun-eq-iff lower-map-map*)
**qed** (*fact liftemb-lower-pd-def liftprj-lower-pd-def liftdefl-lower-pd-def*)+

**end**

**instantiation** *convex-pd* :: (*domain*) *domain*
**begin**

**definition**
*emb = convex-emb oo convex-map·emb*

**definition**
*prj = convex-map·prj oo convex-prj*

**definition**
*defl* (*t*::'*a convex-pd itself*) = *convex-defl*·*DEFL*('*a*)

**definition**
(*liftemb* :: '*a convex-pd u* → *udom u*) = *u-map*·*emb*

**definition**
(*liftprj* :: *udom u* → '*a convex-pd u*) = *u-map*·*prj*

**definition**
*liftdefl* (*t*::'*a convex-pd itself*) = *liftdefl-of*·*DEFL*('*a convex-pd*)

**instance proof**
**show** *ep-pair emb* (*prj* :: *udom* → '*a convex-pd*)
**unfolding** *emb-convex-pd-def prj-convex-pd-def*
**by** (*intro ep-pair-comp ep-pair-convex ep-pair-convex-map ep-pair-emb-prj*)
**next**
**show** *cast*·*DEFL*('*a convex-pd*) = *emb oo* (*prj* :: *udom* → '*a convex-pd*)
**unfolding** *emb-convex-pd-def prj-convex-pd-def defl-convex-pd-def cast-convex-defl*
**by** (*simp add*: *cast-DEFL oo-def cfun-eq-iff convex-map-map*)
**qed** (*fact liftemb-convex-pd-def liftprj-convex-pd-def liftdefl-convex-pd-def*)+

**end**

**lemma** *DEFL-upper*: *DEFL*('*a*::*domain upper-pd*) = *upper-defl*·*DEFL*('*a*)
**by** (*rule defl-upper-pd-def*)

**lemma** *DEFL-lower*: *DEFL*('*a*::*domain lower-pd*) = *lower-defl*·*DEFL*('*a*)
**by** (*rule defl-lower-pd-def*)

**lemma** *DEFL-convex*: *DEFL*('*a*::*domain convex-pd*) = *convex-defl*·*DEFL*('*a*)
**by** (*rule defl-convex-pd-def*)

## 33.4 Isomorphic deflations

**lemma** *isodefl-upper*:
*isodefl d t* ⟹ *isodefl* (*upper-map*·*d*) (*upper-defl*·*t*)
**apply** (*rule isodeflI*)
**apply** (*simp add*: *cast-upper-defl cast-isodefl*)
**apply** (*simp add*: *emb-upper-pd-def prj-upper-pd-def*)
**apply** (*simp add*: *upper-map-map*)
**done**

**lemma** *isodefl-lower*:
*isodefl d t* ⟹ *isodefl* (*lower-map*·*d*) (*lower-defl*·*t*)
**apply** (*rule isodeflI*)
**apply** (*simp add*: *cast-lower-defl cast-isodefl*)
**apply** (*simp add*: *emb-lower-pd-def prj-lower-pd-def*)
**apply** (*simp add*: *lower-map-map*)

**done**

**lemma** *isodefl-convex*:
*isodefl d t* ⟹ *isodefl* (*convex-map*·*d*) (*convex-defl*·*t*)
**apply** (*rule isodeflI*)
**apply** (*simp add*: *cast-convex-defl cast-isodefl*)
**apply** (*simp add*: *emb-convex-pd-def prj-convex-pd-def*)
**apply** (*simp add*: *convex-map-map*)
**done**

## 33.5 Domain package setup for powerdomains

**lemmas** [*domain-defl-simps*] = *DEFL-upper DEFL-lower DEFL-convex*
**lemmas** [*domain-map-ID*] = *upper-map-ID lower-map-ID convex-map-ID*
**lemmas** [*domain-isodefl*] = *isodefl-upper isodefl-lower isodefl-convex*

**lemmas** [*domain-deflation*] =
*deflation-upper-map deflation-lower-map deflation-convex-map*

**setup** ‹
*fold Domain-Take-Proofs.add-rec-type*
[(***type-name*** ‹*upper-pd*›, [*true*]),
(***type-name*** ‹*lower-pd*›, [*true*]),
(***type-name*** ‹*convex-pd*›, [*true*])]
›

**end**

**theory** *HOLCF*
**imports**
*Main*
*Domain*
*Powerdomains*
**begin**

**default-sort** *domain*

**end**