Isabelle/HOL — Higher-Order Logic

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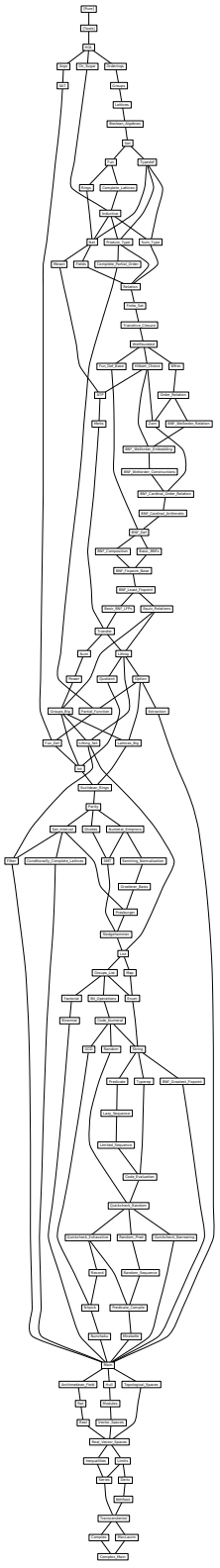

1 Loading the code generator and related modules

theory *Code-Generator*
imports *Pure*
keywords
  print-codeproc code-thms code-deps :: *diag* **and**
  export-code code-identifier code-printing code-reserved
    code-monad code-reflect :: *thy-decl* **and**
  checking **and**
  datatypes functions module-name file file-prefix
    constant type-constructor type-class class-relation class-instance code-module
    :: *quasi-command*
begin

⟨*ML*⟩

code-datatype *TYPE*('*a*::{})

definition *holds* :: *prop* **where**
  holds ≡ ((λ*x*::*prop*. *x*) ≡ (λ*x*. *x*))

lemma *holds*: *PROP holds*
  ⟨*proof*⟩

code-datatype *holds*

lemma *implies-code* [*code*]:
  (*PROP holds* ⟹ *PROP P*) ≡ *PROP P*
  (*PROP P* ⟹ *PROP holds*) ≡ *PROP holds*
⟨*proof*⟩

⟨*ML*⟩

hide-const (**open**) *holds*

end

2 The basis of Higher-Order Logic

theory *HOL*
imports *Pure Tools.Code-Generator*
keywords
  try solve-direct quickcheck print-coercions print-claset
    print-induct-rules :: *diag* **and**
  quickcheck-params :: *thy-decl*
abbrevs *?<* = ∃$_{≤1}$
begin

⟨*ML*⟩

2.1 Primitive logic

The definition of the logic is based on Mike Gordon's technical report [2] that describes the first implementation of HOL. However, there are a number of differences. In particular, we start with the definite description operator and introduce Hilbert's ε operator only much later. Moreover, axiom $(P \longrightarrow Q) \longrightarrow (Q \longrightarrow P) \longrightarrow (P = Q)$ is derived from the other axioms. The fact that this axiom is derivable was first noticed by Bruno Barras (for Mike Gordon's line of HOL systems) and later independently by Alexander Maletzky (for Isabelle/HOL).

2.1.1 Core syntax

⟨*ML*⟩
default-sort *type*
⟨*ML*⟩

axiomatization where *fun-arity*: *OFCLASS*('*a* ⇒ '*b*, *type-class*)
instance *fun* :: (*type*, *type*) *type* ⟨*proof*⟩

axiomatization where *itself-arity*: *OFCLASS*('*a itself*, *type-class*)
instance *itself* :: (*type*) *type* ⟨*proof*⟩

typedecl *bool*

judgment *Trueprop* :: *bool* ⇒ *prop* (-) *5*)

axiomatization *implies* :: [*bool*, *bool*] ⇒ *bool* (**infixr** ⟶ *25*)
and *eq* :: ['*a*, '*a*] ⇒ *bool*
and *The* :: ('*a* ⇒ *bool*) ⇒ '*a*

notation (*input*)
eq (**infixl** = *50*)
notation (**output**)
eq (**infix** = *50*)

The input syntax for *eq* is more permissive than the output syntax because of the large amount of material that relies on infixl.

2.1.2 Defined connectives and quantifiers

definition *True* :: *bool*
where *True* ≡ ((λ*x*::*bool*. *x*) = (λ*x*. *x*))

definition *All* :: ('*a* ⇒ *bool*) ⇒ *bool* (**binder** ∀ *10*)
where *All P* ≡ (*P* = (λ*x*. *True*))

definition *Ex* :: ('*a* ⇒ *bool*) ⇒ *bool* (**binder** ∃ *10*)

where $Ex\ P \equiv \forall Q.\ (\forall x.\ P\ x \longrightarrow Q) \longrightarrow Q$

definition *False* :: *bool*
where $False \equiv (\forall P.\ P)$

definition *Not* :: $bool \Rightarrow bool$ (¬ - [*40*] *40*)
where *not-def*: $\neg\ P \equiv P \longrightarrow False$

definition *conj* :: [*bool*, *bool*] ⇒ *bool* (**infixr** ∧ *35*)
where *and-def*: $P \wedge Q \equiv \forall R.\ (P \longrightarrow Q \longrightarrow R) \longrightarrow R$

definition *disj* :: [*bool*, *bool*] ⇒ *bool* (**infixr** ∨ *30*)
where *or-def*: $P \vee Q \equiv \forall R.\ (P \longrightarrow R) \longrightarrow (Q \longrightarrow R) \longrightarrow R$

definition *Uniq* :: $('a \Rightarrow bool) \Rightarrow bool$
where $Uniq\ P \equiv (\forall x\ y.\ P\ x \longrightarrow P\ y \longrightarrow y = x)$

definition *Ex1* :: $('a \Rightarrow bool) \Rightarrow bool$
where $Ex1\ P \equiv \exists x.\ P\ x \wedge (\forall y.\ P\ y \longrightarrow y = x)$

2.1.3 Additional concrete syntax

syntax (*ASCII*) -*Uniq* :: *pttrn* ⇒ *bool* ⇒ *bool* ((*4?<* -./ -) [*0*, *10*] *10*)
syntax -*Uniq* :: *pttrn* ⇒ *bool* ⇒ *bool* ((*2*∃$_{\leq 1}$ -./ -) [*0*, *10*] *10*)
translations $\exists_{\leq 1} x.\ P \rightleftharpoons CONST\ Uniq\ (\lambda x.\ P)$

⟨*ML*⟩

syntax (*ASCII*)
-*Ex1* :: *pttrn* ⇒ *bool* ⇒ *bool* ((*3EX!* -./ -) [*0*, *10*] *10*)
syntax (*input*)
-*Ex1* :: *pttrn* ⇒ *bool* ⇒ *bool* ((*3?!* -./ -) [*0*, *10*] *10*)
syntax -*Ex1* :: *pttrn* ⇒ *bool* ⇒ *bool* ((*3*∃!-./ -) [*0*, *10*] *10*)
translations $\exists!x.\ P \rightleftharpoons CONST\ Ex1\ (\lambda x.\ P)$

⟨*ML*⟩

syntax
-*Not-Ex* :: *idts* ⇒ *bool* ⇒ *bool* ((*3*∄ -./ -) [*0*, *10*] *10*)
-*Not-Ex1* :: *pttrn* ⇒ *bool* ⇒ *bool* ((*3*∄!-./ -) [*0*, *10*] *10*)
translations
$\nexists x.\ P \rightleftharpoons \neg\ (\exists x.\ P)$
$\nexists!x.\ P \rightleftharpoons \neg\ (\exists!x.\ P)$

abbreviation *not-equal* :: $['a, 'a] \Rightarrow bool$ (**infix** ≠ *50*)
where $x \neq y \equiv \neg\ (x = y)$

notation (*ASCII*)
Not (~ - [*40*] *40*) **and**
conj (**infixr** & *35*) **and**
disj (**infixr** | *30*) **and**
implies (**infixr** −−> *25*) **and**
not-equal (**infix** ~= *50*)

abbreviation (*iff*)
iff :: [*bool*, *bool*] ⇒ *bool* (**infixr** ⟷ *25*)
where $A \longleftrightarrow B \equiv A = B$

syntax -*The* :: [*pttrn*, *bool*] ⇒ *'a* ((*3THE* -./ -) [*0*, *10*] *10*)
translations *THE x. P* ⇌ *CONST The* (λ*x. P*)
⟨*ML*⟩

nonterminal *letbinds* **and** *letbind*
syntax
-*bind* :: [*pttrn*, *'a*] ⇒ *letbind* ((*2*- =/ -) *10*)
:: *letbind* ⇒ *letbinds* (-)
-*binds* :: [*letbind*, *letbinds*] ⇒ *letbinds* (-;/ -)
-*Let* :: [*letbinds*, *'a*] ⇒ *'a* ((*let* (-)/ *in* (-)) [*0*, *10*] *10*)

nonterminal *case-syn* **and** *cases-syn*
syntax
-*case-syntax* :: [*'a*, *cases-syn*] ⇒ *'b* ((*case* - *of*/ -) *10*)
-*case1* :: [*'a*, *'b*] ⇒ *case-syn* ((*2*- ⇒/ -) *10*)
:: *case-syn* ⇒ *cases-syn* (-)
-*case2* :: [*case-syn*, *cases-syn*] ⇒ *cases-syn* (-/ | -)
syntax (*ASCII*)
-*case1* :: [*'a*, *'b*] ⇒ *case-syn* ((*2*- =>/ -) *10*)

notation (*ASCII*)
All (**binder** *ALL* *10*) **and**
Ex (**binder** *EX* *10*)

notation (*input*)
All (**binder** ! *10*) **and**
Ex (**binder** *?* *10*)

2.1.4 Axioms and basic definitions

axiomatization where
refl: $t = (t::'a)$ **and**
subst: $s = t \Longrightarrow P\ s \Longrightarrow P\ t$ **and**
ext: $(\bigwedge x::'a.\ (f\ x ::'b) = g\ x) \Longrightarrow (\lambda x.\ f\ x) = (\lambda x.\ g\ x)$
— Extensionality is built into the meta-logic, and this rule expresses a related property. It is an eta-expanded version of the traditional rule, and similar to the ABS rule of HOL **and**

the-eq-trivial: $(THE\ x.\ x = a) = (a::'a)$

axiomatization where
impI: $(P \Longrightarrow Q) \Longrightarrow P \longrightarrow Q$ **and**
mp: $[\![P \longrightarrow Q;\ P]\!] \Longrightarrow Q$ **and**

True-or-False: $(P = True) \vee (P = False)$

definition *If* :: $bool \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a$ $((if\ (-)/\ then\ (-)/\ else\ (-))\ [0,\ 0,\ 10]\ 10)$
where $If\ P\ x\ y \equiv (THE\ z::'a.\ (P = True \longrightarrow z = x) \wedge (P = False \longrightarrow z = y))$

definition *Let* :: $'a \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'b$
where $Let\ s\ f \equiv f\ s$

translations
-Let (-binds b bs) e $\rightleftharpoons$ *-Let b (-Let bs e)*
let x = a in e $\rightleftharpoons$ *CONST Let a* $(\lambda x.\ e)$

axiomatization *undefined* :: $'a$

class *default* = **fixes** *default* :: $'a$

2.2 Fundamental rules

2.2.1 Equality

lemma *sym*: $s = t \Longrightarrow t = s$
⟨*proof*⟩

lemma *ssubst*: $t = s \Longrightarrow P\ s \Longrightarrow P\ t$
⟨*proof*⟩

lemma *trans*: $[\![r = s;\ s = t]\!] \Longrightarrow r = t$
⟨*proof*⟩

lemma *trans-sym* [*Pure.elim?*]: $r = s \Longrightarrow t = s \Longrightarrow r = t$
⟨*proof*⟩

lemma *meta-eq-to-obj-eq*:
assumes $A \equiv B$
shows $A = B$
⟨*proof*⟩

Useful with *erule* for proving equalities from known equalities.

lemma *box-equals*: $[\![a = b;\ a = c;\ b = d]\!] \Longrightarrow c = d$
⟨*proof*⟩

For calculational reasoning:

lemma *forw-subst*: $a = b \Longrightarrow P\ b \Longrightarrow P\ a$

⟨*proof*⟩

lemma *back-subst*: $P\ a \Longrightarrow a = b \Longrightarrow P\ b$
⟨*proof*⟩

2.2.2 Congruence rules for application

Similar to *AP-THM* in Gordon's HOL.

lemma *fun-cong*: $(f :: 'a \Rightarrow 'b) = g \Longrightarrow f\ x = g\ x$
⟨*proof*⟩

Similar to *AP-TERM* in Gordon's HOL and FOL's *subst-context*.

lemma *arg-cong*: $x = y \Longrightarrow f\ x = f\ y$
⟨*proof*⟩

lemma *arg-cong2*: $[\![a = b;\ c = d]\!] \Longrightarrow f\ a\ c = f\ b\ d$
⟨*proof*⟩

lemma *cong*: $[\![f = g;\ (x::'a) = y]\!] \Longrightarrow f\ x = g\ y$
⟨*proof*⟩

⟨*ML*⟩

2.2.3 Equality of booleans – iff

lemma *iffD2*: $[\![P = Q;\ Q]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *rev-iffD2*: $[\![Q;\ P = Q]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *iffD1*: $Q = P \Longrightarrow Q \Longrightarrow P$
⟨*proof*⟩

lemma *rev-iffD1*: $Q \Longrightarrow Q = P \Longrightarrow P$
⟨*proof*⟩

lemma *iffE*:
assumes *major*: $P = Q$
and *minor*: $[\![P \longrightarrow Q;\ Q \longrightarrow P]\!] \Longrightarrow R$
shows R
⟨*proof*⟩

2.2.4 True (1)

lemma *TrueI*: *True*
⟨*proof*⟩

lemma *eqTrueE*: $P = True \Longrightarrow P$

⟨*proof*⟩

2.2.5 Universal quantifier (1)

lemma *spec*: $\forall x::'a.\ P\ x \Longrightarrow P\ x$
⟨*proof*⟩

lemma *allE*:
assumes *major*: $\forall x.\ P\ x$ **and** *minor*: $P\ x \Longrightarrow R$
shows R
⟨*proof*⟩

lemma *all-dupE*:
assumes *major*: $\forall x.\ P\ x$ **and** *minor*: $[\![P\ x;\ \forall x.\ P\ x]\!] \Longrightarrow R$
shows R
⟨*proof*⟩

2.2.6 False

Depends upon *spec*; it is impossible to do propositional logic before quantifiers!

lemma *FalseE*: $False \Longrightarrow P$
⟨*proof*⟩

lemma *False-neq-True*: $False = True \Longrightarrow P$
⟨*proof*⟩

2.2.7 Negation

lemma *notI*:
assumes $P \Longrightarrow False$
shows $\neg P$
⟨*proof*⟩

lemma *False-not-True*: $False \neq True$
⟨*proof*⟩

lemma *True-not-False*: $True \neq False$
⟨*proof*⟩

lemma *notE*: $[\![\neg P;\ P]\!] \Longrightarrow R$
⟨*proof*⟩

2.2.8 Implication

lemma *impE*:
assumes $P \longrightarrow Q\ P\ Q \Longrightarrow R$
shows R
⟨*proof*⟩

Reduces Q to $P \longrightarrow Q$, allowing substitution in P.

lemma *rev-mp*: $[\![P;\ P \longrightarrow Q]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma *contrapos-nn*:
assumes *major*: $\neg\ Q$
and *minor*: $P \Longrightarrow Q$
shows $\neg\ P$
⟨*proof*⟩

Not used at all, but we already have the other 3 combinations.

lemma *contrapos-pn*:
assumes *major*: Q
and *minor*: $P \Longrightarrow \neg\ Q$
shows $\neg\ P$
⟨*proof*⟩

lemma *not-sym*: $t \neq s \Longrightarrow s \neq t$
⟨*proof*⟩

lemma *eq-neq-eq-imp-neq*: $[\![x = a;\ a \neq b;\ b = y]\!] \Longrightarrow x \neq y$
⟨*proof*⟩

2.2.9 Disjunction (1)

lemma *disjE*:
assumes *major*: $P \vee Q$
and *minorP*: $P \Longrightarrow R$
and *minorQ*: $Q \Longrightarrow R$
shows R
⟨*proof*⟩

2.2.10 Derivation of *iffI*

In an intuitionistic version of HOL *iffI* needs to be an axiom.

lemma *iffI*:
assumes $P \Longrightarrow Q$ **and** $Q \Longrightarrow P$
shows $P = Q$
⟨*proof*⟩

2.2.11 True (2)

lemma *eqTrueI*: $P \Longrightarrow P = True$
⟨*proof*⟩

2.2.12 Universal quantifier (2)

lemma *allI*:
assumes $\bigwedge x::'a.\ P\ x$

shows $\forall x.\ P\ x$
⟨*proof*⟩

2.2.13 Existential quantifier

lemma *exI*: $P\ x \Longrightarrow \exists x::'a.\ P\ x$
⟨*proof*⟩

lemma *exE*:
assumes *major*: $\exists x::'a.\ P\ x$
and *minor*: $\bigwedge x.\ P\ x \Longrightarrow Q$
shows Q
⟨*proof*⟩

2.2.14 Conjunction

lemma *conjI*: $[\![P;\ Q]\!] \Longrightarrow P \wedge Q$
⟨*proof*⟩

lemma *conjunct1*: $[\![P \wedge Q]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *conjunct2*: $[\![P \wedge Q]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma *conjE*:
assumes *major*: $P \wedge Q$
and *minor*: $[\![P;\ Q]\!] \Longrightarrow R$
shows R
⟨*proof*⟩

lemma *context-conjI*:
assumes $P\ P \Longrightarrow Q$
shows $P \wedge Q$
⟨*proof*⟩

2.2.15 Disjunction (2)

lemma *disjI1*: $P \Longrightarrow P \vee Q$
⟨*proof*⟩

lemma *disjI2*: $Q \Longrightarrow P \vee Q$
⟨*proof*⟩

2.2.16 Classical logic

lemma *classical*:
assumes $\neg\ P \Longrightarrow P$
shows P
⟨*proof*⟩

lemmas *ccontr* = *FalseE* [*THEN classical*]

notE with premises exchanged; it discharges ¬ *R* so that it can be used to make elimination rules.

lemma *rev-notE*:
assumes *premp*: *P*
and *premnot*: ¬ *R* ⟹ ¬ *P*
shows *R*
⟨*proof*⟩

Double negation law.

lemma *notnotD*: ¬¬ *P* ⟹ *P*
⟨*proof*⟩

lemma *contrapos-pp*:
assumes *p1*: *Q*
and *p2*: ¬ *P* ⟹ ¬ *Q*
shows *P*
⟨*proof*⟩

2.2.17 Unique existence

lemma *Uniq-I* [*intro?*]:
assumes ⋀*x y*. ⟦*P x*; *P y*⟧ ⟹ *y* = *x*
shows *Uniq P*
⟨*proof*⟩

lemma *Uniq-D* [*dest?*]: ⟦*Uniq P*; *P a*; *P b*⟧ ⟹ *a*=*b*
⟨*proof*⟩

lemma *ex1I*:
assumes *P a* ⋀*x*. *P x* ⟹ *x* = *a*
shows ∃!*x*. *P x*
⟨*proof*⟩

Sometimes easier to use: the premises have no shared variables. Safe!

lemma *ex-ex1I*:
assumes *ex-prem*: ∃ *x*. *P x*
and *eq*: ⋀*x y*. ⟦*P x*; *P y*⟧ ⟹ *x* = *y*
shows ∃!*x*. *P x*
⟨*proof*⟩

lemma *ex1E*:
assumes *major*: ∃!*x*. *P x* **and** *minor*: ⋀*x*. ⟦*P x*; ∀ *y*. *P y* ⟶ *y* = *x*⟧ ⟹ *R*
shows *R*
⟨*proof*⟩

lemma *ex1-implies-ex*: ∃!*x*. *P x* ⟹ ∃ *x*. *P x*

⟨*proof*⟩

2.2.18 Classical intro rules for disjunction and existential quantifiers

lemma *disjCI*:
assumes ¬ $Q \Longrightarrow P$
shows $P \vee Q$
⟨*proof*⟩

lemma *excluded-middle*: ¬ $P \vee P$
⟨*proof*⟩

case distinction as a natural deduction rule. Note that ¬ P is the second case, not the first.

lemma *case-split* [*case-names True False*]:
assumes $P \Longrightarrow Q$ ¬ $P \Longrightarrow Q$
shows Q
⟨*proof*⟩

Classical implies ($\longrightarrow$) elimination.

lemma *impCE*:
assumes *major*: $P \longrightarrow Q$
and *minor*: ¬ $P \Longrightarrow R$ $Q \Longrightarrow R$
shows R
⟨*proof*⟩

This version of $\longrightarrow$ elimination works on Q before P. It works best for those cases in which P holds "almost everywhere". Can't install as default: would break old proofs.

lemma *impCE'*:
assumes *major*: $P \longrightarrow Q$
and *minor*: $Q \Longrightarrow R$ ¬ $P \Longrightarrow R$
shows R
⟨*proof*⟩

Classical $\longleftrightarrow$ elimination.

lemma *iffCE*:
assumes *major*: $P = Q$
and *minor*: $[\![P; Q]\!] \Longrightarrow R$ $[\![¬ P; ¬ Q]\!] \Longrightarrow R$
shows R
⟨*proof*⟩

lemma *exCI*:
assumes $\forall x.$ ¬ $P\ x \Longrightarrow P\ a$
shows $\exists x.\ P\ x$
⟨*proof*⟩

2.2.19 Intuitionistic Reasoning

lemma *impE'*:
assumes *1*: $P \longrightarrow Q$
and *2*: $Q \Longrightarrow R$
and *3*: $P \longrightarrow Q \Longrightarrow P$
shows R
⟨*proof*⟩

lemma *allE'*:
assumes *1*: $\forall x.\ P\ x$
and *2*: $P\ x \Longrightarrow \forall x.\ P\ x \Longrightarrow Q$
shows Q
⟨*proof*⟩

lemma *notE'*:
assumes *1*: $\neg\ P$
and *2*: $\neg\ P \Longrightarrow P$
shows R
⟨*proof*⟩

lemma *TrueE*: *True* $\Longrightarrow P \Longrightarrow P$ ⟨*proof*⟩
lemma *notFalseE*: $\neg$ *False* $\Longrightarrow P \Longrightarrow P$ ⟨*proof*⟩

lemmas [*Pure.elim*!] = *disjE iffE FalseE conjE exE TrueE notFalseE*
and [*Pure.intro*!] = *iffI conjI impI TrueI notI allI refl*
and [*Pure.elim 2*] = *allE notE' impE'*
and [*Pure.intro*] = *exI disjI2 disjI1*

lemmas [*trans*] = *trans*
and [*sym*] = *sym not-sym*
and [*Pure.elim?*] = *iffD1 iffD2 impE*

2.2.20 Atomizing meta-level connectives

axiomatization where
eq-reflection: $x = y \Longrightarrow x \equiv y$ — admissible axiom

lemma *atomize-all* [*atomize*]: $(\bigwedge x.\ P\ x) \equiv$ *Trueprop* $(\forall x.\ P\ x)$
⟨*proof*⟩

lemma *atomize-imp* [*atomize*]: $(A \Longrightarrow B) \equiv$ *Trueprop* $(A \longrightarrow B)$
⟨*proof*⟩

lemma *atomize-not*: $(A \Longrightarrow$ *False*$) \equiv$ *Trueprop* $(\neg\ A)$
⟨*proof*⟩

lemma *atomize-eq* [*atomize*, *code*]: $(x \equiv y) \equiv$ *Trueprop* $(x = y)$
⟨*proof*⟩

lemma *atomize-conj* [*atomize*]: $(A \mathbin{\&\&\&} B) \equiv \mathit{Trueprop}\ (A \wedge B)$
⟨*proof*⟩

lemmas [*symmetric*, *rulify*] = *atomize-all atomize-imp*
and [*symmetric*, *defn*] = *atomize-all atomize-imp atomize-eq*

2.2.21 Atomizing elimination rules

lemma *atomize-exL*[*atomize-elim*]: $(\bigwedge x.\ P\ x \Longrightarrow Q) \equiv ((\exists x.\ P\ x) \Longrightarrow Q)$
⟨*proof*⟩

lemma *atomize-conjL*[*atomize-elim*]: $(A \Longrightarrow B \Longrightarrow C) \equiv (A \wedge B \Longrightarrow C)$
⟨*proof*⟩

lemma *atomize-disjL*[*atomize-elim*]: $((A \Longrightarrow C) \Longrightarrow (B \Longrightarrow C) \Longrightarrow C) \equiv ((A \vee B \Longrightarrow C) \Longrightarrow C)$
⟨*proof*⟩

lemma *atomize-elimL*[*atomize-elim*]: $(\bigwedge B.\ (A \Longrightarrow B) \Longrightarrow B) \equiv \mathit{Trueprop}\ A$ ⟨*proof*⟩

2.3 Package setup

⟨*ML*⟩

2.3.1 Sledgehammer setup

Theorems blacklisted to Sledgehammer. These theorems typically produce clauses that are prolific (match too many equality or membership literals) and relate to seldom-used facts. Some duplicate other rules.

named-theorems *no-atp theorems that should be filtered out by Sledgehammer*

2.3.2 Classical Reasoner setup

lemma *imp-elim*: $P \longrightarrow Q \Longrightarrow (\neg R \Longrightarrow P) \Longrightarrow (Q \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

lemma *swap*: $\neg P \Longrightarrow (\neg R \Longrightarrow P) \Longrightarrow R$
⟨*proof*⟩

lemma *thin-refl*: $[\![x = x;\ \mathit{PROP}\ W]\!] \Longrightarrow \mathit{PROP}\ W$ ⟨*proof*⟩

⟨*ML*⟩

declare *iffI* [*intro*!]
and *notI* [*intro*!]
and *impI* [*intro*!]
and *disjCI* [*intro*!]
and *conjI* [*intro*!]
and *TrueI* [*intro*!]

and *refl* [*intro*!]

declare *iffCE* [*elim*!]
and *FalseE* [*elim*!]
and *impCE* [*elim*!]
and *disjE* [*elim*!]
and *conjE* [*elim*!]

declare *ex-ex1I* [*intro*!]
and *allI* [*intro*!]
and *exI* [*intro*]

declare *exE* [*elim*!]
allE [*elim*]

⟨*ML*⟩

lemma *contrapos-np*: $\neg\ Q \Longrightarrow (\neg\ P \Longrightarrow Q) \Longrightarrow P$
⟨*proof*⟩

declare *ex-ex1I* [*rule del*, *intro*! *2*]
and *ex1I* [*intro*]

declare *ext* [*intro*]

lemmas [*intro?*] = *ext*
and [*elim?*] = *ex1-implies-ex*

Better than *ex1E* for classical reasoner: needs no quantifier duplication!

lemma *alt-ex1E* [*elim*!]:
assumes *major*: $\exists!x.\ P\ x$
and *minor*: $\bigwedge x.\ [\![P\ x;\ \forall\ y\ y'.\ P\ y \wedge P\ y' \longrightarrow y = y']\!] \Longrightarrow R$
shows R
⟨*proof*⟩

And again using Uniq

lemma *alt-ex1E'*:
assumes $\exists!x.\ P\ x$ $\bigwedge x.\ [\![P\ x;\ \exists_{\leq 1}x.\ P\ x]\!] \Longrightarrow R$
shows R
⟨*proof*⟩

lemma *ex1-iff-ex-Uniq*: $(\exists!x.\ P\ x) \longleftrightarrow (\exists\, x.\ P\ x) \wedge (\exists_{\leq 1}x.\ P\ x)$
⟨*proof*⟩

⟨*ML*⟩

2.3.3 THE: definite description operator

lemma *the-equality* [*intro*]:
assumes $P\ a$
and $\bigwedge x.\ P\ x \Longrightarrow x = a$
shows $(THE\ x.\ P\ x) = a$
⟨*proof*⟩

lemma *theI*:
assumes $P\ a$
and $\bigwedge x.\ P\ x \Longrightarrow x = a$
shows $P\ (THE\ x.\ P\ x)$
⟨*proof*⟩

lemma *theI'*: $\exists!x.\ P\ x \Longrightarrow P\ (THE\ x.\ P\ x)$
⟨*proof*⟩

Easier to apply than *theI*: only one occurrence of P.

lemma *theI2*:
assumes $P\ a\ \bigwedge x.\ P\ x \Longrightarrow x = a\ \bigwedge x.\ P\ x \Longrightarrow Q\ x$
shows $Q\ (THE\ x.\ P\ x)$
⟨*proof*⟩

lemma *the1I2*:
assumes $\exists!x.\ P\ x\ \bigwedge x.\ P\ x \Longrightarrow Q\ x$
shows $Q\ (THE\ x.\ P\ x)$
⟨*proof*⟩

lemma *the1-equality* [*elim?*]: $[\![\exists!x.\ P\ x;\ P\ a]\!] \Longrightarrow (THE\ x.\ P\ x) = a$
⟨*proof*⟩

lemma *the1-equality'*: $[\![\exists_{\leq 1}x.\ P\ x;\ P\ a]\!] \Longrightarrow (THE\ x.\ P\ x) = a$
⟨*proof*⟩

lemma *the-sym-eq-trivial*: $(THE\ y.\ x = y) = x$
⟨*proof*⟩

2.3.4 Simplifier

lemma *eta-contract-eq*: $(\lambda s.\ f\ s) = f$ ⟨*proof*⟩

lemma *subst-all*:
$‹(\bigwedge x.\ x = a \Longrightarrow PROP\ P\ x) \equiv PROP\ P\ a›$
$‹(\bigwedge x.\ a = x \Longrightarrow PROP\ P\ x) \equiv PROP\ P\ a›$
⟨*proof*⟩

lemma *simp-thms*:
shows *not-not*: $(\neg\ \neg\ P) = P$
and *Not-eq-iff*: $((\neg\ P) = (\neg\ Q)) = (P = Q)$
and

$(P \neq Q) = (P = (\neg\ Q))$
$(P \vee \neg P) = \mathit{True} \quad (\neg\ P \vee P) = \mathit{True}$
$(x = x) = \mathit{True}$
and *not-True-eq-False* [*code*]: $(\neg\ \mathit{True}) = \mathit{False}$
and *not-False-eq-True* [*code*]: $(\neg\ \mathit{False}) = \mathit{True}$
and
$(\neg\ P) \neq P \ \ P \neq (\neg\ P)$
$(\mathit{True} = P) = P$
and *eq-True*: $(P = \mathit{True}) = P$
and $(\mathit{False} = P) = (\neg\ P)$
and *eq-False*: $(P = \mathit{False}) = (\neg\ P)$
and
$(\mathit{True} \longrightarrow P) = P \ \ (\mathit{False} \longrightarrow P) = \mathit{True}$
$(P \longrightarrow \mathit{True}) = \mathit{True} \ \ (P \longrightarrow P) = \mathit{True}$
$(P \longrightarrow \mathit{False}) = (\neg\ P) \ \ (P \longrightarrow \neg\ P) = (\neg\ P)$
$(P \wedge \mathit{True}) = P \ \ (\mathit{True} \wedge P) = P$
$(P \wedge \mathit{False}) = \mathit{False} \ \ (\mathit{False} \wedge P) = \mathit{False}$
$(P \wedge P) = P \ \ (P \wedge (P \wedge Q)) = (P \wedge Q)$
$(P \wedge \neg\ P) = \mathit{False} \quad (\neg\ P \wedge P) = \mathit{False}$
$(P \vee \mathit{True}) = \mathit{True} \ \ (\mathit{True} \vee P) = \mathit{True}$
$(P \vee \mathit{False}) = P \ \ (\mathit{False} \vee P) = P$
$(P \vee P) = P \ \ (P \vee (P \vee Q)) = (P \vee Q)$ **and**
$(\forall x.\ P) = P \ \ (\exists x.\ P) = P \ \ \exists x.\ x = t \ \ \exists x.\ t = x$
and
$\bigwedge P.\ (\exists x.\ x = t \wedge P\ x) = P\ t$
$\bigwedge P.\ (\exists x.\ t = x \wedge P\ x) = P\ t$
$\bigwedge P.\ (\forall x.\ x = t \longrightarrow P\ x) = P\ t$
$\bigwedge P.\ (\forall x.\ t = x \longrightarrow P\ x) = P\ t$
$(\forall x.\ x \neq t) = \mathit{False} \ \ (\forall x.\ t \neq x) = \mathit{False}$
⟨*proof*⟩

lemma *disj-absorb*: $A \vee A \longleftrightarrow A$
⟨*proof*⟩

lemma *disj-left-absorb*: $A \vee (A \vee B) \longleftrightarrow A \vee B$
⟨*proof*⟩

lemma *conj-absorb*: $A \wedge A \longleftrightarrow A$
⟨*proof*⟩

lemma *conj-left-absorb*: $A \wedge (A \wedge B) \longleftrightarrow A \wedge B$
⟨*proof*⟩

lemma *eq-ac*:
shows *eq-commute*: $a = b \longleftrightarrow b = a$
and *iff-left-commute*: $(P \longleftrightarrow (Q \longleftrightarrow R)) \longleftrightarrow (Q \longleftrightarrow (P \longleftrightarrow R))$
and *iff-assoc*: $((P \longleftrightarrow Q) \longleftrightarrow R) \longleftrightarrow (P \longleftrightarrow (Q \longleftrightarrow R))$
⟨*proof*⟩

lemma *neq-commute*: $a \neq b \longleftrightarrow b \neq a$ ⟨*proof*⟩

lemma *conj-comms*:
shows *conj-commute*: $P \wedge Q \longleftrightarrow Q \wedge P$
and *conj-left-commute*: $P \wedge (Q \wedge R) \longleftrightarrow Q \wedge (P \wedge R)$ ⟨*proof*⟩
lemma *conj-assoc*: $(P \wedge Q) \wedge R \longleftrightarrow P \wedge (Q \wedge R)$ ⟨*proof*⟩

lemmas *conj-ac* = *conj-commute conj-left-commute conj-assoc*

lemma *disj-comms*:
shows *disj-commute*: $P \vee Q \longleftrightarrow Q \vee P$
and *disj-left-commute*: $P \vee (Q \vee R) \longleftrightarrow Q \vee (P \vee R)$ ⟨*proof*⟩
lemma *disj-assoc*: $(P \vee Q) \vee R \longleftrightarrow P \vee (Q \vee R)$ ⟨*proof*⟩

lemmas *disj-ac* = *disj-commute disj-left-commute disj-assoc*

lemma *conj-disj-distribL*: $P \wedge (Q \vee R) \longleftrightarrow P \wedge Q \vee P \wedge R$ ⟨*proof*⟩
lemma *conj-disj-distribR*: $(P \vee Q) \wedge R \longleftrightarrow P \wedge R \vee Q \wedge R$ ⟨*proof*⟩

lemma *disj-conj-distribL*: $P \vee (Q \wedge R) \longleftrightarrow (P \vee Q) \wedge (P \vee R)$ ⟨*proof*⟩
lemma *disj-conj-distribR*: $(P \wedge Q) \vee R \longleftrightarrow (P \vee R) \wedge (Q \vee R)$ ⟨*proof*⟩

lemma *imp-conjR*: $(P \longrightarrow (Q \wedge R)) = ((P \longrightarrow Q) \wedge (P \longrightarrow R))$ ⟨*proof*⟩
lemma *imp-conjL*: $((P \wedge Q) \longrightarrow R) = (P \longrightarrow (Q \longrightarrow R))$ ⟨*proof*⟩
lemma *imp-disjL*: $((P \vee Q) \longrightarrow R) = ((P \longrightarrow R) \wedge (Q \longrightarrow R))$ ⟨*proof*⟩

These two are specialized, but *imp-disj-not1* is useful in *Auth/Yahalom*.

lemma *imp-disj-not1*: $(P \longrightarrow Q \vee R) \longleftrightarrow (\neg Q \longrightarrow P \longrightarrow R)$ ⟨*proof*⟩
lemma *imp-disj-not2*: $(P \longrightarrow Q \vee R) \longleftrightarrow (\neg R \longrightarrow P \longrightarrow Q)$ ⟨*proof*⟩

lemma *imp-disj1*: $((P \longrightarrow Q) \vee R) \longleftrightarrow (P \longrightarrow Q \vee R)$ ⟨*proof*⟩
lemma *imp-disj2*: $(Q \vee (P \longrightarrow R)) \longleftrightarrow (P \longrightarrow Q \vee R)$ ⟨*proof*⟩

lemma *imp-cong*: $(P = P') \Longrightarrow (P' \Longrightarrow (Q = Q')) \Longrightarrow ((P \longrightarrow Q) \longleftrightarrow (P' \longrightarrow Q'))$
⟨*proof*⟩

lemma *de-Morgan-disj*: $\neg (P \vee Q) \longleftrightarrow \neg P \wedge \neg Q$ ⟨*proof*⟩
lemma *de-Morgan-conj*: $\neg (P \wedge Q) \longleftrightarrow \neg P \vee \neg Q$ ⟨*proof*⟩
lemma *not-imp*: $\neg (P \longrightarrow Q) \longleftrightarrow P \wedge \neg Q$ ⟨*proof*⟩
lemma *not-iff*: $P \neq Q \longleftrightarrow (P \longleftrightarrow \neg Q)$ ⟨*proof*⟩
lemma *disj-not1*: $\neg P \vee Q \longleftrightarrow (P \longrightarrow Q)$ ⟨*proof*⟩
lemma *disj-not2*: $P \vee \neg Q \longleftrightarrow (Q \longrightarrow P)$ ⟨*proof*⟩
lemma *imp-conv-disj*: $(P \longrightarrow Q) \longleftrightarrow (\neg P) \vee Q$ ⟨*proof*⟩
lemma *disj-imp*: $P \vee Q \longleftrightarrow \neg P \longrightarrow Q$ ⟨*proof*⟩

lemma *iff-conv-conj-imp*: $(P \longleftrightarrow Q) \longleftrightarrow (P \longrightarrow Q) \wedge (Q \longrightarrow P)$ ⟨*proof*⟩

lemma *cases-simp*: $(P \longrightarrow Q) \wedge (\neg P \longrightarrow Q) \longleftrightarrow Q$
— Avoids duplication of subgoals after *if-split*, when the true and false
— cases boil down to the same thing.
⟨*proof*⟩

lemma *not-all*: $\neg (\forall x.\ P\ x) \longleftrightarrow (\exists x.\ \neg P\ x)$ ⟨*proof*⟩
lemma *imp-all*: $((\forall x.\ P\ x) \longrightarrow Q) \longleftrightarrow (\exists x.\ P\ x \longrightarrow Q)$ ⟨*proof*⟩
lemma *not-ex*: $\neg (\exists x.\ P\ x) \longleftrightarrow (\forall x.\ \neg P\ x)$ ⟨*proof*⟩
lemma *imp-ex*: $((\exists x.\ P\ x) \longrightarrow Q) \longleftrightarrow (\forall x.\ P\ x \longrightarrow Q)$ ⟨*proof*⟩
lemma *all-not-ex*: $(\forall x.\ P\ x) \longleftrightarrow \neg (\exists x.\ \neg P\ x)$ ⟨*proof*⟩

declare *All-def* [*no-atp*]

lemma *ex-disj-distrib*: $(\exists x.\ P\ x \vee Q\ x) \longleftrightarrow (\exists x.\ P\ x) \vee (\exists x.\ Q\ x)$ ⟨*proof*⟩
lemma *all-conj-distrib*: $(\forall x.\ P\ x \wedge Q\ x) \longleftrightarrow (\forall x.\ P\ x) \wedge (\forall x.\ Q\ x)$ ⟨*proof*⟩

The $\wedge$ congruence rule: not included by default! May slow rewrite proofs down by as much as 50%

lemma *conj-cong*: $P = P' \Longrightarrow (P' \Longrightarrow Q = Q') \Longrightarrow (P \wedge Q) = (P' \wedge Q')$
⟨*proof*⟩

lemma *rev-conj-cong*: $Q = Q' \Longrightarrow (Q' \Longrightarrow P = P') \Longrightarrow (P \wedge Q) = (P' \wedge Q')$
⟨*proof*⟩

The | congruence rule: not included by default!

lemma *disj-cong*: $P = P' \Longrightarrow (\neg P' \Longrightarrow Q = Q') \Longrightarrow (P \vee Q) = (P' \vee Q')$
⟨*proof*⟩

if-then-else rules

lemma *if-True* [*code*]: (*if True then x else y*) = *x*
⟨*proof*⟩

lemma *if-False* [*code*]: (*if False then x else y*) = *y*
⟨*proof*⟩

lemma *if-P*: $P \Longrightarrow$ (*if P then x else y*) = *x*
⟨*proof*⟩

lemma *if-not-P*: $\neg P \Longrightarrow$ (*if P then x else y*) = *y*
⟨*proof*⟩

lemma *if-split*: *P* (*if Q then x else y*) = $((Q \longrightarrow P\ x) \wedge (\neg Q \longrightarrow P\ y))$
⟨*proof*⟩

lemma *if-split-asm*: *P* (*if Q then x else y*) = $(\neg ((Q \wedge \neg P\ x) \vee (\neg Q \wedge \neg P\ y)))$
⟨*proof*⟩

lemmas *if-splits* [*no-atp*] = *if-split if-split-asm*

lemma *if-cancel*: (*if c then x else x*) = *x*
⟨*proof*⟩

lemma *if-eq-cancel*: (*if x = y then y else x*) = *x*
⟨*proof*⟩

lemma *if-bool-eq-conj*: (*if P then Q else R*) = ((*P* ⟶ *Q*) ∧ (¬ *P* ⟶ *R*))
— This form is useful for expanding *if*s on the RIGHT of the ⟹ symbol.
⟨*proof*⟩

lemma *if-bool-eq-disj*: (*if P then Q else R*) = ((*P* ∧ *Q*) ∨ (¬ *P* ∧ *R*))
— And this form is useful for expanding *if*s on the LEFT.
⟨*proof*⟩

lemma *Eq-TrueI*: *P* ⟹ *P* ≡ *True* ⟨*proof*⟩
lemma *Eq-FalseI*: ¬ *P* ⟹ *P* ≡ *False* ⟨*proof*⟩

let rules for simproc

lemma *Let-folded*: *f x* ≡ *g x* ⟹ *Let x f* ≡ *Let x g*
⟨*proof*⟩

lemma *Let-unfold*: *f x* ≡ *g* ⟹ *Let x f* ≡ *g*
⟨*proof*⟩

The following copy of the implication operator is useful for fine-tuning congruence rules. It instructs the simplifier to simplify its premise.

definition *simp-implies* :: *prop* ⇒ *prop* ⇒ *prop* (**infixr** =*simp*=> *1*)
where *simp-implies* ≡ (⟹)

lemma *simp-impliesI*:
assumes *PQ*: (*PROP P* ⟹ *PROP Q*)
shows *PROP P* =*simp*=> *PROP Q*
⟨*proof*⟩

lemma *simp-impliesE*:
assumes *PQ*: *PROP P* =*simp*=> *PROP Q*
and *P*: *PROP P*
and *QR*: *PROP Q* ⟹ *PROP R*
shows *PROP R*
⟨*proof*⟩

lemma *simp-implies-cong*:
assumes *PP'* :*PROP P* ≡ *PROP P'*
and *P'QQ'*: *PROP P'* ⟹ (*PROP Q* ≡ *PROP Q'*)
shows (*PROP P* =*simp*=> *PROP Q*) ≡ (*PROP P'* =*simp*=> *PROP Q'*)
⟨*proof*⟩

lemma *uncurry*:
assumes $P \longrightarrow Q \longrightarrow R$
shows $P \wedge Q \longrightarrow R$
⟨*proof*⟩

lemma *iff-allI*:
assumes $\bigwedge x.\ P\ x = Q\ x$
shows $(\forall x.\ P\ x) = (\forall x.\ Q\ x)$
⟨*proof*⟩

lemma *iff-exI*:
assumes $\bigwedge x.\ P\ x = Q\ x$
shows $(\exists x.\ P\ x) = (\exists x.\ Q\ x)$
⟨*proof*⟩

lemma *all-comm*: $(\forall x\ y.\ P\ x\ y) = (\forall y\ x.\ P\ x\ y)$
⟨*proof*⟩

lemma *ex-comm*: $(\exists x\ y.\ P\ x\ y) = (\exists y\ x.\ P\ x\ y)$
⟨*proof*⟩

⟨*ML*⟩

Simproc for proving $(y = x) \equiv \mathit{False}$ from premise $\neg\ (x = y)$:

⟨*ML*⟩

lemma *True-implies-equals*: $(\mathit{True} \Longrightarrow \mathit{PROP}\ P) \equiv \mathit{PROP}\ P$
⟨*proof*⟩

lemma *implies-True-equals*: $(\mathit{PROP}\ P \Longrightarrow \mathit{True}) \equiv \mathit{Trueprop}\ \mathit{True}$
⟨*proof*⟩

lemma *False-implies-equals*: $(\mathit{False} \Longrightarrow P) \equiv \mathit{Trueprop}\ \mathit{True}$
⟨*proof*⟩

lemma *implies-False-swap*:
$(\mathit{False} \Longrightarrow \mathit{PROP}\ P \Longrightarrow \mathit{PROP}\ Q) \equiv (\mathit{PROP}\ P \Longrightarrow \mathit{False} \Longrightarrow \mathit{PROP}\ Q)$
⟨*proof*⟩

⟨*ML*⟩

lemma *ex-simps*:
$\bigwedge P\ Q.\ (\exists x.\ P\ x \wedge Q) = ((\exists x.\ P\ x) \wedge Q)$
$\bigwedge P\ Q.\ (\exists x.\ P \wedge Q\ x) = (P \wedge (\exists x.\ Q\ x))$
$\bigwedge P\ Q.\ (\exists x.\ P\ x \vee Q) = ((\exists x.\ P\ x) \vee Q)$
$\bigwedge P\ Q.\ (\exists x.\ P \vee Q\ x) = (P \vee (\exists x.\ Q\ x))$
$\bigwedge P\ Q.\ (\exists x.\ P\ x \longrightarrow Q) = ((\forall x.\ P\ x) \longrightarrow Q)$

$\bigwedge P\ Q.\ (\exists x.\ P \longrightarrow Q\ x) = (P \longrightarrow (\exists x.\ Q\ x))$
— Miniscoping: pushing in existential quantifiers.
⟨*proof*⟩

lemma *all-simps*:
$\bigwedge P\ Q.\ (\forall x.\ P\ x \wedge Q) \quad = ((\forall x.\ P\ x) \wedge Q)$
$\bigwedge P\ Q.\ (\forall x.\ P \wedge Q\ x) \quad = (P \wedge (\forall x.\ Q\ x))$
$\bigwedge P\ Q.\ (\forall x.\ P\ x \vee Q) \quad = ((\forall x.\ P\ x) \vee Q)$
$\bigwedge P\ Q.\ (\forall x.\ P \vee Q\ x) \quad = (P \vee (\forall x.\ Q\ x))$
$\bigwedge P\ Q.\ (\forall x.\ P\ x \longrightarrow Q) = ((\exists x.\ P\ x) \longrightarrow Q)$
$\bigwedge P\ Q.\ (\forall x.\ P \longrightarrow Q\ x) = (P \longrightarrow (\forall x.\ Q\ x))$
— Miniscoping: pushing in universal quantifiers.
⟨*proof*⟩

lemmas [*simp*] =
triv-forall-equality — prunes params
True-implies-equals implies-True-equals — prune *True* in asms
False-implies-equals — prune *False* in asms
if-True
if-False
if-cancel
if-eq-cancel
imp-disjL — In general it seems wrong to add distributive laws by default: they might cause exponential blow-up. But *imp-disjL* has been in for a while and cannot be removed without affecting existing proofs. Moreover, rewriting by $(P \vee Q \longrightarrow R) = ((P \longrightarrow R) \wedge (Q \longrightarrow R))$ might be justified on the grounds that it allows simplification of R in the two cases.
conj-assoc
disj-assoc
de-Morgan-conj
de-Morgan-disj
imp-disj1
imp-disj2
not-imp
disj-not1
not-all
not-ex
cases-simp
the-eq-trivial
the-sym-eq-trivial
ex-simps
all-simps
simp-thms
subst-all

lemmas [*cong*] = *imp-cong simp-implies-cong*
lemmas [*split*] = *if-split*

⟨*ML*⟩

Simplifies x assuming c and y assuming $\neg\ c$.

lemma *if-cong*:
assumes $b = c$
and $c \Longrightarrow x = u$
and $\neg\ c \Longrightarrow y = v$
shows (*if* b *then* x *else* y) = (*if* c *then* u *else* v)
⟨*proof*⟩

Prevents simplification of x and y: faster and allows the execution of functional programs.

lemma *if-weak-cong* [*cong*]:
assumes $b = c$
shows (*if* b *then* x *else* y) = (*if* c *then* x *else* y)
⟨*proof*⟩

Prevents simplification of t: much faster

lemma *let-weak-cong*:
assumes $a = b$
shows (*let* $x = a$ *in* $t\ x$) = (*let* $x = b$ *in* $t\ x$)
⟨*proof*⟩

To tidy up the result of a simproc. Only the RHS will be simplified.

lemma *eq-cong2*:
assumes $u = u'$
shows $(t \equiv u) \equiv (t \equiv u')$
⟨*proof*⟩

lemma *if-distrib*: f (*if* c *then* x *else* y) = (*if* c *then* $f\ x$ *else* $f\ y$)
⟨*proof*⟩

lemma *if-distribR*: (*if* b *then* f *else* g) x = (*if* b *then* $f\ x$ *else* $g\ x$)
⟨*proof*⟩

lemma *all-if-distrib*: $(\forall\, x.$ *if* $x = a$ *then* $P\ x$ *else* $Q\ x) \longleftrightarrow P\ a \wedge (\forall\, x.\ x \neq a \longrightarrow Q\ x)$
⟨*proof*⟩

lemma *ex-if-distrib*: $(\exists\, x.$ *if* $x = a$ *then* $P\ x$ *else* $Q\ x) \longleftrightarrow P\ a \vee (\exists\, x.\ x \neq a \wedge Q\ x)$
⟨*proof*⟩

lemma *if-if-eq-conj*: (*if* P *then if* Q *then* x *else* y *else* y) = (*if* $P \wedge Q$ *then* x *else* y)
⟨*proof*⟩

As a simplification rule, it replaces all function equalities by first-order equalities.

lemma *fun-eq-iff*: $f = g \longleftrightarrow (\forall\, x.\ f\ x = g\ x)$
⟨*proof*⟩

2.3.5 Generic cases and induction

Rule projections:

⟨*ML*⟩

context
begin

qualified definition *induct-forall* $P \equiv \forall x.\ P\ x$
qualified definition *induct-implies* $A\ B \equiv A \longrightarrow B$
qualified definition *induct-equal* $x\ y \equiv x = y$
qualified definition *induct-conj* $A\ B \equiv A \wedge B$
qualified definition *induct-true* $\equiv$ *True*
qualified definition *induct-false* $\equiv$ *False*

lemma *induct-forall-eq*: $(\bigwedge x.\ P\ x) \equiv$ *Trueprop* (*induct-forall* $(\lambda x.\ P\ x)$)
⟨*proof*⟩

lemma *induct-implies-eq*: $(A \Longrightarrow B) \equiv$ *Trueprop* (*induct-implies* $A\ B$)
⟨*proof*⟩

lemma *induct-equal-eq*: $(x \equiv y) \equiv$ *Trueprop* (*induct-equal* $x\ y$)
⟨*proof*⟩

lemma *induct-conj-eq*: $(A\ \&\&\&\ B) \equiv$ *Trueprop* (*induct-conj* $A\ B$)
⟨*proof*⟩

lemmas *induct-atomize'* = *induct-forall-eq induct-implies-eq induct-conj-eq*
lemmas *induct-atomize* = *induct-atomize' induct-equal-eq*
lemmas *induct-rulify'* [*symmetric*] = *induct-atomize'*
lemmas *induct-rulify* [*symmetric*] = *induct-atomize*
lemmas *induct-rulify-fallback* =
induct-forall-def induct-implies-def induct-equal-def induct-conj-def
induct-true-def induct-false-def

lemma *induct-forall-conj*: *induct-forall* $(\lambda x.$ *induct-conj* $(A\ x)\ (B\ x)) =$
induct-conj (*induct-forall* A) (*induct-forall* B)
⟨*proof*⟩

lemma *induct-implies-conj*: *induct-implies* C (*induct-conj* $A\ B$) =
induct-conj (*induct-implies* $C\ A$) (*induct-implies* $C\ B$)
⟨*proof*⟩

lemma *induct-conj-curry*: (*induct-conj* $A\ B \Longrightarrow$ *PROP* $C) \equiv (A \Longrightarrow B \Longrightarrow$ *PROP* $C)$
⟨*proof*⟩

lemmas *induct-conj* = *induct-forall-conj induct-implies-conj induct-conj-curry*

lemma *induct-trueI*: *induct-true*
⟨*proof*⟩

Method setup.

⟨*ML*⟩

Pre-simplification of induction and cases rules

lemma [*induct-simp*]: $(\bigwedge x.\ induct\text{-}equal\ x\ t \Longrightarrow PROP\ P\ x) \equiv PROP\ P\ t$
⟨*proof*⟩

lemma [*induct-simp*]: $(\bigwedge x.\ induct\text{-}equal\ t\ x \Longrightarrow PROP\ P\ x) \equiv PROP\ P\ t$
⟨*proof*⟩

lemma [*induct-simp*]: $(induct\text{-}false \Longrightarrow P) \equiv Trueprop\ induct\text{-}true$
⟨*proof*⟩

lemma [*induct-simp*]: $(induct\text{-}true \Longrightarrow PROP\ P) \equiv PROP\ P$
⟨*proof*⟩

lemma [*induct-simp*]: $(PROP\ P \Longrightarrow induct\text{-}true) \equiv Trueprop\ induct\text{-}true$
⟨*proof*⟩

lemma [*induct-simp*]: $(\bigwedge x::'a::\{\}.\ induct\text{-}true) \equiv Trueprop\ induct\text{-}true$
⟨*proof*⟩

lemma [*induct-simp*]: $induct\text{-}implies\ induct\text{-}true\ P \equiv P$
⟨*proof*⟩

lemma [*induct-simp*]: $x = x \longleftrightarrow True$
⟨*proof*⟩

end

⟨*ML*⟩

2.3.6 Coherent logic

⟨*ML*⟩

2.3.7 Reorienting equalities

⟨*ML*⟩

2.4 Other simple lemmas and lemma duplicates

lemma *eq-iff-swap*: $(x = y \longleftrightarrow P) \Longrightarrow (y = x \longleftrightarrow P)$
⟨*proof*⟩

lemma *all-cong1*: $(\bigwedge x.\ P\ x = P'\ x) \Longrightarrow (\forall x.\ P\ x) = (\forall x.\ P'\ x)$
⟨*proof*⟩

lemma *ex-cong1*: $(\bigwedge x.\ P\ x = P'\ x) \Longrightarrow (\exists x.\ P\ x) = (\exists x.\ P'\ x)$
⟨*proof*⟩

lemma *all-cong*: $(\bigwedge x.\ Q\ x \Longrightarrow P\ x = P'\ x) \Longrightarrow (\forall x.\ Q\ x \longrightarrow P\ x) = (\forall x.\ Q\ x \longrightarrow P'\ x)$
⟨*proof*⟩

lemma *ex-cong*: $(\bigwedge x.\ Q\ x \Longrightarrow P\ x = P'\ x) \Longrightarrow (\exists x.\ Q\ x \wedge P\ x) = (\exists x.\ Q\ x \wedge P'\ x)$
⟨*proof*⟩

lemma *ex1-eq* [*iff*]: $\exists!x.\ x = t\ \exists!x.\ t = x$
⟨*proof*⟩

lemma *choice-eq*: $(\forall x.\ \exists!y.\ P\ x\ y) = (\exists!f.\ \forall x.\ P\ x\ (f\ x))$ (**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemmas *eq-sym-conv = eq-commute*

lemma *nnf-simps*:
$(\neg\ (P \wedge Q)) = (\neg\ P \vee \neg\ Q)$
$(\neg\ (P \vee Q)) = (\neg\ P \wedge \neg\ Q)$
$(P \longrightarrow Q) = (\neg\ P \vee Q)$
$(P = Q) = ((P \wedge Q) \vee (\neg\ P \wedge \neg\ Q))$
$(\neg\ (P = Q)) = ((P \wedge \neg\ Q) \vee (\neg\ P \wedge Q))$
$(\neg \neg\ P) = P$
⟨*proof*⟩

2.5 Basic ML bindings

⟨*ML*⟩

locale *cnf*
begin

lemma *clause2raw-notE*: $[\![P;\ \neg P]\!] \Longrightarrow False$ ⟨*proof*⟩
lemma *clause2raw-not-disj*: $[\![\neg\ P;\ \neg\ Q]\!] \Longrightarrow \neg\ (P \vee Q)$ ⟨*proof*⟩
lemma *clause2raw-not-not*: $P \Longrightarrow \neg\neg\ P$ ⟨*proof*⟩

lemma *iff-refl*: $(P::bool) = P$ ⟨*proof*⟩
lemma *iff-trans*: $[|\ (P::bool) = Q;\ Q = R\ |] ==> P = R$ ⟨*proof*⟩
lemma *conj-cong*: $[|\ P = P';\ Q = Q'\ |] ==> (P \wedge Q) = (P' \wedge Q')$ ⟨*proof*⟩
lemma *disj-cong*: $[|\ P = P';\ Q = Q'\ |] ==> (P \vee Q) = (P' \vee Q')$ ⟨*proof*⟩

lemma *make-nnf-imp*: $[|\ (\neg P) = P';\ Q = Q'\ |] ==> (P \longrightarrow Q) = (P' \vee Q')$ ⟨*proof*⟩
lemma *make-nnf-iff*: $[|\ P = P';\ (\neg P) = NP;\ Q = Q';\ (\neg Q) = NQ\ |] ==> (P = Q) = ((P' \vee NQ) \wedge (NP \vee Q'))$ ⟨*proof*⟩

lemma *make-nnf-not-false*: $(\neg False) = True$ ⟨*proof*⟩
lemma *make-nnf-not-true*: $(\neg True) = False$ ⟨*proof*⟩
lemma *make-nnf-not-conj*: $[\!|\ (\neg P) = P';\ (\neg Q) = Q'\ |\!] ==> (\neg(P \wedge Q)) = (P' \vee Q')$ ⟨*proof*⟩
lemma *make-nnf-not-disj*: $[\!|\ (\neg P) = P';\ (\neg Q) = Q'\ |\!] ==> (\neg(P \vee Q)) = (P' \wedge Q')$ ⟨*proof*⟩
lemma *make-nnf-not-imp*: $[\!|\ P = P';\ (\neg Q) = Q'\ |\!] ==> (\neg(P \longrightarrow Q)) = (P' \wedge Q')$ ⟨*proof*⟩
lemma *make-nnf-not-iff*: $[\!|\ P = P';\ (\neg P) = NP;\ Q = Q';\ (\neg Q) = NQ\ |\!] ==> (\neg(P = Q)) = ((P' \vee Q') \wedge (NP \vee NQ))$ ⟨*proof*⟩
lemma *make-nnf-not-not*: $P = P' ==> (\neg\neg P) = P'$ ⟨*proof*⟩

lemma *simp-TF-conj-True-l*: $[\!|\ P = True;\ Q = Q'\ |\!] ==> (P \wedge Q) = Q'$ ⟨*proof*⟩
lemma *simp-TF-conj-True-r*: $[\!|\ P = P';\ Q = True\ |\!] ==> (P \wedge Q) = P'$ ⟨*proof*⟩
lemma *simp-TF-conj-False-l*: $P = False ==> (P \wedge Q) = False$ ⟨*proof*⟩
lemma *simp-TF-conj-False-r*: $Q = False ==> (P \wedge Q) = False$ ⟨*proof*⟩
lemma *simp-TF-disj-True-l*: $P = True ==> (P \vee Q) = True$ ⟨*proof*⟩
lemma *simp-TF-disj-True-r*: $Q = True ==> (P \vee Q) = True$ ⟨*proof*⟩
lemma *simp-TF-disj-False-l*: $[\!|\ P = False;\ Q = Q'\ |\!] ==> (P \vee Q) = Q'$ ⟨*proof*⟩
lemma *simp-TF-disj-False-r*: $[\!|\ P = P';\ Q = False\ |\!] ==> (P \vee Q) = P'$ ⟨*proof*⟩

lemma *make-cnf-disj-conj-l*: $[\!|\ (P \vee R) = PR;\ (Q \vee R) = QR\ |\!] ==> ((P \wedge Q) \vee R) = (PR \wedge QR)$ ⟨*proof*⟩
lemma *make-cnf-disj-conj-r*: $[\!|\ (P \vee Q) = PQ;\ (P \vee R) = PR\ |\!] ==> (P \vee (Q \wedge R)) = (PQ \wedge PR)$ ⟨*proof*⟩

lemma *make-cnfx-disj-ex-l*: $((\exists (x{::}bool).\ P\ x) \vee Q) = (\exists x.\ P\ x \vee Q)$ ⟨*proof*⟩
lemma *make-cnfx-disj-ex-r*: $(P \vee (\exists (x{::}bool).\ Q\ x)) = (\exists x.\ P \vee Q\ x)$ ⟨*proof*⟩
lemma *make-cnfx-newlit*: $(P \vee Q) = (\exists x.\ (P \vee x) \wedge (Q \vee \neg x))$ ⟨*proof*⟩
lemma *make-cnfx-ex-cong*: $(\forall (x{::}bool).\ P\ x = Q\ x) \Longrightarrow (\exists x.\ P\ x) = (\exists x.\ Q\ x)$ ⟨*proof*⟩

lemma *weakening-thm*: $[\!|\ P;\ Q\ |\!] ==> Q$ ⟨*proof*⟩

lemma *cnftac-eq-imp*: $[\!|\ P = Q;\ P\ |\!] ==> Q$ ⟨*proof*⟩

end

⟨*ML*⟩

3 *NO-MATCH* simproc

The simplification procedure can be used to avoid simplification of terms of a certain form.

definition *NO-MATCH* :: $'a \Rightarrow 'b \Rightarrow bool$
  where *NO-MATCH pat val* ≡ *True*

lemma *NO-MATCH-cong*[*cong*]: *NO-MATCH pat val* = *NO-MATCH pat val*

⟨*proof*⟩

declare [[*coercion-args NO-MATCH* − −]]

⟨*ML*⟩

This setup ensures that a rewrite rule of the form *NO-MATCH pat val* ⟹ *t* is only applied, if the pattern *pat* does not match the value *val*.

Tagging a premise of a simp rule with ASSUMPTION forces the simplifier not to simplify the argument and to solve it by an assumption.

definition *ASSUMPTION* :: *bool* ⇒ *bool*
  where *ASSUMPTION A* ≡ *A*

lemma *ASSUMPTION-cong*[*cong*]: *ASSUMPTION A* = *ASSUMPTION A*
  ⟨*proof*⟩

lemma *ASSUMPTION-I*: *A* ⟹ *ASSUMPTION A*
  ⟨*proof*⟩

lemma *ASSUMPTION-D*: *ASSUMPTION A* ⟹ *A*
  ⟨*proof*⟩

⟨*ML*⟩

3.1 Code generator setup

3.1.1 Generic code generator preprocessor setup

lemma *conj-left-cong*: *P* ⟷ *Q* ⟹ *P* ∧ *R* ⟷ *Q* ∧ *R*
  ⟨*proof*⟩

lemma *disj-left-cong*: *P* ⟷ *Q* ⟹ *P* ∨ *R* ⟷ *Q* ∨ *R*
  ⟨*proof*⟩

⟨*ML*⟩

3.1.2 Equality

class *equal* =
  fixes *equal* :: *'a* ⇒ *'a* ⇒ *bool*
  assumes *equal-eq*: *equal x y* ⟷ *x* = *y*
begin

lemma *equal*: *equal* = (=)
  ⟨*proof*⟩

lemma *equal-refl*: *equal x x* ⟷ *True*
  ⟨*proof*⟩

lemma *eq-equal*: $(=) \equiv equal$
⟨*proof*⟩

end

declare *eq-equal* [*symmetric*, *code-post*]
declare *eq-equal* [*code*]

⟨*ML*⟩

3.1.3 Generic code generator foundation

Datatype *bool*

code-datatype *True False*

lemma [*code*]:
shows $False \wedge P \longleftrightarrow False$
and $True \wedge P \longleftrightarrow P$
and $P \wedge False \longleftrightarrow False$
and $P \wedge True \longleftrightarrow P$
⟨*proof*⟩

lemma [*code*]:
shows $False \vee P \longleftrightarrow P$
and $True \vee P \longleftrightarrow True$
and $P \vee False \longleftrightarrow P$
and $P \vee True \longleftrightarrow True$
⟨*proof*⟩

lemma [*code*]:
shows $(False \longrightarrow P) \longleftrightarrow True$
and $(True \longrightarrow P) \longleftrightarrow P$
and $(P \longrightarrow False) \longleftrightarrow \neg P$
and $(P \longrightarrow True) \longleftrightarrow True$
⟨*proof*⟩

More about *prop*

lemma [*code nbe*]:
shows $(True \Longrightarrow PROP\ Q) \equiv PROP\ Q$
and $(PROP\ Q \Longrightarrow True) \equiv Trueprop\ True$
and $(P \Longrightarrow R) \equiv Trueprop\ (P \longrightarrow R)$
⟨*proof*⟩

lemma *Trueprop-code* [*code*]: $Trueprop\ True \equiv Code\text{-}Generator.holds$
⟨*proof*⟩

declare *Trueprop-code* [*symmetric*, *code-post*]

Equality

declare *simp-thms*(*6*) [*code nbe*]

instantiation *itself* :: (*type*) *equal*
begin

definition *equal-itself* :: *'a itself* ⇒ *'a itself* ⇒ *bool*
  where *equal-itself x y* ⟷ *x* = *y*

instance
  ⟨*proof*⟩

end

lemma *equal-itself-code* [*code*]: *equal TYPE*(*'a*) *TYPE*(*'a*) ⟷ *True*
  ⟨*proof*⟩

⟨*ML*⟩

lemma *equal-alias-cert*: *OFCLASS*(*'a*, *equal-class*) ≡ (((=) :: *'a* ⇒ *'a* ⇒ *bool*) ≡ *equal*)
  (**is** *?ofclass* ≡ *?equal*)
⟨*proof*⟩

⟨*ML*⟩

Cases

lemma *Let-case-cert*:
  assumes *CASE* ≡ (λ*x*. *Let x f*)
  shows *CASE x* ≡ *f x*
  ⟨*proof*⟩

⟨*ML*⟩

declare [[*code abort*: *undefined*]]

3.1.4 Generic code generator target languages

type *bool*

code-printing
  type-constructor *bool* ⇀
    (*SML*) *bool* **and** (*OCaml*) *bool* **and** (*Haskell*) *Bool* **and** (*Scala*) *Boolean*
| **constant** *True* ⇀
    (*SML*) *true* **and** (*OCaml*) *true* **and** (*Haskell*) *True* **and** (*Scala*) *true*
| **constant** *False* ⇀
    (*SML*) *false* **and** (*OCaml*) *false* **and** (*Haskell*) *False* **and** (*Scala*) *false*

code-reserved *SML*
  bool true false

code-reserved *OCaml*
bool

code-reserved *Scala*
Boolean

code-printing
constant *Not* ⇀
(*SML*) *not* **and** (*OCaml*) *not* **and** (*Haskell*) *not* **and** (*Scala*) *'!* -
| **constant** *HOL.conj* ⇀
(*SML*) **infixl** *1 andalso* **and** (*OCaml*) **infixl** *3* && **and** (*Haskell*) **infixr** *3* && **and** (*Scala*) **infixl** *3* &&
| **constant** *HOL.disj* ⇀
(*SML*) **infixl** *0 orelse* **and** (*OCaml*) **infixl** *2* || **and** (*Haskell*) **infixl** *2* || **and** (*Scala*) **infixl** *1* ||
| **constant** *HOL.implies* ⇀
(*SML*) !(*if* (-)/ *then* (-)/ *else true*)
and (*OCaml*) !(*if* (-)/ *then* (-)/ *else true*)
and (*Haskell*) !(*if* (-)/ *then* (-)/ *else True*)
and (*Scala*) !(*if* ((-))/ (-)/ *else true*)
| **constant** *If* ⇀
(*SML*) !(*if* (-)/ *then* (-)/ *else* (-))
and (*OCaml*) !(*if* (-)/ *then* (-)/ *else* (-))
and (*Haskell*) !(*if* (-)/ *then* (-)/ *else* (-))
and (*Scala*) !(*if* ((-))/ (-)/ *else* (-))

code-reserved *SML*
not

code-reserved *OCaml*
not

code-identifier
code-module *Pure* ⇀
(*SML*) *HOL* **and** (*OCaml*) *HOL* **and** (*Haskell*) *HOL* **and** (*Scala*) *HOL*

Using built-in Haskell equality.

code-printing
type-class *equal* ⇀ (*Haskell*) *Eq*
| **constant** *HOL.equal* ⇀ (*Haskell*) **infix** *4* ==
| **constant** *HOL.eq* ⇀ (*Haskell*) **infix** *4* ==

undefined

code-printing
constant *undefined* ⇀
(*SML*) !(*raise*/ *Fail*/ *undefined*)
and (*OCaml*) *failwith*/ *undefined*
and (*Haskell*) *error*/ *undefined*
and (*Scala*) !*sys.error*(*undefined*)

3.1.5 Evaluation and normalization by evaluation

⟨*ML*⟩

3.2 Counterexample Search Units

3.2.1 Quickcheck

quickcheck-params [*size* = *5*, *iterations* = *50*]

3.2.2 Nitpick setup

named-theorems *nitpick-unfold alternative definitions of constants as needed by Nitpick*
and *nitpick-simp equational specification of constants as needed by Nitpick*
and *nitpick-psimp partial equational specification of constants as needed by Nitpick*
and *nitpick-choice-spec choice specification of constants as needed by Nitpick*

declare *if-bool-eq-conj* [*nitpick-unfold*, *no-atp*]
and *if-bool-eq-disj* [*no-atp*]

3.3 Preprocessing for the predicate compiler

named-theorems *code-pred-def alternative definitions of constants for the Predicate Compiler*
and *code-pred-inline inlining definitions for the Predicate Compiler*
and *code-pred-simp simplification rules for the optimisations in the Predicate Compiler*

3.4 Legacy tactics and ML bindings

⟨*ML*⟩

hide-const (**open**) *eq equal*

end

4 Abstract orderings

theory *Orderings*
imports *HOL*
keywords *print-orders* :: *diag*
begin

4.1 Abstract ordering

locale *partial-preordering* =
fixes *less-eq* :: ‹'*a* ⇒ '*a* ⇒ *bool*› (**infix** ‹≤› *50*)
assumes *refl*: ‹*a* ≤ *a*› — not *iff*: makes problems due to multiple (dual) interpretations

and *trans*: ‹$a \leq b \Longrightarrow b \leq c \Longrightarrow a \leq c$›

locale *preordering* = *partial-preordering* +
fixes *less* :: ‹$'a \Rightarrow 'a \Rightarrow bool$› (**infix** ‹$<$› *50*)
assumes *strict-iff-not*: ‹$a < b \longleftrightarrow a \leq b \wedge \neg\, b \leq a$›
begin

lemma *strict-implies-order*:
‹$a < b \Longrightarrow a \leq b$›
⟨*proof*⟩

lemma *irrefl*: — not *iff*: makes problems due to multiple (dual) interpretations
‹$\neg\, a < a$›
⟨*proof*⟩

lemma *asym*:
‹$a < b \Longrightarrow b < a \Longrightarrow False$›
⟨*proof*⟩

lemma *strict-trans1*:
‹$a \leq b \Longrightarrow b < c \Longrightarrow a < c$›
⟨*proof*⟩

lemma *strict-trans2*:
‹$a < b \Longrightarrow b \leq c \Longrightarrow a < c$›
⟨*proof*⟩

lemma *strict-trans*:
‹$a < b \Longrightarrow b < c \Longrightarrow a < c$›
⟨*proof*⟩

end

lemma *preordering-strictI*: — Alternative introduction rule with bias towards strict order
fixes *less-eq* (**infix** ‹$\leq$› *50*)
and *less* (**infix** ‹$<$› *50*)
assumes *less-eq-less*: ‹$\bigwedge a\ b.\ a \leq b \longleftrightarrow a < b \vee a = b$›
assumes *asym*: ‹$\bigwedge a\ b.\ a < b \Longrightarrow \neg\, b < a$›
assumes *irrefl*: ‹$\bigwedge a.\ \neg\, a < a$›
assumes *trans*: ‹$\bigwedge a\ b\ c.\ a < b \Longrightarrow b < c \Longrightarrow a < c$›
shows ‹*preordering* $(\leq)$ $(<)$›
⟨*proof*⟩

lemma *preordering-dualI*:
fixes *less-eq* (**infix** ‹$\leq$› *50*)
and *less* (**infix** ‹$<$› *50*)
assumes ‹*preordering* $(\lambda a\ b.\ b \leq a)$ $(\lambda a\ b.\ b < a)$›
shows ‹*preordering* $(\leq)$ $(<)$›

⟨*proof*⟩

locale *ordering* = *partial-preordering* +
  fixes *less* :: ‹'*a* ⇒ '*a* ⇒ *bool*› (**infix** ‹<› *50*)
  assumes *strict-iff-order*: ‹*a* < *b* ⟷ *a* ≤ *b* ∧ *a* ≠ *b*›
  assumes *antisym*: ‹*a* ≤ *b* ⟹ *b* ≤ *a* ⟹ *a* = *b*›
begin

sublocale *preordering* ‹(≤)› ‹(<)›
⟨*proof*⟩

lemma *strict-implies-not-eq*:
  ‹*a* < *b* ⟹ *a* ≠ *b*›
  ⟨*proof*⟩

lemma *not-eq-order-implies-strict*:
  ‹*a* ≠ *b* ⟹ *a* ≤ *b* ⟹ *a* < *b*›
  ⟨*proof*⟩

lemma *order-iff-strict*:
  ‹*a* ≤ *b* ⟷ *a* < *b* ∨ *a* = *b*›
  ⟨*proof*⟩

lemma *eq-iff*: ‹*a* = *b* ⟷ *a* ≤ *b* ∧ *b* ≤ *a*›
  ⟨*proof*⟩

end

lemma *ordering-strictI*: — Alternative introduction rule with bias towards strict order
  fixes *less-eq* (**infix** ‹≤› *50*)
    and *less* (**infix** ‹<› *50*)
  assumes *less-eq-less*: ‹⋀*a b*. *a* ≤ *b* ⟷ *a* < *b* ∨ *a* = *b*›
    assumes *asym*: ‹⋀*a b*. *a* < *b* ⟹ ¬ *b* < *a*›
  assumes *irrefl*: ‹⋀*a*. ¬ *a* < *a*›
  assumes *trans*: ‹⋀*a b c*. *a* < *b* ⟹ *b* < *c* ⟹ *a* < *c*›
  shows ‹*ordering* (≤) (<)›
⟨*proof*⟩

lemma *ordering-dualI*:
  fixes *less-eq* (**infix** ‹≤› *50*)
    and *less* (**infix** ‹<› *50*)
  assumes ‹*ordering* (λ*a b*. *b* ≤ *a*) (λ*a b*. *b* < *a*)›
  shows ‹*ordering* (≤) (<)›
⟨*proof*⟩

locale *ordering-top* = *ordering* +
  fixes *top* :: ‹'*a*› (‹⊤›)
  assumes *extremum* [*simp*]: ‹*a* ≤ ⊤›

begin

lemma *extremum-uniqueI*:
‹$\top \leq a \Longrightarrow a = \top$›
⟨*proof*⟩

lemma *extremum-unique*:
‹$\top \leq a \longleftrightarrow a = \top$›
⟨*proof*⟩

lemma *extremum-strict* [*simp*]:
‹$\neg (\top < a)$›
⟨*proof*⟩

lemma *not-eq-extremum*:
‹$a \neq \top \longleftrightarrow a < \top$›
⟨*proof*⟩

end

4.2 Syntactic orders

class *ord* =
fixes *less-eq* :: $'a \Rightarrow 'a \Rightarrow$ *bool*
and *less* :: $'a \Rightarrow 'a \Rightarrow$ *bool*
begin

notation
less-eq ($'(\leq')$) **and**
less-eq ((-/ $\leq$ -) [*51*, *51*] *50*) **and**
less ($'(<')$) **and**
less ((-/ $<$ -) [*51*, *51*] *50*)

abbreviation (*input*)
greater-eq (**infix** $\geq$ *50*)
where $x \geq y \equiv y \leq x$

abbreviation (*input*)
greater (**infix** $>$ *50*)
where $x > y \equiv y < x$

notation (*ASCII*)
less-eq ($'(<=')$) **and**
less-eq ((-/ <= -) [*51*, *51*] *50*)

notation (*input*)
greater-eq (**infix** >= *50*)

end

4.3 Quasi orders

class *preorder* = *ord* +
  assumes *less-le-not-le*: $x < y \longleftrightarrow x \leq y \wedge \neg (y \leq x)$
  and *order-refl* [*iff*]: $x \leq x$
  and *order-trans*: $x \leq y \Longrightarrow y \leq z \Longrightarrow x \leq z$
begin

sublocale *order*: *preordering less-eq less* + *dual-order*: *preordering greater-eq greater*
⟨*proof*⟩

Reflexivity.

lemma *eq-refl*: $x = y \Longrightarrow x \leq y$
  — This form is useful with the classical reasoner.
⟨*proof*⟩

lemma *less-irrefl* [*iff*]: $\neg\ x < x$
⟨*proof*⟩

lemma *less-imp-le*: $x < y \Longrightarrow x \leq y$
⟨*proof*⟩

Asymmetry.

lemma *less-not-sym*: $x < y \Longrightarrow \neg\ (y < x)$
⟨*proof*⟩

lemma *less-asym*: $x < y \Longrightarrow (\neg\ P \Longrightarrow y < x) \Longrightarrow P$
⟨*proof*⟩

Transitivity.

lemma *less-trans*: $x < y \Longrightarrow y < z \Longrightarrow x < z$
⟨*proof*⟩

lemma *le-less-trans*: $x \leq y \Longrightarrow y < z \Longrightarrow x < z$
⟨*proof*⟩

lemma *less-le-trans*: $x < y \Longrightarrow y \leq z \Longrightarrow x < z$
⟨*proof*⟩

Useful for simplification, but too risky to include by default.

lemma *less-imp-not-less*: $x < y \Longrightarrow (\neg\ y < x) \longleftrightarrow \mathit{True}$
⟨*proof*⟩

lemma *less-imp-triv*: $x < y \Longrightarrow (y < x \longrightarrow P) \longleftrightarrow \mathit{True}$
⟨*proof*⟩

Transitivity rules for calculational reasoning

lemma *less-asym'*: $a < b \Longrightarrow b < a \Longrightarrow P$
⟨*proof*⟩

Dual order

lemma *dual-preorder*:
‹*class.preorder* (≥) (>)›
⟨*proof*⟩

end

lemma *preordering-preorderI*:
‹*class.preorder* (≤) (<)› **if** ‹*preordering* (≤) (<)›
for *less-eq* (**infix** ‹≤› *50*) **and** *less* (**infix** ‹<› *50*)
⟨*proof*⟩

4.4 Partial orders

class *order* = *preorder* +
assumes *order-antisym*: $x \leq y \Longrightarrow y \leq x \Longrightarrow x = y$
begin

lemma *less-le*: $x < y \longleftrightarrow x \leq y \wedge x \neq y$
⟨*proof*⟩

sublocale *order*: *ordering less-eq less* + *dual-order*: *ordering greater-eq greater*
⟨*proof*⟩

Reflexivity.

lemma *le-less*: $x \leq y \longleftrightarrow x < y \vee x = y$
— NOT suitable for iff, since it can cause PROOF FAILED.
⟨*proof*⟩

lemma *le-imp-less-or-eq*: $x \leq y \Longrightarrow x < y \vee x = y$
⟨*proof*⟩

Useful for simplification, but too risky to include by default.

lemma *less-imp-not-eq*: $x < y \Longrightarrow (x = y) \longleftrightarrow \mathit{False}$
⟨*proof*⟩

lemma *less-imp-not-eq2*: $x < y \Longrightarrow (y = x) \longleftrightarrow \mathit{False}$
⟨*proof*⟩

Transitivity rules for calculational reasoning

lemma *neq-le-trans*: $a \neq b \Longrightarrow a \leq b \Longrightarrow a < b$
⟨*proof*⟩

lemma *le-neq-trans*: $a \leq b \Longrightarrow a \neq b \Longrightarrow a < b$
⟨*proof*⟩

Asymmetry.

lemma *order-eq-iff*: $x = y \longleftrightarrow x \leq y \wedge y \leq x$

⟨*proof*⟩

lemma *antisym-conv*: $y \leq x \Longrightarrow x \leq y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *less-imp-neq*: $x < y \Longrightarrow x \neq y$
⟨*proof*⟩

lemma *antisym-conv1*: $\neg\ x < y \Longrightarrow x \leq y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *antisym-conv2*: $x \leq y \Longrightarrow \neg\ x < y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *leD*: $y \leq x \Longrightarrow \neg\ x < y$
⟨*proof*⟩

Least value operator

definition (**in** *ord*)
Least :: $('a \Rightarrow bool) \Rightarrow 'a$ (**binder** *LEAST* 10) **where**
Least $P = (THE\ x.\ P\ x \wedge (\forall y.\ P\ y \longrightarrow x \leq y))$

lemma *Least-equality*:
assumes $P\ x$
and $\bigwedge y.\ P\ y \Longrightarrow x \leq y$
shows *Least* $P = x$
⟨*proof*⟩

lemma *LeastI2-order*:
assumes $P\ x$
and $\bigwedge y.\ P\ y \Longrightarrow x \leq y$
and $\bigwedge x.\ P\ x \Longrightarrow \forall y.\ P\ y \longrightarrow x \leq y \Longrightarrow Q\ x$
shows Q (*Least* P)
⟨*proof*⟩

lemma *Least-ex1*:
assumes $\exists!x.\ P\ x \wedge (\forall y.\ P\ y \longrightarrow x \leq y)$
shows *Least1I*: P (*Least* P) **and** *Least1-le*: $P\ z \Longrightarrow$ *Least* $P \leq z$
⟨*proof*⟩

Greatest value operator

definition *Greatest* :: $('a \Rightarrow bool) \Rightarrow 'a$ (**binder** *GREATEST* 10) **where**
Greatest $P = (THE\ x.\ P\ x \wedge (\forall y.\ P\ y \longrightarrow x \geq y))$

lemma *GreatestI2-order*:
$[\![\ P\ x;$
$\bigwedge y.\ P\ y \Longrightarrow x \geq y;$
$\bigwedge x.\ [\![\ P\ x; \forall y.\ P\ y \longrightarrow x \geq y\ ]\!] \Longrightarrow Q\ x\ ]\!]$
$\Longrightarrow Q$ (*Greatest* P)

⟨*proof*⟩

lemma *Greatest-equality*:
⟦ *P x*; ⋀*y*. *P y* ⟹ *x* ≥ *y* ⟧ ⟹ *Greatest P* = *x*
⟨*proof*⟩

end

lemma *ordering-orderI*:
fixes *less-eq* (**infix** ≤ *50*)
and *less* (**infix** < *50*)
assumes *ordering less-eq less*
shows *class.order less-eq less*
⟨*proof*⟩

lemma *order-strictI*:
fixes *less* (**infix** < *50*)
and *less-eq* (**infix** ≤ *50*)
assumes ⋀*a b*. *a* ≤ *b* ⟷ *a* < *b* ∨ *a* = *b*
assumes ⋀*a b*. *a* < *b* ⟹ ¬ *b* < *a*
assumes ⋀*a*. ¬ *a* < *a*
assumes ⋀*a b c*. *a* < *b* ⟹ *b* < *c* ⟹ *a* < *c*
shows *class.order less-eq less*
⟨*proof*⟩

context *order*
begin

Dual order

lemma *dual-order*:
class.order (≥) (>)
⟨*proof*⟩

end

4.5 Linear (total) orders

class *linorder* = *order* +
assumes *linear*: *x* ≤ *y* ∨ *y* ≤ *x*
begin

lemma *less-linear*: *x* < *y* ∨ *x* = *y* ∨ *y* < *x*
⟨*proof*⟩

lemma *le-less-linear*: *x* ≤ *y* ∨ *y* < *x*
⟨*proof*⟩

lemma *le-cases* [*case-names le ge*]:
(*x* ≤ *y* ⟹ *P*) ⟹ (*y* ≤ *x* ⟹ *P*) ⟹ *P*

⟨*proof*⟩

lemma (**in** *linorder*) *le-cases3*:
$[\![[\![x \le y;\ y \le z]\!] \Longrightarrow P;\ [\![y \le x;\ x \le z]\!] \Longrightarrow P;\ [\![x \le z;\ z \le y]\!] \Longrightarrow P;$
$[\![z \le y;\ y \le x]\!] \Longrightarrow P;\ [\![y \le z;\ z \le x]\!] \Longrightarrow P;\ [\![z \le x;\ x \le y]\!] \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *linorder-cases* [*case-names less equal greater*]:
$(x < y \Longrightarrow P) \Longrightarrow (x = y \Longrightarrow P) \Longrightarrow (y < x \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *linorder-wlog*[*case-names le sym*]:
$(\bigwedge a\ b.\ a \le b \Longrightarrow P\ a\ b) \Longrightarrow (\bigwedge a\ b.\ P\ b\ a \Longrightarrow P\ a\ b) \Longrightarrow P\ a\ b$
⟨*proof*⟩

lemma *not-less*: $\neg\ x < y \longleftrightarrow y \le x$
⟨*proof*⟩

lemma *not-less-iff-gr-or-eq*: $\neg(x < y) \longleftrightarrow (x > y \vee x = y)$
⟨*proof*⟩

lemma *not-le*: $\neg\ x \le y \longleftrightarrow y < x$
⟨*proof*⟩

lemma *neq-iff*: $x \ne y \longleftrightarrow x < y \vee y < x$
⟨*proof*⟩

lemma *neqE*: $x \ne y \Longrightarrow (x < y \Longrightarrow R) \Longrightarrow (y < x \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

lemma *antisym-conv3*: $\neg\ y < x \Longrightarrow \neg\ x < y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *leI*: $\neg\ x < y \Longrightarrow y \le x$
⟨*proof*⟩

lemma *not-le-imp-less*: $\neg\ y \le x \Longrightarrow x < y$
⟨*proof*⟩

lemma *linorder-less-wlog*[*case-names less refl sym*]:
$[\![\bigwedge a\ b.\ a < b \Longrightarrow P\ a\ b;\ \bigwedge a.\ P\ a\ a;\ \bigwedge a\ b.\ P\ b\ a \Longrightarrow P\ a\ b]\!] \Longrightarrow P\ a\ b$
⟨*proof*⟩

Dual order

lemma *dual-linorder*:
class.linorder $(\ge)$ $(>)$
⟨*proof*⟩

end

Alternative introduction rule with bias towards strict order

lemma *linorder-strictI*:
  fixes *less-eq* (**infix** $\leq$ *50*)
    and *less* (**infix** $<$ *50*)
  assumes *class.order less-eq less*
  assumes *trichotomy*: $\bigwedge a\ b.\ a < b \vee a = b \vee b < a$
  shows *class.linorder less-eq less*
⟨*proof*⟩

4.6 Reasoning tools setup

⟨*ML*⟩

The method *order* allows one to use the order tactic located in `../Provers/order_tac.ML` in a standalone fashion.

The tactic rearranges the goal to prove *False*, then retrieves order literals of partial and linear orders (i.e. $x = y$, $x \leq y$, $x < y$, and their negated versions) from the premises and finally tries to derive a contradiction. Its main use case is as a solver to *simp* (see below), where it e.g. solves premises of conditional rewrite rules.

The tactic has two configuration attributes that control its behaviour:

- *order-trace* toggles tracing for the solver.
- *order-split-limit* limits the number of order literals of the form $\neg\ x < y$ that are passed to the tactic. This is helpful since these literals lead to case splitting and thus exponential runtime. This only applies to partial orders.

  We setup the solver for HOL with the structure `HOL_Order_Tac` here but the prover is agnostic to the object logic. It is possible to register orders with the prover using the functions `HOL_Order_Tac.declare_order` and `HOL_Order_Tac.declare_linorder`, which we do below for the type classes *order* and *linorder*. If possible, one should instantiate these type classes instead of registering new orders with the solver. One can also interpret the type class locales *order* and *linorder*. An example can be seen in `Library/Sublist.thy`, which contains e.g. the prefix order on lists.

  The diagnostic command **print-orders** shows all orders known to the tactic in the current context.

Declarations to set up transitivity reasoner of partial and linear orders.

context *order*
begin

lemma *nless-le*: $(\neg\ a < b) \longleftrightarrow (\neg\ a \leq b) \vee a = b$

⟨*proof*⟩

⟨*ML*⟩

end

context *linorder*
begin

lemma *nle-le*: $(\neg\ a \leq b) \longleftrightarrow b \leq a \wedge b \neq a$
⟨*proof*⟩

⟨*ML*⟩

end

⟨*ML*⟩

4.7 Bounded quantifiers

syntax (*ASCII*)
-All-less :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL -<-./ -*) [*0*, *0*, *10*] *10*)
-Ex-less :: [*idt*, *'a*, *bool*] => *bool* ((*3EX -<-./ -*) [*0*, *0*, *10*] *10*)
-All-less-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL -<=-./ -*) [*0*, *0*, *10*] *10*)
-Ex-less-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3EX -<=-./ -*) [*0*, *0*, *10*] *10*)

-All-greater :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL ->-./ -*) [*0*, *0*, *10*] *10*)
-Ex-greater :: [*idt*, *'a*, *bool*] => *bool* ((*3EX ->-./ -*) [*0*, *0*, *10*] *10*)
-All-greater-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL ->=-./ -*) [*0*, *0*, *10*] *10*)
-Ex-greater-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3EX ->=-./ -*) [*0*, *0*, *10*] *10*)

-All-neq :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL -~=-./ -*) [*0*, *0*, *10*] *10*)
-Ex-neq :: [*idt*, *'a*, *bool*] => *bool* ((*3EX -~=-./ -*) [*0*, *0*, *10*] *10*)

syntax
-All-less :: [*idt*, *'a*, *bool*] => *bool* ((*3∀ -<-./ -*) [*0*, *0*, *10*] *10*)
-Ex-less :: [*idt*, *'a*, *bool*] => *bool* ((*3∃ -<-./ -*) [*0*, *0*, *10*] *10*)
-All-less-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3∀ -≤-./ -*) [*0*, *0*, *10*] *10*)
-Ex-less-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3∃ -≤-./ -*) [*0*, *0*, *10*] *10*)

-All-greater :: [*idt*, *'a*, *bool*] => *bool* ((*3∀ ->-./ -*) [*0*, *0*, *10*] *10*)
-Ex-greater :: [*idt*, *'a*, *bool*] => *bool* ((*3∃ ->-./ -*) [*0*, *0*, *10*] *10*)
-All-greater-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3∀ -≥-./ -*) [*0*, *0*, *10*] *10*)
-Ex-greater-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3∃ -≥-./ -*) [*0*, *0*, *10*] *10*)

-All-neq :: [*idt*, *'a*, *bool*] => *bool* ((*3∀ -≠-./ -*) [*0*, *0*, *10*] *10*)
-Ex-neq :: [*idt*, *'a*, *bool*] => *bool* ((*3∃ -≠-./ -*) [*0*, *0*, *10*] *10*)

syntax (*input*)

-All-less :: [*idt*, *'a*, *bool*] => *bool* ((*3*! -<-./ -) [*0*, *0*, *10*] *10*)
-Ex-less :: [*idt*, *'a*, *bool*] => *bool* ((*3*? -<-./ -) [*0*, *0*, *10*] *10*)
-All-less-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3*! -<=-./ -) [*0*, *0*, *10*] *10*)
-Ex-less-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3*? -<=-./ -) [*0*, *0*, *10*] *10*)
-All-neq :: [*idt*, *'a*, *bool*] => *bool* ((*3*! -~=-./ -) [*0*, *0*, *10*] *10*)
-Ex-neq :: [*idt*, *'a*, *bool*] => *bool* ((*3*? -~=-./ -) [*0*, *0*, *10*] *10*)

translations
$\forall x{<}y.\ P \rightharpoonup \forall x.\ x < y \longrightarrow P$
$\exists x{<}y.\ P \rightharpoonup \exists x.\ x < y \wedge P$
$\forall x{\leq}y.\ P \rightharpoonup \forall x.\ x \leq y \longrightarrow P$
$\exists x{\leq}y.\ P \rightharpoonup \exists x.\ x \leq y \wedge P$
$\forall x{>}y.\ P \rightharpoonup \forall x.\ x > y \longrightarrow P$
$\exists x{>}y.\ P \rightharpoonup \exists x.\ x > y \wedge P$
$\forall x{\geq}y.\ P \rightharpoonup \forall x.\ x \geq y \longrightarrow P$
$\exists x{\geq}y.\ P \rightharpoonup \exists x.\ x \geq y \wedge P$
$\forall x{\neq}y.\ P \rightharpoonup \forall x.\ x \neq y \longrightarrow P$
$\exists x{\neq}y.\ P \rightharpoonup \exists x.\ x \neq y \wedge P$

⟨*ML*⟩

4.8 Transitivity reasoning

context *ord*
begin

lemma *ord-le-eq-trans*: $a \leq b \Longrightarrow b = c \Longrightarrow a \leq c$
⟨*proof*⟩

lemma *ord-eq-le-trans*: $a = b \Longrightarrow b \leq c \Longrightarrow a \leq c$
⟨*proof*⟩

lemma *ord-less-eq-trans*: $a < b \Longrightarrow b = c \Longrightarrow a < c$
⟨*proof*⟩

lemma *ord-eq-less-trans*: $a = b \Longrightarrow b < c \Longrightarrow a < c$
⟨*proof*⟩

end

lemma *order-less-subst2*: $(a{::}'a{::}order) < b \Longrightarrow f\ b < (c{::}'c{::}order) \Longrightarrow$
$(!!x\ y.\ x < y \Longrightarrow f\ x < f\ y) \Longrightarrow f\ a < c$
⟨*proof*⟩

lemma *order-less-subst1*: $(a{::}'a{::}order) < f\ b \Longrightarrow (b{::}'b{::}order) < c \Longrightarrow$
$(!!x\ y.\ x < y \Longrightarrow f\ x < f\ y) \Longrightarrow a < f\ c$
⟨*proof*⟩

lemma *order-le-less-subst2*: $(a{::}'a{::}order) <= b \Longrightarrow f\ b < (c{::}'c{::}order) \Longrightarrow$

$(!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y) \Longrightarrow f\ a < c$
⟨*proof*⟩

lemma *order-le-less-subst1*: $(a::'a::order) <= f\ b \Longrightarrow (b::'b::order) < c \Longrightarrow$
$(!!x\ y.\ x < y \Longrightarrow f\ x < f\ y) \Longrightarrow a < f\ c$
⟨*proof*⟩

lemma *order-less-le-subst2*: $(a::'a::order) < b \Longrightarrow f\ b <= (c::'c::order) \Longrightarrow$
$(!!x\ y.\ x < y \Longrightarrow f\ x < f\ y) \Longrightarrow f\ a < c$
⟨*proof*⟩

lemma *order-less-le-subst1*: $(a::'a::order) < f\ b \Longrightarrow (b::'b::order) <= c \Longrightarrow$
$(!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y) \Longrightarrow a < f\ c$
⟨*proof*⟩

lemma *order-subst1*: $(a::'a::order) <= f\ b \Longrightarrow (b::'b::order) <= c \Longrightarrow$
$(!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y) \Longrightarrow a <= f\ c$
⟨*proof*⟩

lemma *order-subst2*: $(a::'a::order) <= b \Longrightarrow f\ b <= (c::'c::order) \Longrightarrow$
$(!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y) \Longrightarrow f\ a <= c$
⟨*proof*⟩

lemma *ord-le-eq-subst*: $a <= b \Longrightarrow f\ b = c \Longrightarrow$
$(!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y) \Longrightarrow f\ a <= c$
⟨*proof*⟩

lemma *ord-eq-le-subst*: $a = f\ b \Longrightarrow b <= c \Longrightarrow$
$(!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y) \Longrightarrow a <= f\ c$
⟨*proof*⟩

lemma *ord-less-eq-subst*: $a < b \Longrightarrow f\ b = c \Longrightarrow$
$(!!x\ y.\ x < y \Longrightarrow f\ x < f\ y) \Longrightarrow f\ a < c$
⟨*proof*⟩

lemma *ord-eq-less-subst*: $a = f\ b \Longrightarrow b < c \Longrightarrow$
$(!!x\ y.\ x < y \Longrightarrow f\ x < f\ y) \Longrightarrow a < f\ c$
⟨*proof*⟩

Note that this list of rules is in reverse order of priorities.

lemmas [*trans*] =
order-less-subst2
order-less-subst1
order-le-less-subst2
order-le-less-subst1
order-less-le-subst2
order-less-le-subst1
order-subst2
order-subst1

ord-le-eq-subst
ord-eq-le-subst
ord-less-eq-subst
ord-eq-less-subst
forw-subst
back-subst
rev-mp
mp

lemmas (**in** *order*) [*trans*] =
neq-le-trans
le-neq-trans

lemmas (**in** *preorder*) [*trans*] =
less-trans
less-asym'
le-less-trans
less-le-trans
order-trans

lemmas (**in** *order*) [*trans*] =
order.antisym

lemmas (**in** *ord*) [*trans*] =
ord-le-eq-trans
ord-eq-le-trans
ord-less-eq-trans
ord-eq-less-trans

lemmas [*trans*] =
trans

lemmas *order-trans-rules* =
order-less-subst2
order-less-subst1
order-le-less-subst2
order-le-less-subst1
order-less-le-subst2
order-less-le-subst1
order-subst2
order-subst1
ord-le-eq-subst
ord-eq-le-subst
ord-less-eq-subst
ord-eq-less-subst
forw-subst
back-subst
rev-mp
mp

neq-le-trans
le-neq-trans
less-trans
less-asym'
le-less-trans
less-le-trans
order-trans
order.antisym
ord-le-eq-trans
ord-eq-le-trans
ord-less-eq-trans
ord-eq-less-trans
trans

These support proving chains of decreasing inequalities a ≥ b ≥ c ... in Isar proofs.

lemma *xt1* [*no-atp*]:
$a = b \Longrightarrow b > c \Longrightarrow a > c$
$a > b \Longrightarrow b = c \Longrightarrow a > c$
$a = b \Longrightarrow b \geq c \Longrightarrow a \geq c$
$a \geq b \Longrightarrow b = c \Longrightarrow a \geq c$
$(x::'a::order) \geq y \Longrightarrow y \geq x \Longrightarrow x = y$
$(x::'a::order) \geq y \Longrightarrow y \geq z \Longrightarrow x \geq z$
$(x::'a::order) > y \Longrightarrow y \geq z \Longrightarrow x > z$
$(x::'a::order) \geq y \Longrightarrow y > z \Longrightarrow x > z$
$(a::'a::order) > b \Longrightarrow b > a \Longrightarrow P$
$(x::'a::order) > y \Longrightarrow y > z \Longrightarrow x > z$
$(a::'a::order) \geq b \Longrightarrow a \neq b \Longrightarrow a > b$
$(a::'a::order) \neq b \Longrightarrow a \geq b \Longrightarrow a > b$
$a = f\ b \Longrightarrow b > c \Longrightarrow (\bigwedge x\ y.\ x > y \Longrightarrow f\ x > f\ y) \Longrightarrow a > f\ c$
$a > b \Longrightarrow f\ b = c \Longrightarrow (\bigwedge x\ y.\ x > y \Longrightarrow f\ x > f\ y) \Longrightarrow f\ a > c$
$a = f\ b \Longrightarrow b \geq c \Longrightarrow (\bigwedge x\ y.\ x \geq y \Longrightarrow f\ x \geq f\ y) \Longrightarrow a \geq f\ c$
$a \geq b \Longrightarrow f\ b = c \Longrightarrow (\bigwedge x\ y.\ x \geq y \Longrightarrow f\ x \geq f\ y) \Longrightarrow f\ a \geq c$
⟨*proof*⟩

lemma *xt2* [*no-atp*]:
assumes $(a::'a::order) \geq f\ b$
and $b \geq c$
and $\bigwedge x\ y.\ x \geq y \Longrightarrow f\ x \geq f\ y$
shows $a \geq f\ c$
⟨*proof*⟩

lemma *xt3* [*no-atp*]:
assumes $(a::'a::order) \geq b$
and $(f\ b::'b::order) \geq c$
and $\bigwedge x\ y.\ x \geq y \Longrightarrow f\ x \geq f\ y$
shows $f\ a \geq c$
⟨*proof*⟩

lemma *xt4* [*no-atp*]:
assumes $(a::'a::order) > f\ b$
and $(b::'b::order) \geq c$
and $\bigwedge x\ y.\ x \geq y \Longrightarrow f\ x \geq f\ y$
shows $a > f\ c$
⟨*proof*⟩

lemma *xt5* [*no-atp*]:
assumes $(a::'a::order) > b$
and $(f\ b::'b::order) \geq c$
and $\bigwedge x\ y.\ x > y \Longrightarrow f\ x > f\ y$
shows $f\ a > c$
⟨*proof*⟩

lemma *xt6* [*no-atp*]:
assumes $(a::'a::order) \geq f\ b$
and $b > c$
and $\bigwedge x\ y.\ x > y \Longrightarrow f\ x > f\ y$
shows $a > f\ c$
⟨*proof*⟩

lemma *xt7* [*no-atp*]:
assumes $(a::'a::order) \geq b$
and $(f\ b::'b::order) > c$
and $\bigwedge x\ y.\ x \geq y \Longrightarrow f\ x \geq f\ y$
shows $f\ a > c$
⟨*proof*⟩

lemma *xt8* [*no-atp*]:
assumes $(a::'a::order) > f\ b$
and $(b::'b::order) > c$
and $\bigwedge x\ y.\ x > y \Longrightarrow f\ x > f\ y$
shows $a > f\ c$
⟨*proof*⟩

lemma *xt9* [*no-atp*]:
assumes $(a::'a::order) > b$
and $(f\ b::'b::order) > c$
and $\bigwedge x\ y.\ x > y \Longrightarrow f\ x > f\ y$
shows $f\ a > c$
⟨*proof*⟩

lemmas *xtrans* = *xt1 xt2 xt3 xt4 xt5 xt6 xt7 xt8 xt9*

4.9 min and max – fundamental

definition (**in** *ord*) $min :: 'a \Rightarrow 'a \Rightarrow 'a$ **where**
$min\ a\ b = (if\ a \leq b\ then\ a\ else\ b)$

definition (**in** *ord*) *max* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
max a b = (*if a* $\leq$ *b then b else a*)

lemma *min-absorb1*: $x \leq y \Longrightarrow min\ x\ y = x$
⟨*proof*⟩

lemma *max-absorb2*: $x \leq y \Longrightarrow max\ x\ y = y$
⟨*proof*⟩

lemma *min-absorb2*: $(y::'a::order) \leq x \Longrightarrow min\ x\ y = y$
⟨*proof*⟩

lemma *max-absorb1*: $(y::'a::order) \leq x \Longrightarrow max\ x\ y = x$
⟨*proof*⟩

lemma *max-min-same* [*simp*]:
fixes $x\ y :: 'a :: linorder$
shows $max\ x\ (min\ x\ y) = x\ max\ (min\ x\ y)\ x = x\ max\ (min\ x\ y)\ y = y\ max\ y\ (min\ x\ y) = y$
⟨*proof*⟩

4.10 (Unique) top and bottom elements

class *bot* =
fixes *bot* :: $'a$ ($\bot$)

class *order-bot* = *order* + *bot* +
assumes *bot-least*: $\bot \leq a$
begin

sublocale *bot*: *ordering-top greater-eq greater bot*
⟨*proof*⟩

lemma *le-bot*:
$a \leq \bot \Longrightarrow a = \bot$
⟨*proof*⟩

lemma *bot-unique*:
$a \leq \bot \longleftrightarrow a = \bot$
⟨*proof*⟩

lemma *not-less-bot*:
$\neg\ a < \bot$
⟨*proof*⟩

lemma *bot-less*:
$a \neq \bot \longleftrightarrow \bot < a$
⟨*proof*⟩

lemma *max-bot*[*simp*]: *max bot x* = *x*
⟨*proof*⟩

lemma *max-bot2*[*simp*]: *max x bot* = *x*
⟨*proof*⟩

lemma *min-bot*[*simp*]: *min bot x* = *bot*
⟨*proof*⟩

lemma *min-bot2*[*simp*]: *min x bot* = *bot*
⟨*proof*⟩

end

class *top* =
  fixes *top* :: *'a* (⊤)

class *order-top* = *order* + *top* +
  assumes *top-greatest*: *a* ≤ ⊤
begin

sublocale *top*: *ordering-top less-eq less top*
  ⟨*proof*⟩

lemma *top-le*:
  ⊤ ≤ *a* ⟹ *a* = ⊤
  ⟨*proof*⟩

lemma *top-unique*:
  ⊤ ≤ *a* ⟷ *a* = ⊤
  ⟨*proof*⟩

lemma *not-top-less*:
  ¬ ⊤ < *a*
  ⟨*proof*⟩

lemma *less-top*:
  a ≠ ⊤ ⟷ *a* < ⊤
  ⟨*proof*⟩

lemma *max-top*[*simp*]: *max top x* = *top*
⟨*proof*⟩

lemma *max-top2*[*simp*]: *max x top* = *top*
⟨*proof*⟩

lemma *min-top*[*simp*]: *min top x* = *x*
⟨*proof*⟩

lemma *min-top2*[*simp*]: $min\ x\ top = x$
⟨*proof*⟩

end

4.11 Dense orders

class *dense-order* = *order* +
  assumes *dense*: $x < y \Longrightarrow (\exists\, z.\ x < z \wedge z < y)$

class *dense-linorder* = *linorder* + *dense-order*
begin

lemma *dense-le*:
  fixes $y\ z :: 'a$
  assumes $\bigwedge x.\ x < y \Longrightarrow x \leq z$
  shows $y \leq z$
⟨*proof*⟩

lemma *dense-le-bounded*:
  fixes $x\ y\ z :: 'a$
  assumes $x < y$
  assumes $*$: $\bigwedge w.\ [\![\ x < w\ ;\ w < y\ ]\!] \Longrightarrow w \leq z$
  shows $y \leq z$
⟨*proof*⟩

lemma *dense-ge*:
  fixes $y\ z :: 'a$
  assumes $\bigwedge x.\ z < x \Longrightarrow y \leq x$
  shows $y \leq z$
⟨*proof*⟩

lemma *dense-ge-bounded*:
  fixes $x\ y\ z :: 'a$
  assumes $z < x$
  assumes $*$: $\bigwedge w.\ [\![\ z < w\ ;\ w < x\ ]\!] \Longrightarrow y \leq w$
  shows $y \leq z$
⟨*proof*⟩

end

class *no-top* = *order* +
  assumes *gt-ex*: $\exists\, y.\ x < y$

class *no-bot* = *order* +
  assumes *lt-ex*: $\exists\, y.\ y < x$

class *unbounded-dense-linorder* = *dense-linorder* + *no-top* + *no-bot*

4.12 Wellorders

class *wellorder* = *linorder* +
  assumes *less-induct* [*case-names less*]: $(\bigwedge x.\ (\bigwedge y.\ y < x \Longrightarrow P\ y) \Longrightarrow P\ x) \Longrightarrow P\ a$
begin

lemma *wellorder-Least-lemma*:
  fixes $k :: 'a$
  assumes $P\ k$
  shows *LeastI*: $P\ (LEAST\ x.\ P\ x)$ **and** *Least-le*: $(LEAST\ x.\ P\ x) \leq k$
⟨*proof*⟩
lemma *LeastI-ex*: $\exists\, x.\ P\ x \Longrightarrow P\ (Least\ P)$
  ⟨*proof*⟩

lemma *LeastI2*:
  $P\ a \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow Q\ x) \Longrightarrow Q\ (Least\ P)$
  ⟨*proof*⟩

lemma *LeastI2-ex*:
  $\exists\, a.\ P\ a \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow Q\ x) \Longrightarrow Q\ (Least\ P)$
  ⟨*proof*⟩

lemma *LeastI2-wellorder*:
  assumes $P\ a$
  and $\bigwedge a.\ [\![\ P\ a;\ \forall\, b.\ P\ b \longrightarrow a \leq b\ ]\!] \Longrightarrow Q\ a$
  shows $Q\ (Least\ P)$
⟨*proof*⟩

lemma *LeastI2-wellorder-ex*:
  assumes $\exists\, x.\ P\ x$
  and $\bigwedge a.\ [\![\ P\ a;\ \forall\, b.\ P\ b \longrightarrow a \leq b\ ]\!] \Longrightarrow Q\ a$
  shows $Q\ (Least\ P)$
⟨*proof*⟩

lemma *not-less-Least*: $k < (LEAST\ x.\ P\ x) \Longrightarrow \neg\ P\ k$
⟨*proof*⟩

lemma *exists-least-iff*: $(\exists\, n.\ P\ n) \longleftrightarrow (\exists\, n.\ P\ n \wedge (\forall\, m < n.\ \neg\ P\ m))$ (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

end

4.13 Order on *bool*

instantiation *bool* :: {*order-bot*, *order-top*, *linorder*}
begin

definition

le-bool-def [*simp*]: $P \leq Q \longleftrightarrow P \longrightarrow Q$

definition
[*simp*]: $(P::bool) < Q \longleftrightarrow \neg\ P \wedge Q$

definition
[*simp*]: $\bot \longleftrightarrow False$

definition
[*simp*]: $\top \longleftrightarrow True$

instance ⟨*proof*⟩

end

lemma *le-boolI*: $(P \Longrightarrow Q) \Longrightarrow P \leq Q$
⟨*proof*⟩

lemma *le-boolI'*: $P \longrightarrow Q \Longrightarrow P \leq Q$
⟨*proof*⟩

lemma *le-boolE*: $P \leq Q \Longrightarrow P \Longrightarrow (Q \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

lemma *le-boolD*: $P \leq Q \Longrightarrow P \longrightarrow Q$
⟨*proof*⟩

lemma *bot-boolE*: $\bot \Longrightarrow P$
⟨*proof*⟩

lemma *top-boolI*: $\top$
⟨*proof*⟩

lemma [*code*]:
$False \leq b \longleftrightarrow True$
$True \leq b \longleftrightarrow b$
$False < b \longleftrightarrow b$
$True < b \longleftrightarrow False$
⟨*proof*⟩

4.14 Order on - ⇒ -

instantiation *fun* :: (*type*, *ord*) *ord*
begin

definition
le-fun-def: $f \leq g \longleftrightarrow (\forall x.\ f\ x \leq g\ x)$

definition

$(f::'a \Rightarrow 'b) < g \longleftrightarrow f \leq g \wedge \neg\ (g \leq f)$

instance ⟨*proof*⟩

end

instance *fun* :: (*type*, *preorder*) *preorder* ⟨*proof*⟩

instance *fun* :: (*type*, *order*) *order* ⟨*proof*⟩

instantiation *fun* :: (*type*, *bot*) *bot*
begin

definition
$\bot = (\lambda x.\ \bot)$

instance ⟨*proof*⟩

end

instantiation *fun* :: (*type*, *order-bot*) *order-bot*
begin

lemma *bot-apply* [*simp*, *code*]:
$\bot\ x = \bot$
⟨*proof*⟩

instance ⟨*proof*⟩

end

instantiation *fun* :: (*type*, *top*) *top*
begin

definition
[*no-atp*]: $\top = (\lambda x.\ \top)$

instance ⟨*proof*⟩

end

instantiation *fun* :: (*type*, *order-top*) *order-top*
begin

lemma *top-apply* [*simp*, *code*]:
$\top\ x = \top$
⟨*proof*⟩

instance ⟨*proof*⟩

end

lemma *le-funI*: $(\bigwedge x.\ f\ x \leq g\ x) \Longrightarrow f \leq g$
⟨*proof*⟩

lemma *le-funE*: $f \leq g \Longrightarrow (f\ x \leq g\ x \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *le-funD*: $f \leq g \Longrightarrow f\ x \leq g\ x$
⟨*proof*⟩

4.15 Order on unary and binary predicates

lemma *predicate1I*:
assumes *PQ*: $\bigwedge x.\ P\ x \Longrightarrow Q\ x$
shows $P \leq Q$
⟨*proof*⟩

lemma *predicate1D*:
$P \leq Q \Longrightarrow P\ x \Longrightarrow Q\ x$
⟨*proof*⟩

lemma *rev-predicate1D*:
$P\ x \Longrightarrow P \leq Q \Longrightarrow Q\ x$
⟨*proof*⟩

lemma *predicate2I*:
assumes *PQ*: $\bigwedge x\ y.\ P\ x\ y \Longrightarrow Q\ x\ y$
shows $P \leq Q$
⟨*proof*⟩

lemma *predicate2D*:
$P \leq Q \Longrightarrow P\ x\ y \Longrightarrow Q\ x\ y$
⟨*proof*⟩

lemma *rev-predicate2D*:
$P\ x\ y \Longrightarrow P \leq Q \Longrightarrow Q\ x\ y$
⟨*proof*⟩

lemma *bot1E* [*no-atp*]: $\bot\ x \Longrightarrow P$
⟨*proof*⟩

lemma *bot2E*: $\bot\ x\ y \Longrightarrow P$
⟨*proof*⟩

lemma *top1I*: $\top\ x$
⟨*proof*⟩

lemma *top2I*: $\top\ x\ y$
⟨*proof*⟩

4.16 Name duplicates

lemmas *antisym* = *order.antisym*
lemmas *eq-iff* = *order.eq-iff*

lemmas *order-eq-refl* = *preorder-class.eq-refl*
lemmas *order-less-irrefl* = *preorder-class.less-irrefl*
lemmas *order-less-imp-le* = *preorder-class.less-imp-le*
lemmas *order-less-not-sym* = *preorder-class.less-not-sym*
lemmas *order-less-asym* = *preorder-class.less-asym*
lemmas *order-less-trans* = *preorder-class.less-trans*
lemmas *order-le-less-trans* = *preorder-class.le-less-trans*
lemmas *order-less-le-trans* = *preorder-class.less-le-trans*
lemmas *order-less-imp-not-less* = *preorder-class.less-imp-not-less*
lemmas *order-less-imp-triv* = *preorder-class.less-imp-triv*
lemmas *order-less-asym'* = *preorder-class.less-asym'*

lemmas *order-less-le* = *order-class.less-le*
lemmas *order-le-less* = *order-class.le-less*
lemmas *order-le-imp-less-or-eq* = *order-class.le-imp-less-or-eq*
lemmas *order-less-imp-not-eq* = *order-class.less-imp-not-eq*
lemmas *order-less-imp-not-eq2* = *order-class.less-imp-not-eq2*
lemmas *order-neq-le-trans* = *order-class.neq-le-trans*
lemmas *order-le-neq-trans* = *order-class.le-neq-trans*
lemmas *order-eq-iff* = *order-class.order.eq-iff*
lemmas *order-antisym-conv* = *order-class.antisym-conv*

lemmas *linorder-linear* = *linorder-class.linear*
lemmas *linorder-less-linear* = *linorder-class.less-linear*
lemmas *linorder-le-less-linear* = *linorder-class.le-less-linear*
lemmas *linorder-le-cases* = *linorder-class.le-cases*
lemmas *linorder-not-less* = *linorder-class.not-less*
lemmas *linorder-not-le* = *linorder-class.not-le*
lemmas *linorder-neq-iff* = *linorder-class.neq-iff*
lemmas *linorder-neqE* = *linorder-class.neqE*

end

5 Groups, also combined with orderings

theory *Groups*
imports *Orderings*
begin

5.1 Dynamic facts

named-theorems *ac-simps associativity and commutativity simplification rules*
and *algebra-simps algebra simplification rules for rings*
and *algebra-split-simps algebra simplification rules for rings, with potential goal splitting*
and *field-simps algebra simplification rules for fields*
and *field-split-simps algebra simplification rules for fields, with potential goal splitting*

The rewrites accumulated in *algebra-simps* deal with the classical algebraic structures of groups, rings and family. They simplify terms by multiplying everything out (in case of a ring) and bringing sums and products into a canonical form (by ordered rewriting). As a result it decides group and ring equalities but also helps with inequalities.

Of course it also works for fields, but it knows nothing about multiplicative inverses or division. This is catered for by *field-simps*.

Facts in *field-simps* multiply with denominators in (in)equations if they can be proved to be non-zero (for equations) or positive/negative (for inequalities). Can be too aggressive and is therefore separate from the more benign *algebra-simps*.

Collections *algebra-split-simps* and *field-split-simps* correspond to *algebra-simps* and *field-simps* but contain more aggresive rules that may lead to goal splitting.

5.2 Abstract structures

These locales provide basic structures for interpretation into bigger structures; extensions require careful thinking, otherwise undesired effects may occur due to interpretation.

locale *semigroup* =
fixes $f :: 'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** $*$ *70*)
assumes *assoc* [*ac-simps*]: $a * b * c = a * (b * c)$

locale *abel-semigroup* = *semigroup* +
assumes *commute* [*ac-simps*]: $a * b = b * a$
begin

lemma *left-commute* [*ac-simps*]: $b * (a * c) = a * (b * c)$
⟨*proof*⟩

end

locale *monoid* = *semigroup* +
fixes $z :: 'a$ (**1**)
assumes *left-neutral* [*simp*]: $\mathbf{1} * a = a$

assumes *right-neutral* [*simp*]: $a * \mathbf{1} = a$

locale *comm-monoid* = *abel-semigroup* +
fixes $z :: 'a$ (**1**)
assumes *comm-neutral*: $a * \mathbf{1} = a$
begin

sublocale *monoid*
⟨*proof*⟩

end

locale *group* = *semigroup* +
fixes $z :: 'a$ (**1**)
fixes *inverse* :: $'a \Rightarrow 'a$
assumes *group-left-neutral*: $\mathbf{1} * a = a$
assumes *left-inverse* [*simp*]: $inverse\ a * a = \mathbf{1}$
begin

lemma *left-cancel*: $a * b = a * c \longleftrightarrow b = c$
⟨*proof*⟩

sublocale *monoid*
⟨*proof*⟩

lemma *inverse-unique*:
assumes $a * b = \mathbf{1}$
shows $inverse\ a = b$
⟨*proof*⟩

lemma *inverse-neutral* [*simp*]: $inverse\ \mathbf{1} = \mathbf{1}$
⟨*proof*⟩

lemma *inverse-inverse* [*simp*]: $inverse\ (inverse\ a) = a$
⟨*proof*⟩

lemma *right-inverse* [*simp*]: $a * inverse\ a = \mathbf{1}$
⟨*proof*⟩

lemma *inverse-distrib-swap*: $inverse\ (a * b) = inverse\ b * inverse\ a$
⟨*proof*⟩

lemma *right-cancel*: $b * a = c * a \longleftrightarrow b = c$
⟨*proof*⟩

end

5.3 Generic operations

class *zero* =
  fixes *zero* :: *'a* (*0*)

class *one* =
  fixes *one* :: *'a* (*1*)

hide-const (**open**) *zero one*

lemma *Let-0* [*simp*]: *Let 0 f = f 0*
  ⟨*proof*⟩

lemma *Let-1* [*simp*]: *Let 1 f = f 1*
  ⟨*proof*⟩

⟨*ML*⟩

class *plus* =
  fixes *plus* :: *'a ⇒ 'a ⇒ 'a* (**infixl** + *65*)

class *minus* =
  fixes *minus* :: *'a ⇒ 'a ⇒ 'a* (**infixl** − *65*)

class *uminus* =
  fixes *uminus* :: *'a ⇒ 'a* (− - [*81*] *80*)

class *times* =
  fixes *times* :: *'a ⇒ 'a ⇒ 'a* (**infixl** ∗ *70*)

5.4 Semigroups and Monoids

class *semigroup-add* = *plus* +
 assumes *add-assoc* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
   (*a* + *b*) + *c* = *a* + (*b* + *c*)
begin

sublocale *add*: *semigroup plus*
  ⟨*proof*⟩

end

hide-fact *add-assoc*

class *ab-semigroup-add* = *semigroup-add* +
 assumes *add-commute* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
   a + *b* = *b* + *a*
begin

sublocale *add*: *abel-semigroup plus*

⟨*proof*⟩

declare *add.left-commute* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]

lemmas *add-ac* = *add.assoc add.commute add.left-commute*

end

hide-fact *add-commute*

lemmas *add-ac* = *add.assoc add.commute add.left-commute*

class *semigroup-mult* = *times* +
 assumes *mult-assoc* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
  $(a * b) * c = a * (b * c)$
begin

sublocale *mult*: *semigroup times*
 ⟨*proof*⟩

end

hide-fact *mult-assoc*

class *ab-semigroup-mult* = *semigroup-mult* +
 assumes *mult-commute* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
  $a * b = b * a$
begin

sublocale *mult*: *abel-semigroup times*
 ⟨*proof*⟩

declare *mult.left-commute* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]

lemmas *mult-ac* = *mult.assoc mult.commute mult.left-commute*

end

hide-fact *mult-commute*

lemmas *mult-ac* = *mult.assoc mult.commute mult.left-commute*

class *monoid-add* = *zero* + *semigroup-add* +
 assumes *add-0-left*: $0 + a = a$
  and *add-0-right*: $a + 0 = a$
begin

sublocale *add*: *monoid plus 0*
 ⟨*proof*⟩

end

lemma *zero-reorient*: $0 = x \longleftrightarrow x = 0$
⟨*proof*⟩

class *comm-monoid-add* = *zero* + *ab-semigroup-add* +
assumes *add-0*: $0 + a = a$
begin

subclass *monoid-add*
⟨*proof*⟩

sublocale *add*: *comm-monoid plus 0*
⟨*proof*⟩

end

class *monoid-mult* = *one* + *semigroup-mult* +
assumes *mult-1-left*: $1 * a = a$
and *mult-1-right*: $a * 1 = a$
begin

sublocale *mult*: *monoid times 1*
⟨*proof*⟩

end

lemma *one-reorient*: $1 = x \longleftrightarrow x = 1$
⟨*proof*⟩

class *comm-monoid-mult* = *one* + *ab-semigroup-mult* +
assumes *mult-1*: $1 * a = a$
begin

subclass *monoid-mult*
⟨*proof*⟩

sublocale *mult*: *comm-monoid times 1*
⟨*proof*⟩

end

class *cancel-semigroup-add* = *semigroup-add* +
assumes *add-left-imp-eq*: $a + b = a + c \Longrightarrow b = c$
assumes *add-right-imp-eq*: $b + a = c + a \Longrightarrow b = c$
begin

lemma *add-left-cancel* [*simp*]: $a + b = a + c \longleftrightarrow b = c$

⟨*proof*⟩

lemma *add-right-cancel* [*simp*]: $b + a = c + a \longleftrightarrow b = c$
⟨*proof*⟩

end

class *cancel-ab-semigroup-add* = *ab-semigroup-add* + *minus* +
assumes *add-diff-cancel-left'* [*simp*]: $(a + b) - a = b$
assumes *diff-diff-add* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
$a - b - c = a - (b + c)$
begin

lemma *add-diff-cancel-right'* [*simp*]: $(a + b) - b = a$
⟨*proof*⟩

subclass *cancel-semigroup-add*
⟨*proof*⟩

lemma *add-diff-cancel-left* [*simp*]: $(c + a) - (c + b) = a - b$
⟨*proof*⟩

lemma *add-diff-cancel-right* [*simp*]: $(a + c) - (b + c) = a - b$
⟨*proof*⟩

lemma *diff-right-commute*: $a - c - b = a - b - c$
⟨*proof*⟩

end

class *cancel-comm-monoid-add* = *cancel-ab-semigroup-add* + *comm-monoid-add*
begin

lemma *diff-zero* [*simp*]: $a - 0 = a$
⟨*proof*⟩

lemma *diff-cancel* [*simp*]: $a - a = 0$
⟨*proof*⟩

lemma *add-implies-diff*:
assumes $c + b = a$
shows $c = a - b$
⟨*proof*⟩

lemma *add-cancel-right-right* [*simp*]: $a = a + b \longleftrightarrow b = 0$
(**is** *?P* $\longleftrightarrow$ *?Q*)
⟨*proof*⟩

lemma *add-cancel-right-left* [*simp*]: $a = b + a \longleftrightarrow b = 0$

⟨*proof*⟩

lemma *add-cancel-left-right* [*simp*]: $a + b = a \longleftrightarrow b = 0$
⟨*proof*⟩

lemma *add-cancel-left-left* [*simp*]: $b + a = a \longleftrightarrow b = 0$
⟨*proof*⟩

end

class *comm-monoid-diff* = *cancel-comm-monoid-add* +
assumes *zero-diff* [*simp*]: $0 - a = 0$
begin

lemma *diff-add-zero* [*simp*]: $a - (a + b) = 0$
⟨*proof*⟩

end

5.5 Groups

class *group-add* = *minus* + *uminus* + *monoid-add* +
assumes *left-minus*: $- a + a = 0$
assumes *add-uminus-conv-diff* [*simp*]: $a + (- b) = a - b$
begin

lemma *diff-conv-add-uminus*: $a - b = a + (- b)$
⟨*proof*⟩

sublocale *add*: *group plus 0 uminus*
⟨*proof*⟩

lemma *minus-unique*: $a + b = 0 \Longrightarrow - a = b$
⟨*proof*⟩

lemma *minus-zero*: $- 0 = 0$
⟨*proof*⟩

lemma *minus-minus*: $- (- a) = a$
⟨*proof*⟩

lemma *right-minus*: $a + - a = 0$
⟨*proof*⟩

lemma *diff-self* [*simp*]: $a - a = 0$
⟨*proof*⟩

subclass *cancel-semigroup-add*
⟨*proof*⟩

lemma *minus-add-cancel* [*simp*]: $- a + (a + b) = b$
⟨*proof*⟩

lemma *add-minus-cancel* [*simp*]: $a + (- a + b) = b$
⟨*proof*⟩

lemma *diff-add-cancel* [*simp*]: $a - b + b = a$
⟨*proof*⟩

lemma *add-diff-cancel* [*simp*]: $a + b - b = a$
⟨*proof*⟩

lemma *minus-add*: $- (a + b) = - b + - a$
⟨*proof*⟩

lemma *right-minus-eq* [*simp*]: $a - b = 0 \longleftrightarrow a = b$
⟨*proof*⟩

lemma *eq-iff-diff-eq-0*: $a = b \longleftrightarrow a - b = 0$
⟨*proof*⟩

lemma *diff-0* [*simp*]: $0 - a = - a$
⟨*proof*⟩

lemma *diff-0-right* [*simp*]: $a - 0 = a$
⟨*proof*⟩

lemma *diff-minus-eq-add* [*simp*]: $a - - b = a + b$
⟨*proof*⟩

lemma *neg-equal-iff-equal* [*simp*]: $- a = - b \longleftrightarrow a = b$
⟨*proof*⟩

lemma *neg-equal-0-iff-equal* [*simp*]: $- a = 0 \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *neg-0-equal-iff-equal* [*simp*]: $0 = - a \longleftrightarrow 0 = a$
⟨*proof*⟩

The next two equations can make the simplifier loop!

lemma *equation-minus-iff*: $a = - b \longleftrightarrow b = - a$
⟨*proof*⟩

lemma *minus-equation-iff*: $- a = b \longleftrightarrow - b = a$
⟨*proof*⟩

lemma *eq-neg-iff-add-eq-0*: $a = - b \longleftrightarrow a + b = 0$
⟨*proof*⟩

lemma *add-eq-0-iff2*: $a + b = 0 \longleftrightarrow a = - b$
⟨*proof*⟩

lemma *neg-eq-iff-add-eq-0*: $- a = b \longleftrightarrow a + b = 0$
⟨*proof*⟩

lemma *add-eq-0-iff*: $a + b = 0 \longleftrightarrow b = - a$
⟨*proof*⟩

lemma *minus-diff-eq* [*simp*]: $- (a - b) = b - a$
⟨*proof*⟩

lemma *add-diff-eq* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
$a + (b - c) = (a + b) - c$
⟨*proof*⟩

lemma *diff-add-eq-diff-diff-swap*: $a - (b + c) = a - c - b$
⟨*proof*⟩

lemma *diff-eq-eq* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
$a - b = c \longleftrightarrow a = c + b$
⟨*proof*⟩

lemma *eq-diff-eq* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
$a = c - b \longleftrightarrow a + b = c$
⟨*proof*⟩

lemma *diff-diff-eq2* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
$a - (b - c) = (a + c) - b$
⟨*proof*⟩

lemma *diff-eq-diff-eq*: $a - b = c - d \Longrightarrow a = b \longleftrightarrow c = d$
⟨*proof*⟩

end

class *ab-group-add* = *minus* + *uminus* + *comm-monoid-add* +
assumes *ab-left-minus*: $- a + a = 0$
assumes *ab-diff-conv-add-uminus*: $a - b = a + (- b)$
begin

subclass *group-add*
⟨*proof*⟩

subclass *cancel-comm-monoid-add*
⟨*proof*⟩

lemma *uminus-add-conv-diff* [*simp*]: $- a + b = b - a$

⟨*proof*⟩

lemma *minus-add-distrib* [*simp*]: $-(a + b) = -a + -b$
⟨*proof*⟩

lemma *diff-add-eq* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
$(a - b) + c = (a + c) - b$
⟨*proof*⟩

lemma *minus-diff-commute*:
$-b - a = -a - b$
⟨*proof*⟩

end

5.6 (Partially) Ordered Groups

The theory of partially ordered groups is taken from the books:

- *Lattice Theory* by Garret Birkhoff, American Mathematical Society, 1979
- *Partially Ordered Algebraic Systems*, Pergamon Press, 1963

Most of the used notions can also be looked up in

- http://www.mathworld.com by Eric Weisstein et. al.
- *Algebra I* by van der Waerden, Springer

class *ordered-ab-semigroup-add* = *order* + *ab-semigroup-add* +
assumes *add-left-mono*: $a \leq b \Longrightarrow c + a \leq c + b$
begin

lemma *add-right-mono*: $a \leq b \Longrightarrow a + c \leq b + c$
⟨*proof*⟩

non-strict, in both arguments

lemma *add-mono*: $a \leq b \Longrightarrow c \leq d \Longrightarrow a + c \leq b + d$
⟨*proof*⟩

end

Strict monotonicity in both arguments

class *strict-ordered-ab-semigroup-add* = *ordered-ab-semigroup-add* +
assumes *add-strict-mono*: $a < b \Longrightarrow c < d \Longrightarrow a + c < b + d$

class *ordered-cancel-ab-semigroup-add* =

ordered-ab-semigroup-add + *cancel-ab-semigroup-add*
begin

lemma *add-strict-left-mono*: $a < b \Longrightarrow c + a < c + b$
⟨*proof*⟩

lemma *add-strict-right-mono*: $a < b \Longrightarrow a + c < b + c$
⟨*proof*⟩

subclass *strict-ordered-ab-semigroup-add*
⟨*proof*⟩

lemma *add-less-le-mono*: $a < b \Longrightarrow c \leq d \Longrightarrow a + c < b + d$
⟨*proof*⟩

lemma *add-le-less-mono*: $a \leq b \Longrightarrow c < d \Longrightarrow a + c < b + d$
⟨*proof*⟩

end

class *ordered-ab-semigroup-add-imp-le* = *ordered-cancel-ab-semigroup-add* +
assumes *add-le-imp-le-left*: $c + a \leq c + b \Longrightarrow a \leq b$
begin

lemma *add-less-imp-less-left*:
assumes *less*: $c + a < c + b$
shows $a < b$
⟨*proof*⟩

lemma *add-less-imp-less-right*: $a + c < b + c \Longrightarrow a < b$
⟨*proof*⟩

lemma *add-less-cancel-left* [*simp*]: $c + a < c + b \longleftrightarrow a < b$
⟨*proof*⟩

lemma *add-less-cancel-right* [*simp*]: $a + c < b + c \longleftrightarrow a < b$
⟨*proof*⟩

lemma *add-le-cancel-left* [*simp*]: $c + a \leq c + b \longleftrightarrow a \leq b$
⟨*proof*⟩

lemma *add-le-cancel-right* [*simp*]: $a + c \leq b + c \longleftrightarrow a \leq b$
⟨*proof*⟩

lemma *add-le-imp-le-right*: $a + c \leq b + c \Longrightarrow a \leq b$
⟨*proof*⟩

lemma *max-add-distrib-left*: $max\ x\ y + z = max\ (x + z)\ (y + z)$
⟨*proof*⟩

lemma *min-add-distrib-left*: $min\ x\ y + z = min\ (x + z)\ (y + z)$
⟨*proof*⟩

lemma *max-add-distrib-right*: $x + max\ y\ z = max\ (x + y)\ (x + z)$
⟨*proof*⟩

lemma *min-add-distrib-right*: $x + min\ y\ z = min\ (x + y)\ (x + z)$
⟨*proof*⟩

end

5.7 Support for reasoning about signs

class *ordered-comm-monoid-add* = *comm-monoid-add* + *ordered-ab-semigroup-add*
begin

lemma *add-nonneg-nonneg* [*simp*]: $0 \leq a \Longrightarrow 0 \leq b \Longrightarrow 0 \leq a + b$
⟨*proof*⟩

lemma *add-nonpos-nonpos*: $a \leq 0 \Longrightarrow b \leq 0 \Longrightarrow a + b \leq 0$
⟨*proof*⟩

lemma *add-nonneg-eq-0-iff*: $0 \leq x \Longrightarrow 0 \leq y \Longrightarrow x + y = 0 \longleftrightarrow x = 0 \wedge y = 0$
⟨*proof*⟩

lemma *add-nonpos-eq-0-iff*: $x \leq 0 \Longrightarrow y \leq 0 \Longrightarrow x + y = 0 \longleftrightarrow x = 0 \wedge y = 0$
⟨*proof*⟩

lemma *add-increasing*: $0 \leq a \Longrightarrow b \leq c \Longrightarrow b \leq a + c$
⟨*proof*⟩

lemma *add-increasing2*: $0 \leq c \Longrightarrow b \leq a \Longrightarrow b \leq a + c$
⟨*proof*⟩

lemma *add-decreasing*: $a \leq 0 \Longrightarrow c \leq b \Longrightarrow a + c \leq b$
⟨*proof*⟩

lemma *add-decreasing2*: $c \leq 0 \Longrightarrow a \leq b \Longrightarrow a + c \leq b$
⟨*proof*⟩

lemma *add-pos-nonneg*: $0 < a \Longrightarrow 0 \leq b \Longrightarrow 0 < a + b$
⟨*proof*⟩

lemma *add-pos-pos*: $0 < a \Longrightarrow 0 < b \Longrightarrow 0 < a + b$
⟨*proof*⟩

lemma *add-nonneg-pos*: $0 \leq a \Longrightarrow 0 < b \Longrightarrow 0 < a + b$
⟨*proof*⟩

lemma *add-neg-nonpos*: $a < 0 \Longrightarrow b \leq 0 \Longrightarrow a + b < 0$
⟨*proof*⟩

lemma *add-neg-neg*: $a < 0 \Longrightarrow b < 0 \Longrightarrow a + b < 0$
⟨*proof*⟩

lemma *add-nonpos-neg*: $a \leq 0 \Longrightarrow b < 0 \Longrightarrow a + b < 0$
⟨*proof*⟩

lemmas *add-sign-intros* =
add-pos-nonneg add-pos-pos add-nonneg-pos add-nonneg-nonneg
add-neg-nonpos add-neg-neg add-nonpos-neg add-nonpos-nonpos

end

class *strict-ordered-comm-monoid-add* = *comm-monoid-add* + *strict-ordered-ab-semigroup-add*
begin

lemma *pos-add-strict*: $0 < a \Longrightarrow b < c \Longrightarrow b < a + c$
⟨*proof*⟩

end

class *ordered-cancel-comm-monoid-add* = *ordered-comm-monoid-add* + *cancel-ab-semigroup-add*
begin

subclass *ordered-cancel-ab-semigroup-add* ⟨*proof*⟩
subclass *strict-ordered-comm-monoid-add* ⟨*proof*⟩

lemma *add-strict-increasing*: $0 < a \Longrightarrow b \leq c \Longrightarrow b < a + c$
⟨*proof*⟩

lemma *add-strict-increasing2*: $0 \leq a \Longrightarrow b < c \Longrightarrow b < a + c$
⟨*proof*⟩

end

class *ordered-ab-semigroup-monoid-add-imp-le* = *monoid-add* + *ordered-ab-semigroup-add-imp-le*
begin

lemma *add-less-same-cancel1* [*simp*]: $b + a < b \longleftrightarrow a < 0$
⟨*proof*⟩

lemma *add-less-same-cancel2* [*simp*]: $a + b < b \longleftrightarrow a < 0$
⟨*proof*⟩

lemma *less-add-same-cancel1* [*simp*]: $a < a + b \longleftrightarrow 0 < b$
⟨*proof*⟩

lemma *less-add-same-cancel2* [*simp*]: $a < b + a \longleftrightarrow 0 < b$
⟨*proof*⟩

lemma *add-le-same-cancel1* [*simp*]: $b + a \leq b \longleftrightarrow a \leq 0$
⟨*proof*⟩

lemma *add-le-same-cancel2* [*simp*]: $a + b \leq b \longleftrightarrow a \leq 0$
⟨*proof*⟩

lemma *le-add-same-cancel1* [*simp*]: $a \leq a + b \longleftrightarrow 0 \leq b$
⟨*proof*⟩

lemma *le-add-same-cancel2* [*simp*]: $a \leq b + a \longleftrightarrow 0 \leq b$
⟨*proof*⟩

subclass *cancel-comm-monoid-add*
⟨*proof*⟩

subclass *ordered-cancel-comm-monoid-add*
⟨*proof*⟩

end

class *ordered-ab-group-add* = *ab-group-add* + *ordered-ab-semigroup-add*
begin

subclass *ordered-cancel-ab-semigroup-add* ⟨*proof*⟩

subclass *ordered-ab-semigroup-monoid-add-imp-le*
⟨*proof*⟩

lemma *max-diff-distrib-left*: $max\ x\ y - z = max\ (x - z)\ (y - z)$
⟨*proof*⟩

lemma *min-diff-distrib-left*: $min\ x\ y - z = min\ (x - z)\ (y - z)$
⟨*proof*⟩

lemma *le-imp-neg-le*:
assumes $a \leq b$
shows $- b \leq - a$
⟨*proof*⟩

lemma *neg-le-iff-le* [*simp*]: $- b \leq - a \longleftrightarrow a \leq b$
⟨*proof*⟩

lemma *neg-le-0-iff-le* [*simp*]: $- a \leq 0 \longleftrightarrow 0 \leq a$
⟨*proof*⟩

lemma *neg-0-le-iff-le* [*simp*]: $0 \leq - a \longleftrightarrow a \leq 0$
⟨*proof*⟩

lemma *neg-less-iff-less* [*simp*]: $- b < - a \longleftrightarrow a < b$
⟨*proof*⟩

lemma *neg-less-0-iff-less* [*simp*]: $- a < 0 \longleftrightarrow 0 < a$
⟨*proof*⟩

lemma *neg-0-less-iff-less* [*simp*]: $0 < - a \longleftrightarrow a < 0$
⟨*proof*⟩

The next several equations can make the simplifier loop!

lemma *less-minus-iff*: $a < - b \longleftrightarrow b < - a$
⟨*proof*⟩

lemma *minus-less-iff*: $- a < b \longleftrightarrow - b < a$
⟨*proof*⟩

lemma *le-minus-iff*: $a \leq - b \longleftrightarrow b \leq - a$
⟨*proof*⟩

lemma *minus-le-iff*: $- a \leq b \longleftrightarrow - b \leq a$
⟨*proof*⟩

lemma *diff-less-0-iff-less* [*simp*]: $a - b < 0 \longleftrightarrow a < b$
⟨*proof*⟩

lemmas *less-iff-diff-less-0* = *diff-less-0-iff-less* [*symmetric*]

lemma *diff-less-eq* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
$a - b < c \longleftrightarrow a < c + b$
⟨*proof*⟩

lemma *less-diff-eq* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
$a < c - b \longleftrightarrow a + b < c$
⟨*proof*⟩

lemma *diff-gt-0-iff-gt* [*simp*]: $a - b > 0 \longleftrightarrow a > b$
⟨*proof*⟩

lemma *diff-le-eq* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
$a - b \leq c \longleftrightarrow a \leq c + b$
⟨*proof*⟩

lemma *le-diff-eq* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*]:
$a \leq c - b \longleftrightarrow a + b \leq c$
⟨*proof*⟩

lemma *diff-le-0-iff-le* [*simp*]: $a - b \leq 0 \longleftrightarrow a \leq b$
⟨*proof*⟩

lemmas *le-iff-diff-le-0* = *diff-le-0-iff-le* [*symmetric*]

lemma *diff-ge-0-iff-ge* [*simp*]: $a - b \geq 0 \longleftrightarrow a \geq b$
⟨*proof*⟩

lemma *diff-eq-diff-less*: $a - b = c - d \Longrightarrow a < b \longleftrightarrow c < d$
⟨*proof*⟩

lemma *diff-eq-diff-less-eq*: $a - b = c - d \Longrightarrow a \leq b \longleftrightarrow c \leq d$
⟨*proof*⟩

lemma *diff-mono*: $a \leq b \Longrightarrow d \leq c \Longrightarrow a - c \leq b - d$
⟨*proof*⟩

lemma *diff-left-mono*: $b \leq a \Longrightarrow c - a \leq c - b$
⟨*proof*⟩

lemma *diff-right-mono*: $a \leq b \Longrightarrow a - c \leq b - c$
⟨*proof*⟩

lemma *diff-strict-mono*: $a < b \Longrightarrow d < c \Longrightarrow a - c < b - d$
⟨*proof*⟩

lemma *diff-strict-left-mono*: $b < a \Longrightarrow c - a < c - b$
⟨*proof*⟩

lemma *diff-strict-right-mono*: $a < b \Longrightarrow a - c < b - c$
⟨*proof*⟩

end

locale *group-cancel*
begin

lemma *add1*: $(A::'a::comm\text{-}monoid\text{-}add) \equiv k + a \Longrightarrow A + b \equiv k + (a + b)$
⟨*proof*⟩

lemma *add2*: $(B::'a::comm\text{-}monoid\text{-}add) \equiv k + b \Longrightarrow a + B \equiv k + (a + b)$
⟨*proof*⟩

lemma *sub1*: $(A::'a::ab\text{-}group\text{-}add) \equiv k + a \Longrightarrow A - b \equiv k + (a - b)$
⟨*proof*⟩

lemma *sub2*: $(B::'a::ab\text{-}group\text{-}add) \equiv k + b \Longrightarrow a - B \equiv - k + (a - b)$
⟨*proof*⟩

lemma *neg1*: $(A::'a::ab\text{-}group\text{-}add) \equiv k + a \Longrightarrow - A \equiv - k + - a$
⟨*proof*⟩

lemma *rule0*: $(a::'a::comm\text{-}monoid\text{-}add) \equiv a + 0$
⟨*proof*⟩

end

⟨*ML*⟩

class *linordered-ab-semigroup-add* =
linorder + *ordered-ab-semigroup-add*

class *linordered-cancel-ab-semigroup-add* =
linorder + *ordered-cancel-ab-semigroup-add*
begin

subclass *linordered-ab-semigroup-add* ⟨*proof*⟩

subclass *ordered-ab-semigroup-add-imp-le*
⟨*proof*⟩

end

class *linordered-ab-group-add* = *linorder* + *ordered-ab-group-add*
begin

subclass *linordered-cancel-ab-semigroup-add* ⟨*proof*⟩

lemma *equal-neg-zero* [*simp*]: $a = - a \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *neg-equal-zero* [*simp*]: $- a = a \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *neg-less-eq-nonneg* [*simp*]: $- a \leq a \longleftrightarrow 0 \leq a$
⟨*proof*⟩

lemma *neg-less-pos* [*simp*]: $- a < a \longleftrightarrow 0 < a$
⟨*proof*⟩

lemma *less-eq-neg-nonpos* [*simp*]: $a \leq - a \longleftrightarrow a \leq 0$
⟨*proof*⟩

lemma *less-neg-neg* [*simp*]: $a < - a \longleftrightarrow a < 0$
⟨*proof*⟩

lemma *double-zero* [*simp*]: $a + a = 0 \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *double-zero-sym* [*simp*]: $0 = a + a \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *zero-less-double-add-iff-zero-less-single-add* [*simp*]: $0 < a + a \longleftrightarrow 0 < a$
⟨*proof*⟩

lemma *zero-le-double-add-iff-zero-le-single-add* [*simp*]: $0 \leq a + a \longleftrightarrow 0 \leq a$
⟨*proof*⟩

lemma *double-add-less-zero-iff-single-add-less-zero* [*simp*]: $a + a < 0 \longleftrightarrow a < 0$
⟨*proof*⟩

lemma *double-add-le-zero-iff-single-add-le-zero* [*simp*]: $a + a \leq 0 \longleftrightarrow a \leq 0$
⟨*proof*⟩

lemma *minus-max-eq-min*: $- \max x\ y = \min\ (-x)\ (-y)$
⟨*proof*⟩

lemma *minus-min-eq-max*: $- \min x\ y = \max\ (-x)\ (-y)$
⟨*proof*⟩

end

class *abs* =
fixes *abs* :: $'a \Rightarrow 'a$ (|-|)

class *sgn* =
fixes *sgn* :: $'a \Rightarrow 'a$

class *ordered-ab-group-add-abs* = *ordered-ab-group-add* + *abs* +
assumes *abs-ge-zero* [*simp*]: $|a| \geq 0$
and *abs-ge-self*: $a \leq |a|$
and *abs-leI*: $a \leq b \Longrightarrow -a \leq b \Longrightarrow |a| \leq b$
and *abs-minus-cancel* [*simp*]: $|-a| = |a|$
and *abs-triangle-ineq*: $|a + b| \leq |a| + |b|$
begin

lemma *abs-minus-le-zero*: $-|a| \leq 0$
⟨*proof*⟩

lemma *abs-of-nonneg* [*simp*]:
assumes *nonneg*: $0 \leq a$
shows $|a| = a$
⟨*proof*⟩

lemma *abs-idempotent* [*simp*]: $||a|| = |a|$
⟨*proof*⟩

lemma *abs-eq-0* [*simp*]: $|a| = 0 \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *abs-zero* [*simp*]: $|0| = 0$
⟨*proof*⟩

lemma *abs-0-eq* [*simp*]: $0 = |a| \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *abs-le-zero-iff* [*simp*]: $|a| \leq 0 \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *abs-le-self-iff* [*simp*]: $|a| \leq a \longleftrightarrow 0 \leq a$
⟨*proof*⟩

lemma *zero-less-abs-iff* [*simp*]: $0 < |a| \longleftrightarrow a \neq 0$
⟨*proof*⟩

lemma *abs-not-less-zero* [*simp*]: $\neg\ |a| < 0$
⟨*proof*⟩

lemma *abs-ge-minus-self*: $-\ a \leq |a|$
⟨*proof*⟩

lemma *abs-minus-commute*: $|a - b| = |b - a|$
⟨*proof*⟩

lemma *abs-of-pos*: $0 < a \Longrightarrow |a| = a$
⟨*proof*⟩

lemma *abs-of-nonpos* [*simp*]:
assumes $a \leq 0$
shows $|a| = -\ a$
⟨*proof*⟩

lemma *abs-of-neg*: $a < 0 \Longrightarrow |a| = -\ a$
⟨*proof*⟩

lemma *abs-le-D1*: $|a| \leq b \Longrightarrow a \leq b$
⟨*proof*⟩

lemma *abs-le-D2*: $|a| \leq b \Longrightarrow -\ a \leq b$
⟨*proof*⟩

lemma *abs-le-iff*: $|a| \leq b \longleftrightarrow a \leq b \wedge -\ a \leq b$
⟨*proof*⟩

lemma *abs-triangle-ineq2*: $|a| - |b| \leq |a - b|$
⟨*proof*⟩

lemma *abs-triangle-ineq2-sym*: $|a| - |b| \leq |b - a|$
⟨*proof*⟩

lemma *abs-triangle-ineq3*: $||a| - |b|| \leq |a - b|$
⟨*proof*⟩

lemma *abs-triangle-ineq4*: $|a - b| \leq |a| + |b|$
⟨*proof*⟩

lemma *abs-diff-triangle-ineq*: $|a + b - (c + d)| \leq |a - c| + |b - d|$
⟨*proof*⟩

lemma *abs-add-abs* [*simp*]: $||a| + |b|| = |a| + |b|$
(**is** *?L = ?R*)
⟨*proof*⟩

end

lemma *dense-eq0-I*:
fixes *x*::'*a*::{*dense-linorder*,*ordered-ab-group-add-abs*}
assumes $\bigwedge e.\ 0 < e \Longrightarrow |x| \leq e$
shows $x = 0$
⟨*proof*⟩

hide-fact (**open**) *ab-diff-conv-add-uminus add-0 mult-1 ab-left-minus*

lemmas *add-0 = add-0-left*
lemmas *mult-1 = mult-1-left*
lemmas *ab-left-minus = left-minus*
lemmas *diff-diff-eq = diff-diff-add*

5.8 Canonically ordered monoids

Canonically ordered monoids are never groups.

class *canonically-ordered-monoid-add = comm-monoid-add + order +*
assumes *le-iff-add*: $a \leq b \longleftrightarrow (\exists\, c.\ b = a + c)$
begin

lemma *zero-le*[*simp*]: $0 \leq x$
⟨*proof*⟩

lemma *le-zero-eq*[*simp*]: $n \leq 0 \longleftrightarrow n = 0$
⟨*proof*⟩

lemma *not-less-zero*[*simp*]: $\neg\ n < 0$
⟨*proof*⟩

lemma *zero-less-iff-neq-zero*: $0 < n \longleftrightarrow n \neq 0$

⟨*proof*⟩

This theorem is useful with *blast*

lemma *gr-zeroI*: $(n = 0 \Longrightarrow \mathit{False}) \Longrightarrow 0 < n$
⟨*proof*⟩

lemma *not-gr-zero*[*simp*]: $\neg\ 0 < n \longleftrightarrow n = 0$
⟨*proof*⟩

subclass *ordered-comm-monoid-add*
⟨*proof*⟩

lemma *gr-implies-not-zero*: $m < n \Longrightarrow n \neq 0$
⟨*proof*⟩

lemma *add-eq-0-iff-both-eq-0*[*simp*]: $x + y = 0 \longleftrightarrow x = 0 \wedge y = 0$
⟨*proof*⟩

lemma *zero-eq-add-iff-both-eq-0*[*simp*]: $0 = x + y \longleftrightarrow x = 0 \wedge y = 0$
⟨*proof*⟩

lemma *less-eqE*:
assumes ‹$a \leq b$›
obtains c **where** ‹$b = a + c$›
⟨*proof*⟩

lemma *lessE*:
assumes ‹$a < b$›
obtains c **where** ‹$b = a + c$› **and** ‹$c \neq 0$›
⟨*proof*⟩

lemmas *zero-order* = *zero-le le-zero-eq not-less-zero zero-less-iff-neq-zero not-gr-zero*
— This should be attributed with [*iff*], but then *blast* fails in *Set*.

end

class *ordered-cancel-comm-monoid-diff* =
canonically-ordered-monoid-add + *comm-monoid-diff* + *ordered-ab-semigroup-add-imp-le*
begin

context
fixes $a\ b :: 'a$
assumes *le*: $a \leq b$
begin

lemma *add-diff-inverse*: $a + (b - a) = b$
⟨*proof*⟩

lemma *add-diff-assoc*: $c + (b - a) = c + b - a$

⟨*proof*⟩

lemma *add-diff-assoc2*: $b - a + c = b + c - a$
⟨*proof*⟩

lemma *diff-add-assoc*: $c + b - a = c + (b - a)$
⟨*proof*⟩

lemma *diff-add-assoc2*: $b + c - a = b - a + c$
⟨*proof*⟩

lemma *diff-diff-right*: $c - (b - a) = c + a - b$
⟨*proof*⟩

lemma *diff-add*: $b - a + a = b$
⟨*proof*⟩

lemma *le-add-diff*: $c \leq b + c - a$
⟨*proof*⟩

lemma *le-imp-diff-is-add*: $a \leq b \Longrightarrow b - a = c \longleftrightarrow b = c + a$
⟨*proof*⟩

lemma *le-diff-conv2*: $c \leq b - a \longleftrightarrow c + a \leq b$
(**is** *?P* $\longleftrightarrow$ *?Q*)
⟨*proof*⟩

end

end

5.9 Tools setup

lemma *add-mono-thms-linordered-semiring*:
fixes $i\ j\ k$ :: $'a$::*ordered-ab-semigroup-add*
shows $i \leq j \wedge k \leq l \Longrightarrow i + k \leq j + l$
and $i = j \wedge k \leq l \Longrightarrow i + k \leq j + l$
and $i \leq j \wedge k = l \Longrightarrow i + k \leq j + l$
and $i = j \wedge k = l \Longrightarrow i + k = j + l$
⟨*proof*⟩

lemma *add-mono-thms-linordered-field*:
fixes $i\ j\ k$ :: $'a$::*ordered-cancel-ab-semigroup-add*
shows $i < j \wedge k = l \Longrightarrow i + k < j + l$
and $i = j \wedge k < l \Longrightarrow i + k < j + l$
and $i < j \wedge k \leq l \Longrightarrow i + k < j + l$
and $i \leq j \wedge k < l \Longrightarrow i + k < j + l$
and $i < j \wedge k < l \Longrightarrow i + k < j + l$
⟨*proof*⟩

code-identifier
code-module *Groups* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

end

6 Abstract lattices

theory *Lattices*
imports *Groups*
begin

6.1 Abstract semilattice

These locales provide a basic structure for interpretation into bigger structures; extensions require careful thinking, otherwise undesired effects may occur due to interpretation.

locale *semilattice* = *abel-semigroup* +
assumes *idem* [*simp*]: $a * a = a$
begin

lemma *left-idem* [*simp*]: $a * (a * b) = a * b$
⟨*proof*⟩

lemma *right-idem* [*simp*]: $(a * b) * b = a * b$
⟨*proof*⟩

end

locale *semilattice-neutr* = *semilattice* + *comm-monoid*

locale *semilattice-order* = *semilattice* +
fixes *less-eq* :: $'a \Rightarrow 'a \Rightarrow bool$ (**infix** $\leq$ *50*)
and *less* :: $'a \Rightarrow 'a \Rightarrow bool$ (**infix** $<$ *50*)
assumes *order-iff*: $a \leq b \longleftrightarrow a = a * b$
and *strict-order-iff*: $a < b \longleftrightarrow a = a * b \wedge a \neq b$
begin

lemma *orderI*: $a = a * b \Longrightarrow a \leq b$
⟨*proof*⟩

lemma *orderE*:
assumes $a \leq b$
obtains $a = a * b$
⟨*proof*⟩

sublocale *ordering less-eq less*
⟨*proof*⟩

lemma *cobounded1* [*simp*]: $a * b \leq a$
⟨*proof*⟩

lemma *cobounded2* [*simp*]: $a * b \leq b$
⟨*proof*⟩

lemma *boundedI*:
assumes $a \leq b$ **and** $a \leq c$
shows $a \leq b * c$
⟨*proof*⟩

lemma *boundedE*:
assumes $a \leq b * c$
obtains $a \leq b$ **and** $a \leq c$
⟨*proof*⟩

lemma *bounded-iff* [*simp*]: $a \leq b * c \longleftrightarrow a \leq b \wedge a \leq c$
⟨*proof*⟩

lemma *strict-boundedE*:
assumes $a < b * c$
obtains $a < b$ **and** $a < c$
⟨*proof*⟩

lemma *coboundedI1*: $a \leq c \Longrightarrow a * b \leq c$
⟨*proof*⟩

lemma *coboundedI2*: $b \leq c \Longrightarrow a * b \leq c$
⟨*proof*⟩

lemma *strict-coboundedI1*: $a < c \Longrightarrow a * b < c$
⟨*proof*⟩

lemma *strict-coboundedI2*: $b < c \Longrightarrow a * b < c$
⟨*proof*⟩

lemma *mono*: $a \leq c \Longrightarrow b \leq d \Longrightarrow a * b \leq c * d$
⟨*proof*⟩

lemma *absorb1*: $a \leq b \Longrightarrow a * b = a$
⟨*proof*⟩

lemma *absorb2*: $b \leq a \Longrightarrow a * b = b$
⟨*proof*⟩

lemma *absorb3*: $a < b \Longrightarrow a * b = a$
⟨*proof*⟩

lemma *absorb4*: $b < a \Longrightarrow a * b = b$
⟨*proof*⟩

lemma *absorb-iff1*: $a \leq b \longleftrightarrow a * b = a$
⟨*proof*⟩

lemma *absorb-iff2*: $b \leq a \longleftrightarrow a * b = b$
⟨*proof*⟩

end

locale *semilattice-neutr-order* = *semilattice-neutr* + *semilattice-order*
begin

sublocale *ordering-top less-eq less* **1**
⟨*proof*⟩

lemma *eq-neutr-iff* [*simp*]: ‹$a * b = \mathbf{1} \longleftrightarrow a = \mathbf{1} \wedge b = \mathbf{1}$›
⟨*proof*⟩

lemma *neutr-eq-iff* [*simp*]: ‹$\mathbf{1} = a * b \longleftrightarrow a = \mathbf{1} \wedge b = \mathbf{1}$›
⟨*proof*⟩

end

Interpretations for boolean operators

interpretation *conj*: *semilattice-neutr* ‹(∧)› *True*
⟨*proof*⟩

interpretation *disj*: *semilattice-neutr* ‹(∨)› *False*
⟨*proof*⟩

declare *conj-assoc* [*ac-simps del*] *disj-assoc* [*ac-simps del*] — already simp by default

6.2 Syntactic infimum and supremum operations

class *inf* =
fixes *inf* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** $\sqcap$ *70*)

class *sup* =
fixes *sup* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** $\sqcup$ *65*)

6.3 Concrete lattices

class *semilattice-inf* = *order* + *inf* +
assumes *inf-le1* [*simp*]: $x \sqcap y \leq x$
and *inf-le2* [*simp*]: $x \sqcap y \leq y$
and *inf-greatest*: $x \leq y \Longrightarrow x \leq z \Longrightarrow x \leq y \sqcap z$

class *semilattice-sup* = *order* + *sup* +
  assumes *sup-ge1* [*simp*]: $x \leq x \sqcup y$
  and *sup-ge2* [*simp*]: $y \leq x \sqcup y$
  and *sup-least*: $y \leq x \Longrightarrow z \leq x \Longrightarrow y \sqcup z \leq x$
begin

Dual lattice.

lemma *dual-semilattice*: *class.semilattice-inf sup greater-eq greater*
  ⟨*proof*⟩

end

class *lattice* = *semilattice-inf* + *semilattice-sup*

6.3.1 Intro and elim rules

context *semilattice-inf*
begin

lemma *le-infI1*: $a \leq x \Longrightarrow a \sqcap b \leq x$
  ⟨*proof*⟩

lemma *le-infI2*: $b \leq x \Longrightarrow a \sqcap b \leq x$
  ⟨*proof*⟩

lemma *le-infI*: $x \leq a \Longrightarrow x \leq b \Longrightarrow x \leq a \sqcap b$
  ⟨*proof*⟩

lemma *le-infE*: $x \leq a \sqcap b \Longrightarrow (x \leq a \Longrightarrow x \leq b \Longrightarrow P) \Longrightarrow P$
  ⟨*proof*⟩

lemma *le-inf-iff*: $x \leq y \sqcap z \longleftrightarrow x \leq y \wedge x \leq z$
  ⟨*proof*⟩

lemma *le-iff-inf*: $x \leq y \longleftrightarrow x \sqcap y = x$
  ⟨*proof*⟩

lemma *inf-mono*: $a \leq c \Longrightarrow b \leq d \Longrightarrow a \sqcap b \leq c \sqcap d$
  ⟨*proof*⟩

end

context *semilattice-sup*
begin

lemma *le-supI1*: $x \leq a \Longrightarrow x \leq a \sqcup b$
  ⟨*proof*⟩

lemma *le-supI2*: $x \leq b \Longrightarrow x \leq a \sqcup b$
⟨*proof*⟩

lemma *le-supI*: $a \leq x \Longrightarrow b \leq x \Longrightarrow a \sqcup b \leq x$
⟨*proof*⟩

lemma *le-supE*: $a \sqcup b \leq x \Longrightarrow (a \leq x \Longrightarrow b \leq x \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *le-sup-iff*: $x \sqcup y \leq z \longleftrightarrow x \leq z \wedge y \leq z$
⟨*proof*⟩

lemma *le-iff-sup*: $x \leq y \longleftrightarrow x \sqcup y = y$
⟨*proof*⟩

lemma *sup-mono*: $a \leq c \Longrightarrow b \leq d \Longrightarrow a \sqcup b \leq c \sqcup d$
⟨*proof*⟩

end

6.3.2 Equational laws

context *semilattice-inf*
begin

sublocale *inf*: *semilattice inf*
⟨*proof*⟩

sublocale *inf*: *semilattice-order inf less-eq less*
⟨*proof*⟩

lemma *inf-assoc*: $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
⟨*proof*⟩

lemma *inf-commute*: $(x \sqcap y) = (y \sqcap x)$
⟨*proof*⟩

lemma *inf-left-commute*: $x \sqcap (y \sqcap z) = y \sqcap (x \sqcap z)$
⟨*proof*⟩

lemma *inf-idem*: $x \sqcap x = x$
⟨*proof*⟩

lemma *inf-left-idem*: $x \sqcap (x \sqcap y) = x \sqcap y$
⟨*proof*⟩

lemma *inf-right-idem*: $(x \sqcap y) \sqcap y = x \sqcap y$
⟨*proof*⟩

lemma *inf-absorb1*: $x \leq y \Longrightarrow x \sqcap y = x$
⟨*proof*⟩

lemma *inf-absorb2*: $y \leq x \Longrightarrow x \sqcap y = y$
⟨*proof*⟩

lemmas *inf-aci* = *inf-commute inf-assoc inf-left-commute inf-left-idem*

end

context *semilattice-sup*
begin

sublocale *sup*: *semilattice sup*
⟨*proof*⟩

sublocale *sup*: *semilattice-order sup greater-eq greater*
⟨*proof*⟩

lemma *sup-assoc*: $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
⟨*proof*⟩

lemma *sup-commute*: $(x \sqcup y) = (y \sqcup x)$
⟨*proof*⟩

lemma *sup-left-commute*: $x \sqcup (y \sqcup z) = y \sqcup (x \sqcup z)$
⟨*proof*⟩

lemma *sup-idem*: $x \sqcup x = x$
⟨*proof*⟩

lemma *sup-left-idem* [*simp*]: $x \sqcup (x \sqcup y) = x \sqcup y$
⟨*proof*⟩

lemma *sup-absorb1*: $y \leq x \Longrightarrow x \sqcup y = x$
⟨*proof*⟩

lemma *sup-absorb2*: $x \leq y \Longrightarrow x \sqcup y = y$
⟨*proof*⟩

lemmas *sup-aci* = *sup-commute sup-assoc sup-left-commute sup-left-idem*

end

context *lattice*
begin

lemma *dual-lattice*: *class.lattice sup* $(\geq)$ $(>)$ *inf*
⟨*proof*⟩

lemma *inf-sup-absorb* [*simp*]: $x \sqcap (x \sqcup y) = x$
⟨*proof*⟩

lemma *sup-inf-absorb* [*simp*]: $x \sqcup (x \sqcap y) = x$
⟨*proof*⟩

lemmas *inf-sup-aci* = *inf-aci sup-aci*

lemmas *inf-sup-ord* = *inf-le1 inf-le2 sup-ge1 sup-ge2*

Towards distributivity.

lemma *distrib-sup-le*: $x \sqcup (y \sqcap z) \leq (x \sqcup y) \sqcap (x \sqcup z)$
⟨*proof*⟩

lemma *distrib-inf-le*: $(x \sqcap y) \sqcup (x \sqcap z) \leq x \sqcap (y \sqcup z)$
⟨*proof*⟩

If you have one of them, you have them all.

lemma *distrib-imp1*:
assumes *distrib*: $\bigwedge x\ y\ z.\ x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$
shows $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$
⟨*proof*⟩

lemma *distrib-imp2*:
assumes *distrib*: $\bigwedge x\ y\ z.\ x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$
shows $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$
⟨*proof*⟩

end

6.3.3 Strict order

context *semilattice-inf*
begin

lemma *less-infI1*: $a < x \Longrightarrow a \sqcap b < x$
⟨*proof*⟩

lemma *less-infI2*: $b < x \Longrightarrow a \sqcap b < x$
⟨*proof*⟩

end

context *semilattice-sup*
begin

lemma *less-supI1*: $x < a \Longrightarrow x < a \sqcup b$
⟨*proof*⟩

lemma *less-supI2*: $x < b \Longrightarrow x < a \sqcup b$
⟨*proof*⟩

end

6.4 Distributive lattices

class *distrib-lattice* = *lattice* +
assumes *sup-inf-distrib1*: $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$

context *distrib-lattice*
begin

lemma *sup-inf-distrib2*: $(y \sqcap z) \sqcup x = (y \sqcup x) \sqcap (z \sqcup x)$
⟨*proof*⟩

lemma *inf-sup-distrib1*: $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$
⟨*proof*⟩

lemma *inf-sup-distrib2*: $(y \sqcup z) \sqcap x = (y \sqcap x) \sqcup (z \sqcap x)$
⟨*proof*⟩

lemma *dual-distrib-lattice*: *class.distrib-lattice sup* $(\geq)$ $(>)$ *inf*
⟨*proof*⟩

lemmas *sup-inf-distrib* = *sup-inf-distrib1 sup-inf-distrib2*

lemmas *inf-sup-distrib* = *inf-sup-distrib1 inf-sup-distrib2*

lemmas *distrib* = *sup-inf-distrib1 sup-inf-distrib2 inf-sup-distrib1 inf-sup-distrib2*

end

6.5 Bounded lattices

class *bounded-semilattice-inf-top* = *semilattice-inf* + *order-top*
begin

sublocale *inf-top*: *semilattice-neutr inf top*
+ *inf-top*: *semilattice-neutr-order inf top less-eq less*
⟨*proof*⟩

lemma *inf-top-left*: $\top \sqcap x = x$
⟨*proof*⟩

lemma *inf-top-right*: $x \sqcap \top = x$
⟨*proof*⟩

lemma *inf-eq-top-iff*: $x \sqcap y = \top \longleftrightarrow x = \top \wedge y = \top$

⟨*proof*⟩

lemma *top-eq-inf-iff*: $\top = x \sqcap y \longleftrightarrow x = \top \wedge y = \top$
⟨*proof*⟩

end

class *bounded-semilattice-sup-bot* = *semilattice-sup* + *order-bot*
begin

sublocale *sup-bot*: *semilattice-neutr sup bot*
+ *sup-bot*: *semilattice-neutr-order sup bot greater-eq greater*
⟨*proof*⟩

lemma *sup-bot-left*: $\bot \sqcup x = x$
⟨*proof*⟩

lemma *sup-bot-right*: $x \sqcup \bot = x$
⟨*proof*⟩

lemma *sup-eq-bot-iff*: $x \sqcup y = \bot \longleftrightarrow x = \bot \wedge y = \bot$
⟨*proof*⟩

lemma *bot-eq-sup-iff*: $\bot = x \sqcup y \longleftrightarrow x = \bot \wedge y = \bot$
⟨*proof*⟩

end

class *bounded-lattice-bot* = *lattice* + *order-bot*
begin

subclass *bounded-semilattice-sup-bot* ⟨*proof*⟩

lemma *inf-bot-left* [*simp*]: $\bot \sqcap x = \bot$
⟨*proof*⟩

lemma *inf-bot-right* [*simp*]: $x \sqcap \bot = \bot$
⟨*proof*⟩

end

class *bounded-lattice-top* = *lattice* + *order-top*
begin

subclass *bounded-semilattice-inf-top* ⟨*proof*⟩

lemma *sup-top-left* [*simp*]: $\top \sqcup x = \top$
⟨*proof*⟩

lemma *sup-top-right* [*simp*]: $x \sqcup \top = \top$
⟨*proof*⟩

end

class *bounded-lattice* = *lattice* + *order-bot* + *order-top*
begin

subclass *bounded-lattice-bot* ⟨*proof*⟩
subclass *bounded-lattice-top* ⟨*proof*⟩

lemma *dual-bounded-lattice*: *class.bounded-lattice sup greater-eq greater inf* $\top$ $\bot$
⟨*proof*⟩

end

6.6 *min*/*max* as special case of lattice

context *linorder*
begin

sublocale *min*: *semilattice-order min less-eq less*
+ *max*: *semilattice-order max greater-eq greater*
⟨*proof*⟩

declare *min.absorb1* [*simp*] *min.absorb2* [*simp*]
min.absorb3 [*simp*] *min.absorb4* [*simp*]
max.absorb1 [*simp*] *max.absorb2* [*simp*]
max.absorb3 [*simp*] *max.absorb4* [*simp*]

lemma *min-le-iff-disj*: $min\ x\ y \leq z \longleftrightarrow x \leq z \vee y \leq z$
⟨*proof*⟩

lemma *le-max-iff-disj*: $z \leq max\ x\ y \longleftrightarrow z \leq x \vee z \leq y$
⟨*proof*⟩

lemma *min-less-iff-disj*: $min\ x\ y < z \longleftrightarrow x < z \vee y < z$
⟨*proof*⟩

lemma *less-max-iff-disj*: $z < max\ x\ y \longleftrightarrow z < x \vee z < y$
⟨*proof*⟩

lemma *min-less-iff-conj* [*simp*]: $z < min\ x\ y \longleftrightarrow z < x \wedge z < y$
⟨*proof*⟩

lemma *max-less-iff-conj* [*simp*]: $max\ x\ y < z \longleftrightarrow x < z \wedge y < z$
⟨*proof*⟩

lemma *min-max-distrib1*: $min\ (max\ b\ c)\ a = max\ (min\ b\ a)\ (min\ c\ a)$

⟨*proof*⟩

lemma *min-max-distrib2*: $min\ a\ (max\ b\ c) = max\ (min\ a\ b)\ (min\ a\ c)$
⟨*proof*⟩

lemma *max-min-distrib1*: $max\ (min\ b\ c)\ a = min\ (max\ b\ a)\ (max\ c\ a)$
⟨*proof*⟩

lemma *max-min-distrib2*: $max\ a\ (min\ b\ c) = min\ (max\ a\ b)\ (max\ a\ c)$
⟨*proof*⟩

lemmas *min-max-distribs = min-max-distrib1 min-max-distrib2 max-min-distrib1 max-min-distrib2*

lemma *split-min* [*no-atp*]: $P\ (min\ i\ j) \longleftrightarrow (i \leq j \longrightarrow P\ i) \wedge (\neg\ i \leq j \longrightarrow P\ j)$
⟨*proof*⟩

lemma *split-max* [*no-atp*]: $P\ (max\ i\ j) \longleftrightarrow (i \leq j \longrightarrow P\ j) \wedge (\neg\ i \leq j \longrightarrow P\ i)$
⟨*proof*⟩

lemma *split-min-lin* [*no-atp*]:
$\langle P\ (min\ a\ b) \longleftrightarrow (b = a \longrightarrow P\ a) \wedge (a < b \longrightarrow P\ a) \wedge (b < a \longrightarrow P\ b)\rangle$
⟨*proof*⟩

lemma *split-max-lin* [*no-atp*]:
$\langle P\ (max\ a\ b) \longleftrightarrow (b = a \longrightarrow P\ a) \wedge (a < b \longrightarrow P\ b) \wedge (b < a \longrightarrow P\ a)\rangle$
⟨*proof*⟩

end

lemma *inf-min*: $inf = (min :: 'a::\{semilattice\text{-}inf, linorder\} \Rightarrow 'a \Rightarrow 'a)$
⟨*proof*⟩

lemma *sup-max*: $sup = (max :: 'a::\{semilattice\text{-}sup, linorder\} \Rightarrow 'a \Rightarrow 'a)$
⟨*proof*⟩

6.7 Uniqueness of inf and sup

lemma (**in** *semilattice-inf*) *inf-unique*:
fixes f (**infixl** $\triangle$ *70*)
assumes *le1*: $\bigwedge x\ y.\ x \triangle y \leq x$
and *le2*: $\bigwedge x\ y.\ x \triangle y \leq y$
and *greatest*: $\bigwedge x\ y\ z.\ x \leq y \Longrightarrow x \leq z \Longrightarrow x \leq y \triangle z$
shows $x \sqcap y = x \triangle y$
⟨*proof*⟩

lemma (**in** *semilattice-sup*) *sup-unique*:
fixes f (**infixl** ∇ *70*)
assumes *ge1* [*simp*]: $\bigwedge x\ y.\ x \leq x \nabla y$

and *ge2*: $\bigwedge x\ y.\ y \leq x \nabla y$
and *least*: $\bigwedge x\ y\ z.\ y \leq x \Longrightarrow z \leq x \Longrightarrow y \nabla z \leq x$
shows $x \sqcup y = x \nabla y$
⟨*proof*⟩

6.8 Lattice on - ⇒ -

instantiation *fun* :: (*type*, *semilattice-sup*) *semilattice-sup*
begin

definition $f \sqcup g = (\lambda x.\ f\ x \sqcup g\ x)$

lemma *sup-apply* [*simp*, *code*]: $(f \sqcup g)\ x = f\ x \sqcup g\ x$
⟨*proof*⟩

instance
⟨*proof*⟩

end

instantiation *fun* :: (*type*, *semilattice-inf*) *semilattice-inf*
begin

definition $f \sqcap g = (\lambda x.\ f\ x \sqcap g\ x)$

lemma *inf-apply* [*simp*, *code*]: $(f \sqcap g)\ x = f\ x \sqcap g\ x$
⟨*proof*⟩

instance ⟨*proof*⟩

end

instance *fun* :: (*type*, *lattice*) *lattice* ⟨*proof*⟩

instance *fun* :: (*type*, *distrib-lattice*) *distrib-lattice*
⟨*proof*⟩

instance *fun* :: (*type*, *bounded-lattice*) *bounded-lattice* ⟨*proof*⟩

instantiation *fun* :: (*type*, *uminus*) *uminus*
begin

definition *fun-Compl-def*: $-A = (\lambda x.\ -A\ x)$

lemma *uminus-apply* [*simp*, *code*]: $(-A)\ x = -(A\ x)$
⟨*proof*⟩

instance ⟨*proof*⟩

end

instantiation *fun* :: (*type*, *minus*) *minus*
begin

definition *fun-diff-def*: $A - B = (\lambda x.\ A\ x - B\ x)$

lemma *minus-apply* [*simp*, *code*]: $(A - B)\ x = A\ x - B\ x$
⟨*proof*⟩

instance ⟨*proof*⟩

end

end

7 Boolean Algebras

theory *Boolean-Algebras*
imports *Lattices*
begin

7.1 Abstract boolean algebra

locale *abstract-boolean-algebra* = *conj*: *abel-semigroup* ‹(⊓)› + *disj*: *abel-semigroup* ‹(⊔)›
for *conj* :: ‹'*a* ⇒ '*a* ⇒ '*a*› (**infixr** ‹⊓› *70*)
and *disj* :: ‹'*a* ⇒ '*a* ⇒ '*a*› (**infixr** ‹⊔› *65*) +
fixes *compl* :: ‹'*a* ⇒ '*a*› (‹− -› [*81*] *80*)
and *zero* :: ‹'*a*› (‹**0**›)
and *one* :: ‹'*a*› (‹**1**›)
assumes *conj-disj-distrib*: ‹*x* ⊓ (*y* ⊔ *z*) = (*x* ⊓ *y*) ⊔ (*x* ⊓ *z*)›
and *disj-conj-distrib*: ‹*x* ⊔ (*y* ⊓ *z*) = (*x* ⊔ *y*) ⊓ (*x* ⊔ *z*)›
and *conj-one-right*: ‹*x* ⊓ **1** = *x*›
and *disj-zero-right*: ‹*x* ⊔ **0** = *x*›
and *conj-cancel-right* [*simp*]: ‹*x* ⊓ − *x* = **0**›
and *disj-cancel-right* [*simp*]: ‹*x* ⊔ − *x* = **1**›
begin

sublocale *conj*: *semilattice-neutr* ‹(⊓)› ‹**1**›
⟨*proof*⟩

sublocale *disj*: *semilattice-neutr* ‹(⊔)› ‹**0**›
⟨*proof*⟩

7.1.1 Complement

lemma *complement-unique*:
assumes *1*: *a* ⊓ *x* = **0**

assumes *2*: $a \sqcup x = \mathbf{1}$
assumes *3*: $a \sqcap y = \mathbf{0}$
assumes *4*: $a \sqcup y = \mathbf{1}$
shows $x = y$
⟨*proof*⟩

lemma *compl-unique*: $x \sqcap y = \mathbf{0} \Longrightarrow x \sqcup y = \mathbf{1} \Longrightarrow - x = y$
⟨*proof*⟩

lemma *double-compl* [*simp*]: $- (- x) = x$
⟨*proof*⟩

lemma *compl-eq-compl-iff* [*simp*]:
‹$- x = - y \longleftrightarrow x = y$› (**is** ‹*?P* $\longleftrightarrow$ *?Q*›)
⟨*proof*⟩

7.1.2 Conjunction

lemma *conj-zero-right* [*simp*]: $x \sqcap \mathbf{0} = \mathbf{0}$
⟨*proof*⟩

lemma *compl-one* [*simp*]: $- \mathbf{1} = \mathbf{0}$
⟨*proof*⟩

lemma *conj-zero-left* [*simp*]: $\mathbf{0} \sqcap x = \mathbf{0}$
⟨*proof*⟩

lemma *conj-cancel-left* [*simp*]: $- x \sqcap x = \mathbf{0}$
⟨*proof*⟩

lemma *conj-disj-distrib2*: $(y \sqcup z) \sqcap x = (y \sqcap x) \sqcup (z \sqcap x)$
⟨*proof*⟩

lemmas *conj-disj-distribs* = *conj-disj-distrib conj-disj-distrib2*

7.1.3 Disjunction

context
begin

interpretation *dual*: *abstract-boolean-algebra* ‹(⊔)› ‹(⊓)› *compl* ‹**1**› ‹**0**›
⟨*proof*⟩

lemma *disj-one-right* [*simp*]: $x \sqcup \mathbf{1} = \mathbf{1}$
⟨*proof*⟩

lemma *compl-zero* [*simp*]: $- \mathbf{0} = \mathbf{1}$
⟨*proof*⟩

lemma *disj-one-left* [*simp*]: $\mathbf{1} \sqcup x = \mathbf{1}$

⟨*proof*⟩

lemma *disj-cancel-left* [*simp*]: $- x \sqcup x = \mathbf{1}$
⟨*proof*⟩

lemma *disj-conj-distrib2*: $(y \sqcap z) \sqcup x = (y \sqcup x) \sqcap (z \sqcup x)$
⟨*proof*⟩

lemmas *disj-conj-distribs* = *disj-conj-distrib disj-conj-distrib2*

end

7.1.4 De Morgan's Laws

lemma *de-Morgan-conj* [*simp*]: $- (x \sqcap y) = - x \sqcup - y$
⟨*proof*⟩

context
begin

interpretation *dual*: *abstract-boolean-algebra* ‹(⊔)› ‹(⊓)› *compl* ‹**1**› ‹**0**›
⟨*proof*⟩

lemma *de-Morgan-disj* [*simp*]: $- (x \sqcup y) = - x \sqcap - y$
⟨*proof*⟩

end

end

7.2 Symmetric Difference

locale *abstract-boolean-algebra-sym-diff* = *abstract-boolean-algebra* +
fixes *xor* :: ‹'*a* ⇒ '*a* ⇒ '*a*› (**infixr** ‹⊖› *65*)
assumes *xor-def* : ‹$x \ominus y = (x \sqcap - y) \sqcup (- x \sqcap y)$›
begin

sublocale *xor*: *comm-monoid xor* ‹**0**›
⟨*proof*⟩

lemma *xor-def2*:
‹$x \ominus y = (x \sqcup y) \sqcap (- x \sqcup - y)$›
⟨*proof*⟩

lemma *xor-one-right* [*simp*]: $x \ominus \mathbf{1} = - x$
⟨*proof*⟩

lemma *xor-one-left* [*simp*]: $\mathbf{1} \ominus x = - x$
⟨*proof*⟩

lemma *xor-self* [*simp*]: $x \ominus x = \mathbf{0}$
⟨*proof*⟩

lemma *xor-left-self* [*simp*]: $x \ominus (x \ominus y) = y$
⟨*proof*⟩

lemma *xor-compl-left* [*simp*]: $- x \ominus y = - (x \ominus y)$
⟨*proof*⟩

lemma *xor-compl-right* [*simp*]: $x \ominus - y = - (x \ominus y)$
⟨*proof*⟩

lemma *xor-cancel-right* [*simp*]: $x \ominus - x = \mathbf{1}$
⟨*proof*⟩

lemma *xor-cancel-left* [*simp*]: $- x \ominus x = \mathbf{1}$
⟨*proof*⟩

lemma *conj-xor-distrib*: $x \sqcap (y \ominus z) = (x \sqcap y) \ominus (x \sqcap z)$
⟨*proof*⟩

lemma *conj-xor-distrib2*: $(y \ominus z) \sqcap x = (y \sqcap x) \ominus (z \sqcap x)$
⟨*proof*⟩

lemmas *conj-xor-distribs* = *conj-xor-distrib conj-xor-distrib2*

end

7.3 Type classes

class *boolean-algebra* = *distrib-lattice* + *bounded-lattice* + *minus* + *uminus* +
assumes *inf-compl-bot*: ‹$x \sqcap - x = \bot$›
and *sup-compl-top*: ‹$x \sqcup - x = \top$›
assumes *diff-eq*: ‹$x - y = x \sqcap - y$›
begin

sublocale *boolean-algebra*: *abstract-boolean-algebra* ‹(⊓)› ‹(⊔)› *uminus* $\bot$ $\top$
⟨*proof*⟩

lemma *compl-inf-bot*: $- x \sqcap x = \bot$
⟨*proof*⟩

lemma *compl-sup-top*: $- x \sqcup x = \top$
⟨*proof*⟩

lemma *compl-unique*:
assumes $x \sqcap y = \bot$
and $x \sqcup y = \top$
shows $- x = y$

⟨*proof*⟩

lemma *double-compl*: $- (- x) = x$
⟨*proof*⟩

lemma *compl-eq-compl-iff*: $- x = - y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *compl-bot-eq*: $- \bot = \top$
⟨*proof*⟩

lemma *compl-top-eq*: $- \top = \bot$
⟨*proof*⟩

lemma *compl-inf*: $- (x \sqcap y) = - x \sqcup - y$
⟨*proof*⟩

lemma *compl-sup*: $- (x \sqcup y) = - x \sqcap - y$
⟨*proof*⟩

lemma *compl-mono*:
assumes $x \leq y$
shows $- y \leq - x$
⟨*proof*⟩

lemma *compl-le-compl-iff* [*simp*]: $- x \leq - y \longleftrightarrow y \leq x$
⟨*proof*⟩

lemma *compl-le-swap1*:
assumes $y \leq - x$
shows $x \leq -y$
⟨*proof*⟩

lemma *compl-le-swap2*:
assumes $- y \leq x$
shows $- x \leq y$
⟨*proof*⟩

lemma *compl-less-compl-iff* [*simp*]: $- x < - y \longleftrightarrow y < x$
⟨*proof*⟩

lemma *compl-less-swap1*:
assumes $y < - x$
shows $x < - y$
⟨*proof*⟩

lemma *compl-less-swap2*:
assumes $- y < x$
shows $- x < y$

⟨*proof*⟩

lemma *sup-cancel-left1*: ‹$x \sqcup a \sqcup (- x \sqcup b) = \top$›
⟨*proof*⟩

lemma *sup-cancel-left2*: ‹$- x \sqcup a \sqcup (x \sqcup b) = \top$›
⟨*proof*⟩

lemma *inf-cancel-left1*: ‹$x \sqcap a \sqcap (- x \sqcap b) = \bot$›
⟨*proof*⟩

lemma *inf-cancel-left2*: ‹$- x \sqcap a \sqcap (x \sqcap b) = \bot$›
⟨*proof*⟩

lemma *sup-compl-top-left1* [*simp*]: ‹$- x \sqcup (x \sqcup y) = \top$›
⟨*proof*⟩

lemma *sup-compl-top-left2* [*simp*]: ‹$x \sqcup (- x \sqcup y) = \top$›
⟨*proof*⟩

lemma *inf-compl-bot-left1* [*simp*]: ‹$- x \sqcap (x \sqcap y) = \bot$›
⟨*proof*⟩

lemma *inf-compl-bot-left2* [*simp*]: ‹$x \sqcap (- x \sqcap y) = \bot$›
⟨*proof*⟩

lemma *inf-compl-bot-right* [*simp*]: ‹$x \sqcap (y \sqcap - x) = \bot$›
⟨*proof*⟩

end

7.4 Lattice on *bool*

instantiation *bool* :: *boolean-algebra*
begin

definition *bool-Compl-def* [*simp*]: *uminus* = *Not*

definition *bool-diff-def* [*simp*]: $A - B \longleftrightarrow A \wedge \neg B$

definition [*simp*]: $P \sqcap Q \longleftrightarrow P \wedge Q$

definition [*simp*]: $P \sqcup Q \longleftrightarrow P \vee Q$

instance ⟨*proof*⟩

end

lemma *sup-boolI1*: $P \Longrightarrow P \sqcup Q$

⟨*proof*⟩

lemma *sup-boolI2*: $Q \Longrightarrow P \sqcup Q$
⟨*proof*⟩

lemma *sup-boolE*: $P \sqcup Q \Longrightarrow (P \Longrightarrow R) \Longrightarrow (Q \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

instance *fun* :: (*type*, *boolean-algebra*) *boolean-algebra*
⟨*proof*⟩

7.5 Lattice on unary and binary predicates

lemma *inf1I*: $A\ x \Longrightarrow B\ x \Longrightarrow (A \sqcap B)\ x$
⟨*proof*⟩

lemma *inf2I*: $A\ x\ y \Longrightarrow B\ x\ y \Longrightarrow (A \sqcap B)\ x\ y$
⟨*proof*⟩

lemma *inf1E*: $(A \sqcap B)\ x \Longrightarrow (A\ x \Longrightarrow B\ x \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *inf2E*: $(A \sqcap B)\ x\ y \Longrightarrow (A\ x\ y \Longrightarrow B\ x\ y \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *inf1D1*: $(A \sqcap B)\ x \Longrightarrow A\ x$
⟨*proof*⟩

lemma *inf2D1*: $(A \sqcap B)\ x\ y \Longrightarrow A\ x\ y$
⟨*proof*⟩

lemma *inf1D2*: $(A \sqcap B)\ x \Longrightarrow B\ x$
⟨*proof*⟩

lemma *inf2D2*: $(A \sqcap B)\ x\ y \Longrightarrow B\ x\ y$
⟨*proof*⟩

lemma *sup1I1*: $A\ x \Longrightarrow (A \sqcup B)\ x$
⟨*proof*⟩

lemma *sup2I1*: $A\ x\ y \Longrightarrow (A \sqcup B)\ x\ y$
⟨*proof*⟩

lemma *sup1I2*: $B\ x \Longrightarrow (A \sqcup B)\ x$
⟨*proof*⟩

lemma *sup2I2*: $B\ x\ y \Longrightarrow (A \sqcup B)\ x\ y$
⟨*proof*⟩

lemma *sup1E*: $(A \sqcup B)\ x \Longrightarrow (A\ x \Longrightarrow P) \Longrightarrow (B\ x \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *sup2E*: $(A \sqcup B)\ x\ y \Longrightarrow (A\ x\ y \Longrightarrow P) \Longrightarrow (B\ x\ y \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

Classical introduction rule: no commitment to A vs B.

lemma *sup1CI*: $(\neg\ B\ x \Longrightarrow A\ x) \Longrightarrow (A \sqcup B)\ x$
⟨*proof*⟩

lemma *sup2CI*: $(\neg\ B\ x\ y \Longrightarrow A\ x\ y) \Longrightarrow (A \sqcup B)\ x\ y$
⟨*proof*⟩

7.6 Simproc setup

locale *boolean-algebra-cancel*
begin

lemma *sup1*: $(A::'a::\textit{semilattice-sup}) \equiv \textit{sup}\ k\ a \Longrightarrow \textit{sup}\ A\ b \equiv \textit{sup}\ k\ (\textit{sup}\ a\ b)$
⟨*proof*⟩

lemma *sup2*: $(B::'a::\textit{semilattice-sup}) \equiv \textit{sup}\ k\ b \Longrightarrow \textit{sup}\ a\ B \equiv \textit{sup}\ k\ (\textit{sup}\ a\ b)$
⟨*proof*⟩

lemma *sup0*: $(a::'a::\textit{bounded-semilattice-sup-bot}) \equiv \textit{sup}\ a\ \textit{bot}$
⟨*proof*⟩

lemma *inf1*: $(A::'a::\textit{semilattice-inf}) \equiv \textit{inf}\ k\ a \Longrightarrow \textit{inf}\ A\ b \equiv \textit{inf}\ k\ (\textit{inf}\ a\ b)$
⟨*proof*⟩

lemma *inf2*: $(B::'a::\textit{semilattice-inf}) \equiv \textit{inf}\ k\ b \Longrightarrow \textit{inf}\ a\ B \equiv \textit{inf}\ k\ (\textit{inf}\ a\ b)$
⟨*proof*⟩

lemma *inf0*: $(a::'a::\textit{bounded-semilattice-inf-top}) \equiv \textit{inf}\ a\ \textit{top}$
⟨*proof*⟩

end

⟨*ML*⟩

context *boolean-algebra*
begin

lemma *shunt1*: $(x \sqcap y \leq z) \longleftrightarrow (x \leq -y \sqcup z)$
⟨*proof*⟩

lemma *shunt2*: $(x \sqcap -y \leq z) \longleftrightarrow (x \leq y \sqcup z)$
⟨*proof*⟩

lemma *inf-shunt*: $(x \sqcap y = \bot) \longleftrightarrow (x \leq - y)$
⟨*proof*⟩

lemma *sup-shunt*: $(x \sqcup y = \top) \longleftrightarrow (- x \leq y)$
⟨*proof*⟩

lemma *diff-shunt-var*: $(x - y = \bot) \longleftrightarrow (x \leq y)$
⟨*proof*⟩

lemma *sup-neg-inf*:
$‹p \leq q \sqcup r \longleftrightarrow p \sqcap - q \leq r›$ (**is** $‹?P \longleftrightarrow ?Q›$)
⟨*proof*⟩

end

end

8 Set theory for higher-order logic

theory *Set*
imports *Lattices Boolean-Algebras*
begin

8.1 Sets as predicates

typedecl *'a set*

axiomatization *Collect* :: $('a \Rightarrow bool) \Rightarrow 'a\ set$ — comprehension
and *member* :: $'a \Rightarrow 'a\ set \Rightarrow bool$ — membership
where *mem-Collect-eq* [*iff*, *code-unfold*]: $member\ a\ (Collect\ P) = P\ a$
and *Collect-mem-eq* [*simp*]: $Collect\ (\lambda x.\ member\ x\ A) = A$

notation
member ($'(\in')$) **and**
member ((-/ ∈ -) [*51*, *51*] *50*)

abbreviation *not-member*
where $not\text{-}member\ x\ A \equiv \neg\ (x \in A)$ — non-membership
notation
not-member ($'(\notin')$) **and**
not-member ((-/ ∉ -) [*51*, *51*] *50*)

notation (*ASCII*)
member ($'(:')$) **and**
member ((-/ : -) [*51*, *51*] *50*) **and**
not-member ($'(\sim:')$) **and**
not-member ((-/ ~: -) [*51*, *51*] *50*)

Set comprehensions

syntax
  -*Coll* :: *pttrn* ⇒ *bool* ⇒ *'a set* ((*1*{-./ -}))
translations
  {*x*. *P*} ⇌ *CONST Collect* (λ*x*. *P*)

syntax (*ASCII*)
  -*Collect* :: *pttrn* ⇒ *'a set* ⇒ *bool* ⇒ *'a set* ((*1*{(-/: -)./ -}))
syntax
  -*Collect* :: *pttrn* ⇒ *'a set* ⇒ *bool* ⇒ *'a set* ((*1*{(-/ ∈ -)./ -}))
translations
  {*p*:*A*. *P*} ⇀ *CONST Collect* (λ*p*. *p* ∈ *A* ∧ *P*)

lemma *CollectI*: *P a* ⟹ *a* ∈ {*x*. *P x*}
  ⟨*proof*⟩

lemma *CollectD*: *a* ∈ {*x*. *P x*} ⟹ *P a*
  ⟨*proof*⟩

lemma *Collect-cong*: (⋀*x*. *P x* = *Q x*) ⟹ {*x*. *P x*} = {*x*. *Q x*}
  ⟨*proof*⟩

Simproc for pulling $x = t$ in $\{x.\ \ldots \wedge x = t \wedge \ldots\}$ to the front (and similarly for $t = x$):

⟨*ML*⟩

lemmas *CollectE* = *CollectD* [*elim-format*]

lemma *set-eqI*:
  assumes ⋀*x*. *x* ∈ *A* ⟷ *x* ∈ *B*
  shows *A* = *B*
⟨*proof*⟩

lemma *set-eq-iff*: *A* = *B* ⟷ (∀ *x*. *x* ∈ *A* ⟷ *x* ∈ *B*)
  ⟨*proof*⟩

lemma *Collect-eqI*:
  assumes ⋀*x*. *P x* = *Q x*
  shows *Collect P* = *Collect Q*
  ⟨*proof*⟩

Lifting of predicate class instances

instantiation *set* :: (*type*) *boolean-algebra*
begin

definition *less-eq-set*
  where *A* ≤ *B* ⟷ (λ*x*. *member x A*) ≤ (λ*x*. *member x B*)

definition *less-set*
  where $A < B \longleftrightarrow (\lambda x.\ member\ x\ A) < (\lambda x.\ member\ x\ B)$

definition *inf-set*
  where $A \sqcap B = Collect\ ((\lambda x.\ member\ x\ A) \sqcap (\lambda x.\ member\ x\ B))$

definition *sup-set*
  where $A \sqcup B = Collect\ ((\lambda x.\ member\ x\ A) \sqcup (\lambda x.\ member\ x\ B))$

definition *bot-set*
  where $\bot = Collect\ \bot$

definition *top-set*
  where $\top = Collect\ \top$

definition *uminus-set*
  where $-\ A = Collect\ (-\ (\lambda x.\ member\ x\ A))$

definition *minus-set*
  where $A - B = Collect\ ((\lambda x.\ member\ x\ A) - (\lambda x.\ member\ x\ B))$

instance
  ⟨*proof*⟩

end

Set enumerations

abbreviation *empty* :: *'a set* ({})
  where $\{\} \equiv bot$

definition *insert* :: *'a* ⇒ *'a set* ⇒ *'a set*
  where *insert-compr*: $insert\ a\ B = \{x.\ x = a \vee x \in B\}$

syntax
  -Finset :: *args* ⇒ *'a set* ({(-)})
translations
  $\{x, xs\} \rightleftharpoons CONST\ insert\ x\ \{xs\}$
  $\{x\} \rightleftharpoons CONST\ insert\ x\ \{\}$

8.2 Subsets and bounded quantifiers

abbreviation *subset* :: *'a set* ⇒ *'a set* ⇒ *bool*
  where $subset \equiv less$

abbreviation *subset-eq* :: *'a set* ⇒ *'a set* ⇒ *bool*
  where $subset\text{-}eq \equiv less\text{-}eq$

notation
  subset (*'*(⊂*'*)) **and**

subset ((-/ ⊂ -) [*51*, *51*] *50*) **and**
subset-eq ('(⊆')) **and**
subset-eq ((-/ ⊆ -) [*51*, *51*] *50*)

abbreviation (*input*)
supset :: *'a set* ⇒ *'a set* ⇒ *bool* **where**
supset ≡ *greater*

abbreviation (*input*)
supset-eq :: *'a set* ⇒ *'a set* ⇒ *bool* **where**
supset-eq ≡ *greater-eq*

notation
supset ('(⊃')) **and**
supset ((-/ ⊃ -) [*51*, *51*] *50*) **and**
supset-eq ('(⊇')) **and**
supset-eq ((-/ ⊇ -) [*51*, *51*] *50*)

notation (*ASCII* **output**)
subset ('(<')) **and**
subset ((-/ < -) [*51*, *51*] *50*) **and**
subset-eq ('(<=')) **and**
subset-eq ((-/ <= -) [*51*, *51*] *50*)

definition *Ball* :: *'a set* ⇒ (*'a* ⇒ *bool*) ⇒ *bool*
where *Ball A P* ⟷ (∀ *x*. *x* ∈ *A* ⟶ *P x*) — bounded universal quantifiers

definition *Bex* :: *'a set* ⇒ (*'a* ⇒ *bool*) ⇒ *bool*
where *Bex A P* ⟷ (∃ *x*. *x* ∈ *A* ∧ *P x*) — bounded existential quantifiers

syntax (*ASCII*)
-Ball :: *pttrn* ⇒ *'a set* ⇒ *bool* ⇒ *bool* ((*3ALL* (-/:-)./ -) [*0*, *0*, *10*] *10*)
-Bex :: *pttrn* ⇒ *'a set* ⇒ *bool* ⇒ *bool* ((*3EX* (-/:-)./ -) [*0*, *0*, *10*] *10*)
-Bex1 :: *pttrn* ⇒ *'a set* ⇒ *bool* ⇒ *bool* ((*3EX*! (-/:-)./ -) [*0*, *0*, *10*] *10*)
-Bleast :: *id* ⇒ *'a set* ⇒ *bool* ⇒ *'a* ((*3LEAST* (-/:-)./ -) [*0*, *0*, *10*] *10*)

syntax (*input*)
-Ball :: *pttrn* ⇒ *'a set* ⇒ *bool* ⇒ *bool* ((*3*! (-/:-)./ -) [*0*, *0*, *10*] *10*)
-Bex :: *pttrn* ⇒ *'a set* ⇒ *bool* ⇒ *bool* ((*3?* (-/:-)./ -) [*0*, *0*, *10*] *10*)
-Bex1 :: *pttrn* ⇒ *'a set* ⇒ *bool* ⇒ *bool* ((*3?*! (-/:-)./ -) [*0*, *0*, *10*] *10*)

syntax
-Ball :: *pttrn* ⇒ *'a set* ⇒ *bool* ⇒ *bool* ((*3*∀ (-/∈-)./ -) [*0*, *0*, *10*] *10*)
-Bex :: *pttrn* ⇒ *'a set* ⇒ *bool* ⇒ *bool* ((*3*∃ (-/∈-)./ -) [*0*, *0*, *10*] *10*)
-Bex1 :: *pttrn* ⇒ *'a set* ⇒ *bool* ⇒ *bool* ((*3*∃!(-/∈-)./ -) [*0*, *0*, *10*] *10*)
-Bleast :: *id* ⇒ *'a set* ⇒ *bool* ⇒ *'a* ((*3LEAST*(-/∈-)./ -) [*0*, *0*, *10*] *10*)

translations
∀ *x*∈*A*. *P* ⇌ *CONST Ball A* (λ*x*. *P*)

$\exists\, x{\in}A.\ P \rightleftharpoons CONST\ Bex\ A\ (\lambda x.\ P)$
$\exists!x{\in}A.\ P \rightharpoonup \exists!x.\ x \in A \wedge P$
$LEAST\ x{:}A.\ P \rightharpoonup LEAST\ x.\ x \in A \wedge P$

syntax (*ASCII* **output**)
-setlessAll :: $[idt,\ 'a,\ bool] \Rightarrow bool$ *((3ALL -<-./ -) [0, 0, 10] 10)*
-setlessEx :: $[idt,\ 'a,\ bool] \Rightarrow bool$ *((3EX -<-./ -) [0, 0, 10] 10)*
-setleAll :: $[idt,\ 'a,\ bool] \Rightarrow bool$ *((3ALL -<=-./ -) [0, 0, 10] 10)*
-setleEx :: $[idt,\ 'a,\ bool] \Rightarrow bool$ *((3EX -<=-./ -) [0, 0, 10] 10)*
-setleEx1 :: $[idt,\ 'a,\ bool] \Rightarrow bool$ *((3EX! -<=-./ -) [0, 0, 10] 10)*

syntax
-setlessAll :: $[idt,\ 'a,\ bool] \Rightarrow bool$ *((3∀ -⊂-./ -) [0, 0, 10] 10)*
-setlessEx :: $[idt,\ 'a,\ bool] \Rightarrow bool$ *((3∃ -⊂-./ -) [0, 0, 10] 10)*
-setleAll :: $[idt,\ 'a,\ bool] \Rightarrow bool$ *((3∀ -⊆-./ -) [0, 0, 10] 10)*
-setleEx :: $[idt,\ 'a,\ bool] \Rightarrow bool$ *((3∃ -⊆-./ -) [0, 0, 10] 10)*
-setleEx1 :: $[idt,\ 'a,\ bool] \Rightarrow bool$ *((3∃!-⊆-./ -) [0, 0, 10] 10)*

translations
$\forall\, A{\subset}B.\ P \rightharpoonup \forall\, A.\ A \subset B \longrightarrow P$
$\exists\, A{\subset}B.\ P \rightharpoonup \exists\, A.\ A \subset B \wedge P$
$\forall\, A{\subseteq}B.\ P \rightharpoonup \forall\, A.\ A \subseteq B \longrightarrow P$
$\exists\, A{\subseteq}B.\ P \rightharpoonup \exists\, A.\ A \subseteq B \wedge P$
$\exists!A{\subseteq}B.\ P \rightharpoonup \exists!A.\ A \subseteq B \wedge P$

⟨*ML*⟩

Translate between $\{e \mid x1 \ldots xn.\ P\}$ and $\{u.\ \exists\, x1 \ldots xn.\ u = e \wedge P\}$; $\{y.\ \exists\, x1 \ldots xn.\ y = e \wedge P\}$ is only translated if $[0..n] \subseteq bvs\ e$.

syntax
-Setcompr :: $'a \Rightarrow idts \Rightarrow bool \Rightarrow 'a\ set$ *((1{- |/-./ -}))*

⟨*ML*⟩

lemma *ballI* [*intro*!]: $(\bigwedge x.\ x \in A \Longrightarrow P\ x) \Longrightarrow \forall\, x{\in}A.\ P\ x$
⟨*proof*⟩

lemmas *strip* = *impI allI ballI*

lemma *bspec* [*dest?*]: $\forall\, x{\in}A.\ P\ x \Longrightarrow x \in A \Longrightarrow P\ x$
⟨*proof*⟩

Gives better instantiation for bound:

⟨*ML*⟩

lemma *ballE* [*elim*]: $\forall\, x{\in}A.\ P\ x \Longrightarrow (P\ x \Longrightarrow Q) \Longrightarrow (x \notin A \Longrightarrow Q) \Longrightarrow Q$
⟨*proof*⟩

lemma *bexI* [*intro*]: $P\ x \Longrightarrow x \in A \Longrightarrow \exists\, x{\in}A.\ P\ x$
— Normally the best argument order: $P\ x$ constrains the choice of $x \in A$.
⟨*proof*⟩

lemma *rev-bexI* [*intro?*]: $x \in A \Longrightarrow P\ x \Longrightarrow \exists\, x{\in}A.\ P\ x$
— The best argument order when there is only one $x \in A$.
⟨*proof*⟩

lemma *bexCI*: $(\forall\, x{\in}A.\ \neg\ P\ x \Longrightarrow P\ a) \Longrightarrow a \in A \Longrightarrow \exists\, x{\in}A.\ P\ x$
⟨*proof*⟩

lemma *bexE* [*elim!*]: $\exists\, x{\in}A.\ P\ x \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow P\ x \Longrightarrow Q) \Longrightarrow Q$
⟨*proof*⟩

lemma *ball-triv* [*simp*]: $(\forall\, x{\in}A.\ P) \longleftrightarrow ((\exists\, x.\ x \in A) \longrightarrow P)$
— trivial rewrite rule.
⟨*proof*⟩

lemma *bex-triv* [*simp*]: $(\exists\, x{\in}A.\ P) \longleftrightarrow ((\exists\, x.\ x \in A) \wedge P)$
— Dual form for existentials.
⟨*proof*⟩

lemma *bex-triv-one-point1* [*simp*]: $(\exists\, x{\in}A.\ x = a) \longleftrightarrow a \in A$
⟨*proof*⟩

lemma *bex-triv-one-point2* [*simp*]: $(\exists\, x{\in}A.\ a = x) \longleftrightarrow a \in A$
⟨*proof*⟩

lemma *bex-one-point1* [*simp*]: $(\exists\, x{\in}A.\ x = a \wedge P\ x) \longleftrightarrow a \in A \wedge P\ a$
⟨*proof*⟩

lemma *bex-one-point2* [*simp*]: $(\exists\, x{\in}A.\ a = x \wedge P\ x) \longleftrightarrow a \in A \wedge P\ a$
⟨*proof*⟩

lemma *ball-one-point1* [*simp*]: $(\forall\, x{\in}A.\ x = a \longrightarrow P\ x) \longleftrightarrow (a \in A \longrightarrow P\ a)$
⟨*proof*⟩

lemma *ball-one-point2* [*simp*]: $(\forall\, x{\in}A.\ a = x \longrightarrow P\ x) \longleftrightarrow (a \in A \longrightarrow P\ a)$
⟨*proof*⟩

lemma *ball-conj-distrib*: $(\forall\, x{\in}A.\ P\ x \wedge Q\ x) \longleftrightarrow (\forall\, x{\in}A.\ P\ x) \wedge (\forall\, x{\in}A.\ Q\ x)$
⟨*proof*⟩

lemma *bex-disj-distrib*: $(\exists\, x{\in}A.\ P\ x \vee Q\ x) \longleftrightarrow (\exists\, x{\in}A.\ P\ x) \vee (\exists\, x{\in}A.\ Q\ x)$
⟨*proof*⟩

Congruence rules

lemma *ball-cong*:
$[\![\ A = B;\ \bigwedge x.\ x \in B \Longrightarrow P\ x \longleftrightarrow Q\ x\ ]\!] \Longrightarrow$

$(\forall x{\in}A.\ P\ x) \longleftrightarrow (\forall x{\in}B.\ Q\ x)$
⟨*proof*⟩

lemma *ball-cong-simp* [*cong*]:
$[\![\ A = B;\ \bigwedge x.\ x \in B \Longrightarrow P\ x \longleftrightarrow Q\ x\ ]\!] \Longrightarrow$
$(\forall x{\in}A.\ P\ x) \longleftrightarrow (\forall x{\in}B.\ Q\ x)$
⟨*proof*⟩

lemma *bex-cong*:
$[\![\ A = B;\ \bigwedge x.\ x \in B \Longrightarrow P\ x \longleftrightarrow Q\ x\ ]\!] \Longrightarrow$
$(\exists x{\in}A.\ P\ x) \longleftrightarrow (\exists x{\in}B.\ Q\ x)$
⟨*proof*⟩

lemma *bex-cong-simp* [*cong*]:
$[\![\ A = B;\ \bigwedge x.\ x \in B \Longrightarrow P\ x \longleftrightarrow Q\ x\ ]\!] \Longrightarrow$
$(\exists x{\in}A.\ P\ x) \longleftrightarrow (\exists x{\in}B.\ Q\ x)$
⟨*proof*⟩

lemma *bex1-def*: $(\exists!x{\in}X.\ P\ x) \longleftrightarrow (\exists x{\in}X.\ P\ x) \wedge (\forall x{\in}X.\ \forall y{\in}X.\ P\ x \longrightarrow P\ y \longrightarrow x = y)$
⟨*proof*⟩

8.3 Basic operations

8.3.1 Subsets

lemma *subsetI* [*intro*!]: $(\bigwedge x.\ x \in A \Longrightarrow x \in B) \Longrightarrow A \subseteq B$
⟨*proof*⟩

Map the type $'a\ set \Rightarrow anything$ to just $'a$; for overloading constants whose first argument has type $'a\ set$.

lemma *subsetD* [*elim*, *intro?*]: $A \subseteq B \Longrightarrow c \in A \Longrightarrow c \in B$
⟨*proof*⟩

lemma *rev-subsetD* [*intro?*,*no-atp*]: $c \in A \Longrightarrow A \subseteq B \Longrightarrow c \in B$
— The same, with reversed premises for use with *erule* – cf. $[\![?P;\ ?P \longrightarrow ?Q]\!] \Longrightarrow ?Q$.
⟨*proof*⟩

lemma *subsetCE* [*elim*,*no-atp*]: $A \subseteq B \Longrightarrow (c \notin A \Longrightarrow P) \Longrightarrow (c \in B \Longrightarrow P) \Longrightarrow P$
— Classical elimination rule.
⟨*proof*⟩

lemma *subset-eq*: $A \subseteq B \longleftrightarrow (\forall x{\in}A.\ x \in B)$
⟨*proof*⟩

lemma *contra-subsetD* [*no-atp*]: $A \subseteq B \Longrightarrow c \notin B \Longrightarrow c \notin A$
⟨*proof*⟩

lemma *subset-refl*: $A \subseteq A$
⟨*proof*⟩

lemma *subset-trans*: $A \subseteq B \Longrightarrow B \subseteq C \Longrightarrow A \subseteq C$
⟨*proof*⟩

lemma *subset-not-subset-eq* [*code*]: $A \subset B \longleftrightarrow A \subseteq B \wedge \neg\, B \subseteq A$
⟨*proof*⟩

lemma *eq-mem-trans*: $a = b \Longrightarrow b \in A \Longrightarrow a \in A$
⟨*proof*⟩

lemmas *basic-trans-rules* [*trans*] =
order-trans-rules rev-subsetD subsetD eq-mem-trans

8.3.2 Equality

lemma *subset-antisym* [*intro*!]: $A \subseteq B \Longrightarrow B \subseteq A \Longrightarrow A = B$
— Anti-symmetry of the subset relation.
⟨*proof*⟩

Equality rules from ZF set theory – are they appropriate here?

lemma *equalityD1*: $A = B \Longrightarrow A \subseteq B$
⟨*proof*⟩

lemma *equalityD2*: $A = B \Longrightarrow B \subseteq A$
⟨*proof*⟩

Be careful when adding this to the claset as *subset-empty* is in the simpset: $A = \{\}$ goes to $\{\} \subseteq A$ and $A \subseteq \{\}$ and then back to $A = \{\}$!

lemma *equalityE*: $A = B \Longrightarrow (A \subseteq B \Longrightarrow B \subseteq A \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *equalityCE* [*elim*]: $A = B \Longrightarrow (c \in A \Longrightarrow c \in B \Longrightarrow P) \Longrightarrow (c \notin A \Longrightarrow c \notin B \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *eqset-imp-iff*: $A = B \Longrightarrow x \in A \longleftrightarrow x \in B$
⟨*proof*⟩

lemma *eqelem-imp-iff*: $x = y \Longrightarrow x \in A \longleftrightarrow y \in A$
⟨*proof*⟩

8.3.3 The empty set

lemma *empty-def*: $\{\} = \{x.\ \mathit{False}\}$
⟨*proof*⟩

lemma *empty-iff* [*simp*]: $c \in \{\} \longleftrightarrow \mathit{False}$
⟨*proof*⟩

lemma *emptyE* [*elim*!]: $a \in \{\} \Longrightarrow P$
⟨*proof*⟩

lemma *empty-subsetI* [*iff*]: $\{\} \subseteq A$
— One effect is to delete the ASSUMPTION $\{\} \subseteq A$
⟨*proof*⟩

lemma *equals0I*: $(\bigwedge y.\ y \in A \Longrightarrow \mathit{False}) \Longrightarrow A = \{\}$
⟨*proof*⟩

lemma *equals0D*: $A = \{\} \Longrightarrow a \notin A$
— Use for reasoning about disjointness: $A \cap B = \{\}$
⟨*proof*⟩

lemma *ball-empty* [*simp*]: $\mathit{Ball}\ \{\}\ P \longleftrightarrow \mathit{True}$
⟨*proof*⟩

lemma *bex-empty* [*simp*]: $\mathit{Bex}\ \{\}\ P \longleftrightarrow \mathit{False}$
⟨*proof*⟩

8.3.4 The universal set – UNIV

abbreviation *UNIV* :: *'a set*
where $\mathit{UNIV} \equiv \mathit{top}$

lemma *UNIV-def*: $\mathit{UNIV} = \{x.\ \mathit{True}\}$
⟨*proof*⟩

lemma *UNIV-I* [*simp*]: $x \in \mathit{UNIV}$
⟨*proof*⟩

declare *UNIV-I* [*intro*] — unsafe makes it less likely to cause problems

lemma *UNIV-witness* [*intro?*]: $\exists x.\ x \in \mathit{UNIV}$
⟨*proof*⟩

lemma *subset-UNIV*: $A \subseteq \mathit{UNIV}$
⟨*proof*⟩

Eta-contracting these two rules (to remove P) causes them to be ignored because of their interaction with congruence rules.

lemma *ball-UNIV* [*simp*]: $\mathit{Ball}\ \mathit{UNIV}\ P \longleftrightarrow \mathit{All}\ P$
⟨*proof*⟩

lemma *bex-UNIV* [*simp*]: $\mathit{Bex}\ \mathit{UNIV}\ P \longleftrightarrow \mathit{Ex}\ P$

⟨*proof*⟩

lemma *UNIV-eq-I*: $(\bigwedge x.\ x \in A) \Longrightarrow UNIV = A$
⟨*proof*⟩

lemma *UNIV-not-empty* [*iff*]: $UNIV \neq \{\}$
⟨*proof*⟩

lemma *empty-not-UNIV*[*simp*]: $\{\} \neq UNIV$
⟨*proof*⟩

8.3.5 The Powerset operator – Pow

definition $Pow :: 'a\ set \Rightarrow 'a\ set\ set$
where *Pow-def*: $Pow\ A = \{B.\ B \subseteq A\}$

lemma *Pow-iff* [*iff*]: $A \in Pow\ B \longleftrightarrow A \subseteq B$
⟨*proof*⟩

lemma *PowI*: $A \subseteq B \Longrightarrow A \in Pow\ B$
⟨*proof*⟩

lemma *PowD*: $A \in Pow\ B \Longrightarrow A \subseteq B$
⟨*proof*⟩

lemma *Pow-bottom*: $\{\} \in Pow\ B$
⟨*proof*⟩

lemma *Pow-top*: $A \in Pow\ A$
⟨*proof*⟩

lemma *Pow-not-empty*: $Pow\ A \neq \{\}$
⟨*proof*⟩

8.3.6 Set complement

lemma *Compl-iff* [*simp*]: $c \in -A \longleftrightarrow c \notin A$
⟨*proof*⟩

lemma *ComplI* [*intro!*]: $(c \in A \Longrightarrow False) \Longrightarrow c \in -A$
⟨*proof*⟩

This form, with negated conclusion, works well with the Classical prover. Negated assumptions behave like formulae on the right side of the notional turnstile ...

lemma *ComplD* [*dest!*]: $c \in -A \Longrightarrow c \notin A$
⟨*proof*⟩

lemmas *ComplE* = *ComplD* [*elim-format*]

lemma *Compl-eq*: $- A = \{x.\ \neg\ x \in A\}$
⟨*proof*⟩

8.3.7 Binary intersection

abbreviation *inter* :: *'a set* ⇒ *'a set* ⇒ *'a set* (**infixl** ∩ *70*)
where $(\cap) \equiv inf$

notation (*ASCII*)
inter (**infixl** *Int* *70*)

lemma *Int-def*: $A \cap B = \{x.\ x \in A \wedge x \in B\}$
⟨*proof*⟩

lemma *Int-iff* [*simp*]: $c \in A \cap B \longleftrightarrow c \in A \wedge c \in B$
⟨*proof*⟩

lemma *IntI* [*intro*!]: $c \in A \Longrightarrow c \in B \Longrightarrow c \in A \cap B$
⟨*proof*⟩

lemma *IntD1*: $c \in A \cap B \Longrightarrow c \in A$
⟨*proof*⟩

lemma *IntD2*: $c \in A \cap B \Longrightarrow c \in B$
⟨*proof*⟩

lemma *IntE* [*elim*!]: $c \in A \cap B \Longrightarrow (c \in A \Longrightarrow c \in B \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

8.3.8 Binary union

abbreviation *union* :: *'a set* ⇒ *'a set* ⇒ *'a set* (**infixl** ∪ *65*)
where $union \equiv sup$

notation (*ASCII*)
union (**infixl** *Un* *65*)

lemma *Un-def*: $A \cup B = \{x.\ x \in A \vee x \in B\}$
⟨*proof*⟩

lemma *Un-iff* [*simp*]: $c \in A \cup B \longleftrightarrow c \in A \vee c \in B$
⟨*proof*⟩

lemma *UnI1* [*elim?*]: $c \in A \Longrightarrow c \in A \cup B$
⟨*proof*⟩

lemma *UnI2* [*elim?*]: $c \in B \Longrightarrow c \in A \cup B$
⟨*proof*⟩

Classical introduction rule: no commitment to A vs. B.

lemma *UnCI* [*intro*!]: $(c \notin B \Longrightarrow c \in A) \Longrightarrow c \in A \cup B$
⟨*proof*⟩

lemma *UnE* [*elim*!]: $c \in A \cup B \Longrightarrow (c \in A \Longrightarrow P) \Longrightarrow (c \in B \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *insert-def*: *insert a B* $= \{x.\ x = a\} \cup B$
⟨*proof*⟩

8.3.9 Set difference

lemma *Diff-iff* [*simp*]: $c \in A - B \longleftrightarrow c \in A \wedge c \notin B$
⟨*proof*⟩

lemma *DiffI* [*intro*!]: $c \in A \Longrightarrow c \notin B \Longrightarrow c \in A - B$
⟨*proof*⟩

lemma *DiffD1*: $c \in A - B \Longrightarrow c \in A$
⟨*proof*⟩

lemma *DiffD2*: $c \in A - B \Longrightarrow c \in B \Longrightarrow P$
⟨*proof*⟩

lemma *DiffE* [*elim*!]: $c \in A - B \Longrightarrow (c \in A \Longrightarrow c \notin B \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *set-diff-eq*: $A - B = \{x.\ x \in A \wedge x \notin B\}$
⟨*proof*⟩

lemma *Compl-eq-Diff-UNIV*: $-A = (UNIV - A)$
⟨*proof*⟩

abbreviation *sym-diff* :: *'a set* ⇒ *'a set* ⇒ *'a set* **where**
sym-diff A B ≡ $((A - B) \cup (B - A))$

8.3.10 Augmenting a set − *insert*

lemma *insert-iff* [*simp*]: $a \in$ *insert b A* $\longleftrightarrow a = b \vee a \in A$
⟨*proof*⟩

lemma *insertI1*: $a \in$ *insert a B*
⟨*proof*⟩

lemma *insertI2*: $a \in B \Longrightarrow a \in$ *insert b B*
⟨*proof*⟩

lemma *insertE* [*elim*!]: $a \in$ *insert b A* $\Longrightarrow (a = b \Longrightarrow P) \Longrightarrow (a \in A \Longrightarrow P) \Longrightarrow$
P

⟨*proof*⟩

lemma *insertCI* [*intro*!]: $(a \notin B \Longrightarrow a = b) \Longrightarrow a \in \mathit{insert}\ b\ B$
— Classical introduction rule.
⟨*proof*⟩

lemma *subset-insert-iff*: $A \subseteq \mathit{insert}\ x\ B \longleftrightarrow (\mathit{if}\ x \in A\ \mathit{then}\ A - \{x\} \subseteq B\ \mathit{else}\ A \subseteq B)$
⟨*proof*⟩

lemma *set-insert*:
assumes $x \in A$
obtains B **where** $A = \mathit{insert}\ x\ B$ **and** $x \notin B$
⟨*proof*⟩

lemma *insert-ident*: $x \notin A \Longrightarrow x \notin B \Longrightarrow \mathit{insert}\ x\ A = \mathit{insert}\ x\ B \longleftrightarrow A = B$
⟨*proof*⟩

lemma *insert-eq-iff*:
assumes $a \notin A\ b \notin B$
shows $\mathit{insert}\ a\ A = \mathit{insert}\ b\ B \longleftrightarrow$
$(\mathit{if}\ a = b\ \mathit{then}\ A = B\ \mathit{else}\ \exists C.\ A = \mathit{insert}\ b\ C \wedge b \notin C \wedge B = \mathit{insert}\ a\ C \wedge a \notin C)$
(**is** *?L* $\longleftrightarrow$ *?R*)
⟨*proof*⟩

lemma *insert-UNIV*[*simp*]: $\mathit{insert}\ x\ \mathit{UNIV} = \mathit{UNIV}$
⟨*proof*⟩

8.3.11 Singletons, using insert

lemma *singletonI* [*intro*!]: $a \in \{a\}$
— Redundant? But unlike *insertCI*, it proves the subgoal immediately!
⟨*proof*⟩

lemma *singletonD* [*dest*!]: $b \in \{a\} \Longrightarrow b = a$
⟨*proof*⟩

lemmas *singletonE* = *singletonD* [*elim-format*]

lemma *singleton-iff*: $b \in \{a\} \longleftrightarrow b = a$
⟨*proof*⟩

lemma *singleton-inject* [*dest*!]: $\{a\} = \{b\} \Longrightarrow a = b$
⟨*proof*⟩

lemma *singleton-insert-inj-eq* [*iff*]: $\{b\} = \mathit{insert}\ a\ A \longleftrightarrow a = b \wedge A \subseteq \{b\}$
⟨*proof*⟩

lemma *singleton-insert-inj-eq′* [*iff*]: *insert a A* = {*b*} ⟷ *a* = *b* ∧ *A* ⊆ {*b*}
⟨*proof*⟩

lemma *subset-singletonD*: *A* ⊆ {*x*} ⟹ *A* = {} ∨ *A* = {*x*}
⟨*proof*⟩

lemma *subset-singleton-iff*: *X* ⊆ {*a*} ⟷ *X* = {} ∨ *X* = {*a*}
⟨*proof*⟩

lemma *subset-singleton-iff-Uniq*: (∃ *a*. *A* ⊆ {*a*}) ⟷ (∃$_{\leq 1}$*x*. *x* ∈ *A*)
⟨*proof*⟩

lemma *singleton-conv* [*simp*]: {*x*. *x* = *a*} = {*a*}
⟨*proof*⟩

lemma *singleton-conv2* [*simp*]: {*x*. *a* = *x*} = {*a*}
⟨*proof*⟩

lemma *Diff-single-insert*: *A* − {*x*} ⊆ *B* ⟹ *A* ⊆ *insert x B*
⟨*proof*⟩

lemma *subset-Diff-insert*: *A* ⊆ *B* − *insert x C* ⟷ *A* ⊆ *B* − *C* ∧ *x* ∉ *A*
⟨*proof*⟩

lemma *doubleton-eq-iff*: {*a*, *b*} = {*c*, *d*} ⟷ *a* = *c* ∧ *b* = *d* ∨ *a* = *d* ∧ *b* = *c*
⟨*proof*⟩

lemma *Un-singleton-iff*: *A* ∪ *B* = {*x*} ⟷ *A* = {} ∧ *B* = {*x*} ∨ *A* = {*x*} ∧ *B* = {} ∨ *A* = {*x*} ∧ *B* = {*x*}
⟨*proof*⟩

lemma *singleton-Un-iff*: {*x*} = *A* ∪ *B* ⟷ *A* = {} ∧ *B* = {*x*} ∨ *A* = {*x*} ∧ *B* = {} ∨ *A* = {*x*} ∧ *B* = {*x*}
⟨*proof*⟩

8.3.12 Image of a set under a function

Frequently *b* does not have the syntactic form of *f x*.

definition *image* :: (′*a* ⇒ ′*b*) ⇒ ′*a set* ⇒ ′*b set* (**infixr** ‘ *90*)
where *f* ‘ *A* = {*y*. ∃ *x*∈*A*. *y* = *f x*}

lemma *image-eqI* [*simp*, *intro*]: *b* = *f x* ⟹ *x* ∈ *A* ⟹ *b* ∈ *f* ‘ *A*
⟨*proof*⟩

lemma *imageI*: *x* ∈ *A* ⟹ *f x* ∈ *f* ‘ *A*
⟨*proof*⟩

lemma *rev-image-eqI*: *x* ∈ *A* ⟹ *b* = *f x* ⟹ *b* ∈ *f* ‘ *A*
— This version's more effective when we already have the required *x*.

⟨*proof*⟩

lemma *imageE* [*elim*!]:
assumes $b \in (\lambda x.\ f\,x)\ `\ A$ — The eta-expansion gives variable-name preservation.
obtains x **where** $b = f\,x$ **and** $x \in A$
⟨*proof*⟩

lemma *Compr-image-eq*: $\{x \in f\ `\ A.\ P\,x\} = f\ `\ \{x \in A.\ P\ (f\,x)\}$
⟨*proof*⟩

lemma *image-Un*: $f\ `\ (A \cup B) = f\ `\ A \cup f\ `\ B$
⟨*proof*⟩

lemma *image-iff*: $z \in f\ `\ A \longleftrightarrow (\exists\, x{\in}A.\ z = f\,x)$
⟨*proof*⟩

lemma *image-subsetI*: $(\bigwedge x.\ x \in A \Longrightarrow f\,x \in B) \Longrightarrow f\ `\ A \subseteq B$
— Replaces the three steps *subsetI*, *imageE*, *hypsubst*, but breaks too many existing proofs.
⟨*proof*⟩

lemma *image-subset-iff*: $f\ `\ A \subseteq B \longleftrightarrow (\forall\, x{\in}A.\ f\,x \in B)$
— This rewrite rule would confuse users if made default.
⟨*proof*⟩

lemma *subset-imageE*:
assumes $B \subseteq f\ `\ A$
obtains C **where** $C \subseteq A$ **and** $B = f\ `\ C$
⟨*proof*⟩

lemma *subset-image-iff*: $B \subseteq f\ `\ A \longleftrightarrow (\exists\, AA{\subseteq}A.\ B = f\ `\ AA)$
⟨*proof*⟩

lemma *image-ident* [*simp*]: $(\lambda x.\ x)\ `\ Y = Y$
⟨*proof*⟩

lemma *image-empty* [*simp*]: $f\ `\ \{\} = \{\}$
⟨*proof*⟩

lemma *image-insert* [*simp*]: $f\ `\ insert\ a\ B = insert\ (f\ a)\ (f\ `\ B)$
⟨*proof*⟩

lemma *image-constant*: $x \in A \Longrightarrow (\lambda x.\ c)\ `\ A = \{c\}$
⟨*proof*⟩

lemma *image-constant-conv*: $(\lambda x.\ c)\ `\ A = (if\ A = \{\}\ then\ \{\}\ else\ \{c\})$
⟨*proof*⟩

lemma *image-image*: $f\ `\ (g\ `\ A) = (\lambda x.\ f\ (g\ x))\ `\ A$

⟨*proof*⟩

lemma *insert-image* [*simp*]: $x \in A \Longrightarrow$ *insert* $(f\ x)\ (f\ `\ A) = f\ `\ A$
⟨*proof*⟩

lemma *image-is-empty* [*iff*]: $f\ `\ A = \{\} \longleftrightarrow A = \{\}$
⟨*proof*⟩

lemma *empty-is-image* [*iff*]: $\{\} = f\ `\ A \longleftrightarrow A = \{\}$
⟨*proof*⟩

lemma *image-Collect*: $f\ `\ \{x.\ P\ x\} = \{f\ x \mid x.\ P\ x\}$
— NOT suitable as a default simp rule: the RHS isn't simpler than the LHS, with its implicit quantifier and conjunction. Also image enjoys better equational properties than does the RHS.
⟨*proof*⟩

lemma *if-image-distrib* [*simp*]:
$(\lambda x.\ \textit{if}\ P\ x\ \textit{then}\ f\ x\ \textit{else}\ g\ x)\ `\ S = f\ `\ (S \cap \{x.\ P\ x\}) \cup g\ `\ (S \cap \{x.\ \neg\ P\ x\})$
⟨*proof*⟩

lemma *image-cong*:
$f\ `\ M = g\ `\ N$ **if** $M = N\ \bigwedge x.\ x \in N \Longrightarrow f\ x = g\ x$
⟨*proof*⟩

lemma *image-cong-simp* [*cong*]:
$f\ `\ M = g\ `\ N$ **if** $M = N\ \bigwedge x.\ x \in N$ =*simp*=> $f\ x = g\ x$
⟨*proof*⟩

lemma *image-Int-subset*: $f\ `\ (A \cap B) \subseteq f\ `\ A \cap f\ `\ B$
⟨*proof*⟩

lemma *image-diff-subset*: $f\ `\ A - f\ `\ B \subseteq f\ `\ (A - B)$
⟨*proof*⟩

lemma *Setcompr-eq-image*: $\{f\ x \mid x.\ x \in A\} = f\ `\ A$
⟨*proof*⟩

lemma *setcompr-eq-image*: $\{f\ x \mid x.\ P\ x\} = f\ `\ \{x.\ P\ x\}$
⟨*proof*⟩

lemma *ball-imageD*: $\forall x \in f\ `\ A.\ P\ x \Longrightarrow \forall x \in A.\ P\ (f\ x)$
⟨*proof*⟩

lemma *bex-imageD*: $\exists x \in f\ `\ A.\ P\ x \Longrightarrow \exists x \in A.\ P\ (f\ x)$
⟨*proof*⟩

lemma *image-add-0* [*simp*]: $(+)\ (0::'a::\textit{comm-monoid-add})\ `\ S = S$
⟨*proof*⟩

theorem *Cantors-theorem*: $\nexists f.\ f\ `\ A = Pow\ A$
⟨*proof*⟩

Range of a function – just an abbreviation for image!

abbreviation *range* :: $('a \Rightarrow 'b) \Rightarrow 'b\ set$ — of function
where $range\ f \equiv f\ `\ UNIV$

lemma *range-eqI*: $b = f\ x \Longrightarrow b \in range\ f$
⟨*proof*⟩

lemma *rangeI*: $f\ x \in range\ f$
⟨*proof*⟩

lemma *rangeE* [*elim?*]: $b \in range\ (\lambda x.\ f\ x) \Longrightarrow (\bigwedge x.\ b = f\ x \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *range-subsetD*: $range\ f \subseteq B \Longrightarrow f\ i \in B$
⟨*proof*⟩

lemma *full-SetCompr-eq*: $\{u.\ \exists x.\ u = f\ x\} = range\ f$
⟨*proof*⟩

lemma *range-composition*: $range\ (\lambda x.\ f\ (g\ x)) = f\ `\ range\ g$
⟨*proof*⟩

lemma *range-constant* [*simp*]: $range\ (\lambda\text{-}.\ x) = \{x\}$
⟨*proof*⟩

lemma *range-eq-singletonD*: $range\ f = \{a\} \Longrightarrow f\ x = a$
⟨*proof*⟩

8.3.13 Some rules with *if*

Elimination of $\{x.\ \ldots \wedge x = t \wedge \ldots\}$.

lemma *Collect-conv-if*: $\{x.\ x = a \wedge P\ x\} = (if\ P\ a\ then\ \{a\}\ else\ \{\})$
⟨*proof*⟩

lemma *Collect-conv-if2*: $\{x.\ a = x \wedge P\ x\} = (if\ P\ a\ then\ \{a\}\ else\ \{\})$
⟨*proof*⟩

Rewrite rules for boolean case-splitting: faster than *if-split* [*split*].

lemma *if-split-eq1*: $(if\ Q\ then\ x\ else\ y) = b \longleftrightarrow (Q \longrightarrow x = b) \wedge (\neg\ Q \longrightarrow y = b)$
⟨*proof*⟩

lemma *if-split-eq2*: $a = (if\ Q\ then\ x\ else\ y) \longleftrightarrow (Q \longrightarrow a = x) \wedge (\neg\ Q \longrightarrow a = y)$

⟨*proof*⟩

Split ifs on either side of the membership relation. Not for [*simp*] – can cause goals to blow up!

lemma *if-split-mem1*: (*if Q then x else y*) ∈ *b* ⟷ (*Q* ⟶ *x* ∈ *b*) ∧ (¬ *Q* ⟶ *y* ∈ *b*)
⟨*proof*⟩

lemma *if-split-mem2*: (*a* ∈ (*if Q then x else y*)) ⟷ (*Q* ⟶ *a* ∈ *x*) ∧ (¬ *Q* ⟶ *a* ∈ *y*)
⟨*proof*⟩

lemmas *split-ifs* = *if-bool-eq-conj if-split-eq1 if-split-eq2 if-split-mem1 if-split-mem2*

8.4 Further operations and lemmas

8.4.1 The "proper subset" relation

lemma *psubsetI* [*intro*!]: *A* ⊆ *B* ⟹ *A* ≠ *B* ⟹ *A* ⊂ *B*
⟨*proof*⟩

lemma *psubsetE* [*elim*!]: *A* ⊂ *B* ⟹ (*A* ⊆ *B* ⟹ ¬ *B* ⊆ *A* ⟹ *R*) ⟹ *R*
⟨*proof*⟩

lemma *psubset-insert-iff*:
A ⊂ *insert x B* ⟷ (*if x* ∈ *B then A* ⊂ *B else if x* ∈ *A then A* − {*x*} ⊂ *B else A* ⊆ *B*)
⟨*proof*⟩

lemma *psubset-eq*: *A* ⊂ *B* ⟷ *A* ⊆ *B* ∧ *A* ≠ *B*
⟨*proof*⟩

lemma *psubset-imp-subset*: *A* ⊂ *B* ⟹ *A* ⊆ *B*
⟨*proof*⟩

lemma *psubset-trans*: *A* ⊂ *B* ⟹ *B* ⊂ *C* ⟹ *A* ⊂ *C*
⟨*proof*⟩

lemma *psubsetD*: *A* ⊂ *B* ⟹ *c* ∈ *A* ⟹ *c* ∈ *B*
⟨*proof*⟩

lemma *psubset-subset-trans*: *A* ⊂ *B* ⟹ *B* ⊆ *C* ⟹ *A* ⊂ *C*
⟨*proof*⟩

lemma *subset-psubset-trans*: *A* ⊆ *B* ⟹ *B* ⊂ *C* ⟹ *A* ⊂ *C*
⟨*proof*⟩

lemma *psubset-imp-ex-mem*: *A* ⊂ *B* ⟹ ∃ *b*. *b* ∈ *B* − *A*
⟨*proof*⟩

lemma *atomize-ball*: $(\bigwedge x.\ x \in A \Longrightarrow P\ x) \equiv \mathit{Trueprop}\ (\forall x{\in}A.\ P\ x)$
⟨*proof*⟩

lemmas [*symmetric*, *rulify*] = *atomize-ball*
and [*symmetric*, *defn*] = *atomize-ball*

lemma *image-Pow-mono*: $f\ `\ A \subseteq B \Longrightarrow \mathit{image}\ f\ `\ \mathit{Pow}\ A \subseteq \mathit{Pow}\ B$
⟨*proof*⟩

lemma *image-Pow-surj*: $f\ `\ A = B \Longrightarrow \mathit{image}\ f\ `\ \mathit{Pow}\ A = \mathit{Pow}\ B$
⟨*proof*⟩

8.4.2 Derived rules involving subsets.

insert.

lemma *subset-insertI*: $B \subseteq \mathit{insert}\ a\ B$
⟨*proof*⟩

lemma *subset-insertI2*: $A \subseteq B \Longrightarrow A \subseteq \mathit{insert}\ b\ B$
⟨*proof*⟩

lemma *subset-insert*: $x \notin A \Longrightarrow A \subseteq \mathit{insert}\ x\ B \longleftrightarrow A \subseteq B$
⟨*proof*⟩

Finite Union – the least upper bound of two sets.

lemma *Un-upper1*: $A \subseteq A \cup B$
⟨*proof*⟩

lemma *Un-upper2*: $B \subseteq A \cup B$
⟨*proof*⟩

lemma *Un-least*: $A \subseteq C \Longrightarrow B \subseteq C \Longrightarrow A \cup B \subseteq C$
⟨*proof*⟩

Finite Intersection – the greatest lower bound of two sets.

lemma *Int-lower1*: $A \cap B \subseteq A$
⟨*proof*⟩

lemma *Int-lower2*: $A \cap B \subseteq B$
⟨*proof*⟩

lemma *Int-greatest*: $C \subseteq A \Longrightarrow C \subseteq B \Longrightarrow C \subseteq A \cap B$
⟨*proof*⟩

Set difference.

lemma *Diff-subset*[*simp*]: $A - B \subseteq A$
⟨*proof*⟩

lemma *Diff-subset-conv*: $A - B \subseteq C \longleftrightarrow A \subseteq B \cup C$
⟨*proof*⟩

8.4.3 Equalities involving union, intersection, inclusion, etc.

{}.

lemma *Collect-const* [*simp*]: $\{s.\ P\} = (\mathit{if}\ P\ \mathit{then}\ \mathit{UNIV}\ \mathit{else}\ \{\})$
— supersedes *Collect-False-empty*
⟨*proof*⟩

lemma *subset-empty* [*simp*]: $A \subseteq \{\} \longleftrightarrow A = \{\}$
⟨*proof*⟩

lemma *not-psubset-empty* [*iff*]: $\neg\ (A < \{\})$
⟨*proof*⟩

lemma *Collect-subset* [*simp*]: $\{x{\in}A.\ P\ x\} \subseteq A$ ⟨*proof*⟩

lemma *Collect-empty-eq* [*simp*]: $\mathit{Collect}\ P = \{\} \longleftrightarrow (\forall x.\ \neg\ P\ x)$
⟨*proof*⟩

lemma *empty-Collect-eq* [*simp*]: $\{\} = \mathit{Collect}\ P \longleftrightarrow (\forall x.\ \neg\ P\ x)$
⟨*proof*⟩

lemma *Collect-neg-eq*: $\{x.\ \neg\ P\ x\} = -\ \{x.\ P\ x\}$
⟨*proof*⟩

lemma *Collect-disj-eq*: $\{x.\ P\ x \vee Q\ x\} = \{x.\ P\ x\} \cup \{x.\ Q\ x\}$
⟨*proof*⟩

lemma *Collect-imp-eq*: $\{x.\ P\ x \longrightarrow Q\ x\} = -\ \{x.\ P\ x\} \cup \{x.\ Q\ x\}$
⟨*proof*⟩

lemma *Collect-conj-eq*: $\{x.\ P\ x \wedge Q\ x\} = \{x.\ P\ x\} \cap \{x.\ Q\ x\}$
⟨*proof*⟩

lemma *Collect-mono-iff*: $\mathit{Collect}\ P \subseteq \mathit{Collect}\ Q \longleftrightarrow (\forall x.\ P\ x \longrightarrow Q\ x)$
⟨*proof*⟩

insert.

lemma *insert-is-Un*: $\mathit{insert}\ a\ A = \{a\} \cup A$
— NOT SUITABLE FOR REWRITING since $\{a\} \equiv \mathit{insert}\ a\ \{\}$
⟨*proof*⟩

lemma *insert-not-empty* [*simp*]: $\mathit{insert}\ a\ A \neq \{\}$
and *empty-not-insert* [*simp*]: $\{\} \neq \mathit{insert}\ a\ A$
⟨*proof*⟩

lemma *insert-absorb*: $a \in A \Longrightarrow \mathit{insert}\ a\ A = A$
— [*simp*] causes recursive calls when there are nested inserts
— with *quadratic* running time
⟨*proof*⟩

lemma *insert-absorb2* [*simp*]: $\mathit{insert}\ x\ (\mathit{insert}\ x\ A) = \mathit{insert}\ x\ A$
⟨*proof*⟩

lemma *insert-commute*: $\mathit{insert}\ x\ (\mathit{insert}\ y\ A) = \mathit{insert}\ y\ (\mathit{insert}\ x\ A)$
⟨*proof*⟩

lemma *insert-subset* [*simp*]: $\mathit{insert}\ x\ A \subseteq B \longleftrightarrow x \in B \wedge A \subseteq B$
⟨*proof*⟩

lemma *mk-disjoint-insert*: $a \in A \Longrightarrow \exists B.\ A = \mathit{insert}\ a\ B \wedge a \notin B$
— use new B rather than $A - \{a\}$ to avoid infinite unfolding
⟨*proof*⟩

lemma *insert-Collect*: $\mathit{insert}\ a\ (\mathit{Collect}\ P) = \{u.\ u \neq a \longrightarrow P\ u\}$
⟨*proof*⟩

lemma *insert-inter-insert* [*simp*]: $\mathit{insert}\ a\ A \cap \mathit{insert}\ a\ B = \mathit{insert}\ a\ (A \cap B)$
⟨*proof*⟩

lemma *insert-disjoint* [*simp*]:
$\mathit{insert}\ a\ A \cap B = \{\} \longleftrightarrow a \notin B \wedge A \cap B = \{\}$
$\{\} = \mathit{insert}\ a\ A \cap B \longleftrightarrow a \notin B \wedge \{\} = A \cap B$
⟨*proof*⟩

lemma *disjoint-insert* [*simp*]:
$B \cap \mathit{insert}\ a\ A = \{\} \longleftrightarrow a \notin B \wedge B \cap A = \{\}$
$\{\} = A \cap \mathit{insert}\ b\ B \longleftrightarrow b \notin A \wedge \{\} = A \cap B$
⟨*proof*⟩

Int

lemma *Int-absorb*: $A \cap A = A$
⟨*proof*⟩

lemma *Int-left-absorb*: $A \cap (A \cap B) = A \cap B$
⟨*proof*⟩

lemma *Int-commute*: $A \cap B = B \cap A$
⟨*proof*⟩

lemma *Int-left-commute*: $A \cap (B \cap C) = B \cap (A \cap C)$
⟨*proof*⟩

lemma *Int-assoc*: $(A \cap B) \cap C = A \cap (B \cap C)$

⟨*proof*⟩

lemmas *Int-ac* = *Int-assoc Int-left-absorb Int-commute Int-left-commute*
— Intersection is an AC-operator

lemma *Int-absorb1*: $B \subseteq A \Longrightarrow A \cap B = B$
⟨*proof*⟩

lemma *Int-absorb2*: $A \subseteq B \Longrightarrow A \cap B = A$
⟨*proof*⟩

lemma *Int-empty-left*: $\{\} \cap B = \{\}$
⟨*proof*⟩

lemma *Int-empty-right*: $A \cap \{\} = \{\}$
⟨*proof*⟩

lemma *disjoint-eq-subset-Compl*: $A \cap B = \{\} \longleftrightarrow A \subseteq - B$
⟨*proof*⟩

lemma *disjoint-iff*: $A \cap B = \{\} \longleftrightarrow (\forall x.\ x{\in}A \longrightarrow x \notin B)$
⟨*proof*⟩

lemma *disjoint-iff-not-equal*: $A \cap B = \{\} \longleftrightarrow (\forall x{\in}A.\ \forall y{\in}B.\ x \neq y)$
⟨*proof*⟩

lemma *Int-UNIV-left*: $UNIV \cap B = B$
⟨*proof*⟩

lemma *Int-UNIV-right*: $A \cap UNIV = A$
⟨*proof*⟩

lemma *Int-Un-distrib*: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
⟨*proof*⟩

lemma *Int-Un-distrib2*: $(B \cup C) \cap A = (B \cap A) \cup (C \cap A)$
⟨*proof*⟩

lemma *Int-UNIV* [*simp*]: $A \cap B = UNIV \longleftrightarrow A = UNIV \wedge B = UNIV$
⟨*proof*⟩

lemma *Int-subset-iff* [*simp*]: $C \subseteq A \cap B \longleftrightarrow C \subseteq A \wedge C \subseteq B$
⟨*proof*⟩

lemma *Int-Collect*: $x \in A \cap \{x.\ P\ x\} \longleftrightarrow x \in A \wedge P\ x$
⟨*proof*⟩

Un.

lemma *Un-absorb*: $A \cup A = A$

⟨*proof*⟩

lemma *Un-left-absorb*: $A \cup (A \cup B) = A \cup B$
⟨*proof*⟩

lemma *Un-commute*: $A \cup B = B \cup A$
⟨*proof*⟩

lemma *Un-left-commute*: $A \cup (B \cup C) = B \cup (A \cup C)$
⟨*proof*⟩

lemma *Un-assoc*: $(A \cup B) \cup C = A \cup (B \cup C)$
⟨*proof*⟩

lemmas *Un-ac = Un-assoc Un-left-absorb Un-commute Un-left-commute*
— Union is an AC-operator

lemma *Un-absorb1*: $A \subseteq B \Longrightarrow A \cup B = B$
⟨*proof*⟩

lemma *Un-absorb2*: $B \subseteq A \Longrightarrow A \cup B = A$
⟨*proof*⟩

lemma *Un-empty-left*: $\{\} \cup B = B$
⟨*proof*⟩

lemma *Un-empty-right*: $A \cup \{\} = A$
⟨*proof*⟩

lemma *Un-UNIV-left*: $UNIV \cup B = UNIV$
⟨*proof*⟩

lemma *Un-UNIV-right*: $A \cup UNIV = UNIV$
⟨*proof*⟩

lemma *Un-insert-left* [*simp*]: $(insert\ a\ B) \cup C = insert\ a\ (B \cup C)$
⟨*proof*⟩

lemma *Un-insert-right* [*simp*]: $A \cup (insert\ a\ B) = insert\ a\ (A \cup B)$
⟨*proof*⟩

lemma *Int-insert-left*: $(insert\ a\ B) \cap C = (if\ a \in C\ then\ insert\ a\ (B \cap C)\ else\ B \cap C)$
⟨*proof*⟩

lemma *Int-insert-left-if0* [*simp*]: $a \notin C \Longrightarrow (insert\ a\ B) \cap C = B \cap C$
⟨*proof*⟩

lemma *Int-insert-left-if1* [*simp*]: $a \in C \Longrightarrow (insert\ a\ B) \cap C = insert\ a\ (B \cap C)$

⟨*proof*⟩

lemma *Int-insert-right*: $A \cap (\mathit{insert}\ a\ B) = (\mathit{if}\ a \in A\ \mathit{then}\ \mathit{insert}\ a\ (A \cap B)\ \mathit{else}\ A \cap B)$
⟨*proof*⟩

lemma *Int-insert-right-if0* [*simp*]: $a \notin A \Longrightarrow A \cap (\mathit{insert}\ a\ B) = A \cap B$
⟨*proof*⟩

lemma *Int-insert-right-if1* [*simp*]: $a \in A \Longrightarrow A \cap (\mathit{insert}\ a\ B) = \mathit{insert}\ a\ (A \cap B)$
⟨*proof*⟩

lemma *Un-Int-distrib*: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
⟨*proof*⟩

lemma *Un-Int-distrib2*: $(B \cap C) \cup A = (B \cup A) \cap (C \cup A)$
⟨*proof*⟩

lemma *Un-Int-crazy*: $(A \cap B) \cup (B \cap C) \cup (C \cap A) = (A \cup B) \cap (B \cup C) \cap (C \cup A)$
⟨*proof*⟩

lemma *subset-Un-eq*: $A \subseteq B \longleftrightarrow A \cup B = B$
⟨*proof*⟩

lemma *Un-empty* [*iff*]: $A \cup B = \{\} \longleftrightarrow A = \{\} \wedge B = \{\}$
⟨*proof*⟩

lemma *Un-subset-iff* [*simp*]: $A \cup B \subseteq C \longleftrightarrow A \subseteq C \wedge B \subseteq C$
⟨*proof*⟩

lemma *Un-Diff-Int*: $(A - B) \cup (A \cap B) = A$
⟨*proof*⟩

lemma *Diff-Int2*: $A \cap C - B \cap C = A \cap C - B$
⟨*proof*⟩

lemma *subset-UnE*:
assumes $C \subseteq A \cup B$
obtains $A'\ B'$ **where** $A' \subseteq A\ B' \subseteq B\ C = A' \cup B'$
⟨*proof*⟩

lemma *Un-Int-eq* [*simp*]: $(S \cup T) \cap S = S\ (S \cup T) \cap T = T\ S \cap (S \cup T) = S\ T \cap (S \cup T) = T$
⟨*proof*⟩

lemma *Int-Un-eq* [*simp*]: $(S \cap T) \cup S = S\ (S \cap T) \cup T = T\ S \cup (S \cap T) = S\ T \cup (S \cap T) = T$

⟨*proof*⟩

Set complement

lemma *Compl-disjoint* [*simp*]: $A \cap - A = \{\}$
⟨*proof*⟩

lemma *Compl-disjoint2* [*simp*]: $- A \cap A = \{\}$
⟨*proof*⟩

lemma *Compl-partition*: $A \cup - A = UNIV$
⟨*proof*⟩

lemma *Compl-partition2*: $- A \cup A = UNIV$
⟨*proof*⟩

lemma *double-complement*: $- (-A) = A$ **for** $A :: 'a\ set$
⟨*proof*⟩

lemma *Compl-Un*: $- (A \cup B) = (- A) \cap (- B)$
⟨*proof*⟩

lemma *Compl-Int*: $- (A \cap B) = (- A) \cup (- B)$
⟨*proof*⟩

lemma *subset-Compl-self-eq*: $A \subseteq - A \longleftrightarrow A = \{\}$
⟨*proof*⟩

lemma *Un-Int-assoc-eq*: $(A \cap B) \cup C = A \cap (B \cup C) \longleftrightarrow C \subseteq A$
— Halmos, Naive Set Theory, page 16.
⟨*proof*⟩

lemma *Compl-UNIV-eq*: $- UNIV = \{\}$
⟨*proof*⟩

lemma *Compl-empty-eq*: $- \{\} = UNIV$
⟨*proof*⟩

lemma *Compl-subset-Compl-iff* [*iff*]: $- A \subseteq - B \longleftrightarrow B \subseteq A$
⟨*proof*⟩

lemma *Compl-eq-Compl-iff* [*iff*]: $- A = - B \longleftrightarrow A = B$
for $A\ B :: 'a\ set$
⟨*proof*⟩

lemma *Compl-insert*: $- insert\ x\ A = (- A) - \{x\}$
⟨*proof*⟩

Bounded quantifiers.

The following are not added to the default simpset because (a) they duplicate the body and (b) there are no similar rules for *Int*.

lemma *ball-Un*: $(\forall x \in A \cup B.\ P\ x) \longleftrightarrow (\forall x{\in}A.\ P\ x) \wedge (\forall x{\in}B.\ P\ x)$
⟨*proof*⟩

lemma *bex-Un*: $(\exists x \in A \cup B.\ P\ x) \longleftrightarrow (\exists x{\in}A.\ P\ x) \vee (\exists x{\in}B.\ P\ x)$
⟨*proof*⟩

Set difference.

lemma *Diff-eq*: $A - B = A \cap (-\ B)$
⟨*proof*⟩

lemma *Diff-eq-empty-iff* [*simp*]: $A - B = \{\} \longleftrightarrow A \subseteq B$
⟨*proof*⟩

lemma *Diff-cancel* [*simp*]: $A - A = \{\}$
⟨*proof*⟩

lemma *Diff-idemp* [*simp*]: $(A - B) - B = A - B$
for $A\ B :: 'a\ set$
⟨*proof*⟩

lemma *Diff-triv*: $A \cap B = \{\} \Longrightarrow A - B = A$
⟨*proof*⟩

lemma *empty-Diff* [*simp*]: $\{\} - A = \{\}$
⟨*proof*⟩

lemma *Diff-empty* [*simp*]: $A - \{\} = A$
⟨*proof*⟩

lemma *Diff-UNIV* [*simp*]: $A - UNIV = \{\}$
⟨*proof*⟩

lemma *Diff-insert0* [*simp*]: $x \notin A \Longrightarrow A - insert\ x\ B = A - B$
⟨*proof*⟩

lemma *Diff-insert*: $A - insert\ a\ B = A - B - \{a\}$
— NOT SUITABLE FOR REWRITING since $\{a\} \equiv insert\ a\ 0$
⟨*proof*⟩

lemma *Diff-insert2*: $A - insert\ a\ B = A - \{a\} - B$
— NOT SUITABLE FOR REWRITING since $\{a\} \equiv insert\ a\ 0$
⟨*proof*⟩

lemma *insert-Diff-if*: $insert\ x\ A - B = (if\ x \in B\ then\ A - B\ else\ insert\ x\ (A - B))$
⟨*proof*⟩

lemma *insert-Diff1* [*simp*]: $x \in B \Longrightarrow \mathit{insert}\ x\ A - B = A - B$
⟨*proof*⟩

lemma *insert-Diff-single*[*simp*]: $\mathit{insert}\ a\ (A - \{a\}) = \mathit{insert}\ a\ A$
⟨*proof*⟩

lemma *insert-Diff*: $a \in A \Longrightarrow \mathit{insert}\ a\ (A - \{a\}) = A$
⟨*proof*⟩

lemma *Diff-insert-absorb*: $x \notin A \Longrightarrow (\mathit{insert}\ x\ A) - \{x\} = A$
⟨*proof*⟩

lemma *Diff-disjoint* [*simp*]: $A \cap (B - A) = \{\}$
⟨*proof*⟩

lemma *Diff-partition*: $A \subseteq B \Longrightarrow A \cup (B - A) = B$
⟨*proof*⟩

lemma *double-diff*: $A \subseteq B \Longrightarrow B \subseteq C \Longrightarrow B - (C - A) = A$
⟨*proof*⟩

lemma *Un-Diff-cancel* [*simp*]: $A \cup (B - A) = A \cup B$
⟨*proof*⟩

lemma *Un-Diff-cancel2* [*simp*]: $(B - A) \cup A = B \cup A$
⟨*proof*⟩

lemma *Diff-Un*: $A - (B \cup C) = (A - B) \cap (A - C)$
⟨*proof*⟩

lemma *Diff-Int*: $A - (B \cap C) = (A - B) \cup (A - C)$
⟨*proof*⟩

lemma *Diff-Diff-Int*: $A - (A - B) = A \cap B$
⟨*proof*⟩

lemma *Un-Diff*: $(A \cup B) - C = (A - C) \cup (B - C)$
⟨*proof*⟩

lemma *Int-Diff*: $(A \cap B) - C = A \cap (B - C)$
⟨*proof*⟩

lemma *Diff-Int-distrib*: $C \cap (A - B) = (C \cap A) - (C \cap B)$
⟨*proof*⟩

lemma *Diff-Int-distrib2*: $(A - B) \cap C = (A \cap C) - (B \cap C)$
⟨*proof*⟩

lemma *Diff-Compl* [*simp*]: $A - (- B) = A \cap B$
⟨*proof*⟩

lemma *Compl-Diff-eq* [*simp*]: $- (A - B) = - A \cup B$
⟨*proof*⟩

lemma *subset-Compl-singleton* [*simp*]: $A \subseteq - \{b\} \longleftrightarrow b \notin A$
⟨*proof*⟩

Quantification over type *bool*.

lemma *bool-induct*: $P\ True \Longrightarrow P\ False \Longrightarrow P\ x$
⟨*proof*⟩

lemma *all-bool-eq*: $(\forall b.\ P\ b) \longleftrightarrow P\ True \wedge P\ False$
⟨*proof*⟩

lemma *bool-contrapos*: $P\ x \Longrightarrow \neg\ P\ False \Longrightarrow P\ True$
⟨*proof*⟩

lemma *ex-bool-eq*: $(\exists b.\ P\ b) \longleftrightarrow P\ True \vee P\ False$
⟨*proof*⟩

lemma *UNIV-bool*: $UNIV = \{False, True\}$
⟨*proof*⟩

Pow

lemma *Pow-empty* [*simp*]: $Pow\ \{\} = \{\{\}\}$
⟨*proof*⟩

lemma *Pow-singleton-iff* [*simp*]: $Pow\ X = \{Y\} \longleftrightarrow X = \{\} \wedge Y = \{\}$
⟨*proof*⟩

lemma *Pow-insert*: $Pow\ (insert\ a\ A) = Pow\ A \cup (insert\ a\ `\ Pow\ A)$
⟨*proof*⟩

lemma *Pow-Compl*: $Pow\ (- A) = \{- B \mid B.\ A \in Pow\ B\}$
⟨*proof*⟩

lemma *Pow-UNIV* [*simp*]: $Pow\ UNIV = UNIV$
⟨*proof*⟩

lemma *Un-Pow-subset*: $Pow\ A \cup Pow\ B \subseteq Pow\ (A \cup B)$
⟨*proof*⟩

lemma *Pow-Int-eq* [*simp*]: $Pow\ (A \cap B) = Pow\ A \cap Pow\ B$
⟨*proof*⟩

Miscellany.

lemma *Int-Diff-disjoint*: $A \cap B \cap (A - B) = \{\}$
⟨*proof*⟩

lemma *Int-Diff-Un*: $A \cap B \cup (A - B) = A$
⟨*proof*⟩

lemma *set-eq-subset*: $A = B \longleftrightarrow A \subseteq B \wedge B \subseteq A$
⟨*proof*⟩

lemma *subset-iff*: $A \subseteq B \longleftrightarrow (\forall t.\ t \in A \longrightarrow t \in B)$
⟨*proof*⟩

lemma *subset-iff-psubset-eq*: $A \subseteq B \longleftrightarrow A \subset B \vee A = B$
⟨*proof*⟩

lemma *all-not-in-conv* [*simp*]: $(\forall x.\ x \notin A) \longleftrightarrow A = \{\}$
⟨*proof*⟩

lemma *ex-in-conv*: $(\exists x.\ x \in A) \longleftrightarrow A \neq \{\}$
⟨*proof*⟩

lemma *ball-simps* [*simp*, *no-atp*]:
$\bigwedge A\ P\ Q.\ (\forall x{\in}A.\ P\ x \vee Q) \longleftrightarrow ((\forall x{\in}A.\ P\ x) \vee Q)$
$\bigwedge A\ P\ Q.\ (\forall x{\in}A.\ P \vee Q\ x) \longleftrightarrow (P \vee (\forall x{\in}A.\ Q\ x))$
$\bigwedge A\ P\ Q.\ (\forall x{\in}A.\ P \longrightarrow Q\ x) \longleftrightarrow (P \longrightarrow (\forall x{\in}A.\ Q\ x))$
$\bigwedge A\ P\ Q.\ (\forall x{\in}A.\ P\ x \longrightarrow Q) \longleftrightarrow ((\exists x{\in}A.\ P\ x) \longrightarrow Q)$
$\bigwedge P.\ (\forall x{\in}\{\}.\ P\ x) \longleftrightarrow \mathit{True}$
$\bigwedge P.\ (\forall x{\in}\mathit{UNIV}.\ P\ x) \longleftrightarrow (\forall x.\ P\ x)$
$\bigwedge a\ B\ P.\ (\forall x{\in}\mathit{insert}\ a\ B.\ P\ x) \longleftrightarrow (P\ a \wedge (\forall x{\in}B.\ P\ x))$
$\bigwedge P\ Q.\ (\forall x{\in}\mathit{Collect}\ Q.\ P\ x) \longleftrightarrow (\forall x.\ Q\ x \longrightarrow P\ x)$
$\bigwedge A\ P\ f.\ (\forall x{\in}f`A.\ P\ x) \longleftrightarrow (\forall x{\in}A.\ P\ (f\ x))$
$\bigwedge A\ P.\ (\neg\ (\forall x{\in}A.\ P\ x)) \longleftrightarrow (\exists x{\in}A.\ \neg\ P\ x)$
⟨*proof*⟩

lemma *bex-simps* [*simp*, *no-atp*]:
$\bigwedge A\ P\ Q.\ (\exists x{\in}A.\ P\ x \wedge Q) \longleftrightarrow ((\exists x{\in}A.\ P\ x) \wedge Q)$
$\bigwedge A\ P\ Q.\ (\exists x{\in}A.\ P \wedge Q\ x) \longleftrightarrow (P \wedge (\exists x{\in}A.\ Q\ x))$
$\bigwedge P.\ (\exists x{\in}\{\}.\ P\ x) \longleftrightarrow \mathit{False}$
$\bigwedge P.\ (\exists x{\in}\mathit{UNIV}.\ P\ x) \longleftrightarrow (\exists x.\ P\ x)$
$\bigwedge a\ B\ P.\ (\exists x{\in}\mathit{insert}\ a\ B.\ P\ x) \longleftrightarrow (P\ a \vee (\exists x{\in}B.\ P\ x))$
$\bigwedge P\ Q.\ (\exists x{\in}\mathit{Collect}\ Q.\ P\ x) \longleftrightarrow (\exists x.\ Q\ x \wedge P\ x)$
$\bigwedge A\ P\ f.\ (\exists x{\in}f`A.\ P\ x) \longleftrightarrow (\exists x{\in}A.\ P\ (f\ x))$
$\bigwedge A\ P.\ (\neg(\exists x{\in}A.\ P\ x)) \longleftrightarrow (\forall x{\in}A.\ \neg\ P\ x)$
⟨*proof*⟩

lemma *ex-image-cong-iff* [*simp*, *no-atp*]:
$(\exists x.\ x{\in}f`A) \longleftrightarrow A \neq \{\}$ $(\exists x.\ x{\in}f`A \wedge P\ x) \longleftrightarrow (\exists x{\in}A.\ P\ (f\ x))$
⟨*proof*⟩

8.4.4 Monotonicity of various operations

lemma *image-mono*: $A \subseteq B \Longrightarrow f\ `\ A \subseteq f\ `\ B$
⟨*proof*⟩

lemma *Pow-mono*: $A \subseteq B \Longrightarrow Pow\ A \subseteq Pow\ B$
⟨*proof*⟩

lemma *insert-mono*: $C \subseteq D \Longrightarrow insert\ a\ C \subseteq insert\ a\ D$
⟨*proof*⟩

lemma *Un-mono*: $A \subseteq C \Longrightarrow B \subseteq D \Longrightarrow A \cup B \subseteq C \cup D$
⟨*proof*⟩

lemma *Int-mono*: $A \subseteq C \Longrightarrow B \subseteq D \Longrightarrow A \cap B \subseteq C \cap D$
⟨*proof*⟩

lemma *Diff-mono*: $A \subseteq C \Longrightarrow D \subseteq B \Longrightarrow A - B \subseteq C - D$
⟨*proof*⟩

lemma *Compl-anti-mono*: $A \subseteq B \Longrightarrow -B \subseteq -A$
⟨*proof*⟩

Monotonicity of implications.

lemma *in-mono*: $A \subseteq B \Longrightarrow x \in A \longrightarrow x \in B$
⟨*proof*⟩

lemma *conj-mono*: $P1 \longrightarrow Q1 \Longrightarrow P2 \longrightarrow Q2 \Longrightarrow (P1 \wedge P2) \longrightarrow (Q1 \wedge Q2)$
⟨*proof*⟩

lemma *disj-mono*: $P1 \longrightarrow Q1 \Longrightarrow P2 \longrightarrow Q2 \Longrightarrow (P1 \vee P2) \longrightarrow (Q1 \vee Q2)$
⟨*proof*⟩

lemma *imp-mono*: $Q1 \longrightarrow P1 \Longrightarrow P2 \longrightarrow Q2 \Longrightarrow (P1 \longrightarrow P2) \longrightarrow (Q1 \longrightarrow Q2)$
⟨*proof*⟩

lemma *imp-refl*: $P \longrightarrow P$ ⟨*proof*⟩

lemma *not-mono*: $Q \longrightarrow P \Longrightarrow \neg P \longrightarrow \neg Q$
⟨*proof*⟩

lemma *ex-mono*: $(\bigwedge x.\ P\ x \longrightarrow Q\ x) \Longrightarrow (\exists x.\ P\ x) \longrightarrow (\exists x.\ Q\ x)$
⟨*proof*⟩

lemma *all-mono*: $(\bigwedge x.\ P\ x \longrightarrow Q\ x) \Longrightarrow (\forall x.\ P\ x) \longrightarrow (\forall x.\ Q\ x)$
⟨*proof*⟩

lemma *Collect-mono*: $(\bigwedge x.\ P\ x \longrightarrow Q\ x) \Longrightarrow Collect\ P \subseteq Collect\ Q$

⟨*proof*⟩

lemma *Int-Collect-mono*: $A \subseteq B \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow P\ x \longrightarrow Q\ x) \Longrightarrow A \cap Collect\ P \subseteq B \cap Collect\ Q$
⟨*proof*⟩

lemmas *basic-monos* =
subset-refl imp-refl disj-mono conj-mono ex-mono Collect-mono in-mono

lemma *eq-to-mono*: $a = b \Longrightarrow c = d \Longrightarrow b \longrightarrow d \Longrightarrow a \longrightarrow c$
⟨*proof*⟩

8.4.5 Inverse image of a function

definition *vimage* :: $('a \Rightarrow 'b) \Rightarrow 'b\ set \Rightarrow 'a\ set$ (**infixr** $-`$ *90*)
where $f\ -`\ B \equiv \{x.\ f\ x \in B\}$

lemma *vimage-eq* [*simp*]: $a \in f\ -`\ B \longleftrightarrow f\ a \in B$
⟨*proof*⟩

lemma *vimage-singleton-eq*: $a \in f\ -`\ \{b\} \longleftrightarrow f\ a = b$
⟨*proof*⟩

lemma *vimageI* [*intro*]: $f\ a = b \Longrightarrow b \in B \Longrightarrow a \in f\ -`\ B$
⟨*proof*⟩

lemma *vimageI2*: $f\ a \in A \Longrightarrow a \in f\ -`\ A$
⟨*proof*⟩

lemma *vimageE* [*elim!*]: $a \in f\ -`\ B \Longrightarrow (\bigwedge x.\ f\ a = x \Longrightarrow x \in B \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *vimageD*: $a \in f\ -`\ A \Longrightarrow f\ a \in A$
⟨*proof*⟩

lemma *vimage-empty* [*simp*]: $f\ -`\ \{\} = \{\}$
⟨*proof*⟩

lemma *vimage-Compl*: $f\ -`\ (-\ A) = -\ (f\ -`\ A)$
⟨*proof*⟩

lemma *vimage-Un* [*simp*]: $f\ -`\ (A \cup B) = (f\ -`\ A) \cup (f\ -`\ B)$
⟨*proof*⟩

lemma *vimage-Int* [*simp*]: $f\ -`\ (A \cap B) = (f\ -`\ A) \cap (f\ -`\ B)$
⟨*proof*⟩

lemma *vimage-Collect-eq* [*simp*]: $f\ -`\ Collect\ P = \{y.\ P\ (f\ y)\}$
⟨*proof*⟩

lemma *vimage-Collect*: $(\bigwedge x.\ P\ (f\ x) = Q\ x) \Longrightarrow f\ -`\ (Collect\ P) = Collect\ Q$
⟨*proof*⟩

lemma *vimage-insert*: $f\ -`\ (insert\ a\ B) = (f\ -`\ \{a\}) \cup (f\ -`\ B)$
— NOT suitable for rewriting because of the recurrence of $\{a\}$.
⟨*proof*⟩

lemma *vimage-Diff*: $f\ -`\ (A - B) = (f\ -`\ A) - (f\ -`\ B)$
⟨*proof*⟩

lemma *vimage-UNIV* [*simp*]: $f\ -`\ UNIV = UNIV$
⟨*proof*⟩

lemma *vimage-mono*: $A \subseteq B \Longrightarrow f\ -`\ A \subseteq f\ -`\ B$
— monotonicity
⟨*proof*⟩

lemma *vimage-image-eq*: $f\ -`\ (f\ `\ A) = \{y.\ \exists x{\in}A.\ f\ x = f\ y\}$
⟨*proof*⟩

lemma *image-vimage-subset*: $f\ `\ (f\ -`\ A) \subseteq A$
⟨*proof*⟩

lemma *image-vimage-eq* [*simp*]: $f\ `\ (f\ -`\ A) = A \cap range\ f$
⟨*proof*⟩

lemma *image-subset-iff-subset-vimage*: $f\ `\ A \subseteq B \longleftrightarrow A \subseteq f\ -`\ B$
⟨*proof*⟩

lemma *subset-vimage-iff*: $A \subseteq f\ -`\ B \longleftrightarrow (\forall x{\in}A.\ f\ x \in B)$
⟨*proof*⟩

lemma *vimage-const* [*simp*]: $((\lambda x.\ c)\ -`\ A) = (if\ c \in A\ then\ UNIV\ else\ \{\})$
⟨*proof*⟩

lemma *vimage-if* [*simp*]: $((\lambda x.\ if\ x \in B\ then\ c\ else\ d)\ -`\ A) =$
$(if\ c \in A\ then\ (if\ d \in A\ then\ UNIV\ else\ B)$
$else\ if\ d \in A\ then - B\ else\ \{\})$
⟨*proof*⟩

lemma *vimage-inter-cong*: $(\bigwedge\ w.\ w \in S \Longrightarrow f\ w = g\ w) \Longrightarrow f\ -`\ y \cap S = g\ -`\ y \cap S$
⟨*proof*⟩

lemma *vimage-ident* [*simp*]: $(\lambda x.\ x)\ -`\ Y = Y$
⟨*proof*⟩

8.4.6 Singleton sets

definition *is-singleton* :: *'a set ⇒ bool*
  where *is-singleton A ⟷ (∃ x. A = {x})*

lemma *is-singletonI* [*simp*, *intro!*]: *is-singleton {x}*
  ⟨*proof*⟩

lemma *is-singletonI'*: *A ≠ {} ⟹ (⋀x y. x ∈ A ⟹ y ∈ A ⟹ x = y) ⟹ is-singleton A*
  ⟨*proof*⟩

lemma *is-singletonE*: *is-singleton A ⟹ (⋀x. A = {x} ⟹ P) ⟹ P*
  ⟨*proof*⟩

8.4.7 Getting the contents of a singleton set

definition *the-elem* :: *'a set ⇒ 'a*
  where *the-elem X = (THE x. X = {x})*

lemma *the-elem-eq* [*simp*]: *the-elem {x} = x*
  ⟨*proof*⟩

lemma *is-singleton-the-elem*: *is-singleton A ⟷ A = {the-elem A}*
  ⟨*proof*⟩

lemma *the-elem-image-unique*:
  assumes *A ≠ {}*
    and *∗*: *⋀y. y ∈ A ⟹ f y = f x*
  shows *the-elem (f ‘ A) = f x*
  ⟨*proof*⟩

8.4.8 Monad operation

definition *bind* :: *'a set ⇒ ('a ⇒ 'b set) ⇒ 'b set*
  where *bind A f = {x. ∃ B ∈ f‘A. x ∈ B}*

hide-const (**open**) *bind*

lemma *bind-bind*: *Set.bind (Set.bind A B) C = Set.bind A (λx. Set.bind (B x) C)*
  for *A* :: *'a set*
  ⟨*proof*⟩

lemma *empty-bind* [*simp*]: *Set.bind {} f = {}*
  ⟨*proof*⟩

lemma *nonempty-bind-const*: *A ≠ {} ⟹ Set.bind A (λ-. B) = B*
  ⟨*proof*⟩

lemma *bind-const*: *Set.bind A* (λ-. *B*) = (*if A* = {} *then* {} *else B*)
⟨*proof*⟩

lemma *bind-singleton-conv-image*: *Set.bind A* (λ*x*. {*f x*}) = *f* ' *A*
⟨*proof*⟩

8.4.9 Operations for execution

definition *is-empty* :: *'a set* ⇒ *bool*
where [*code-abbrev*]: *is-empty A* ⟷ *A* = {}

hide-const (**open**) *is-empty*

definition *remove* :: *'a* ⇒ *'a set* ⇒ *'a set*
where [*code-abbrev*]: *remove x A* = *A* − {*x*}

hide-const (**open**) *remove*

lemma *member-remove* [*simp*]: *x* ∈ *Set.remove y A* ⟷ *x* ∈ *A* ∧ *x* ≠ *y*
⟨*proof*⟩

definition *filter* :: (*'a* ⇒ *bool*) ⇒ *'a set* ⇒ *'a set*
where [*code-abbrev*]: *filter P A* = {*a* ∈ *A*. *P a*}

hide-const (**open**) *filter*

lemma *member-filter* [*simp*]: *x* ∈ *Set.filter P A* ⟷ *x* ∈ *A* ∧ *P x*
⟨*proof*⟩

instantiation *set* :: (*equal*) *equal*
begin

definition *HOL.equal A B* ⟷ *A* ⊆ *B* ∧ *B* ⊆ *A*

instance ⟨*proof*⟩

end

Misc

definition *pairwise* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *'a set* ⇒ *bool*
where *pairwise R S* ⟷ (∀ *x* ∈ *S*. ∀ *y* ∈ *S*. *x* ≠ *y* ⟶ *R x y*)

lemma *pairwise-alt*: *pairwise R S* ⟷ (∀ *x*∈*S*. ∀ *y*∈*S*−{*x*}. *R x y*)
⟨*proof*⟩

lemma *pairwise-trivial* [*simp*]: *pairwise* (λ*i j*. *j* ≠ *i*) *I*
⟨*proof*⟩

lemma *pairwiseI* [*intro?*]:

pairwise R S **if** $\bigwedge x\ y.\ x \in S \Longrightarrow y \in S \Longrightarrow x \neq y \Longrightarrow R\ x\ y$
⟨*proof*⟩

lemma *pairwiseD*:
$R\ x\ y$ **and** $R\ y\ x$
if *pairwise R S* $x \in S$ **and** $y \in S$ **and** $x \neq y$
⟨*proof*⟩

lemma *pairwise-empty* [*simp*]: *pairwise P* $\{\}$
⟨*proof*⟩

lemma *pairwise-singleton* [*simp*]: *pairwise P* $\{A\}$
⟨*proof*⟩

lemma *pairwise-insert*:
pairwise r (*insert x s*) $\longleftrightarrow$ $(\forall\ y.\ y \in s \wedge y \neq x \longrightarrow r\ x\ y \wedge r\ y\ x) \wedge$ *pairwise r s*
⟨*proof*⟩

lemma *pairwise-subset*: *pairwise P S* $\Longrightarrow$ $T \subseteq S \Longrightarrow$ *pairwise P T*
⟨*proof*⟩

lemma *pairwise-mono*: $[\![$*pairwise P A*; $\bigwedge x\ y.\ P\ x\ y \Longrightarrow Q\ x\ y$; $B \subseteq A]\!] \Longrightarrow$ *pairwise Q B*
⟨*proof*⟩

lemma *pairwise-imageI*:
pairwise P (*f* ‘ *A*)
if $\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow x \neq y \Longrightarrow f\ x \neq f\ y \Longrightarrow P\ (f\ x)\ (f\ y)$
⟨*proof*⟩

lemma *pairwise-image*: *pairwise r* (*f* ‘ *s*) $\longleftrightarrow$ *pairwise* $(\lambda x\ y.\ (f\ x \neq f\ y) \longrightarrow r\ (f\ x)\ (f\ y))\ s$
⟨*proof*⟩

definition *disjnt* :: *'a set* $\Rightarrow$ *'a set* $\Rightarrow$ *bool*
where *disjnt A B* $\longleftrightarrow$ $A \cap B = \{\}$

lemma *disjnt-self-iff-empty* [*simp*]: *disjnt S S* $\longleftrightarrow$ $S = \{\}$
⟨*proof*⟩

lemma *disjnt-iff*: *disjnt A B* $\longleftrightarrow$ $(\forall\ x.\ \neg\ (x \in A \wedge x \in B))$
⟨*proof*⟩

lemma *disjnt-sym*: *disjnt A B* $\Longrightarrow$ *disjnt B A*
⟨*proof*⟩

lemma *disjnt-empty1* [*simp*]: *disjnt* $\{\}$ *A* **and** *disjnt-empty2* [*simp*]: *disjnt A* $\{\}$
⟨*proof*⟩

lemma *disjnt-insert1* [*simp*]: *disjnt* (*insert a X*) *Y* ⟷ *a* ∉ *Y* ∧ *disjnt X Y*
⟨*proof*⟩

lemma *disjnt-insert2* [*simp*]: *disjnt Y* (*insert a X*) ⟷ *a* ∉ *Y* ∧ *disjnt Y X*
⟨*proof*⟩

lemma *disjnt-subset1* : ⟦*disjnt X Y*; *Z* ⊆ *X*⟧ ⟹ *disjnt Z Y*
⟨*proof*⟩

lemma *disjnt-subset2* : ⟦*disjnt X Y*; *Z* ⊆ *Y*⟧ ⟹ *disjnt X Z*
⟨*proof*⟩

lemma *disjnt-Un1* [*simp*]: *disjnt* (*A* ∪ *B*) *C* ⟷ *disjnt A C* ∧ *disjnt B C*
⟨*proof*⟩

lemma *disjnt-Un2* [*simp*]: *disjnt C* (*A* ∪ *B*) ⟷ *disjnt C A* ∧ *disjnt C B*
⟨*proof*⟩

lemma *disjnt-Diff1*: *disjnt* (*X*−*Y*) (*U*−*V*) **and** *disjnt-Diff2*: *disjnt* (*U*−*V*) (*X*−*Y*)
if *X* ⊆ *V*
⟨*proof*⟩

lemma *disjoint-image-subset*: ⟦*pairwise disjnt* 𝒜; ⋀*X*. *X* ∈ 𝒜 ⟹ *f X* ⊆ *X*⟧ ⟹
pairwise disjnt (*f* ‘𝒜)
⟨*proof*⟩

lemma *pairwise-disjnt-iff*: *pairwise disjnt* 𝒜 ⟷ (∀ *x*. ∃≤1 *X*. *X* ∈ 𝒜 ∧ *x* ∈ *X*)
⟨*proof*⟩

lemma *disjnt-insert*:
‹*disjnt* (*insert x M*) *N*› **if** ‹*x* ∉ *N*› ‹*disjnt M N*›
⟨*proof*⟩

lemma *Int-emptyI*: (⋀*x*. *x* ∈ *A* ⟹ *x* ∈ *B* ⟹ *False*) ⟹ *A* ∩ *B* = {}
⟨*proof*⟩

lemma *in-image-insert-iff*:
assumes ⋀*C*. *C* ∈ *B* ⟹ *x* ∉ *C*
shows *A* ∈ *insert x* ‘ *B* ⟷ *x* ∈ *A* ∧ *A* − {*x*} ∈ *B* (**is** *?P* ⟷ *?Q*)
⟨*proof*⟩

hide-const (**open**) *member not-member*

lemmas *equalityI* = *subset-antisym*
lemmas *set-mp* = *subsetD*
lemmas *set-rev-mp* = *rev-subsetD*

⟨*ML*⟩

end

9 HOL type definitions

theory *Typedef*
imports *Set*
keywords
typedef :: *thy-goal-defn* **and**
morphisms :: *quasi-command*
begin

locale *type-definition* =
fixes *Rep* **and** *Abs* **and** *A*
assumes *Rep*: $Rep\ x \in A$
and *Rep-inverse*: $Abs\ (Rep\ x) = x$
and *Abs-inverse*: $y \in A \Longrightarrow Rep\ (Abs\ y) = y$
— This will be axiomatized for each typedef!
begin

lemma *Rep-inject*: $Rep\ x = Rep\ y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *Abs-inject*:
assumes $x \in A$ **and** $y \in A$
shows $Abs\ x = Abs\ y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *Rep-cases* [*cases set*]:
assumes $y \in A$
and *hyp*: $\bigwedge x.\ y = Rep\ x \Longrightarrow P$
shows P
⟨*proof*⟩

lemma *Abs-cases* [*cases type*]:
assumes r: $\bigwedge y.\ x = Abs\ y \Longrightarrow y \in A \Longrightarrow P$
shows P
⟨*proof*⟩

lemma *Rep-induct* [*induct set*]:
assumes y: $y \in A$
and *hyp*: $\bigwedge x.\ P\ (Rep\ x)$
shows $P\ y$
⟨*proof*⟩

lemma *Abs-induct* [*induct type*]:
assumes r: $\bigwedge y.\ y \in A \Longrightarrow P\ (Abs\ y)$
shows $P\ x$
⟨*proof*⟩

lemma *Rep-range*: *range Rep* = *A*
⟨*proof*⟩

lemma *Abs-image*: *Abs* ' *A* = *UNIV*
⟨*proof*⟩

end

⟨*ML*⟩

end

10 Notions about functions

theory *Fun*
imports *Set*
keywords *functor* :: *thy-goal-defn*
begin

lemma *apply-inverse*: *f x* = *u* ⟹ (⋀*x*. *P x* ⟹ *g* (*f x*) = *x*) ⟹ *P x* ⟹ *x* = *g u*
⟨*proof*⟩

Uniqueness, so NOT the axiom of choice.

lemma *uniq-choice*: ∀ *x*. ∃!*y*. *Q x y* ⟹ ∃ *f*. ∀ *x*. *Q x* (*f x*)
⟨*proof*⟩

lemma *b-uniq-choice*: ∀ *x*∈*S*. ∃!*y*. *Q x y* ⟹ ∃ *f*. ∀ *x*∈*S*. *Q x* (*f x*)
⟨*proof*⟩

10.1 The Identity Function *id*

definition *id* :: *'a* ⇒ *'a*
where *id* = (λ*x*. *x*)

lemma *id-apply* [*simp*]: *id x* = *x*
⟨*proof*⟩

lemma *image-id* [*simp*]: *image id* = *id*
⟨*proof*⟩

lemma *vimage-id* [*simp*]: *vimage id* = *id*
⟨*proof*⟩

lemma *eq-id-iff*: (∀ *x*. *f x* = *x*) ⟷ *f* = *id*
⟨*proof*⟩

code-printing
constant *id* ⤳ (*Haskell*) *id*

10.2 The Composition Operator $f \circ g$

definition *comp* :: $('b \Rightarrow 'c) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'c$ (**infixl** $\circ$ *55*)
where $f \circ g = (\lambda x.\ f\ (g\ x))$

notation (*ASCII*)
comp (**infixl** *o 55*)

lemma *comp-apply* [*simp*]: $(f \circ g)\ x = f\ (g\ x)$
⟨*proof*⟩

lemma *comp-assoc*: $(f \circ g) \circ h = f \circ (g \circ h)$
⟨*proof*⟩

lemma *id-comp* [*simp*]: $id \circ g = g$
⟨*proof*⟩

lemma *comp-id* [*simp*]: $f \circ id = f$
⟨*proof*⟩

lemma *comp-eq-dest*: $a \circ b = c \circ d \Longrightarrow a\ (b\ v) = c\ (d\ v)$
⟨*proof*⟩

lemma *comp-eq-elim*: $a \circ b = c \circ d \Longrightarrow ((\bigwedge v.\ a\ (b\ v) = c\ (d\ v)) \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

lemma *comp-eq-dest-lhs*: $a \circ b = c \Longrightarrow a\ (b\ v) = c\ v$
⟨*proof*⟩

lemma *comp-eq-id-dest*: $a \circ b = id \circ c \Longrightarrow a\ (b\ v) = c\ v$
⟨*proof*⟩

lemma *image-comp*: $f\ `\ (g\ `\ r) = (f \circ g)\ `\ r$
⟨*proof*⟩

lemma *vimage-comp*: $f\ -`\ (g\ -`\ x) = (g \circ f)\ -`\ x$
⟨*proof*⟩

lemma *image-eq-imp-comp*: $f\ `\ A = g\ `\ B \Longrightarrow (h \circ f)\ `\ A = (h \circ g)\ `\ B$
⟨*proof*⟩

lemma *image-bind*: $f\ `\ (Set.bind\ A\ g) = Set.bind\ A\ ((`)\ f \circ g)$
⟨*proof*⟩

lemma *bind-image*: $Set.bind\ (f\ `\ A)\ g = Set.bind\ A\ (g \circ f)$
⟨*proof*⟩

lemma (**in** *group-add*) *minus-comp-minus* [*simp*]: $uminus \circ uminus = id$
⟨*proof*⟩

lemma (**in** *boolean-algebra*) *minus-comp-minus* [*simp*]: *uminus* ∘ *uminus* = *id*
⟨*proof*⟩

code-printing
constant *comp* ⇀ (*SML*) **infixl** *5 o* **and** (*Haskell*) **infixr** *9* .

10.3 The Forward Composition Operator *fcomp*

definition *fcomp* :: ('*a* ⇒ '*b*) ⇒ ('*b* ⇒ '*c*) ⇒ '*a* ⇒ '*c* (**infixl** ∘> *60*)
where *f* ∘> *g* = (λ*x*. *g* (*f x*))

lemma *fcomp-apply* [*simp*]: (*f* ∘> *g*) *x* = *g* (*f x*)
⟨*proof*⟩

lemma *fcomp-assoc*: (*f* ∘> *g*) ∘> *h* = *f* ∘> (*g* ∘> *h*)
⟨*proof*⟩

lemma *id-fcomp* [*simp*]: *id* ∘> *g* = *g*
⟨*proof*⟩

lemma *fcomp-id* [*simp*]: *f* ∘> *id* = *f*
⟨*proof*⟩

lemma *fcomp-comp*: *fcomp f g* = *comp g f*
⟨*proof*⟩

code-printing
constant *fcomp* ⇀ (*Eval*) **infixl** *1* #>

no-notation *fcomp* (**infixl** ∘> *60*)

10.4 Mapping functions

definition *map-fun* :: ('*c* ⇒ '*a*) ⇒ ('*b* ⇒ '*d*) ⇒ ('*a* ⇒ '*b*) ⇒ '*c* ⇒ '*d*
where *map-fun f g h* = *g* ∘ *h* ∘ *f*

lemma *map-fun-apply* [*simp*]: *map-fun f g h x* = *g* (*h* (*f x*))
⟨*proof*⟩

10.5 Injectivity and Bijectivity

definition *inj-on* :: ('*a* ⇒ '*b*) ⇒ '*a set* ⇒ *bool* — injective
where *inj-on f A* ⟷ (∀ *x*∈*A*. ∀ *y*∈*A*. *f x* = *f y* ⟶ *x* = *y*)

definition *bij-betw* :: ('*a* ⇒ '*b*) ⇒ '*a set* ⇒ '*b set* ⇒ *bool* — bijective
where *bij-betw f A B* ⟷ *inj-on f A* ∧ *f* ‘ *A* = *B*

A common special case: functions injective, surjective or bijective over the entire domain type.

abbreviation *inj* :: ('*a* ⇒ '*b*) ⇒ *bool*

where $inj\ f \equiv inj\text{-}on\ f\ UNIV$

abbreviation $surj :: ('a \Rightarrow 'b) \Rightarrow bool$
where $surj\ f \equiv range\ f = UNIV$

translations — The negated case:
$\neg\ CONST\ surj\ f \leftharpoonup CONST\ range\ f \neq CONST\ UNIV$

abbreviation $bij :: ('a \Rightarrow 'b) \Rightarrow bool$
where $bij\ f \equiv bij\text{-}betw\ f\ UNIV\ UNIV$

lemma *inj-def*: $inj\ f \longleftrightarrow (\forall x\ y.\ f\ x = f\ y \longrightarrow x = y)$
⟨*proof*⟩

lemma *injI*: $(\bigwedge x\ y.\ f\ x = f\ y \Longrightarrow x = y) \Longrightarrow inj\ f$
⟨*proof*⟩

theorem *range-ex1-eq*: $inj\ f \Longrightarrow b \in range\ f \longleftrightarrow (\exists!x.\ b = f\ x)$
⟨*proof*⟩

lemma *injD*: $inj\ f \Longrightarrow f\ x = f\ y \Longrightarrow x = y$
⟨*proof*⟩

lemma *inj-on-eq-iff*: $inj\text{-}on\ f\ A \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow f\ x = f\ y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *inj-on-cong*: $(\bigwedge a.\ a \in A \Longrightarrow f\ a = g\ a) \Longrightarrow inj\text{-}on\ f\ A \longleftrightarrow inj\text{-}on\ g\ A$
⟨*proof*⟩

lemma *image-strict-mono*: $inj\text{-}on\ f\ B \Longrightarrow A \subset B \Longrightarrow f\ `\ A \subset f\ `\ B$
⟨*proof*⟩

lemma *inj-compose*: $inj\ f \Longrightarrow inj\ g \Longrightarrow inj\ (f \circ g)$
⟨*proof*⟩

lemma *inj-fun*: $inj\ f \Longrightarrow inj\ (\lambda x\ y.\ f\ x)$
⟨*proof*⟩

lemma *inj-eq*: $inj\ f \Longrightarrow f\ x = f\ y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *inj-on-iff-Uniq*: $inj\text{-}on\ f\ A \longleftrightarrow (\forall x{\in}A.\ \exists_{\leq 1} y.\ y{\in}A \wedge f\ x = f\ y)$
⟨*proof*⟩

lemma *inj-on-id*[*simp*]: $inj\text{-}on\ id\ A$
⟨*proof*⟩

lemma *inj-on-id2*[*simp*]: $inj\text{-}on\ (\lambda x.\ x)\ A$
⟨*proof*⟩

lemma *inj-on-Int*: *inj-on f A* ∨ *inj-on f B* ⟹ *inj-on f* (*A* ∩ *B*)
⟨*proof*⟩

lemma *surj-id*: *surj id*
⟨*proof*⟩

lemma *bij-id*[*simp*]: *bij id*
⟨*proof*⟩

lemma *bij-uminus*: *bij* (*uminus* :: *'a* ⇒ *'a*::*group-add*)
⟨*proof*⟩

lemma *bij-betwE*: *bij-betw f A B* ⟹ ∀ *a*∈*A*. *f a* ∈ *B*
⟨*proof*⟩

lemma *inj-onI* [*intro?*]: (⋀*x y*. *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ *f x* = *f y* ⟹ *x* = *y*) ⟹ *inj-on f A*
⟨*proof*⟩

For those frequent proofs by contradiction

lemma *inj-onCI*: (⋀*x y*. *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ *f x* = *f y* ⟹ *x* ≠ *y* ⟹ *False*) ⟹ *inj-on f A*
⟨*proof*⟩

lemma *inj-on-inverseI*: (⋀*x*. *x* ∈ *A* ⟹ *g* (*f x*) = *x*) ⟹ *inj-on f A*
⟨*proof*⟩

lemma *inj-onD*: *inj-on f A* ⟹ *f x* = *f y* ⟹ *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ *x* = *y*
⟨*proof*⟩

lemma *inj-on-subset*:
assumes *inj-on f A*
and *B* ⊆ *A*
shows *inj-on f B*
⟨*proof*⟩

lemma *comp-inj-on*: *inj-on f A* ⟹ *inj-on g* (*f* ‘ *A*) ⟹ *inj-on* (*g* ∘ *f*) *A*
⟨*proof*⟩

lemma *inj-on-imageI*: *inj-on* (*g* ∘ *f*) *A* ⟹ *inj-on g* (*f* ‘ *A*)
⟨*proof*⟩

lemma *inj-on-image-iff*:
∀ *x*∈*A*. ∀ *y*∈*A*. *g* (*f x*) = *g* (*f y*) ⟷ *g x* = *g y* ⟹ *inj-on f A* ⟹ *inj-on g* (*f* ‘ *A*) ⟷ *inj-on g A*
⟨*proof*⟩

lemma *inj-on-contraD*: *inj-on f A* ⟹ *x* ≠ *y* ⟹ *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ *f x* ≠ *f y*

⟨*proof*⟩

lemma *inj-singleton* [*simp*]: *inj-on* (λ*x*. {*x*}) *A*
⟨*proof*⟩

lemma *inj-on-empty*[*iff*]: *inj-on f* {}
⟨*proof*⟩

lemma *subset-inj-on*: *inj-on f B* ⟹ *A* ⊆ *B* ⟹ *inj-on f A*
⟨*proof*⟩

lemma *inj-on-Un*: *inj-on f* (*A* ∪ *B*) ⟷ *inj-on f A* ∧ *inj-on f B* ∧ *f* ‘ (*A* − *B*) ∩ *f* ‘ (*B* − *A*) = {}
⟨*proof*⟩

lemma *inj-on-insert* [*iff*]: *inj-on f* (*insert a A*) ⟷ *inj-on f A* ∧ *f a* ∉ *f* ‘ (*A* − {*a*})
⟨*proof*⟩

lemma *inj-on-diff*: *inj-on f A* ⟹ *inj-on f* (*A* − *B*)
⟨*proof*⟩

lemma *comp-inj-on-iff*: *inj-on f A* ⟹ *inj-on f′* (*f* ‘ *A*) ⟷ *inj-on* (*f′* ∘ *f*) *A*
⟨*proof*⟩

lemma *inj-on-imageI2*: *inj-on* (*f′* ∘ *f*) *A* ⟹ *inj-on f A*
⟨*proof*⟩

lemma *inj-img-insertE*:
assumes *inj-on f A*
assumes *x* ∉ *B*
and *insert x B* = *f* ‘ *A*
obtains *x′ A′* **where** *x′* ∉ *A′* **and** *A* = *insert x′ A′* **and** *x* = *f x′* **and** *B* = *f* ‘ *A′*
⟨*proof*⟩

lemma *linorder-inj-onI*:
fixes *A* :: *′a*::*order set*
assumes *ne*: ⋀*x y*. [*x* < *y*; *x*∈*A*; *y*∈*A*] ⟹ *f x* ≠ *f y* **and** *lin*: ⋀*x y*. [*x*∈*A*; *y*∈*A*] ⟹ *x*≤*y* ∨ *y*≤*x*
shows *inj-on f A*
⟨*proof*⟩

lemma *linorder-inj-onI′*:
fixes *A* :: *′a* :: *linorder set*
assumes ⋀*i j*. *i* ∈ *A* ⟹ *j* ∈ *A* ⟹ *i* < *j* ⟹ *f i* ≠ *f j*
shows *inj-on f A*
⟨*proof*⟩

lemma *linorder-injI*:
assumes $\bigwedge x\ y{::}'a{::}linorder.\ x < y \Longrightarrow f\ x \neq f\ y$
shows *inj f*
— Courtesy of Stephan Merz
⟨*proof*⟩

lemma *inj-on-image-Pow*: *inj-on f A* ⟹*inj-on* (*image f*) (*Pow A*)
⟨*proof*⟩

lemma *bij-betw-image-Pow*: *bij-betw f A B* ⟹ *bij-betw* (*image f*) (*Pow A*) (*Pow B*)
⟨*proof*⟩

lemma *surj-def*: *surj f* ⟷ $(\forall y.\ \exists x.\ y = f\ x)$
⟨*proof*⟩

lemma *surjI*:
assumes $\bigwedge x.\ g\ (f\ x) = x$
shows *surj g*
⟨*proof*⟩

lemma *surjD*: *surj f* ⟹ $\exists x.\ y = f\ x$
⟨*proof*⟩

lemma *surjE*: *surj f* ⟹ $(\bigwedge x.\ y = f\ x \Longrightarrow C) \Longrightarrow C$
⟨*proof*⟩

lemma *comp-surj*: *surj f* ⟹ *surj g* ⟹ *surj* $(g \circ f)$
⟨*proof*⟩

lemma *bij-betw-imageI*: *inj-on f A* ⟹ *f ' A = B* ⟹ *bij-betw f A B*
⟨*proof*⟩

lemma *bij-betw-imp-surj-on*: *bij-betw f A B* ⟹ *f ' A = B*
⟨*proof*⟩

lemma *bij-betw-imp-surj*: *bij-betw f A UNIV* ⟹ *surj f*
⟨*proof*⟩

lemma *bij-betw-empty1*: *bij-betw f* {} *A* ⟹ *A* = {}
⟨*proof*⟩

lemma *bij-betw-empty2*: *bij-betw f A* {} ⟹ *A* = {}
⟨*proof*⟩

lemma *inj-on-imp-bij-betw*: *inj-on f A* ⟹ *bij-betw f A* (*f ' A*)
⟨*proof*⟩

lemma *bij-betw-DiffI*:

assumes *bij-betw f A B bij-betw f C D C* $\subseteq$ *A D* $\subseteq$ *B*
shows *bij-betw f* (*A* − *C*) (*B* − *D*)
⟨*proof*⟩

lemma *bij-betw-singleton-iff* [*simp*]: *bij-betw f* {*x*} {*y*} $\longleftrightarrow$ *f x* = *y*
⟨*proof*⟩

lemma *bij-betw-singletonI* [*intro*]: *f x* = *y* $\Longrightarrow$ *bij-betw f* {*x*} {*y*}
⟨*proof*⟩

lemma *bij-betw-apply*: ⟦*bij-betw f A B*; *a* ∈ *A*⟧ $\Longrightarrow$ *f a* ∈ *B*
⟨*proof*⟩

lemma *bij-def*: *bij f* $\longleftrightarrow$ *inj f* ∧ *surj f*
⟨*proof*⟩

lemma *bijI*: *inj f* $\Longrightarrow$ *surj f* $\Longrightarrow$ *bij f*
⟨*proof*⟩

lemma *bij-is-inj*: *bij f* $\Longrightarrow$ *inj f*
⟨*proof*⟩

lemma *bij-is-surj*: *bij f* $\Longrightarrow$ *surj f*
⟨*proof*⟩

lemma *bij-betw-imp-inj-on*: *bij-betw f A B* $\Longrightarrow$ *inj-on f A*
⟨*proof*⟩

lemma *bij-betw-trans*: *bij-betw f A B* $\Longrightarrow$ *bij-betw g B C* $\Longrightarrow$ *bij-betw* (*g* ∘ *f*) *A C*
⟨*proof*⟩

lemma *bij-comp*: *bij f* $\Longrightarrow$ *bij g* $\Longrightarrow$ *bij* (*g* ∘ *f*)
⟨*proof*⟩

lemma *bij-betw-comp-iff*: *bij-betw f A A′* $\Longrightarrow$ *bij-betw f′ A′ A″* $\longleftrightarrow$ *bij-betw* (*f′* ∘ *f*) *A A″*
⟨*proof*⟩

lemma *bij-betw-comp-iff2*:
assumes *bij*: *bij-betw f′ A′ A″*
and *img*: *f* ‘ *A* ≤ *A′*
shows *bij-betw f A A′* $\longleftrightarrow$ *bij-betw* (*f′* ∘ *f*) *A A″* (**is** *?L* $\longleftrightarrow$ *?R*)
⟨*proof*⟩

lemma *bij-betw-inv*:
assumes *bij-betw f A B*
shows ∃ *g*. *bij-betw g B A*
⟨*proof*⟩

lemma *bij-betw-cong*: $(\bigwedge a.\ a \in A \Longrightarrow f\ a = g\ a) \Longrightarrow$ *bij-betw f A A′* = *bij-betw g A A′*
⟨*proof*⟩

lemma *bij-betw-id*[*intro*, *simp*]: *bij-betw id A A*
⟨*proof*⟩

lemma *bij-betw-id-iff*: *bij-betw id A B* $\longleftrightarrow$ $A = B$
⟨*proof*⟩

lemma *bij-betw-combine*:
bij-betw f A B $\Longrightarrow$ *bij-betw f C D* $\Longrightarrow$ $B \cap D = \{\}$ $\Longrightarrow$ *bij-betw f* $(A \cup C)$ $(B \cup D)$
⟨*proof*⟩

lemma *bij-betw-subset*: *bij-betw f A A′* $\Longrightarrow$ $B \subseteq A \Longrightarrow$ f ‘ $B = B' \Longrightarrow$ *bij-betw f B B′*
⟨*proof*⟩

lemma *bij-betw-ball*: *bij-betw f A B* $\Longrightarrow$ $(\forall\, b \in B.\ phi\ b) = (\forall\, a \in A.\ phi\ (f\ a))$
⟨*proof*⟩

lemma *bij-pointE*:
assumes *bij f*
obtains x **where** $y = f\ x$ **and** $\bigwedge x'.\ y = f\ x' \Longrightarrow x' = x$
⟨*proof*⟩

lemma *bij-iff*:
‹*bij f* $\longleftrightarrow$ $(\forall\, x.\ \exists!y.\ f\ y = x)$› (**is** ‹*?P* $\longleftrightarrow$ *?Q*›)
⟨*proof*⟩

lemma *bij-betw-partition*:
‹*bij-betw f A B*›
if ‹*bij-betw f* $(A \cup C)$ $(B \cup D)$› ‹*bij-betw f C D*› ‹$A \cap C = \{\}$› ‹$B \cap D = \{\}$›
⟨*proof*⟩

lemma *surj-image-vimage-eq*: *surj f* $\Longrightarrow$ f ‘ $(f - `\ A) = A$
⟨*proof*⟩

lemma *surj-vimage-empty*:
assumes *surj f*
shows $f - `\ A = \{\} \longleftrightarrow A = \{\}$
⟨*proof*⟩

lemma *inj-vimage-image-eq*: *inj f* $\Longrightarrow$ $f - `\ (f\ `\ A) = A$
⟨*proof*⟩

lemma *vimage-subsetD*: *surj f* $\Longrightarrow$ $f - `\ B \subseteq A \Longrightarrow B \subseteq f\ `\ A$
⟨*proof*⟩

lemma *vimage-subsetI*: $inj\ f \Longrightarrow B \subseteq f\ `\ A \Longrightarrow f\ -`\ B \subseteq A$
⟨*proof*⟩

lemma *vimage-subset-eq*: $bij\ f \Longrightarrow f\ -`\ B \subseteq A \longleftrightarrow B \subseteq f\ `\ A$
⟨*proof*⟩

lemma *inj-on-image-eq-iff*: $inj\text{-}on\ f\ C \Longrightarrow A \subseteq C \Longrightarrow B \subseteq C \Longrightarrow f\ `\ A = f\ `\ B \longleftrightarrow A = B$
⟨*proof*⟩

lemma *inj-on-Un-image-eq-iff*: $inj\text{-}on\ f\ (A \cup B) \Longrightarrow f\ `\ A = f\ `\ B \longleftrightarrow A = B$
⟨*proof*⟩

lemma *inj-on-image-Int*: $inj\text{-}on\ f\ C \Longrightarrow A \subseteq C \Longrightarrow B \subseteq C \Longrightarrow f\ `\ (A \cap B) = f\ `\ A \cap f\ `\ B$
⟨*proof*⟩

lemma *inj-on-image-set-diff*: $inj\text{-}on\ f\ C \Longrightarrow A - B \subseteq C \Longrightarrow B \subseteq C \Longrightarrow f\ `\ (A - B) = f\ `\ A - f\ `\ B$
⟨*proof*⟩

lemma *image-Int*: $inj\ f \Longrightarrow f\ `\ (A \cap B) = f\ `\ A \cap f\ `\ B$
⟨*proof*⟩

lemma *image-set-diff*: $inj\ f \Longrightarrow f\ `\ (A - B) = f\ `\ A - f\ `\ B$
⟨*proof*⟩

lemma *inj-on-image-mem-iff*: $inj\text{-}on\ f\ B \Longrightarrow a \in B \Longrightarrow A \subseteq B \Longrightarrow f\ a \in f\ `\ A \longleftrightarrow a \in A$
⟨*proof*⟩

lemma *inj-image-mem-iff*: $inj\ f \Longrightarrow f\ a \in f\ `\ A \longleftrightarrow a \in A$
⟨*proof*⟩

lemma *inj-image-subset-iff*: $inj\ f \Longrightarrow f\ `\ A \subseteq f\ `\ B \longleftrightarrow A \subseteq B$
⟨*proof*⟩

lemma *inj-image-eq-iff*: $inj\ f \Longrightarrow f\ `\ A = f\ `\ B \longleftrightarrow A = B$
⟨*proof*⟩

lemma *surj-Compl-image-subset*: $surj\ f \Longrightarrow -\ (f\ `\ A) \subseteq f\ `\ (-\ A)$
⟨*proof*⟩

lemma *inj-image-Compl-subset*: $inj\ f \Longrightarrow f\ `\ (-\ A) \subseteq -\ (f\ `\ A)$
⟨*proof*⟩

lemma *bij-image-Compl-eq*: $bij\ f \Longrightarrow f\ `\ (-\ A) = -\ (f\ `\ A)$
⟨*proof*⟩

lemma *inj-vimage-singleton*: *inj f* $\Longrightarrow$ *f* $-`$ {*a*} $\subseteq$ {*THE x. f x = a*}
— The inverse image of a singleton under an injective function is included in a singleton.
⟨*proof*⟩

lemma *inj-on-vimage-singleton*: *inj-on f A* $\Longrightarrow$ *f* $-`$ {*a*} $\cap$ *A* $\subseteq$ {*THE x. x* $\in$ *A* $\wedge$ *f x = a*}
⟨*proof*⟩

lemma *bij-betw-byWitness*:
assumes *left*: $\forall a \in A.\ f'\ (f\ a) = a$
and *right*: $\forall a' \in A'.\ f\ (f'\ a') = a'$
and *f* $`$ *A* $\subseteq$ *A′*
and *img2*: *f′* $`$ *A′* $\subseteq$ *A*
shows *bij-betw f A A′*
⟨*proof*⟩

corollary *notIn-Un-bij-betw*:
assumes *b* $\notin$ *A*
and *f b* $\notin$ *A′*
and *bij-betw f A A′*
shows *bij-betw f* (*A* $\cup$ {*b*}) (*A′* $\cup$ {*f b*})
⟨*proof*⟩

lemma *notIn-Un-bij-betw3*:
assumes *b* $\notin$ *A*
and *f b* $\notin$ *A′*
shows *bij-betw f A A′* = *bij-betw f* (*A* $\cup$ {*b*}) (*A′* $\cup$ {*f b*})
⟨*proof*⟩

lemma *inj-on-disjoint-Un*:
assumes *inj-on f A* **and** *inj-on g B*
and *f* $`$ *A* $\cap$ *g* $`$ *B* = {}
shows *inj-on* ($\lambda x.$ *if x* $\in$ *A then f x else g x*) (*A* $\cup$ *B*)
⟨*proof*⟩

lemma *bij-betw-disjoint-Un*:
assumes *bij-betw f A C* **and** *bij-betw g B D*
and *A* $\cap$ *B* = {}
and *C* $\cap$ *D* = {}
shows *bij-betw* ($\lambda x.$ *if x* $\in$ *A then f x else g x*) (*A* $\cup$ *B*) (*C* $\cup$ *D*)
⟨*proof*⟩

lemma *involuntory-imp-bij*:
‹*bij f*› **if** ‹$\bigwedge x.$ *f* (*f x*) = *x*›
⟨*proof*⟩

10.5.1 Inj/surj/bij of Algebraic Operations

context *cancel-semigroup-add*
begin

lemma *inj-on-add* [*simp*]:
inj-on ((+) *a*) *A*
⟨*proof*⟩

lemma *inj-on-add'* [*simp*]:
inj-on (λ*b*. *b* + *a*) *A*
⟨*proof*⟩

lemma *bij-betw-add* [*simp*]:
bij-betw ((+) *a*) *A B* ⟷ (+) *a* ‘ *A* = *B*
⟨*proof*⟩

end

context *group-add*
begin

lemma *diff-left-imp-eq*: *a* − *b* = *a* − *c* ⟹ *b* = *c*
⟨*proof*⟩

lemma *inj-uminus*[*simp*, *intro*]: *inj-on uminus A*
⟨*proof*⟩

lemma *surj-uminus*[*simp*]: *surj uminus*
⟨*proof*⟩

lemma *surj-plus* [*simp*]:
surj ((+) *a*)
⟨*proof*⟩

lemma *surj-plus-right* [*simp*]:
surj (λ*b*. *b*+*a*)
⟨*proof*⟩

lemma *inj-on-diff-left* [*simp*]:
‹*inj-on* ((−) *a*) *A*›
⟨*proof*⟩

lemma *inj-on-diff-right* [*simp*]:
‹*inj-on* (λ*b*. *b* − *a*) *A*›
⟨*proof*⟩

lemma *surj-diff* [*simp*]:
surj ((−) *a*)
⟨*proof*⟩

lemma *surj-diff-right* [*simp*]:
  surj (λ*x*. *x* − *a*)
⟨*proof*⟩

lemma shows *bij-plus*: *bij* ((+) *a*) **and** *bij-plus-right*: *bij* (λ*x*. *x* + *a*)
  and *bij-uminus*: *bij uminus*
  and *bij-diff*: *bij* ((−) *a*) **and** *bij-diff-right*: *bij* (λ*x*. *x* − *a*)
⟨*proof*⟩

lemma *translation-subtract-Compl*:
  (λ*x*. *x* − *a*) ‘ (− *t*) = − ((λ*x*. *x* − *a*) ‘ *t*)
⟨*proof*⟩

lemma *translation-diff*:
  (+) *a* ‘ (*s* − *t*) = ((+) *a* ‘ *s*) − ((+) *a* ‘ *t*)
  ⟨*proof*⟩

lemma *translation-subtract-diff*:
  (λ*x*. *x* − *a*) ‘ (*s* − *t*) = ((λ*x*. *x* − *a*) ‘ *s*) − ((λ*x*. *x* − *a*) ‘ *t*)
⟨*proof*⟩

lemma *translation-Int*:
  (+) *a* ‘ (*s* ∩ *t*) = ((+) *a* ‘ *s*) ∩ ((+) *a* ‘ *t*)
  ⟨*proof*⟩

lemma *translation-subtract-Int*:
  (λ*x*. *x* − *a*) ‘ (*s* ∩ *t*) = ((λ*x*. *x* − *a*) ‘ *s*) ∩ ((λ*x*. *x* − *a*) ‘ *t*)
⟨*proof*⟩

end

context *ab-group-add*
begin

lemma *translation-Compl*:
  (+) *a* ‘ (− *t*) = − ((+) *a* ‘ *t*)
⟨*proof*⟩

end

10.6 Function Updating

definition *fun-upd* :: (′*a* ⇒ ′*b*) ⇒ ′*a* ⇒ ′*b* ⇒ (′*a* ⇒ ′*b*)
  where *fun-upd f a b* = (λ*x*. *if x* = *a then b else f x*)

nonterminal *updbinds* **and** *updbind*

syntax
-updbind :: *'a* ⇒ *'a* ⇒ *updbind* ((*2*- :=/ -))
:: *updbind* ⇒ *updbinds* (-)
-updbinds:: *updbind* ⇒ *updbinds* ⇒ *updbinds* (-,/ -)
-Update :: *'a* ⇒ *updbinds* ⇒ *'a* (-/'((-)') [*1000*, *0*] *900*)

translations
-Update f (*-updbinds b bs*) ⇌ *-Update* (*-Update f b*) *bs*
f(*x*:=*y*) ⇌ *CONST fun-upd f x y*

lemma *fun-upd-idem-iff*: *f*(*x*:=*y*) = *f* ⟷ *f x* = *y*
⟨*proof*⟩

lemma *fun-upd-idem*: *f x* = *y* ⟹ *f*(*x* := *y*) = *f*
⟨*proof*⟩

lemma *fun-upd-triv* [*iff*]: *f*(*x* := *f x*) = *f*
⟨*proof*⟩

lemma *fun-upd-apply* [*simp*]: (*f*(*x* := *y*)) *z* = (*if z* = *x then y else f z*)
⟨*proof*⟩

lemma *fun-upd-same*: (*f*(*x* := *y*)) *x* = *y*
⟨*proof*⟩

lemma *fun-upd-other*: *z* ≠ *x* ⟹ (*f*(*x* := *y*)) *z* = *f z*
⟨*proof*⟩

lemma *fun-upd-upd* [*simp*]: *f*(*x* := *y*, *x* := *z*) = *f*(*x* := *z*)
⟨*proof*⟩

lemma *fun-upd-twist*: *a* ≠ *c* ⟹ (*m*(*a* := *b*))(*c* := *d*) = (*m*(*c* := *d*))(*a* := *b*)
⟨*proof*⟩

lemma *inj-on-fun-updI*: *inj-on f A* ⟹ *y* ∉ *f* ‘ *A* ⟹ *inj-on* (*f*(*x* := *y*)) *A*
⟨*proof*⟩

lemma *fun-upd-image*: *f*(*x* := *y*) ‘ *A* = (*if x* ∈ *A then insert y* (*f* ‘ (*A* − {*x*}))
else f ‘ *A*)
⟨*proof*⟩

lemma *fun-upd-comp*: *f* ∘ (*g*(*x* := *y*)) = (*f* ∘ *g*)(*x* := *f y*)
⟨*proof*⟩

lemma *fun-upd-eqD*: *f*(*x* := *y*) = *g*(*x* := *z*) ⟹ *y* = *z*
⟨*proof*⟩

10.7 *override-on*

definition *override-on* :: $('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a\ set \Rightarrow 'a \Rightarrow 'b$
where *override-on f g A* = $(\lambda a.\ if\ a \in A\ then\ g\ a\ else\ f\ a)$

lemma *override-on-emptyset*[*simp*]: *override-on f g* $\{\}$ = *f*
⟨*proof*⟩

lemma *override-on-apply-notin*[*simp*]: $a \notin A \Longrightarrow$ (*override-on f g A*) *a* = *f a*
⟨*proof*⟩

lemma *override-on-apply-in*[*simp*]: $a \in A \Longrightarrow$ (*override-on f g A*) *a* = *g a*
⟨*proof*⟩

lemma *override-on-insert*: *override-on f g* (*insert x X*) = (*override-on f g X*)(*x*:=*g x*)
⟨*proof*⟩

lemma *override-on-insert'*: *override-on f g* (*insert x X*) = (*override-on* (*f*(*x*:=*g x*)) *g X*)
⟨*proof*⟩

10.8 Inversion of injective functions

definition *the-inv-into* :: $'a\ set \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('b \Rightarrow 'a)$
where *the-inv-into A f* = $(\lambda x.\ THE\ y.\ y \in A \wedge f\ y = x)$

lemma *the-inv-into-f-f*: *inj-on f A* $\Longrightarrow x \in A \Longrightarrow$ *the-inv-into A f* (*f x*) = *x*
⟨*proof*⟩

lemma *f-the-inv-into-f*: *inj-on f A* $\Longrightarrow y \in f\ `\ A \Longrightarrow f$ (*the-inv-into A f y*) = *y*
⟨*proof*⟩

lemma *f-the-inv-into-f-bij-betw*:
bij-betw f A B $\Longrightarrow$ (*bij-betw f A B* $\Longrightarrow x \in B) \Longrightarrow f$ (*the-inv-into A f x*) = *x*
⟨*proof*⟩

lemma *the-inv-into-into*: *inj-on f A* $\Longrightarrow x \in f\ `\ A \Longrightarrow A \subseteq B \Longrightarrow$ *the-inv-into A f x* $\in B$
⟨*proof*⟩

lemma *the-inv-into-onto* [*simp*]: *inj-on f A* $\Longrightarrow$ *the-inv-into A f* ` (*f* ` *A*) = *A*
⟨*proof*⟩

lemma *the-inv-into-f-eq*: *inj-on f A* $\Longrightarrow f\ x = y \Longrightarrow x \in A \Longrightarrow$ *the-inv-into A f y* = *x*
⟨*proof*⟩

lemma *the-inv-into-comp*:
inj-on f (*g* ` *A*) $\Longrightarrow$ *inj-on g A* $\Longrightarrow x \in f\ `\ g\ `\ A \Longrightarrow$

the-inv-into A (f ∘ g) x = (the-inv-into A g ∘ the-inv-into (g ' A) f) x
⟨*proof*⟩

lemma *inj-on-the-inv-into*: *inj-on f A ⟹ inj-on (the-inv-into A f) (f ' A)*
⟨*proof*⟩

lemma *bij-betw-the-inv-into*: *bij-betw f A B ⟹ bij-betw (the-inv-into A f) B A*
⟨*proof*⟩

lemma *bij-betw-iff-bijections*:
bij-betw f A B ⟷ (∃ g. (∀ x ∈ A. f x ∈ B ∧ g(f x) = x) ∧ (∀ y ∈ B. g y ∈ A ∧ f(g y) = y))
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

abbreviation *the-inv* :: *('a ⇒ 'b) ⇒ ('b ⇒ 'a)*
where *the-inv f ≡ the-inv-into UNIV f*

lemma *the-inv-f-f*: *the-inv f (f x) = x* **if** *inj f*
⟨*proof*⟩

10.9 Monotonicity

definition *monotone-on* :: *'a set ⇒ ('a ⇒ 'a ⇒ bool) ⇒ ('b ⇒ 'b ⇒ bool) ⇒ ('a ⇒ 'b) ⇒ bool*
where *monotone-on A orda ordb f ⟷ (∀ x∈A. ∀ y∈A. orda x y ⟶ ordb (f x) (f y))*

abbreviation *monotone* :: *('a ⇒ 'a ⇒ bool) ⇒ ('b ⇒ 'b ⇒ bool) ⇒ ('a ⇒ 'b) ⇒ bool*
where *monotone ≡ monotone-on UNIV*

lemma *monotone-def*[*no-atp*]: *monotone orda ordb f ⟷ (∀ x y. orda x y ⟶ ordb (f x) (f y))*
⟨*proof*⟩

Lemma *monotone-def* is provided for backward compatibility.

lemma *monotone-onI*:
(⋀x y. x ∈ A ⟹ y ∈ A ⟹ orda x y ⟹ ordb (f x) (f y)) ⟹ monotone-on A orda ordb f
⟨*proof*⟩

lemma *monotoneI*[*intro?*]: *(⋀x y. orda x y ⟹ ordb (f x) (f y)) ⟹ monotone orda ordb f*
⟨*proof*⟩

lemma *monotone-onD*:
monotone-on A orda ordb f ⟹ x ∈ A ⟹ y ∈ A ⟹ orda x y ⟹ ordb (f x) (f y)

⟨*proof*⟩

lemma *monotoneD*[*dest?*]: *monotone orda ordb f* ⟹ *orda x y* ⟹ *ordb* (*f x*) (*f y*)
⟨*proof*⟩

lemma *monotone-on-subset*: *monotone-on A orda ordb f* ⟹ *B* ⊆ *A* ⟹ *monotone-on B orda ordb f*
⟨*proof*⟩

lemma *monotone-on-empty*[*simp*]: *monotone-on* {} *orda ordb f*
⟨*proof*⟩

lemma *monotone-on-o*:
assumes
mono-f: *monotone-on A orda ordb f* **and**
mono-g: *monotone-on B ordc orda g* **and**
g ‘ *B* ⊆ *A*
shows *monotone-on B ordc ordb* (*f* ∘ *g*)
⟨*proof*⟩

10.9.1 Specializations For *ord* Type Class And More

context *ord* **begin**

abbreviation *mono-on* :: *'a set* ⇒ (*'a* ⇒ *'b* :: *ord*) ⇒ *bool*
where *mono-on A* ≡ *monotone-on A* (≤) (≤)

abbreviation *strict-mono-on* :: *'a set* ⇒ (*'a* ⇒ *'b* :: *ord*) ⇒ *bool*
where *strict-mono-on A* ≡ *monotone-on A* (<) (<)

abbreviation *antimono-on* :: *'a set* ⇒ (*'a* ⇒ *'b* :: *ord*) ⇒ *bool*
where *antimono-on A* ≡ *monotone-on A* (≤) (≥)

abbreviation *strict-antimono-on* :: *'a set* ⇒ (*'a* ⇒ *'b* :: *ord*) ⇒ *bool*
where *strict-antimono-on A* ≡ *monotone-on A* (<) (>)

lemma *mono-on-def*[*no-atp*]: *mono-on A f* ⟷ (∀ *r s*. *r* ∈ *A* ∧ *s* ∈ *A* ∧ *r* ≤ *s* ⟶ *f r* ≤ *f s*)
⟨*proof*⟩

lemma *strict-mono-on-def*[*no-atp*]:
strict-mono-on A f ⟷ (∀ *r s*. *r* ∈ *A* ∧ *s* ∈ *A* ∧ *r* < *s* ⟶ *f r* < *f s*)
⟨*proof*⟩

Lemmas *mono-on-def* and *strict-mono-on-def* are provided for backward compatibility.

lemma *mono-onI*:
(⋀*r s*. *r* ∈ *A* ⟹ *s* ∈ *A* ⟹ *r* ≤ *s* ⟹ *f r* ≤ *f s*) ⟹ *mono-on A f*
⟨*proof*⟩

lemma *strict-mono-onI*:
$(\bigwedge r\ s.\ r \in A \Longrightarrow s \in A \Longrightarrow r < s \Longrightarrow f\ r < f\ s) \Longrightarrow$ *strict-mono-on* $A\ f$
⟨*proof*⟩

lemma *mono-onD*: $[\![$*mono-on* $A\ f$; $r \in A$; $s \in A$; $r \leq s]\!] \Longrightarrow f\ r \leq f\ s$
⟨*proof*⟩

lemma *strict-mono-onD*: $[\![$*strict-mono-on* $A\ f$; $r \in A$; $s \in A$; $r < s]\!] \Longrightarrow f\ r < f\ s$
⟨*proof*⟩

lemma *mono-on-subset*: *mono-on* $A\ f \Longrightarrow B \subseteq A \Longrightarrow$ *mono-on* $B\ f$
⟨*proof*⟩

end

lemma *mono-on-greaterD*:
assumes *mono-on* $A\ g\ x \in A\ y \in A\ g\ x > (g\ (y::-::linorder) :: - :: linorder)$
shows $x > y$
⟨*proof*⟩

context *order* **begin**

abbreviation *mono* :: $('a \Rightarrow 'b::order) \Rightarrow bool$
where *mono* $\equiv$ *mono-on UNIV*

abbreviation *strict-mono* :: $('a \Rightarrow 'b::order) \Rightarrow bool$
where *strict-mono* $\equiv$ *strict-mono-on UNIV*

abbreviation *antimono* :: $('a \Rightarrow 'b::order) \Rightarrow bool$
where *antimono* $\equiv$ *monotone* $(\leq)$ $(\lambda x\ y.\ y \leq x)$

lemma *mono-def*[*no-atp*]: *mono* $f \longleftrightarrow (\forall x\ y.\ x \leq y \longrightarrow f\ x \leq f\ y)$
⟨*proof*⟩

lemma *strict-mono-def*[*no-atp*]: *strict-mono* $f \longleftrightarrow (\forall x\ y.\ x < y \longrightarrow f\ x < f\ y)$
⟨*proof*⟩

lemma *antimono-def*[*no-atp*]: *antimono* $f \longleftrightarrow (\forall x\ y.\ x \leq y \longrightarrow f\ x \geq f\ y)$
⟨*proof*⟩

Lemmas *mono-def*, *strict-mono-def*, and *antimono-def* are provided for backward compatibility.

lemma *monoI* [*intro?*]: $(\bigwedge x\ y.\ x \leq y \Longrightarrow f\ x \leq f\ y) \Longrightarrow$ *mono* f
⟨*proof*⟩

lemma *strict-monoI* [*intro?*]: $(\bigwedge x\ y.\ x < y \Longrightarrow f\ x < f\ y) \Longrightarrow$ *strict-mono* f
⟨*proof*⟩

lemma *antimonoI* [*intro?*]: $(\bigwedge x\ y.\ x \leq y \Longrightarrow f\ x \geq f\ y) \Longrightarrow$ *antimono* f
⟨*proof*⟩

lemma *monoD* [*dest?*]: *mono* $f \Longrightarrow x \leq y \Longrightarrow f\ x \leq f\ y$
⟨*proof*⟩

lemma *strict-monoD* [*dest?*]: *strict-mono* $f \Longrightarrow x < y \Longrightarrow f\ x < f\ y$
⟨*proof*⟩

lemma *antimonoD* [*dest?*]: *antimono* $f \Longrightarrow x \leq y \Longrightarrow f\ x \geq f\ y$
⟨*proof*⟩

lemma *monoE*:
assumes *mono* f
assumes $x \leq y$
obtains $f\ x \leq f\ y$
⟨*proof*⟩

lemma *antimonoE*:
fixes $f :: 'a \Rightarrow 'b$::*order*
assumes *antimono* f
assumes $x \leq y$
obtains $f\ x \geq f\ y$
⟨*proof*⟩

lemma *mono-imp-mono-on*: *mono* $f \Longrightarrow$ *mono-on* $A\ f$
⟨*proof*⟩

lemma *strict-mono-mono* [*dest?*]:
assumes *strict-mono* f
shows *mono* f
⟨*proof*⟩

end

context *linorder* **begin**

lemma *mono-invE*:
fixes $f :: 'a \Rightarrow 'b$::*order*
assumes *mono* f
assumes $f\ x < f\ y$
obtains $x \leq y$
⟨*proof*⟩

lemma *mono-strict-invE*:
fixes $f :: 'a \Rightarrow 'b$::*order*
assumes *mono* f
assumes $f\ x < f\ y$
obtains $x < y$

⟨*proof*⟩

lemma *strict-mono-eq*:
assumes *strict-mono f*
shows $f\ x = f\ y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *strict-mono-less-eq*:
assumes *strict-mono f*
shows $f\ x \leq f\ y \longleftrightarrow x \leq y$
⟨*proof*⟩

lemma *strict-mono-less*:
assumes *strict-mono f*
shows $f\ x < f\ y \longleftrightarrow x < y$
⟨*proof*⟩

end

lemma *strict-mono-inv*:
fixes $f :: ('a::linorder) \Rightarrow ('b::linorder)$
assumes *strict-mono f* **and** *surj f* **and** *inv*: $\bigwedge x.\ g\ (f\ x) = x$
shows *strict-mono g*
⟨*proof*⟩

lemma *strict-mono-on-imp-inj-on*:
assumes *strict-mono-on A* $(f :: (- :: linorder) \Rightarrow (- :: preorder))$
shows *inj-on f A*
⟨*proof*⟩

lemma *strict-mono-on-leD*:
assumes *strict-mono-on A* $(f :: (- :: linorder) \Rightarrow - :: preorder)\ x \in A\ y \in A\ x \leq y$
shows $f\ x \leq f\ y$
⟨*proof*⟩

lemma *strict-mono-on-eqD*:
fixes $f :: (- :: linorder) \Rightarrow (- :: preorder)$
assumes *strict-mono-on A f* $f\ x = f\ y\ x \in A\ y \in A$
shows $y = x$
⟨*proof*⟩

lemma *strict-mono-on-imp-mono-on*:
strict-mono-on A $(f :: (- :: linorder) \Rightarrow - :: preorder) \Longrightarrow$ *mono-on A f*
⟨*proof*⟩

lemma *mono-imp-strict-mono*:
fixes $f :: 'a::order \Rightarrow 'b::order$
shows $[\![$*mono-on S f*; *inj-on f S*$]\!] \Longrightarrow$ *strict-mono-on S f*

⟨*proof*⟩

lemma *strict-mono-iff-mono*:
fixes $f :: 'a::linorder \Rightarrow 'b::order$
shows *strict-mono-on* $S\ f \longleftrightarrow$ *mono-on* $S\ f \wedge$ *inj-on* $f\ S$
⟨*proof*⟩

lemma *antimono-imp-strict-antimono*:
fixes $f :: 'a::order \Rightarrow 'b::order$
shows $[\![$*antimono-on* $S\ f$; *inj-on* $f\ S]\!] \Longrightarrow$ *strict-antimono-on* $S\ f$
⟨*proof*⟩

lemma *strict-antimono-iff-antimono*:
fixes $f :: 'a::linorder \Rightarrow 'b::order$
shows *strict-antimono-on* $S\ f \longleftrightarrow$ *antimono-on* $S\ f \wedge$ *inj-on* $f\ S$
⟨*proof*⟩

lemma *mono-compose*: *mono* $Q \Longrightarrow$ *mono* $(\lambda i\ x.\ Q\ i\ (f\ x))$
⟨*proof*⟩

lemma *mono-add*:
fixes $a :: 'a::ordered\text{-}ab\text{-}semigroup\text{-}add$
shows *mono* $((+)\ a)$
⟨*proof*⟩

lemma (**in** *semilattice-inf*) *mono-inf*: *mono* $f \Longrightarrow f\ (A \sqcap B) \leq f\ A \sqcap f\ B$
for $f :: 'a \Rightarrow 'b::semilattice\text{-}inf$
⟨*proof*⟩

lemma (**in** *semilattice-sup*) *mono-sup*: *mono* $f \Longrightarrow f\ A \sqcup f\ B \leq f\ (A \sqcup B)$
for $f :: 'a \Rightarrow 'b::semilattice\text{-}sup$
⟨*proof*⟩

lemma (**in** *linorder*) *min-of-mono*: *mono* $f \Longrightarrow min\ (f\ m)\ (f\ n) = f\ (min\ m\ n)$
⟨*proof*⟩

lemma (**in** *linorder*) *max-of-mono*: *mono* $f \Longrightarrow max\ (f\ m)\ (f\ n) = f\ (max\ m\ n)$
⟨*proof*⟩

lemma (**in** *linorder*)
max-of-antimono: *antimono* $f \Longrightarrow max\ (f\ x)\ (f\ y) = f\ (min\ x\ y)$ **and**
min-of-antimono: *antimono* $f \Longrightarrow min\ (f\ x)\ (f\ y) = f\ (max\ x\ y)$
⟨*proof*⟩

lemma (**in** *linorder*) *strict-mono-imp-inj-on*: *strict-mono* $f \Longrightarrow$ *inj-on* $f\ A$
⟨*proof*⟩

lemma *mono-Int*: *mono* $f \Longrightarrow f\ (A \cap B) \subseteq f\ A \cap f\ B$
⟨*proof*⟩

lemma *mono-Un*: *mono f* $\Longrightarrow$ $f\ A \cup f\ B \subseteq f\ (A \cup B)$
⟨*proof*⟩

10.9.2 Least value operator

lemma *Least-mono*: *mono f* $\Longrightarrow$ $\exists x{\in}S.\ \forall y{\in}S.\ x \leq y \Longrightarrow (LEAST\ y.\ y \in f\ `\ S) = f\ (LEAST\ x.\ x \in S)$
for $f :: 'a{::}order \Rightarrow 'b{::}order$
— Courtesy of Stephan Merz
⟨*proof*⟩

10.10 Setup

10.10.1 Proof tools

Simplify terms of the form $f(\ldots,x{:=}y,\ldots,x{:=}z,\ldots)$ to $f(\ldots,x{:=}z,\ldots)$

⟨*ML*⟩

10.10.2 Functorial structure of types

⟨*ML*⟩

functor *map-fun*: *map-fun*
⟨*proof*⟩

functor *vimage*
⟨*proof*⟩

Legacy theorem names

lemmas *o-def* = *comp-def*
lemmas *o-apply* = *comp-apply*
lemmas *o-assoc* = *comp-assoc* [*symmetric*]
lemmas *id-o* = *id-comp*
lemmas *o-id* = *comp-id*
lemmas *o-eq-dest* = *comp-eq-dest*
lemmas *o-eq-elim* = *comp-eq-elim*
lemmas *o-eq-dest-lhs* = *comp-eq-dest-lhs*
lemmas *o-eq-id-dest* = *comp-eq-id-dest*

end

11 Complete lattices

theory *Complete-Lattices*
imports *Fun*
begin

11.1 Syntactic infimum and supremum operations

class *Inf* =
  fixes *Inf* :: *'a set* ⇒ *'a* (⨅ - [*900*] *900*)

class *Sup* =
  fixes *Sup* :: *'a set* ⇒ *'a* (⨆ - [*900*] *900*)

syntax
  -INF1 :: *pttrns* ⇒ *'b* ⇒ *'b* ((*3INF -./ -*) [*0*, *10*] *10*)
  -INF :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* ((*3INF -∈-./ -*) [*0*, *0*, *10*] *10*)
  -SUP1 :: *pttrns* ⇒ *'b* ⇒ *'b* ((*3SUP -./ -*) [*0*, *10*] *10*)
  -SUP :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* ((*3SUP -∈-./ -*) [*0*, *0*, *10*] *10*)

syntax
  -INF1 :: *pttrns* ⇒ *'b* ⇒ *'b* ((*3*⨅ *-./ -*) [*0*, *10*] *10*)
  -INF :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* ((*3*⨅ *-∈-./ -*) [*0*, *0*, *10*] *10*)
  -SUP1 :: *pttrns* ⇒ *'b* ⇒ *'b* ((*3*⨆ *-./ -*) [*0*, *10*] *10*)
  -SUP :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* ((*3*⨆ *-∈-./ -*) [*0*, *0*, *10*] *10*)

translations
  ⨅*x y. f* ⇌ ⨅*x*. ⨅*y. f*
  ⨅*x. f* ⇌ ⨅(*CONST range* (λ*x. f*))
  ⨅*x*∈*A. f* ⇌ *CONST Inf* ((λ*x. f*) ‘ *A*)
  ⨆*x y. f* ⇌ ⨆*x*. ⨆*y. f*
  ⨆*x. f* ⇌ ⨆(*CONST range* (λ*x. f*))
  ⨆*x*∈*A. f* ⇌ *CONST Sup* ((λ*x. f*) ‘ *A*)

context *Inf*
begin

lemma *INF-image*: ⨅ (*g* ‘ *f* ‘ *A*) = ⨅ ((*g* ∘ *f*) ‘ *A*)
  ⟨*proof*⟩

lemma *INF-identity-eq* [*simp*]: (⨅*x*∈*A. x*) = ⨅*A*
  ⟨*proof*⟩

lemma *INF-id-eq* [*simp*]: ⨅(*id* ‘ *A*) = ⨅*A*
  ⟨*proof*⟩

lemma *INF-cong*: *A* = *B* ⟹ (⋀*x. x* ∈ *B* ⟹ *C x* = *D x*) ⟹ ⨅(*C* ‘ *A*) = ⨅(*D* ‘ *B*)
  ⟨*proof*⟩

lemma *INF-cong-simp*:
  A = *B* ⟹ (⋀*x. x* ∈ *B* =*simp*=> *C x* = *D x*) ⟹ ⨅(*C* ‘ *A*) = ⨅(*D* ‘ *B*)
  ⟨*proof*⟩

end

context *Sup*
begin

lemma *SUP-image*: $\bigsqcup\ (g\ `\ f\ `\ A) = \bigsqcup\ ((g \circ f)\ `\ A)$
⟨*proof*⟩

lemma *SUP-identity-eq* [*simp*]: $(\bigsqcup x{\in}A.\ x) = \bigsqcup A$
⟨*proof*⟩

lemma *SUP-id-eq* [*simp*]: $\bigsqcup(id\ `\ A) = \bigsqcup A$
⟨*proof*⟩

lemma *SUP-cong*: $A = B \Longrightarrow (\bigwedge x.\ x \in B \Longrightarrow C\ x = D\ x) \Longrightarrow \bigsqcup(C\ `\ A) = \bigsqcup(D\ `\ B)$
⟨*proof*⟩

lemma *SUP-cong-simp*:
$A = B \Longrightarrow (\bigwedge x.\ x \in B =simp=> C\ x = D\ x) \Longrightarrow \bigsqcup(C\ `\ A) = \bigsqcup(D\ `\ B)$
⟨*proof*⟩

end

11.2 Abstract complete lattices

A complete lattice always has a bottom and a top, so we include them into the following type class, along with assumptions that define bottom and top in terms of infimum and supremum.

class *complete-lattice* = *lattice* + *Inf* + *Sup* + *bot* + *top* +
assumes *Inf-lower*: $x \in A \Longrightarrow \bigsqcap A \leq x$
and *Inf-greatest*: $(\bigwedge x.\ x \in A \Longrightarrow z \leq x) \Longrightarrow z \leq \bigsqcap A$
and *Sup-upper*: $x \in A \Longrightarrow x \leq \bigsqcup A$
and *Sup-least*: $(\bigwedge x.\ x \in A \Longrightarrow x \leq z) \Longrightarrow \bigsqcup A \leq z$
and *Inf-empty* [*simp*]: $\bigsqcap\{\} = \top$
and *Sup-empty* [*simp*]: $\bigsqcup\{\} = \bot$
begin

subclass *bounded-lattice*
⟨*proof*⟩

lemma *dual-complete-lattice*: *class.complete-lattice Sup Inf sup* (≥) (>) *inf* ⊤ ⊥
⟨*proof*⟩

end

context *complete-lattice*
begin

lemma *Sup-eqI*:

$(\bigwedge y.\ y \in A \Longrightarrow y \leq x) \Longrightarrow (\bigwedge y.\ (\bigwedge z.\ z \in A \Longrightarrow z \leq y) \Longrightarrow x \leq y) \Longrightarrow \bigsqcup A = x$
⟨*proof*⟩

lemma *Inf-eqI*:
$(\bigwedge i.\ i \in A \Longrightarrow x \leq i) \Longrightarrow (\bigwedge y.\ (\bigwedge i.\ i \in A \Longrightarrow y \leq i) \Longrightarrow y \leq x) \Longrightarrow \bigsqcap A = x$
⟨*proof*⟩

lemma *SUP-eqI*:
$(\bigwedge i.\ i \in A \Longrightarrow f\ i \leq x) \Longrightarrow (\bigwedge y.\ (\bigwedge i.\ i \in A \Longrightarrow f\ i \leq y) \Longrightarrow x \leq y) \Longrightarrow (\bigsqcup i{\in}A.\ f\ i) = x$
⟨*proof*⟩

lemma *INF-eqI*:
$(\bigwedge i.\ i \in A \Longrightarrow x \leq f\ i) \Longrightarrow (\bigwedge y.\ (\bigwedge i.\ i \in A \Longrightarrow f\ i \geq y) \Longrightarrow x \geq y) \Longrightarrow (\bigsqcap i{\in}A.\ f\ i) = x$
⟨*proof*⟩

lemma *INF-lower*: $i \in A \Longrightarrow (\bigsqcap i{\in}A.\ f\ i) \leq f\ i$
⟨*proof*⟩

lemma *INF-greatest*: $(\bigwedge i.\ i \in A \Longrightarrow u \leq f\ i) \Longrightarrow u \leq (\bigsqcap i{\in}A.\ f\ i)$
⟨*proof*⟩

lemma *SUP-upper*: $i \in A \Longrightarrow f\ i \leq (\bigsqcup i{\in}A.\ f\ i)$
⟨*proof*⟩

lemma *SUP-least*: $(\bigwedge i.\ i \in A \Longrightarrow f\ i \leq u) \Longrightarrow (\bigsqcup i{\in}A.\ f\ i) \leq u$
⟨*proof*⟩

lemma *Inf-lower2*: $u \in A \Longrightarrow u \leq v \Longrightarrow \bigsqcap A \leq v$
⟨*proof*⟩

lemma *INF-lower2*: $i \in A \Longrightarrow f\ i \leq u \Longrightarrow (\bigsqcap i{\in}A.\ f\ i) \leq u$
⟨*proof*⟩

lemma *Sup-upper2*: $u \in A \Longrightarrow v \leq u \Longrightarrow v \leq \bigsqcup A$
⟨*proof*⟩

lemma *SUP-upper2*: $i \in A \Longrightarrow u \leq f\ i \Longrightarrow u \leq (\bigsqcup i{\in}A.\ f\ i)$
⟨*proof*⟩

lemma *le-Inf-iff*: $b \leq \bigsqcap A \longleftrightarrow (\forall\, a{\in}A.\ b \leq a)$
⟨*proof*⟩

lemma *le-INF-iff*: $u \leq (\bigsqcap i{\in}A.\ f\ i) \longleftrightarrow (\forall\, i{\in}A.\ u \leq f\ i)$
⟨*proof*⟩

lemma *Sup-le-iff*: $\bigsqcup A \leq b \longleftrightarrow (\forall\, a{\in}A.\ a \leq b)$

⟨*proof*⟩

lemma *SUP-le-iff*: $(\bigsqcup i{\in}A.\ f\ i) \leq u \longleftrightarrow (\forall\, i{\in}A.\ f\ i \leq u)$
⟨*proof*⟩

lemma *Inf-insert* [*simp*]: $\bigsqcap (insert\ a\ A) = a \sqcap \bigsqcap A$
⟨*proof*⟩

lemma *INF-insert*: $(\bigsqcap x{\in}insert\ a\ A.\ f\ x) = f\ a \sqcap \bigsqcap (f\ `\ A)$
⟨*proof*⟩

lemma *Sup-insert* [*simp*]: $\bigsqcup (insert\ a\ A) = a \sqcup \bigsqcup A$
⟨*proof*⟩

lemma *SUP-insert*: $(\bigsqcup x{\in}insert\ a\ A.\ f\ x) = f\ a \sqcup \bigsqcup (f\ `\ A)$
⟨*proof*⟩

lemma *INF-empty*: $(\bigsqcap x{\in}\{\}.\ f\ x) = \top$
⟨*proof*⟩

lemma *SUP-empty*: $(\bigsqcup x{\in}\{\}.\ f\ x) = \bot$
⟨*proof*⟩

lemma *Inf-UNIV* [*simp*]: $\bigsqcap UNIV = \bot$
⟨*proof*⟩

lemma *Sup-UNIV* [*simp*]: $\bigsqcup UNIV = \top$
⟨*proof*⟩

lemma *Inf-eq-Sup*: $\bigsqcap A = \bigsqcup \{b.\ \forall\, a \in A.\ b \leq a\}$
⟨*proof*⟩

lemma *Sup-eq-Inf*: $\bigsqcup A = \bigsqcap \{b.\ \forall\, a \in A.\ a \leq b\}$
⟨*proof*⟩

lemma *Inf-superset-mono*: $B \subseteq A \Longrightarrow \bigsqcap A \leq \bigsqcap B$
⟨*proof*⟩

lemma *Sup-subset-mono*: $A \subseteq B \Longrightarrow \bigsqcup A \leq \bigsqcup B$
⟨*proof*⟩

lemma *Inf-mono*:
assumes $\bigwedge b.\ b \in B \Longrightarrow \exists\, a{\in}A.\ a \leq b$
shows $\bigsqcap A \leq \bigsqcap B$
⟨*proof*⟩

lemma *INF-mono*: $(\bigwedge m.\ m \in B \Longrightarrow \exists\, n{\in}A.\ f\ n \leq g\ m) \Longrightarrow (\bigsqcap n{\in}A.\ f\ n) \leq (\bigsqcap n{\in}B.\ g\ n)$
⟨*proof*⟩

lemma *INF-mono'*: $(\bigwedge x.\ f\ x \leq g\ x) \Longrightarrow (\bigsqcap x{\in}A.\ f\ x) \leq (\bigsqcap x{\in}A.\ g\ x)$
⟨*proof*⟩

lemma *Sup-mono*:
assumes $\bigwedge a.\ a \in A \Longrightarrow \exists\, b{\in}B.\ a \leq b$
shows $\bigsqcup A \leq \bigsqcup B$
⟨*proof*⟩

lemma *SUP-mono*: $(\bigwedge n.\ n \in A \Longrightarrow \exists\, m{\in}B.\ f\ n \leq g\ m) \Longrightarrow (\bigsqcup n{\in}A.\ f\ n) \leq (\bigsqcup n{\in}B.\ g\ n)$
⟨*proof*⟩

lemma *SUP-mono'*: $(\bigwedge x.\ f\ x \leq g\ x) \Longrightarrow (\bigsqcup x{\in}A.\ f\ x) \leq (\bigsqcup x{\in}A.\ g\ x)$
⟨*proof*⟩

lemma *INF-superset-mono*: $B \subseteq A \Longrightarrow (\bigwedge x.\ x \in B \Longrightarrow f\ x \leq g\ x) \Longrightarrow (\bigsqcap x{\in}A.\ f\ x) \leq (\bigsqcap x{\in}B.\ g\ x)$
— The last inclusion is POSITIVE!
⟨*proof*⟩

lemma *SUP-subset-mono*: $A \subseteq B \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow f\ x \leq g\ x) \Longrightarrow (\bigsqcup x{\in}A.\ f\ x) \leq (\bigsqcup x{\in}B.\ g\ x)$
⟨*proof*⟩

lemma *Inf-less-eq*:
assumes $\bigwedge v.\ v \in A \Longrightarrow v \leq u$
and $A \neq \{\}$
shows $\bigsqcap A \leq u$
⟨*proof*⟩

lemma *less-eq-Sup*:
assumes $\bigwedge v.\ v \in A \Longrightarrow u \leq v$
and $A \neq \{\}$
shows $u \leq \bigsqcup A$
⟨*proof*⟩

lemma *INF-eq*:
assumes $\bigwedge i.\ i \in A \Longrightarrow \exists\, j{\in}B.\ f\ i \geq g\ j$
and $\bigwedge j.\ j \in B \Longrightarrow \exists\, i{\in}A.\ g\ j \geq f\ i$
shows $\bigsqcap (f\ `\ A) = \bigsqcap (g\ `\ B)$
⟨*proof*⟩

lemma *SUP-eq*:
assumes $\bigwedge i.\ i \in A \Longrightarrow \exists\, j{\in}B.\ f\ i \leq g\ j$
and $\bigwedge j.\ j \in B \Longrightarrow \exists\, i{\in}A.\ g\ j \leq f\ i$
shows $\bigsqcup (f\ `\ A) = \bigsqcup (g\ `\ B)$
⟨*proof*⟩

lemma *less-eq-Inf-inter*: $\bigsqcap A \sqcup \bigsqcap B \leq \bigsqcap (A \cap B)$
⟨*proof*⟩

lemma *Sup-inter-less-eq*: $\bigsqcup (A \cap B) \leq \bigsqcup A \sqcap \bigsqcup B$
⟨*proof*⟩

lemma *Inf-union-distrib*: $\bigsqcap (A \cup B) = \bigsqcap A \sqcap \bigsqcap B$
⟨*proof*⟩

lemma *INF-union*: $(\bigsqcap i \in A \cup B.\ M\ i) = (\bigsqcap i \in A.\ M\ i) \sqcap (\bigsqcap i{\in}B.\ M\ i)$
⟨*proof*⟩

lemma *Sup-union-distrib*: $\bigsqcup (A \cup B) = \bigsqcup A \sqcup \bigsqcup B$
⟨*proof*⟩

lemma *SUP-union*: $(\bigsqcup i \in A \cup B.\ M\ i) = (\bigsqcup i \in A.\ M\ i) \sqcup (\bigsqcup i{\in}B.\ M\ i)$
⟨*proof*⟩

lemma *INF-inf-distrib*: $(\bigsqcap a{\in}A.\ f\ a) \sqcap (\bigsqcap a{\in}A.\ g\ a) = (\bigsqcap a{\in}A.\ f\ a \sqcap g\ a)$
⟨*proof*⟩

lemma *SUP-sup-distrib*: $(\bigsqcup a{\in}A.\ f\ a) \sqcup (\bigsqcup a{\in}A.\ g\ a) = (\bigsqcup a{\in}A.\ f\ a \sqcup g\ a)$
(**is** *?L = ?R*)
⟨*proof*⟩

lemma *Inf-top-conv* [*simp*]:
$\bigsqcap A = \top \longleftrightarrow (\forall x{\in}A.\ x = \top)$
$\top = \bigsqcap A \longleftrightarrow (\forall x{\in}A.\ x = \top)$
⟨*proof*⟩

lemma *INF-top-conv* [*simp*]:
$(\bigsqcap x{\in}A.\ B\ x) = \top \longleftrightarrow (\forall x{\in}A.\ B\ x = \top)$
$\top = (\bigsqcap x{\in}A.\ B\ x) \longleftrightarrow (\forall x{\in}A.\ B\ x = \top)$
⟨*proof*⟩

lemma *Sup-bot-conv* [*simp*]:
$\bigsqcup A = \bot \longleftrightarrow (\forall x{\in}A.\ x = \bot)$
$\bot = \bigsqcup A \longleftrightarrow (\forall x{\in}A.\ x = \bot)$
⟨*proof*⟩

lemma *SUP-bot-conv* [*simp*]:
$(\bigsqcup x{\in}A.\ B\ x) = \bot \longleftrightarrow (\forall x{\in}A.\ B\ x = \bot)$
$\bot = (\bigsqcup x{\in}A.\ B\ x) \longleftrightarrow (\forall x{\in}A.\ B\ x = \bot)$
⟨*proof*⟩

lemma *INF-constant*: $(\bigsqcap y{\in}A.\ c) = (\mathit{if}\ A = \{\}\ \mathit{then}\ \top\ \mathit{else}\ c)$
⟨*proof*⟩

lemma *SUP-constant*: $(\bigsqcup y{\in}A.\ c) = (\mathit{if}\ A = \{\}\ \mathit{then}\ \bot\ \mathit{else}\ c)$

⟨*proof*⟩

lemma *INF-const* [*simp*]: $A \neq \{\} \Longrightarrow (\bigsqcap i{\in}A.\ f) = f$
⟨*proof*⟩

lemma *SUP-const* [*simp*]: $A \neq \{\} \Longrightarrow (\bigsqcup i{\in}A.\ f) = f$
⟨*proof*⟩

lemma *INF-top* [*simp*]: $(\bigsqcap x{\in}A.\ \top) = \top$
⟨*proof*⟩

lemma *SUP-bot* [*simp*]: $(\bigsqcup x{\in}A.\ \bot) = \bot$
⟨*proof*⟩

lemma *INF-commute*: $(\bigsqcap i{\in}A.\ \bigsqcap j{\in}B.\ f\ i\ j) = (\bigsqcap j{\in}B.\ \bigsqcap i{\in}A.\ f\ i\ j)$
⟨*proof*⟩

lemma *SUP-commute*: $(\bigsqcup i{\in}A.\ \bigsqcup j{\in}B.\ f\ i\ j) = (\bigsqcup j{\in}B.\ \bigsqcup i{\in}A.\ f\ i\ j)$
⟨*proof*⟩

lemma *INF-absorb*:
assumes $k \in I$
shows $A\ k \sqcap (\bigsqcap i{\in}I.\ A\ i) = (\bigsqcap i{\in}I.\ A\ i)$
⟨*proof*⟩

lemma *SUP-absorb*:
assumes $k \in I$
shows $A\ k \sqcup (\bigsqcup i{\in}I.\ A\ i) = (\bigsqcup i{\in}I.\ A\ i)$
⟨*proof*⟩

lemma *INF-inf-const1*: $I \neq \{\} \Longrightarrow (\bigsqcap i{\in}I.\ inf\ x\ (f\ i)) = inf\ x\ (\bigsqcap i{\in}I.\ f\ i)$
⟨*proof*⟩

lemma *INF-inf-const2*: $I \neq \{\} \Longrightarrow (\bigsqcap i{\in}I.\ inf\ (f\ i)\ x) = inf\ (\bigsqcap i{\in}I.\ f\ i)\ x$
⟨*proof*⟩

lemma *less-INF-D*:
assumes $y < (\bigsqcap i{\in}A.\ f\ i)\ i \in A$
shows $y < f\ i$
⟨*proof*⟩

lemma *SUP-lessD*:
assumes $(\bigsqcup i{\in}A.\ f\ i) < y\ i \in A$
shows $f\ i < y$
⟨*proof*⟩

lemma *INF-UNIV-bool-expand*: $(\bigsqcap b.\ A\ b) = A\ True \sqcap A\ False$
⟨*proof*⟩

lemma *SUP-UNIV-bool-expand*: $(\bigsqcup b.\ A\ b) = A\ True \sqcup A\ False$
⟨*proof*⟩

lemma *Inf-le-Sup*: $A \neq \{\} \Longrightarrow Inf\ A \leq Sup\ A$
⟨*proof*⟩

lemma *INF-le-SUP*: $A \neq \{\} \Longrightarrow \bigsqcap(f\ `\ A) \leq \bigsqcup(f\ `\ A)$
⟨*proof*⟩

lemma *INF-eq-const*: $I \neq \{\} \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow f\ i = x) \Longrightarrow \bigsqcap(f\ `\ I) = x$
⟨*proof*⟩

lemma *SUP-eq-const*: $I \neq \{\} \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow f\ i = x) \Longrightarrow \bigsqcup(f\ `\ I) = x$
⟨*proof*⟩

lemma *INF-eq-iff*: $I \neq \{\} \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow f\ i \leq c) \Longrightarrow \bigsqcap(f\ `\ I) = c \longleftrightarrow (\forall\, i{\in}I.\ f\ i = c)$
⟨*proof*⟩

lemma *SUP-eq-iff*: $I \neq \{\} \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow c \leq f\ i) \Longrightarrow \bigsqcup(f\ `\ I) = c \longleftrightarrow (\forall\, i{\in}I.\ f\ i = c)$
⟨*proof*⟩

end

context *complete-lattice*
begin
lemma *Sup-Inf-le*: $Sup\ (Inf\ `\ \{f\ `\ A \mid f.\ (\forall\ Y \in A.\ f\ Y \in Y)\}) \leq Inf\ (Sup\ `\ A)$
⟨*proof*⟩
end

class *complete-distrib-lattice* = *complete-lattice* +
assumes *Inf-Sup-le*: $Inf\ (Sup\ `\ A) \leq Sup\ (Inf\ `\ \{f\ `\ A \mid f.\ (\forall\ Y \in A.\ f\ Y \in Y)\})$
begin

lemma *Inf-Sup*: $Inf\ (Sup\ `\ A) = Sup\ (Inf\ `\ \{f\ `\ A \mid f.\ (\forall\ Y \in A.\ f\ Y \in Y)\})$
⟨*proof*⟩

subclass *distrib-lattice*
⟨*proof*⟩
end

context *complete-lattice*
begin
context
fixes $f :: 'a \Rightarrow 'b{::}complete\text{-}lattice$
assumes *mono f*
begin

lemma *mono-Inf*: $f\ (\bigsqcap A) \leq (\bigsqcap x{\in}A.\ f\ x)$
⟨*proof*⟩

lemma *mono-Sup*: $(\bigsqcup x{\in}A.\ f\ x) \leq f\ (\bigsqcup A)$
⟨*proof*⟩

lemma *mono-INF*: $f\ (\bigsqcap i{\in}I.\ A\ i) \leq (\bigsqcap x{\in}I.\ f\ (A\ x))$
⟨*proof*⟩

lemma *mono-SUP*: $(\bigsqcup x{\in}I.\ f\ (A\ x)) \leq f\ (\bigsqcup i{\in}I.\ A\ i)$
⟨*proof*⟩

end

end

class *complete-boolean-algebra* = *boolean-algebra* + *complete-distrib-lattice*
begin

lemma *uminus-Inf*: $-\ (\bigsqcap A) = \bigsqcup (uminus\ `\ A)$
⟨*proof*⟩

lemma *uminus-INF*: $-\ (\bigsqcap x{\in}A.\ B\ x) = (\bigsqcup x{\in}A.\ -\ B\ x)$
⟨*proof*⟩

lemma *uminus-Sup*: $-\ (\bigsqcup A) = \bigsqcap (uminus\ `\ A)$
⟨*proof*⟩

lemma *uminus-SUP*: $-\ (\bigsqcup x{\in}A.\ B\ x) = (\bigsqcap x{\in}A.\ -\ B\ x)$
⟨*proof*⟩

end

class *complete-linorder* = *linorder* + *complete-lattice*
begin

lemma *dual-complete-linorder*:
class.complete-linorder Sup Inf sup $(\geq)$ $(>)$ *inf* $\top$ $\bot$
⟨*proof*⟩

lemma *complete-linorder-inf-min*: *inf* = *min*
⟨*proof*⟩

lemma *complete-linorder-sup-max*: *sup* = *max*
⟨*proof*⟩

lemma *Inf-less-iff*: $\bigsqcap S < a \longleftrightarrow (\exists\, x{\in}S.\ x < a)$
⟨*proof*⟩

lemma *INF-less-iff*: $(\bigsqcap i{\in}A.\ f\ i) < a \longleftrightarrow (\exists\, x{\in}A.\ f\ x < a)$
⟨*proof*⟩

lemma *less-Sup-iff*: $a < \bigsqcup S \longleftrightarrow (\exists\, x{\in}S.\ a < x)$
⟨*proof*⟩

lemma *less-SUP-iff*: $a < (\bigsqcup i{\in}A.\ f\ i) \longleftrightarrow (\exists\, x{\in}A.\ a < f\ x)$
⟨*proof*⟩

lemma *Sup-eq-top-iff* [*simp*]: $\bigsqcup A = \top \longleftrightarrow (\forall\, x{<}\top.\ \exists\, i{\in}A.\ x < i)$
⟨*proof*⟩

lemma *SUP-eq-top-iff* [*simp*]: $(\bigsqcup i{\in}A.\ f\ i) = \top \longleftrightarrow (\forall\, x{<}\top.\ \exists\, i{\in}A.\ x < f\ i)$
⟨*proof*⟩

lemma *Inf-eq-bot-iff* [*simp*]: $\bigsqcap A = \bot \longleftrightarrow (\forall\, x{>}\bot.\ \exists\, i{\in}A.\ i < x)$
⟨*proof*⟩

lemma *INF-eq-bot-iff* [*simp*]: $(\bigsqcap i{\in}A.\ f\ i) = \bot \longleftrightarrow (\forall\, x{>}\bot.\ \exists\, i{\in}A.\ f\ i < x)$
⟨*proof*⟩

lemma *Inf-le-iff*: $\bigsqcap A \leq x \longleftrightarrow (\forall\, y{>}x.\ \exists\, a{\in}A.\ y > a)$
⟨*proof*⟩

lemma *INF-le-iff*: $\bigsqcap (f\ `\ A) \leq x \longleftrightarrow (\forall\, y{>}x.\ \exists\, i{\in}A.\ y > f\ i)$
⟨*proof*⟩

lemma *le-Sup-iff*: $x \leq \bigsqcup A \longleftrightarrow (\forall\, y{<}x.\ \exists\, a{\in}A.\ y < a)$
⟨*proof*⟩

lemma *le-SUP-iff*: $x \leq \bigsqcup (f\ `\ A) \longleftrightarrow (\forall\, y{<}x.\ \exists\, i{\in}A.\ y < f\ i)$
⟨*proof*⟩

end

11.3 Complete lattice on *bool*

instantiation *bool* :: *complete-lattice*
begin

definition [*simp*, *code*]: $\bigsqcap A \longleftrightarrow \mathit{False} \notin A$

definition [*simp*, *code*]: $\bigsqcup A \longleftrightarrow \mathit{True} \in A$

instance
⟨*proof*⟩

end

lemma *not-False-in-image-Ball* [*simp*]: $False \notin P \mathbin{`} A \longleftrightarrow Ball\ A\ P$
⟨*proof*⟩

lemma *True-in-image-Bex* [*simp*]: $True \in P \mathbin{`} A \longleftrightarrow Bex\ A\ P$
⟨*proof*⟩

lemma *INF-bool-eq* [*simp*]: $(\lambda A\ f.\ \bigsqcap(f \mathbin{`} A)) = Ball$
⟨*proof*⟩

lemma *SUP-bool-eq* [*simp*]: $(\lambda A\ f.\ \bigsqcup(f \mathbin{`} A)) = Bex$
⟨*proof*⟩

instance *bool* :: *complete-boolean-algebra*
⟨*proof*⟩

11.4 Complete lattice on - ⇒ -

instantiation *fun* :: (*type*, *Inf*) *Inf*
begin

definition $\bigsqcap A = (\lambda x.\ \bigsqcap f{\in}A.\ f\ x)$

lemma *Inf-apply* [*simp*, *code*]: $(\bigsqcap A)\ x = (\bigsqcap f{\in}A.\ f\ x)$
⟨*proof*⟩

instance ⟨*proof*⟩

end

instantiation *fun* :: (*type*, *Sup*) *Sup*
begin

definition $\bigsqcup A = (\lambda x.\ \bigsqcup f{\in}A.\ f\ x)$

lemma *Sup-apply* [*simp*, *code*]: $(\bigsqcup A)\ x = (\bigsqcup f{\in}A.\ f\ x)$
⟨*proof*⟩

instance ⟨*proof*⟩

end

instantiation *fun* :: (*type*, *complete-lattice*) *complete-lattice*
begin

instance
⟨*proof*⟩

end

lemma *INF-apply* [*simp*]: $(\bigsqcap y{\in}A.\ f\ y)\ x = (\bigsqcap y{\in}A.\ f\ y\ x)$
⟨*proof*⟩

lemma *SUP-apply* [*simp*]: $(\bigsqcup y{\in}A.\ f\ y)\ x = (\bigsqcup y{\in}A.\ f\ y\ x)$
⟨*proof*⟩

11.5 Complete lattice on unary and binary predicates

lemma *Inf1-I*: $(\bigwedge P.\ P \in A \Longrightarrow P\ a) \Longrightarrow (\bigsqcap A)\ a$
⟨*proof*⟩

lemma *INF1-I*: $(\bigwedge x.\ x \in A \Longrightarrow B\ x\ b) \Longrightarrow (\bigsqcap x{\in}A.\ B\ x)\ b$
⟨*proof*⟩

lemma *INF2-I*: $(\bigwedge x.\ x \in A \Longrightarrow B\ x\ b\ c) \Longrightarrow (\bigsqcap x{\in}A.\ B\ x)\ b\ c$
⟨*proof*⟩

lemma *Inf2-I*: $(\bigwedge r.\ r \in A \Longrightarrow r\ a\ b) \Longrightarrow (\bigsqcap A)\ a\ b$
⟨*proof*⟩

lemma *Inf1-D*: $(\bigsqcap A)\ a \Longrightarrow P \in A \Longrightarrow P\ a$
⟨*proof*⟩

lemma *INF1-D*: $(\bigsqcap x{\in}A.\ B\ x)\ b \Longrightarrow a \in A \Longrightarrow B\ a\ b$
⟨*proof*⟩

lemma *Inf2-D*: $(\bigsqcap A)\ a\ b \Longrightarrow r \in A \Longrightarrow r\ a\ b$
⟨*proof*⟩

lemma *INF2-D*: $(\bigsqcap x{\in}A.\ B\ x)\ b\ c \Longrightarrow a \in A \Longrightarrow B\ a\ b\ c$
⟨*proof*⟩

lemma *Inf1-E*:
assumes $(\bigsqcap A)\ a$
obtains $P\ a \mid P \notin A$
⟨*proof*⟩

lemma *INF1-E*:
assumes $(\bigsqcap x{\in}A.\ B\ x)\ b$
obtains $B\ a\ b \mid a \notin A$
⟨*proof*⟩

lemma *Inf2-E*:
assumes $(\bigsqcap A)\ a\ b$
obtains $r\ a\ b \mid r \notin A$
⟨*proof*⟩

lemma *INF2-E*:

assumes $(\bigsqcap x{\in}A.\ B\ x)\ b\ c$
obtains $B\ a\ b\ c \mid a \notin A$
⟨*proof*⟩

lemma *Sup1-I*: $P \in A \Longrightarrow P\ a \Longrightarrow (\bigsqcup A)\ a$
⟨*proof*⟩

lemma *SUP1-I*: $a \in A \Longrightarrow B\ a\ b \Longrightarrow (\bigsqcup x{\in}A.\ B\ x)\ b$
⟨*proof*⟩

lemma *Sup2-I*: $r \in A \Longrightarrow r\ a\ b \Longrightarrow (\bigsqcup A)\ a\ b$
⟨*proof*⟩

lemma *SUP2-I*: $a \in A \Longrightarrow B\ a\ b\ c \Longrightarrow (\bigsqcup x{\in}A.\ B\ x)\ b\ c$
⟨*proof*⟩

lemma *Sup1-E*:
assumes $(\bigsqcup A)\ a$
obtains P **where** $P \in A$ **and** $P\ a$
⟨*proof*⟩

lemma *SUP1-E*:
assumes $(\bigsqcup x{\in}A.\ B\ x)\ b$
obtains x **where** $x \in A$ **and** $B\ x\ b$
⟨*proof*⟩

lemma *Sup2-E*:
assumes $(\bigsqcup A)\ a\ b$
obtains r **where** $r \in A\ r\ a\ b$
⟨*proof*⟩

lemma *SUP2-E*:
assumes $(\bigsqcup x{\in}A.\ B\ x)\ b\ c$
obtains x **where** $x \in A\ B\ x\ b\ c$
⟨*proof*⟩

11.6 Complete lattice on - *set*

instantiation *set* :: (*type*) *complete-lattice*
begin

definition $\bigsqcap A = \{x.\ \bigsqcap((\lambda B.\ x \in B)\ `\ A)\}$

definition $\bigsqcup A = \{x.\ \bigsqcup((\lambda B.\ x \in B)\ `\ A)\}$

instance
⟨*proof*⟩

end

11.6.1 Inter

abbreviation *Inter* :: *'a set set* ⇒ *'a set* $(\bigcap)$
where $\bigcap S \equiv \prod S$

lemma *Inter-eq*: $\bigcap A = \{x.\ \forall B \in A.\ x \in B\}$
⟨*proof*⟩

lemma *Inter-iff* [*simp*]: $A \in \bigcap C \longleftrightarrow (\forall X{\in}C.\ A \in X)$
⟨*proof*⟩

lemma *InterI* [*intro*!]: $(\bigwedge X.\ X \in C \Longrightarrow A \in X) \Longrightarrow A \in \bigcap C$
⟨*proof*⟩

A "destruct" rule – every X in C contains A as an element, but $A \in X$ can hold when $X \in C$ does not! This rule is analogous to *spec*.

lemma *InterD* [*elim*, *Pure.elim*]: $A \in \bigcap C \Longrightarrow X \in C \Longrightarrow A \in X$
⟨*proof*⟩

lemma *InterE* [*elim*]: $A \in \bigcap C \Longrightarrow (X \notin C \Longrightarrow R) \Longrightarrow (A \in X \Longrightarrow R) \Longrightarrow R$
— "Classical" elimination rule – does not require proving $X \in C$.
⟨*proof*⟩

lemma *Inter-lower*: $B \in A \Longrightarrow \bigcap A \subseteq B$
⟨*proof*⟩

lemma *Inter-subset*: $(\bigwedge X.\ X \in A \Longrightarrow X \subseteq B) \Longrightarrow A \neq \{\} \Longrightarrow \bigcap A \subseteq B$
⟨*proof*⟩

lemma *Inter-greatest*: $(\bigwedge X.\ X \in A \Longrightarrow C \subseteq X) \Longrightarrow C \subseteq \bigcap A$
⟨*proof*⟩

lemma *Inter-empty*: $\bigcap \{\} = UNIV$
⟨*proof*⟩

lemma *Inter-UNIV*: $\bigcap UNIV = \{\}$
⟨*proof*⟩

lemma *Inter-insert*: $\bigcap (insert\ a\ B) = a \cap \bigcap B$
⟨*proof*⟩

lemma *Inter-Un-subset*: $\bigcap A \cup \bigcap B \subseteq \bigcap (A \cap B)$
⟨*proof*⟩

lemma *Inter-Un-distrib*: $\bigcap (A \cup B) = \bigcap A \cap \bigcap B$
⟨*proof*⟩

lemma *Inter-UNIV-conv* [*simp*]:
$\bigcap A = UNIV \longleftrightarrow (\forall x{\in}A.\ x = UNIV)$

$UNIV = \bigcap A \longleftrightarrow (\forall x{\in}A.\ x = UNIV)$
⟨*proof*⟩

lemma *Inter-anti-mono*: $B \subseteq A \Longrightarrow \bigcap A \subseteq \bigcap B$
⟨*proof*⟩

11.6.2 Intersections of families

syntax (*ASCII*)
-INTER1 :: *pttrns* ⇒ *'b set* ⇒ *'b set* (*(3INT -./ -) [0, 10] 10*)
-INTER :: *pttrn* ⇒ *'a set* ⇒ *'b set* ⇒ *'b set* (*(3INT -:-./ -) [0, 0, 10] 10*)

syntax
-INTER1 :: *pttrns* ⇒ *'b set* ⇒ *'b set* (*(3⋂ -./ -) [0, 10] 10*)
-INTER :: *pttrn* ⇒ *'a set* ⇒ *'b set* ⇒ *'b set* (*(3⋂ -∈-./ -) [0, 0, 10] 10*)

syntax (*latex* **output**)
-INTER1 :: *pttrns* ⇒ *'b set* ⇒ *'b set* (*(3⋂(‹unbreakable›_)/ -) [0, 10] 10*)
-INTER :: *pttrn* ⇒ *'a set* ⇒ *'b set* ⇒ *'b set* (*(3⋂(‹unbreakable›_∈_)/ -) [0, 0, 10] 10*)

translations
$\bigcap x\ y.\ f \rightleftharpoons \bigcap x.\ \bigcap y.\ f$
$\bigcap x.\ f \rightleftharpoons \bigcap (CONST\ range\ (\lambda x.\ f))$
$\bigcap x{\in}A.\ f \rightleftharpoons CONST\ Inter\ ((\lambda x.\ f)\ `\ A)$

lemma *INTER-eq*: $(\bigcap x{\in}A.\ B\ x) = \{y.\ \forall x{\in}A.\ y \in B\ x\}$
⟨*proof*⟩

lemma *INT-iff* [*simp*]: $b \in (\bigcap x{\in}A.\ B\ x) \longleftrightarrow (\forall x{\in}A.\ b \in B\ x)$
⟨*proof*⟩

lemma *INT-I* [*intro!*]: $(\bigwedge x.\ x \in A \Longrightarrow b \in B\ x) \Longrightarrow b \in (\bigcap x{\in}A.\ B\ x)$
⟨*proof*⟩

lemma *INT-D* [*elim*, *Pure.elim*]: $b \in (\bigcap x{\in}A.\ B\ x) \Longrightarrow a \in A \Longrightarrow b \in B\ a$
⟨*proof*⟩

lemma *INT-E* [*elim*]: $b \in (\bigcap x{\in}A.\ B\ x) \Longrightarrow (b \in B\ a \Longrightarrow R) \Longrightarrow (a \notin A \Longrightarrow R) \Longrightarrow R$
— "Classical" elimination – by the Excluded Middle on $a \in A$.
⟨*proof*⟩

lemma *Collect-ball-eq*: $\{x.\ \forall y{\in}A.\ P\ x\ y\} = (\bigcap y{\in}A.\ \{x.\ P\ x\ y\})$
⟨*proof*⟩

lemma *Collect-all-eq*: $\{x.\ \forall y.\ P\ x\ y\} = (\bigcap y.\ \{x.\ P\ x\ y\})$
⟨*proof*⟩

lemma *INT-lower*: $a \in A \Longrightarrow (\bigcap x{\in}A.\ B\ x) \subseteq B\ a$
⟨*proof*⟩

lemma *INT-greatest*: $(\bigwedge x.\ x \in A \Longrightarrow C \subseteq B\ x) \Longrightarrow C \subseteq (\bigcap x{\in}A.\ B\ x)$
⟨*proof*⟩

lemma *INT-empty*: $(\bigcap x{\in}\{\}.\ B\ x) = UNIV$
⟨*proof*⟩

lemma *INT-absorb*: $k \in I \Longrightarrow A\ k \cap (\bigcap i{\in}I.\ A\ i) = (\bigcap i{\in}I.\ A\ i)$
⟨*proof*⟩

lemma *INT-subset-iff*: $B \subseteq (\bigcap i{\in}I.\ A\ i) \longleftrightarrow (\forall\, i{\in}I.\ B \subseteq A\ i)$
⟨*proof*⟩

lemma *INT-insert* [*simp*]: $(\bigcap x \in insert\ a\ A.\ B\ x) = B\ a \cap \bigcap\ (B\ `\ A)$
⟨*proof*⟩

lemma *INT-Un*: $(\bigcap i \in A \cup B.\ M\ i) = (\bigcap i \in A.\ M\ i) \cap (\bigcap i{\in}B.\ M\ i)$
⟨*proof*⟩

lemma *INT-insert-distrib*: $u \in A \Longrightarrow (\bigcap x{\in}A.\ insert\ a\ (B\ x)) = insert\ a\ (\bigcap x{\in}A.\ B\ x)$
⟨*proof*⟩

lemma *INT-constant* [*simp*]: $(\bigcap y{\in}A.\ c) = (if\ A = \{\}\ then\ UNIV\ else\ c)$
⟨*proof*⟩

lemma *INTER-UNIV-conv*:
$(UNIV = (\bigcap x{\in}A.\ B\ x)) = (\forall\, x{\in}A.\ B\ x = UNIV)$
$((\bigcap x{\in}A.\ B\ x) = UNIV) = (\forall\, x{\in}A.\ B\ x = UNIV)$
⟨*proof*⟩

lemma *INT-bool-eq*: $(\bigcap b.\ A\ b) = A\ True \cap A\ False$
⟨*proof*⟩

lemma *INT-anti-mono*: $A \subseteq B \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow f\ x \subseteq g\ x) \Longrightarrow (\bigcap x{\in}B.\ f\ x) \subseteq (\bigcap x{\in}A.\ g\ x)$
— The last inclusion is POSITIVE!
⟨*proof*⟩

lemma *Pow-INT-eq*: $Pow\ (\bigcap x{\in}A.\ B\ x) = (\bigcap x{\in}A.\ Pow\ (B\ x))$
⟨*proof*⟩

lemma *vimage-INT*: $f\ -`\ (\bigcap x{\in}A.\ B\ x) = (\bigcap x{\in}A.\ f\ -`\ B\ x)$
⟨*proof*⟩

11.6.3 Union

abbreviation *Union* :: *'a set set* ⇒ *'a set* $(\bigcup)$
where $\bigcup S \equiv \bigsqcup S$

lemma *Union-eq*: $\bigcup A = \{x.\ \exists B \in A.\ x \in B\}$
⟨*proof*⟩

lemma *Union-iff* [*simp*]: $A \in \bigcup C \longleftrightarrow (\exists X{\in}C.\ A{\in}X)$
⟨*proof*⟩

lemma *UnionI* [*intro*]: $X \in C \Longrightarrow A \in X \Longrightarrow A \in \bigcup C$
— The order of the premises presupposes that C is rigid; A may be flexible.
⟨*proof*⟩

lemma *UnionE* [*elim!*]: $A \in \bigcup C \Longrightarrow (\bigwedge X.\ A \in X \Longrightarrow X \in C \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

lemma *Union-upper*: $B \in A \Longrightarrow B \subseteq \bigcup A$
⟨*proof*⟩

lemma *Union-least*: $(\bigwedge X.\ X \in A \Longrightarrow X \subseteq C) \Longrightarrow \bigcup A \subseteq C$
⟨*proof*⟩

lemma *Union-empty*: $\bigcup \{\} = \{\}$
⟨*proof*⟩

lemma *Union-UNIV*: $\bigcup UNIV = UNIV$
⟨*proof*⟩

lemma *Union-insert*: $\bigcup (insert\ a\ B) = a \cup \bigcup B$
⟨*proof*⟩

lemma *Union-Un-distrib* [*simp*]: $\bigcup (A \cup B) = \bigcup A \cup \bigcup B$
⟨*proof*⟩

lemma *Union-Int-subset*: $\bigcup (A \cap B) \subseteq \bigcup A \cap \bigcup B$
⟨*proof*⟩

lemma *Union-empty-conv*: $(\bigcup A = \{\}) \longleftrightarrow (\forall x{\in}A.\ x = \{\})$
⟨*proof*⟩

lemma *empty-Union-conv*: $(\{\} = \bigcup A) \longleftrightarrow (\forall x{\in}A.\ x = \{\})$
⟨*proof*⟩

lemma *subset-Pow-Union*: $A \subseteq Pow\ (\bigcup A)$
⟨*proof*⟩

lemma *Union-Pow-eq* [*simp*]: $\bigcup (Pow\ A) = A$
⟨*proof*⟩

lemma *Union-mono*: $A \subseteq B \Longrightarrow \bigcup A \subseteq \bigcup B$
⟨*proof*⟩

lemma *Union-subsetI*: $(\bigwedge x.\ x \in A \Longrightarrow \exists y.\ y \in B \wedge x \subseteq y) \Longrightarrow \bigcup A \subseteq \bigcup B$
⟨*proof*⟩

lemma *disjnt-inj-on-iff*:
$[\![inj\text{-}on\ f\ (\bigcup \mathcal{A});\ X \in \mathcal{A};\ Y \in \mathcal{A}]\!] \Longrightarrow disjnt\ (f\ `\ X)\ (f\ `\ Y) \longleftrightarrow disjnt\ X\ Y$
⟨*proof*⟩

lemma *disjnt-Union1* [*simp*]: $disjnt\ (\bigcup \mathcal{A})\ B \longleftrightarrow (\forall A \in \mathcal{A}.\ disjnt\ A\ B)$
⟨*proof*⟩

lemma *disjnt-Union2* [*simp*]: $disjnt\ B\ (\bigcup \mathcal{A}) \longleftrightarrow (\forall A \in \mathcal{A}.\ disjnt\ B\ A)$
⟨*proof*⟩

11.6.4 Unions of families

syntax (*ASCII*)
-UNION1 :: *pttrns* => *'b set* => *'b set* (*(3UN -./ -) [0, 10] 10*)
-UNION :: *pttrn* => *'a set* => *'b set* => *'b set* (*(3UN -:-./ -) [0, 0, 10] 10*)

syntax
-UNION1 :: *pttrns* => *'b set* => *'b set* (*(3⋃ -./ -) [0, 10] 10*)
-UNION :: *pttrn* => *'a set* => *'b set* => *'b set* (*(3⋃ -∈-./ -) [0, 0, 10] 10*)

syntax (*latex* **output**)
-UNION1 :: *pttrns* => *'b set* => *'b set* (*(3⋃(‹unbreakable›-)/ -) [0, 10] 10*)
-UNION :: *pttrn* => *'a set* => *'b set* => *'b set* (*(3⋃(‹unbreakable›-∈-)/ -) [0, 0, 10] 10*)

translations
$\bigcup x\ y.\ f \rightleftharpoons \bigcup x.\ \bigcup y.\ f$
$\bigcup x.\ f \rightleftharpoons \bigcup (CONST\ range\ (\lambda x.\ f))$
$\bigcup x \in A.\ f \rightleftharpoons CONST\ Union\ ((\lambda x.\ f)\ `\ A)$

Note the difference between ordinary syntax of indexed unions and intersections (e.g. $\bigcup a_1 \in A_1.\ B$) and their LaTeX rendition: $\bigcup_{a_1 \in A_1} B$.

lemma *disjoint-UN-iff*: $disjnt\ A\ (\bigcup i \in I.\ B\ i) \longleftrightarrow (\forall i \in I.\ disjnt\ A\ (B\ i))$
⟨*proof*⟩

lemma *UNION-eq*: $(\bigcup x \in A.\ B\ x) = \{y.\ \exists x \in A.\ y \in B\ x\}$
⟨*proof*⟩

lemma *bind-UNION* [*code*]: $Set.bind\ A\ f = \bigcup (f\ `\ A)$
⟨*proof*⟩

lemma *member-bind* [*simp*]: $x \in \mathit{Set.bind}\ A\ f \longleftrightarrow x \in \bigcup (f\ `\ A)$
⟨*proof*⟩

lemma *Union-SetCompr-eq*: $\bigcup \{f\ x |\ x.\ P\ x\} = \{a.\ \exists x.\ P\ x \wedge a \in f\ x\}$
⟨*proof*⟩

lemma *UN-iff* [*simp*]: $b \in (\bigcup x{\in}A.\ B\ x) \longleftrightarrow (\exists x{\in}A.\ b \in B\ x)$
⟨*proof*⟩

lemma *UN-I* [*intro*]: $a \in A \Longrightarrow b \in B\ a \Longrightarrow b \in (\bigcup x{\in}A.\ B\ x)$
— The order of the premises presupposes that A is rigid; b may be flexible.
⟨*proof*⟩

lemma *UN-E* [*elim*!]: $b \in (\bigcup x{\in}A.\ B\ x) \Longrightarrow (\bigwedge x.\ x{\in}A \Longrightarrow b \in B\ x \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

lemma *UN-upper*: $a \in A \Longrightarrow B\ a \subseteq (\bigcup x{\in}A.\ B\ x)$
⟨*proof*⟩

lemma *UN-least*: $(\bigwedge x.\ x \in A \Longrightarrow B\ x \subseteq C) \Longrightarrow (\bigcup x{\in}A.\ B\ x) \subseteq C$
⟨*proof*⟩

lemma *Collect-bex-eq*: $\{x.\ \exists y{\in}A.\ P\ x\ y\} = (\bigcup y{\in}A.\ \{x.\ P\ x\ y\})$
⟨*proof*⟩

lemma *UN-insert-distrib*: $u \in A \Longrightarrow (\bigcup x{\in}A.\ \mathit{insert}\ a\ (B\ x)) = \mathit{insert}\ a\ (\bigcup x{\in}A.\ B\ x)$
⟨*proof*⟩

lemma *UN-empty*: $(\bigcup x{\in}\{\}.\ B\ x) = \{\}$
⟨*proof*⟩

lemma *UN-empty2*: $(\bigcup x{\in}A.\ \{\}) = \{\}$
⟨*proof*⟩

lemma *UN-absorb*: $k \in I \Longrightarrow A\ k \cup (\bigcup i{\in}I.\ A\ i) = (\bigcup i{\in}I.\ A\ i)$
⟨*proof*⟩

lemma *UN-insert* [*simp*]: $(\bigcup x{\in}\mathit{insert}\ a\ A.\ B\ x) = B\ a \cup \bigcup (B\ `\ A)$
⟨*proof*⟩

lemma *UN-Un* [*simp*]: $(\bigcup i \in A \cup B.\ M\ i) = (\bigcup i{\in}A.\ M\ i) \cup (\bigcup i{\in}B.\ M\ i)$
⟨*proof*⟩

lemma *UN-UN-flatten*: $(\bigcup x \in (\bigcup y{\in}A.\ B\ y).\ C\ x) = (\bigcup y{\in}A.\ \bigcup x{\in}B\ y.\ C\ x)$
⟨*proof*⟩

lemma *UN-subset-iff*: $((\bigcup i{\in}I.\ A\ i) \subseteq B) = (\forall i{\in}I.\ A\ i \subseteq B)$

⟨*proof*⟩

lemma *UN-constant* [*simp*]: $(\bigcup y{\in}A.\ c) = (\mathit{if}\ A = \{\}\ \mathit{then}\ \{\}\ \mathit{else}\ c)$
⟨*proof*⟩

lemma *UNION-singleton-eq-range*: $(\bigcup x{\in}A.\ \{f\ x\}) = f\ `\ A$
⟨*proof*⟩

lemma *image-Union*: $f\ `\ \bigcup S = (\bigcup x{\in}S.\ f\ `\ x)$
⟨*proof*⟩

lemma *UNION-empty-conv*:
$\{\} = (\bigcup x{\in}A.\ B\ x) \longleftrightarrow (\forall x{\in}A.\ B\ x = \{\})$
$(\bigcup x{\in}A.\ B\ x) = \{\} \longleftrightarrow (\forall x{\in}A.\ B\ x = \{\})$
⟨*proof*⟩

lemma *Collect-ex-eq*: $\{x.\ \exists y.\ P\ x\ y\} = (\bigcup y.\ \{x.\ P\ x\ y\})$
⟨*proof*⟩

lemma *ball-UN*: $(\forall z \in \bigcup(B\ `\ A).\ P\ z) \longleftrightarrow (\forall x{\in}A.\ \forall z \in B\ x.\ P\ z)$
⟨*proof*⟩

lemma *bex-UN*: $(\exists z \in \bigcup(B\ `\ A).\ P\ z) \longleftrightarrow (\exists x{\in}A.\ \exists z{\in}B\ x.\ P\ z)$
⟨*proof*⟩

lemma *Un-eq-UN*: $A \cup B = (\bigcup b.\ \mathit{if}\ b\ \mathit{then}\ A\ \mathit{else}\ B)$
⟨*proof*⟩

lemma *UN-bool-eq*: $(\bigcup b.\ A\ b) = (A\ \mathit{True} \cup A\ \mathit{False})$
⟨*proof*⟩

lemma *UN-Pow-subset*: $(\bigcup x{\in}A.\ \mathit{Pow}\ (B\ x)) \subseteq \mathit{Pow}\ (\bigcup x{\in}A.\ B\ x)$
⟨*proof*⟩

lemma *UN-mono*:
$A \subseteq B \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow f\ x \subseteq g\ x) \Longrightarrow$
$(\bigcup x{\in}A.\ f\ x) \subseteq (\bigcup x{\in}B.\ g\ x)$
⟨*proof*⟩

lemma *vimage-Union*: $f\ -`\ (\bigcup A) = (\bigcup X{\in}A.\ f\ -`\ X)$
⟨*proof*⟩

lemma *vimage-UN*: $f\ -`\ (\bigcup x{\in}A.\ B\ x) = (\bigcup x{\in}A.\ f\ -`\ B\ x)$
⟨*proof*⟩

lemma *vimage-eq-UN*: $f\ -`\ B = (\bigcup y{\in}B.\ f\ -`\ \{y\})$
— NOT suitable for rewriting
⟨*proof*⟩

lemma *image-UN*: $f\ `\ \bigcup(B\ `\ A) = (\bigcup x{\in}A.\ f\ `\ B\ x)$
⟨*proof*⟩

lemma *UN-singleton* [*simp*]: $(\bigcup x{\in}A.\ \{x\}) = A$
⟨*proof*⟩

lemma *inj-on-image*: *inj-on* $f\ (\bigcup A) \Longrightarrow$ *inj-on* $((`)\ f)\ A$
⟨*proof*⟩

11.6.5 Distributive laws

lemma *Int-Union*: $A \cap \bigcup B = (\bigcup C{\in}B.\ A \cap C)$
⟨*proof*⟩

lemma *Un-Inter*: $A \cup \bigcap B = (\bigcap C{\in}B.\ A \cup C)$
⟨*proof*⟩

lemma *Int-Union2*: $\bigcup B \cap A = (\bigcup C{\in}B.\ C \cap A)$
⟨*proof*⟩

lemma *INT-Int-distrib*: $(\bigcap i{\in}I.\ A\ i \cap B\ i) = (\bigcap i{\in}I.\ A\ i) \cap (\bigcap i{\in}I.\ B\ i)$
⟨*proof*⟩

lemma *UN-Un-distrib*: $(\bigcup i{\in}I.\ A\ i \cup B\ i) = (\bigcup i{\in}I.\ A\ i) \cup (\bigcup i{\in}I.\ B\ i)$
⟨*proof*⟩

lemma *Int-Inter-image*: $(\bigcap x{\in}C.\ A\ x \cap B\ x) = \bigcap(A\ `\ C) \cap \bigcap(B\ `\ C)$
⟨*proof*⟩

lemma *Int-Inter-eq*: $A \cap \bigcap\mathcal{B} = (\textit{if}\ \mathcal{B}{=}\{\}\ \textit{then}\ A\ \textit{else}\ (\bigcap B{\in}\mathcal{B}.\ A \cap B))$
$\bigcap\mathcal{B} \cap A = (\textit{if}\ \mathcal{B}{=}\{\}\ \textit{then}\ A\ \textit{else}\ (\bigcap B{\in}\mathcal{B}.\ B \cap A))$
⟨*proof*⟩

lemma *Un-Union-image*: $(\bigcup x{\in}C.\ A\ x \cup B\ x) = \bigcup(A\ `\ C) \cup \bigcup(B\ `\ C)$
— Devlin, Fundamentals of Contemporary Set Theory, page 12, exercise 5:
— Union of a family of unions
⟨*proof*⟩

lemma *Un-INT-distrib*: $B \cup (\bigcap i{\in}I.\ A\ i) = (\bigcap i{\in}I.\ B \cup A\ i)$
⟨*proof*⟩

lemma *Int-UN-distrib*: $B \cap (\bigcup i{\in}I.\ A\ i) = (\bigcup i{\in}I.\ B \cap A\ i)$
— Halmos, Naive Set Theory, page 35.
⟨*proof*⟩

lemma *Int-UN-distrib2*: $(\bigcup i{\in}I.\ A\ i) \cap (\bigcup j{\in}J.\ B\ j) = (\bigcup i{\in}I.\ \bigcup j{\in}J.\ A\ i \cap B\ j)$
⟨*proof*⟩

lemma *Un-INT-distrib2*: $(\bigcap i{\in}I.\ A\ i) \cup (\bigcap j{\in}J.\ B\ j) = (\bigcap i{\in}I.\ \bigcap j{\in}J.\ A\ i \cup B\ j)$
⟨*proof*⟩

lemma *Union-disjoint*: $(\bigcup C \cap A = \{\}) \longleftrightarrow (\forall B{\in}C.\ B \cap A = \{\})$
⟨*proof*⟩

lemma *SUP-UNION*: $(\bigsqcup x{\in}(\bigcup y{\in}A.\ g\ y).\ f\ x) = (\bigsqcup y{\in}A.\ \bigsqcup x{\in}g\ y.\ f\ x$ :: - :: *complete-lattice*$)$
⟨*proof*⟩

11.7 Injections and bijections

lemma *inj-on-Inter*: $S \neq \{\} \Longrightarrow (\bigwedge A.\ A \in S \Longrightarrow$ *inj-on f A*$) \Longrightarrow$ *inj-on f* $(\bigcap S)$
⟨*proof*⟩

lemma *inj-on-INTER*: $I \neq \{\} \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow$ *inj-on f* $(A\ i)) \Longrightarrow$ *inj-on f* $(\bigcap i \in I.\ A\ i)$
⟨*proof*⟩

lemma *inj-on-UNION-chain*:
assumes *chain*: $\bigwedge i\ j.\ i \in I \Longrightarrow j \in I \Longrightarrow A\ i \leq A\ j \vee A\ j \leq A\ i$
and *inj*: $\bigwedge i.\ i \in I \Longrightarrow$ *inj-on f* $(A\ i)$
shows *inj-on f* $(\bigcup i \in I.\ A\ i)$
⟨*proof*⟩

lemma *bij-betw-UNION-chain*:
assumes *chain*: $\bigwedge i\ j.\ i \in I \Longrightarrow j \in I \Longrightarrow A\ i \leq A\ j \vee A\ j \leq A\ i$
and *bij*: $\bigwedge i.\ i \in I \Longrightarrow$ *bij-betw f* $(A\ i)\ (A'\ i)$
shows *bij-betw f* $(\bigcup i \in I.\ A\ i)\ (\bigcup i \in I.\ A'\ i)$
⟨*proof*⟩

lemma *image-INT*: *inj-on f C* $\Longrightarrow \forall x{\in}A.\ B\ x \subseteq C \Longrightarrow j \in A \Longrightarrow f\ `\ (\bigcap (B\ `\ A)) = (\bigcap x{\in}A.\ f\ `\ B\ x)$
⟨*proof*⟩

lemma *bij-image-INT*: *bij f* $\Longrightarrow f\ `\ (\bigcap (B\ `\ A)) = (\bigcap x{\in}A.\ f\ `\ B\ x)$
⟨*proof*⟩

lemma *UNION-fun-upd*: $\bigcup (A(i := B)\ `\ J) = \bigcup (A\ `\ (J - \{i\})) \cup$ (*if* $i \in J$ *then* B *else* $\{\}$)
⟨*proof*⟩

lemma *bij-betw-Pow*:
assumes *bij-betw f A B*
shows *bij-betw* (*image f*) (*Pow A*) (*Pow B*)
⟨*proof*⟩

11.7.1 Complement

lemma *Compl-INT* [*simp*]: $-(\bigcap x{\in}A.\ B\ x) = (\bigcup x{\in}A.\ -B\ x)$
⟨*proof*⟩

lemma *Compl-UN* [*simp*]: $-(\bigcup x{\in}A.\ B\ x) = (\bigcap x{\in}A.\ -B\ x)$
⟨*proof*⟩

11.7.2 Miniscoping and maxiscoping

Miniscoping: pushing in quantifiers and big Unions and Intersections.

lemma *UN-simps* [*simp*]:
$\bigwedge a\ B\ C.\ (\bigcup x{\in}C.\ \mathit{insert}\ a\ (B\ x)) = (\mathit{if}\ C{=}\{\}\ \mathit{then}\ \{\}\ \mathit{else}\ \mathit{insert}\ a\ (\bigcup x{\in}C.\ B\ x))$
$\bigwedge A\ B\ C.\ (\bigcup x{\in}C.\ A\ x \cup B) = ((\mathit{if}\ C{=}\{\}\ \mathit{then}\ \{\}\ \mathit{else}\ (\bigcup x{\in}C.\ A\ x) \cup B))$
$\bigwedge A\ B\ C.\ (\bigcup x{\in}C.\ A \cup B\ x) = ((\mathit{if}\ C{=}\{\}\ \mathit{then}\ \{\}\ \mathit{else}\ A \cup (\bigcup x{\in}C.\ B\ x)))$
$\bigwedge A\ B\ C.\ (\bigcup x{\in}C.\ A\ x \cap B) = ((\bigcup x{\in}C.\ A\ x) \cap B)$
$\bigwedge A\ B\ C.\ (\bigcup x{\in}C.\ A \cap B\ x) = (A \cap (\bigcup x{\in}C.\ B\ x))$
$\bigwedge A\ B\ C.\ (\bigcup x{\in}C.\ A\ x - B) = ((\bigcup x{\in}C.\ A\ x) - B)$
$\bigwedge A\ B\ C.\ (\bigcup x{\in}C.\ A - B\ x) = (A - (\bigcap x{\in}C.\ B\ x))$
$\bigwedge A\ B.\ (\bigcup x{\in}\bigcup A.\ B\ x) = (\bigcup y{\in}A.\ \bigcup x{\in}y.\ B\ x)$
$\bigwedge A\ B\ C.\ (\bigcup z{\in}(\bigcup (B\ `\ A)).\ C\ z) = (\bigcup x{\in}A.\ \bigcup z{\in}B\ x.\ C\ z)$
$\bigwedge A\ B\ f.\ (\bigcup x{\in}f`A.\ B\ x) = (\bigcup a{\in}A.\ B\ (f\ a))$
⟨*proof*⟩

lemma *INT-simps* [*simp*]:
$\bigwedge A\ B\ C.\ (\bigcap x{\in}C.\ A\ x \cap B) = (\mathit{if}\ C{=}\{\}\ \mathit{then}\ \mathit{UNIV}\ \mathit{else}\ (\bigcap x{\in}C.\ A\ x) \cap B)$
$\bigwedge A\ B\ C.\ (\bigcap x{\in}C.\ A \cap B\ x) = (\mathit{if}\ C{=}\{\}\ \mathit{then}\ \mathit{UNIV}\ \mathit{else}\ A \cap (\bigcap x{\in}C.\ B\ x))$
$\bigwedge A\ B\ C.\ (\bigcap x{\in}C.\ A\ x - B) = (\mathit{if}\ C{=}\{\}\ \mathit{then}\ \mathit{UNIV}\ \mathit{else}\ (\bigcap x{\in}C.\ A\ x) - B)$
$\bigwedge A\ B\ C.\ (\bigcap x{\in}C.\ A - B\ x) = (\mathit{if}\ C{=}\{\}\ \mathit{then}\ \mathit{UNIV}\ \mathit{else}\ A - (\bigcup x{\in}C.\ B\ x))$
$\bigwedge a\ B\ C.\ (\bigcap x{\in}C.\ \mathit{insert}\ a\ (B\ x)) = \mathit{insert}\ a\ (\bigcap x{\in}C.\ B\ x)$
$\bigwedge A\ B\ C.\ (\bigcap x{\in}C.\ A\ x \cup B) = ((\bigcap x{\in}C.\ A\ x) \cup B)$
$\bigwedge A\ B\ C.\ (\bigcap x{\in}C.\ A \cup B\ x) = (A \cup (\bigcap x{\in}C.\ B\ x))$
$\bigwedge A\ B.\ (\bigcap x{\in}\bigcup A.\ B\ x) = (\bigcap y{\in}A.\ \bigcap x{\in}y.\ B\ x)$
$\bigwedge A\ B\ C.\ (\bigcap z{\in}(\bigcup (B\ `\ A)).\ C\ z) = (\bigcap x{\in}A.\ \bigcap z{\in}B\ x.\ C\ z)$
$\bigwedge A\ B\ f.\ (\bigcap x{\in}f`A.\ B\ x) = (\bigcap a{\in}A.\ B\ (f\ a))$
⟨*proof*⟩

lemma *UN-ball-bex-simps* [*simp*]:
$\bigwedge A\ P.\ (\forall x{\in}\bigcup A.\ P\ x) \longleftrightarrow (\forall y{\in}A.\ \forall x{\in}y.\ P\ x)$
$\bigwedge A\ B\ P.\ (\forall x{\in}(\bigcup (B\ `\ A)).\ P\ x) = (\forall a{\in}A.\ \forall x{\in}\ B\ a.\ P\ x)$
$\bigwedge A\ P.\ (\exists x{\in}\bigcup A.\ P\ x) \longleftrightarrow (\exists y{\in}A.\ \exists x{\in}y.\ P\ x)$
$\bigwedge A\ B\ P.\ (\exists x{\in}(\bigcup (B\ `\ A)).\ P\ x) \longleftrightarrow (\exists a{\in}A.\ \exists x{\in}B\ a.\ P\ x)$
⟨*proof*⟩

Maxiscoping: pulling out big Unions and Intersections.

lemma *UN-extend-simps*:
$\bigwedge a\ B\ C.\ \mathit{insert}\ a\ (\bigcup x{\in}C.\ B\ x) = (\mathit{if}\ C{=}\{\}\ \mathit{then}\ \{a\}\ \mathit{else}\ (\bigcup x{\in}C.\ \mathit{insert}\ a\ (B\ x)))$

$\bigwedge A\ B\ C.\ (\bigcup x{\in}C.\ A\ x) \cup B = (\mathit{if}\ C{=}\{\}\ \mathit{then}\ B\ \mathit{else}\ (\bigcup x{\in}C.\ A\ x \cup B))$
$\bigwedge A\ B\ C.\ A \cup (\bigcup x{\in}C.\ B\ x) = (\mathit{if}\ C{=}\{\}\ \mathit{then}\ A\ \mathit{else}\ (\bigcup x{\in}C.\ A \cup B\ x))$
$\bigwedge A\ B\ C.\ ((\bigcup x{\in}C.\ A\ x) \cap B) = (\bigcup x{\in}C.\ A\ x \cap B)$
$\bigwedge A\ B\ C.\ (A \cap (\bigcup x{\in}C.\ B\ x)) = (\bigcup x{\in}C.\ A \cap B\ x)$
$\bigwedge A\ B\ C.\ ((\bigcup x{\in}C.\ A\ x) - B) = (\bigcup x{\in}C.\ A\ x - B)$
$\bigwedge A\ B\ C.\ (A - (\bigcap x{\in}C.\ B\ x)) = (\bigcup x{\in}C.\ A - B\ x)$
$\bigwedge A\ B.\ (\bigcup y{\in}A.\ \bigcup x{\in}y.\ B\ x) = (\bigcup x{\in}\bigcup A.\ B\ x)$
$\bigwedge A\ B\ C.\ (\bigcup x{\in}A.\ \bigcup z{\in}B\ x.\ C\ z) = (\bigcup z{\in}(\bigcup (B\ `\ A)).\ C\ z)$
$\bigwedge A\ B\ f.\ (\bigcup a{\in}A.\ B\ (f\ a)) = (\bigcup x{\in}f`A.\ B\ x)$
⟨*proof*⟩

lemma *INT-extend-simps*:
$\bigwedge A\ B\ C.\ (\bigcap x{\in}C.\ A\ x) \cap B = (\mathit{if}\ C{=}\{\}\ \mathit{then}\ B\ \mathit{else}\ (\bigcap x{\in}C.\ A\ x \cap B))$
$\bigwedge A\ B\ C.\ A \cap (\bigcap x{\in}C.\ B\ x) = (\mathit{if}\ C{=}\{\}\ \mathit{then}\ A\ \mathit{else}\ (\bigcap x{\in}C.\ A \cap B\ x))$
$\bigwedge A\ B\ C.\ (\bigcap x{\in}C.\ A\ x) - B = (\mathit{if}\ C{=}\{\}\ \mathit{then}\ \mathit{UNIV} - B\ \mathit{else}\ (\bigcap x{\in}C.\ A\ x - B))$
$\bigwedge A\ B\ C.\ A - (\bigcup x{\in}C.\ B\ x) = (\mathit{if}\ C{=}\{\}\ \mathit{then}\ A\ \mathit{else}\ (\bigcap x{\in}C.\ A - B\ x))$
$\bigwedge a\ B\ C.\ \mathit{insert}\ a\ (\bigcap x{\in}C.\ B\ x) = (\bigcap x{\in}C.\ \mathit{insert}\ a\ (B\ x))$
$\bigwedge A\ B\ C.\ ((\bigcap x{\in}C.\ A\ x) \cup B) = (\bigcap x{\in}C.\ A\ x \cup B)$
$\bigwedge A\ B\ C.\ A \cup (\bigcap x{\in}C.\ B\ x) = (\bigcap x{\in}C.\ A \cup B\ x)$
$\bigwedge A\ B.\ (\bigcap y{\in}A.\ \bigcap x{\in}y.\ B\ x) = (\bigcap x{\in}\bigcup A.\ B\ x)$
$\bigwedge A\ B\ C.\ (\bigcap x{\in}A.\ \bigcap z{\in}B\ x.\ C\ z) = (\bigcap z{\in}(\bigcup (B\ `\ A)).\ C\ z)$
$\bigwedge A\ B\ f.\ (\bigcap a{\in}A.\ B\ (f\ a)) = (\bigcap x{\in}f`A.\ B\ x)$
⟨*proof*⟩

Finally

lemmas *mem-simps* =
insert-iff empty-iff Un-iff Int-iff Compl-iff Diff-iff
mem-Collect-eq UN-iff Union-iff INT-iff Inter-iff
— Each of these has ALREADY been added [*simp*] above.

end

12 Wrapping Existing Freely Generated Type's Constructors

theory *Ctr-Sugar*
imports *HOL*
keywords
print-case-translations :: *diag* **and**
free-constructors :: *thy-goal*
begin

consts
case-guard :: *bool* ⇒ $'a$ ⇒ ($'a$ ⇒ $'b$) ⇒ $'b$
case-nil :: $'a$ ⇒ $'b$
case-cons :: ($'a$ ⇒ $'b$) ⇒ ($'a$ ⇒ $'b$) ⇒ $'a$ ⇒ $'b$
case-elem :: $'a$ ⇒ $'b$ ⇒ $'a$ ⇒ $'b$

case-abs :: $('c \Rightarrow 'b) \Rightarrow 'b$

declare [[*coercion-args case-guard* − + −]]
declare [[*coercion-args case-cons* − −]]
declare [[*coercion-args case-abs* −]]
declare [[*coercion-args case-elem* − +]]

⟨*ML*⟩

lemma *iffI-np*: $[\![x \Longrightarrow \neg\ y;\ \neg\ x \Longrightarrow y]\!] \Longrightarrow \neg\ x \longleftrightarrow y$
⟨*proof*⟩

lemma *iff-contradict*:
$\neg\ P \Longrightarrow P \longleftrightarrow Q \Longrightarrow Q \Longrightarrow R$
$\neg\ Q \Longrightarrow P \longleftrightarrow Q \Longrightarrow P \Longrightarrow R$
⟨*proof*⟩

⟨*ML*⟩

Coinduction method that avoids some boilerplate compared with coinduct.

⟨*ML*⟩

end

13 Knaster-Tarski Fixpoint Theorem and inductive definitions

theory *Inductive*
imports *Complete-Lattices Ctr-Sugar*
keywords
inductive coinductive inductive-cases inductive-simps :: *thy-defn* **and**
monos **and**
print-inductives :: *diag* **and**
old-rep-datatype :: *thy-goal* **and**
primrec :: *thy-defn*
begin

13.1 Least fixed points

context *complete-lattice*
begin

definition *lfp* :: $('a \Rightarrow 'a) \Rightarrow 'a$
where $lfp\ f = Inf\ \{u.\ f\ u \leq u\}$

lemma *lfp-lowerbound*: $f\ A \leq A \Longrightarrow lfp\ f \leq A$
⟨*proof*⟩

lemma *lfp-greatest*: $(\bigwedge u.\ f\ u \leq u \Longrightarrow A \leq u) \Longrightarrow A \leq lfp\ f$
⟨*proof*⟩

end

lemma *lfp-fixpoint*:
assumes *mono f*
shows $f\ (lfp\ f) = lfp\ f$
⟨*proof*⟩

lemma *lfp-unfold*: $mono\ f \Longrightarrow lfp\ f = f\ (lfp\ f)$
⟨*proof*⟩

lemma *lfp-const*: $lfp\ (\lambda x.\ t) = t$
⟨*proof*⟩

lemma *lfp-eqI*: $mono\ F \Longrightarrow F\ x = x \Longrightarrow (\bigwedge z.\ F\ z = z \Longrightarrow x \leq z) \Longrightarrow lfp\ F = x$
⟨*proof*⟩

13.2 General induction rules for least fixed points

lemma *lfp-ordinal-induct* [*case-names mono step union*]:
fixes $f :: 'a{::}complete\text{-}lattice \Rightarrow 'a$
assumes *mono*: *mono f*
and *P-f*: $\bigwedge S.\ P\ S \Longrightarrow S \leq lfp\ f \Longrightarrow P\ (f\ S)$
and *P-Union*: $\bigwedge M.\ \forall S{\in}M.\ P\ S \Longrightarrow P\ (Sup\ M)$
shows $P\ (lfp\ f)$
⟨*proof*⟩

theorem *lfp-induct*:
assumes *mono*: *mono f*
and *ind*: $f\ (inf\ (lfp\ f)\ P) \leq P$
shows $lfp\ f \leq P$
⟨*proof*⟩

lemma *lfp-induct-set*:
assumes *lfp*: $a \in lfp\ f$
and *mono*: *mono f*
and *hyp*: $\bigwedge x.\ x \in f\ (lfp\ f \cap \{x.\ P\ x\}) \Longrightarrow P\ x$
shows $P\ a$
⟨*proof*⟩

lemma *lfp-ordinal-induct-set*:
assumes *mono*: *mono f*
and *P-f*: $\bigwedge S.\ P\ S \Longrightarrow P\ (f\ S)$
and *P-Union*: $\bigwedge M.\ \forall S{\in}M.\ P\ S \Longrightarrow P\ (\bigcup M)$
shows $P\ (lfp\ f)$
⟨*proof*⟩

Definition forms of *lfp-unfold* and *lfp-induct*, to control unfolding.

lemma *def-lfp-unfold*: $h \equiv lfp\ f \Longrightarrow mono\ f \Longrightarrow h = f\ h$
⟨*proof*⟩

lemma *def-lfp-induct*: $A \equiv lfp\ f \Longrightarrow mono\ f \Longrightarrow f\ (inf\ A\ P) \leq P \Longrightarrow A \leq P$
⟨*proof*⟩

lemma *def-lfp-induct-set*:
$A \equiv lfp\ f \Longrightarrow mono\ f \Longrightarrow a \in A \Longrightarrow (\bigwedge x.\ x \in f\ (A \cap \{x.\ P\ x\}) \Longrightarrow P\ x) \Longrightarrow P\ a$
⟨*proof*⟩

Monotonicity of *lfp*!

lemma *lfp-mono*: $(\bigwedge Z.\ f\ Z \leq g\ Z) \Longrightarrow lfp\ f \leq lfp\ g$
⟨*proof*⟩

13.3 Greatest fixed points

context *complete-lattice*
begin

definition *gfp* :: $('a \Rightarrow 'a) \Rightarrow 'a$
where $gfp\ f = Sup\ \{u.\ u \leq f\ u\}$

lemma *gfp-upperbound*: $X \leq f\ X \Longrightarrow X \leq gfp\ f$
⟨*proof*⟩

lemma *gfp-least*: $(\bigwedge u.\ u \leq f\ u \Longrightarrow u \leq X) \Longrightarrow gfp\ f \leq X$
⟨*proof*⟩

end

lemma *lfp-le-gfp*: $mono\ f \Longrightarrow lfp\ f \leq gfp\ f$
⟨*proof*⟩

lemma *gfp-fixpoint*:
assumes *mono f*
shows $f\ (gfp\ f) = gfp\ f$
⟨*proof*⟩

lemma *gfp-unfold*: $mono\ f \Longrightarrow gfp\ f = f\ (gfp\ f)$
⟨*proof*⟩

lemma *gfp-const*: $gfp\ (\lambda x.\ t) = t$
⟨*proof*⟩

lemma *gfp-eqI*: $mono\ F \Longrightarrow F\ x = x \Longrightarrow (\bigwedge z.\ F\ z = z \Longrightarrow z \leq x) \Longrightarrow gfp\ F = x$
⟨*proof*⟩

13.4 Coinduction rules for greatest fixed points

Weak version.

lemma *weak-coinduct*: $a \in X \Longrightarrow X \subseteq f\ X \Longrightarrow a \in gfp\ f$
⟨*proof*⟩

lemma *weak-coinduct-image*: $a \in X \Longrightarrow g`X \subseteq f\ (g`X) \Longrightarrow g\ a \in gfp\ f$
⟨*proof*⟩

lemma *coinduct-lemma*: $X \leq f\ (sup\ X\ (gfp\ f)) \Longrightarrow mono\ f \Longrightarrow sup\ X\ (gfp\ f) \leq f\ (sup\ X\ (gfp\ f))$
⟨*proof*⟩

Strong version, thanks to Coen and Frost.

lemma *coinduct-set*: $mono\ f \Longrightarrow a \in X \Longrightarrow X \subseteq f\ (X \cup gfp\ f) \Longrightarrow a \in gfp\ f$
⟨*proof*⟩

lemma *gfp-fun-UnI2*: $mono\ f \Longrightarrow a \in gfp\ f \Longrightarrow a \in f\ (X \cup gfp\ f)$
⟨*proof*⟩

lemma *gfp-ordinal-induct*[*case-names mono step union*]:
fixes $f :: 'a::complete\text{-}lattice \Rightarrow 'a$
assumes *mono*: $mono\ f$
and *P-f*: $\bigwedge S.\ P\ S \Longrightarrow gfp\ f \leq S \Longrightarrow P\ (f\ S)$
and *P-Union*: $\bigwedge M.\ \forall S{\in}M.\ P\ S \Longrightarrow P\ (Inf\ M)$
shows $P\ (gfp\ f)$
⟨*proof*⟩

lemma *coinduct*:
assumes *mono*: $mono\ f$
and *ind*: $X \leq f\ (sup\ X\ (gfp\ f))$
shows $X \leq gfp\ f$
⟨*proof*⟩

13.5 Even Stronger Coinduction Rule, by Martin Coen

Weakens the condition $X \subseteq f\ X$ to one expressed using both *lfp* and *gfp*

lemma *coinduct3-mono-lemma*: $mono\ f \Longrightarrow mono\ (\lambda x.\ f\ x \cup X \cup B)$
⟨*proof*⟩

lemma *coinduct3-lemma*:
$X \subseteq f\ (lfp\ (\lambda x.\ f\ x \cup X \cup gfp\ f)) \Longrightarrow mono\ f \Longrightarrow$
$lfp\ (\lambda x.\ f\ x \cup X \cup gfp\ f) \subseteq f\ (lfp\ (\lambda x.\ f\ x \cup X \cup gfp\ f))$
⟨*proof*⟩

lemma *coinduct3*: $mono\ f \Longrightarrow a \in X \Longrightarrow X \subseteq f\ (lfp\ (\lambda x.\ f\ x \cup X \cup gfp\ f)) \Longrightarrow a \in gfp\ f$
⟨*proof*⟩

Definition forms of *gfp-unfold* and *coinduct*, to control unfolding.

lemma *def-gfp-unfold*: $A \equiv gfp\ f \Longrightarrow mono\ f \Longrightarrow A = f\ A$
⟨*proof*⟩

lemma *def-coinduct*: $A \equiv gfp\ f \Longrightarrow mono\ f \Longrightarrow X \leq f\ (sup\ X\ A) \Longrightarrow X \leq A$
⟨*proof*⟩

lemma *def-coinduct-set*: $A \equiv gfp\ f \Longrightarrow mono\ f \Longrightarrow a \in X \Longrightarrow X \subseteq f\ (X \cup A) \Longrightarrow a \in A$
⟨*proof*⟩

lemma *def-Collect-coinduct*:
$A \equiv gfp\ (\lambda w.\ Collect\ (P\ w)) \Longrightarrow mono\ (\lambda w.\ Collect\ (P\ w)) \Longrightarrow a \in X \Longrightarrow$
$(\bigwedge z.\ z \in X \Longrightarrow P\ (X \cup A)\ z) \Longrightarrow a \in A$
⟨*proof*⟩

lemma *def-coinduct3*: $A \equiv gfp\ f \Longrightarrow mono\ f \Longrightarrow a \in X \Longrightarrow X \subseteq f\ (lfp\ (\lambda x.\ f\ x \cup X \cup A)) \Longrightarrow a \in A$
⟨*proof*⟩

Monotonicity of *gfp*!

lemma *gfp-mono*: $(\bigwedge Z.\ f\ Z \leq g\ Z) \Longrightarrow gfp\ f \leq gfp\ g$
⟨*proof*⟩

13.6 Rules for fixed point calculus

lemma *lfp-rolling*:
assumes *mono g mono f*
shows $g\ (lfp\ (\lambda x.\ f\ (g\ x))) = lfp\ (\lambda x.\ g\ (f\ x))$
⟨*proof*⟩

lemma *lfp-lfp*:
assumes *f*: $\bigwedge x\ y\ w\ z.\ x \leq y \Longrightarrow w \leq z \Longrightarrow f\ x\ w \leq f\ y\ z$
shows $lfp\ (\lambda x.\ lfp\ (f\ x)) = lfp\ (\lambda x.\ f\ x\ x)$
⟨*proof*⟩

lemma *gfp-rolling*:
assumes *mono g mono f*
shows $g\ (gfp\ (\lambda x.\ f\ (g\ x))) = gfp\ (\lambda x.\ g\ (f\ x))$
⟨*proof*⟩

lemma *gfp-gfp*:
assumes *f*: $\bigwedge x\ y\ w\ z.\ x \leq y \Longrightarrow w \leq z \Longrightarrow f\ x\ w \leq f\ y\ z$
shows $gfp\ (\lambda x.\ gfp\ (f\ x)) = gfp\ (\lambda x.\ f\ x\ x)$
⟨*proof*⟩

13.7 Inductive predicates and sets

Package setup.

lemmas *basic-monos* =
subset-refl imp-refl disj-mono conj-mono ex-mono all-mono if-bool-eq-conj
Collect-mono in-mono vimage-mono

lemma *le-rel-bool-arg-iff*: $X \leq Y \longleftrightarrow X\ False \leq Y\ False \wedge X\ True \leq Y\ True$
⟨*proof*⟩

lemma *imp-conj-iff*: $((P \longrightarrow Q) \wedge P) = (P \wedge Q)$
⟨*proof*⟩

lemma *meta-fun-cong*: $P \equiv Q \Longrightarrow P\ a \equiv Q\ a$
⟨*proof*⟩

⟨*ML*⟩

lemmas [*mono*] =
imp-refl disj-mono conj-mono ex-mono all-mono if-bool-eq-conj
imp-mono not-mono
Ball-def Bex-def
induct-rulify-fallback

13.8 The Schroeder-Bernstein Theorem

See also:

- `$ISABELLE_HOME/src/HOL/ex/Set_Theory.thy`
- http://planetmath.org/proofofschroederbernsteintheoremusingtarskiknastertheorem
- Springer LNCS 828 (cover page)

theorem *Schroeder-Bernstein*:
fixes $f :: 'a \Rightarrow 'b$ **and** $g :: 'b \Rightarrow 'a$
and $A :: 'a\ set$ **and** $B :: 'b\ set$
assumes *inj1*: *inj-on f A* **and** *sub1*: $f\ `\ A \subseteq B$
and *inj2*: *inj-on g B* **and** *sub2*: $g\ `\ B \subseteq A$
shows $\exists h.\ bij\text{-}betw\ h\ A\ B$
⟨*proof*⟩

13.9 Inductive datatypes and primitive recursion

Package setup.

⟨*ML*⟩

Lambda-abstractions with pattern matching:

syntax (*ASCII*)
-lam-pats-syntax :: *cases-syn* ⇒ $'a \Rightarrow 'b$ ((%-) *10*)
syntax

-lam-pats-syntax :: *cases-syn* ⇒ *'a* ⇒ *'b* ((λ-) *10*)
⟨*ML*⟩

end

14 Cartesian products

theory *Product-Type*
imports *Typedef Inductive Fun*
keywords *inductive-set coinductive-set* :: *thy-defn*
begin

14.1 *bool* is a datatype

free-constructors (*discs-sels*) *case-bool* **for** *True* | *False*
⟨*proof*⟩

Avoid name clashes by prefixing the output of *old-rep-datatype* with *old*.

⟨*ML*⟩

old-rep-datatype *True False* ⟨*proof*⟩

⟨*ML*⟩

But erase the prefix for properties that are not generated by *free-constructors*.

⟨*ML*⟩

lemmas *induct* = *old.bool.induct*
lemmas *inducts* = *old.bool.inducts*
lemmas *rec* = *old.bool.rec*
lemmas *simps* = *bool.distinct bool.case bool.rec*

⟨*ML*⟩

declare *case-split* [*cases type*: *bool*]
— prefer plain propositional version

lemma [*code*]: *HOL.equal False P* ⟷ ¬ *P*
and [*code*]: *HOL.equal True P* ⟷ *P*
and [*code*]: *HOL.equal P False* ⟷ ¬ *P*
and [*code*]: *HOL.equal P True* ⟷ *P*
and [*code nbe*]: *HOL.equal P P* ⟷ *True*
⟨*proof*⟩

lemma *If-case-cert*:
assumes *CASE* ≡ (λ*b*. *If b f g*)
shows (*CASE True* ≡ *f*) &&& (*CASE False* ≡ *g*)
⟨*proof*⟩

⟨*ML*⟩

code-printing
  constant *HOL.equal* :: *bool* ⇒ *bool* ⇒ *bool* ⇀ (*Haskell*) **infix** *4* ==
| **class-instance** *bool* :: *equal* ⇀ (*Haskell*) –

14.2 The *unit* type

typedef *unit* = {*True*}
  ⟨*proof*⟩

definition *Unity* :: *unit* ('())
  where () = *Abs-unit True*

lemma *unit-eq* [*no-atp*]: *u* = ()
  ⟨*proof*⟩

Simplification procedure for *unit-eq*. Cannot use this rule directly — it loops!
⟨*ML*⟩

free-constructors *case-unit* **for** ()
  ⟨*proof*⟩

Avoid name clashes by prefixing the output of *old-rep-datatype* with *old*.
⟨*ML*⟩

old-rep-datatype () ⟨*proof*⟩

⟨*ML*⟩

But erase the prefix for properties that are not generated by *free-constructors*.
⟨*ML*⟩

lemmas *induct* = *old.unit.induct*
lemmas *inducts* = *old.unit.inducts*
lemmas *rec* = *old.unit.rec*
lemmas *simps* = *unit.case unit.rec*

⟨*ML*⟩

lemma *unit-all-eq1*: (⋀*x*::*unit*. *PROP P x*) ≡ *PROP P* ()
  ⟨*proof*⟩

lemma *unit-all-eq2*: (⋀*x*::*unit*. *PROP P*) ≡ *PROP P*
  ⟨*proof*⟩

This rewrite counters the effect of simproc *unit-eq* on λ*u*::*unit*. *f u*, replacing it by *f* rather than by λ*u*. *f* ().

lemma *unit-abs-eta-conv* [*simp*]: $(\lambda u{::}unit.\ f\ ()) = f$
⟨*proof*⟩

lemma *UNIV-unit*: $UNIV = \{()\}$
⟨*proof*⟩

instantiation *unit* :: *default*
begin

definition *default* = ()

instance ⟨*proof*⟩

end

instantiation *unit* :: {*complete-boolean-algebra*,*complete-linorder*,*wellorder*}
begin

definition *less-eq-unit* :: *unit* ⇒ *unit* ⇒ *bool*
where (-::*unit*) ≤ - ⟷ *True*

lemma *less-eq-unit* [*iff*]: $u \leq v$ **for** $u\ v$ :: *unit*
⟨*proof*⟩

definition *less-unit* :: *unit* ⇒ *unit* ⇒ *bool*
where (-::*unit*) < - ⟷ *False*

lemma *less-unit* [*iff*]: $\neg\ u < v$ **for** $u\ v$ :: *unit*
⟨*proof*⟩

definition *bot-unit* :: *unit*
where [*code-unfold*]: $\bot$ = ()

definition *top-unit* :: *unit*
where [*code-unfold*]: $\top$ = ()

definition *inf-unit* :: *unit* ⇒ *unit* ⇒ *unit*
where [*simp*]: - ⊓ - = ()

definition *sup-unit* :: *unit* ⇒ *unit* ⇒ *unit*
where [*simp*]: - ⊔ - = ()

definition *Inf-unit* :: *unit set* ⇒ *unit*
where [*simp*]: $\bigsqcap$ - = ()

definition *Sup-unit* :: *unit set* ⇒ *unit*
where [*simp*]: $\bigsqcup$ - = ()

definition *uminus-unit* :: *unit* ⇒ *unit*

where [*simp*]: $- \text{ - } = ()$

declare *less-eq-unit-def* [*abs-def*, *code-unfold*]
less-unit-def [*abs-def*, *code-unfold*]
inf-unit-def [*abs-def*, *code-unfold*]
sup-unit-def [*abs-def*, *code-unfold*]
Inf-unit-def [*abs-def*, *code-unfold*]
Sup-unit-def [*abs-def*, *code-unfold*]
uminus-unit-def [*abs-def*, *code-unfold*]

instance
⟨*proof*⟩

end

lemma [*code*]: $HOL.equal\ u\ v \longleftrightarrow True$ **for** $u\ v :: unit$
⟨*proof*⟩

code-printing
type-constructor *unit* ⇀
(*SML*) *unit*
and (*OCaml*) *unit*
and (*Haskell*) ()
and (*Scala*) *Unit*
| **constant** *Unity* ⇀
(*SML*) ()
and (*OCaml*) ()
and (*Haskell*) ()
and (*Scala*) ()
| **class-instance** *unit* :: *equal* ⇀
(*Haskell*) –
| **constant** $HOL.equal :: unit \Rightarrow unit \Rightarrow bool$ ⇀
(*Haskell*) **infix** 4 ==

code-reserved *SML*
unit

code-reserved *OCaml*
unit

code-reserved *Scala*
Unit

14.3 The product type

14.3.1 Type definition

definition $Pair\text{-}Rep :: 'a \Rightarrow 'b \Rightarrow 'a \Rightarrow 'b \Rightarrow bool$
where $Pair\text{-}Rep\ a\ b = (\lambda x\ y.\ x = a \wedge y = b)$

definition *prod* = $\{f.\ \exists\, a\ b.\ f = Pair\text{-}Rep\ (a{::}'a)\ (b{::}'b)\}$

typedef $('a, 'b)$ *prod* ((- ×/ -) [*21*, *20*] *20*) = *prod* :: $('a \Rightarrow 'b \Rightarrow bool)$ *set*
⟨*proof*⟩

type-notation (*ASCII*)
prod (**infixr** ∗ *20*)

definition *Pair* :: $'a \Rightarrow 'b \Rightarrow 'a \times 'b$
where *Pair a b* = *Abs-prod* (*Pair-Rep a b*)

lemma *prod-cases*: $(\bigwedge a\ b.\ P\ (Pair\ a\ b)) \Longrightarrow P\ p$
⟨*proof*⟩

free-constructors *case-prod* **for** *Pair fst snd*
⟨*proof*⟩

Avoid name clashes by prefixing the output of *old-rep-datatype* with *old*.

⟨*ML*⟩

old-rep-datatype *Pair*
⟨*proof*⟩

⟨*ML*⟩

But erase the prefix for properties that are not generated by *free-constructors*.

⟨*ML*⟩

declare *old.prod.inject* [*iff del*]

lemmas *induct* = *old.prod.induct*
lemmas *inducts* = *old.prod.inducts*
lemmas *rec* = *old.prod.rec*
lemmas *simps* = *prod.inject prod.case prod.rec*

⟨*ML*⟩

declare *prod.case* [*nitpick-simp del*]
declare *old.prod.case-cong-weak* [*cong del*]
declare *prod.case-eq-if* [*mono*]
declare *prod.split* [*no-atp*]
declare *prod.split-asm* [*no-atp*]

prod.split could be declared as [*split*] done after the Splitter has been speeded up significantly; precompute the constants involved and don't do anything unless the current goal contains one of those constants.

14.3.2 Tuple syntax

Patterns – extends pre-defined type *pttrn* used in abstractions.

nonterminal *tuple-args* **and** *patterns*
syntax
-tuple :: $'a \Rightarrow$ *tuple-args* $\Rightarrow 'a \times 'b$ ((1 '(-,/ -')))
-tuple-arg :: $'a \Rightarrow$ *tuple-args* (-)
-tuple-args :: $'a \Rightarrow$ *tuple-args* $\Rightarrow$ *tuple-args* (-,/ -)
-pattern :: *pttrn* $\Rightarrow$ *patterns* $\Rightarrow$ *pttrn* ('(-,/ -'))
:: *pttrn* $\Rightarrow$ *patterns* (-)
-patterns :: *pttrn* $\Rightarrow$ *patterns* $\Rightarrow$ *patterns* (-,/ -)
-unit :: *pttrn* ('()')
translations
$(x, y) \rightleftharpoons$ *CONST Pair x y*
-pattern x y $\rightleftharpoons$ *CONST Pair x y*
-patterns x y $\rightleftharpoons$ *CONST Pair x y*
-tuple x (*-tuple-args y z*) $\rightleftharpoons$ *-tuple x* (*-tuple-arg* (*-tuple y z*))
$\lambda(x, y, zs).\ b \rightleftharpoons$ *CONST case-prod* $(\lambda x\ (y, zs).\ b)$
$\lambda(x, y).\ b \rightleftharpoons$ *CONST case-prod* $(\lambda x\ y.\ b)$
-abs (*CONST Pair x y*) $t \rightharpoonup \lambda(x, y).\ t$
— This rule accommodates tuples in *case C* ... (x, y) ... $\Rightarrow$ ...: The (x, y) is parsed as *Pair x y* because it is *logic*, not *pttrn*.
$\lambda().\ b \rightleftharpoons$ *CONST case-unit b*
-abs (*CONST Unity*) $t \rightharpoonup \lambda().\ t$

print *case-prod f* as *case-prod f* and *case-prod f* as *case-prod f*

⟨*ML*⟩
Reconstruct pattern from (nested) *case-prod*s, avoiding eta-contraction of body; required for enclosing "let", if "let" does not avoid eta-contraction, which has been observed to occur.

⟨*ML*⟩

14.3.3 Code generator setup

code-printing
type-constructor *prod* $\rightharpoonup$
(*SML*) **infix** *2* $*$
and (*OCaml*) **infix** *2* $*$
and (*Haskell*) !((-),/ (-))
and (*Scala*) ((-),/ (-))
| **constant** *Pair* $\rightharpoonup$
(*SML*) !((-),/ (-))
and (*OCaml*) !((-),/ (-))
and (*Haskell*) !((-),/ (-))
and (*Scala*) !((-),/ (-))
| **class-instance** *prod* :: *equal* $\rightharpoonup$
(*Haskell*) –
| **constant** *HOL.equal* :: $'a \times 'b \Rightarrow 'a \times 'b \Rightarrow$ *bool* $\rightharpoonup$

(*Haskell*) **infix** *4* ==
| **constant** *fst* ⇀ (*Haskell*) *fst*
| **constant** *snd* ⇀ (*Haskell*) *snd*

14.3.4 Fundamental operations and properties

lemma *Pair-inject*: $(a,\ b) = (a',\ b') \Longrightarrow (a = a' \Longrightarrow b = b' \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

lemma *surj-pair* [*simp*]: $\exists\, x\ y.\ p = (x,\ y)$
⟨*proof*⟩

lemma *fst-eqD*: $fst\ (x,\ y) = a \Longrightarrow x = a$
⟨*proof*⟩

lemma *snd-eqD*: $snd\ (x,\ y) = a \Longrightarrow y = a$
⟨*proof*⟩

lemma *case-prod-unfold* [*nitpick-unfold*]: $case\text{-}prod = (\lambda c\ p.\ c\ (fst\ p)\ (snd\ p))$
⟨*proof*⟩

lemma *case-prod-conv* [*simp*, *code*]: $(case\ (a,\ b)\ of\ (c,\ d) \Rightarrow f\ c\ d) = f\ a\ b$
⟨*proof*⟩

lemmas *surjective-pairing* = *prod.collapse* [*symmetric*]

lemma *prod-eq-iff*: $s = t \longleftrightarrow fst\ s = fst\ t \wedge snd\ s = snd\ t$
⟨*proof*⟩

lemma *prod-eqI* [*intro?*]: $fst\ p = fst\ q \Longrightarrow snd\ p = snd\ q \Longrightarrow p = q$
⟨*proof*⟩

lemma *case-prodI*: $f\ a\ b \Longrightarrow case\ (a,\ b)\ of\ (c,\ d) \Rightarrow f\ c\ d$
⟨*proof*⟩

lemma *case-prodD*: $(case\ (a,\ b)\ of\ (c,\ d) \Rightarrow f\ c\ d) \Longrightarrow f\ a\ b$
⟨*proof*⟩

lemma *case-prod-Pair* [*simp*]: $case\text{-}prod\ Pair = id$
⟨*proof*⟩

lemma *case-prod-eta*: $(\lambda(x,\ y).\ f\ (x,\ y)) = f$
— Subsumes the old *split-Pair* when f is the identity function.
⟨*proof*⟩

lemma *case-prod-comp*: $(case\ x\ of\ (a,\ b) \Rightarrow (f \circ g)\ a\ b) = f\ (g\ (fst\ x))\ (snd\ x)$
⟨*proof*⟩

lemma *The-case-prod*: *The* (*case-prod P*) = (*THE xy*. *P* (*fst xy*) (*snd xy*))
⟨*proof*⟩

lemma *cond-case-prod-eta*: (⋀*x y*. *f x y* = *g* (*x*, *y*)) ⟹ (λ(*x*, *y*). *f x y*) = *g*
⟨*proof*⟩

lemma *split-paired-all* [*no-atp*]: (⋀*x*. *PROP P x*) ≡ (⋀*a b*. *PROP P* (*a*, *b*))
⟨*proof*⟩

The rule *split-paired-all* does not work with the Simplifier because it also affects premises in congrence rules, where this can lead to premises of the form ⋀*a b*. ... = *?P*(*a*, *b*) which cannot be solved by reflexivity.

lemmas *split-tupled-all* = *split-paired-all unit-all-eq2*

⟨*ML*⟩

lemma *split-paired-All* [*simp*, *no-atp*]: (∀ *x*. *P x*) ⟷ (∀ *a b*. *P* (*a*, *b*))
— [*iff*] is not a good idea because it makes *blast* loop
⟨*proof*⟩

lemma *split-paired-Ex* [*simp*, *no-atp*]: (∃ *x*. *P x*) ⟷ (∃ *a b*. *P* (*a*, *b*))
⟨*proof*⟩

lemma *split-paired-The* [*no-atp*]: (*THE x*. *P x*) = (*THE* (*a*, *b*). *P* (*a*, *b*))
— Can't be added to simpset: loops!
⟨*proof*⟩

Simplification procedure for *cond-case-prod-eta*. Using *case-prod-eta* as a rewrite rule is not general enough, and using *cond-case-prod-eta* directly would render some existing proofs very inefficient; similarly for *prod.case-eq-if*.

⟨*ML*⟩

lemma *case-prod-beta′*: (λ(*x*,*y*). *f x y*) = (λ*x*. *f* (*fst x*) (*snd x*))
⟨*proof*⟩

case-prod used as a logical connective or set former.

These rules are for use with *blast*; could instead call *simp* using *prod.split* as rewrite.

lemma *case-prodI2*:
⋀*p*. (⋀*a b*. *p* = (*a*, *b*) ⟹ *c a b*) ⟹ *case p of* (*a*, *b*) ⇒ *c a b*
⟨*proof*⟩

lemma *case-prodI2′*:
⋀*p*. (⋀*a b*. (*a*, *b*) = *p* ⟹ *c a b x*) ⟹ (*case p of* (*a*, *b*) ⇒ *c a b*) *x*
⟨*proof*⟩

lemma *case-prodE* [*elim*!]:
(*case p of* (*a*, *b*) ⇒ *c a b*) ⟹ (⋀*x y*. *p* = (*x*, *y*) ⟹ *c x y* ⟹ *Q*) ⟹ *Q*
⟨*proof*⟩

lemma *case-prodE'* [*elim*!]:
(*case p of* (*a*, *b*) ⇒ *c a b*) *z* ⟹ (⋀*x y*. *p* = (*x*, *y*) ⟹ *c x y z* ⟹ *Q*) ⟹ *Q*
⟨*proof*⟩

lemma *case-prodE2*:
assumes *q*: *Q* (*case z of* (*a*, *b*) ⇒ *P a b*)
and *r*: ⋀*x y*. *z* = (*x*, *y*) ⟹ *Q* (*P x y*) ⟹ *R*
shows *R*
⟨*proof*⟩

lemma *case-prodD'*: (*case* (*a*, *b*) *of* (*c*, *d*) ⇒ *R c d*) *c* ⟹ *R a b c*
⟨*proof*⟩

lemma *mem-case-prodI*: *z* ∈ *c a b* ⟹ *z* ∈ (*case* (*a*, *b*) *of* (*d*, *e*) ⇒ *c d e*)
⟨*proof*⟩

lemma *mem-case-prodI2* [*intro*!]:
⋀*p*. (⋀*a b*. *p* = (*a*, *b*) ⟹ *z* ∈ *c a b*) ⟹ *z* ∈ (*case p of* (*a*, *b*) ⇒ *c a b*)
⟨*proof*⟩

declare *mem-case-prodI* [*intro*!] — postponed to maintain traditional declaration order!
declare *case-prodI2'* [*intro*!] — postponed to maintain traditional declaration order!
declare *case-prodI2* [*intro*!] — postponed to maintain traditional declaration order!
declare *case-prodI* [*intro*!] — postponed to maintain traditional declaration order!

lemma *mem-case-prodE* [*elim*!]:
assumes *z* ∈ *case-prod c p*
obtains *x y* **where** *p* = (*x*, *y*) **and** *z* ∈ *c x y*
⟨*proof*⟩

⟨*ML*⟩

lemma *split-eta-SetCompr* [*simp*, *no-atp*]: (λ*u*. ∃ *x y*. *u* = (*x*, *y*) ∧ *P* (*x*, *y*)) = *P*
⟨*proof*⟩

lemma *split-eta-SetCompr2* [*simp*, *no-atp*]: (λ*u*. ∃ *x y*. *u* = (*x*, *y*) ∧ *P x y*) = *case-prod P*
⟨*proof*⟩

lemma *split-part* [*simp*]: (λ(*a*,*b*). *P* ∧ *Q a b*) = (λ*ab*. *P* ∧ *case-prod Q ab*)
— Allows simplifications of nested splits in case of independent predicates.
⟨*proof*⟩

lemma *split-comp-eq*:
fixes $f :: 'a \Rightarrow 'b \Rightarrow 'c$
and $g :: 'd \Rightarrow 'a$
shows $(\lambda u.\ f\ (g\ (fst\ u))\ (snd\ u)) = case\text{-}prod\ (\lambda x.\ f\ (g\ x))$
⟨*proof*⟩

lemma *pair-imageI* [*intro*]: $(a,\ b) \in A \Longrightarrow f\ a\ b \in (\lambda(a,\ b).\ f\ a\ b)\ `\ A$
⟨*proof*⟩

lemma *Collect-const-case-prod*[*simp*]: $\{(a,b).\ P\} = (if\ P\ then\ UNIV\ else\ \{\})$
⟨*proof*⟩

lemma *The-split-eq* [*simp*]: $(THE\ (x',y').\ x = x' \wedge y = y') = (x,\ y)$
⟨*proof*⟩

lemma *case-prod-beta*: $case\text{-}prod\ f\ p = f\ (fst\ p)\ (snd\ p)$
⟨*proof*⟩

lemma *prod-cases3* [*cases type*]:
obtains (*fields*) $a\ b\ c$ **where** $y = (a,\ b,\ c)$
⟨*proof*⟩

lemma *prod-induct3* [*case-names fields*, *induct type*]:
$(\bigwedge a\ b\ c.\ P\ (a,\ b,\ c)) \Longrightarrow P\ x$
⟨*proof*⟩

lemma *prod-cases4* [*cases type*]:
obtains (*fields*) $a\ b\ c\ d$ **where** $y = (a,\ b,\ c,\ d)$
⟨*proof*⟩

lemma *prod-induct4* [*case-names fields*, *induct type*]:
$(\bigwedge a\ b\ c\ d.\ P\ (a,\ b,\ c,\ d)) \Longrightarrow P\ x$
⟨*proof*⟩

lemma *prod-cases5* [*cases type*]:
obtains (*fields*) $a\ b\ c\ d\ e$ **where** $y = (a,\ b,\ c,\ d,\ e)$
⟨*proof*⟩

lemma *prod-induct5* [*case-names fields*, *induct type*]:
$(\bigwedge a\ b\ c\ d\ e.\ P\ (a,\ b,\ c,\ d,\ e)) \Longrightarrow P\ x$
⟨*proof*⟩

lemma *prod-cases6* [*cases type*]:
obtains (*fields*) $a\ b\ c\ d\ e\ f$ **where** $y = (a,\ b,\ c,\ d,\ e,\ f)$
⟨*proof*⟩

lemma *prod-induct6* [*case-names fields*, *induct type*]:
(⋀*a b c d e f*. *P* (*a*, *b*, *c*, *d*, *e*, *f*)) ⟹ *P x*
⟨*proof*⟩

lemma *prod-cases7* [*cases type*]:
obtains (*fields*) *a b c d e f g* **where** *y* = (*a*, *b*, *c*, *d*, *e*, *f*, *g*)
⟨*proof*⟩

lemma *prod-induct7* [*case-names fields*, *induct type*]:
(⋀*a b c d e f g*. *P* (*a*, *b*, *c*, *d*, *e*, *f*, *g*)) ⟹ *P x*
⟨*proof*⟩

definition *internal-case-prod* :: ('*a* ⇒ '*b* ⇒ '*c*) ⇒ '*a* × '*b* ⇒ '*c*
where *internal-case-prod* ≡ *case-prod*

lemma *internal-case-prod-conv*: *internal-case-prod c* (*a*, *b*) = *c a b*
⟨*proof*⟩

⟨*ML*⟩

hide-const *internal-case-prod*

14.3.5 Derived operations

definition *curry* :: ('*a* × '*b* ⇒ '*c*) ⇒ '*a* ⇒ '*b* ⇒ '*c*
where *curry* = (λ*c x y*. *c* (*x*, *y*))

lemma *curry-conv* [*simp*, *code*]: *curry f a b* = *f* (*a*, *b*)
⟨*proof*⟩

lemma *curryI* [*intro*!]: *f* (*a*, *b*) ⟹ *curry f a b*
⟨*proof*⟩

lemma *curryD* [*dest*!]: *curry f a b* ⟹ *f* (*a*, *b*)
⟨*proof*⟩

lemma *curryE*: *curry f a b* ⟹ (*f* (*a*, *b*) ⟹ *Q*) ⟹ *Q*
⟨*proof*⟩

lemma *curry-case-prod* [*simp*]: *curry* (*case-prod f*) = *f*
⟨*proof*⟩

lemma *case-prod-curry* [*simp*]: *case-prod* (*curry f*) = *f*
⟨*proof*⟩

lemma *curry-K*: *curry* (λ*x*. *c*) = (λ*x y*. *c*)
⟨*proof*⟩

The composition-uncurry combinator.

definition *scomp* :: $('a \Rightarrow 'b \times 'c) \Rightarrow ('b \Rightarrow 'c \Rightarrow 'd) \Rightarrow 'a \Rightarrow 'd$ (**infixl** $\circ\!\!\rightarrow$ *60*)
  where $f \circ\!\!\rightarrow g = (\lambda x.\ \mathit{case\text{-}prod}\ g\ (f\ x))$

no-notation *scomp* (**infixl** $\circ\!\!\rightarrow$ *60*)

bundle *state-combinator-syntax*
begin

notation *fcomp* (**infixl** $\circ\!\!>$ *60*)
notation *scomp* (**infixl** $\circ\!\!\rightarrow$ *60*)

end

context
  includes *state-combinator-syntax*
begin

lemma *scomp-unfold*: $(\circ\!\!\rightarrow) = (\lambda f\ g\ x.\ g\ (\mathit{fst}\ (f\ x))\ (\mathit{snd}\ (f\ x)))$
  ⟨*proof*⟩

lemma *scomp-apply* [*simp*]: $(f \circ\!\!\rightarrow g)\ x = \mathit{case\text{-}prod}\ g\ (f\ x)$
  ⟨*proof*⟩

lemma *Pair-scomp*: $\mathit{Pair}\ x \circ\!\!\rightarrow f = f\ x$
  ⟨*proof*⟩

lemma *scomp-Pair*: $x \circ\!\!\rightarrow \mathit{Pair} = x$
  ⟨*proof*⟩

lemma *scomp-scomp*: $(f \circ\!\!\rightarrow g) \circ\!\!\rightarrow h = f \circ\!\!\rightarrow (\lambda x.\ g\ x \circ\!\!\rightarrow h)$
  ⟨*proof*⟩

lemma *scomp-fcomp*: $(f \circ\!\!\rightarrow g) \circ\!\!> h = f \circ\!\!\rightarrow (\lambda x.\ g\ x \circ\!\!> h)$
  ⟨*proof*⟩

lemma *fcomp-scomp*: $(f \circ\!\!> g) \circ\!\!\rightarrow h = f \circ\!\!> (g \circ\!\!\rightarrow h)$
  ⟨*proof*⟩

end

code-printing
  constant *scomp* ⇀ (*Eval*) **infixl** *3* #−>

map-prod — action of the product functor upon functions.

definition *map-prod* :: $('a \Rightarrow 'c) \Rightarrow ('b \Rightarrow 'd) \Rightarrow 'a \times 'b \Rightarrow 'c \times 'd$
  where $\mathit{map\text{-}prod}\ f\ g = (\lambda(x,\ y).\ (f\ x,\ g\ y))$

lemma *map-prod-simp* [*simp*, *code*]: $\mathit{map\text{-}prod}\ f\ g\ (a,\ b) = (f\ a,\ g\ b)$

⟨*proof*⟩

functor *map-prod*: *map-prod*
⟨*proof*⟩

lemma *fst-map-prod* [*simp*]: *fst* (*map-prod f g x*) = *f* (*fst x*)
⟨*proof*⟩

lemma *snd-map-prod* [*simp*]: *snd* (*map-prod f g x*) = *g* (*snd x*)
⟨*proof*⟩

lemma *fst-comp-map-prod* [*simp*]: *fst* ∘ *map-prod f g* = *f* ∘ *fst*
⟨*proof*⟩

lemma *snd-comp-map-prod* [*simp*]: *snd* ∘ *map-prod f g* = *g* ∘ *snd*
⟨*proof*⟩

lemma *map-prod-compose*: *map-prod* (*f1* ∘ *f2*) (*g1* ∘ *g2*) = (*map-prod f1 g1* ∘ *map-prod f2 g2*)
⟨*proof*⟩

lemma *map-prod-ident* [*simp*]: *map-prod* (λ*x*. *x*) (λ*y*. *y*) = (λ*z*. *z*)
⟨*proof*⟩

lemma *map-prod-imageI* [*intro*]: (*a*, *b*) ∈ *R* ⟹ (*f a*, *g b*) ∈ *map-prod f g* ‘ *R*
⟨*proof*⟩

lemma *prod-fun-imageE* [*elim*!]:
assumes *major*: *c* ∈ *map-prod f g* ‘ *R*
and *cases*: ⋀*x y*. *c* = (*f x*, *g y*) ⟹ (*x*, *y*) ∈ *R* ⟹ *P*
shows *P*
⟨*proof*⟩

definition *apfst* :: ('*a* ⇒ '*c*) ⇒ '*a* × '*b* ⇒ '*c* × '*b*
where *apfst f* = *map-prod f id*

definition *apsnd* :: ('*b* ⇒ '*c*) ⇒ '*a* × '*b* ⇒ '*a* × '*c*
where *apsnd f* = *map-prod id f*

lemma *apfst-conv* [*simp*, *code*]: *apfst f* (*x*, *y*) = (*f x*, *y*)
⟨*proof*⟩

lemma *apsnd-conv* [*simp*, *code*]: *apsnd f* (*x*, *y*) = (*x*, *f y*)
⟨*proof*⟩

lemma *fst-apfst* [*simp*]: *fst* (*apfst f x*) = *f* (*fst x*)
⟨*proof*⟩

lemma *fst-comp-apfst* [*simp*]: *fst* ∘ *apfst f* = *f* ∘ *fst*

⟨*proof*⟩

lemma *fst-apsnd* [*simp*]: *fst* (*apsnd f x*) = *fst x*
⟨*proof*⟩

lemma *fst-comp-apsnd* [*simp*]: *fst* ∘ *apsnd f* = *fst*
⟨*proof*⟩

lemma *snd-apfst* [*simp*]: *snd* (*apfst f x*) = *snd x*
⟨*proof*⟩

lemma *snd-comp-apfst* [*simp*]: *snd* ∘ *apfst f* = *snd*
⟨*proof*⟩

lemma *snd-apsnd* [*simp*]: *snd* (*apsnd f x*) = *f* (*snd x*)
⟨*proof*⟩

lemma *snd-comp-apsnd* [*simp*]: *snd* ∘ *apsnd f* = *f* ∘ *snd*
⟨*proof*⟩

lemma *apfst-compose*: *apfst f* (*apfst g x*) = *apfst* (*f* ∘ *g*) *x*
⟨*proof*⟩

lemma *apsnd-compose*: *apsnd f* (*apsnd g x*) = *apsnd* (*f* ∘ *g*) *x*
⟨*proof*⟩

lemma *apfst-apsnd* [*simp*]: *apfst f* (*apsnd g x*) = (*f* (*fst x*), *g* (*snd x*))
⟨*proof*⟩

lemma *apsnd-apfst* [*simp*]: *apsnd f* (*apfst g x*) = (*g* (*fst x*), *f* (*snd x*))
⟨*proof*⟩

lemma *apfst-id* [*simp*]: *apfst id* = *id*
⟨*proof*⟩

lemma *apsnd-id* [*simp*]: *apsnd id* = *id*
⟨*proof*⟩

lemma *apfst-eq-conv* [*simp*]: *apfst f x* = *apfst g x* ⟷ *f* (*fst x*) = *g* (*fst x*)
⟨*proof*⟩

lemma *apsnd-eq-conv* [*simp*]: *apsnd f x* = *apsnd g x* ⟷ *f* (*snd x*) = *g* (*snd x*)
⟨*proof*⟩

lemma *apsnd-apfst-commute*: *apsnd f* (*apfst g p*) = *apfst g* (*apsnd f p*)
⟨*proof*⟩

context
begin

⟨*ML*⟩

definition *swap* :: $'a \times 'b \Rightarrow 'b \times 'a$
  where *swap p* = (*snd p*, *fst p*)

end

lemma *swap-simp* [*simp*]: *prod.swap* (*x*, *y*) = (*y*, *x*)
  ⟨*proof*⟩

lemma *swap-swap* [*simp*]: *prod.swap* (*prod.swap p*) = *p*
  ⟨*proof*⟩

lemma *swap-comp-swap* [*simp*]: *prod.swap* ∘ *prod.swap* = *id*
  ⟨*proof*⟩

lemma *pair-in-swap-image* [*simp*]: (*y*, *x*) ∈ *prod.swap* ‘ *A* ⟷ (*x*, *y*) ∈ *A*
  ⟨*proof*⟩

lemma *inj-swap* [*simp*]: *inj-on prod.swap A*
  ⟨*proof*⟩

lemma *swap-inj-on*: *inj-on* (λ(*i*, *j*). (*j*, *i*)) *A*
  ⟨*proof*⟩

lemma *surj-swap* [*simp*]: *surj prod.swap*
  ⟨*proof*⟩

lemma *bij-swap* [*simp*]: *bij prod.swap*
  ⟨*proof*⟩

lemma *case-swap* [*simp*]: (*case prod.swap p of* (*y*, *x*) ⇒ *f x y*) = (*case p of* (*x*, *y*)
⇒ *f x y*)
  ⟨*proof*⟩

lemma *fst-swap* [*simp*]: *fst* (*prod.swap x*) = *snd x*
  ⟨*proof*⟩

lemma *snd-swap* [*simp*]: *snd* (*prod.swap x*) = *fst x*
  ⟨*proof*⟩

Disjoint union of a family of sets – Sigma.

definition *Sigma* :: $'a$ *set* ⇒ ($'a$ ⇒ $'b$ *set*) ⇒ ($'a \times 'b$) *set*
  where *Sigma A B* ≡ $\bigcup x{\in}A.\ \bigcup y{\in}B\ x.\ \{Pair\ x\ y\}$

abbreviation *Times* :: $'a$ *set* ⇒ $'b$ *set* ⇒ ($'a \times 'b$) *set* (**infixr** × *80*)
  where $A \times B \equiv Sigma\ A\ (\lambda\text{-}.\ B)$

hide-const (**open**) *Times*

bundle *no-Set-Product-syntax* **begin**
no-notation *Product-Type.Times* (**infixr** × *80*)
end
bundle *Set-Product-syntax* **begin**
notation *Product-Type.Times* (**infixr** × *80*)
end

syntax
-Sigma :: *pttrn* ⇒ *'a set* ⇒ *'b set* ⇒ (*'a* × *'b*) *set* ((*3SIGMA* -:-./ -) [*0*, *0*, *10*] *10*)
translations
SIGMA x:*A*. *B* ⇌ *CONST Sigma A* (λ*x*. *B*)

lemma *SigmaI* [*intro*!]: *a* ∈ *A* ⟹ *b* ∈ *B a* ⟹ (*a*, *b*) ∈ *Sigma A B*
⟨*proof*⟩

lemma *SigmaE* [*elim*!]: *c* ∈ *Sigma A B* ⟹ (⋀*x y*. *x* ∈ *A* ⟹ *y* ∈ *B x* ⟹ *c* = (*x*, *y*) ⟹ *P*) ⟹ *P*
— The general elimination rule.
⟨*proof*⟩

Elimination of (*a*, *b*) ∈ *A* × *B* – introduces no eigenvariables.

lemma *SigmaD1*: (*a*, *b*) ∈ *Sigma A B* ⟹ *a* ∈ *A*
⟨*proof*⟩

lemma *SigmaD2*: (*a*, *b*) ∈ *Sigma A B* ⟹ *b* ∈ *B a*
⟨*proof*⟩

lemma *SigmaE2*: (*a*, *b*) ∈ *Sigma A B* ⟹ (*a* ∈ *A* ⟹ *b* ∈ *B a* ⟹ *P*) ⟹ *P*
⟨*proof*⟩

lemma *Sigma-cong*: *A* = *B* ⟹ (⋀*x*. *x* ∈ *B* ⟹ *C x* = *D x*) ⟹ (*SIGMA x*:*A*. *C x*) = (*SIGMA x*:*B*. *D x*)
⟨*proof*⟩

lemma *Sigma-mono*: *A* ⊆ *C* ⟹ (⋀*x*. *x* ∈ *A* ⟹ *B x* ⊆ *D x*) ⟹ *Sigma A B* ⊆ *Sigma C D*
⟨*proof*⟩

lemma *Sigma-empty1* [*simp*]: *Sigma* {} *B* = {}
⟨*proof*⟩

lemma *Sigma-empty2* [*simp*]: *A* × {} = {}
⟨*proof*⟩

lemma *UNIV-Times-UNIV* [*simp*]: *UNIV* × *UNIV* = *UNIV*
⟨*proof*⟩

lemma *Compl-Times-UNIV1* [*simp*]: $-(UNIV \times A) = UNIV \times (-A)$
⟨*proof*⟩

lemma *Compl-Times-UNIV2* [*simp*]: $-(A \times UNIV) = (-A) \times UNIV$
⟨*proof*⟩

lemma *mem-Sigma-iff* [*iff*]: $(a, b) \in Sigma\ A\ B \longleftrightarrow a \in A \wedge b \in B\ a$
⟨*proof*⟩

lemma *mem-Times-iff*: $x \in A \times B \longleftrightarrow fst\ x \in A \wedge snd\ x \in B$
⟨*proof*⟩

lemma *Sigma-empty-iff*: $(SIGMA\ i{:}I.\ X\ i) = \{\} \longleftrightarrow (\forall i{\in}I.\ X\ i = \{\})$
⟨*proof*⟩

lemma *Times-subset-cancel2*: $x \in C \Longrightarrow A \times C \subseteq B \times C \longleftrightarrow A \subseteq B$
⟨*proof*⟩

lemma *Times-eq-cancel2*: $x \in C \Longrightarrow A \times C = B \times C \longleftrightarrow A = B$
⟨*proof*⟩

lemma *Collect-case-prod-Sigma*: $\{(x, y).\ P\ x \wedge Q\ x\ y\} = (SIGMA\ x{:}Collect\ P.\ Collect\ (Q\ x))$
⟨*proof*⟩

lemma *Collect-case-prod* [*simp*]: $\{(a, b).\ P\ a \wedge Q\ b\} = Collect\ P \times Collect\ Q$
⟨*proof*⟩

lemma *Collect-case-prodD*: $x \in Collect\ (case\text{-}prod\ A) \Longrightarrow A\ (fst\ x)\ (snd\ x)$
⟨*proof*⟩

lemma *Collect-case-prod-mono*: $A \leq B \Longrightarrow Collect\ (case\text{-}prod\ A) \subseteq Collect\ (case\text{-}prod\ B)$
⟨*proof*⟩

lemma *Collect-split-mono-strong*:
$X = fst\ `\ A \Longrightarrow Y = snd\ `\ A \Longrightarrow \forall a{\in}X.\ \forall b \in Y.\ P\ a\ b \longrightarrow Q\ a\ b$
$\Longrightarrow A \subseteq Collect\ (case\text{-}prod\ P) \Longrightarrow A \subseteq Collect\ (case\text{-}prod\ Q)$
⟨*proof*⟩

lemma *UN-Times-distrib*: $(\bigcup (a, b){\in}A \times B.\ E\ a \times F\ b) = \bigcup (E\ `\ A) \times \bigcup (F\ `\ B)$
— Suggested by Pierre Chartier
⟨*proof*⟩

lemma *split-paired-Ball-Sigma* [*simp*, *no-atp*]: $(\forall z{\in}Sigma\ A\ B.\ P\ z) \longleftrightarrow (\forall x{\in}A.\ \forall y{\in}B\ x.\ P\ (x, y))$
⟨*proof*⟩

lemma *split-paired-Bex-Sigma* [*simp*, *no-atp*]: $(\exists\, z{\in}Sigma\ A\ B.\ P\ z) \longleftrightarrow (\exists\, x{\in}A.\ \exists\, y{\in}B\ x.\ P\ (x,\ y))$
⟨*proof*⟩

lemma *Sigma-Un-distrib1*: $Sigma\ (I \cup J)\ C = Sigma\ I\ C \cup Sigma\ J\ C$
⟨*proof*⟩

lemma *Sigma-Un-distrib2*: $(SIGMA\ i{:}I.\ A\ i \cup B\ i) = Sigma\ I\ A \cup Sigma\ I\ B$
⟨*proof*⟩

lemma *Sigma-Int-distrib1*: $Sigma\ (I \cap J)\ C = Sigma\ I\ C \cap Sigma\ J\ C$
⟨*proof*⟩

lemma *Sigma-Int-distrib2*: $(SIGMA\ i{:}I.\ A\ i \cap B\ i) = Sigma\ I\ A \cap Sigma\ I\ B$
⟨*proof*⟩

lemma *Sigma-Diff-distrib1*: $Sigma\ (I - J)\ C = Sigma\ I\ C - Sigma\ J\ C$
⟨*proof*⟩

lemma *Sigma-Diff-distrib2*: $(SIGMA\ i{:}I.\ A\ i - B\ i) = Sigma\ I\ A - Sigma\ I\ B$
⟨*proof*⟩

lemma *Sigma-Union*: $Sigma\ (\bigcup X)\ B = (\bigcup A{\in}X.\ Sigma\ A\ B)$
⟨*proof*⟩

lemma *Pair-vimage-Sigma*: $Pair\ x\ -`\ Sigma\ A\ f = (if\ x \in A\ then\ f\ x\ else\ \{\})$
⟨*proof*⟩

Non-dependent versions are needed to avoid the need for higher-order matching, especially when the rules are re-oriented.

lemma *Times-Un-distrib1*: $(A \cup B) \times C = A \times C \cup B \times C$
⟨*proof*⟩

lemma *Times-Int-distrib1*: $(A \cap B) \times C = A \times C \cap B \times C$
⟨*proof*⟩

lemma *Times-Diff-distrib1*: $(A - B) \times C = A \times C - B \times C$
⟨*proof*⟩

lemma *Times-empty* [*simp*]: $A \times B = \{\} \longleftrightarrow A = \{\} \vee B = \{\}$
⟨*proof*⟩

lemma *times-subset-iff*: $A \times C \subseteq B \times D \longleftrightarrow A{=}\{\} \vee C{=}\{\} \vee A \subseteq B \wedge C \subseteq D$
⟨*proof*⟩

lemma *times-eq-iff*: $A \times B = C \times D \longleftrightarrow A = C \wedge B = D \vee (A = \{\} \vee B = \{\}) \wedge (C = \{\} \vee D = \{\})$
⟨*proof*⟩

lemma *fst-image-times* [*simp*]: *fst* ‘ (*A* × *B*) = (*if B* = {} *then* {} *else A*)
⟨*proof*⟩

lemma *snd-image-times* [*simp*]: *snd* ‘ (*A* × *B*) = (*if A* = {} *then* {} *else B*)
⟨*proof*⟩

lemma *fst-image-Sigma*: *fst* ‘ (*Sigma A B*) = {*x* ∈ *A*. *B*(*x*) ≠ {}}
⟨*proof*⟩

lemma *snd-image-Sigma*: *snd* ‘ (*Sigma A B*) = (⋃ *x* ∈ *A*. *B x*)
⟨*proof*⟩

lemma *vimage-fst*: *fst* −‘ *A* = *A* × *UNIV*
⟨*proof*⟩

lemma *vimage-snd*: *snd* −‘ *A* = *UNIV* × *A*
⟨*proof*⟩

lemma *insert-Times-insert* [*simp*]:
insert a A × *insert b B* = *insert* (*a*,*b*) (*A* × *insert b B* ∪ {*a*} × *B*)
⟨*proof*⟩

lemma *vimage-Times*: *f* −‘ (*A* × *B*) = (*fst* ∘ *f*) −‘ *A* ∩ (*snd* ∘ *f*) −‘ *B*
⟨*proof*⟩

lemma *Times-Int-Times*: *A* × *B* ∩ *C* × *D* = (*A* ∩ *C*) × (*B* ∩ *D*)
⟨*proof*⟩

lemma *image-paired-Times*:
(λ(*x*,*y*). (*f x*, *g y*)) ‘ (*A* × *B*) = (*f* ‘ *A*) × (*g* ‘ *B*)
⟨*proof*⟩

lemma *product-swap*: *prod.swap* ‘ (*A* × *B*) = *B* × *A*
⟨*proof*⟩

lemma *swap-product*: (λ(*i*, *j*). (*j*, *i*)) ‘ (*A* × *B*) = *B* × *A*
⟨*proof*⟩

lemma *image-split-eq-Sigma*: (λ*x*. (*f x*, *g x*)) ‘ *A* = *Sigma* (*f* ‘ *A*) (λ*x*. *g* ‘ (*f* −‘ {*x*} ∩ *A*))
⟨*proof*⟩

lemma *subset-fst-snd*: *A* ⊆ (*fst* ‘ *A* × *snd* ‘ *A*)
⟨*proof*⟩

lemma *inj-on-apfst* [*simp*]: *inj-on* (*apfst f*) (*A* × *UNIV*) ⟷ *inj-on f A*
⟨*proof*⟩

lemma *inj-apfst* [*simp*]: *inj* (*apfst f*) ⟷ *inj f*

⟨*proof*⟩

lemma *inj-on-apsnd* [*simp*]: *inj-on* (*apsnd f*) (*UNIV* × *A*) ⟷ *inj-on f A*
⟨*proof*⟩

lemma *inj-apsnd* [*simp*]: *inj* (*apsnd f*) ⟷ *inj f*
⟨*proof*⟩

context
begin

qualified definition *product* :: *'a set* ⇒ *'b set* ⇒ (*'a* × *'b*) *set*
where [*code-abbrev*]: *product A B* = *A* × *B*

lemma *member-product*: *x* ∈ *Product-Type.product A B* ⟷ *x* ∈ *A* × *B*
⟨*proof*⟩

end

The following *map-prod* lemmas are due to Joachim Breitner:

lemma *map-prod-inj-on*:
assumes *inj-on f A*
and *inj-on g B*
shows *inj-on* (*map-prod f g*) (*A* × *B*)
⟨*proof*⟩

lemma *map-prod-surj*:
fixes *f* :: *'a* ⇒ *'b*
and *g* :: *'c* ⇒ *'d*
assumes *surj f* **and** *surj g*
shows *surj* (*map-prod f g*)
⟨*proof*⟩

lemma *map-prod-surj-on*:
assumes *f* ‘ *A* = *A'* **and** *g* ‘ *B* = *B'*
shows *map-prod f g* ‘ (*A* × *B*) = *A'* × *B'*
⟨*proof*⟩

14.4 Simproc for rewriting a set comprehension into a point-free expression

⟨*ML*⟩

14.5 Lemmas about disjointness

lemma *disjnt-Times1-iff* [*simp*]: *disjnt* (*C* × *A*) (*C* × *B*) ⟷ *C* = {} ∨ *disjnt A B*
⟨*proof*⟩

lemma *disjnt-Times2-iff* [*simp*]: *disjnt* $(A \times C)$ $(B \times C) \longleftrightarrow C = \{\} \vee$ *disjnt* A B
⟨*proof*⟩

lemma *disjnt-Sigma-iff*: *disjnt* $(Sigma\ A\ C)$ $(Sigma\ B\ C) \longleftrightarrow (\forall i \in A \cap B.\ C\ i = \{\}) \vee$ *disjnt* A B
⟨*proof*⟩

14.6 Inductively defined sets

⟨*ML*⟩

14.7 Legacy theorem bindings and duplicates

lemmas *fst-conv* = *prod.sel*(*1*)
lemmas *snd-conv* = *prod.sel*(*2*)
lemmas *split-def* = *case-prod-unfold*
lemmas *split-beta'* = *case-prod-beta'*
lemmas *split-beta* = *prod.case-eq-if*
lemmas *split-conv* = *case-prod-conv*
lemmas *split* = *case-prod-conv*

hide-const (**open**) *prod*

end

15 The Disjoint Sum of Two Types

theory *Sum-Type*
imports *Typedef Inductive Fun*
begin

15.1 Construction of the sum type and its basic abstract operations

definition *Inl-Rep* :: $'a \Rightarrow 'a \Rightarrow 'b \Rightarrow bool \Rightarrow bool$
where *Inl-Rep* $a\ x\ y\ p \longleftrightarrow x = a \wedge p$

definition *Inr-Rep* :: $'b \Rightarrow 'a \Rightarrow 'b \Rightarrow bool \Rightarrow bool$
where *Inr-Rep* $b\ x\ y\ p \longleftrightarrow y = b \wedge \neg\ p$

definition $sum = \{f.\ (\exists a.\ f =$ *Inl-Rep* $(a::'a)) \vee (\exists b.\ f =$ *Inr-Rep* $(b::'b))\}$

typedef $('a, 'b)$ *sum* (**infixr** + *10*) $=$ *sum* :: $('a \Rightarrow 'b \Rightarrow bool \Rightarrow bool)$ *set*
⟨*proof*⟩

lemma *Inl-RepI*: *Inl-Rep* $a \in$ *sum*
⟨*proof*⟩

lemma *Inr-RepI*: *Inr-Rep b* ∈ *sum*
⟨*proof*⟩

lemma *inj-on-Abs-sum*: *A* ⊆ *sum* ⟹ *inj-on Abs-sum A*
⟨*proof*⟩

lemma *Inl-Rep-inject*: *inj-on Inl-Rep A*
⟨*proof*⟩

lemma *Inr-Rep-inject*: *inj-on Inr-Rep A*
⟨*proof*⟩

lemma *Inl-Rep-not-Inr-Rep*: *Inl-Rep a* ≠ *Inr-Rep b*
⟨*proof*⟩

definition *Inl* :: *'a* ⇒ *'a* + *'b*
where *Inl* = *Abs-sum* ∘ *Inl-Rep*

definition *Inr* :: *'b* ⇒ *'a* + *'b*
where *Inr* = *Abs-sum* ∘ *Inr-Rep*

lemma *inj-Inl* [*simp*]: *inj-on Inl A*
⟨*proof*⟩

lemma *Inl-inject*: *Inl x* = *Inl y* ⟹ *x* = *y*
⟨*proof*⟩

lemma *inj-Inr* [*simp*]: *inj-on Inr A*
⟨*proof*⟩

lemma *Inr-inject*: *Inr x* = *Inr y* ⟹ *x* = *y*
⟨*proof*⟩

lemma *Inl-not-Inr*: *Inl a* ≠ *Inr b*
⟨*proof*⟩

lemma *Inr-not-Inl*: *Inr b* ≠ *Inl a*
⟨*proof*⟩

lemma *sumE*:
assumes ⋀*x*::*'a*. *s* = *Inl x* ⟹ *P*
and ⋀*y*::*'b*. *s* = *Inr y* ⟹ *P*
shows *P*
⟨*proof*⟩

free-constructors *case-sum* **for**
isl: *Inl projl*
| *Inr projr*
⟨*proof*⟩

Avoid name clashes by prefixing the output of *old-rep-datatype* with *old*.

⟨*ML*⟩

old-rep-datatype *Inl Inr*
⟨*proof*⟩

⟨*ML*⟩

But erase the prefix for properties that are not generated by *free-constructors*.

⟨*ML*⟩

declare
  old.sum.inject[*iff del*]
  old.sum.distinct(*1*)[*simp del*, *induct-simp del*]

lemmas *induct* = *old.sum.induct*
lemmas *inducts* = *old.sum.inducts*
lemmas *rec* = *old.sum.rec*
lemmas *simps* = *sum.inject sum.distinct sum.case sum.rec*

⟨*ML*⟩

primrec *map-sum* :: $('a \Rightarrow 'c) \Rightarrow ('b \Rightarrow 'd) \Rightarrow 'a + 'b \Rightarrow 'c + 'd$
  where
    map-sum f1 f2 (*Inl a*) = *Inl* (*f1 a*)
  | *map-sum f1 f2* (*Inr a*) = *Inr* (*f2 a*)

functor *map-sum*: *map-sum*
⟨*proof*⟩

lemma *split-sum-all*: $(\forall x.\ P\ x) \longleftrightarrow (\forall x.\ P\ (Inl\ x)) \wedge (\forall x.\ P\ (Inr\ x))$
  ⟨*proof*⟩

lemma *split-sum-ex*: $(\exists x.\ P\ x) \longleftrightarrow (\exists x.\ P\ (Inl\ x)) \vee (\exists x.\ P\ (Inr\ x))$
  ⟨*proof*⟩

15.2 Projections

lemma *case-sum-KK* [*simp*]: *case-sum* ($\lambda x.\ a$) ($\lambda x.\ a$) = ($\lambda x.\ a$)
  ⟨*proof*⟩

lemma *surjective-sum*: *case-sum* ($\lambda x::'a.\ f\ (Inl\ x)$) ($\lambda y::'b.\ f\ (Inr\ y)$) = *f*
⟨*proof*⟩

lemma *case-sum-inject*:
  assumes *a*: *case-sum f1 f2* = *case-sum g1 g2*
    and *r*: *f1* = *g1* $\Longrightarrow$ *f2* = *g2* $\Longrightarrow$ *P*
  shows *P*
⟨*proof*⟩

primrec *Suml* :: $('a \Rightarrow 'c) \Rightarrow 'a + 'b \Rightarrow 'c$
  where $Suml\ f\ (Inl\ x) = f\ x$

primrec *Sumr* :: $('b \Rightarrow 'c) \Rightarrow 'a + 'b \Rightarrow 'c$
  where $Sumr\ f\ (Inr\ x) = f\ x$

lemma *Suml-inject*:
  assumes $Suml\ f = Suml\ g$
  shows $f = g$
⟨*proof*⟩

lemma *Sumr-inject*:
  assumes $Sumr\ f = Sumr\ g$
  shows $f = g$
⟨*proof*⟩

15.3 The Disjoint Sum of Sets

definition *Plus* :: $'a\ set \Rightarrow 'b\ set \Rightarrow ('a + 'b)\ set$ (**infixr** <+> *65*)
  where $A <+> B = Inl\ `\ A \cup Inr\ `\ B$

hide-const (**open**) *Plus* — Valuable identifier

lemma *InlI* [*intro*!]: $a \in A \Longrightarrow Inl\ a \in A <+> B$
  ⟨*proof*⟩

lemma *InrI* [*intro*!]: $b \in B \Longrightarrow Inr\ b \in A <+> B$
  ⟨*proof*⟩

Exhaustion rule for sums, a degenerate form of induction

lemma *PlusE* [*elim*!]:
  $u \in A <+> B \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow u = Inl\ x \Longrightarrow P) \Longrightarrow (\bigwedge y.\ y \in B \Longrightarrow u = Inr\ y \Longrightarrow P) \Longrightarrow P$
  ⟨*proof*⟩

lemma *Plus-eq-empty-conv* [*simp*]: $A <+> B = \{\} \longleftrightarrow A = \{\} \wedge B = \{\}$
  ⟨*proof*⟩

lemma *UNIV-Plus-UNIV* [*simp*]: $UNIV <+> UNIV = UNIV$
⟨*proof*⟩

lemma *UNIV-sum*: $UNIV = Inl\ `\ UNIV \cup Inr\ `\ UNIV$
⟨*proof*⟩

hide-const (**open**) *Suml Sumr sum*

end

16 Rings

theory *Rings*
imports *Groups Set Fun*
begin

16.1 Semirings and rings

class *semiring* = *ab-semigroup-add* + *semigroup-mult* +
assumes *distrib-right* [*algebra-simps*, *algebra-split-simps*]: $(a + b) * c = a * c + b * c$
assumes *distrib-left* [*algebra-simps*, *algebra-split-simps*]: $a * (b + c) = a * b + a * c$
begin

For the *combine-numerals* simproc

lemma *combine-common-factor*: $a * e + (b * e + c) = (a + b) * e + c$
⟨*proof*⟩

end

class *mult-zero* = *times* + *zero* +
assumes *mult-zero-left* [*simp*]: $0 * a = 0$
assumes *mult-zero-right* [*simp*]: $a * 0 = 0$
begin

lemma *mult-not-zero*: $a * b \neq 0 \Longrightarrow a \neq 0 \wedge b \neq 0$
⟨*proof*⟩

end

class *semiring-0* = *semiring* + *comm-monoid-add* + *mult-zero*

class *semiring-0-cancel* = *semiring* + *cancel-comm-monoid-add*
begin

subclass *semiring-0*
⟨*proof*⟩

end

class *comm-semiring* = *ab-semigroup-add* + *ab-semigroup-mult* +
assumes *distrib*: $(a + b) * c = a * c + b * c$
begin

subclass *semiring*
⟨*proof*⟩

end

class *comm-semiring-0* = *comm-semiring* + *comm-monoid-add* + *mult-zero*
begin

subclass *semiring-0* ⟨*proof*⟩

end

class *comm-semiring-0-cancel* = *comm-semiring* + *cancel-comm-monoid-add*
begin

subclass *semiring-0-cancel* ⟨*proof*⟩

subclass *comm-semiring-0* ⟨*proof*⟩

end

class *zero-neq-one* = *zero* + *one* +
  assumes *zero-neq-one* [*simp*]: *0* ≠ *1*
begin

lemma *one-neq-zero* [*simp*]: *1* ≠ *0*
  ⟨*proof*⟩

definition *of-bool* :: *bool* ⇒ *'a*
  where *of-bool p* = (*if p then 1 else 0*)

lemma *of-bool-eq* [*simp*, *code*]:
  of-bool False = *0*
  of-bool True = *1*
  ⟨*proof*⟩

lemma *of-bool-eq-iff*: *of-bool p* = *of-bool q* ⟷ *p* = *q*
  ⟨*proof*⟩

lemma *split-of-bool* [*split*]: *P* (*of-bool p*) ⟷ (*p* ⟶ *P 1*) ∧ (¬ *p* ⟶ *P 0*)
  ⟨*proof*⟩

lemma *split-of-bool-asm*: *P* (*of-bool p*) ⟷ ¬ (*p* ∧ ¬ *P 1* ∨ ¬ *p* ∧ ¬ *P 0*)
  ⟨*proof*⟩

lemma *of-bool-eq-0-iff* [*simp*]:
  ‹*of-bool P* = *0* ⟷ ¬ *P*›
  ⟨*proof*⟩

lemma *of-bool-eq-1-iff* [*simp*]:
  ‹*of-bool P* = *1* ⟷ *P*›
  ⟨*proof*⟩

end

class *semiring-1* = *zero-neq-one* + *semiring-0* + *monoid-mult*
begin

lemma *of-bool-conj*:
of-bool (*P* ∧ *Q*) = *of-bool P* ∗ *of-bool Q*
⟨*proof*⟩

end

lemma *lambda-zero*: (λ*h*::′*a*::*mult-zero*. *0*) = (∗) *0*
⟨*proof*⟩

lemma *lambda-one*: (λ*x*::′*a*::*monoid-mult*. *x*) = (∗) *1*
⟨*proof*⟩

16.2 Abstract divisibility

class *dvd* = *times*
begin

definition *dvd* :: ′*a* ⇒ ′*a* ⇒ *bool* (**infix** *dvd 50*)
where *b dvd a* ⟷ (∃ *k*. *a* = *b* ∗ *k*)

lemma *dvdI* [*intro?*]: *a* = *b* ∗ *k* ⟹ *b dvd a*
⟨*proof*⟩

lemma *dvdE* [*elim*]: *b dvd a* ⟹ (⋀*k*. *a* = *b* ∗ *k* ⟹ *P*) ⟹ *P*
⟨*proof*⟩

end

context *comm-monoid-mult*
begin

subclass *dvd* ⟨*proof*⟩

lemma *dvd-refl* [*simp*]: *a dvd a*
⟨*proof*⟩

lemma *dvd-trans* [*trans*]:
assumes *a dvd b* **and** *b dvd c*
shows *a dvd c*
⟨*proof*⟩

lemma *subset-divisors-dvd*: {*c*. *c dvd a*} ⊆ {*c*. *c dvd b*} ⟷ *a dvd b*
⟨*proof*⟩

lemma *strict-subset-divisors-dvd*: {*c*. *c dvd a*} ⊂ {*c*. *c dvd b*} ⟷ *a dvd b* ∧ ¬ *b*

dvd a
⟨*proof*⟩

lemma *one-dvd* [*simp*]: *1 dvd a*
⟨*proof*⟩

lemma *dvd-mult* [*simp*]: *a dvd* (*b* ∗ *c*) **if** *a dvd c*
⟨*proof*⟩

lemma *dvd-mult2* [*simp*]: *a dvd* (*b* ∗ *c*) **if** *a dvd b*
⟨*proof*⟩

lemma *dvd-triv-right* [*simp*]: *a dvd b* ∗ *a*
⟨*proof*⟩

lemma *dvd-triv-left* [*simp*]: *a dvd a* ∗ *b*
⟨*proof*⟩

lemma *mult-dvd-mono*:
assumes *a dvd b*
and *c dvd d*
shows *a* ∗ *c dvd b* ∗ *d*
⟨*proof*⟩

lemma *dvd-mult-left*: *a* ∗ *b dvd c* ⟹ *a dvd c*
⟨*proof*⟩

lemma *dvd-mult-right*: *a* ∗ *b dvd c* ⟹ *b dvd c*
⟨*proof*⟩

end

class *comm-semiring-1* = *zero-neq-one* + *comm-semiring-0* + *comm-monoid-mult*
begin

subclass *semiring-1* ⟨*proof*⟩

lemma *dvd-0-left-iff* [*simp*]: *0 dvd a* ⟷ *a* = *0*
⟨*proof*⟩

lemma *dvd-0-right* [*iff*]: *a dvd 0*
⟨*proof*⟩

lemma *dvd-0-left*: *0 dvd a* ⟹ *a* = *0*
⟨*proof*⟩

lemma *dvd-add* [*simp*]:
assumes *a dvd b* **and** *a dvd c*
shows *a dvd* (*b* + *c*)

⟨*proof*⟩

end

class *semiring-1-cancel* = *semiring* + *cancel-comm-monoid-add*
+ *zero-neq-one* + *monoid-mult*
begin

subclass *semiring-0-cancel* ⟨*proof*⟩

subclass *semiring-1* ⟨*proof*⟩

end

class *comm-semiring-1-cancel* =
comm-semiring + *cancel-comm-monoid-add* + *zero-neq-one* + *comm-monoid-mult*
+
assumes *right-diff-distrib'* [*algebra-simps*, *algebra-split-simps*]:
$a * (b - c) = a * b - a * c$
begin

subclass *semiring-1-cancel* ⟨*proof*⟩
subclass *comm-semiring-0-cancel* ⟨*proof*⟩
subclass *comm-semiring-1* ⟨*proof*⟩

lemma *left-diff-distrib'* [*algebra-simps*, *algebra-split-simps*]:
$(b - c) * a = b * a - c * a$
⟨*proof*⟩

lemma *dvd-add-times-triv-left-iff* [*simp*]: *a dvd c* $*$ *a* + *b* $\longleftrightarrow$ *a dvd b*
⟨*proof*⟩

lemma *dvd-add-times-triv-right-iff* [*simp*]: *a dvd b* + *c* $*$ *a* $\longleftrightarrow$ *a dvd b*
⟨*proof*⟩

lemma *dvd-add-triv-left-iff* [*simp*]: *a dvd a* + *b* $\longleftrightarrow$ *a dvd b*
⟨*proof*⟩

lemma *dvd-add-triv-right-iff* [*simp*]: *a dvd b* + *a* $\longleftrightarrow$ *a dvd b*
⟨*proof*⟩

lemma *dvd-add-right-iff*:
assumes *a dvd b*
shows *a dvd b* + *c* $\longleftrightarrow$ *a dvd c* (**is** *?P* $\longleftrightarrow$ *?Q*)
⟨*proof*⟩

lemma *dvd-add-left-iff*: *a dvd c* $\Longrightarrow$ *a dvd b* + *c* $\longleftrightarrow$ *a dvd b*
⟨*proof*⟩

end

class *ring* = *semiring* + *ab-group-add*
begin

subclass *semiring-0-cancel* ⟨*proof*⟩

Distribution rules

lemma *minus-mult-left*: $-(a * b) = -a * b$
⟨*proof*⟩

lemma *minus-mult-right*: $-(a * b) = a * -b$
⟨*proof*⟩

Extract signs from products

lemmas *mult-minus-left* [*simp*] = *minus-mult-left* [*symmetric*]
lemmas *mult-minus-right* [*simp*] = *minus-mult-right* [*symmetric*]

lemma *minus-mult-minus* [*simp*]: $-a * -b = a * b$
⟨*proof*⟩

lemma *minus-mult-commute*: $-a * b = a * -b$
⟨*proof*⟩

lemma *right-diff-distrib* [*algebra-simps*, *algebra-split-simps*]:
$a * (b - c) = a * b - a * c$
⟨*proof*⟩

lemma *left-diff-distrib* [*algebra-simps*, *algebra-split-simps*]:
$(a - b) * c = a * c - b * c$
⟨*proof*⟩

lemmas *ring-distribs* = *distrib-left distrib-right left-diff-distrib right-diff-distrib*

lemma *eq-add-iff1*: $a * e + c = b * e + d \longleftrightarrow (a - b) * e + c = d$
⟨*proof*⟩

lemma *eq-add-iff2*: $a * e + c = b * e + d \longleftrightarrow c = (b - a) * e + d$
⟨*proof*⟩

end

lemmas *ring-distribs* = *distrib-left distrib-right left-diff-distrib right-diff-distrib*

class *comm-ring* = *comm-semiring* + *ab-group-add*
begin

subclass *ring* ⟨*proof*⟩
subclass *comm-semiring-0-cancel* ⟨*proof*⟩

lemma *square-diff-square-factored*: $x * x - y * y = (x + y) * (x - y)$
⟨*proof*⟩

end

class *ring-1* = *ring* + *zero-neq-one* + *monoid-mult*
begin

subclass *semiring-1-cancel* ⟨*proof*⟩

lemma *of-bool-not-iff*:
‹*of-bool* (¬ *P*) = *1* − *of-bool P›*
⟨*proof*⟩

lemma *square-diff-one-factored*: $x * x - 1 = (x + 1) * (x - 1)$
⟨*proof*⟩

end

class *comm-ring-1* = *comm-ring* + *zero-neq-one* + *comm-monoid-mult*
begin

subclass *ring-1* ⟨*proof*⟩
subclass *comm-semiring-1-cancel*
⟨*proof*⟩

lemma *dvd-minus-iff* [*simp*]: *x dvd* − *y* ⟷ *x dvd y*
⟨*proof*⟩

lemma *minus-dvd-iff* [*simp*]: − *x dvd y* ⟷ *x dvd y*
⟨*proof*⟩

lemma *dvd-diff* [*simp*]: *x dvd y* ⟹ *x dvd z* ⟹ *x dvd* (*y* − *z*)
⟨*proof*⟩

end

16.3 Towards integral domains

class *semiring-no-zero-divisors* = *semiring-0* +
assumes *no-zero-divisors*: $a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow a * b \neq 0$
begin

lemma *divisors-zero*:
assumes $a * b = 0$
shows $a = 0 \vee b = 0$
⟨*proof*⟩

lemma *mult-eq-0-iff* [*simp*]: $a * b = 0 \longleftrightarrow a = 0 \vee b = 0$
⟨*proof*⟩

end

class *semiring-1-no-zero-divisors* = *semiring-1* + *semiring-no-zero-divisors*

class *semiring-no-zero-divisors-cancel* = *semiring-no-zero-divisors* +
assumes *mult-cancel-right* [*simp*]: $a * c = b * c \longleftrightarrow c = 0 \vee a = b$
and *mult-cancel-left* [*simp*]: $c * a = c * b \longleftrightarrow c = 0 \vee a = b$
begin

lemma *mult-left-cancel*: $c \neq 0 \Longrightarrow c * a = c * b \longleftrightarrow a = b$
⟨*proof*⟩

lemma *mult-right-cancel*: $c \neq 0 \Longrightarrow a * c = b * c \longleftrightarrow a = b$
⟨*proof*⟩

end

class *ring-no-zero-divisors* = *ring* + *semiring-no-zero-divisors*
begin

subclass *semiring-no-zero-divisors-cancel*
⟨*proof*⟩

end

class *ring-1-no-zero-divisors* = *ring-1* + *ring-no-zero-divisors*
begin

subclass *semiring-1-no-zero-divisors* ⟨*proof*⟩

lemma *square-eq-1-iff*: $x * x = 1 \longleftrightarrow x = 1 \vee x = -1$
⟨*proof*⟩

lemma *mult-cancel-right1* [*simp*]: $c = b * c \longleftrightarrow c = 0 \vee b = 1$
⟨*proof*⟩

lemma *mult-cancel-right2* [*simp*]: $a * c = c \longleftrightarrow c = 0 \vee a = 1$
⟨*proof*⟩

lemma *mult-cancel-left1* [*simp*]: $c = c * b \longleftrightarrow c = 0 \vee b = 1$
⟨*proof*⟩

lemma *mult-cancel-left2* [*simp*]: $c * a = c \longleftrightarrow c = 0 \vee a = 1$
⟨*proof*⟩

end

class *semidom* = *comm-semiring-1-cancel* + *semiring-no-zero-divisors*
begin

subclass *semiring-1-no-zero-divisors* ⟨*proof*⟩

end

class *idom* = *comm-ring-1* + *semiring-no-zero-divisors*
begin

subclass *semidom* ⟨*proof*⟩

subclass *ring-1-no-zero-divisors* ⟨*proof*⟩

lemma *dvd-mult-cancel-right* [*simp*]:
$a * c \ dvd \ b * c \longleftrightarrow c = 0 \vee a \ dvd \ b$
⟨*proof*⟩

lemma *dvd-mult-cancel-left* [*simp*]:
$c * a \ dvd \ c * b \longleftrightarrow c = 0 \vee a \ dvd \ b$
⟨*proof*⟩

lemma *square-eq-iff*: $a * a = b * b \longleftrightarrow a = b \vee a = - b$
⟨*proof*⟩

lemma *inj-mult-left* [*simp*]: ‹*inj* $((*) \ a) \longleftrightarrow a \neq 0$› (**is** ‹*?P* ⟷ *?Q*›)
⟨*proof*⟩

end

class *idom-abs-sgn* = *idom* + *abs* + *sgn* +
assumes *sgn-mult-abs*: $sgn \ a * |a| = a$
and *sgn-sgn* [*simp*]: $sgn \ (sgn \ a) = sgn \ a$
and *abs-abs* [*simp*]: $||a|| = |a|$
and *abs-0* [*simp*]: $|0| = 0$
and *sgn-0* [*simp*]: $sgn \ 0 = 0$
and *sgn-1* [*simp*]: $sgn \ 1 = 1$
and *sgn-minus-1*: $sgn \ (- \ 1) = - \ 1$
and *sgn-mult*: $sgn \ (a * b) = sgn \ a * sgn \ b$
begin

lemma *sgn-eq-0-iff*:
$sgn \ a = 0 \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *abs-eq-0-iff*:
$|a| = 0 \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *abs-mult-sgn*:
$|a| * sgn\ a = a$
⟨*proof*⟩

lemma *abs-1* [*simp*]:
$|1| = 1$
⟨*proof*⟩

lemma *sgn-abs* [*simp*]:
$|sgn\ a| = of\text{-}bool\ (a \neq 0)$
⟨*proof*⟩

lemma *abs-sgn* [*simp*]:
$sgn\ |a| = of\text{-}bool\ (a \neq 0)$
⟨*proof*⟩

lemma *abs-mult*:
$|a * b| = |a| * |b|$
⟨*proof*⟩

lemma *sgn-minus* [*simp*]:
$sgn\ (-\ a) = -\ sgn\ a$
⟨*proof*⟩

lemma *abs-minus* [*simp*]:
$|-\ a| = |a|$
⟨*proof*⟩

end

16.4 (Partial) Division

class *divide* =
fixes *divide* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** *div 70*)

⟨*ML*⟩

context *semiring*
begin

lemma [*field-simps*, *field-split-simps*]:
shows *distrib-left-NO-MATCH*: $NO\text{-}MATCH\ (x\ div\ y)\ a \Longrightarrow a * (b + c) = a * b + a * c$
and *distrib-right-NO-MATCH*: $NO\text{-}MATCH\ (x\ div\ y)\ c \Longrightarrow (a + b) * c = a * c + b * c$
⟨*proof*⟩

end

context *ring*
begin

lemma [*field-simps*, *field-split-simps*]:
shows *left-diff-distrib-NO-MATCH*: *NO-MATCH* $(x\ div\ y)\ c \Longrightarrow (a - b) * c = a * c - b * c$
and *right-diff-distrib-NO-MATCH*: *NO-MATCH* $(x\ div\ y)\ a \Longrightarrow a * (b - c) = a * b - a * c$
⟨*proof*⟩

end

⟨*ML*⟩

Algebraic classes with division

class *semidom-divide* = *semidom* + *divide* +
assumes *nonzero-mult-div-cancel-right* [*simp*]: $b \neq 0 \Longrightarrow (a * b)\ div\ b = a$
assumes *div-by-0* [*simp*]: $a\ div\ 0 = 0$
begin

lemma *nonzero-mult-div-cancel-left* [*simp*]: $a \neq 0 \Longrightarrow (a * b)\ div\ a = b$
⟨*proof*⟩

subclass *semiring-no-zero-divisors-cancel*
⟨*proof*⟩

lemma *div-self* [*simp*]: $a \neq 0 \Longrightarrow a\ div\ a = 1$
⟨*proof*⟩

lemma *div-0* [*simp*]: $0\ div\ a = 0$
⟨*proof*⟩

lemma *div-by-1* [*simp*]: $a\ div\ 1 = a$
⟨*proof*⟩

lemma *dvd-div-eq-0-iff*:
assumes *b dvd a*
shows $a\ div\ b = 0 \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *dvd-div-eq-cancel*:
$a\ div\ c = b\ div\ c \Longrightarrow c\ dvd\ a \Longrightarrow c\ dvd\ b \Longrightarrow a = b$
⟨*proof*⟩

lemma *dvd-div-eq-iff*:
$c\ dvd\ a \Longrightarrow c\ dvd\ b \Longrightarrow a\ div\ c = b\ div\ c \longleftrightarrow a = b$
⟨*proof*⟩

lemma *inj-on-mult*:
inj-on ((∗) *a*) *A* **if** *a* ≠ *0*
⟨*proof*⟩

end

class *idom-divide* = *idom* + *semidom-divide*
begin

lemma *dvd-neg-div*:
assumes *b dvd a*
shows − *a div b* = − (*a div b*)
⟨*proof*⟩

lemma *dvd-div-neg*:
assumes *b dvd a*
shows *a div* − *b* = − (*a div b*)
⟨*proof*⟩

end

class *algebraic-semidom* = *semidom-divide*
begin

Class *algebraic-semidom* enriches a integral domain by notions from algebra, like units in a ring. It is a separate class to avoid spoiling fields with notions which are degenerated there.

lemma *dvd-times-left-cancel-iff* [*simp*]:
assumes *a* ≠ *0*
shows *a* ∗ *b dvd a* ∗ *c* ⟷ *b dvd c*
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *dvd-times-right-cancel-iff* [*simp*]:
assumes *a* ≠ *0*
shows *b* ∗ *a dvd c* ∗ *a* ⟷ *b dvd c*
⟨*proof*⟩

lemma *div-dvd-iff-mult*:
assumes *b* ≠ *0* **and** *b dvd a*
shows *a div b dvd c* ⟷ *a dvd c* ∗ *b*
⟨*proof*⟩

lemma *dvd-div-iff-mult*:
assumes *c* ≠ *0* **and** *c dvd b*
shows *a dvd b div c* ⟷ *a* ∗ *c dvd b*
⟨*proof*⟩

lemma *div-dvd-div* [*simp*]:

assumes *a dvd b* **and** *a dvd c*
shows *b div a dvd c div a* $\longleftrightarrow$ *b dvd c*
⟨*proof*⟩

lemma *div-add* [*simp*]:
assumes *c dvd a* **and** *c dvd b*
shows $(a + b)$ *div c* $=$ *a div c* $+$ *b div c*
⟨*proof*⟩

lemma *div-mult-div-if-dvd*:
assumes *b dvd a* **and** *d dvd c*
shows $(a$ *div b*$) * (c$ *div d*$) = (a * c)$ *div* $(b * d)$
⟨*proof*⟩

lemma *dvd-div-eq-mult*:
assumes $a \neq 0$ **and** *a dvd b*
shows *b div a* $=$ *c* $\longleftrightarrow$ *b* $=$ *c* $*$ *a*
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *dvd-div-mult-self* [*simp*]: *a dvd b* $\Longrightarrow$ *b div a* $*$ *a* $=$ *b*
⟨*proof*⟩

lemma *dvd-mult-div-cancel* [*simp*]: *a dvd b* $\Longrightarrow$ $a * (b$ *div a*$) = b$
⟨*proof*⟩

lemma *div-mult-swap*:
assumes *c dvd b*
shows $a * (b$ *div c*$) = (a * b)$ *div c*
⟨*proof*⟩

lemma *dvd-div-mult*: *c dvd b* $\Longrightarrow$ *b div c* $*$ *a* $= (b * a)$ *div c*
⟨*proof*⟩

lemma *dvd-div-mult2-eq*:
assumes $b * c$ *dvd a*
shows *a div* $(b * c) =$ *a div b div c*
⟨*proof*⟩

lemma *dvd-div-div-eq-mult*:
assumes $a \neq 0$ $c \neq 0$ **and** *a dvd b c dvd d*
shows *b div a* $=$ *d div c* $\longleftrightarrow$ $b * c = a * d$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *dvd-mult-imp-div*:
assumes $a * c$ *dvd b*
shows *a dvd b div c*
⟨*proof*⟩

lemma *div-div-eq-right*:
assumes *c dvd b b dvd a*
shows *a div (b div c) = a div b * c*
⟨*proof*⟩

lemma *div-div-div-same*:
assumes *d dvd b b dvd a*
shows *(a div d) div (b div d) = a div b*
⟨*proof*⟩

Units: invertible elements in a ring

abbreviation *is-unit* :: *'a ⇒ bool*
where *is-unit a ≡ a dvd 1*

lemma *not-is-unit-0* [*simp*]: *¬ is-unit 0*
⟨*proof*⟩

lemma *unit-imp-dvd* [*dest*]: *is-unit b ⟹ b dvd a*
⟨*proof*⟩

lemma *unit-dvdE*:
assumes *is-unit a*
obtains *c* **where** *a ≠ 0* **and** *b = a * c*
⟨*proof*⟩

lemma *dvd-unit-imp-unit*: *a dvd b ⟹ is-unit b ⟹ is-unit a*
⟨*proof*⟩

lemma *unit-div-1-unit* [*simp*, *intro*]:
assumes *is-unit a*
shows *is-unit (1 div a)*
⟨*proof*⟩

lemma *is-unitE* [*elim?*]:
assumes *is-unit a*
obtains *b* **where** *a ≠ 0* **and** *b ≠ 0*
and *is-unit b* **and** *1 div a = b* **and** *1 div b = a*
and *a * b = 1* **and** *c div a = c * b*
⟨*proof*⟩

lemma *unit-prod* [*intro*]: *is-unit a ⟹ is-unit b ⟹ is-unit (a * b)*
⟨*proof*⟩

lemma *is-unit-mult-iff*: *is-unit (a * b) ⟷ is-unit a ∧ is-unit b*
⟨*proof*⟩

lemma *unit-div* [*intro*]: *is-unit a ⟹ is-unit b ⟹ is-unit (a div b)*
⟨*proof*⟩

lemma *mult-unit-dvd-iff*:
  assumes *is-unit b*
  shows *a * b dvd c ⟷ a dvd c*
⟨*proof*⟩

lemma *mult-unit-dvd-iff'*: *is-unit a ⟹ (a * b) dvd c ⟷ b dvd c*
  ⟨*proof*⟩

lemma *dvd-mult-unit-iff*:
  assumes *is-unit b*
  shows *a dvd c * b ⟷ a dvd c*
⟨*proof*⟩

lemma *dvd-mult-unit-iff'*: *is-unit b ⟹ a dvd b * c ⟷ a dvd c*
  ⟨*proof*⟩

lemma *div-unit-dvd-iff*: *is-unit b ⟹ a div b dvd c ⟷ a dvd c*
  ⟨*proof*⟩

lemma *dvd-div-unit-iff*: *is-unit b ⟹ a dvd c div b ⟷ a dvd c*
  ⟨*proof*⟩

lemmas *unit-dvd-iff = mult-unit-dvd-iff mult-unit-dvd-iff'*
  dvd-mult-unit-iff dvd-mult-unit-iff'
  div-unit-dvd-iff dvd-div-unit-iff

lemma *unit-mult-div-div* [*simp*]: *is-unit a ⟹ b * (1 div a) = b div a*
  ⟨*proof*⟩

lemma *unit-div-mult-self* [*simp*]: *is-unit a ⟹ b div a * a = b*
  ⟨*proof*⟩

lemma *unit-div-1-div-1* [*simp*]: *is-unit a ⟹ 1 div (1 div a) = a*
  ⟨*proof*⟩

lemma *unit-div-mult-swap*: *is-unit c ⟹ a * (b div c) = (a * b) div c*
  ⟨*proof*⟩

lemma *unit-div-commute*: *is-unit b ⟹ (a div b) * c = (a * c) div b*
  ⟨*proof*⟩

lemma *unit-eq-div1*: *is-unit b ⟹ a div b = c ⟷ a = c * b*
  ⟨*proof*⟩

lemma *unit-eq-div2*: *is-unit b ⟹ a = c div b ⟷ a * b = c*
  ⟨*proof*⟩

lemma *unit-mult-left-cancel*: *is-unit a ⟹ a * b = a * c ⟷ b = c*

⟨*proof*⟩

lemma *unit-mult-right-cancel*: *is-unit a* ⟹ *b* ∗ *a* = *c* ∗ *a* ⟷ *b* = *c*
⟨*proof*⟩

lemma *unit-div-cancel*:
assumes *is-unit a*
shows *b div a* = *c div a* ⟷ *b* = *c*
⟨*proof*⟩

lemma *is-unit-div-mult2-eq*:
assumes *is-unit b* **and** *is-unit c*
shows *a div* (*b* ∗ *c*) = *a div b div c*
⟨*proof*⟩

lemma *is-unit-div-mult-cancel-left*:
assumes *a* ≠ *0* **and** *is-unit b*
shows *a div* (*a* ∗ *b*) = *1 div b*
⟨*proof*⟩

lemma *is-unit-div-mult-cancel-right*:
assumes *a* ≠ *0* **and** *is-unit b*
shows *a div* (*b* ∗ *a*) = *1 div b*
⟨*proof*⟩

lemma *unit-div-eq-0-iff*:
assumes *is-unit b*
shows *a div b* = *0* ⟷ *a* = *0*
⟨*proof*⟩

lemma *div-mult-unit2*:
is-unit c ⟹ *b dvd a* ⟹ *a div* (*b* ∗ *c*) = *a div b div c*
⟨*proof*⟩

Coprimality

definition *coprime* :: *'a* ⇒ *'a* ⇒ *bool*
where *coprime a b* ⟷ (∀ *c*. *c dvd a* ⟶ *c dvd b* ⟶ *is-unit c*)

lemma *coprimeI*:
assumes ⋀*c*. *c dvd a* ⟹ *c dvd b* ⟹ *is-unit c*
shows *coprime a b*
⟨*proof*⟩

lemma *not-coprimeI*:
assumes *c dvd a* **and** *c dvd b* **and** ¬ *is-unit c*
shows ¬ *coprime a b*
⟨*proof*⟩

lemma *coprime-common-divisor*:

is-unit c **if** *coprime a b* **and** *c dvd a* **and** *c dvd b*
⟨*proof*⟩

lemma *not-coprimeE*:
assumes ¬ *coprime a b*
obtains *c* **where** *c dvd a* **and** *c dvd b* **and** ¬ *is-unit c*
⟨*proof*⟩

lemma *coprime-imp-coprime*:
coprime a b **if** *coprime c d*
and ⋀*e*. ¬ *is-unit e* ⟹ *e dvd a* ⟹ *e dvd b* ⟹ *e dvd c*
and ⋀*e*. ¬ *is-unit e* ⟹ *e dvd a* ⟹ *e dvd b* ⟹ *e dvd d*
⟨*proof*⟩

lemma *coprime-divisors*:
coprime a b **if** *a dvd c b dvd d* **and** *coprime c d*
⟨*proof*⟩

lemma *coprime-self* [*simp*]:
coprime a a ⟷ *is-unit a* (**is** *?P* ⟷ *?Q*)
⟨*proof*⟩

lemma *coprime-commute* [*ac-simps*]:
coprime b a ⟷ *coprime a b*
⟨*proof*⟩

lemma *is-unit-left-imp-coprime*:
coprime a b **if** *is-unit a*
⟨*proof*⟩

lemma *is-unit-right-imp-coprime*:
coprime a b **if** *is-unit b*
⟨*proof*⟩

lemma *coprime-1-left* [*simp*]:
coprime 1 a
⟨*proof*⟩

lemma *coprime-1-right* [*simp*]:
coprime a 1
⟨*proof*⟩

lemma *coprime-0-left-iff* [*simp*]:
coprime 0 a ⟷ *is-unit a*
⟨*proof*⟩

lemma *coprime-0-right-iff* [*simp*]:
coprime a 0 ⟷ *is-unit a*
⟨*proof*⟩

lemma *coprime-mult-self-left-iff* [*simp*]:
coprime (*c* ∗ *a*) (*c* ∗ *b*) ⟷ *is-unit c* ∧ *coprime a b*
⟨*proof*⟩

lemma *coprime-mult-self-right-iff* [*simp*]:
coprime (*a* ∗ *c*) (*b* ∗ *c*) ⟷ *is-unit c* ∧ *coprime a b*
⟨*proof*⟩

lemma *coprime-absorb-left*:
assumes *x dvd y*
shows *coprime x y* ⟷ *is-unit x*
⟨*proof*⟩

lemma *coprime-absorb-right*:
assumes *y dvd x*
shows *coprime x y* ⟷ *is-unit y*
⟨*proof*⟩

end

class *unit-factor* =
fixes *unit-factor* :: *'a* ⇒ *'a*

class *semidom-divide-unit-factor* = *semidom-divide* + *unit-factor* +
assumes *unit-factor-0* [*simp*]: *unit-factor 0* = *0*
and *is-unit-unit-factor*: *a dvd 1* ⟹ *unit-factor a* = *a*
and *unit-factor-is-unit*: *a* ≠ *0* ⟹ *unit-factor a dvd 1*
and *unit-factor-mult-unit-left*: *a dvd 1* ⟹ *unit-factor* (*a* ∗ *b*) = *a* ∗ *unit-factor b*
— This fine-grained hierarchy will later on allow lean normalization of polynomials
begin

lemma *unit-factor-mult-unit-right*: *a dvd 1* ⟹ *unit-factor* (*b* ∗ *a*) = *unit-factor b* ∗ *a*
⟨*proof*⟩

lemmas [*simp*] = *unit-factor-mult-unit-left unit-factor-mult-unit-right*

end

class *normalization-semidom* = *algebraic-semidom* + *semidom-divide-unit-factor* +
fixes *normalize* :: *'a* ⇒ *'a*
assumes *unit-factor-mult-normalize* [*simp*]: *unit-factor a* ∗ *normalize a* = *a*
and *normalize-0* [*simp*]: *normalize 0* = *0*
begin

Class *normalization-semidom* cultivates the idea that each integral domain

can be split into equivalence classes whose representants are associated, i.e. divide each other. *normalize* specifies a canonical representant for each equivalence class. The rationale behind this is that it is easier to reason about equality than equivalences, hence we prefer to think about equality of normalized values rather than associated elements.

declare *unit-factor-is-unit* [*iff*]

lemma *unit-factor-dvd* [*simp*]: $a \neq 0 \Longrightarrow$ *unit-factor a dvd b*
⟨*proof*⟩

lemma *unit-factor-self* [*simp*]: *unit-factor a dvd a*
⟨*proof*⟩

lemma *normalize-mult-unit-factor* [*simp*]: *normalize a* $*$ *unit-factor a* $=$ *a*
⟨*proof*⟩

lemma *normalize-eq-0-iff* [*simp*]: *normalize a* $= 0 \longleftrightarrow$ *a* $= 0$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *unit-factor-eq-0-iff* [*simp*]: *unit-factor a* $= 0 \longleftrightarrow$ *a* $= 0$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *div-unit-factor* [*simp*]: *a div unit-factor a* $=$ *normalize a*
⟨*proof*⟩

lemma *normalize-div* [*simp*]: *normalize a div a* $= 1$ *div unit-factor a*
⟨*proof*⟩

lemma *is-unit-normalize*:
assumes *is-unit a*
shows *normalize a* $= 1$
⟨*proof*⟩

lemma *unit-factor-1* [*simp*]: *unit-factor* $1 = 1$
⟨*proof*⟩

lemma *normalize-1* [*simp*]: *normalize* $1 = 1$
⟨*proof*⟩

lemma *normalize-1-iff*: *normalize a* $= 1 \longleftrightarrow$ *is-unit a*
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *div-normalize* [*simp*]: *a div normalize a* $=$ *unit-factor a*
⟨*proof*⟩

lemma *mult-one-div-unit-factor* [*simp*]: $a * (1\ div\ unit\text{-}factor\ b) = a\ div\ unit\text{-}factor\ b$
⟨*proof*⟩

lemma *inv-unit-factor-eq-0-iff* [*simp*]:
$1\ div\ unit\text{-}factor\ a = 0 \longleftrightarrow a = 0$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *unit-factor-idem* [*simp*]: $unit\text{-}factor\ (unit\text{-}factor\ a) = unit\text{-}factor\ a$
⟨*proof*⟩

lemma *normalize-unit-factor* [*simp*]: $a \neq 0 \Longrightarrow normalize\ (unit\text{-}factor\ a) = 1$
⟨*proof*⟩

lemma *normalize-mult-unit-left* [*simp*]:
assumes *a dvd 1*
shows $normalize\ (a * b) = normalize\ b$
⟨*proof*⟩

lemma *normalize-mult-unit-right* [*simp*]:
assumes *b dvd 1*
shows $normalize\ (a * b) = normalize\ a$
⟨*proof*⟩

lemma *normalize-idem* [*simp*]: $normalize\ (normalize\ a) = normalize\ a$
⟨*proof*⟩

lemma *unit-factor-normalize* [*simp*]:
assumes $a \neq 0$
shows $unit\text{-}factor\ (normalize\ a) = 1$
⟨*proof*⟩

lemma *normalize-dvd-iff* [*simp*]: $normalize\ a\ dvd\ b \longleftrightarrow a\ dvd\ b$
⟨*proof*⟩

lemma *dvd-normalize-iff* [*simp*]: $a\ dvd\ normalize\ b \longleftrightarrow a\ dvd\ b$
⟨*proof*⟩

lemma *normalize-idem-imp-unit-factor-eq*:
assumes $normalize\ a = a$
shows $unit\text{-}factor\ a = of\text{-}bool\ (a \neq 0)$
⟨*proof*⟩

lemma *normalize-idem-imp-is-unit-iff*:
assumes $normalize\ a = a$
shows $is\text{-}unit\ a \longleftrightarrow a = 1$
⟨*proof*⟩

lemma *coprime-normalize-left-iff* [*simp*]:
coprime (*normalize a*) *b* ⟷ *coprime a b*
⟨*proof*⟩

lemma *coprime-normalize-right-iff* [*simp*]:
coprime a (*normalize b*) ⟷ *coprime a b*
⟨*proof*⟩

We avoid an explicit definition of associated elements but prefer explicit normalisation instead. In theory we could define an abbreviation like *associated a b* = (*normalize a* = *normalize b*) but this is counterproductive without suggestive infix syntax, which we do not want to sacrifice for this purpose here.

lemma *associatedI*:
assumes *a dvd b* **and** *b dvd a*
shows *normalize a* = *normalize b*
⟨*proof*⟩

lemma *associatedD1*: *normalize a* = *normalize b* ⟹ *a dvd b*
⟨*proof*⟩

lemma *associatedD2*: *normalize a* = *normalize b* ⟹ *b dvd a*
⟨*proof*⟩

lemma *associated-unit*: *normalize a* = *normalize b* ⟹ *is-unit a* ⟹ *is-unit b*
⟨*proof*⟩

lemma *associated-iff-dvd*: *normalize a* = *normalize b* ⟷ *a dvd b* ∧ *b dvd a*
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *associated-eqI*:
assumes *a dvd b* **and** *b dvd a*
assumes *normalize a* = *a* **and** *normalize b* = *b*
shows *a* = *b*
⟨*proof*⟩

lemma *normalize-unit-factor-eqI*:
assumes *normalize a* = *normalize b*
and *unit-factor a* = *unit-factor b*
shows *a* = *b*
⟨*proof*⟩

lemma *normalize-mult-normalize-left* [*simp*]: *normalize* (*normalize a* ∗ *b*) = *normalize* (*a* ∗ *b*)
⟨*proof*⟩

lemma *normalize-mult-normalize-right* [*simp*]: *normalize* (*a* ∗ *normalize b*) = *normalize* (*a* ∗ *b*)

⟨*proof*⟩

end

class *normalization-semidom-multiplicative* = *normalization-semidom* +
  assumes *unit-factor-mult*: *unit-factor* (*a* ∗ *b*) = *unit-factor a* ∗ *unit-factor b*
begin

lemma *normalize-mult*: *normalize* (*a* ∗ *b*) = *normalize a* ∗ *normalize b*
⟨*proof*⟩

lemma *dvd-unit-factor-div*:
  assumes *b dvd a*
  shows *unit-factor* (*a div b*) = *unit-factor a div unit-factor b*
⟨*proof*⟩

lemma *dvd-normalize-div*:
  assumes *b dvd a*
  shows *normalize* (*a div b*) = *normalize a div normalize b*
⟨*proof*⟩

end

Syntactic division remainder operator

class *modulo* = *dvd* + *divide* +
  fixes *modulo* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** *mod 70*)

Arbitrary quotient and remainder partitions

class *semiring-modulo* = *comm-semiring-1-cancel* + *divide* + *modulo* +
  assumes *div-mult-mod-eq*: *a div b* ∗ *b* + *a mod b* = *a*
begin

lemma *mod-div-decomp*:
  fixes *a b*
  obtains *q r* **where** *q* = *a div b* **and** *r* = *a mod b*
    and *a* = *q* ∗ *b* + *r*
⟨*proof*⟩

lemma *mult-div-mod-eq*: *b* ∗ (*a div b*) + *a mod b* = *a*
  ⟨*proof*⟩

lemma *mod-div-mult-eq*: *a mod b* + *a div b* ∗ *b* = *a*
  ⟨*proof*⟩

lemma *mod-mult-div-eq*: *a mod b* + *b* ∗ (*a div b*) = *a*
  ⟨*proof*⟩

lemma *minus-div-mult-eq-mod*: *a* − *a div b* ∗ *b* = *a mod b*

⟨*proof*⟩

lemma *minus-mult-div-eq-mod*: $a - b * (a \ div \ b) = a \ mod \ b$
⟨*proof*⟩

lemma *minus-mod-eq-div-mult*: $a - a \ mod \ b = a \ div \ b * b$
⟨*proof*⟩

lemma *minus-mod-eq-mult-div*: $a - a \ mod \ b = b * (a \ div \ b)$
⟨*proof*⟩

lemma *mod-0-imp-dvd* [*dest*!]:
$b \ dvd \ a$ **if** $a \ mod \ b = 0$
⟨*proof*⟩

lemma [*nitpick-unfold*]:
$a \ mod \ b = a - a \ div \ b * b$
⟨*proof*⟩

end

16.5 Quotient and remainder in integral domains

class *semidom-modulo* = *algebraic-semidom* + *semiring-modulo*
begin

lemma *mod-0* [*simp*]: $0 \ mod \ a = 0$
⟨*proof*⟩

lemma *mod-by-0* [*simp*]: $a \ mod \ 0 = a$
⟨*proof*⟩

lemma *mod-by-1* [*simp*]:
$a \ mod \ 1 = 0$
⟨*proof*⟩

lemma *mod-self* [*simp*]:
$a \ mod \ a = 0$
⟨*proof*⟩

lemma *dvd-imp-mod-0* [*simp*]:
$b \ mod \ a = 0$ **if** $a \ dvd \ b$
⟨*proof*⟩

lemma *mod-eq-0-iff-dvd*:
$a \ mod \ b = 0 \longleftrightarrow b \ dvd \ a$
⟨*proof*⟩

lemma *dvd-eq-mod-eq-0* [*nitpick-unfold*, *code*]:

a dvd b $\longleftrightarrow$ *b mod a = 0*
⟨*proof*⟩

lemma *dvd-mod-iff*:
assumes *c dvd b*
shows *c dvd a mod b* $\longleftrightarrow$ *c dvd a*
⟨*proof*⟩

lemma *dvd-mod-imp-dvd*:
assumes *c dvd a mod b* **and** *c dvd b*
shows *c dvd a*
⟨*proof*⟩

lemma *dvd-minus-mod* [*simp*]:
b dvd a − a mod b
⟨*proof*⟩

lemma *cancel-div-mod-rules*:
*((a div b) * b + a mod b) + c = a + c*
*(b * (a div b) + a mod b) + c = a + c*
⟨*proof*⟩

end

class *idom-modulo = idom + semidom-modulo*
begin

subclass *idom-divide* ⟨*proof*⟩

lemma *div-diff* [*simp*]:
c dvd a $\Longrightarrow$ *c dvd b* $\Longrightarrow$ *(a − b) div c = a div c − b div c*
⟨*proof*⟩

end

16.6 Interlude: basic tool support for algebraic and arithmetic calculations

named-theorems *arith arith facts −− only ground formulas*
⟨*ML*⟩

16.7 Ordered semirings and rings

The theory of partially ordered rings is taken from the books:

- *Lattice Theory* by Garret Birkhoff, American Mathematical Society, 1979
- *Partially Ordered Algebraic Systems*, Pergamon Press, 1963

Most of the used notions can also be looked up in

- http://www.mathworld.com by Eric Weisstein et. al.
- *Algebra I* by van der Waerden, Springer

class *ordered-semiring* = *semiring* + *ordered-comm-monoid-add* +
  assumes *mult-left-mono*: $a \leq b \Longrightarrow 0 \leq c \Longrightarrow c * a \leq c * b$
  assumes *mult-right-mono*: $a \leq b \Longrightarrow 0 \leq c \Longrightarrow a * c \leq b * c$
begin

lemma *mult-mono*: $a \leq b \Longrightarrow c \leq d \Longrightarrow 0 \leq b \Longrightarrow 0 \leq c \Longrightarrow a * c \leq b * d$
  ⟨*proof*⟩

lemma *mult-mono'*: $a \leq b \Longrightarrow c \leq d \Longrightarrow 0 \leq a \Longrightarrow 0 \leq c \Longrightarrow a * c \leq b * d$
  ⟨*proof*⟩

end

lemma *mono-mult*:
  fixes a :: $'a$::*ordered-semiring*
  shows $a \geq 0 \Longrightarrow$ *mono* $((*)\ a)$
  ⟨*proof*⟩

class *ordered-semiring-0* = *semiring-0* + *ordered-semiring*
begin

lemma *mult-nonneg-nonneg* [*simp*]: $0 \leq a \Longrightarrow 0 \leq b \Longrightarrow 0 \leq a * b$
  ⟨*proof*⟩

lemma *mult-nonneg-nonpos*: $0 \leq a \Longrightarrow b \leq 0 \Longrightarrow a * b \leq 0$
  ⟨*proof*⟩

lemma *mult-nonpos-nonneg*: $a \leq 0 \Longrightarrow 0 \leq b \Longrightarrow a * b \leq 0$
  ⟨*proof*⟩

Legacy – use *mult-nonpos-nonneg*.

lemma *mult-nonneg-nonpos2*: $0 \leq a \Longrightarrow b \leq 0 \Longrightarrow b * a \leq 0$
  ⟨*proof*⟩

lemma *split-mult-neg-le*: $(0 \leq a \wedge b \leq 0) \vee (a \leq 0 \wedge 0 \leq b) \Longrightarrow a * b \leq 0$
  ⟨*proof*⟩

end

class *ordered-cancel-semiring* = *ordered-semiring* + *cancel-comm-monoid-add*
begin

subclass *semiring-0-cancel* ⟨*proof*⟩

subclass *ordered-semiring-0* ⟨*proof*⟩

end

class *linordered-semiring* = *ordered-semiring* + *linordered-cancel-ab-semigroup-add*
begin

subclass *ordered-cancel-semiring* ⟨*proof*⟩

subclass *ordered-cancel-comm-monoid-add* ⟨*proof*⟩

subclass *ordered-ab-semigroup-monoid-add-imp-le* ⟨*proof*⟩

lemma *mult-left-less-imp-less*: $c * a < c * b \Longrightarrow 0 \leq c \Longrightarrow a < b$
⟨*proof*⟩

lemma *mult-right-less-imp-less*: $a * c < b * c \Longrightarrow 0 \leq c \Longrightarrow a < b$
⟨*proof*⟩

end

class *zero-less-one* = *order* + *zero* + *one* +
assumes *zero-less-one* [*simp*]: $0 < 1$
begin

subclass *zero-neq-one*
⟨*proof*⟩

lemma *zero-le-one* [*simp*]:
‹$0 \leq 1$› ⟨*proof*⟩

end

class *linordered-semiring-1* = *linordered-semiring* + *semiring-1* + *zero-less-one*
begin

lemma *convex-bound-le*:
assumes $x \leq a$ $y \leq a$ $0 \leq u$ $0 \leq v$ $u + v = 1$
shows $u * x + v * y \leq a$
⟨*proof*⟩

end

class *linordered-semiring-strict* = *semiring* + *comm-monoid-add* + *linordered-cancel-ab-semigroup-add*
+
assumes *mult-strict-left-mono*: $a < b \Longrightarrow 0 < c \Longrightarrow c * a < c * b$
assumes *mult-strict-right-mono*: $a < b \Longrightarrow 0 < c \Longrightarrow a * c < b * c$
begin

subclass *semiring-0-cancel* ⟨*proof*⟩

subclass *linordered-semiring*
⟨*proof*⟩

lemma *mult-left-le-imp-le*: $c * a \leq c * b \Longrightarrow 0 < c \Longrightarrow a \leq b$
⟨*proof*⟩

lemma *mult-right-le-imp-le*: $a * c \leq b * c \Longrightarrow 0 < c \Longrightarrow a \leq b$
⟨*proof*⟩

lemma *mult-pos-pos*[*simp*]: $0 < a \Longrightarrow 0 < b \Longrightarrow 0 < a * b$
⟨*proof*⟩

lemma *mult-pos-neg*: $0 < a \Longrightarrow b < 0 \Longrightarrow a * b < 0$
⟨*proof*⟩

lemma *mult-neg-pos*: $a < 0 \Longrightarrow 0 < b \Longrightarrow a * b < 0$
⟨*proof*⟩

Legacy – use *mult-neg-pos*.

lemma *mult-pos-neg2*: $0 < a \Longrightarrow b < 0 \Longrightarrow b * a < 0$
⟨*proof*⟩

lemma *zero-less-mult-pos*:
assumes $0 < a * b$ $0 < a$ **shows** $0 < b$
⟨*proof*⟩

lemma *zero-less-mult-pos2*:
assumes $0 < b * a$ $0 < a$ **shows** $0 < b$
⟨*proof*⟩

Strict monotonicity in both arguments

lemma *mult-strict-mono*:
assumes $a < b$ $c < d$ $0 < b$ $0 \leq c$
shows $a * c < b * d$
⟨*proof*⟩

This weaker variant has more natural premises

lemma *mult-strict-mono'*:
assumes $a < b$ **and** $c < d$ **and** $0 \leq a$ **and** $0 \leq c$
shows $a * c < b * d$
⟨*proof*⟩

lemma *mult-less-le-imp-less*:
assumes $a < b$ **and** $c \leq d$ **and** $0 \leq a$ **and** $0 < c$
shows $a * c < b * d$

⟨*proof*⟩

lemma *mult-le-less-imp-less*:
assumes $a \leq b$ **and** $c < d$ **and** $0 < a$ **and** $0 \leq c$
shows $a * c < b * d$
⟨*proof*⟩

end

class *linordered-semiring-1-strict* = *linordered-semiring-strict* + *semiring-1* + *zero-less-one*
begin

subclass *linordered-semiring-1* ⟨*proof*⟩

lemma *convex-bound-lt*:
assumes $x < a$ $y < a$ $0 \leq u$ $0 \leq v$ $u + v = 1$
shows $u * x + v * y < a$
⟨*proof*⟩

end

class *ordered-comm-semiring* = *comm-semiring-0* + *ordered-ab-semigroup-add* +
assumes *comm-mult-left-mono*: $a \leq b \Longrightarrow 0 \leq c \Longrightarrow c * a \leq c * b$
begin

subclass *ordered-semiring*
⟨*proof*⟩

end

class *ordered-cancel-comm-semiring* = *ordered-comm-semiring* + *cancel-comm-monoid-add*
begin

subclass *comm-semiring-0-cancel* ⟨*proof*⟩
subclass *ordered-comm-semiring* ⟨*proof*⟩
subclass *ordered-cancel-semiring* ⟨*proof*⟩

end

class *linordered-comm-semiring-strict* = *comm-semiring-0* + *linordered-cancel-ab-semigroup-add*
+
assumes *comm-mult-strict-left-mono*: $a < b \Longrightarrow 0 < c \Longrightarrow c * a < c * b$
begin

subclass *linordered-semiring-strict*
⟨*proof*⟩

subclass *ordered-cancel-comm-semiring*
⟨*proof*⟩

end

class *ordered-ring* = *ring* + *ordered-cancel-semiring*
begin

subclass *ordered-ab-group-add* ⟨*proof*⟩

lemma *less-add-iff1*: $a * e + c < b * e + d \longleftrightarrow (a - b) * e + c < d$
⟨*proof*⟩

lemma *less-add-iff2*: $a * e + c < b * e + d \longleftrightarrow c < (b - a) * e + d$
⟨*proof*⟩

lemma *le-add-iff1*: $a * e + c \leq b * e + d \longleftrightarrow (a - b) * e + c \leq d$
⟨*proof*⟩

lemma *le-add-iff2*: $a * e + c \leq b * e + d \longleftrightarrow c \leq (b - a) * e + d$
⟨*proof*⟩

lemma *mult-left-mono-neg*: $b \leq a \Longrightarrow c \leq 0 \Longrightarrow c * a \leq c * b$
⟨*proof*⟩

lemma *mult-right-mono-neg*: $b \leq a \Longrightarrow c \leq 0 \Longrightarrow a * c \leq b * c$
⟨*proof*⟩

lemma *mult-nonpos-nonpos*: $a \leq 0 \Longrightarrow b \leq 0 \Longrightarrow 0 \leq a * b$
⟨*proof*⟩

lemma *split-mult-pos-le*: $(0 \leq a \wedge 0 \leq b) \vee (a \leq 0 \wedge b \leq 0) \Longrightarrow 0 \leq a * b$
⟨*proof*⟩

end

class *abs-if* = *minus* + *uminus* + *ord* + *zero* + *abs* +
assumes *abs-if*: $|a| = (\mathit{if}\ a < 0\ \mathit{then} - a\ \mathit{else}\ a)$

class *linordered-ring* = *ring* + *linordered-semiring* + *linordered-ab-group-add* + *abs-if*
begin

subclass *ordered-ring* ⟨*proof*⟩

subclass *ordered-ab-group-add-abs*
⟨*proof*⟩

lemma *zero-le-square* [*simp*]: $0 \leq a * a$
⟨*proof*⟩

lemma *not-square-less-zero* [*simp*]: $\neg\ (a * a < 0)$
⟨*proof*⟩

proposition *abs-eq-iff*: $|x| = |y| \longleftrightarrow x = y \vee x = -y$
⟨*proof*⟩

lemma *abs-eq-iff'*:
$|a| = b \longleftrightarrow b \geq 0 \wedge (a = b \vee a = - b)$
⟨*proof*⟩

lemma *eq-abs-iff'*:
$a = |b| \longleftrightarrow a \geq 0 \wedge (b = a \vee b = - a)$
⟨*proof*⟩

lemma *sum-squares-ge-zero*: $0 \leq x * x + y * y$
⟨*proof*⟩

lemma *not-sum-squares-lt-zero*: $\neg\ x * x + y * y < 0$
⟨*proof*⟩

end

class *linordered-ring-strict* = *ring* + *linordered-semiring-strict*
+ *ordered-ab-group-add* + *abs-if*
begin

subclass *linordered-ring* ⟨*proof*⟩

lemma *mult-strict-left-mono-neg*: $b < a \Longrightarrow c < 0 \Longrightarrow c * a < c * b$
⟨*proof*⟩

lemma *mult-strict-right-mono-neg*: $b < a \Longrightarrow c < 0 \Longrightarrow a * c < b * c$
⟨*proof*⟩

lemma *mult-neg-neg*: $a < 0 \Longrightarrow b < 0 \Longrightarrow 0 < a * b$
⟨*proof*⟩

subclass *ring-no-zero-divisors*
⟨*proof*⟩

lemma *zero-less-mult-iff* [*algebra-split-simps*, *field-split-simps*]:
$0 < a * b \longleftrightarrow 0 < a \wedge 0 < b \vee a < 0 \wedge b < 0$
⟨*proof*⟩

lemma *zero-le-mult-iff* [*algebra-split-simps*, *field-split-simps*]:
$0 \leq a * b \longleftrightarrow 0 \leq a \wedge 0 \leq b \vee a \leq 0 \wedge b \leq 0$
⟨*proof*⟩

lemma *mult-less-0-iff* [*algebra-split-simps*, *field-split-simps*]:

$a * b < 0 \longleftrightarrow 0 < a \wedge b < 0 \vee a < 0 \wedge 0 < b$
⟨*proof*⟩

lemma *mult-le-0-iff* [*algebra-split-simps*, *field-split-simps*]:
$a * b \leq 0 \longleftrightarrow 0 \leq a \wedge b \leq 0 \vee a \leq 0 \wedge 0 \leq b$
⟨*proof*⟩

Cancellation laws for $c * a < c * b$ and $a * c < b * c$, also with the relations $\leq$ and equality.

These "disjunction" versions produce two cases when the comparison is an assumption, but effectively four when the comparison is a goal.

lemma *mult-less-cancel-right-disj*: $a * c < b * c \longleftrightarrow 0 < c \wedge a < b \vee c < 0 \wedge b < a$
⟨*proof*⟩

lemma *mult-less-cancel-left-disj*: $c * a < c * b \longleftrightarrow 0 < c \wedge a < b \vee c < 0 \wedge b < a$
⟨*proof*⟩

The "conjunction of implication" lemmas produce two cases when the comparison is a goal, but give four when the comparison is an assumption.

lemma *mult-less-cancel-right*: $a * c < b * c \longleftrightarrow (0 \leq c \longrightarrow a < b) \wedge (c \leq 0 \longrightarrow b < a)$
⟨*proof*⟩

lemma *mult-less-cancel-left*: $c * a < c * b \longleftrightarrow (0 \leq c \longrightarrow a < b) \wedge (c \leq 0 \longrightarrow b < a)$
⟨*proof*⟩

lemma *mult-le-cancel-right*: $a * c \leq b * c \longleftrightarrow (0 < c \longrightarrow a \leq b) \wedge (c < 0 \longrightarrow b \leq a)$
⟨*proof*⟩

lemma *mult-le-cancel-left*: $c * a \leq c * b \longleftrightarrow (0 < c \longrightarrow a \leq b) \wedge (c < 0 \longrightarrow b \leq a)$
⟨*proof*⟩

lemma *mult-le-cancel-left-pos*: $0 < c \Longrightarrow c * a \leq c * b \longleftrightarrow a \leq b$
⟨*proof*⟩

lemma *mult-le-cancel-left-neg*: $c < 0 \Longrightarrow c * a \leq c * b \longleftrightarrow b \leq a$
⟨*proof*⟩

lemma *mult-less-cancel-left-pos*: $0 < c \Longrightarrow c * a < c * b \longleftrightarrow a < b$
⟨*proof*⟩

lemma *mult-less-cancel-left-neg*: $c < 0 \Longrightarrow c * a < c * b \longleftrightarrow b < a$
⟨*proof*⟩

end

lemmas *mult-sign-intros* =
mult-nonneg-nonneg mult-nonneg-nonpos
mult-nonpos-nonneg mult-nonpos-nonpos
mult-pos-pos mult-pos-neg
mult-neg-pos mult-neg-neg

class *ordered-comm-ring* = *comm-ring* + *ordered-comm-semiring*
begin

subclass *ordered-ring* ⟨*proof*⟩
subclass *ordered-cancel-comm-semiring* ⟨*proof*⟩

end

class *linordered-nonzero-semiring* = *ordered-comm-semiring* + *monoid-mult* + *linorder*
+ *zero-less-one* +
assumes *add-mono1*: $a < b \Longrightarrow a + 1 < b + 1$
begin

subclass *zero-neq-one*
⟨*proof*⟩

subclass *comm-semiring-1*
⟨*proof*⟩

lemma *zero-le-one* [*simp*]: $0 \leq 1$
⟨*proof*⟩

lemma *not-one-le-zero* [*simp*]: $\neg\ 1 \leq 0$
⟨*proof*⟩

lemma *not-one-less-zero* [*simp*]: $\neg\ 1 < 0$
⟨*proof*⟩

lemma *of-bool-less-eq-iff* [*simp*]:
‹*of-bool* $P \leq$ *of-bool* $Q \longleftrightarrow (P \longrightarrow Q)$›
⟨*proof*⟩

lemma *of-bool-less-iff* [*simp*]:
‹*of-bool* $P <$ *of-bool* $Q \longleftrightarrow \neg\ P \wedge Q$›
⟨*proof*⟩

lemma *mult-left-le*: $c \leq 1 \Longrightarrow 0 \leq a \Longrightarrow a * c \leq a$
⟨*proof*⟩

lemma *mult-le-one*: $a \leq 1 \Longrightarrow 0 \leq b \Longrightarrow b \leq 1 \Longrightarrow a * b \leq 1$

⟨*proof*⟩

lemma *zero-less-two*: $0 < 1 + 1$
⟨*proof*⟩

end

class *linordered-semidom* = *semidom* + *linordered-comm-semiring-strict* + *zero-less-one* +
assumes *le-add-diff-inverse2* [*simp*]: $b \leq a \Longrightarrow a - b + b = a$
begin

subclass *linordered-nonzero-semiring*
⟨*proof*⟩

lemma *zero-less-eq-of-bool* [*simp*]:
‹$0 \leq$ *of-bool* P›
⟨*proof*⟩

lemma *zero-less-of-bool-iff* [*simp*]:
‹$0 <$ *of-bool* $P \longleftrightarrow P$›
⟨*proof*⟩

lemma *of-bool-less-eq-one* [*simp*]:
‹*of-bool* $P \leq 1$›
⟨*proof*⟩

lemma *of-bool-less-one-iff* [*simp*]:
‹*of-bool* $P < 1 \longleftrightarrow \neg P$›
⟨*proof*⟩

lemma *of-bool-or-iff* [*simp*]:
‹*of-bool* $(P \vee Q) =$ *max* (*of-bool* P) (*of-bool* Q)›
⟨*proof*⟩

Addition is the inverse of subtraction.

lemma *le-add-diff-inverse* [*simp*]: $b \leq a \Longrightarrow b + (a - b) = a$
⟨*proof*⟩

lemma *add-diff-inverse*: $\neg\ a < b \Longrightarrow b + (a - b) = a$
⟨*proof*⟩

lemma *add-le-imp-le-diff*:
assumes $i + k \leq n$ **shows** $i \leq n - k$
⟨*proof*⟩

lemma *add-le-add-imp-diff-le*:
assumes *1*: $i + k \leq n$
and *2*: $n \leq j + k$

shows $i + k \leq n \Longrightarrow n \leq j + k \Longrightarrow n - k \leq j$
⟨*proof*⟩

lemma *less-1-mult*: $1 < m \Longrightarrow 1 < n \Longrightarrow 1 < m * n$
⟨*proof*⟩

end

class *linordered-idom* = *comm-ring-1* + *linordered-comm-semiring-strict* +
ordered-ab-group-add + *abs-if* + *sgn* +
assumes *sgn-if*: *sgn x* = (*if x* = *0 then 0 else if 0* < *x then 1 else* − *1*)
begin

subclass *linordered-ring-strict* ⟨*proof*⟩

subclass *linordered-semiring-1-strict*
⟨*proof*⟩

subclass *ordered-comm-ring* ⟨*proof*⟩
subclass *idom* ⟨*proof*⟩

subclass *linordered-semidom*
⟨*proof*⟩

subclass *idom-abs-sgn*
⟨*proof*⟩

lemma *abs-bool-eq* [*simp*]:
‹|*of-bool P*| = *of-bool P*›
⟨*proof*⟩

lemma *linorder-neqE-linordered-idom*:
assumes $x \neq y$
obtains $x < y \mid y < x$
⟨*proof*⟩

These cancellation simp rules also produce two cases when the comparison is a goal.

lemma *mult-le-cancel-right1*: $c \leq b * c \longleftrightarrow (0 < c \longrightarrow 1 \leq b) \wedge (c < 0 \longrightarrow b \leq 1)$
⟨*proof*⟩

lemma *mult-le-cancel-right2*: $a * c \leq c \longleftrightarrow (0 < c \longrightarrow a \leq 1) \wedge (c < 0 \longrightarrow 1 \leq a)$
⟨*proof*⟩

lemma *mult-le-cancel-left1*: $c \leq c * b \longleftrightarrow (0 < c \longrightarrow 1 \leq b) \wedge (c < 0 \longrightarrow b \leq 1)$
⟨*proof*⟩

lemma *mult-le-cancel-left2*: $c * a \leq c \longleftrightarrow (0 < c \longrightarrow a \leq 1) \wedge (c < 0 \longrightarrow 1 \leq a)$
⟨*proof*⟩

lemma *mult-less-cancel-right1*: $c < b * c \longleftrightarrow (0 \leq c \longrightarrow 1 < b) \wedge (c \leq 0 \longrightarrow b < 1)$
⟨*proof*⟩

lemma *mult-less-cancel-right2*: $a * c < c \longleftrightarrow (0 \leq c \longrightarrow a < 1) \wedge (c \leq 0 \longrightarrow 1 < a)$
⟨*proof*⟩

lemma *mult-less-cancel-left1*: $c < c * b \longleftrightarrow (0 \leq c \longrightarrow 1 < b) \wedge (c \leq 0 \longrightarrow b < 1)$
⟨*proof*⟩

lemma *mult-less-cancel-left2*: $c * a < c \longleftrightarrow (0 \leq c \longrightarrow a < 1) \wedge (c \leq 0 \longrightarrow 1 < a)$
⟨*proof*⟩

lemma *sgn-0-0*: $sgn\ a = 0 \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *sgn-1-pos*: $sgn\ a = 1 \longleftrightarrow a > 0$
⟨*proof*⟩

lemma *sgn-1-neg*: $sgn\ a = -1 \longleftrightarrow a < 0$
⟨*proof*⟩

lemma *sgn-pos* [*simp*]: $0 < a \Longrightarrow sgn\ a = 1$
⟨*proof*⟩

lemma *sgn-neg* [*simp*]: $a < 0 \Longrightarrow sgn\ a = -1$
⟨*proof*⟩

lemma *abs-sgn*: $|k| = k * sgn\ k$
⟨*proof*⟩

lemma *sgn-greater* [*simp*]: $0 < sgn\ a \longleftrightarrow 0 < a$
⟨*proof*⟩

lemma *sgn-less* [*simp*]: $sgn\ a < 0 \longleftrightarrow a < 0$
⟨*proof*⟩

lemma *abs-sgn-eq-1* [*simp*]:
$a \neq 0 \Longrightarrow |sgn\ a| = 1$
⟨*proof*⟩

lemma *abs-sgn-eq*: $|sgn\ a| = (if\ a = 0\ then\ 0\ else\ 1)$
⟨*proof*⟩

lemma *sgn-mult-self-eq* [*simp*]:
$sgn\ a * sgn\ a = of\text{-}bool\ (a \neq 0)$
⟨*proof*⟩

lemma *left-sgn-mult-self-eq* [*simp*]:
‹$sgn\ a * (sgn\ a * b) = of\text{-}bool\ (a \neq 0) * b$›
⟨*proof*⟩

lemma *abs-mult-self-eq* [*simp*]:
$|a| * |a| = a * a$
⟨*proof*⟩

lemma *same-sgn-sgn-add*:
$sgn\ (a + b) = sgn\ a$ **if** $sgn\ b = sgn\ a$
⟨*proof*⟩

lemma *same-sgn-abs-add*:
$|a + b| = |a| + |b|$ **if** $sgn\ b = sgn\ a$
⟨*proof*⟩

lemma *sgn-not-eq-imp*:
$sgn\ a = - sgn\ b$ **if** $sgn\ b \neq sgn\ a$ **and** $sgn\ a \neq 0$ **and** $sgn\ b \neq 0$
⟨*proof*⟩

lemma *abs-dvd-iff* [*simp*]: $|m|\ dvd\ k \longleftrightarrow m\ dvd\ k$
⟨*proof*⟩

lemma *dvd-abs-iff* [*simp*]: $m\ dvd\ |k| \longleftrightarrow m\ dvd\ k$
⟨*proof*⟩

lemma *dvd-if-abs-eq*: $|l| = |k| \Longrightarrow l\ dvd\ k$
⟨*proof*⟩

The following lemmas can be proven in more general structures, but are dangerous as simp rules in absence of $(-\ ?a = ?a) = (?a = (0::'a))$, $(-\ ?a < ?a) = ((0::'a) < ?a)$, $(-\ ?a \leq ?a) = ((0::'a) \leq ?a)$.

lemma *equation-minus-iff-1* [*simp*, *no-atp*]: $1 = - a \longleftrightarrow a = - 1$
⟨*proof*⟩

lemma *minus-equation-iff-1* [*simp*, *no-atp*]: $- a = 1 \longleftrightarrow a = - 1$
⟨*proof*⟩

lemma *le-minus-iff-1* [*simp*, *no-atp*]: $1 \leq - b \longleftrightarrow b \leq - 1$
⟨*proof*⟩

lemma *minus-le-iff-1* [*simp*, *no-atp*]: $- a \leq 1 \longleftrightarrow - 1 \leq a$

⟨*proof*⟩

lemma *less-minus-iff-1* [*simp*, *no-atp*]: $1 < - b \longleftrightarrow b < - 1$
⟨*proof*⟩

lemma *minus-less-iff-1* [*simp*, *no-atp*]: $- a < 1 \longleftrightarrow - 1 < a$
⟨*proof*⟩

lemma *add-less-zeroD*:
shows $x+y < 0 \Longrightarrow x<0 \vee y<0$
⟨*proof*⟩

Is this really better than just rewriting with *abs-if*?

lemma *abs-split* [*no-atp*]: $‹P\ |a| \longleftrightarrow (0 \leq a \longrightarrow P\ a) \wedge (a < 0 \longrightarrow P\ (- a))›$
⟨*proof*⟩

end

Reasoning about inequalities with division

context *linordered-semidom*
begin

lemma *less-add-one*: $a < a + 1$
⟨*proof*⟩

end

context *linordered-idom*
begin

lemma *mult-right-le-one-le*: $0 \leq x \Longrightarrow 0 \leq y \Longrightarrow y \leq 1 \Longrightarrow x * y \leq x$
⟨*proof*⟩

lemma *mult-left-le-one-le*: $0 \leq x \Longrightarrow 0 \leq y \Longrightarrow y \leq 1 \Longrightarrow y * x \leq x$
⟨*proof*⟩

end

Absolute Value

context *linordered-idom*
begin

lemma *mult-sgn-abs*: $sgn\ x * |x| = x$
⟨*proof*⟩

lemma *abs-one*: $|1| = 1$
⟨*proof*⟩

end

class *ordered-ring-abs* = *ordered-ring* + *ordered-ab-group-add-abs* +
  assumes *abs-eq-mult*:
    $(0 \leq a \vee a \leq 0) \wedge (0 \leq b \vee b \leq 0) \Longrightarrow |a * b| = |a| * |b|$

context *linordered-idom*
begin

subclass *ordered-ring-abs*
  ⟨*proof*⟩

lemma *abs-mult-self*: $|a| * |a| = a * a$
  ⟨*proof*⟩

lemma *abs-mult-less*:
  assumes *ac*: $|a| < c$
    and *bd*: $|b| < d$
  shows $|a| * |b| < c * d$
⟨*proof*⟩

lemma *abs-less-iff*: $|a| < b \longleftrightarrow a < b \wedge -a < b$
  ⟨*proof*⟩

lemma *abs-mult-pos*: $0 \leq x \Longrightarrow |y| * x = |y * x|$
  ⟨*proof*⟩

lemma *abs-mult-pos'*: $0 \leq x \Longrightarrow x * |y| = |x * y|$
  ⟨*proof*⟩

lemma *abs-diff-less-iff*: $|x - a| < r \longleftrightarrow a - r < x \wedge x < a + r$
  ⟨*proof*⟩

lemma *abs-diff-le-iff*: $|x - a| \leq r \longleftrightarrow a - r \leq x \wedge x \leq a + r$
  ⟨*proof*⟩

lemma *abs-add-one-gt-zero*: $0 < 1 + |x|$
  ⟨*proof*⟩

end

16.8 Dioids

Dioids are the alternative extensions of semirings, a semiring can either be a ring or a dioid but never both.

class *dioid* = *semiring-1* + *canonically-ordered-monoid-add*
begin

subclass *ordered-semiring*
  ⟨*proof*⟩

end

hide-fact (**open**) *comm-mult-left-mono comm-mult-strict-left-mono distrib*

code-identifier
  code-module *Rings* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

end

17 Natural numbers

theory *Nat*
imports *Inductive Typedef Fun Rings*
begin

17.1 Type *ind*

typedecl *ind*

axiomatization *Zero-Rep* :: *ind* **and** *Suc-Rep* :: *ind* ⇒ *ind*
  — The axiom of infinity in 2 parts:
  where *Suc-Rep-inject*: *Suc-Rep x* = *Suc-Rep y* ⟹ *x* = *y*
    and *Suc-Rep-not-Zero-Rep*: *Suc-Rep x* ≠ *Zero-Rep*

17.2 Type nat

Type definition

inductive *Nat* :: *ind* ⇒ *bool*
  where
    Zero-RepI: *Nat Zero-Rep*
  | *Suc-RepI*: *Nat i* ⟹ *Nat* (*Suc-Rep i*)

typedef *nat* = {*n*. *Nat n*}
  morphisms *Rep-Nat Abs-Nat*
  ⟨*proof*⟩

lemma *Nat-Rep-Nat*: *Nat* (*Rep-Nat n*)
  ⟨*proof*⟩

lemma *Nat-Abs-Nat-inverse*: *Nat n* ⟹ *Rep-Nat* (*Abs-Nat n*) = *n*
  ⟨*proof*⟩

lemma *Nat-Abs-Nat-inject*: *Nat n* ⟹ *Nat m* ⟹ *Abs-Nat n* = *Abs-Nat m* ⟷ *n* = *m*
  ⟨*proof*⟩

instantiation *nat* :: *zero*
begin

definition *Zero-nat-def*: *0* = *Abs-Nat Zero-Rep*

instance ⟨*proof*⟩

end

definition *Suc* :: *nat* ⇒ *nat*
  where *Suc n* = *Abs-Nat* (*Suc-Rep* (*Rep-Nat n*))

lemma *Suc-not-Zero*: *Suc m* ≠ *0*
  ⟨*proof*⟩

lemma *Zero-not-Suc*: *0* ≠ *Suc m*
  ⟨*proof*⟩

lemma *Suc-Rep-inject'*: *Suc-Rep x* = *Suc-Rep y* ⟷ *x* = *y*
  ⟨*proof*⟩

lemma *nat-induct0*:
  assumes *P 0* **and** ⋀*n*. *P n* ⟹ *P* (*Suc n*)
  shows *P n*
⟨*proof*⟩

free-constructors *case-nat* **for** *0* :: *nat* | *Suc pred*
  where *pred* (*0* :: *nat*) = (*0* :: *nat*)
⟨*proof*⟩
⟨*ML*⟩

old-rep-datatype *0* :: *nat Suc*
  ⟨*proof*⟩

⟨*ML*⟩

declare *old.nat.inject*[*iff del*]
  and *old.nat.distinct*(*1*)[*simp del*, *induct-simp del*]

lemmas *induct* = *old.nat.induct*
lemmas *inducts* = *old.nat.inducts*
lemmas *rec* = *old.nat.rec*
lemmas *simps* = *nat.inject nat.distinct nat.case nat.rec*

⟨*ML*⟩

abbreviation *rec-nat* :: *'a* ⇒ (*nat* ⇒ *'a* ⇒ *'a*) ⇒ *nat* ⇒ *'a*
  where *rec-nat* ≡ *old.rec-nat*

declare *nat.sel*[*code del*]

hide-const (**open**) *Nat.pred* — hide everything related to the selector
hide-fact
nat.case-eq-if
nat.collapse
nat.expand
nat.sel
nat.exhaust-sel
nat.split-sel
nat.split-sel-asm

lemma *nat-exhaust* [*case-names 0 Suc*, *cases type*: *nat*]:
$(y = 0 \Longrightarrow P) \Longrightarrow (\bigwedge nat.\ y = Suc\ nat \Longrightarrow P) \Longrightarrow P$
— for backward compatibility – names of variables differ
⟨*proof*⟩

lemma *nat-induct* [*case-names 0 Suc*, *induct type*: *nat*]:
fixes *n*
assumes *P 0* **and** $\bigwedge n.\ P\ n \Longrightarrow P\ (Suc\ n)$
shows *P n*
— for backward compatibility – names of variables differ
⟨*proof*⟩

hide-fact
nat-exhaust
nat-induct0

⟨*ML*⟩

Injectiveness and distinctness lemmas

lemma *inj-Suc* [*simp*]:
inj-on Suc N
⟨*proof*⟩

lemma *bij-betw-Suc* [*simp*]:
$bij\text{-}betw\ Suc\ M\ N \longleftrightarrow Suc\ `\ M = N$
⟨*proof*⟩

lemma *Suc-neq-Zero*: $Suc\ m = 0 \Longrightarrow R$
⟨*proof*⟩

lemma *Zero-neq-Suc*: $0 = Suc\ m \Longrightarrow R$
⟨*proof*⟩

lemma *Suc-inject*: $Suc\ x = Suc\ y \Longrightarrow x = y$
⟨*proof*⟩

lemma *n-not-Suc-n*: $n \neq Suc\ n$

⟨*proof*⟩

lemma *Suc-n-not-n*: $Suc\ n \neq n$
⟨*proof*⟩

A special form of induction for reasoning about $m < n$ and $m - n$.

lemma *diff-induct*:
assumes $\bigwedge x.\ P\ x\ 0$
and $\bigwedge y.\ P\ 0\ (Suc\ y)$
and $\bigwedge x\ y.\ P\ x\ y \Longrightarrow P\ (Suc\ x)\ (Suc\ y)$
shows $P\ m\ n$
⟨*proof*⟩

17.3 Arithmetic operators

instantiation *nat* :: *comm-monoid-diff*
begin

primrec *plus-nat*
where
add-0: $0 + n = (n::nat)$
| *add-Suc*: $Suc\ m + n = Suc\ (m + n)$

lemma *add-0-right* [*simp*]: $m + 0 = m$
for m :: *nat*
⟨*proof*⟩

lemma *add-Suc-right* [*simp*]: $m + Suc\ n = Suc\ (m + n)$
⟨*proof*⟩

declare *add-0* [*code*]

lemma *add-Suc-shift* [*code*]: $Suc\ m + n = m + Suc\ n$
⟨*proof*⟩

primrec *minus-nat*
where
diff-0 [*code*]: $m - 0 = (m::nat)$
| *diff-Suc*: $m - Suc\ n = (case\ m - n\ of\ 0 \Rightarrow 0 \mid Suc\ k \Rightarrow k)$

declare *diff-Suc* [*simp del*]

lemma *diff-0-eq-0* [*simp*, *code*]: $0 - n = 0$
for n :: *nat*
⟨*proof*⟩

lemma *diff-Suc-Suc* [*simp*, *code*]: $Suc\ m - Suc\ n = m - n$
⟨*proof*⟩

instance
⟨*proof*⟩

end

hide-fact (**open**) *add-0 add-0-right diff-0*

instantiation *nat* :: *comm-semiring-1-cancel*
begin

definition *One-nat-def* [*simp*]: *1 = Suc 0*

primrec *times-nat*
  where
    mult-0: *0 * n = (0::nat)*
  | *mult-Suc*: *Suc m * n = n + (m * n)*

lemma *mult-0-right* [*simp*]: *m * 0 = 0*
  for *m* :: *nat*
  ⟨*proof*⟩

lemma *mult-Suc-right* [*simp*]: *m * Suc n = m + (m * n)*
  ⟨*proof*⟩

lemma *add-mult-distrib*: *(m + n) * k = (m * k) + (n * k)*
  for *m n k* :: *nat*
  ⟨*proof*⟩

instance
⟨*proof*⟩

end

17.3.1 Addition

Reasoning about *m + 0 = 0*, etc.

lemma *add-is-0* [*iff*]: *m + n = 0 ⟷ m = 0 ∧ n = 0*
  for *m n* :: *nat*
  ⟨*proof*⟩

lemma *add-is-1*: *m + n = Suc 0 ⟷ m = Suc 0 ∧ n = 0 ∨ m = 0 ∧ n = Suc 0*
  ⟨*proof*⟩

lemma *one-is-add*: *Suc 0 = m + n ⟷ m = Suc 0 ∧ n = 0 ∨ m = 0 ∧ n = Suc 0*
  ⟨*proof*⟩

lemma *add-eq-self-zero*: *m + n = m ⟹ n = 0*
  for *m n* :: *nat*

⟨*proof*⟩

lemma *plus-1-eq-Suc*:
plus 1 = Suc
⟨*proof*⟩

lemma *Suc-eq-plus1*: *Suc n = n + 1*
⟨*proof*⟩

lemma *Suc-eq-plus1-left*: *Suc n = 1 + n*
⟨*proof*⟩

17.3.2 Difference

lemma *Suc-diff-diff* [*simp*]: *(Suc m − n) − Suc k = m − n − k*
⟨*proof*⟩

lemma *diff-Suc-1* [*simp*]: *Suc n − 1 = n*
⟨*proof*⟩

17.3.3 Multiplication

lemma *mult-is-0* [*simp*]: *m * n = 0 ⟷ m = 0 ∨ n = 0* **for** *m n :: nat*
⟨*proof*⟩

lemma *mult-eq-1-iff* [*simp*]: *m * n = Suc 0 ⟷ m = Suc 0 ∧ n = Suc 0*
⟨*proof*⟩

lemma *one-eq-mult-iff* [*simp*]: *Suc 0 = m * n ⟷ m = Suc 0 ∧ n = Suc 0*
⟨*proof*⟩

lemma *nat-mult-eq-1-iff* [*simp*]: *m * n = 1 ⟷ m = 1 ∧ n = 1*
and *nat-1-eq-mult-iff* [*simp*]: *1 = m * n ⟷ m = 1 ∧ n = 1* **for** *m n :: nat*
⟨*proof*⟩

lemma *mult-cancel1* [*simp*]: *k * m = k * n ⟷ m = n ∨ k = 0*
for *k m n :: nat*
⟨*proof*⟩

lemma *mult-cancel2* [*simp*]: *m * k = n * k ⟷ m = n ∨ k = 0*
for *k m n :: nat*
⟨*proof*⟩

lemma *Suc-mult-cancel1*: *Suc k * m = Suc k * n ⟷ m = n*
⟨*proof*⟩

17.4 Orders on *nat*

17.4.1 Operation definition

instantiation *nat* :: *linorder*
begin

primrec *less-eq-nat*
  where
    $(0::nat) \leq n \longleftrightarrow True$
  | $Suc\ m \leq n \longleftrightarrow (case\ n\ of\ 0 \Rightarrow False \mid Suc\ n \Rightarrow m \leq n)$

declare *less-eq-nat.simps* [*simp del*]

lemma *le0* [*iff*]: $0 \leq n$ **for**
  $n :: nat$
  ⟨*proof*⟩

lemma [*code*]: $0 \leq n \longleftrightarrow True$
  for $n :: nat$
  ⟨*proof*⟩

definition *less-nat*
  where *less-eq-Suc-le*: $n < m \longleftrightarrow Suc\ n \leq m$

lemma *Suc-le-mono* [*iff*]: $Suc\ n \leq Suc\ m \longleftrightarrow n \leq m$
  ⟨*proof*⟩

lemma *Suc-le-eq* [*code*]: $Suc\ m \leq n \longleftrightarrow m < n$
  ⟨*proof*⟩

lemma *le-0-eq* [*iff*]: $n \leq 0 \longleftrightarrow n = 0$
  for $n :: nat$
  ⟨*proof*⟩

lemma *not-less0* [*iff*]: $\neg\ n < 0$
  for $n :: nat$
  ⟨*proof*⟩

lemma *less-nat-zero-code* [*code*]: $n < 0 \longleftrightarrow False$
  for $n :: nat$
  ⟨*proof*⟩

lemma *Suc-less-eq* [*iff*]: $Suc\ m < Suc\ n \longleftrightarrow m < n$
  ⟨*proof*⟩

lemma *less-Suc-eq-le* [*code*]: $m < Suc\ n \longleftrightarrow m \leq n$
  ⟨*proof*⟩

lemma *Suc-less-eq2*: $Suc\ n < m \longleftrightarrow (\exists\, m'.\ m = Suc\ m' \wedge n < m')$

⟨*proof*⟩

lemma *le-SucI*: $m \leq n \Longrightarrow m \leq Suc\ n$
⟨*proof*⟩

lemma *Suc-leD*: $Suc\ m \leq n \Longrightarrow m \leq n$
⟨*proof*⟩

lemma *less-SucI*: $m < n \Longrightarrow m < Suc\ n$
⟨*proof*⟩

lemma *Suc-lessD*: $Suc\ m < n \Longrightarrow m < n$
⟨*proof*⟩

instance
⟨*proof*⟩

end

instantiation *nat* :: *order-bot*
begin

definition *bot-nat* :: *nat*
where *bot-nat* = *0*

instance
⟨*proof*⟩

end

instance *nat* :: *no-top*
⟨*proof*⟩

17.4.2 Introduction properties

lemma *lessI* [*iff*]: $n < Suc\ n$
⟨*proof*⟩

lemma *zero-less-Suc* [*iff*]: $0 < Suc\ n$
⟨*proof*⟩

17.4.3 Elimination properties

lemma *less-not-refl*: $\neg\ n < n$
for n :: *nat*
⟨*proof*⟩

lemma *less-not-refl2*: $n < m \Longrightarrow m \neq n$
for $m\ n$ :: *nat*
⟨*proof*⟩

lemma *less-not-refl3*: $s < t \Longrightarrow s \neq t$
for $s\ t$ :: *nat*
⟨*proof*⟩

lemma *less-irrefl-nat*: $n < n \Longrightarrow R$
for n :: *nat*
⟨*proof*⟩

lemma *less-zeroE*: $n < 0 \Longrightarrow R$
for n :: *nat*
⟨*proof*⟩

lemma *less-Suc-eq*: $m < Suc\ n \longleftrightarrow m < n \vee m = n$
⟨*proof*⟩

lemma *less-Suc0* [*iff*]: $(n < Suc\ 0) = (n = 0)$
⟨*proof*⟩

lemma *less-one* [*iff*]: $n < 1 \longleftrightarrow n = 0$
for n :: *nat*
⟨*proof*⟩

lemma *Suc-mono*: $m < n \Longrightarrow Suc\ m < Suc\ n$
⟨*proof*⟩

"Less than" is antisymmetric, sort of.

lemma *less-antisym*: $\neg\ n < m \Longrightarrow n < Suc\ m \Longrightarrow m = n$
⟨*proof*⟩

lemma *nat-neq-iff*: $m \neq n \longleftrightarrow m < n \vee n < m$
for $m\ n$ :: *nat*
⟨*proof*⟩

17.4.4 Inductive (?) properties

lemma *Suc-lessI*: $m < n \Longrightarrow Suc\ m \neq n \Longrightarrow Suc\ m < n$
⟨*proof*⟩

lemma *lessE*:
assumes *major*: $i < k$
and *1*: $k = Suc\ i \Longrightarrow P$
and *2*: $\bigwedge j.\ i < j \Longrightarrow k = Suc\ j \Longrightarrow P$
shows P
⟨*proof*⟩

lemma *less-SucE*:
assumes *major*: $m < Suc\ n$
and *less*: $m < n \Longrightarrow P$

and *eq*: $m = n \Longrightarrow P$
shows P
⟨*proof*⟩

lemma *Suc-lessE*:
assumes *major*: $Suc\ i < k$
and *minor*: $\bigwedge j.\ i < j \Longrightarrow k = Suc\ j \Longrightarrow P$
shows P
⟨*proof*⟩

lemma *Suc-less-SucD*: $Suc\ m < Suc\ n \Longrightarrow m < n$
⟨*proof*⟩

lemma *less-trans-Suc*:
assumes *le*: $i < j$
shows $j < k \Longrightarrow Suc\ i < k$
⟨*proof*⟩

Can be used with *less-Suc-eq* to get $n = m \vee n < m$.

lemma *not-less-eq*: $\neg\ m < n \longleftrightarrow n < Suc\ m$
⟨*proof*⟩

lemma *not-less-eq-eq*: $\neg\ m \leq n \longleftrightarrow Suc\ n \leq m$
⟨*proof*⟩

Properties of "less than or equal".

lemma *le-imp-less-Suc*: $m \leq n \Longrightarrow m < Suc\ n$
⟨*proof*⟩

lemma *Suc-n-not-le-n*: $\neg\ Suc\ n \leq n$
⟨*proof*⟩

lemma *le-Suc-eq*: $m \leq Suc\ n \longleftrightarrow m \leq n \vee m = Suc\ n$
⟨*proof*⟩

lemma *le-SucE*: $m \leq Suc\ n \Longrightarrow (m \leq n \Longrightarrow R) \Longrightarrow (m = Suc\ n \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

lemma *Suc-leI*: $m < n \Longrightarrow Suc\ m \leq n$
⟨*proof*⟩

Stronger version of *Suc-leD*.

lemma *Suc-le-lessD*: $Suc\ m \leq n \Longrightarrow m < n$
⟨*proof*⟩

lemma *less-imp-le-nat*: $m < n \Longrightarrow m \leq n$ **for** $m\ n :: nat$
⟨*proof*⟩

For instance, $(Suc\ m < Suc\ n) = (Suc\ m \leq n) = (m < n)$

lemmas *le-simps* = *less-imp-le-nat less-Suc-eq-le Suc-le-eq*

Equivalence of $m \leq n$ and $m < n \vee m = n$

lemma *less-or-eq-imp-le*: $m < n \vee m = n \Longrightarrow m \leq n$
for m n :: *nat*
⟨*proof*⟩

lemma *le-eq-less-or-eq*: $m \leq n \longleftrightarrow m < n \vee m = n$
for m n :: *nat*
⟨*proof*⟩

Useful with *blast*.

lemma *eq-imp-le*: $m = n \Longrightarrow m \leq n$
for m n :: *nat*
⟨*proof*⟩

lemma *le-refl*: $n \leq n$
for n :: *nat*
⟨*proof*⟩

lemma *le-trans*: $i \leq j \Longrightarrow j \leq k \Longrightarrow i \leq k$
for i j k :: *nat*
⟨*proof*⟩

lemma *le-antisym*: $m \leq n \Longrightarrow n \leq m \Longrightarrow m = n$
for m n :: *nat*
⟨*proof*⟩

lemma *nat-less-le*: $m < n \longleftrightarrow m \leq n \wedge m \neq n$
for m n :: *nat*
⟨*proof*⟩

lemma *le-neq-implies-less*: $m \leq n \Longrightarrow m \neq n \Longrightarrow m < n$
for m n :: *nat*
⟨*proof*⟩

lemma *nat-le-linear*: $m \leq n \vee n \leq m$
for m n :: *nat*
⟨*proof*⟩

lemmas *linorder-neqE-nat* = *linorder-neqE* [**where** *'a* = *nat*]

lemma *le-less-Suc-eq*: $m \leq n \Longrightarrow n < Suc\ m \longleftrightarrow n = m$
⟨*proof*⟩

lemma *not-less-less-Suc-eq*: $\neg\ n < m \Longrightarrow n < Suc\ m \longleftrightarrow n = m$
⟨*proof*⟩

lemmas *not-less-simps* = *not-less-less-Suc-eq le-less-Suc-eq*

lemma *not0-implies-Suc*: $n \neq 0 \Longrightarrow \exists\, m.\ n = Suc\ m$
⟨*proof*⟩

lemma *gr0-implies-Suc*: $n > 0 \Longrightarrow \exists\, m.\ n = Suc\ m$
⟨*proof*⟩

lemma *gr-implies-not0*: $m < n \Longrightarrow n \neq 0$
for $m\ n :: nat$
⟨*proof*⟩

lemma *neq0-conv*[*iff*]: $n \neq 0 \longleftrightarrow 0 < n$
for $n :: nat$
⟨*proof*⟩

This theorem is useful with *blast*

lemma *gr0I*: $(n = 0 \Longrightarrow False) \Longrightarrow 0 < n$
for $n :: nat$
⟨*proof*⟩

lemma *gr0-conv-Suc*: $0 < n \longleftrightarrow (\exists\, m.\ n = Suc\ m)$
⟨*proof*⟩

lemma *not-gr0* [*iff*]: $\neg\ 0 < n \longleftrightarrow n = 0$
for $n :: nat$
⟨*proof*⟩

lemma *Suc-le-D*: $Suc\ n \leq m' \Longrightarrow \exists\, m.\ m' = Suc\ m$
⟨*proof*⟩

Useful in certain inductive arguments

lemma *less-Suc-eq-0-disj*: $m < Suc\ n \longleftrightarrow m = 0 \vee (\exists\, j.\ m = Suc\ j \wedge j < n)$
⟨*proof*⟩

lemma *All-less-Suc*: $(\forall\, i < Suc\ n.\ P\ i) = (P\ n \wedge (\forall\, i < n.\ P\ i))$
⟨*proof*⟩

lemma *All-less-Suc2*: $(\forall\, i < Suc\ n.\ P\ i) = (P\ 0 \wedge (\forall\, i < n.\ P(Suc\ i)))$
⟨*proof*⟩

lemma *Ex-less-Suc*: $(\exists\, i < Suc\ n.\ P\ i) = (P\ n \vee (\exists\, i < n.\ P\ i))$
⟨*proof*⟩

lemma *Ex-less-Suc2*: $(\exists\, i < Suc\ n.\ P\ i) = (P\ 0 \vee (\exists\, i < n.\ P(Suc\ i)))$
⟨*proof*⟩

mono (non-strict) doesn't imply increasing, as the function could be constant

lemma *strict-mono-imp-increasing*:
fixes n::nat

assumes *strict-mono f* **shows** $f\ n \geq n$
⟨*proof*⟩

17.4.5 Monotonicity of Addition

lemma *Suc-pred* [*simp*]: $n > 0 \Longrightarrow Suc\ (n - Suc\ 0) = n$
⟨*proof*⟩

lemma *Suc-diff-1* [*simp*]: $0 < n \Longrightarrow Suc\ (n - 1) = n$
⟨*proof*⟩

lemma *nat-add-left-cancel-le* [*simp*]: $k + m \leq k + n \longleftrightarrow m \leq n$
for $k\ m\ n :: nat$
⟨*proof*⟩

lemma *nat-add-left-cancel-less* [*simp*]: $k + m < k + n \longleftrightarrow m < n$
for $k\ m\ n :: nat$
⟨*proof*⟩

lemma *add-gr-0* [*iff*]: $m + n > 0 \longleftrightarrow m > 0 \vee n > 0$
for $m\ n :: nat$
⟨*proof*⟩

strict, in 1st argument

lemma *add-less-mono1*: $i < j \Longrightarrow i + k < j + k$
for $i\ j\ k :: nat$
⟨*proof*⟩

strict, in both arguments

lemma *add-less-mono*:
fixes $i\ j\ k\ l :: nat$
assumes $i < j\ k < l$ **shows** $i + k < j + l$
⟨*proof*⟩

lemma *less-imp-Suc-add*: $m < n \Longrightarrow \exists k.\ n = Suc\ (m + k)$
⟨*proof*⟩

lemma *le-Suc-ex*: $k \leq l \Longrightarrow (\exists n.\ l = k + n)$
for $k\ l :: nat$
⟨*proof*⟩

lemma *less-natE*:
assumes ‹$m < n$›
obtains q **where** ‹$n = Suc\ (m + q)$›
⟨*proof*⟩

strict, in 1st argument; proof is by induction on $k > 0$

lemma *mult-less-mono2*:
fixes $i\ j :: nat$

assumes $i < j$ **and** $0 < k$
shows $k * i < k * j$
⟨*proof*⟩

Addition is the inverse of subtraction: if $n \leq m$ then $n + (m - n) = m$.

lemma *add-diff-inverse-nat*: $\neg\ m < n \Longrightarrow n + (m - n) = m$
for $m\ n$ :: *nat*
⟨*proof*⟩

lemma *nat-le-iff-add*: $m \leq n \longleftrightarrow (\exists k.\ n = m + k)$
for $m\ n$ :: *nat*
⟨*proof*⟩

The naturals form an ordered *semidom* and a *dioid*.

instance *nat* :: *linordered-semidom*
⟨*proof*⟩

instance *nat* :: *dioid*
⟨*proof*⟩

declare *le0*[*simp del*] — This is now $(0::?'a) \leq ?x$
declare *le-0-eq*[*simp del*] — This is now $(?n \leq (0::?'a)) = (?n = (0::?'a))$
declare *not-less0*[*simp del*] — This is now $\neg\ ?n < (0::?'a)$
declare *not-gr0*[*simp del*] — This is now $(\neg\ (0::?'a) < ?n) = (?n = (0::?'a))$

instance *nat* :: *ordered-cancel-comm-monoid-add* ⟨*proof*⟩
instance *nat* :: *ordered-cancel-comm-monoid-diff* ⟨*proof*⟩

17.4.6 *min* and *max*

global-interpretation *bot-nat-0*: *ordering-top* ‹(≥)› ‹(>)› ‹0::*nat*›
⟨*proof*⟩

global-interpretation *max-nat*: *semilattice-neutr-order max* ‹0::*nat*› ‹(≥)› ‹(>)›
⟨*proof*⟩

lemma *mono-Suc*: *mono Suc*
⟨*proof*⟩

lemma *min-0L* [*simp*]: *min 0 n = 0*
for n :: *nat*
⟨*proof*⟩

lemma *min-0R* [*simp*]: *min n 0 = 0*
for n :: *nat*
⟨*proof*⟩

lemma *min-Suc-Suc* [*simp*]: *min* (*Suc m*) (*Suc n*) = *Suc* (*min m n*)
⟨*proof*⟩

lemma *min-Suc1*: *min* (*Suc n*) *m* = (*case m of 0* ⇒ *0* | *Suc m′* ⇒ *Suc*(*min n m′*))
⟨*proof*⟩

lemma *min-Suc2*: *min m* (*Suc n*) = (*case m of 0* ⇒ *0* | *Suc m′* ⇒ *Suc*(*min m′ n*))
⟨*proof*⟩

lemma *max-0L* [*simp*]: *max 0 n* = *n*
for *n* :: *nat*
⟨*proof*⟩

lemma *max-0R* [*simp*]: *max n 0* = *n*
for *n* :: *nat*
⟨*proof*⟩

lemma *max-Suc-Suc* [*simp*]: *max* (*Suc m*) (*Suc n*) = *Suc* (*max m n*)
⟨*proof*⟩

lemma *max-Suc1*: *max* (*Suc n*) *m* = (*case m of 0* ⇒ *Suc n* | *Suc m′* ⇒ *Suc* (*max n m′*))
⟨*proof*⟩

lemma *max-Suc2*: *max m* (*Suc n*) = (*case m of 0* ⇒ *Suc n* | *Suc m′* ⇒ *Suc* (*max m′ n*))
⟨*proof*⟩

lemma *nat-mult-min-left*: *min m n* ∗ *q* = *min* (*m* ∗ *q*) (*n* ∗ *q*)
for *m n q* :: *nat*
⟨*proof*⟩

lemma *nat-mult-min-right*: *m* ∗ *min n q* = *min* (*m* ∗ *n*) (*m* ∗ *q*)
for *m n q* :: *nat*
⟨*proof*⟩

lemma *nat-add-max-left*: *max m n* + *q* = *max* (*m* + *q*) (*n* + *q*)
for *m n q* :: *nat*
⟨*proof*⟩

lemma *nat-add-max-right*: *m* + *max n q* = *max* (*m* + *n*) (*m* + *q*)
for *m n q* :: *nat*
⟨*proof*⟩

lemma *nat-mult-max-left*: *max m n* ∗ *q* = *max* (*m* ∗ *q*) (*n* ∗ *q*)
for *m n q* :: *nat*
⟨*proof*⟩

lemma *nat-mult-max-right*: *m* ∗ *max n q* = *max* (*m* ∗ *n*) (*m* ∗ *q*)

for m n q :: nat
⟨*proof*⟩

17.4.7 Additional theorems about (≤)

Complete induction, aka course-of-values induction

instance nat :: $wellorder$
⟨*proof*⟩

lemma *Least-eq-0*[*simp*]: $P\ 0 \Longrightarrow Least\ P = 0$
for P :: $nat \Rightarrow bool$
⟨*proof*⟩

lemma *Least-Suc*:
assumes $P\ n\ \neg\ P\ 0$
shows $(LEAST\ n.\ P\ n) = Suc\ (LEAST\ m.\ P\ (Suc\ m))$
⟨*proof*⟩

lemma *Least-Suc2*: $P\ n \Longrightarrow Q\ m \Longrightarrow \neg\ P\ 0 \Longrightarrow \forall k.\ P\ (Suc\ k) = Q\ k \Longrightarrow Least\ P = Suc\ (Least\ Q)$
⟨*proof*⟩

lemma *ex-least-nat-le*:
fixes P :: $nat \Rightarrow bool$
assumes $P\ n\ \neg\ P\ 0$
shows $\exists k{\leq}n.\ (\forall i{<}k.\ \neg\ P\ i) \wedge P\ k$
⟨*proof*⟩

lemma *ex-least-nat-less*:
fixes P :: $nat \Rightarrow bool$
assumes $P\ n\ \neg\ P\ 0$
shows $\exists k{<}n.\ (\forall i{\leq}k.\ \neg\ P\ i) \wedge P\ (Suc\ k)$
⟨*proof*⟩

lemma *nat-less-induct*:
fixes P :: $nat \Rightarrow bool$
assumes $\bigwedge n.\ \forall m.\ m < n \longrightarrow P\ m \Longrightarrow P\ n$
shows $P\ n$
⟨*proof*⟩

lemma *measure-induct-rule* [*case-names less*]:
fixes f :: $'a \Rightarrow 'b$::$wellorder$
assumes *step*: $\bigwedge x.\ (\bigwedge y.\ f\ y < f\ x \Longrightarrow P\ y) \Longrightarrow P\ x$
shows $P\ a$
⟨*proof*⟩

old style induction rules:

lemma *measure-induct*:
fixes $f :: 'a \Rightarrow 'b$::*wellorder*
shows $(\bigwedge x.\ \forall y.\ f\ y < f\ x \longrightarrow P\ y \Longrightarrow P\ x) \Longrightarrow P\ a$
⟨*proof*⟩

lemma *full-nat-induct*:
assumes *step*: $\bigwedge n.\ (\forall m.\ Suc\ m \leq n \longrightarrow P\ m) \Longrightarrow P\ n$
shows $P\ n$
⟨*proof*⟩

An induction rule for establishing binary relations

lemma *less-Suc-induct* [*consumes 1*]:
assumes *less*: $i < j$
and *step*: $\bigwedge i.\ P\ i\ (Suc\ i)$
and *trans*: $\bigwedge i\ j\ k.\ i < j \Longrightarrow j < k \Longrightarrow P\ i\ j \Longrightarrow P\ j\ k \Longrightarrow P\ i\ k$
shows $P\ i\ j$
⟨*proof*⟩

The method of infinite descent, frequently used in number theory. Provided by Roelof Oosterhuis. $P\ n$ is true for all natural numbers if

- case "0": given $n = 0$ prove $P\ n$
- case "smaller": given $n > 0$ and $\neg\ P\ n$ prove there exists a smaller natural number m such that $\neg\ P\ m$.

lemma *infinite-descent*: $(\bigwedge n.\ \neg\ P\ n \Longrightarrow \exists\, m{<}n.\ \neg\ P\ m) \Longrightarrow P\ n$ **for** $P :: nat \Rightarrow bool$
— compact version without explicit base case
⟨*proof*⟩

lemma *infinite-descent0* [*case-names 0 smaller*]:
fixes $P :: nat \Rightarrow bool$
assumes $P\ 0$
and $\bigwedge n.\ n > 0 \Longrightarrow \neg\ P\ n \Longrightarrow \exists\, m.\ m < n \wedge \neg\ P\ m$
shows $P\ n$
⟨*proof*⟩

Infinite descent using a mapping to *nat*: $P\ x$ is true for all $x \in D$ if there exists a $V \in D \Rightarrow nat$ and

- case "0": given $V\ x = 0$ prove $P\ x$
- "smaller": given $V\ x > 0$ and $\neg\ P\ x$ prove there exists a $y \in D$ such that $V\ y < V\ x$ and $\neg\ P\ y$.

corollary *infinite-descent0-measure* [*case-names 0 smaller*]:
fixes $V :: 'a \Rightarrow nat$

assumes *1*: $\bigwedge x.\ V\ x = 0 \Longrightarrow P\ x$
and *2*: $\bigwedge x.\ V\ x > 0 \Longrightarrow \neg\ P\ x \Longrightarrow \exists\, y.\ V\ y < V\ x \wedge \neg\ P\ y$
shows $P\ x$
⟨*proof*⟩

Again, without explicit base case:

lemma *infinite-descent-measure*:
fixes $V :: 'a \Rightarrow nat$
assumes $\bigwedge x.\ \neg\ P\ x \Longrightarrow \exists\, y.\ V\ y < V\ x \wedge \neg\ P\ y$
shows $P\ x$
⟨*proof*⟩

A (clumsy) way of lifting $<$ monotonicity to $\leq$ monotonicity

lemma *less-mono-imp-le-mono*:
fixes $f :: nat \Rightarrow nat$
and $i\ j :: nat$
assumes $\bigwedge i\ j{::}nat.\ i < j \Longrightarrow f\ i < f\ j$
and $i \leq j$
shows $f\ i \leq f\ j$
⟨*proof*⟩

non-strict, in 1st argument

lemma *add-le-mono1*: $i \leq j \Longrightarrow i + k \leq j + k$
for $i\ j\ k :: nat$
⟨*proof*⟩

non-strict, in both arguments

lemma *add-le-mono*: $i \leq j \Longrightarrow k \leq l \Longrightarrow i + k \leq j + l$
for $i\ j\ k\ l :: nat$
⟨*proof*⟩

lemma *le-add2*: $n \leq m + n$
for $m\ n :: nat$
⟨*proof*⟩

lemma *le-add1*: $n \leq n + m$
for $m\ n :: nat$
⟨*proof*⟩

lemma *less-add-Suc1*: $i < Suc\ (i + m)$
⟨*proof*⟩

lemma *less-add-Suc2*: $i < Suc\ (m + i)$
⟨*proof*⟩

lemma *less-iff-Suc-add*: $m < n \longleftrightarrow (\exists\, k.\ n = Suc\ (m + k))$
⟨*proof*⟩

lemma *trans-le-add1*: $i \leq j \Longrightarrow i \leq j + m$
for $i\ j\ m$ :: *nat*
⟨*proof*⟩

lemma *trans-le-add2*: $i \leq j \Longrightarrow i \leq m + j$
for $i\ j\ m$ :: *nat*
⟨*proof*⟩

lemma *trans-less-add1*: $i < j \Longrightarrow i < j + m$
for $i\ j\ m$ :: *nat*
⟨*proof*⟩

lemma *trans-less-add2*: $i < j \Longrightarrow i < m + j$
for $i\ j\ m$ :: *nat*
⟨*proof*⟩

lemma *add-lessD1*: $i + j < k \Longrightarrow i < k$
for $i\ j\ k$ :: *nat*
⟨*proof*⟩

lemma *not-add-less1* [*iff*]: $\neg\ i + j < i$
for $i\ j$ :: *nat*
⟨*proof*⟩

lemma *not-add-less2* [*iff*]: $\neg\ j + i < i$
for $i\ j$ :: *nat*
⟨*proof*⟩

lemma *add-leD1*: $m + k \leq n \Longrightarrow m \leq n$
for $k\ m\ n$ :: *nat*
⟨*proof*⟩

lemma *add-leD2*: $m + k \leq n \Longrightarrow k \leq n$
for $k\ m\ n$ :: *nat*
⟨*proof*⟩

lemma *add-leE*: $m + k \leq n \Longrightarrow (m \leq n \Longrightarrow k \leq n \Longrightarrow R) \Longrightarrow R$
for $k\ m\ n$ :: *nat*
⟨*proof*⟩

needs $\bigwedge k$ for *ac-simps* to work

lemma *less-add-eq-less*: $\bigwedge k.\ k < l \Longrightarrow m + l = k + n \Longrightarrow m < n$
for $l\ m\ n$ :: *nat*
⟨*proof*⟩

17.4.8 More results about difference

lemma *Suc-diff-le*: $n \leq m \Longrightarrow Suc\ m - n = Suc\ (m - n)$
⟨*proof*⟩

lemma *diff-less-Suc*: $m - n < Suc\ m$
⟨*proof*⟩

lemma *diff-le-self* [*simp*]: $m - n \leq m$
for $m\ n :: nat$
⟨*proof*⟩

lemma *less-imp-diff-less*: $j < k \Longrightarrow j - n < k$
for $j\ k\ n :: nat$
⟨*proof*⟩

lemma *diff-Suc-less* [*simp*]: $0 < n \Longrightarrow n - Suc\ i < n$
⟨*proof*⟩

lemma *diff-add-assoc*: $k \leq j \Longrightarrow (i + j) - k = i + (j - k)$
for $i\ j\ k :: nat$
⟨*proof*⟩

lemma *add-diff-assoc* [*simp*]: $k \leq j \Longrightarrow i + (j - k) = i + j - k$
for $i\ j\ k :: nat$
⟨*proof*⟩

lemma *diff-add-assoc2*: $k \leq j \Longrightarrow (j + i) - k = (j - k) + i$
for $i\ j\ k :: nat$
⟨*proof*⟩

lemma *add-diff-assoc2* [*simp*]: $k \leq j \Longrightarrow j - k + i = j + i - k$
for $i\ j\ k :: nat$
⟨*proof*⟩

lemma *le-imp-diff-is-add*: $i \leq j \Longrightarrow (j - i = k) = (j = k + i)$
for $i\ j\ k :: nat$
⟨*proof*⟩

lemma *diff-is-0-eq* [*simp*]: $m - n = 0 \longleftrightarrow m \leq n$
for $m\ n :: nat$
⟨*proof*⟩

lemma *diff-is-0-eq'* [*simp*]: $m \leq n \Longrightarrow m - n = 0$
for $m\ n :: nat$
⟨*proof*⟩

lemma *zero-less-diff* [*simp*]: $0 < n - m \longleftrightarrow m < n$
for $m\ n :: nat$
⟨*proof*⟩

lemma *less-imp-add-positive*:
assumes $i < j$

shows $\exists k{::}nat.\ 0 < k \wedge i + k = j$
⟨*proof*⟩

a nice rewrite for bounded subtraction

lemma *nat-minus-add-max*: $n - m + m = max\ n\ m$
for $m\ n :: nat$
⟨*proof*⟩

lemma *nat-diff-split*: $P\ (a - b) \longleftrightarrow (a < b \longrightarrow P\ 0) \wedge (\forall d.\ a = b + d \longrightarrow P\ d)$
for $a\ b :: nat$
— elimination of $-$ on *nat*
⟨*proof*⟩

lemma *nat-diff-split-asm*: $P\ (a - b) \longleftrightarrow \neg\ (a < b \wedge \neg\ P\ 0 \vee (\exists d.\ a = b + d \wedge \neg\ P\ d))$
for $a\ b :: nat$
— elimination of $-$ on *nat* in assumptions
⟨*proof*⟩

lemma *Suc-pred'*: $0 < n \Longrightarrow n = Suc(n - 1)$
⟨*proof*⟩

lemma *add-eq-if*: $m + n = (if\ m = 0\ then\ n\ else\ Suc\ ((m - 1) + n))$
⟨*proof*⟩

lemma *mult-eq-if*: $m * n = (if\ m = 0\ then\ 0\ else\ n + ((m - 1) * n))$
for $m\ n :: nat$
⟨*proof*⟩

lemma *Suc-diff-eq-diff-pred*: $0 < n \Longrightarrow Suc\ m - n = m - (n - 1)$
⟨*proof*⟩

lemma *diff-Suc-eq-diff-pred*: $m - Suc\ n = (m - 1) - n$
⟨*proof*⟩

lemma *Let-Suc* [*simp*]: $Let\ (Suc\ n)\ f \equiv f\ (Suc\ n)$
⟨*proof*⟩

17.4.9 Monotonicity of multiplication

lemma *mult-le-mono1*: $i \leq j \Longrightarrow i * k \leq j * k$
for $i\ j\ k :: nat$
⟨*proof*⟩

lemma *mult-le-mono2*: $i \leq j \Longrightarrow k * i \leq k * j$
for $i\ j\ k :: nat$
⟨*proof*⟩

$\leq$ monotonicity, BOTH arguments

lemma *mult-le-mono*: $i \leq j \Longrightarrow k \leq l \Longrightarrow i * k \leq j * l$
for $i\ j\ k\ l$ :: *nat*
⟨*proof*⟩

lemma *mult-less-mono1*: $i < j \Longrightarrow 0 < k \Longrightarrow i * k < j * k$
for $i\ j\ k$ :: *nat*
⟨*proof*⟩

Differs from the standard *zero-less-mult-iff* in that there are no negative numbers.

lemma *nat-0-less-mult-iff* [*simp*]: $0 < m * n \longleftrightarrow 0 < m \wedge 0 < n$
for $m\ n$ :: *nat*
⟨*proof*⟩

lemma *one-le-mult-iff* [*simp*]: $Suc\ 0 \leq m * n \longleftrightarrow Suc\ 0 \leq m \wedge Suc\ 0 \leq n$
⟨*proof*⟩

lemma *mult-less-cancel2* [*simp*]: $m * k < n * k \longleftrightarrow 0 < k \wedge m < n$
for $k\ m\ n$ :: *nat*
⟨*proof*⟩

lemma *mult-less-cancel1* [*simp*]: $k * m < k * n \longleftrightarrow 0 < k \wedge m < n$
for $k\ m\ n$ :: *nat*
⟨*proof*⟩

lemma *mult-le-cancel1* [*simp*]: $k * m \leq k * n \longleftrightarrow (0 < k \longrightarrow m \leq n)$
for $k\ m\ n$ :: *nat*
⟨*proof*⟩

lemma *mult-le-cancel2* [*simp*]: $m * k \leq n * k \longleftrightarrow (0 < k \longrightarrow m \leq n)$
for $k\ m\ n$ :: *nat*
⟨*proof*⟩

lemma *Suc-mult-less-cancel1*: $Suc\ k * m < Suc\ k * n \longleftrightarrow m < n$
⟨*proof*⟩

lemma *Suc-mult-le-cancel1*: $Suc\ k * m \leq Suc\ k * n \longleftrightarrow m \leq n$
⟨*proof*⟩

lemma *le-square*: $m \leq m * m$
for m :: *nat*
⟨*proof*⟩

lemma *le-cube*: $m \leq m * (m * m)$
for m :: *nat*
⟨*proof*⟩

Lemma for *gcd*

lemma *mult-eq-self-implies-10*:

fixes $m\ n :: nat$
assumes $m = m * n$ **shows** $n = 1 \vee m = 0$
⟨*proof*⟩

lemma *mono-times-nat*:
fixes $n :: nat$
assumes $n > 0$
shows *mono* (*times n*)
⟨*proof*⟩

The lattice order on *nat*.

instantiation *nat* :: *distrib-lattice*
begin

definition (*inf* :: *nat* ⇒ *nat* ⇒ *nat*) = *min*

definition (*sup* :: *nat* ⇒ *nat* ⇒ *nat*) = *max*

instance
⟨*proof*⟩

end

17.5 Natural operation of natural numbers on functions

We use the same logical constant for the power operations on functions and relations, in order to share the same syntax.

consts *compow* :: *nat* ⇒ *'a* ⇒ *'a*

abbreviation *compower* :: *'a* ⇒ *nat* ⇒ *'a* (**infixr** ^^ *80*)
where $f \hat{\ }\hat{\ } n \equiv compow\ n\ f$

notation (*latex* **output**)
compower ((-[^]) [*1000*] *1000*)

$f \hat{\ }\hat{\ } n = f \circ \ldots \circ f$, the n-fold composition of f

overloading
funpow ≡ *compow* :: *nat* ⇒ (*'a* ⇒ *'a*) ⇒ (*'a* ⇒ *'a*)
begin

primrec *funpow* :: *nat* ⇒ (*'a* ⇒ *'a*) ⇒ *'a* ⇒ *'a*
where
funpow 0 f = *id*
| *funpow* (*Suc n*) *f* = *f* ∘ *funpow n f*

end

lemma *funpow-0* [*simp*]: $(f \hat{\ }\hat{\ } 0)\ x = x$

⟨*proof*⟩

lemma *funpow-Suc-right*: $f \mathbin{\frown\frown} Suc\ n = f \mathbin{\frown\frown} n \circ f$
⟨*proof*⟩

lemmas *funpow-simps-right* = *funpow.simps*(*1*) *funpow-Suc-right*

For code generation.

context
begin

qualified definition *funpow* :: *nat* ⇒ ('*a* ⇒ '*a*) ⇒ '*a* ⇒ '*a*
  where *funpow-code-def* [*code-abbrev*]: *funpow* = *compow*

lemma [*code*]:
  funpow (*Suc n*) *f* = *f* ∘ *funpow n f*
  funpow 0 f = *id*
  ⟨*proof*⟩

end

lemma *funpow-add*: $f \mathbin{\frown\frown} (m + n) = f \mathbin{\frown\frown} m \circ f \mathbin{\frown\frown} n$
  ⟨*proof*⟩

lemma *funpow-mult*: $(f \mathbin{\frown\frown} m) \mathbin{\frown\frown} n = f \mathbin{\frown\frown} (m * n)$
  for *f* :: '*a* ⇒ '*a*
  ⟨*proof*⟩

lemma *funpow-swap1*: $f\ ((f \mathbin{\frown\frown} n)\ x) = (f \mathbin{\frown\frown} n)\ (f\ x)$
⟨*proof*⟩

lemma *comp-funpow*: $comp\ f \mathbin{\frown\frown} n = comp\ (f \mathbin{\frown\frown} n)$
  for *f* :: '*a* ⇒ '*a*
  ⟨*proof*⟩

lemma *Suc-funpow*[*simp*]: $Suc \mathbin{\frown\frown} n = ((+)\ n)$
  ⟨*proof*⟩

lemma *id-funpow*[*simp*]: $id \mathbin{\frown\frown} n = id$
  ⟨*proof*⟩

lemma *funpow-mono*: *mono f* ⟹ *A* ≤ *B* ⟹ $(f \mathbin{\frown\frown} n)\ A \leq (f \mathbin{\frown\frown} n)\ B$
  for *f* :: '*a* ⇒ ('*a*::*order*)
  ⟨*proof*⟩

lemma *funpow-mono2*:
  assumes *mono f*
    and *i* ≤ *j*
    and *x* ≤ *y*

and $x \leq f\ x$
shows $(f \frown\frown i)\ x \leq (f \frown\frown j)\ y$
⟨*proof*⟩

lemma *inj-fn*[*simp*]:
fixes $f::'a \Rightarrow 'a$
assumes *inj f*
shows *inj* $(f\frown\frown n)$
⟨*proof*⟩

lemma *surj-fn*[*simp*]:
fixes $f::'a \Rightarrow 'a$
assumes *surj f*
shows *surj* $(f\frown\frown n)$
⟨*proof*⟩

lemma *bij-fn*[*simp*]:
fixes $f::'a \Rightarrow 'a$
assumes *bij f*
shows *bij* $(f\frown\frown n)$
⟨*proof*⟩

lemma *bij-betw-funpow*:
assumes *bij-betw f S S* **shows** *bij-betw* $(f \frown\frown n)$ *S S*
⟨*proof*⟩

17.6 Kleene iteration

lemma *Kleene-iter-lpfp*:
fixes $f :: 'a::order\text{-}bot \Rightarrow 'a$
assumes *mono f*
and $f\ p \leq p$
shows $(f \frown\frown k)\ bot \leq p$
⟨*proof*⟩

lemma *lfp-Kleene-iter*:
assumes *mono f*
and $(f \frown\frown Suc\ k)\ bot = (f \frown\frown k)\ bot$
shows $lfp\ f = (f \frown\frown k)\ bot$
⟨*proof*⟩

lemma *mono-pow*: $mono\ f \Longrightarrow mono\ (f \frown\frown n)$
for $f :: 'a \Rightarrow 'a::complete\text{-}lattice$
⟨*proof*⟩

lemma *lfp-funpow*:
assumes *f*: *mono f*
shows $lfp\ (f \frown\frown Suc\ n) = lfp\ f$
⟨*proof*⟩

lemma *gfp-funpow*:
assumes *f*: *mono f*
shows *gfp* (*f* ^^ *Suc n*) = *gfp f*
⟨*proof*⟩

lemma *Kleene-iter-gpfp*:
fixes *f* :: *'a*::*order-top* ⇒ *'a*
assumes *mono f*
and *p* ≤ *f p*
shows *p* ≤ (*f* ^^ *k*) *top*
⟨*proof*⟩

lemma *gfp-Kleene-iter*:
assumes *mono f*
and (*f* ^^ *Suc k*) *top* = (*f* ^^ *k*) *top*
shows *gfp f* = (*f* ^^ *k*) *top*
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

17.7 Embedding of the naturals into any *semiring-1*: *of-nat*

context *semiring-1*
begin

definition *of-nat* :: *nat* ⇒ *'a*
where *of-nat n* = (*plus 1* ^^ *n*) *0*

lemma *of-nat-simps* [*simp*]:
shows *of-nat-0*: *of-nat 0* = *0*
and *of-nat-Suc*: *of-nat* (*Suc m*) = *1* + *of-nat m*
⟨*proof*⟩

lemma *of-nat-1* [*simp*]: *of-nat 1* = *1*
⟨*proof*⟩

lemma *of-nat-add* [*simp*]: *of-nat* (*m* + *n*) = *of-nat m* + *of-nat n*
⟨*proof*⟩

lemma *of-nat-mult* [*simp*]: *of-nat* (*m* ∗ *n*) = *of-nat m* ∗ *of-nat n*
⟨*proof*⟩

lemma *mult-of-nat-commute*: *of-nat x* ∗ *y* = *y* ∗ *of-nat x*
⟨*proof*⟩

primrec *of-nat-aux* :: (*'a* ⇒ *'a*) ⇒ *nat* ⇒ *'a* ⇒ *'a*
where
of-nat-aux inc 0 i = *i*
| *of-nat-aux inc* (*Suc n*) *i* = *of-nat-aux inc n* (*inc i*) — tail recursive

lemma *of-nat-code*: *of-nat n = of-nat-aux (λi. i + 1) n 0*
⟨*proof*⟩

lemma *of-nat-of-bool* [*simp*]:
of-nat (of-bool P) = of-bool P
⟨*proof*⟩

end

declare *of-nat-code* [*code*]

context *semiring-1-cancel*
begin

lemma *of-nat-diff*:
‹*of-nat (m − n) = of-nat m − of-nat n*› **if** ‹*n ≤ m*›
⟨*proof*⟩

end

Class for unital semirings with characteristic zero. Includes non-ordered rings like the complex numbers.

class *semiring-char-0 = semiring-1 +*
assumes *inj-of-nat*: *inj of-nat*
begin

lemma *of-nat-eq-iff* [*simp*]: *of-nat m = of-nat n ⟷ m = n*
⟨*proof*⟩

Special cases where either operand is zero

lemma *of-nat-0-eq-iff* [*simp*]: *0 = of-nat n ⟷ 0 = n*
⟨*proof*⟩

lemma *of-nat-eq-0-iff* [*simp*]: *of-nat m = 0 ⟷ m = 0*
⟨*proof*⟩

lemma *of-nat-1-eq-iff* [*simp*]: *1 = of-nat n ⟷ n=1*
⟨*proof*⟩

lemma *of-nat-eq-1-iff* [*simp*]: *of-nat n = 1 ⟷ n=1*
⟨*proof*⟩

lemma *of-nat-neq-0* [*simp*]: *of-nat (Suc n) ≠ 0*
⟨*proof*⟩

lemma *of-nat-0-neq* [*simp*]: *0 ≠ of-nat (Suc n)*
⟨*proof*⟩

end

class *ring-char-0* = *ring-1* + *semiring-char-0*

context *linordered-nonzero-semiring*
begin

lemma *of-nat-0-le-iff* [*simp*]: $0 \leq$ *of-nat* n
⟨*proof*⟩

lemma *of-nat-less-0-iff* [*simp*]: ¬ *of-nat* $m < 0$
⟨*proof*⟩

lemma *of-nat-mono*[*simp*]: $i \leq j \Longrightarrow$ *of-nat* $i \leq$ *of-nat* j
⟨*proof*⟩

lemma *of-nat-less-iff* [*simp*]: *of-nat* $m <$ *of-nat* $n \longleftrightarrow m < n$
⟨*proof*⟩

lemma *of-nat-le-iff* [*simp*]: *of-nat* $m \leq$ *of-nat* $n \longleftrightarrow m \leq n$
⟨*proof*⟩

lemma *less-imp-of-nat-less*: $m < n \Longrightarrow$ *of-nat* $m <$ *of-nat* n
⟨*proof*⟩

lemma *of-nat-less-imp-less*: *of-nat* $m <$ *of-nat* $n \Longrightarrow m < n$
⟨*proof*⟩

Every *linordered-nonzero-semiring* has characteristic zero.

subclass *semiring-char-0*
⟨*proof*⟩

Special cases where either operand is zero

lemma *of-nat-le-0-iff* [*simp*]: *of-nat* $m \leq 0 \longleftrightarrow m = 0$
⟨*proof*⟩

lemma *of-nat-0-less-iff* [*simp*]: $0 <$ *of-nat* $n \longleftrightarrow 0 < n$
⟨*proof*⟩

end

context *linordered-nonzero-semiring*
begin

lemma *of-nat-max*: *of-nat* (*max* x y) = *max* (*of-nat* x) (*of-nat* y)
⟨*proof*⟩

lemma *of-nat-min*: *of-nat* (*min* x y) = *min* (*of-nat* x) (*of-nat* y)
⟨*proof*⟩

end

context *linordered-semidom*
begin

subclass *linordered-nonzero-semiring* ⟨*proof*⟩

subclass *semiring-char-0* ⟨*proof*⟩

end

context *linordered-idom*
begin

lemma *abs-of-nat* [*simp*]:
|*of-nat n*| = *of-nat n*
⟨*proof*⟩

lemma *sgn-of-nat* [*simp*]:
sgn (*of-nat n*) = *of-bool* (*n* > *0*)
⟨*proof*⟩

end

lemma *of-nat-id* [*simp*]: *of-nat n* = *n*
⟨*proof*⟩

lemma *of-nat-eq-id* [*simp*]: *of-nat* = *id*
⟨*proof*⟩

17.8 The set of natural numbers

context *semiring-1*
begin

definition *Nats* :: *'a set* (ℕ)
where ℕ = *range of-nat*

lemma *of-nat-in-Nats* [*simp*]: *of-nat n* ∈ ℕ
⟨*proof*⟩

lemma *Nats-0* [*simp*]: *0* ∈ ℕ
⟨*proof*⟩

lemma *Nats-1* [*simp*]: *1* ∈ ℕ
⟨*proof*⟩

lemma *Nats-add* [*simp*]: *a* ∈ ℕ ⟹ *b* ∈ ℕ ⟹ *a* + *b* ∈ ℕ

⟨*proof*⟩

lemma *Nats-mult* [*simp*]: $a \in \mathbb{N} \Longrightarrow b \in \mathbb{N} \Longrightarrow a * b \in \mathbb{N}$
⟨*proof*⟩

lemma *Nats-cases* [*cases set*: *Nats*]:
assumes $x \in \mathbb{N}$
obtains (*of-nat*) n **where** $x = \mathit{of\text{-}nat}\ n$
⟨*proof*⟩

lemma *Nats-induct* [*case-names of-nat*, *induct set*: *Nats*]: $x \in \mathbb{N} \Longrightarrow (\bigwedge n.\ P\ (\mathit{of\text{-}nat}\ n)) \Longrightarrow P\ x$
⟨*proof*⟩

end

lemma *Nats-diff* [*simp*]:
fixes a:: $'a$::*linordered-idom*
assumes $a \in \mathbb{N}$ $b \in \mathbb{N}$ $b \leq a$ **shows** $a - b \in \mathbb{N}$
⟨*proof*⟩

17.9 Further arithmetic facts concerning the natural numbers

lemma *subst-equals*:
assumes $t = s$ **and** $u = t$
shows $u = s$
⟨*proof*⟩

locale *nat-arith*
begin

lemma *add1*: $(A$::$'a$::*comm-monoid-add*$) \equiv k + a \Longrightarrow A + b \equiv k + (a + b)$
⟨*proof*⟩

lemma *add2*: $(B$::$'a$::*comm-monoid-add*$) \equiv k + b \Longrightarrow a + B \equiv k + (a + b)$
⟨*proof*⟩

lemma *suc1*: $A == k + a \Longrightarrow \mathit{Suc}\ A \equiv k + \mathit{Suc}\ a$
⟨*proof*⟩

lemma *rule0*: $(a$::$'a$::*comm-monoid-add*$) \equiv a + 0$
⟨*proof*⟩

end

⟨*ML*⟩

context *order*

begin

lemma *lift-Suc-mono-le*:
assumes *mono*: $\bigwedge n.\ f\ n \leq f\ (Suc\ n)$
and $n \leq n'$
shows $f\ n \leq f\ n'$
⟨*proof*⟩

lemma *lift-Suc-antimono-le*:
assumes *mono*: $\bigwedge n.\ f\ n \geq f\ (Suc\ n)$
and $n \leq n'$
shows $f\ n \geq f\ n'$
⟨*proof*⟩

lemma *lift-Suc-mono-less*:
assumes *mono*: $\bigwedge n.\ f\ n < f\ (Suc\ n)$
and $n < n'$
shows $f\ n < f\ n'$
⟨*proof*⟩

lemma *lift-Suc-mono-less-iff*: $(\bigwedge n.\ f\ n < f\ (Suc\ n)) \Longrightarrow f\ n < f\ m \longleftrightarrow n < m$
⟨*proof*⟩

end

lemma *mono-iff-le-Suc*: $mono\ f \longleftrightarrow (\forall\, n.\ f\ n \leq f\ (Suc\ n))$
⟨*proof*⟩

lemma *antimono-iff-le-Suc*: $antimono\ f \longleftrightarrow (\forall\, n.\ f\ (Suc\ n) \leq f\ n)$
⟨*proof*⟩

lemma *strict-mono-Suc-iff*: $strict\text{-}mono\ f \longleftrightarrow (\forall\, n.\ f\ n < f\ (Suc\ n))$
⟨*proof*⟩

lemma *strict-mono-add*: $strict\text{-}mono\ (\lambda n{::}'a{::}linordered\text{-}semidom.\ n + k)$
⟨*proof*⟩

lemma *mono-nat-linear-lb*:
fixes $f :: nat \Rightarrow nat$
assumes $\bigwedge m\ n.\ m < n \Longrightarrow f\ m < f\ n$
shows $f\ m + k \leq f\ (m + k)$
⟨*proof*⟩

Subtraction laws, mostly by Clemens Ballarin

lemma *diff-less-mono*:
fixes $a\ b\ c :: nat$
assumes $a < b$ **and** $c \leq a$
shows $a - c < b - c$
⟨*proof*⟩

lemma *less-diff-conv*: $i < j - k \longleftrightarrow i + k < j$
for $i\ j\ k$ :: *nat*
⟨*proof*⟩

lemma *less-diff-conv2*: $k \leq j \Longrightarrow j - k < i \longleftrightarrow j < i + k$
for $j\ k\ i$ :: *nat*
⟨*proof*⟩

lemma *le-diff-conv*: $j - k \leq i \longleftrightarrow j \leq i + k$
for $j\ k\ i$ :: *nat*
⟨*proof*⟩

lemma *diff-diff-cancel* [*simp*]: $i \leq n \Longrightarrow n - (n - i) = i$
for $i\ n$ :: *nat*
⟨*proof*⟩

lemma *diff-less* [*simp*]: $0 < n \Longrightarrow 0 < m \Longrightarrow m - n < m$
for $i\ n$ :: *nat*
⟨*proof*⟩

Simplification of relational expressions involving subtraction

lemma *diff-diff-eq*: $k \leq m \Longrightarrow k \leq n \Longrightarrow m - k - (n - k) = m - n$
for $m\ n\ k$ :: *nat*
⟨*proof*⟩

hide-fact (**open**) *diff-diff-eq*

lemma *eq-diff-iff*: $k \leq m \Longrightarrow k \leq n \Longrightarrow m - k = n - k \longleftrightarrow m = n$
for $m\ n\ k$ :: *nat*
⟨*proof*⟩

lemma *less-diff-iff*: $k \leq m \Longrightarrow k \leq n \Longrightarrow m - k < n - k \longleftrightarrow m < n$
for $m\ n\ k$ :: *nat*
⟨*proof*⟩

lemma *le-diff-iff*: $k \leq m \Longrightarrow k \leq n \Longrightarrow m - k \leq n - k \longleftrightarrow m \leq n$
for $m\ n\ k$ :: *nat*
⟨*proof*⟩

lemma *le-diff-iff'*: $a \leq c \Longrightarrow b \leq c \Longrightarrow c - a \leq c - b \longleftrightarrow b \leq a$
for $a\ b\ c$ :: *nat*
⟨*proof*⟩

(Anti)Monotonicity of subtraction – by Stephan Merz

lemma *diff-le-mono*: $m \leq n \Longrightarrow m - l \leq n - l$
for $m\ n\ l$ :: *nat*
⟨*proof*⟩

lemma *diff-le-mono2*: $m \leq n \Longrightarrow l - n \leq l - m$
for $m\ n\ l :: nat$
⟨*proof*⟩

lemma *diff-less-mono2*: $m < n \Longrightarrow m < l \Longrightarrow l - n < l - m$
for $m\ n\ l :: nat$
⟨*proof*⟩

lemma *diffs0-imp-equal*: $m - n = 0 \Longrightarrow n - m = 0 \Longrightarrow m = n$
for $m\ n :: nat$
⟨*proof*⟩

lemma *min-diff*: $min\ (m - i)\ (n - i) = min\ m\ n - i$
for $m\ n\ i :: nat$
⟨*proof*⟩

lemma *inj-on-diff-nat*:
fixes $k :: nat$
assumes $\bigwedge n.\ n \in N \Longrightarrow k \leq n$
shows $inj\text{-}on\ (\lambda n.\ n - k)\ N$
⟨*proof*⟩

Rewriting to pull differences out

lemma *diff-diff-right* [*simp*]: $k \leq j \Longrightarrow i - (j - k) = i + k - j$
for $i\ j\ k :: nat$
⟨*proof*⟩

lemma *diff-Suc-diff-eq1* [*simp*]:
assumes $k \leq j$
shows $i - Suc\ (j - k) = i + k - Suc\ j$
⟨*proof*⟩

lemma *diff-Suc-diff-eq2* [*simp*]:
assumes $k \leq j$
shows $Suc\ (j - k) - i = Suc\ j - (k + i)$
⟨*proof*⟩

lemma *Suc-diff-Suc*:
assumes $n < m$
shows $Suc\ (m - Suc\ n) = m - n$
⟨*proof*⟩

lemma *one-less-mult*: $Suc\ 0 < n \Longrightarrow Suc\ 0 < m \Longrightarrow Suc\ 0 < m * n$
⟨*proof*⟩

lemma *n-less-m-mult-n*: $0 < n \Longrightarrow Suc\ 0 < m \Longrightarrow n < m * n$
⟨*proof*⟩

lemma *n-less-n-mult-m*: $0 < n \Longrightarrow Suc\ 0 < m \Longrightarrow n < n * m$

⟨*proof*⟩

Induction starting beyond zero

lemma *nat-induct-at-least* [*consumes 1*, *case-names base Suc*]:
P n **if** $n \geq m$ *P m* $\bigwedge n.\ n \geq m \Longrightarrow P\ n \Longrightarrow P\ (Suc\ n)$
⟨*proof*⟩

lemma *nat-induct-non-zero* [*consumes 1*, *case-names 1 Suc*]:
P n **if** $n > 0$ *P 1* $\bigwedge n.\ n > 0 \Longrightarrow P\ n \Longrightarrow P\ (Suc\ n)$
⟨*proof*⟩

Specialized induction principles that work "backwards":

lemma *inc-induct* [*consumes 1*, *case-names base step*]:
assumes *less*: $i \leq j$
and *base*: *P j*
and *step*: $\bigwedge n.\ i \leq n \Longrightarrow n < j \Longrightarrow P\ (Suc\ n) \Longrightarrow P\ n$
shows *P i*
⟨*proof*⟩

lemma *strict-inc-induct* [*consumes 1*, *case-names base step*]:
assumes *less*: $i < j$
and *base*: $\bigwedge i.\ j = Suc\ i \Longrightarrow P\ i$
and *step*: $\bigwedge i.\ i < j \Longrightarrow P\ (Suc\ i) \Longrightarrow P\ i$
shows *P i*
⟨*proof*⟩

lemma *zero-induct-lemma*: $P\ k \Longrightarrow (\bigwedge n.\ P\ (Suc\ n) \Longrightarrow P\ n) \Longrightarrow P\ (k - i)$
⟨*proof*⟩

lemma *zero-induct*: $P\ k \Longrightarrow (\bigwedge n.\ P\ (Suc\ n) \Longrightarrow P\ n) \Longrightarrow P\ 0$
⟨*proof*⟩

Further induction rule similar to $⟦?i \leq ?j;\ ?P\ ?j;\ \bigwedge n.\ ⟦?i \leq n;\ n < ?j;\ ?P\ (Suc\ n)⟧ \Longrightarrow ?P\ n⟧ \Longrightarrow ?P\ ?i$.

lemma *dec-induct* [*consumes 1*, *case-names base step*]:
$i \leq j \Longrightarrow P\ i \Longrightarrow (\bigwedge n.\ i \leq n \Longrightarrow n < j \Longrightarrow P\ n \Longrightarrow P\ (Suc\ n)) \Longrightarrow P\ j$
⟨*proof*⟩

lemma *transitive-stepwise-le*:
assumes $m \leq n$ $\bigwedge x.\ R\ x\ x$ $\bigwedge x\ y\ z.\ R\ x\ y \Longrightarrow R\ y\ z \Longrightarrow R\ x\ z$ **and** $\bigwedge n.\ R\ n\ (Suc\ n)$
shows *R m n*
⟨*proof*⟩

17.9.1 Greatest operator

lemma *ex-has-greatest-nat*:
$P\ (k::nat) \Longrightarrow \forall y.\ P\ y \longrightarrow y \leq b \Longrightarrow \exists x.\ P\ x \wedge (\forall y.\ P\ y \longrightarrow y \leq x)$
⟨*proof*⟩

lemma
fixes *k*::*nat*
assumes *P k* **and** *minor*: $\bigwedge y.\ P\ y \Longrightarrow y \leq b$
shows *GreatestI-nat*: *P* (*Greatest P*)
and *Greatest-le-nat*: $k \leq$ *Greatest P*
⟨*proof*⟩

lemma *GreatestI-ex-nat*:
$[\![\ \exists k{::}nat.\ P\ k;\ \ \bigwedge y.\ P\ y \Longrightarrow y \leq b\ ]\!] \Longrightarrow P\ (Greatest\ P)$
⟨*proof*⟩

17.10 Monotonicity of *funpow*

lemma *funpow-increasing*: $m \leq n \Longrightarrow mono\ f \Longrightarrow (f \frown n)\ \top \leq (f \frown m)\ \top$
for *f* :: *'a*::{*lattice*,*order-top*} ⇒ *'a*
⟨*proof*⟩

lemma *funpow-decreasing*: $m \leq n \Longrightarrow mono\ f \Longrightarrow (f \frown m)\ \bot \leq (f \frown n)\ \bot$
for *f* :: *'a*::{*lattice*,*order-bot*} ⇒ *'a*
⟨*proof*⟩

lemma *mono-funpow*: $mono\ Q \Longrightarrow mono\ (\lambda i.\ (Q \frown i)\ \bot)$
for *Q* :: *'a*::{*lattice*,*order-bot*} ⇒ *'a*
⟨*proof*⟩

lemma *antimono-funpow*: $mono\ Q \Longrightarrow antimono\ (\lambda i.\ (Q \frown i)\ \top)$
for *Q* :: *'a*::{*lattice*,*order-top*} ⇒ *'a*
⟨*proof*⟩

17.11 The divides relation on *nat*

lemma *dvd-1-left* [*iff*]: *Suc 0 dvd k*
⟨*proof*⟩

lemma *dvd-1-iff-1* [*simp*]: *m dvd Suc 0* ⟷ *m* = *Suc 0*
⟨*proof*⟩

lemma *nat-dvd-1-iff-1* [*simp*]: *m dvd 1* ⟷ *m* = *1*
for *m* :: *nat*
⟨*proof*⟩

lemma *dvd-antisym*: *m dvd n* ⟹ *n dvd m* ⟹ *m* = *n*
for *m n* :: *nat*
⟨*proof*⟩

lemma *dvd-diff-nat* [*simp*]: *k dvd m* ⟹ *k dvd n* ⟹ *k dvd* (*m* − *n*)
for *k m n* :: *nat*
⟨*proof*⟩

lemma *dvd-diffD*:
fixes $k\ m\ n :: nat$
assumes $k\ dvd\ m - n\ k\ dvd\ n\ n \leq m$
shows $k\ dvd\ m$
⟨*proof*⟩

lemma *dvd-diffD1*: $k\ dvd\ m - n \Longrightarrow k\ dvd\ m \Longrightarrow n \leq m \Longrightarrow k\ dvd\ n$
for $k\ m\ n :: nat$
⟨*proof*⟩

lemma *dvd-mult-cancel*:
fixes $m\ n\ k :: nat$
assumes $k * m\ dvd\ k * n$ **and** $0 < k$
shows $m\ dvd\ n$
⟨*proof*⟩

lemma *dvd-mult-cancel1*:
fixes $m\ n :: nat$
assumes $0 < m$
shows $m * n\ dvd\ m \longleftrightarrow n = 1$
⟨*proof*⟩

lemma *dvd-mult-cancel2*: $0 < m \Longrightarrow n * m\ dvd\ m \longleftrightarrow n = 1$
for $m\ n :: nat$
⟨*proof*⟩

lemma *dvd-imp-le*: $k\ dvd\ n \Longrightarrow 0 < n \Longrightarrow k \leq n$
for $k\ n :: nat$
⟨*proof*⟩

lemma *nat-dvd-not-less*: $0 < m \Longrightarrow m < n \Longrightarrow \neg\ n\ dvd\ m$
for $m\ n :: nat$
⟨*proof*⟩

lemma *less-eq-dvd-minus*:
fixes $m\ n :: nat$
assumes $m \leq n$
shows $m\ dvd\ n \longleftrightarrow m\ dvd\ n - m$
⟨*proof*⟩

lemma *dvd-minus-self*: $m\ dvd\ n - m \longleftrightarrow n < m \vee m\ dvd\ n$
for $m\ n :: nat$
⟨*proof*⟩

lemma *dvd-minus-add*:
fixes $m\ n\ q\ r :: nat$
assumes $q \leq n\ q \leq r * m$
shows $m\ dvd\ n - q \longleftrightarrow m\ dvd\ n + (r * m - q)$
⟨*proof*⟩

17.12 Aliasses

lemma *nat-mult-1*: $1 * n = n$
for n :: *nat*
⟨*proof*⟩

lemma *nat-mult-1-right*: $n * 1 = n$
for n :: *nat*
⟨*proof*⟩

lemma *diff-mult-distrib*: $(m - n) * k = (m * k) - (n * k)$
for $k\ m\ n$ :: *nat*
⟨*proof*⟩

lemma *diff-mult-distrib2*: $k * (m - n) = (k * m) - (k * n)$
for $k\ m\ n$ :: *nat*
⟨*proof*⟩

lemma *le-diff-conv2*: $k \leq j \Longrightarrow (i \leq j - k) = (i + k \leq j)$
for $i\ j\ k$ :: *nat*
⟨*proof*⟩

lemma *diff-self-eq-0* [*simp*]: $m - m = 0$
for m :: *nat*
⟨*proof*⟩

lemma *diff-diff-left* [*simp*]: $i - j - k = i - (j + k)$
for $i\ j\ k$ :: *nat*
⟨*proof*⟩

lemma *diff-commute*: $i - j - k = i - k - j$
for $i\ j\ k$ :: *nat*
⟨*proof*⟩

lemma *diff-add-inverse*: $(n + m) - n = m$
for $m\ n$ :: *nat*
⟨*proof*⟩

lemma *diff-add-inverse2*: $(m + n) - n = m$
for $m\ n$ :: *nat*
⟨*proof*⟩

lemma *diff-cancel*: $(k + m) - (k + n) = m - n$
for $k\ m\ n$ :: *nat*
⟨*proof*⟩

lemma *diff-cancel2*: $(m + k) - (n + k) = m - n$
for $k\ m\ n$ :: *nat*
⟨*proof*⟩

lemma *diff-add-0*: $n - (n + m) = 0$
for m n :: *nat*
⟨*proof*⟩

lemma *add-mult-distrib2*: $k * (m + n) = (k * m) + (k * n)$
for k m n :: *nat*
⟨*proof*⟩

lemmas *nat-distrib* =
add-mult-distrib distrib-left diff-mult-distrib diff-mult-distrib2

17.13 Size of a datatype value

class *size* =
fixes *size* :: $'a \Rightarrow$ *nat* — see further theory *Wellfounded*

instantiation *nat* :: *size*
begin

definition *size-nat* **where** [*simp*, *code*]: *size* (n::*nat*) = n

instance ⟨*proof*⟩

end

lemmas *size-nat* = *size-nat-def*

lemma *size-neq-size-imp-neq*: *size* $x \neq$ *size* $y \Longrightarrow x \neq y$
⟨*proof*⟩

17.14 Code module namespace

code-identifier
code-module *Nat* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

hide-const (**open**) *of-nat-aux*

end

18 Fields

theory *Fields*
imports *Nat*
begin

18.1 Division rings

A division ring is like a field, but without the commutativity requirement.

class *inverse* = *divide* +
  fixes *inverse* :: *'a* ⇒ *'a*
begin

abbreviation *inverse-divide* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** *'/ 70*)
where
  inverse-divide ≡ *divide*

end

Setup for linear arithmetic prover

⟨*ML*⟩

lemmas [*linarith-split*] = *nat-diff-split split-min split-max abs-split*

Lemmas *divide-simps* move division to the outside and eliminates them on (in)equalities.

named-theorems *divide-simps rewrite rules to eliminate divisions*

class *division-ring* = *ring-1* + *inverse* +
  assumes *left-inverse* [*simp*]: *a* ≠ *0* ⟹ *inverse a* ∗ *a* = *1*
  assumes *right-inverse* [*simp*]: *a* ≠ *0* ⟹ *a* ∗ *inverse a* = *1*
  assumes *divide-inverse*: *a* / *b* = *a* ∗ *inverse b*
  assumes *inverse-zero* [*simp*]: *inverse 0* = *0*
begin

subclass *ring-1-no-zero-divisors*
⟨*proof*⟩

lemma *nonzero-imp-inverse-nonzero*:
  a ≠ *0* ⟹ *inverse a* ≠ *0*
⟨*proof*⟩

lemma *inverse-zero-imp-zero*:
  assumes *inverse a* = *0* **shows** *a* = *0*
⟨*proof*⟩

lemma *inverse-unique*:
  assumes *ab*: *a* ∗ *b* = *1*
  shows *inverse a* = *b*
⟨*proof*⟩

lemma *nonzero-inverse-minus-eq*:
  a ≠ *0* ⟹ *inverse* (− *a*) = − *inverse a*
⟨*proof*⟩

lemma *nonzero-inverse-inverse-eq*:
  a ≠ *0* ⟹ *inverse* (*inverse a*) = *a*
⟨*proof*⟩

lemma *nonzero-inverse-eq-imp-eq*:
assumes *inverse a* = *inverse b* **and** $a \neq 0$ **and** $b \neq 0$
shows $a = b$
⟨*proof*⟩

lemma *inverse-1* [*simp*]: *inverse 1* = *1*
⟨*proof*⟩

lemma *nonzero-inverse-mult-distrib*:
assumes $a \neq 0$ **and** $b \neq 0$
shows *inverse* $(a * b)$ = *inverse b* * *inverse a*
⟨*proof*⟩

lemma *division-ring-inverse-add*:
$a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow$ *inverse a* + *inverse b* = *inverse a* * $(a + b)$ * *inverse b*
⟨*proof*⟩

lemma *division-ring-inverse-diff*:
$a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow$ *inverse a* − *inverse b* = *inverse a* * $(b - a)$ * *inverse b*
⟨*proof*⟩

lemma *right-inverse-eq*: $b \neq 0 \Longrightarrow a \,/\, b = 1 \longleftrightarrow a = b$
⟨*proof*⟩

lemma *nonzero-inverse-eq-divide*: $a \neq 0 \Longrightarrow$ *inverse a* = *1* / *a*
⟨*proof*⟩

lemma *divide-self* [*simp*]: $a \neq 0 \Longrightarrow a \,/\, a = 1$
⟨*proof*⟩

lemma *inverse-eq-divide* [*field-simps*, *field-split-simps*, *divide-simps*]: *inverse a* = *1* / *a*
⟨*proof*⟩

lemma *add-divide-distrib*: $(a+b) \,/\, c = a/c + b/c$
⟨*proof*⟩

lemma *times-divide-eq-right* [*simp*]: $a * (b \,/\, c) = (a * b) \,/\, c$
⟨*proof*⟩

lemma *minus-divide-left*: $-\,(a \,/\, b) = (-a) \,/\, b$
⟨*proof*⟩

lemma *nonzero-minus-divide-right*: $b \neq 0 \Longrightarrow -\,(a \,/\, b) = a \,/\, (-\,b)$
⟨*proof*⟩

lemma *nonzero-minus-divide-divide*: $b \neq 0 \Longrightarrow (-a) \,/\, (-b) = a \,/\, b$
⟨*proof*⟩

lemma *divide-minus-left* [*simp*]: $(-a) \mathbin{/} b = - (a \mathbin{/} b)$
⟨*proof*⟩

lemma *diff-divide-distrib*: $(a - b) \mathbin{/} c = a \mathbin{/} c - b \mathbin{/} c$
⟨*proof*⟩

lemma *nonzero-eq-divide-eq* [*field-simps*]: $c \neq 0 \Longrightarrow a = b \mathbin{/} c \longleftrightarrow a * c = b$
⟨*proof*⟩

lemma *nonzero-divide-eq-eq* [*field-simps*]: $c \neq 0 \Longrightarrow b \mathbin{/} c = a \longleftrightarrow b = a * c$
⟨*proof*⟩

lemma *nonzero-neg-divide-eq-eq* [*field-simps*]: $b \neq 0 \Longrightarrow - (a \mathbin{/} b) = c \longleftrightarrow - a = c * b$
⟨*proof*⟩

lemma *nonzero-neg-divide-eq-eq2* [*field-simps*]: $b \neq 0 \Longrightarrow c = - (a \mathbin{/} b) \longleftrightarrow c * b = - a$
⟨*proof*⟩

lemma *divide-eq-imp*: $c \neq 0 \Longrightarrow b = a * c \Longrightarrow b \mathbin{/} c = a$
⟨*proof*⟩

lemma *eq-divide-imp*: $c \neq 0 \Longrightarrow a * c = b \Longrightarrow a = b \mathbin{/} c$
⟨*proof*⟩

lemma *add-divide-eq-iff* [*field-simps*]:
$z \neq 0 \Longrightarrow x + y \mathbin{/} z = (x * z + y) \mathbin{/} z$
⟨*proof*⟩

lemma *divide-add-eq-iff* [*field-simps*]:
$z \neq 0 \Longrightarrow x \mathbin{/} z + y = (x + y * z) \mathbin{/} z$
⟨*proof*⟩

lemma *diff-divide-eq-iff* [*field-simps*]:
$z \neq 0 \Longrightarrow x - y \mathbin{/} z = (x * z - y) \mathbin{/} z$
⟨*proof*⟩

lemma *minus-divide-add-eq-iff* [*field-simps*]:
$z \neq 0 \Longrightarrow - (x \mathbin{/} z) + y = (- x + y * z) \mathbin{/} z$
⟨*proof*⟩

lemma *divide-diff-eq-iff* [*field-simps*]:
$z \neq 0 \Longrightarrow x \mathbin{/} z - y = (x - y * z) \mathbin{/} z$
⟨*proof*⟩

lemma *minus-divide-diff-eq-iff* [*field-simps*]:
$z \neq 0 \Longrightarrow - (x \mathbin{/} z) - y = (- x - y * z) \mathbin{/} z$

⟨*proof*⟩

lemma *division-ring-divide-zero* [*simp*]:
a / 0 = 0
⟨*proof*⟩

lemma *divide-self-if* [*simp*]:
a / a = (if a = 0 then 0 else 1)
⟨*proof*⟩

lemma *inverse-nonzero-iff-nonzero* [*simp*]:
inverse a = 0 ⟷ a = 0
⟨*proof*⟩

lemma *inverse-minus-eq* [*simp*]:
inverse (− a) = − inverse a
⟨*proof*⟩

lemma *inverse-inverse-eq* [*simp*]:
inverse (inverse a) = a
⟨*proof*⟩

lemma *inverse-eq-imp-eq*:
inverse a = inverse b ⟹ a = b
⟨*proof*⟩

lemma *inverse-eq-iff-eq* [*simp*]:
inverse a = inverse b ⟷ a = b
⟨*proof*⟩

lemma *mult-commute-imp-mult-inverse-commute*:
assumes *y * x = x * y*
shows *inverse y * x = x * inverse y*
⟨*proof*⟩

lemmas *mult-inverse-of-nat-commute =*
mult-commute-imp-mult-inverse-commute[*OF mult-of-nat-commute*]

lemma *divide-divide-eq-left'*:
*(a / b) / c = a / (c * b)*
⟨*proof*⟩

lemma *add-divide-eq-if-simps* [*field-split-simps*, *divide-simps*]:
*a + b / z = (if z = 0 then a else (a * z + b) / z)*
*a / z + b = (if z = 0 then b else (a + b * z) / z)*
*− (a / z) + b = (if z = 0 then b else (−a + b * z) / z)*
*a − b / z = (if z = 0 then a else (a * z − b) / z)*
*a / z − b = (if z = 0 then −b else (a − b * z) / z)*
*− (a / z) − b = (if z = 0 then −b else (− a − b * z) / z)*

⟨*proof*⟩

lemma [*field-split-simps*, *divide-simps*]:
shows *divide-eq-eq*: $b / c = a \longleftrightarrow (\textit{if } c \neq 0 \textit{ then } b = a * c \textit{ else } a = 0)$
and *eq-divide-eq*: $a = b / c \longleftrightarrow (\textit{if } c \neq 0 \textit{ then } a * c = b \textit{ else } a = 0)$
and *minus-divide-eq-eq*: $-(b / c) = a \longleftrightarrow (\textit{if } c \neq 0 \textit{ then } -b = a * c \textit{ else } a = 0)$
and *eq-minus-divide-eq*: $a = -(b / c) \longleftrightarrow (\textit{if } c \neq 0 \textit{ then } a * c = -b \textit{ else } a = 0)$
⟨*proof*⟩

end

18.2 Fields

class *field* = *comm-ring-1* + *inverse* +
assumes *field-inverse*: $a \neq 0 \Longrightarrow \textit{inverse } a * a = 1$
assumes *field-divide-inverse*: $a / b = a * \textit{inverse } b$
assumes *field-inverse-zero*: $\textit{inverse } 0 = 0$
begin

subclass *division-ring*
⟨*proof*⟩

subclass *idom-divide*
⟨*proof*⟩

There is no slick version using division by zero.

lemma *inverse-add*:
$a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow \textit{inverse } a + \textit{inverse } b = (a + b) * \textit{inverse } a * \textit{inverse } b$
⟨*proof*⟩

lemma *nonzero-mult-divide-mult-cancel-left* [*simp*]:
assumes [*simp*]: $c \neq 0$
shows $(c * a) / (c * b) = a / b$
⟨*proof*⟩

lemma *nonzero-mult-divide-mult-cancel-right* [*simp*]:
$c \neq 0 \Longrightarrow (a * c) / (b * c) = a / b$
⟨*proof*⟩

lemma *times-divide-eq-left* [*simp*]: $(b / c) * a = (b * a) / c$
⟨*proof*⟩

lemma *divide-inverse-commute*: $a / b = \textit{inverse } b * a$
⟨*proof*⟩

lemma *add-frac-eq*:
assumes $y \neq 0$ **and** $z \neq 0$

shows $x \,/\, y + w \,/\, z = (x * z + w * y) \,/\, (y * z)$
⟨*proof*⟩

Special Cancellation Simprules for Division

lemma *nonzero-divide-mult-cancel-right* [*simp*]:
$b \neq 0 \Longrightarrow b \,/\, (a * b) = 1 \,/\, a$
⟨*proof*⟩

lemma *nonzero-divide-mult-cancel-left* [*simp*]:
$a \neq 0 \Longrightarrow a \,/\, (a * b) = 1 \,/\, b$
⟨*proof*⟩

lemma *nonzero-mult-divide-mult-cancel-left2* [*simp*]:
$c \neq 0 \Longrightarrow (c * a) \,/\, (b * c) = a \,/\, b$
⟨*proof*⟩

lemma *nonzero-mult-divide-mult-cancel-right2* [*simp*]:
$c \neq 0 \Longrightarrow (a * c) \,/\, (c * b) = a \,/\, b$
⟨*proof*⟩

lemma *diff-frac-eq*:
$y \neq 0 \Longrightarrow z \neq 0 \Longrightarrow x \,/\, y - w \,/\, z = (x * z - w * y) \,/\, (y * z)$
⟨*proof*⟩

lemma *frac-eq-eq*:
$y \neq 0 \Longrightarrow z \neq 0 \Longrightarrow (x \,/\, y = w \,/\, z) = (x * z = w * y)$
⟨*proof*⟩

lemma *divide-minus1* [*simp*]: $x \,/\, -1 = -x$
⟨*proof*⟩

This version builds in division by zero while also re-orienting the right-hand side.

lemma *inverse-mult-distrib* [*simp*]:
$inverse\ (a * b) = inverse\ a * inverse\ b$
⟨*proof*⟩

lemma *inverse-divide* [*simp*]:
$inverse\ (a \,/\, b) = b \,/\, a$
⟨*proof*⟩

Calculations with fractions

There is a whole bunch of simp-rules just for class *field* but none for class *field* and *nonzero-divides* because the latter are covered by a simproc.

lemmas *mult-divide-mult-cancel-left* = *nonzero-mult-divide-mult-cancel-left*

lemmas *mult-divide-mult-cancel-right* = *nonzero-mult-divide-mult-cancel-right*

lemma *divide-divide-eq-right* [*simp*]:
$a \mathbin{/} (b \mathbin{/} c) = (a * c) \mathbin{/} b$
⟨*proof*⟩

lemma *divide-divide-eq-left* [*simp*]:
$(a \mathbin{/} b) \mathbin{/} c = a \mathbin{/} (b * c)$
⟨*proof*⟩

lemma *divide-divide-times-eq*:
$(x \mathbin{/} y) \mathbin{/} (z \mathbin{/} w) = (x * w) \mathbin{/} (y * z)$
⟨*proof*⟩

Special Cancellation Simprules for Division

lemma *mult-divide-mult-cancel-left-if* [*simp*]:
shows $(c * a) \mathbin{/} (c * b) = (\textit{if } c = 0 \textit{ then } 0 \textit{ else } a \mathbin{/} b)$
⟨*proof*⟩

Division and Unary Minus

lemma *minus-divide-right*:
$-(a \mathbin{/} b) = a \mathbin{/} -b$
⟨*proof*⟩

lemma *divide-minus-right* [*simp*]:
$a \mathbin{/} -b = -(a \mathbin{/} b)$
⟨*proof*⟩

lemma *minus-divide-divide*:
$(-a) \mathbin{/} (-b) = a \mathbin{/} b$
⟨*proof*⟩

lemma *inverse-eq-1-iff* [*simp*]:
$\textit{inverse } x = 1 \longleftrightarrow x = 1$
⟨*proof*⟩

lemma *divide-eq-0-iff* [*simp*]:
$a \mathbin{/} b = 0 \longleftrightarrow a = 0 \vee b = 0$
⟨*proof*⟩

lemma *divide-cancel-right* [*simp*]:
$a \mathbin{/} c = b \mathbin{/} c \longleftrightarrow c = 0 \vee a = b$
⟨*proof*⟩

lemma *divide-cancel-left* [*simp*]:
$c \mathbin{/} a = c \mathbin{/} b \longleftrightarrow c = 0 \vee a = b$
⟨*proof*⟩

lemma *divide-eq-1-iff* [*simp*]:
$a \mathbin{/} b = 1 \longleftrightarrow b \neq 0 \wedge a = b$
⟨*proof*⟩

lemma *one-eq-divide-iff* [*simp*]:
$1 = a \,/\, b \longleftrightarrow b \neq 0 \wedge a = b$
⟨*proof*⟩

lemma *divide-eq-minus-1-iff*:
$(a \,/\, b = -\,1) \longleftrightarrow b \neq 0 \wedge a = -\,b$
⟨*proof*⟩

lemma *times-divide-times-eq*:
$(x \,/\, y) * (z \,/\, w) = (x * z) \,/\, (y * w)$
⟨*proof*⟩

lemma *add-frac-num*:
$y \neq 0 \Longrightarrow x \,/\, y + z = (x + z * y) \,/\, y$
⟨*proof*⟩

lemma *add-num-frac*:
$y \neq 0 \Longrightarrow z + x \,/\, y = (x + z * y) \,/\, y$
⟨*proof*⟩

lemma *dvd-field-iff*:
$a \ dvd \ b \longleftrightarrow (a = 0 \longrightarrow b = 0)$
⟨*proof*⟩

lemma *inj-divide-right* [*simp*]:
$inj \ (\lambda b.\ b \,/\, a) \longleftrightarrow a \neq 0$
⟨*proof*⟩

end

class *field-char-0* = *field* + *ring-char-0*

18.3 Ordered fields

class *field-abs-sgn* = *field* + *idom-abs-sgn*
begin

lemma *sgn-inverse* [*simp*]:
$sgn \ (inverse \ a) = inverse \ (sgn \ a)$
⟨*proof*⟩

lemma *abs-inverse* [*simp*]:
$|inverse \ a| = inverse \ |a|$
⟨*proof*⟩

lemma *sgn-divide* [*simp*]:
$sgn \ (a \,/\, b) = sgn \ a \,/\, sgn \ b$
⟨*proof*⟩

lemma *abs-divide* [*simp*]:
$|a\ /\ b| = |a|\ /\ |b|$
⟨*proof*⟩

end

class *linordered-field* = *field* + *linordered-idom*
begin

lemma *positive-imp-inverse-positive*:
assumes *a-gt-0*: $0 < a$
shows $0 <$ *inverse* a
⟨*proof*⟩

lemma *negative-imp-inverse-negative*:
$a < 0 \Longrightarrow$ *inverse* $a < 0$
⟨*proof*⟩

lemma *inverse-le-imp-le*:
assumes *invle*: *inverse* $a \leq$ *inverse* b **and** *apos*: $0 < a$
shows $b \leq a$
⟨*proof*⟩

lemma *inverse-positive-imp-positive*:
assumes *inv-gt-0*: $0 <$ *inverse* a **and** *nz*: $a \neq 0$
shows $0 < a$
⟨*proof*⟩

lemma *inverse-negative-imp-negative*:
assumes *inv-less-0*: *inverse* $a < 0$ **and** *nz*: $a \neq 0$
shows $a < 0$
⟨*proof*⟩

lemma *linordered-field-no-lb*:
$\forall x.\ \exists y.\ y < x$
⟨*proof*⟩

lemma *linordered-field-no-ub*:
$\forall\ x.\ \exists y.\ y > x$
⟨*proof*⟩

lemma *less-imp-inverse-less*:
assumes *less*: $a < b$ **and** *apos*: $0 < a$
shows *inverse* $b <$ *inverse* a
⟨*proof*⟩

lemma *inverse-less-imp-less*:
assumes *inverse* $a <$ *inverse* b $0 < a$

shows $b < a$
⟨*proof*⟩

Both premises are essential. Consider -1 and 1.

lemma *inverse-less-iff-less* [*simp*]:
$0 < a \Longrightarrow 0 < b \Longrightarrow inverse\ a < inverse\ b \longleftrightarrow b < a$
⟨*proof*⟩

lemma *le-imp-inverse-le*:
$a \leq b \Longrightarrow 0 < a \Longrightarrow inverse\ b \leq inverse\ a$
⟨*proof*⟩

lemma *inverse-le-iff-le* [*simp*]:
$0 < a \Longrightarrow 0 < b \Longrightarrow inverse\ a \leq inverse\ b \longleftrightarrow b \leq a$
⟨*proof*⟩

These results refer to both operands being negative. The opposite-sign case is trivial, since inverse preserves signs.

lemma *inverse-le-imp-le-neg*:
assumes $inverse\ a \leq inverse\ b\ b < 0$
shows $b \leq a$
⟨*proof*⟩

lemma *less-imp-inverse-less-neg*:
assumes $a < b\ b < 0$
shows $inverse\ b < inverse\ a$
⟨*proof*⟩

lemma *inverse-less-imp-less-neg*:
assumes $inverse\ a < inverse\ b\ b < 0$
shows $b < a$
⟨*proof*⟩

lemma *inverse-less-iff-less-neg* [*simp*]:
$a < 0 \Longrightarrow b < 0 \Longrightarrow inverse\ a < inverse\ b \longleftrightarrow b < a$
⟨*proof*⟩

lemma *le-imp-inverse-le-neg*:
$a \leq b \Longrightarrow b < 0 \Longrightarrow inverse\ b \leq inverse\ a$
⟨*proof*⟩

lemma *inverse-le-iff-le-neg* [*simp*]:
$a < 0 \Longrightarrow b < 0 \Longrightarrow inverse\ a \leq inverse\ b \longleftrightarrow b \leq a$
⟨*proof*⟩

lemma *one-less-inverse*:
$0 < a \Longrightarrow a < 1 \Longrightarrow 1 < inverse\ a$
⟨*proof*⟩

lemma *one-le-inverse*:
$0 < a \Longrightarrow a \leq 1 \Longrightarrow 1 \leq inverse\ a$
⟨*proof*⟩

lemma *pos-le-divide-eq* [*field-simps*]:
assumes $0 < c$
shows $a \leq b\ /\ c \longleftrightarrow a * c \leq b$
⟨*proof*⟩

lemma *pos-less-divide-eq* [*field-simps*]:
assumes $0 < c$
shows $a < b\ /\ c \longleftrightarrow a * c < b$
⟨*proof*⟩

lemma *neg-less-divide-eq* [*field-simps*]:
assumes $c < 0$
shows $a < b\ /\ c \longleftrightarrow b < a * c$
⟨*proof*⟩

lemma *neg-le-divide-eq* [*field-simps*]:
assumes $c < 0$
shows $a \leq b\ /\ c \longleftrightarrow b \leq a * c$
⟨*proof*⟩

lemma *pos-divide-le-eq* [*field-simps*]:
assumes $0 < c$
shows $b\ /\ c \leq a \longleftrightarrow b \leq a * c$
⟨*proof*⟩

lemma *pos-divide-less-eq* [*field-simps*]:
assumes $0 < c$
shows $b\ /\ c < a \longleftrightarrow b < a * c$
⟨*proof*⟩

lemma *neg-divide-le-eq* [*field-simps*]:
assumes $c < 0$
shows $b\ /\ c \leq a \longleftrightarrow a * c \leq b$
⟨*proof*⟩

lemma *neg-divide-less-eq* [*field-simps*]:
assumes $c < 0$
shows $b\ /\ c < a \longleftrightarrow a * c < b$
⟨*proof*⟩

The following *field-simps* rules are necessary, as minus is always moved atop of division but we want to get rid of division.

lemma *pos-le-minus-divide-eq* [*field-simps*]: $0 < c \Longrightarrow a \leq -\ (b\ /\ c) \longleftrightarrow a * c \leq -\ b$
⟨*proof*⟩

lemma *neg-le-minus-divide-eq* [*field-simps*]: $c < 0 \Longrightarrow a \leq - (b \mathbin{/} c) \longleftrightarrow - b \leq a * c$
⟨*proof*⟩

lemma *pos-less-minus-divide-eq* [*field-simps*]: $0 < c \Longrightarrow a < - (b \mathbin{/} c) \longleftrightarrow a * c < - b$
⟨*proof*⟩

lemma *neg-less-minus-divide-eq* [*field-simps*]: $c < 0 \Longrightarrow a < - (b \mathbin{/} c) \longleftrightarrow - b < a * c$
⟨*proof*⟩

lemma *pos-minus-divide-less-eq* [*field-simps*]: $0 < c \Longrightarrow - (b \mathbin{/} c) < a \longleftrightarrow - b < a * c$
⟨*proof*⟩

lemma *neg-minus-divide-less-eq* [*field-simps*]: $c < 0 \Longrightarrow - (b \mathbin{/} c) < a \longleftrightarrow a * c < - b$
⟨*proof*⟩

lemma *pos-minus-divide-le-eq* [*field-simps*]: $0 < c \Longrightarrow - (b \mathbin{/} c) \leq a \longleftrightarrow - b \leq a * c$
⟨*proof*⟩

lemma *neg-minus-divide-le-eq* [*field-simps*]: $c < 0 \Longrightarrow - (b \mathbin{/} c) \leq a \longleftrightarrow a * c \leq - b$
⟨*proof*⟩

lemma *frac-less-eq*:
$y \neq 0 \Longrightarrow z \neq 0 \Longrightarrow x \mathbin{/} y < w \mathbin{/} z \longleftrightarrow (x * z - w * y) \mathbin{/} (y * z) < 0$
⟨*proof*⟩

lemma *frac-le-eq*:
$y \neq 0 \Longrightarrow z \neq 0 \Longrightarrow x \mathbin{/} y \leq w \mathbin{/} z \longleftrightarrow (x * z - w * y) \mathbin{/} (y * z) \leq 0$
⟨*proof*⟩

lemma *divide-pos-pos*[*simp*]:
$0 < x \Longrightarrow 0 < y \Longrightarrow 0 < x \mathbin{/} y$
⟨*proof*⟩

lemma *divide-nonneg-pos*:
$0 \leq x \Longrightarrow 0 < y \Longrightarrow 0 \leq x \mathbin{/} y$
⟨*proof*⟩

lemma *divide-neg-pos*:
$x < 0 \Longrightarrow 0 < y \Longrightarrow x \mathbin{/} y < 0$
⟨*proof*⟩

lemma *divide-nonpos-pos*:
$x \leq 0 \Longrightarrow 0 < y \Longrightarrow x \mathbin{/} y \leq 0$
⟨*proof*⟩

lemma *divide-pos-neg*:
$0 < x \Longrightarrow y < 0 \Longrightarrow x \mathbin{/} y < 0$
⟨*proof*⟩

lemma *divide-nonneg-neg*:
$0 \leq x \Longrightarrow y < 0 \Longrightarrow x \mathbin{/} y \leq 0$
⟨*proof*⟩

lemma *divide-neg-neg*:
$x < 0 \Longrightarrow y < 0 \Longrightarrow 0 < x \mathbin{/} y$
⟨*proof*⟩

lemma *divide-nonpos-neg*:
$x \leq 0 \Longrightarrow y < 0 \Longrightarrow 0 \leq x \mathbin{/} y$
⟨*proof*⟩

lemma *divide-strict-right-mono*:
$[\![a < b;\ 0 < c]\!] \Longrightarrow a \mathbin{/} c < b \mathbin{/} c$
⟨*proof*⟩

lemma *divide-strict-right-mono-neg*:
assumes $b < a$ $c < 0$ **shows** $a \mathbin{/} c < b \mathbin{/} c$
⟨*proof*⟩

The last premise ensures that a and b have the same sign

lemma *divide-strict-left-mono*:
$[\![b < a;\ 0 < c;\ 0 < a*b]\!] \Longrightarrow c \mathbin{/} a < c \mathbin{/} b$
⟨*proof*⟩

lemma *divide-left-mono*:
$[\![b \leq a;\ 0 \leq c;\ 0 < a*b]\!] \Longrightarrow c \mathbin{/} a \leq c \mathbin{/} b$
⟨*proof*⟩

lemma *divide-strict-left-mono-neg*:
$[\![a < b;\ c < 0;\ 0 < a*b]\!] \Longrightarrow c \mathbin{/} a < c \mathbin{/} b$
⟨*proof*⟩

lemma *mult-imp-div-pos-le*: $0 < y \Longrightarrow x \leq z * y \Longrightarrow x \mathbin{/} y \leq z$
⟨*proof*⟩

lemma *mult-imp-le-div-pos*: $0 < y \Longrightarrow z * y \leq x \Longrightarrow z \leq x \mathbin{/} y$
⟨*proof*⟩

lemma *mult-imp-div-pos-less*: $0 < y \Longrightarrow x < z * y \Longrightarrow x \mathbin{/} y < z$

⟨*proof*⟩

lemma *mult-imp-less-div-pos*: $0 < y \Longrightarrow z * y < x \Longrightarrow z < x \mathbin{/} y$
⟨*proof*⟩

lemma *frac-le*:
assumes $0 \leq y$ $x \leq y$ $0 < w$ $w \leq z$
shows $x \mathbin{/} z \leq y \mathbin{/} w$
⟨*proof*⟩

lemma *frac-less*:
assumes $0 \leq x$ $x < y$ $0 < w$ $w \leq z$
shows $x \mathbin{/} z < y \mathbin{/} w$
⟨*proof*⟩

lemma *frac-less2*:
assumes $0 < x$ $x \leq y$ $0 < w$ $w < z$
shows $x \mathbin{/} z < y \mathbin{/} w$
⟨*proof*⟩

lemma *less-half-sum*: $a < b \Longrightarrow a < (a+b) \mathbin{/} (1+1)$
⟨*proof*⟩

lemma *gt-half-sum*: $a < b \Longrightarrow (a+b)/(1+1) < b$
⟨*proof*⟩

subclass *unbounded-dense-linorder*
⟨*proof*⟩

subclass *field-abs-sgn* ⟨*proof*⟩

lemma *inverse-sgn* [*simp*]:
$inverse\ (sgn\ a) = sgn\ a$
⟨*proof*⟩

lemma *divide-sgn* [*simp*]:
$a \mathbin{/} sgn\ b = a * sgn\ b$
⟨*proof*⟩

lemma *nonzero-abs-inverse*:
$a \neq 0 \Longrightarrow |inverse\ a| = inverse\ |a|$
⟨*proof*⟩

lemma *nonzero-abs-divide*:
$b \neq 0 \Longrightarrow |a \mathbin{/} b| = |a| \mathbin{/} |b|$
⟨*proof*⟩

lemma *field-le-epsilon*:
assumes e: $\bigwedge e.\ 0 < e \Longrightarrow x \leq y + e$

shows $x \leq y$
⟨*proof*⟩

lemma *inverse-positive-iff-positive* [*simp*]: $(0 < inverse\ a) = (0 < a)$
⟨*proof*⟩

lemma *inverse-negative-iff-negative* [*simp*]: $(inverse\ a < 0) = (a < 0)$
⟨*proof*⟩

lemma *inverse-nonnegative-iff-nonnegative* [*simp*]: $0 \leq inverse\ a \longleftrightarrow 0 \leq a$
⟨*proof*⟩

lemma *inverse-nonpositive-iff-nonpositive* [*simp*]: $inverse\ a \leq 0 \longleftrightarrow a \leq 0$
⟨*proof*⟩

lemma *one-less-inverse-iff*: $1 < inverse\ x \longleftrightarrow 0 < x \wedge x < 1$
⟨*proof*⟩

lemma *one-le-inverse-iff*: $1 \leq inverse\ x \longleftrightarrow 0 < x \wedge x \leq 1$
⟨*proof*⟩

lemma *inverse-less-1-iff*: $inverse\ x < 1 \longleftrightarrow x \leq 0 \vee 1 < x$
⟨*proof*⟩

lemma *inverse-le-1-iff*: $inverse\ x \leq 1 \longleftrightarrow x \leq 0 \vee 1 \leq x$
⟨*proof*⟩

lemma [*field-split-simps*, *divide-simps*]:
shows *le-divide-eq*: $a \leq b\ /\ c \longleftrightarrow (if\ 0 < c\ then\ a * c \leq b\ else\ if\ c < 0\ then\ b \leq a * c\ else\ a \leq 0)$
and *divide-le-eq*: $b\ /\ c \leq a \longleftrightarrow (if\ 0 < c\ then\ b \leq a * c\ else\ if\ c < 0\ then\ a * c \leq b\ else\ 0 \leq a)$
and *less-divide-eq*: $a < b\ /\ c \longleftrightarrow (if\ 0 < c\ then\ a * c < b\ else\ if\ c < 0\ then\ b < a * c\ else\ a < 0)$
and *divide-less-eq*: $b\ /\ c < a \longleftrightarrow (if\ 0 < c\ then\ b < a * c\ else\ if\ c < 0\ then\ a * c < b\ else\ 0 < a)$
and *le-minus-divide-eq*: $a \leq -\ (b\ /\ c) \longleftrightarrow (if\ 0 < c\ then\ a * c \leq -\ b\ else\ if\ c < 0\ then\ -\ b \leq a * c\ else\ a \leq 0)$
and *minus-divide-le-eq*: $-\ (b\ /\ c) \leq a \longleftrightarrow (if\ 0 < c\ then\ -\ b \leq a * c\ else\ if\ c < 0\ then\ a * c \leq -\ b\ else\ 0 \leq a)$
and *less-minus-divide-eq*: $a < -\ (b\ /\ c) \longleftrightarrow (if\ 0 < c\ then\ a * c < -\ b\ else\ if\ c < 0\ then\ -\ b < a * c\ else\ \ a < 0)$
and *minus-divide-less-eq*: $-\ (b\ /\ c) < a \longleftrightarrow (if\ 0 < c\ then\ -\ b < a * c\ else\ if\ c < 0\ then\ a * c < -\ b\ else\ 0 < a)$
⟨*proof*⟩

Division and Signs

lemma
shows *zero-less-divide-iff*: $0 < a\ /\ b \longleftrightarrow 0 < a \wedge 0 < b \vee a < 0 \wedge b < 0$

and *divide-less-0-iff*: $a \mathbin{/} b < 0 \longleftrightarrow 0 < a \wedge b < 0 \vee a < 0 \wedge 0 < b$
and *zero-le-divide-iff*: $0 \leq a \mathbin{/} b \longleftrightarrow 0 \leq a \wedge 0 \leq b \vee a \leq 0 \wedge b \leq 0$
and *divide-le-0-iff*: $a \mathbin{/} b \leq 0 \longleftrightarrow 0 \leq a \wedge b \leq 0 \vee a \leq 0 \wedge 0 \leq b$
⟨*proof*⟩

Division and the Number One

Simplify expressions equated with 1

lemma *zero-eq-1-divide-iff* [*simp*]: $0 = 1 \mathbin{/} a \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *one-divide-eq-0-iff* [*simp*]: $1 \mathbin{/} a = 0 \longleftrightarrow a = 0$
⟨*proof*⟩

Simplify expressions such as $0 < 1/x$ to $0 < x$

lemma *zero-le-divide-1-iff* [*simp*]:
$0 \leq 1 \mathbin{/} a \longleftrightarrow 0 \leq a$
⟨*proof*⟩

lemma *zero-less-divide-1-iff* [*simp*]:
$0 < 1 \mathbin{/} a \longleftrightarrow 0 < a$
⟨*proof*⟩

lemma *divide-le-0-1-iff* [*simp*]:
$1 \mathbin{/} a \leq 0 \longleftrightarrow a \leq 0$
⟨*proof*⟩

lemma *divide-less-0-1-iff* [*simp*]:
$1 \mathbin{/} a < 0 \longleftrightarrow a < 0$
⟨*proof*⟩

lemma *divide-right-mono*:
$[\![a \leq b;\ 0 \leq c]\!] \Longrightarrow a/c \leq b/c$
⟨*proof*⟩

lemma *divide-right-mono-neg*: $a \leq b \Longrightarrow c \leq 0 \Longrightarrow b \mathbin{/} c \leq a \mathbin{/} c$
⟨*proof*⟩

lemma *divide-left-mono-neg*: $a \leq b \Longrightarrow c \leq 0 \Longrightarrow 0 < a * b \Longrightarrow c \mathbin{/} a \leq c \mathbin{/} b$
⟨*proof*⟩

lemma *inverse-le-iff*: $inverse\ a \leq inverse\ b \longleftrightarrow (0 < a * b \longrightarrow b \leq a) \wedge (a * b \leq 0 \longrightarrow a \leq b)$
⟨*proof*⟩

lemma *inverse-less-iff*: $inverse\ a < inverse\ b \longleftrightarrow (0 < a * b \longrightarrow b < a) \wedge (a * b \leq 0 \longrightarrow a < b)$
⟨*proof*⟩

lemma *divide-le-cancel*: $a \mathbin{/} c \leq b \mathbin{/} c \longleftrightarrow (0 < c \longrightarrow a \leq b) \wedge (c < 0 \longrightarrow b \leq a)$
⟨*proof*⟩

lemma *divide-less-cancel*: $a \mathbin{/} c < b \mathbin{/} c \longleftrightarrow (0 < c \longrightarrow a < b) \wedge (c < 0 \longrightarrow b < a) \wedge c \neq 0$
⟨*proof*⟩

Simplify quotients that are compared with the value 1.

lemma *le-divide-eq-1*:
$(1 \leq b \mathbin{/} a) = ((0 < a \wedge a \leq b) \vee (a < 0 \wedge b \leq a))$
⟨*proof*⟩

lemma *divide-le-eq-1*:
$(b \mathbin{/} a \leq 1) = ((0 < a \wedge b \leq a) \vee (a < 0 \wedge a \leq b) \vee a{=}0)$
⟨*proof*⟩

lemma *less-divide-eq-1*:
$(1 < b \mathbin{/} a) = ((0 < a \wedge a < b) \vee (a < 0 \wedge b < a))$
⟨*proof*⟩

lemma *divide-less-eq-1*:
$(b \mathbin{/} a < 1) = ((0 < a \wedge b < a) \vee (a < 0 \wedge a < b) \vee a{=}0)$
⟨*proof*⟩

lemma *divide-nonneg-nonneg* [*simp*]:
$0 \leq x \Longrightarrow 0 \leq y \Longrightarrow 0 \leq x \mathbin{/} y$
⟨*proof*⟩

lemma *divide-nonpos-nonpos*:
$x \leq 0 \Longrightarrow y \leq 0 \Longrightarrow 0 \leq x \mathbin{/} y$
⟨*proof*⟩

lemma *divide-nonneg-nonpos*:
$0 \leq x \Longrightarrow y \leq 0 \Longrightarrow x \mathbin{/} y \leq 0$
⟨*proof*⟩

lemma *divide-nonpos-nonneg*:
$x \leq 0 \Longrightarrow 0 \leq y \Longrightarrow x \mathbin{/} y \leq 0$
⟨*proof*⟩

Conditional Simplification Rules: No Case Splits

lemma *le-divide-eq-1-pos* [*simp*]:
$0 < a \Longrightarrow (1 \leq b/a) = (a \leq b)$
⟨*proof*⟩

lemma *le-divide-eq-1-neg* [*simp*]:
$a < 0 \Longrightarrow (1 \leq b/a) = (b \leq a)$
⟨*proof*⟩

lemma *divide-le-eq-1-pos* [*simp*]:
$0 < a \Longrightarrow (b/a \leq 1) = (b \leq a)$
⟨*proof*⟩

lemma *divide-le-eq-1-neg* [*simp*]:
$a < 0 \Longrightarrow (b/a \leq 1) = (a \leq b)$
⟨*proof*⟩

lemma *less-divide-eq-1-pos* [*simp*]:
$0 < a \Longrightarrow (1 < b/a) = (a < b)$
⟨*proof*⟩

lemma *less-divide-eq-1-neg* [*simp*]:
$a < 0 \Longrightarrow (1 < b/a) = (b < a)$
⟨*proof*⟩

lemma *divide-less-eq-1-pos* [*simp*]:
$0 < a \Longrightarrow (b/a < 1) = (b < a)$
⟨*proof*⟩

lemma *divide-less-eq-1-neg* [*simp*]:
$a < 0 \Longrightarrow b/a < 1 \longleftrightarrow a < b$
⟨*proof*⟩

lemma *eq-divide-eq-1* [*simp*]:
$(1 = b/a) = ((a \neq 0 \wedge a = b))$
⟨*proof*⟩

lemma *divide-eq-eq-1* [*simp*]:
$(b/a = 1) = ((a \neq 0 \wedge a = b))$
⟨*proof*⟩

lemma *abs-div-pos*: $0 < y \Longrightarrow |x| \ / \ y = |x \ / \ y|$
⟨*proof*⟩

lemma *zero-le-divide-abs-iff* [*simp*]: $(0 \leq a \ / \ |b|) = (0 \leq a \vee b = 0)$
⟨*proof*⟩

lemma *divide-le-0-abs-iff* [*simp*]: $(a \ / \ |b| \leq 0) = (a \leq 0 \vee b = 0)$
⟨*proof*⟩

lemma *field-le-mult-one-interval*:
assumes $*$: $\bigwedge z.\ [\![\ 0 < z\ ;\ z < 1\ ]\!] \Longrightarrow z * x \leq y$
shows $x \leq y$
⟨*proof*⟩

For creating values between u and v.

lemma *scaling-mono*:

assumes $u \leq v\ 0 \leq r\ r \leq s$
shows $u + r * (v - u) / s \leq v$
⟨*proof*⟩

end

Min/max Simplification Rules

lemma *min-mult-distrib-left*:
fixes x::$'a$::*linordered-idom*
shows $p * min\ x\ y = (if\ 0 \leq p\ then\ min\ (p*x)\ (p*y)\ else\ max\ (p*x)\ (p*y))$
⟨*proof*⟩

lemma *min-mult-distrib-right*:
fixes x::$'a$::*linordered-idom*
shows $min\ x\ y * p = (if\ 0 \leq p\ then\ min\ (x*p)\ (y*p)\ else\ max\ (x*p)\ (y*p))$
⟨*proof*⟩

lemma *min-divide-distrib-right*:
fixes x::$'a$::*linordered-field*
shows $min\ x\ y\ /\ p = (if\ 0 \leq p\ then\ min\ (x/p)\ (y/p)\ else\ max\ (x/p)\ (y/p))$
⟨*proof*⟩

lemma *max-mult-distrib-left*:
fixes x::$'a$::*linordered-idom*
shows $p * max\ x\ y = (if\ 0 \leq p\ then\ max\ (p*x)\ (p*y)\ else\ min\ (p*x)\ (p*y))$
⟨*proof*⟩

lemma *max-mult-distrib-right*:
fixes x::$'a$::*linordered-idom*
shows $max\ x\ y * p = (if\ 0 \leq p\ then\ max\ (x*p)\ (y*p)\ else\ min\ (x*p)\ (y*p))$
⟨*proof*⟩

lemma *max-divide-distrib-right*:
fixes x::$'a$::*linordered-field*
shows $max\ x\ y\ /\ p = (if\ 0 \leq p\ then\ max\ (x/p)\ (y/p)\ else\ min\ (x/p)\ (y/p))$
⟨*proof*⟩

hide-fact (**open**) *field-inverse field-divide-inverse field-inverse-zero*

code-identifier
code-module *Fields* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

end

19 Relations – as sets of pairs, and binary predicates

theory *Relation*

imports *Product-Type Sum-Type Fields*
begin

A preliminary: classical rules for reasoning on predicates

declare *predicate1I* [*Pure.intro*!, *intro*!]
declare *predicate1D* [*Pure.dest*, *dest*]
declare *predicate2I* [*Pure.intro*!, *intro*!]
declare *predicate2D* [*Pure.dest*, *dest*]
declare *bot1E* [*elim*!]
declare *bot2E* [*elim*!]
declare *top1I* [*intro*!]
declare *top2I* [*intro*!]
declare *inf1I* [*intro*!]
declare *inf2I* [*intro*!]
declare *inf1E* [*elim*!]
declare *inf2E* [*elim*!]
declare *sup1I1* [*intro?*]
declare *sup2I1* [*intro?*]
declare *sup1I2* [*intro?*]
declare *sup2I2* [*intro?*]
declare *sup1E* [*elim*!]
declare *sup2E* [*elim*!]
declare *sup1CI* [*intro*!]
declare *sup2CI* [*intro*!]
declare *Inf1-I* [*intro*!]
declare *INF1-I* [*intro*!]
declare *Inf2-I* [*intro*!]
declare *INF2-I* [*intro*!]
declare *Inf1-D* [*elim*]
declare *INF1-D* [*elim*]
declare *Inf2-D* [*elim*]
declare *INF2-D* [*elim*]
declare *Inf1-E* [*elim*]
declare *INF1-E* [*elim*]
declare *Inf2-E* [*elim*]
declare *INF2-E* [*elim*]
declare *Sup1-I* [*intro*]
declare *SUP1-I* [*intro*]
declare *Sup2-I* [*intro*]
declare *SUP2-I* [*intro*]
declare *Sup1-E* [*elim*!]
declare *SUP1-E* [*elim*!]
declare *Sup2-E* [*elim*!]
declare *SUP2-E* [*elim*!]

19.1 Fundamental

19.1.1 Relations as sets of pairs

type-synonym $'a\ rel = ('a \times 'a)\ set$

lemma *subrelI*: $(\bigwedge x\ y.\ (x,\ y) \in r \Longrightarrow (x,\ y) \in s) \Longrightarrow r \subseteq s$
— Version of *subsetI* for binary relations
⟨*proof*⟩

lemma *lfp-induct2*:
$(a,\ b) \in \mathit{lfp}\ f \Longrightarrow \mathit{mono}\ f \Longrightarrow$
$(\bigwedge a\ b.\ (a,\ b) \in f\ (\mathit{lfp}\ f \cap \{(x,\ y).\ P\ x\ y\}) \Longrightarrow P\ a\ b) \Longrightarrow P\ a\ b$
— Version of *lfp-induct* for binary relations
⟨*proof*⟩

19.1.2 Conversions between set and predicate relations

lemma *pred-equals-eq* [*pred-set-conv*]: $(\lambda x.\ x \in R) = (\lambda x.\ x \in S) \longleftrightarrow R = S$
⟨*proof*⟩

lemma *pred-equals-eq2* [*pred-set-conv*]: $(\lambda x\ y.\ (x,\ y) \in R) = (\lambda x\ y.\ (x,\ y) \in S) \longleftrightarrow R = S$
⟨*proof*⟩

lemma *pred-subset-eq* [*pred-set-conv*]: $(\lambda x.\ x \in R) \leq (\lambda x.\ x \in S) \longleftrightarrow R \subseteq S$
⟨*proof*⟩

lemma *pred-subset-eq2* [*pred-set-conv*]: $(\lambda x\ y.\ (x,\ y) \in R) \leq (\lambda x\ y.\ (x,\ y) \in S) \longleftrightarrow R \subseteq S$
⟨*proof*⟩

lemma *bot-empty-eq* [*pred-set-conv*]: $\bot = (\lambda x.\ x \in \{\})$
⟨*proof*⟩

lemma *bot-empty-eq2* [*pred-set-conv*]: $\bot = (\lambda x\ y.\ (x,\ y) \in \{\})$
⟨*proof*⟩

lemma *top-empty-eq*: $\top = (\lambda x.\ x \in \mathit{UNIV})$
⟨*proof*⟩

lemma *top-empty-eq2*: $\top = (\lambda x\ y.\ (x,\ y) \in \mathit{UNIV})$
⟨*proof*⟩

lemma *inf-Int-eq* [*pred-set-conv*]: $(\lambda x.\ x \in R) \sqcap (\lambda x.\ x \in S) = (\lambda x.\ x \in R \cap S)$
⟨*proof*⟩

lemma *inf-Int-eq2* [*pred-set-conv*]: $(\lambda x\ y.\ (x,\ y) \in R) \sqcap (\lambda x\ y.\ (x,\ y) \in S) = (\lambda x\ y.\ (x,\ y) \in R \cap S)$
⟨*proof*⟩

lemma *sup-Un-eq* [*pred-set-conv*]: $(\lambda x.\ x \in R) \sqcup (\lambda x.\ x \in S) = (\lambda x.\ x \in R \cup S)$
⟨*proof*⟩

lemma *sup-Un-eq2* [*pred-set-conv*]: $(\lambda x\ y.\ (x,\ y) \in R) \sqcup (\lambda x\ y.\ (x,\ y) \in S) = (\lambda x\ y.\ (x,\ y) \in R \cup S)$
⟨*proof*⟩

lemma *INF-INT-eq* [*pred-set-conv*]: $(\bigsqcap i{\in}S.\ (\lambda x.\ x \in r\ i)) = (\lambda x.\ x \in (\bigcap i{\in}S.\ r\ i))$
⟨*proof*⟩

lemma *INF-INT-eq2* [*pred-set-conv*]: $(\bigsqcap i{\in}S.\ (\lambda x\ y.\ (x,\ y) \in r\ i)) = (\lambda x\ y.\ (x,\ y) \in (\bigcap i{\in}S.\ r\ i))$
⟨*proof*⟩

lemma *SUP-UN-eq* [*pred-set-conv*]: $(\bigsqcup i{\in}S.\ (\lambda x.\ x \in r\ i)) = (\lambda x.\ x \in (\bigcup i{\in}S.\ r\ i))$
⟨*proof*⟩

lemma *SUP-UN-eq2* [*pred-set-conv*]: $(\bigsqcup i{\in}S.\ (\lambda x\ y.\ (x,\ y) \in r\ i)) = (\lambda x\ y.\ (x,\ y) \in (\bigcup i{\in}S.\ r\ i))$
⟨*proof*⟩

lemma *Inf-INT-eq* [*pred-set-conv*]: $\bigsqcap S = (\lambda x.\ x \in (\bigcap (Collect\ `\ S)))$
⟨*proof*⟩

lemma *INF-Int-eq* [*pred-set-conv*]: $(\bigsqcap i{\in}S.\ (\lambda x.\ x \in i)) = (\lambda x.\ x \in \bigcap S)$
⟨*proof*⟩

lemma *Inf-INT-eq2* [*pred-set-conv*]: $\bigsqcap S = (\lambda x\ y.\ (x,\ y) \in (\bigcap (Collect\ `\ case\text{-}prod\ `\ S)))$
⟨*proof*⟩

lemma *INF-Int-eq2* [*pred-set-conv*]: $(\bigsqcap i{\in}S.\ (\lambda x\ y.\ (x,\ y) \in i)) = (\lambda x\ y.\ (x,\ y) \in \bigcap S)$
⟨*proof*⟩

lemma *Sup-SUP-eq* [*pred-set-conv*]: $\bigsqcup S = (\lambda x.\ x \in \bigcup (Collect\ `\ S))$
⟨*proof*⟩

lemma *SUP-Sup-eq* [*pred-set-conv*]: $(\bigsqcup i{\in}S.\ (\lambda x.\ x \in i)) = (\lambda x.\ x \in \bigcup S)$
⟨*proof*⟩

lemma *Sup-SUP-eq2* [*pred-set-conv*]: $\bigsqcup S = (\lambda x\ y.\ (x,\ y) \in (\bigcup (Collect\ `\ case\text{-}prod\ `\ S)))$
⟨*proof*⟩

lemma *SUP-Sup-eq2* [*pred-set-conv*]: $(\bigsqcup i{\in}S.\ (\lambda x\ y.\ (x,\ y) \in i)) = (\lambda x\ y.\ (x,\ y) \in \bigcup S)$
⟨*proof*⟩

19.2 Properties of relations

19.2.1 Reflexivity

definition *refl-on* :: *'a set* ⇒ *'a rel* ⇒ *bool*
where *refl-on A r* ⟷ *r* ⊆ *A* × *A* ∧ (∀ *x*∈*A*. (*x*, *x*) ∈ *r*)

abbreviation *refl* :: *'a rel* ⇒ *bool* — reflexivity over a type
where *refl* ≡ *refl-on UNIV*

definition *reflp-on* :: *'a set* ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*
where *reflp-on A R* ⟷ (∀ *x*∈*A*. *R x x*)

abbreviation *reflp* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*
where *reflp* ≡ *reflp-on UNIV*

lemma *reflp-def*[*no-atp*]: *reflp R* ⟷ (∀ *x*. *R x x*)
⟨*proof*⟩

reflp-def is for backward compatibility.

lemma *reflp-refl-eq* [*pred-set-conv*]: *reflp* (λ*x y*. (*x*, *y*) ∈ *r*) ⟷ *refl r*
⟨*proof*⟩

lemma *refl-onI* [*intro?*]: *r* ⊆ *A* × *A* ⟹ (⋀*x*. *x* ∈ *A* ⟹ (*x*, *x*) ∈ *r*) ⟹ *refl-on A r*
⟨*proof*⟩

lemma *reflI*: (⋀*x*. (*x*, *x*) ∈ *r*) ⟹ *refl r*
⟨*proof*⟩

lemma *reflp-onI*:
(⋀*x*. *x* ∈ *A* ⟹ *R x x*) ⟹ *reflp-on A R*
⟨*proof*⟩

lemma *reflpI*[*intro?*]: (⋀*x*. *R x x*) ⟹ *reflp R*
⟨*proof*⟩

lemma *refl-onD*: *refl-on A r* ⟹ *a* ∈ *A* ⟹ (*a*, *a*) ∈ *r*
⟨*proof*⟩

lemma *refl-onD1*: *refl-on A r* ⟹ (*x*, *y*) ∈ *r* ⟹ *x* ∈ *A*
⟨*proof*⟩

lemma *refl-onD2*: *refl-on A r* ⟹ (*x*, *y*) ∈ *r* ⟹ *y* ∈ *A*
⟨*proof*⟩

lemma *reflD*: *refl r* ⟹ (*a*, *a*) ∈ *r*
⟨*proof*⟩

lemma *reflp-onD*:

*refl*p-on *A R* ⟹ *x* ∈ *A* ⟹ *R x x*
⟨*proof*⟩

lemma *reflpD*[*dest?*]: *refl*p *R* ⟹ *R x x*
⟨*proof*⟩

lemma *reflpE*:
assumes *refl*p *r*
obtains *r x x*
⟨*proof*⟩

lemma *refl*p-on-subset: *refl*p-on *A R* ⟹ *B* ⊆ *A* ⟹ *refl*p-on *B R*
⟨*proof*⟩

lemma *refl*-on-Int: *refl*-on *A r* ⟹ *refl*-on *B s* ⟹ *refl*-on (*A* ∩ *B*) (*r* ∩ *s*)
⟨*proof*⟩

lemma *refl*p-on-inf: *refl*p-on *A R* ⟹ *refl*p-on *B S* ⟹ *refl*p-on (*A* ∩ *B*) (*R* ⊓ *S*)
⟨*proof*⟩

lemma *refl*p-inf: *refl*p *r* ⟹ *refl*p *s* ⟹ *refl*p (*r* ⊓ *s*)
⟨*proof*⟩

lemma *refl*-on-Un: *refl*-on *A r* ⟹ *refl*-on *B s* ⟹ *refl*-on (*A* ∪ *B*) (*r* ∪ *s*)
⟨*proof*⟩

lemma *refl*p-on-sup: *refl*p-on *A R* ⟹ *refl*p-on *B S* ⟹ *refl*p-on (*A* ∪ *B*) (*R* ⊔ *S*)
⟨*proof*⟩

lemma *refl*p-sup: *refl*p *r* ⟹ *refl*p *s* ⟹ *refl*p (*r* ⊔ *s*)
⟨*proof*⟩

lemma *refl*-on-INTER: ∀ *x*∈*S*. *refl*-on (*A x*) (*r x*) ⟹ *refl*-on (⋂(*A* ‘ *S*)) (⋂(*r* ‘ *S*))
⟨*proof*⟩

lemma *refl*p-on-Inf: ∀ *x*∈*S*. *refl*p-on (*A x*) (*R x*) ⟹ *refl*p-on (⋂(*A* ‘ *S*)) (⨅(*R* ‘ *S*))
⟨*proof*⟩

lemma *refl*-on-UNION: ∀ *x*∈*S*. *refl*-on (*A x*) (*r x*) ⟹ *refl*-on (⋃(*A* ‘ *S*)) (⋃(*r* ‘ *S*))
⟨*proof*⟩

lemma *refl*p-on-Sup: ∀ *x*∈*S*. *refl*p-on (*A x*) (*R x*) ⟹ *refl*p-on (⋃(*A* ‘ *S*)) (⨆(*R* ‘ *S*))
⟨*proof*⟩

lemma *refl*-on-empty [*simp*]: *refl*-on {} {}

⟨*proof*⟩

lemma *reflp-on-empty* [*simp*]: *reflp-on* {} *R*
⟨*proof*⟩

lemma *refl-on-singleton* [*simp*]: *refl-on* {*x*} {(*x*, *x*)}
⟨*proof*⟩

lemma *refl-on-def'* [*nitpick-unfold*, *code*]:
refl-on A r ⟷ (∀(*x*, *y*) ∈ *r*. *x* ∈ *A* ∧ *y* ∈ *A*) ∧ (∀*x* ∈ *A*. (*x*, *x*) ∈ *r*)
⟨*proof*⟩

lemma *reflp-on-equality* [*simp*]: *reflp-on A* (=)
⟨*proof*⟩

lemma *reflp-on-mono*:
reflp-on A R ⟹ (⋀*x y*. *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ *R x y* ⟹ *Q x y*) ⟹ *reflp-on A Q*
⟨*proof*⟩

lemma *reflp-mono*: *reflp R* ⟹ (⋀*x y*. *R x y* ⟹ *Q x y*) ⟹ *reflp Q*
⟨*proof*⟩

lemma (**in** *preorder*) *reflp-on-le*[*simp*]: *reflp-on A* (≤)
⟨*proof*⟩

lemma (**in** *preorder*) *reflp-on-ge*[*simp*]: *reflp-on A* (≥)
⟨*proof*⟩

19.2.2 Irreflexivity

definition *irrefl-on* :: *'a set* ⇒ *'a rel* ⇒ *bool* **where**
irrefl-on A r ⟷ (∀*a* ∈ *A*. (*a*, *a*) ∉ *r*)

abbreviation *irrefl* :: *'a rel* ⇒ *bool* **where**
irrefl ≡ *irrefl-on UNIV*

definition *irreflp-on* :: *'a set* ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool* **where**
irreflp-on A R ⟷ (∀*a* ∈ *A*. ¬ *R a a*)

abbreviation *irreflp* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool* **where**
irreflp ≡ *irreflp-on UNIV*

lemma *irrefl-def*[*no-atp*]: *irrefl r* ⟷ (∀*a*. (*a*, *a*) ∉ *r*)
⟨*proof*⟩

lemma *irreflp-def*[*no-atp*]: *irreflp R* ⟷ (∀*a*. ¬ *R a a*)
⟨*proof*⟩

irrefl-def and *irreflp-def* are for backward compatibility.

lemma *irreflp-on-irrefl-on-eq* [*pred-set-conv*]: *irreflp-on A* (λ*a b*. (*a*, *b*) ∈ *r*) ⟷ *irrefl-on A r*
⟨*proof*⟩

lemmas *irreflp-irrefl-eq* = *irreflp-on-irrefl-on-eq*[*of UNIV*]

lemma *irrefl-onI*: (⋀*a*. *a* ∈ *A* ⟹ (*a*, *a*) ∉ *r*) ⟹ *irrefl-on A r*
⟨*proof*⟩

lemma *irreflI*[*intro?*]: (⋀*a*. (*a*, *a*) ∉ *r*) ⟹ *irrefl r*
⟨*proof*⟩

lemma *irreflp-onI*: (⋀*a*. *a* ∈ *A* ⟹ ¬ *R a a*) ⟹ *irreflp-on A R*
⟨*proof*⟩

lemma *irreflpI*[*intro?*]: (⋀*a*. ¬ *R a a*) ⟹ *irreflp R*
⟨*proof*⟩

lemma *irrefl-onD*: *irrefl-on A r* ⟹ *a* ∈ *A* ⟹ (*a*, *a*) ∉ *r*
⟨*proof*⟩

lemma *irreflD*: *irrefl r* ⟹ (*x*, *x*) ∉ *r*
⟨*proof*⟩

lemma *irreflp-onD*: *irreflp-on A R* ⟹ *a* ∈ *A* ⟹ ¬ *R a a*
⟨*proof*⟩

lemma *irreflpD*: *irreflp R* ⟹ ¬ *R x x*
⟨*proof*⟩

lemma *irrefl-on-distinct* [*code*]: *irrefl-on A r* ⟷ (∀(*a*, *b*) ∈ *r*. *a* ∈ *A* ⟶ *b* ∈ *A* ⟶ *a* ≠ *b*)
⟨*proof*⟩

lemmas *irrefl-distinct* = *irrefl-on-distinct* — For backward compatibility

lemma *irrefl-on-subset*: *irrefl-on A r* ⟹ *B* ⊆ *A* ⟹ *irrefl-on B r*
⟨*proof*⟩

lemma *irreflp-on-subset*: *irreflp-on A R* ⟹ *B* ⊆ *A* ⟹ *irreflp-on B R*
⟨*proof*⟩

lemma (**in** *preorder*) *irreflp-on-less*[*simp*]: *irreflp-on A* (<)
⟨*proof*⟩

lemma (**in** *preorder*) *irreflp-on-greater*[*simp*]: *irreflp-on A* (>)
⟨*proof*⟩

19.2.3 Asymmetry

definition *asym-on* :: *'a set* ⇒ *'a rel* ⇒ *bool* **where**
asym-on A r ⟷ (∀ *x* ∈ *A*. ∀ *y* ∈ *A*. (*x*, *y*) ∈ *r* ⟶ (*y*, *x*) ∉ *r*)

abbreviation *asym* :: *'a rel* ⇒ *bool* **where**
asym ≡ *asym-on UNIV*

definition *asymp-on* :: *'a set* ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool* **where**
asymp-on A R ⟷ (∀ *x* ∈ *A*. ∀ *y* ∈ *A*. *R x y* ⟶ ¬ *R y x*)

abbreviation *asymp* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool* **where**
asymp ≡ *asymp-on UNIV*

lemma *asymp-on-asym-on-eq*[*pred-set-conv*]: *asymp-on A* (λ*x y*. (*x*, *y*) ∈ *r*) ⟷ *asym-on A r*
⟨*proof*⟩

lemmas *asymp-asym-eq* = *asymp-on-asym-on-eq*[*of UNIV*] — For backward compatibility

lemma *asym-onI*[*intro*]:
(⋀*x y*. *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ (*x*, *y*) ∈ *r* ⟹ (*y*, *x*) ∉ *r*) ⟹ *asym-on A r*
⟨*proof*⟩

lemma *asymI*[*intro*]: (⋀*x y*. (*x*, *y*) ∈ *r* ⟹ (*y*, *x*) ∉ *r*) ⟹ *asym r*
⟨*proof*⟩

lemma *asymp-onI*[*intro*]:
(⋀*x y*. *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ *R x y* ⟹ ¬ *R y x*) ⟹ *asymp-on A R*
⟨*proof*⟩

lemma *asympI*[*intro*]: (⋀*x y*. *R x y* ⟹ ¬ *R y x*) ⟹ *asymp R*
⟨*proof*⟩

lemma *asym-onD*: *asym-on A r* ⟹ *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ (*x*, *y*) ∈ *r* ⟹ (*y*, *x*) ∉ *r*
⟨*proof*⟩

lemma *asymD*: *asym r* ⟹ (*x*, *y*) ∈ *r* ⟹ (*y*, *x*) ∉ *r*
⟨*proof*⟩

lemma *asymp-onD*: *asymp-on A R* ⟹ *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ *R x y* ⟹ ¬ *R y x*
⟨*proof*⟩

lemma *asympD*: *asymp R* ⟹ *R x y* ⟹ ¬ *R y x*
⟨*proof*⟩

lemma *asym-iff*: *asym r* ⟷ (∀ *x y*. (*x*,*y*) ∈ *r* ⟶ (*y*,*x*) ∉ *r*)
⟨*proof*⟩

lemma *asym-on-subset*: *asym-on A r* $\Longrightarrow$ $B \subseteq A$ $\Longrightarrow$ *asym-on B r*
⟨*proof*⟩

lemma *asymp-on-subset*: *asymp-on A R* $\Longrightarrow$ $B \subseteq A$ $\Longrightarrow$ *asymp-on B R*
⟨*proof*⟩

lemma *irrefl-on-if-asym-on*[*simp*]: *asym-on A r* $\Longrightarrow$ *irrefl-on A r*
⟨*proof*⟩

lemma *irreflp-on-if-asymp-on*[*simp*]: *asymp-on A r* $\Longrightarrow$ *irreflp-on A r*
⟨*proof*⟩

lemma (**in** *preorder*) *asymp-on-less*[*simp*]: *asymp-on A* $(<)$
⟨*proof*⟩

lemma (**in** *preorder*) *asymp-on-greater*[*simp*]: *asymp-on A* $(>)$
⟨*proof*⟩

19.2.4 Symmetry

definition *sym-on* :: *'a set* ⇒ *'a rel* ⇒ *bool* **where**
sym-on A r $\longleftrightarrow$ $(\forall x \in A.\ \forall y \in A.\ (x, y) \in r \longrightarrow (y, x) \in r)$

abbreviation *sym* :: *'a rel* ⇒ *bool* **where**
sym ≡ *sym-on UNIV*

definition *symp-on* :: *'a set* ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool* **where**
symp-on A R $\longleftrightarrow$ $(\forall x \in A.\ \forall y \in A.\ R\ x\ y \longrightarrow R\ y\ x)$

abbreviation *symp* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool* **where**
symp ≡ *symp-on UNIV*

lemma *sym-def*[*no-atp*]: *sym r* $\longleftrightarrow$ $(\forall x\ y.\ (x, y) \in r \longrightarrow (y, x) \in r)$
⟨*proof*⟩

lemma *symp-def*[*no-atp*]: *symp R* $\longleftrightarrow$ $(\forall x\ y.\ R\ x\ y \longrightarrow R\ y\ x)$
⟨*proof*⟩

sym-def and *symp-def* are for backward compatibility.

lemma *symp-on-sym-on-eq*[*pred-set-conv*]: *symp-on A* $(\lambda x\ y.\ (x, y) \in r)$ $\longleftrightarrow$ *sym-on A r*
⟨*proof*⟩

lemmas *symp-sym-eq* = *symp-on-sym-on-eq*[*of UNIV*] — For backward compatibility

lemma *sym-on-subset*: *sym-on A r* $\Longrightarrow$ $B \subseteq A$ $\Longrightarrow$ *sym-on B r*
⟨*proof*⟩

lemma *symp-on-subset*: *symp-on A R* $\Longrightarrow$ $B \subseteq A$ $\Longrightarrow$ *symp-on B R*
⟨*proof*⟩

lemma *sym-onI*: $(\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow (x, y) \in r \Longrightarrow (y, x) \in r) \Longrightarrow$ *sym-on A r*
⟨*proof*⟩

lemma *symI* [*intro?*]: $(\bigwedge x\ y.\ (x, y) \in r \Longrightarrow (y, x) \in r) \Longrightarrow$ *sym r*
⟨*proof*⟩

lemma *symp-onI*: $(\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow R\ x\ y \Longrightarrow R\ y\ x) \Longrightarrow$ *symp-on A R*
⟨*proof*⟩

lemma *sympI* [*intro?*]: $(\bigwedge x\ y.\ R\ x\ y \Longrightarrow R\ y\ x) \Longrightarrow$ *symp R*
⟨*proof*⟩

lemma *symE*:
assumes *sym r* **and** $(b, a) \in r$
obtains $(a, b) \in r$
⟨*proof*⟩

lemma *sympE*:
assumes *symp r* **and** *r b a*
obtains *r a b*
⟨*proof*⟩

lemma *sym-onD*: *sym-on A r* $\Longrightarrow$ $x \in A$ $\Longrightarrow$ $y \in A$ $\Longrightarrow$ $(x, y) \in r$ $\Longrightarrow$ $(y, x) \in r$
⟨*proof*⟩

lemma *symD* [*dest?*]: *sym r* $\Longrightarrow$ $(x, y) \in r$ $\Longrightarrow$ $(y, x) \in r$
⟨*proof*⟩

lemma *symp-onD*: *symp-on A R* $\Longrightarrow$ $x \in A$ $\Longrightarrow$ $y \in A$ $\Longrightarrow$ *R x y* $\Longrightarrow$ *R y x*
⟨*proof*⟩

lemma *sympD* [*dest?*]: *symp R* $\Longrightarrow$ *R x y* $\Longrightarrow$ *R y x*
⟨*proof*⟩

lemma *sym-Int*: *sym r* $\Longrightarrow$ *sym s* $\Longrightarrow$ *sym* $(r \cap s)$
⟨*proof*⟩

lemma *symp-inf*: *symp r* $\Longrightarrow$ *symp s* $\Longrightarrow$ *symp* $(r \sqcap s)$
⟨*proof*⟩

lemma *sym-Un*: *sym r* $\Longrightarrow$ *sym s* $\Longrightarrow$ *sym* $(r \cup s)$
⟨*proof*⟩

lemma *symp-sup*: *symp r* ⟹ *symp s* ⟹ *symp* (*r* ⊔ *s*)
⟨*proof*⟩

lemma *sym-INTER*: ∀ *x*∈*S*. *sym* (*r x*) ⟹ *sym* (⋂(*r* ‘ *S*))
⟨*proof*⟩

lemma *symp-INF*: ∀ *x*∈*S*. *symp* (*r x*) ⟹ *symp* (⨅(*r* ‘ *S*))
⟨*proof*⟩

lemma *sym-UNION*: ∀ *x*∈*S*. *sym* (*r x*) ⟹ *sym* (⋃(*r* ‘ *S*))
⟨*proof*⟩

lemma *symp-SUP*: ∀ *x*∈*S*. *symp* (*r x*) ⟹ *symp* (⨆(*r* ‘ *S*))
⟨*proof*⟩

19.2.5 Antisymmetry

definition *antisym-on* :: *'a set* ⇒ *'a rel* ⇒ *bool* **where**
antisym-on A r ⟷ (∀ *x* ∈ *A*. ∀ *y* ∈ *A*. (*x*, *y*) ∈ *r* ⟶ (*y*, *x*) ∈ *r* ⟶ *x* = *y*)

abbreviation *antisym* :: *'a rel* ⇒ *bool* **where**
antisym ≡ *antisym-on UNIV*

definition *antisymp-on* :: *'a set* ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool* **where**
antisymp-on A R ⟷ (∀ *x* ∈ *A*. ∀ *y* ∈ *A*. *R x y* ⟶ *R y x* ⟶ *x* = *y*)

abbreviation *antisymp* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool* **where**
antisymp ≡ *antisymp-on UNIV*

lemma *antisym-def*[*no-atp*]: *antisym r* ⟷ (∀ *x y*. (*x*, *y*) ∈ *r* ⟶ (*y*, *x*) ∈ *r* ⟶ *x* = *y*)
⟨*proof*⟩

lemma *antisymp-def*[*no-atp*]: *antisymp R* ⟷ (∀ *x y*. *R x y* ⟶ *R y x* ⟶ *x* = *y*)
⟨*proof*⟩

antisym-def and *antisymp-def* are for backward compatibility.

lemma *antisymp-on-antisym-on-eq*[*pred-set-conv*]:
antisymp-on A (λ*x y*. (*x*, *y*) ∈ *r*) ⟷ *antisym-on A r*
⟨*proof*⟩

lemmas *antisymp-antisym-eq* = *antisymp-on-antisym-on-eq*[*of UNIV*] — For backward compatibility

lemma *antisym-on-subset*: *antisym-on A r* ⟹ *B* ⊆ *A* ⟹ *antisym-on B r*
⟨*proof*⟩

lemma *antisymp-on-subset*: *antisymp-on A R* ⟹ *B* ⊆ *A* ⟹ *antisymp-on B R*

⟨*proof*⟩

lemma *antisym-onI*:
(⋀*x y*. *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ (*x*, *y*) ∈ *r* ⟹ (*y*, *x*) ∈ *r* ⟹ *x* = *y*) ⟹ *antisym-on A r*
⟨*proof*⟩

lemma *antisymI* [*intro?*]:
(⋀*x y*. (*x*, *y*) ∈ *r* ⟹ (*y*, *x*) ∈ *r* ⟹ *x* = *y*) ⟹ *antisym r*
⟨*proof*⟩

lemma *antisymp-onI*:
(⋀*x y*. *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ *R x y* ⟹ *R y x* ⟹ *x* = *y*) ⟹ *antisymp-on A R*
⟨*proof*⟩

lemma *antisympI* [*intro?*]:
(⋀*x y*. *R x y* ⟹ *R y x* ⟹ *x* = *y*) ⟹ *antisymp R*
⟨*proof*⟩

lemma *antisym-onD*:
antisym-on A r ⟹ *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ (*x*, *y*) ∈ *r* ⟹ (*y*, *x*) ∈ *r* ⟹ *x* = *y*
⟨*proof*⟩

lemma *antisymD* [*dest?*]:
antisym r ⟹ (*x*, *y*) ∈ *r* ⟹ (*y*, *x*) ∈ *r* ⟹ *x* = *y*
⟨*proof*⟩

lemma *antisymp-onD*:
antisymp-on A R ⟹ *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ *R x y* ⟹ *R y x* ⟹ *x* = *y*
⟨*proof*⟩

lemma *antisympD* [*dest?*]:
antisymp R ⟹ *R x y* ⟹ *R y x* ⟹ *x* = *y*
⟨*proof*⟩

lemma *antisym-subset*:
r ⊆ *s* ⟹ *antisym s* ⟹ *antisym r*
⟨*proof*⟩

lemma *antisymp-less-eq*:
r ≤ *s* ⟹ *antisymp s* ⟹ *antisymp r*
⟨*proof*⟩

lemma *antisym-empty* [*simp*]:
antisym {}
⟨*proof*⟩

lemma *antisym-bot* [*simp*]:
antisymp ⊥

⟨*proof*⟩

lemma *antisymp-equality* [*simp*]:
antisymp HOL.eq
⟨*proof*⟩

lemma *antisym-singleton* [*simp*]:
antisym {*x*}
⟨*proof*⟩

lemma *antisym-on-if-asym-on*: *asym-on A r* ⟹ *antisym-on A r*
⟨*proof*⟩

lemma *antisymp-on-if-asymp-on*: *asymp-on A R* ⟹ *antisymp-on A R*
⟨*proof*⟩

lemma (**in** *preorder*) *antisymp-on-less*[*simp*]: *antisymp-on A* (<)
⟨*proof*⟩

lemma (**in** *preorder*) *antisymp-on-greater*[*simp*]: *antisymp-on A* (>)
⟨*proof*⟩

lemma (**in** *order*) *antisymp-on-le*[*simp*]: *antisymp-on A* (≤)
⟨*proof*⟩

lemma (**in** *order*) *antisymp-on-ge*[*simp*]: *antisymp-on A* (≥)
⟨*proof*⟩

19.2.6 Transitivity

definition *trans-on* :: *'a set* ⇒ *'a rel* ⇒ *bool* **where**
trans-on A r ⟷ (∀ *x* ∈ *A*. ∀ *y* ∈ *A*. ∀ *z* ∈ *A*. (*x*, *y*) ∈ *r* ⟶ (*y*, *z*) ∈ *r* ⟶ (*x*, *z*) ∈ *r*)

abbreviation *trans* :: *'a rel* ⇒ *bool* **where**
trans ≡ *trans-on UNIV*

definition *transp-on* :: *'a set* ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool* **where**
transp-on A R ⟷ (∀ *x* ∈ *A*. ∀ *y* ∈ *A*. ∀ *z* ∈ *A*. *R x y* ⟶ *R y z* ⟶ *R x z*)

abbreviation *transp* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool* **where**
transp ≡ *transp-on UNIV*

lemma *trans-def*[*no-atp*]: *trans r* ⟷ (∀ *x y z*. (*x*, *y*) ∈ *r* ⟶ (*y*, *z*) ∈ *r* ⟶ (*x*, *z*) ∈ *r*)
⟨*proof*⟩

lemma *transp-def*: *transp R* ⟷ (∀ *x y z*. *R x y* ⟶ *R y z* ⟶ *R x z*)
⟨*proof*⟩

trans-def and *transp-def* are for backward compatibility.

lemma *transp-on-trans-on-eq*[*pred-set-conv*]: *transp-on A* ($\lambda x\ y.\ (x,\ y) \in r$) $\longleftrightarrow$ *trans-on A r*
⟨*proof*⟩

lemmas *transp-trans-eq* = *transp-on-trans-on-eq*[*of UNIV*] — For backward compatibility

lemma *trans-onI*:
($\bigwedge x\ y\ z.\ x \in A \Longrightarrow y \in A \Longrightarrow z \in A \Longrightarrow (x,\ y) \in r \Longrightarrow (y,\ z) \in r \Longrightarrow (x,\ z) \in r$) $\Longrightarrow$
trans-on A r
⟨*proof*⟩

lemma *transI* [*intro?*]: ($\bigwedge x\ y\ z.\ (x,\ y) \in r \Longrightarrow (y,\ z) \in r \Longrightarrow (x,\ z) \in r$) $\Longrightarrow$ *trans r*
⟨*proof*⟩

lemma *transp-onI*:
($\bigwedge x\ y\ z.\ x \in A \Longrightarrow y \in A \Longrightarrow z \in A \Longrightarrow R\ x\ y \Longrightarrow R\ y\ z \Longrightarrow R\ x\ z$) $\Longrightarrow$ *transp-on A R*
⟨*proof*⟩

lemma *transpI* [*intro?*]: ($\bigwedge x\ y\ z.\ R\ x\ y \Longrightarrow R\ y\ z \Longrightarrow R\ x\ z$) $\Longrightarrow$ *transp R*
⟨*proof*⟩

lemma *transE*:
assumes *trans r* **and** $(x,\ y) \in r$ **and** $(y,\ z) \in r$
obtains $(x,\ z) \in r$
⟨*proof*⟩

lemma *transpE*:
assumes *transp r* **and** *r x y* **and** *r y z*
obtains *r x z*
⟨*proof*⟩

lemma *trans-onD*:
trans-on A r $\Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow z \in A \Longrightarrow (x,\ y) \in r \Longrightarrow (y,\ z) \in r \Longrightarrow (x,\ z) \in r$
⟨*proof*⟩

lemma *transD*[*dest?*]: *trans r* $\Longrightarrow (x,\ y) \in r \Longrightarrow (y,\ z) \in r \Longrightarrow (x,\ z) \in r$
⟨*proof*⟩

lemma *transp-onD*: *transp-on A R* $\Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow z \in A \Longrightarrow R\ x\ y \Longrightarrow R\ y\ z \Longrightarrow R\ x\ z$
⟨*proof*⟩

lemma *transpD*[*dest?*]: *transp R* $\Longrightarrow R\ x\ y \Longrightarrow R\ y\ z \Longrightarrow R\ x\ z$

⟨*proof*⟩

lemma *trans-on-subset*: *trans-on A r* ⟹ *B* ⊆ *A* ⟹ *trans-on B r*
⟨*proof*⟩

lemma *transp-on-subset*: *transp-on A R* ⟹ *B* ⊆ *A* ⟹ *transp-on B R*
⟨*proof*⟩

lemma *trans-Int*: *trans r* ⟹ *trans s* ⟹ *trans* (*r* ∩ *s*)
⟨*proof*⟩

lemma *transp-inf*: *transp r* ⟹ *transp s* ⟹ *transp* (*r* ⊓ *s*)
⟨*proof*⟩

lemma *trans-INTER*: ∀ *x*∈*S*. *trans* (*r x*) ⟹ *trans* (⋂(*r* ‘ *S*))
⟨*proof*⟩

lemma *transp-INF*: ∀ *x*∈*S*. *transp* (*r x*) ⟹ *transp* (⨅(*r* ‘ *S*))
⟨*proof*⟩

lemma *trans-on-join* [*code*]:
trans-on A r ⟷ (∀(*x*, *y1*) ∈ *r*. *x* ∈ *A* ⟶ *y1* ∈ *A* ⟶
(∀(*y2*, *z*) ∈ *r*. *y1* = *y2* ⟶ *z* ∈ *A* ⟶ (*x*, *z*) ∈ *r*))
⟨*proof*⟩

lemma *trans-join*: *trans r* ⟷ (∀(*x*, *y1*) ∈ *r*. ∀(*y2*, *z*) ∈ *r*. *y1* = *y2* ⟶ (*x*, *z*) ∈ *r*)
⟨*proof*⟩

lemma *transp-trans*: *transp r* ⟷ *trans* {(*x*, *y*). *r x y*}
⟨*proof*⟩

lemma *transp-equality* [*simp*]: *transp* (=)
⟨*proof*⟩

lemma *trans-empty* [*simp*]: *trans* {}
⟨*proof*⟩

lemma *transp-empty* [*simp*]: *transp* (λ*x y*. *False*)
⟨*proof*⟩

lemma *trans-singleton* [*simp*]: *trans* {(*a*, *a*)}
⟨*proof*⟩

lemma *transp-singleton* [*simp*]: *transp* (λ*x y*. *x* = *a* ∧ *y* = *a*)
⟨*proof*⟩

lemma *asym-on-iff-irrefl-on-if-trans-on*: *trans-on A r* ⟹ *asym-on A r* ⟷ *irrefl-on A r*

⟨*proof*⟩

lemma *asymp-on-iff-irreflp-on-if-transp-on*: *transp-on A R* ⟹ *asymp-on A R* ⟷ *irreflp-on A R*
⟨*proof*⟩

lemma (**in** *preorder*) *transp-on-le*[*simp*]: *transp-on A* (≤)
⟨*proof*⟩

lemma (**in** *preorder*) *transp-on-less*[*simp*]: *transp-on A* (<)
⟨*proof*⟩

lemma (**in** *preorder*) *transp-on-ge*[*simp*]: *transp-on A* (≥)
⟨*proof*⟩

lemma (**in** *preorder*) *transp-on-greater*[*simp*]: *transp-on A* (>)
⟨*proof*⟩

19.2.7 Totality

definition *total-on* :: *'a set* ⇒ *'a rel* ⇒ *bool* **where**
total-on A r ⟷ (∀ *x*∈*A*. ∀ *y*∈*A*. *x* ≠ *y* ⟶ (*x*, *y*) ∈ *r* ∨ (*y*, *x*) ∈ *r*)

abbreviation *total* :: *'a rel* ⇒ *bool* **where**
total ≡ *total-on UNIV*

definition *totalp-on* :: *'a set* ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool* **where**
totalp-on A R ⟷ (∀ *x* ∈ *A*. ∀ *y* ∈ *A*. *x* ≠ *y* ⟶ *R x y* ∨ *R y x*)

abbreviation *totalp* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool* **where**
totalp ≡ *totalp-on UNIV*

lemma *totalp-on-total-on-eq*[*pred-set-conv*]: *totalp-on A* (λ*x y*. (*x*, *y*) ∈ *r*) ⟷ *total-on A r*
⟨*proof*⟩

lemma *total-onI* [*intro?*]:
(⋀*x y*. *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ *x* ≠ *y* ⟹ (*x*, *y*) ∈ *r* ∨ (*y*, *x*) ∈ *r*) ⟹ *total-on A r*
⟨*proof*⟩

lemma *totalI*: (⋀*x y*. *x* ≠ *y* ⟹ (*x*, *y*) ∈ *r* ∨ (*y*, *x*) ∈ *r*) ⟹ *total r*
⟨*proof*⟩

lemma *totalp-onI*: (⋀*x y*. *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ *x* ≠ *y* ⟹ *R x y* ∨ *R y x*) ⟹ *totalp-on A R*
⟨*proof*⟩

lemma *totalpI*: (⋀*x y*. *x* ≠ *y* ⟹ *R x y* ∨ *R y x*) ⟹ *totalp R*
⟨*proof*⟩

lemma *totalp-onD*:
totalp-on A R $\Longrightarrow$ $x \in A \Longrightarrow y \in A \Longrightarrow x \neq y \Longrightarrow R\ x\ y \vee R\ y\ x$
⟨*proof*⟩

lemma *totalpD*: *totalp R* $\Longrightarrow$ $x \neq y \Longrightarrow R\ x\ y \vee R\ y\ x$
⟨*proof*⟩

lemma *total-on-subset*: *total-on A r* $\Longrightarrow$ $B \subseteq A \Longrightarrow$ *total-on B r*
⟨*proof*⟩

lemma *totalp-on-subset*: *totalp-on A R* $\Longrightarrow$ $B \subseteq A \Longrightarrow$ *totalp-on B R*
⟨*proof*⟩

lemma *total-on-empty* [*simp*]: *total-on* $\{\}$ *r*
⟨*proof*⟩

lemma *totalp-on-empty* [*simp*]: *totalp-on* $\{\}$ *R*
⟨*proof*⟩

lemma *total-on-singleton* [*simp*]: *total-on* $\{x\}$ *r*
⟨*proof*⟩

lemma *totalp-on-singleton* [*simp*]: *totalp-on* $\{x\}$ *R*
⟨*proof*⟩

lemma (**in** *linorder*) *totalp-on-less*[*simp*]: *totalp-on A* $(<)$
⟨*proof*⟩

lemma (**in** *linorder*) *totalp-on-greater*[*simp*]: *totalp-on A* $(>)$
⟨*proof*⟩

lemma (**in** *linorder*) *totalp-on-le*[*simp*]: *totalp-on A* $(\leq)$
⟨*proof*⟩

lemma (**in** *linorder*) *totalp-on-ge*[*simp*]: *totalp-on A* $(\geq)$
⟨*proof*⟩

19.2.8 Single valued relations

definition *single-valued* :: $('a \times 'b)$ *set* $\Rightarrow$ *bool*
where *single-valued r* $\longleftrightarrow$ $(\forall x\ y.\ (x, y) \in r \longrightarrow (\forall z.\ (x, z) \in r \longrightarrow y = z))$

definition *single-valuedp* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$
where *single-valuedp r* $\longleftrightarrow$ $(\forall x\ y.\ r\ x\ y \longrightarrow (\forall z.\ r\ x\ z \longrightarrow y = z))$

lemma *single-valuedp-single-valued-eq* [*pred-set-conv*]:
single-valuedp $(\lambda x\ y.\ (x, y) \in r)$ $\longleftrightarrow$ *single-valued r*
⟨*proof*⟩

lemma *single-valuedp-iff-Uniq*:
single-valuedp r $\longleftrightarrow$ $(\forall x.\ \exists_{\leq 1} y.\ r\ x\ y)$
⟨*proof*⟩

lemma *single-valuedI*:
$(\bigwedge x\ y.\ (x,\ y) \in r \Longrightarrow (\bigwedge z.\ (x,\ z) \in r \Longrightarrow y = z)) \Longrightarrow$ *single-valued r*
⟨*proof*⟩

lemma *single-valuedpI*:
$(\bigwedge x\ y.\ r\ x\ y \Longrightarrow (\bigwedge z.\ r\ x\ z \Longrightarrow y = z)) \Longrightarrow$ *single-valuedp r*
⟨*proof*⟩

lemma *single-valuedD*:
single-valued r $\Longrightarrow (x,\ y) \in r \Longrightarrow (x,\ z) \in r \Longrightarrow y = z$
⟨*proof*⟩

lemma *single-valuedpD*:
single-valuedp r $\Longrightarrow r\ x\ y \Longrightarrow r\ x\ z \Longrightarrow y = z$
⟨*proof*⟩

lemma *single-valued-empty* [*simp*]:
single-valued $\{\}$
⟨*proof*⟩

lemma *single-valuedp-bot* [*simp*]:
single-valuedp $\bot$
⟨*proof*⟩

lemma *single-valued-subset*:
$r \subseteq s \Longrightarrow$ *single-valued s* $\Longrightarrow$ *single-valued r*
⟨*proof*⟩

lemma *single-valuedp-less-eq*:
$r \leq s \Longrightarrow$ *single-valuedp s* $\Longrightarrow$ *single-valuedp r*
⟨*proof*⟩

19.3 Relation operations

19.3.1 The identity relation

definition *Id* :: *'a rel*
where $Id = \{p.\ \exists x.\ p = (x,\ x)\}$

lemma *IdI* [*intro*]: $(a,\ a) \in Id$
⟨*proof*⟩

lemma *IdE* [*elim!*]: $p \in Id \Longrightarrow (\bigwedge x.\ p = (x,\ x) \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *pair-in-Id-conv* [*iff*]: $(a, b) \in Id \longleftrightarrow a = b$
⟨*proof*⟩

lemma *refl-Id*: *refl Id*
⟨*proof*⟩

lemma *antisym-Id*: *antisym Id*
— A strange result, since *Id* is also symmetric.
⟨*proof*⟩

lemma *sym-Id*: *sym Id*
⟨*proof*⟩

lemma *trans-Id*: *trans Id*
⟨*proof*⟩

lemma *single-valued-Id* [*simp*]: *single-valued Id*
⟨*proof*⟩

lemma *irrefl-diff-Id* [*simp*]: *irrefl* $(r - Id)$
⟨*proof*⟩

lemma *trans-diff-Id*: *trans* $r \Longrightarrow$ *antisym* $r \Longrightarrow$ *trans* $(r - Id)$
⟨*proof*⟩

lemma *total-on-diff-Id* [*simp*]: *total-on* A $(r - Id) =$ *total-on* A r
⟨*proof*⟩

lemma *Id-fstsnd-eq*: $Id = \{x.\ fst\ x = snd\ x\}$
⟨*proof*⟩

19.3.2 Diagonal: identity over a set

definition *Id-on* :: *'a set* ⇒ *'a rel*
where *Id-on* $A = (\bigcup x{\in}A.\ \{(x, x)\})$

lemma *Id-on-empty* [*simp*]: *Id-on* $\{\} = \{\}$
⟨*proof*⟩

lemma *Id-on-eqI*: $a = b \Longrightarrow a \in A \Longrightarrow (a, b) \in$ *Id-on* A
⟨*proof*⟩

lemma *Id-onI* [*intro!*]: $a \in A \Longrightarrow (a, a) \in$ *Id-on* A
⟨*proof*⟩

lemma *Id-onE* [*elim!*]: $c \in$ *Id-on* $A \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow c = (x, x) \Longrightarrow P) \Longrightarrow P$
— The general elimination rule.
⟨*proof*⟩

lemma *Id-on-iff*: $(x,\ y) \in \mathit{Id\text{-}on}\ A \longleftrightarrow x = y \wedge x \in A$
⟨*proof*⟩

lemma *Id-on-def'* [*nitpick-unfold*]: $\mathit{Id\text{-}on}\ \{x.\ A\ x\} = \mathit{Collect}\ (\lambda(x,\ y).\ x = y \wedge A\ x)$
⟨*proof*⟩

lemma *Id-on-subset-Times*: $\mathit{Id\text{-}on}\ A \subseteq A \times A$
⟨*proof*⟩

lemma *refl-on-Id-on*: $\mathit{refl\text{-}on}\ A\ (\mathit{Id\text{-}on}\ A)$
⟨*proof*⟩

lemma *antisym-Id-on* [*simp*]: $\mathit{antisym}\ (\mathit{Id\text{-}on}\ A)$
⟨*proof*⟩

lemma *sym-Id-on* [*simp*]: $\mathit{sym}\ (\mathit{Id\text{-}on}\ A)$
⟨*proof*⟩

lemma *trans-Id-on* [*simp*]: $\mathit{trans}\ (\mathit{Id\text{-}on}\ A)$
⟨*proof*⟩

lemma *single-valued-Id-on* [*simp*]: $\mathit{single\text{-}valued}\ (\mathit{Id\text{-}on}\ A)$
⟨*proof*⟩

19.3.3 Composition

inductive-set $\mathit{relcomp}\ ::\ ('a \times 'b)\ \mathit{set} \Rightarrow ('b \times 'c)\ \mathit{set} \Rightarrow ('a \times 'c)\ \mathit{set}$ (**infixr** O 75)
for $r ::\ ('a \times 'b)\ \mathit{set}$ **and** $s ::\ ('b \times 'c)\ \mathit{set}$
where *relcompI* [*intro*]: $(a,\ b) \in r \Longrightarrow (b,\ c) \in s \Longrightarrow (a,\ c) \in r\ O\ s$

notation *relcompp* (**infixr** OO 75)

lemmas $\mathit{relcomppI} = \mathit{relcompp.intros}$

For historic reasons, the elimination rules are not wholly corresponding. Feel free to consolidate this.

inductive-cases *relcompEpair*: $(a,\ c) \in r\ O\ s$
inductive-cases *relcomppE* [*elim*!]: $(r\ OO\ s)\ a\ c$

lemma *relcompE* [*elim*!]: $xz \in r\ O\ s \Longrightarrow$
$(\bigwedge x\ y\ z.\ xz = (x,\ z) \Longrightarrow (x,\ y) \in r \Longrightarrow (y,\ z) \in s\ \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *R-O-Id* [*simp*]: $R\ O\ \mathit{Id} = R$
⟨*proof*⟩

lemma *Id-O-R* [*simp*]: $\mathit{Id}\ O\ R = R$

⟨*proof*⟩

lemma *relcomp-empty1* [*simp*]: $\{\}\ O\ R = \{\}$
⟨*proof*⟩

lemma *relcompp-bot1* [*simp*]: $\bot\ OO\ R = \bot$
⟨*proof*⟩

lemma *relcomp-empty2* [*simp*]: $R\ O\ \{\} = \{\}$
⟨*proof*⟩

lemma *relcompp-bot2* [*simp*]: $R\ OO\ \bot = \bot$
⟨*proof*⟩

lemma *O-assoc*: $(R\ O\ S)\ O\ T = R\ O\ (S\ O\ T)$
⟨*proof*⟩

lemma *relcompp-assoc*: $(r\ OO\ s)\ OO\ t = r\ OO\ (s\ OO\ t)$
⟨*proof*⟩

lemma *trans-O-subset*: $trans\ r \Longrightarrow r\ O\ r \subseteq r$
⟨*proof*⟩

lemma *transp-relcompp-less-eq*: $transp\ r \Longrightarrow r\ OO\ r \leq r$
⟨*proof*⟩

lemma *relcomp-mono*: $r' \subseteq r \Longrightarrow s' \subseteq s \Longrightarrow r'\ O\ s' \subseteq r\ O\ s$
⟨*proof*⟩

lemma *relcompp-mono*: $r' \leq r \Longrightarrow s' \leq s \Longrightarrow r'\ OO\ s' \leq r\ OO\ s$
⟨*proof*⟩

lemma *relcomp-subset-Sigma*: $r \subseteq A \times B \Longrightarrow s \subseteq B \times C \Longrightarrow r\ O\ s \subseteq A \times C$
⟨*proof*⟩

lemma *relcomp-distrib* [*simp*]: $R\ O\ (S \cup T) = (R\ O\ S) \cup (R\ O\ T)$
⟨*proof*⟩

lemma *relcompp-distrib* [*simp*]: $R\ OO\ (S \sqcup T) = R\ OO\ S \sqcup R\ OO\ T$
⟨*proof*⟩

lemma *relcomp-distrib2* [*simp*]: $(S \cup T)\ O\ R = (S\ O\ R) \cup (T\ O\ R)$
⟨*proof*⟩

lemma *relcompp-distrib2* [*simp*]: $(S \sqcup T)\ OO\ R = S\ OO\ R \sqcup T\ OO\ R$
⟨*proof*⟩

lemma *relcomp-UNION-distrib*: $s\ O\ \bigcup(r\ `\ I) = (\bigcup i{\in}I.\ s\ O\ r\ i)$
⟨*proof*⟩

lemma *relcompp-SUP-distrib*: $s\ OO\ \bigsqcup (r\ `\ I) = (\bigsqcup i{\in}I.\ s\ OO\ r\ i)$
⟨*proof*⟩

lemma *relcomp-UNION-distrib2*: $\bigcup (r\ `\ I)\ O\ s = (\bigcup i{\in}I.\ r\ i\ O\ s)$
⟨*proof*⟩

lemma *relcompp-SUP-distrib2*: $\bigsqcup (r\ `\ I)\ OO\ s = (\bigsqcup i{\in}I.\ r\ i\ OO\ s)$
⟨*proof*⟩

lemma *single-valued-relcomp*: *single-valued* $r \Longrightarrow$ *single-valued* $s \Longrightarrow$ *single-valued* $(r\ O\ s)$
⟨*proof*⟩

lemma *relcomp-unfold*: $r\ O\ s = \{(x,\ z).\ \exists y.\ (x,\ y) \in r \wedge (y,\ z) \in s\}$
⟨*proof*⟩

lemma *relcompp-apply*: $(R\ OO\ S)\ a\ c \longleftrightarrow (\exists b.\ R\ a\ b \wedge S\ b\ c)$
⟨*proof*⟩

lemma *eq-OO*: $(=)\ OO\ R = R$
⟨*proof*⟩

lemma *OO-eq*: $R\ OO\ (=) = R$
⟨*proof*⟩

19.3.4 Converse

inductive-set *converse* :: $('a \times 'b)\ set \Rightarrow ('b \times 'a)\ set$ $((\text{-}^{-1})\ [1000]\ 999)$
for r :: $('a \times 'b)\ set$
where $(a,\ b) \in r \Longrightarrow (b,\ a) \in r^{-1}$

notation *conversep* $((\text{-}^{-1-1})\ [1000]\ 1000)$

notation (*ASCII*)
converse $((\text{-}\hat{}{-}1)\ [1000]\ 999)$ **and**
conversep $((\text{-}\hat{}{-}{-}1)\ [1000]\ 1000)$

lemma *converseI* [*sym*]: $(a,\ b) \in r \Longrightarrow (b,\ a) \in r^{-1}$
⟨*proof*⟩

lemma *conversepI* : $r\ a\ b \Longrightarrow r^{-1-1}\ b\ a$
⟨*proof*⟩

lemma *converseD* [*sym*]: $(a,\ b) \in r^{-1} \Longrightarrow (b,\ a) \in r$
⟨*proof*⟩

lemma *conversepD* : $r^{-1-1}\ b\ a \Longrightarrow r\ a\ b$
⟨*proof*⟩

lemma *converseE* [*elim*!]: $yx \in r^{-1} \Longrightarrow (\bigwedge x\ y.\ yx = (y,\ x) \Longrightarrow (x,\ y) \in r \Longrightarrow P) \Longrightarrow P$
— More general than *converseD*, as it "splits" the member of the relation.
⟨*proof*⟩

lemmas *conversepE* [*elim*!] = *conversep.cases*

lemma *converse-iff* [*iff*]: $(a,\ b) \in r^{-1} \longleftrightarrow (b,\ a) \in r$
⟨*proof*⟩

lemma *conversep-iff* [*iff*]: $r^{-1-1}\ a\ b = r\ b\ a$
⟨*proof*⟩

lemma *converse-converse* [*simp*]: $(r^{-1})^{-1} = r$
⟨*proof*⟩

lemma *conversep-conversep* [*simp*]: $(r^{-1-1})^{-1-1} = r$
⟨*proof*⟩

lemma *converse-empty*[*simp*]: $\{\}^{-1} = \{\}$
⟨*proof*⟩

lemma *converse-UNIV*[*simp*]: $UNIV^{-1} = UNIV$
⟨*proof*⟩

lemma *converse-relcomp*: $(r\ O\ s)^{-1} = s^{-1}\ O\ r^{-1}$
⟨*proof*⟩

lemma *converse-relcompp*: $(r\ OO\ s)^{-1-1} = s^{-1-1}\ OO\ r^{-1-1}$
⟨*proof*⟩

lemma *converse-Int*: $(r \cap s)^{-1} = r^{-1} \cap s^{-1}$
⟨*proof*⟩

lemma *converse-meet*: $(r \sqcap s)^{-1-1} = r^{-1-1} \sqcap s^{-1-1}$
⟨*proof*⟩

lemma *converse-Un*: $(r \cup s)^{-1} = r^{-1} \cup s^{-1}$
⟨*proof*⟩

lemma *converse-join*: $(r \sqcup s)^{-1-1} = r^{-1-1} \sqcup s^{-1-1}$
⟨*proof*⟩

lemma *converse-INTER*: $(\bigcap (r\ `\ S))^{-1} = (\bigcap x{\in}S.\ (r\ x)^{-1})$
⟨*proof*⟩

lemma *converse-UNION*: $(\bigcup (r\ `\ S))^{-1} = (\bigcup x{\in}S.\ (r\ x)^{-1})$
⟨*proof*⟩

lemma *converse-mono*[*simp*]: $r^{-1} \subseteq s^{-1} \longleftrightarrow r \subseteq s$
⟨*proof*⟩

lemma *conversep-mono*[*simp*]: $r^{-1-1} \leq s^{-1-1} \longleftrightarrow r \leq s$
⟨*proof*⟩

lemma *converse-inject*[*simp*]: $r^{-1} = s^{-1} \longleftrightarrow r = s$
⟨*proof*⟩

lemma *conversep-inject*[*simp*]: $r^{-1-1} = s^{-1-1} \longleftrightarrow r = s$
⟨*proof*⟩

lemma *converse-subset-swap*: $r \subseteq s^{-1} \longleftrightarrow r^{-1} \subseteq s$
⟨*proof*⟩

lemma *conversep-le-swap*: $r \leq s^{-1-1} \longleftrightarrow r^{-1-1} \leq s$
⟨*proof*⟩

lemma *converse-Id* [*simp*]: $Id^{-1} = Id$
⟨*proof*⟩

lemma *converse-Id-on* [*simp*]: $(Id\text{-}on\ A)^{-1} = Id\text{-}on\ A$
⟨*proof*⟩

lemma *refl-on-converse* [*simp*]: *refl-on A* (r^{-1}) = *refl-on A r*
⟨*proof*⟩

lemma *reflp-on-conversp* [*simp*]: *reflp-on A* $R^{-1-1} \longleftrightarrow$ *reflp-on A R*
⟨*proof*⟩

lemma *irrefl-on-converse* [*simp*]: *irrefl-on A* (r^{-1}) = *irrefl-on A r*
⟨*proof*⟩

lemma *irreflp-on-converse* [*simp*]: *irreflp-on A* (r^{-1-1}) = *irreflp-on A r*
⟨*proof*⟩

lemma *sym-on-converse* [*simp*]: *sym-on A* (r^{-1}) = *sym-on A r*
⟨*proof*⟩

lemma *symp-on-conversep* [*simp*]: *symp-on A* R^{-1-1} = *symp-on A R*
⟨*proof*⟩

lemma *asym-on-converse* [*simp*]: *asym-on A* (r^{-1}) = *asym-on A r*
⟨*proof*⟩

lemma *asymp-on-conversep* [*simp*]: *asymp-on A* R^{-1-1} = *asymp-on A R*
⟨*proof*⟩

lemma *antisym-on-converse* [*simp*]: *antisym-on A* (r^{-1}) = *antisym-on A r*
⟨*proof*⟩

lemma *antisymp-on-conversep* [*simp*]: *antisymp-on A* R^{-1-1} = *antisymp-on A R*
⟨*proof*⟩

lemma *trans-on-converse* [*simp*]: *trans-on A* (r^{-1}) = *trans-on A r*
⟨*proof*⟩

lemma *transp-on-conversep* [*simp*]: *transp-on A* R^{-1-1} = *transp-on A R*
⟨*proof*⟩

lemma *sym-conv-converse-eq*: *sym r* $\longleftrightarrow$ $r^{-1} = r$
⟨*proof*⟩

lemma *sym-Un-converse*: *sym* $(r \cup r^{-1})$
⟨*proof*⟩

lemma *sym-Int-converse*: *sym* $(r \cap r^{-1})$
⟨*proof*⟩

lemma *total-on-converse* [*simp*]: *total-on A* (r^{-1}) = *total-on A r*
⟨*proof*⟩

lemma *totalp-on-converse* [*simp*]: *totalp-on A* R^{-1-1} = *totalp-on A R*
⟨*proof*⟩

lemma *conversep-noteq* [*simp*]: $(\neq)^{-1-1} = (\neq)$
⟨*proof*⟩

lemma *conversep-eq* [*simp*]: $(=)^{-1-1} = (=)$
⟨*proof*⟩

lemma *converse-unfold* [*code*]: $r^{-1} = \{(y, x).\ (x, y) \in r\}$
⟨*proof*⟩

19.3.5 Domain, range and field

inductive-set *Domain* :: $('a \times 'b)$ *set* ⇒ $'a$ *set* **for** *r* :: $('a \times 'b)$ *set*
where *DomainI* [*intro*]: $(a, b) \in r \Longrightarrow a \in$ *Domain r*

lemmas *DomainPI* = *Domainp.DomainI*

inductive-cases *DomainE* [*elim*!]: *a* ∈ *Domain r*
inductive-cases *DomainpE* [*elim*!]: *Domainp r a*

inductive-set *Range* :: $('a \times 'b)$ *set* ⇒ $'b$ *set* **for** *r* :: $('a \times 'b)$ *set*
where *RangeI* [*intro*]: $(a, b) \in r \Longrightarrow b \in$ *Range r*

lemmas *RangePI* = *Rangep.RangeI*

inductive-cases *RangeE* [*elim*!]: $b \in Range\ r$
inductive-cases *RangepE* [*elim*!]: *Rangep r b*

definition *Field* :: *'a rel* ⇒ *'a set*
  where $Field\ r = Domain\ r \cup Range\ r$

lemma *Field-iff*: $x \in Field\ r \longleftrightarrow (\exists\, y.\ (x,y) \in r \vee (y,x) \in r)$
  ⟨*proof*⟩

lemma *FieldI1*: $(i, j) \in R \Longrightarrow i \in Field\ R$
  ⟨*proof*⟩

lemma *FieldI2*: $(i, j) \in R \Longrightarrow j \in Field\ R$
  ⟨*proof*⟩

lemma *Domain-fst* [*code*]: $Domain\ r = fst\ `\ r$
  ⟨*proof*⟩

lemma *Range-snd* [*code*]: $Range\ r = snd\ `\ r$
  ⟨*proof*⟩

lemma *fst-eq-Domain*: $fst\ `\ R = Domain\ R$
  ⟨*proof*⟩

lemma *snd-eq-Range*: $snd\ `\ R = Range\ R$
  ⟨*proof*⟩

lemma *range-fst* [*simp*]: *range fst* = *UNIV*
  ⟨*proof*⟩

lemma *range-snd* [*simp*]: *range snd* = *UNIV*
  ⟨*proof*⟩

lemma *Domain-empty* [*simp*]: $Domain\ \{\} = \{\}$
  ⟨*proof*⟩

lemma *Range-empty* [*simp*]: $Range\ \{\} = \{\}$
  ⟨*proof*⟩

lemma *Field-empty* [*simp*]: $Field\ \{\} = \{\}$
  ⟨*proof*⟩

lemma *Domain-empty-iff*: $Domain\ r = \{\} \longleftrightarrow r = \{\}$
  ⟨*proof*⟩

lemma *Range-empty-iff*: $Range\ r = \{\} \longleftrightarrow r = \{\}$
  ⟨*proof*⟩

lemma *Domain-insert* [*simp*]: *Domain* (*insert* (*a*, *b*) *r*) = *insert a* (*Domain r*)
⟨*proof*⟩

lemma *Range-insert* [*simp*]: *Range* (*insert* (*a*, *b*) *r*) = *insert b* (*Range r*)
⟨*proof*⟩

lemma *Field-insert* [*simp*]: *Field* (*insert* (*a*, *b*) *r*) = {*a*, *b*} ∪ *Field r*
⟨*proof*⟩

lemma *Domain-iff*: *a* ∈ *Domain r* ⟷ (∃ *y*. (*a*, *y*) ∈ *r*)
⟨*proof*⟩

lemma *Range-iff*: *a* ∈ *Range r* ⟷ (∃ *y*. (*y*, *a*) ∈ *r*)
⟨*proof*⟩

lemma *Domain-Id* [*simp*]: *Domain Id* = *UNIV*
⟨*proof*⟩

lemma *Range-Id* [*simp*]: *Range Id* = *UNIV*
⟨*proof*⟩

lemma *Domain-Id-on* [*simp*]: *Domain* (*Id-on A*) = *A*
⟨*proof*⟩

lemma *Range-Id-on* [*simp*]: *Range* (*Id-on A*) = *A*
⟨*proof*⟩

lemma *Domain-Un-eq*: *Domain* (*A* ∪ *B*) = *Domain A* ∪ *Domain B*
⟨*proof*⟩

lemma *Range-Un-eq*: *Range* (*A* ∪ *B*) = *Range A* ∪ *Range B*
⟨*proof*⟩

lemma *Field-Un* [*simp*]: *Field* (*r* ∪ *s*) = *Field r* ∪ *Field s*
⟨*proof*⟩

lemma *Domain-Int-subset*: *Domain* (*A* ∩ *B*) ⊆ *Domain A* ∩ *Domain B*
⟨*proof*⟩

lemma *Range-Int-subset*: *Range* (*A* ∩ *B*) ⊆ *Range A* ∩ *Range B*
⟨*proof*⟩

lemma *Domain-Diff-subset*: *Domain A* − *Domain B* ⊆ *Domain* (*A* − *B*)
⟨*proof*⟩

lemma *Range-Diff-subset*: *Range A* − *Range B* ⊆ *Range* (*A* − *B*)
⟨*proof*⟩

lemma *Domain-Union*: *Domain* $(\bigcup S) = (\bigcup A{\in}S.\ \mathit{Domain}\ A)$
⟨*proof*⟩

lemma *Range-Union*: *Range* $(\bigcup S) = (\bigcup A{\in}S.\ \mathit{Range}\ A)$
⟨*proof*⟩

lemma *Field-Union* [*simp*]: *Field* $(\bigcup R) = \bigcup(\mathit{Field}\ `\ R)$
⟨*proof*⟩

lemma *Domain-converse* [*simp*]: *Domain* $(r^{-1}) = \mathit{Range}\ r$
⟨*proof*⟩

lemma *Range-converse* [*simp*]: *Range* $(r^{-1}) = \mathit{Domain}\ r$
⟨*proof*⟩

lemma *Field-converse* [*simp*]: *Field* $(r^{-1}) = \mathit{Field}\ r$
⟨*proof*⟩

lemma *Domain-Collect-case-prod* [*simp*]: *Domain* $\{(x,\ y).\ P\ x\ y\} = \{x.\ \exists\, y.\ P\ x\ y\}$
⟨*proof*⟩

lemma *Range-Collect-case-prod* [*simp*]: *Range* $\{(x,\ y).\ P\ x\ y\} = \{y.\ \exists\, x.\ P\ x\ y\}$
⟨*proof*⟩

lemma *Domain-mono*: $r \subseteq s \Longrightarrow \mathit{Domain}\ r \subseteq \mathit{Domain}\ s$
⟨*proof*⟩

lemma *Range-mono*: $r \subseteq s \Longrightarrow \mathit{Range}\ r \subseteq \mathit{Range}\ s$
⟨*proof*⟩

lemma *mono-Field*: $r \subseteq s \Longrightarrow \mathit{Field}\ r \subseteq \mathit{Field}\ s$
⟨*proof*⟩

lemma *Domain-unfold*: $\mathit{Domain}\ r = \{x.\ \exists\, y.\ (x,\ y) \in r\}$
⟨*proof*⟩

lemma *Field-square* [*simp*]: *Field* $(x \times x) = x$
⟨*proof*⟩

19.3.6 Image of a set under a relation

definition *Image* :: $('a \times 'b)\ \mathit{set} \Rightarrow 'a\ \mathit{set} \Rightarrow 'b\ \mathit{set}$ (**infixr** `` 90)
where $r\ ``\ s = \{y.\ \exists\, x{\in}s.\ (x,\ y) \in r\}$

lemma *Image-iff*: $b \in r``A \longleftrightarrow (\exists\, x{\in}A.\ (x,\ b) \in r)$
⟨*proof*⟩

lemma *Image-singleton*: $r``\{a\} = \{b.\ (a,\ b) \in r\}$

⟨*proof*⟩

lemma *Image-singleton-iff* [*iff*]: $b \in r``\{a\} \longleftrightarrow (a, b) \in r$
⟨*proof*⟩

lemma *ImageI* [*intro*]: $(a, b) \in r \Longrightarrow a \in A \Longrightarrow b \in r``A$
⟨*proof*⟩

lemma *ImageE* [*elim!*]: $b \in r `` A \Longrightarrow (\bigwedge x.\ (x, b) \in r \Longrightarrow x \in A \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *rev-ImageI*: $a \in A \Longrightarrow (a, b) \in r \Longrightarrow b \in r `` A$
— This version's more effective when we already have the required a
⟨*proof*⟩

lemma *Image-empty1* [*simp*]: $\{\} `` X = \{\}$
⟨*proof*⟩

lemma *Image-empty2* [*simp*]: $R``\{\} = \{\}$
⟨*proof*⟩

lemma *Image-Id* [*simp*]: $Id `` A = A$
⟨*proof*⟩

lemma *Image-Id-on* [*simp*]: $Id\text{-}on\ A `` B = A \cap B$
⟨*proof*⟩

lemma *Image-Int-subset*: $R `` (A \cap B) \subseteq R `` A \cap R `` B$
⟨*proof*⟩

lemma *Image-Int-eq*: $single\text{-}valued\ (converse\ R) \Longrightarrow R `` (A \cap B) = R `` A \cap R `` B$
⟨*proof*⟩

lemma *Image-Un*: $R `` (A \cup B) = R `` A \cup R `` B$
⟨*proof*⟩

lemma *Un-Image*: $(R \cup S) `` A = R `` A \cup S `` A$
⟨*proof*⟩

lemma *Image-subset*: $r \subseteq A \times B \Longrightarrow r``C \subseteq B$
⟨*proof*⟩

lemma *Image-eq-UN*: $r``B = (\bigcup y \in B.\ r``\{y\})$
— NOT suitable for rewriting
⟨*proof*⟩

lemma *Image-mono*: $r' \subseteq r \Longrightarrow A' \subseteq A \Longrightarrow (r' `` A') \subseteq (r `` A)$
⟨*proof*⟩

lemma *Image-UN*: r `` $(\bigcup (B$ ` $A)) = (\bigcup x{\in}A.\ r$ `` $(B\ x))$
⟨*proof*⟩

lemma *UN-Image*: $(\bigcup i{\in}I.\ X\ i)$ `` $S = (\bigcup i{\in}I.\ X\ i$ `` $S)$
⟨*proof*⟩

lemma *Image-INT-subset*: $(r$ `` $(\bigcap (B$ ` $A))) \subseteq (\bigcap x{\in}A.\ r$ `` $(B\ x))$
⟨*proof*⟩

Converse inclusion requires some assumptions

lemma *Image-INT-eq*:
assumes *single-valued* (r^{-1})
and $A \neq \{\}$
shows r `` $(\bigcap (B$ ` $A)) = (\bigcap x{\in}A.\ r$ `` $B\ x)$
⟨*proof*⟩

lemma *Image-subset-eq*: r``$A \subseteq B \longleftrightarrow A \subseteq -\ ((r^{-1})$ `` $(-\ B))$
⟨*proof*⟩

lemma *Image-Collect-case-prod* [*simp*]: $\{(x,\ y).\ P\ x\ y\}$ `` $A = \{y.\ \exists x{\in}A.\ P\ x\ y\}$
⟨*proof*⟩

lemma *Sigma-Image*: $(SIGMA\ x{:}A.\ B\ x)$ `` $X = (\bigcup x{\in}X \cap A.\ B\ x)$
⟨*proof*⟩

lemma *relcomp-Image*: $(X\ O\ Y)$ `` $Z = Y$ `` $(X$ `` $Z)$
⟨*proof*⟩

19.3.7 Inverse image

definition *inv-image* :: $'b\ rel \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a\ rel$
where *inv-image* $r\ f = \{(x,\ y).\ (f\ x,\ f\ y) \in r\}$

definition *inv-imagep* :: $('b \Rightarrow 'b \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$
where *inv-imagep* $r\ f = (\lambda x\ y.\ r\ (f\ x)\ (f\ y))$

lemma [*pred-set-conv*]: *inv-imagep* $(\lambda x\ y.\ (x,\ y) \in r)\ f = (\lambda x\ y.\ (x,\ y) \in$ *inv-image* $r\ f)$
⟨*proof*⟩

lemma *sym-inv-image*: *sym* $r \Longrightarrow$ *sym* $(inv\text{-}image\ r\ f)$
⟨*proof*⟩

lemma *trans-inv-image*: *trans* $r \Longrightarrow$ *trans* $(inv\text{-}image\ r\ f)$
⟨*proof*⟩

lemma *total-inv-image*: $[\![inj\ f;\ total\ r]\!] \Longrightarrow total\ (inv\text{-}image\ r\ f)$
⟨*proof*⟩

lemma *asym-inv-image*: *asym R* $\Longrightarrow$ *asym* (*inv-image R f*)
⟨*proof*⟩

lemma *in-inv-image*[*simp*]: (*x*, *y*) ∈ *inv-image r f* $\longleftrightarrow$ (*f x*, *f y*) ∈ *r*
⟨*proof*⟩

lemma *converse-inv-image*[*simp*]: (*inv-image R f*)$^{-1}$ = *inv-image* (R^{-1}) *f*
⟨*proof*⟩

lemma *in-inv-imagep* [*simp*]: *inv-imagep r f x y* = *r* (*f x*) (*f y*)
⟨*proof*⟩

19.3.8 Powerset

definition *Powp* :: (′*a* ⇒ *bool*) ⇒ ′*a set* ⇒ *bool*
where *Powp A* = (λ*B*. ∀ *x* ∈ *B*. *A x*)

lemma *Powp-Pow-eq* [*pred-set-conv*]: *Powp* (λ*x*. *x* ∈ *A*) = (λ*x*. *x* ∈ *Pow A*)
⟨*proof*⟩

lemmas *Powp-mono* [*mono*] = *Pow-mono* [*to-pred*]

end

20 Finite sets

theory *Finite-Set*
imports *Product-Type Sum-Type Fields Relation*
begin

20.1 Predicate for finite sets

context notes [[*inductive-internals*]]
begin

inductive *finite* :: ′*a set* ⇒ *bool*
where
emptyI [*simp*, *intro*!]: *finite* {}
| *insertI* [*simp*, *intro*!]: *finite A* $\Longrightarrow$ *finite* (*insert a A*)

end

⟨*ML*⟩

declare [[*simproc del*: *finite-Collect*]]

lemma *finite-induct* [*case-names empty insert*, *induct set*: *finite*]:
— Discharging *x* ∉ *F* entails extra work.

assumes *finite F*
assumes *P* {}
and *insert*: $\bigwedge x\ F.\ \mathit{finite}\ F \Longrightarrow x \notin F \Longrightarrow P\ F \Longrightarrow P\ (\mathit{insert}\ x\ F)$
shows *P F*
⟨*proof*⟩

lemma *infinite-finite-induct* [*case-names infinite empty insert*]:
assumes *infinite*: $\bigwedge A.\ \neg\ \mathit{finite}\ A \Longrightarrow P\ A$
and *empty*: *P* {}
and *insert*: $\bigwedge x\ F.\ \mathit{finite}\ F \Longrightarrow x \notin F \Longrightarrow P\ F \Longrightarrow P\ (\mathit{insert}\ x\ F)$
shows *P A*
⟨*proof*⟩

20.1.1 Choice principles

lemma *ex-new-if-finite*: — does not depend on def of finite at all
assumes ¬ *finite* (*UNIV* :: *'a set*) **and** *finite A*
shows $\exists\, a{::}'a.\ a \notin A$
⟨*proof*⟩

A finite choice principle. Does not need the SOME choice operator.

lemma *finite-set-choice*: $\mathit{finite}\ A \Longrightarrow \forall\, x{\in}A.\ \exists\, y.\ P\ x\ y \Longrightarrow \exists f.\ \forall\, x{\in}A.\ P\ x\ (f\ x)$
⟨*proof*⟩

20.1.2 Finite sets are the images of initial segments of natural numbers

lemma *finite-imp-nat-seg-image-inj-on*:
assumes *finite A*
shows $\exists (n{::}\mathit{nat})\ f.\ A = f\ `\ \{i.\ i < n\} \wedge \mathit{inj\text{-}on}\ f\ \{i.\ i < n\}$
⟨*proof*⟩

lemma *nat-seg-image-imp-finite*: $A = f\ `\ \{i{::}\mathit{nat}.\ i < n\} \Longrightarrow \mathit{finite}\ A$
⟨*proof*⟩

lemma *finite-conv-nat-seg-image*: $\mathit{finite}\ A \longleftrightarrow (\exists\, n\ f.\ A = f\ `\ \{i{::}\mathit{nat}.\ i < n\})$
⟨*proof*⟩

lemma *finite-imp-inj-to-nat-seg*:
assumes *finite A*
shows $\exists f\ n.\ f\ `\ A = \{i{::}\mathit{nat}.\ i < n\} \wedge \mathit{inj\text{-}on}\ f\ A$
⟨*proof*⟩

lemma *finite-Collect-less-nat* [*iff*]: $\mathit{finite}\ \{n{::}\mathit{nat}.\ n < k\}$
⟨*proof*⟩

lemma *finite-Collect-le-nat* [*iff*]: $\mathit{finite}\ \{n{::}\mathit{nat}.\ n \leq k\}$
⟨*proof*⟩

20.2 Finiteness and common set operations

lemma *rev-finite-subset*: *finite* $B \Longrightarrow A \subseteq B \Longrightarrow$ *finite* A
⟨*proof*⟩

lemma *finite-subset*: $A \subseteq B \Longrightarrow$ *finite* $B \Longrightarrow$ *finite* A
⟨*proof*⟩

⟨*ML*⟩

declare [[*simproc del*: *finite*]]

lemma *finite-UnI*:
assumes *finite* F **and** *finite* G
shows *finite* $(F \cup G)$
⟨*proof*⟩

lemma *finite-Un* [*iff*]: *finite* $(F \cup G) \longleftrightarrow$ *finite* $F \wedge$ *finite* G
⟨*proof*⟩

lemma *finite-insert* [*simp*]: *finite* $(\mathit{insert}\ a\ A) \longleftrightarrow$ *finite* A
⟨*proof*⟩

lemma *finite-Int* [*simp*, *intro*]: *finite* $F \vee$ *finite* $G \Longrightarrow$ *finite* $(F \cap G)$
⟨*proof*⟩

lemma *finite-Collect-conjI* [*simp*, *intro*]:
finite $\{x.\ P\ x\} \vee$ *finite* $\{x.\ Q\ x\} \Longrightarrow$ *finite* $\{x.\ P\ x \wedge Q\ x\}$
⟨*proof*⟩

lemma *finite-Collect-disjI* [*simp*]:
finite $\{x.\ P\ x \vee Q\ x\} \longleftrightarrow$ *finite* $\{x.\ P\ x\} \wedge$ *finite* $\{x.\ Q\ x\}$
⟨*proof*⟩

lemma *finite-Diff* [*simp*, *intro*]: *finite* $A \Longrightarrow$ *finite* $(A - B)$
⟨*proof*⟩

lemma *finite-Diff2* [*simp*]:
assumes *finite* B
shows *finite* $(A - B) \longleftrightarrow$ *finite* A
⟨*proof*⟩

lemma *finite-Diff-insert* [*iff*]: *finite* $(A - \mathit{insert}\ a\ B) \longleftrightarrow$ *finite* $(A - B)$
⟨*proof*⟩

lemma *finite-compl* [*simp*]:
finite $(A :: 'a\ \mathit{set}) \Longrightarrow$ *finite* $(-A) \longleftrightarrow$ *finite* $(\mathit{UNIV} :: 'a\ \mathit{set})$
⟨*proof*⟩

lemma *finite-Collect-not* [*simp*]:
finite {*x* :: *'a*. *P x*} ⟹ *finite* {*x*. ¬ *P x*} ⟷ *finite* (*UNIV* :: *'a set*)
⟨*proof*⟩

lemma *finite-Union* [*simp*, *intro*]:
finite A ⟹ (⋀*M*. *M* ∈ *A* ⟹ *finite M*) ⟹ *finite* (⋃ *A*)
⟨*proof*⟩

lemma *finite-UN-I* [*intro*]:
finite A ⟹ (⋀*a*. *a* ∈ *A* ⟹ *finite* (*B a*)) ⟹ *finite* (⋃ *a*∈*A*. *B a*)
⟨*proof*⟩

lemma *finite-UN* [*simp*]: *finite A* ⟹ *finite* (⋃ (*B* ‘ *A*)) ⟷ (∀ *x*∈*A*. *finite* (*B x*))
⟨*proof*⟩

lemma *finite-Inter* [*intro*]: ∃ *A*∈*M*. *finite A* ⟹ *finite* (⋂ *M*)
⟨*proof*⟩

lemma *finite-INT* [*intro*]: ∃ *x*∈*I*. *finite* (*A x*) ⟹ *finite* (⋂ *x*∈*I*. *A x*)
⟨*proof*⟩

lemma *finite-imageI* [*simp*, *intro*]: *finite F* ⟹ *finite* (*h* ‘ *F*)
⟨*proof*⟩

lemma *finite-image-set* [*simp*]: *finite* {*x*. *P x*} ⟹ *finite* {*f x* |*x*. *P x*}
⟨*proof*⟩

lemma *finite-image-set2*:
finite {*x*. *P x*} ⟹ *finite* {*y*. *Q y*} ⟹ *finite* {*f x y* |*x y*. *P x* ∧ *Q y*}
⟨*proof*⟩

lemma *finite-imageD*:
assumes *finite* (*f* ‘ *A*) **and** *inj-on f A*
shows *finite A*
⟨*proof*⟩

lemma *finite-image-iff*: *inj-on f A* ⟹ *finite* (*f* ‘ *A*) ⟷ *finite A*
⟨*proof*⟩

lemma *finite-surj*: *finite A* ⟹ *B* ⊆ *f* ‘ *A* ⟹ *finite B*
⟨*proof*⟩

lemma *finite-range-imageI*: *finite* (*range g*) ⟹ *finite* (*range* (λ*x*. *f* (*g x*)))
⟨*proof*⟩

lemma *finite-subset-image*:
assumes *finite B*
shows *B* ⊆ *f* ‘ *A* ⟹ ∃ *C*⊆*A*. *finite C* ∧ *B* = *f* ‘ *C*
⟨*proof*⟩

lemma *all-subset-image*: $(\forall B.\ B \subseteq f\ `\ A \longrightarrow P\ B) \longleftrightarrow (\forall B.\ B \subseteq A \longrightarrow P(f\ `\ B))$
⟨*proof*⟩

lemma *all-finite-subset-image*:
$(\forall B.\ \mathit{finite}\ B \wedge B \subseteq f\ `\ A \longrightarrow P\ B) \longleftrightarrow (\forall B.\ \mathit{finite}\ B \wedge B \subseteq A \longrightarrow P\ (f\ `\ B))$
⟨*proof*⟩

lemma *ex-finite-subset-image*:
$(\exists B.\ \mathit{finite}\ B \wedge B \subseteq f\ `\ A \wedge P\ B) \longleftrightarrow (\exists B.\ \mathit{finite}\ B \wedge B \subseteq A \wedge P\ (f\ `\ B))$
⟨*proof*⟩

lemma *finite-vimage-IntI*: $\mathit{finite}\ F \Longrightarrow \mathit{inj\text{-}on}\ h\ A \Longrightarrow \mathit{finite}\ (h\ -`\ F \cap A)$
⟨*proof*⟩

lemma *finite-finite-vimage-IntI*:
assumes *finite F*
and $\bigwedge y.\ y \in F \Longrightarrow \mathit{finite}\ ((h\ -`\ \{y\}) \cap A)$
shows *finite* $(h\ -`\ F \cap A)$
⟨*proof*⟩

lemma *finite-vimageI*: $\mathit{finite}\ F \Longrightarrow \mathit{inj}\ h \Longrightarrow \mathit{finite}\ (h\ -`\ F)$
⟨*proof*⟩

lemma *finite-vimageD'*: $\mathit{finite}\ (f\ -`\ A) \Longrightarrow A \subseteq \mathit{range}\ f \Longrightarrow \mathit{finite}\ A$
⟨*proof*⟩

lemma *finite-vimageD*: $\mathit{finite}\ (h\ -`\ F) \Longrightarrow \mathit{surj}\ h \Longrightarrow \mathit{finite}\ F$
⟨*proof*⟩

lemma *finite-vimage-iff*: $\mathit{bij}\ h \Longrightarrow \mathit{finite}\ (h\ -`\ F) \longleftrightarrow \mathit{finite}\ F$
⟨*proof*⟩

lemma *finite-inverse-image-gen*:
assumes *finite A inj-on f D*
shows *finite* $\{j \in D.\ f\ j \in A\}$
⟨*proof*⟩

lemma *finite-inverse-image*:
assumes *finite A inj f*
shows *finite* $\{j.\ f\ j \in A\}$
⟨*proof*⟩

lemma *finite-Collect-bex* [*simp*]:
assumes *finite A*
shows *finite* $\{x.\ \exists y \in A.\ Q\ x\ y\} \longleftrightarrow (\forall y \in A.\ \mathit{finite}\ \{x.\ Q\ x\ y\})$
⟨*proof*⟩

lemma *finite-Collect-bounded-ex* [*simp*]:
assumes *finite* {*y*. *P y*}
shows *finite* {*x*. ∃ *y*. *P y* ∧ *Q x y*} ⟷ (∀ *y*. *P y* ⟶ *finite* {*x*. *Q x y*})
⟨*proof*⟩

lemma *finite-Plus*: *finite A* ⟹ *finite B* ⟹ *finite* (*A* <+> *B*)
⟨*proof*⟩

lemma *finite-PlusD*:
fixes *A* :: *'a set* **and** *B* :: *'b set*
assumes *fin*: *finite* (*A* <+> *B*)
shows *finite A finite B*
⟨*proof*⟩

lemma *finite-Plus-iff* [*simp*]: *finite* (*A* <+> *B*) ⟷ *finite A* ∧ *finite B*
⟨*proof*⟩

lemma *finite-Plus-UNIV-iff* [*simp*]:
finite (*UNIV* :: (*'a* + *'b*) *set*) ⟷ *finite* (*UNIV* :: *'a set*) ∧ *finite* (*UNIV* :: *'b set*)
⟨*proof*⟩

lemma *finite-SigmaI* [*simp*, *intro*]:
finite A ⟹ (⋀*a*. *a*∈*A* ⟹ *finite* (*B a*)) ⟹ *finite* (*SIGMA a*:*A*. *B a*)
⟨*proof*⟩

lemma *finite-SigmaI2*:
assumes *finite* {*x*∈*A*. *B x* ≠ {}}
and ⋀*a*. *a* ∈ *A* ⟹ *finite* (*B a*)
shows *finite* (*Sigma A B*)
⟨*proof*⟩

lemma *finite-cartesian-product*: *finite A* ⟹ *finite B* ⟹ *finite* (*A* × *B*)
⟨*proof*⟩

lemma *finite-Prod-UNIV*:
finite (*UNIV* :: *'a set*) ⟹ *finite* (*UNIV* :: *'b set*) ⟹ *finite* (*UNIV* :: (*'a* × *'b*) *set*)
⟨*proof*⟩

lemma *finite-cartesian-productD1*:
assumes *finite* (*A* × *B*) **and** *B* ≠ {}
shows *finite A*
⟨*proof*⟩

lemma *finite-cartesian-productD2*:
assumes *finite* (*A* × *B*) **and** *A* ≠ {}
shows *finite B*
⟨*proof*⟩

lemma *finite-cartesian-product-iff*:
finite $(A \times B) \longleftrightarrow (A = \{\} \vee B = \{\} \vee (\mathit{finite}\ A \wedge \mathit{finite}\ B))$
⟨*proof*⟩

lemma *finite-prod*:
finite $(UNIV :: ('a \times 'b)\ set) \longleftrightarrow \mathit{finite}\ (UNIV :: 'a\ set) \wedge \mathit{finite}\ (UNIV :: 'b\ set)$
⟨*proof*⟩

lemma *finite-Pow-iff* [*iff*]: *finite* $(Pow\ A) \longleftrightarrow \mathit{finite}\ A$
⟨*proof*⟩

corollary *finite-Collect-subsets* [*simp*, *intro*]: *finite* $A \Longrightarrow \mathit{finite}\ \{B.\ B \subseteq A\}$
⟨*proof*⟩

lemma *finite-set*: *finite* $(UNIV :: 'a\ set\ set) \longleftrightarrow \mathit{finite}\ (UNIV :: 'a\ set)$
⟨*proof*⟩

lemma *finite-UnionD*: *finite* $(\bigcup A) \Longrightarrow \mathit{finite}\ A$
⟨*proof*⟩

lemma *finite-bind*:
assumes *finite* S
assumes $\forall x \in S.\ \mathit{finite}\ (f\ x)$
shows *finite* $(Set.bind\ S\ f)$
⟨*proof*⟩

lemma *finite-filter* [*simp*]: *finite* $S \Longrightarrow \mathit{finite}\ (Set.filter\ P\ S)$
⟨*proof*⟩

lemma *finite-set-of-finite-funs*:
assumes *finite* A *finite* B
shows *finite* $\{f.\ \forall x.\ (x \in A \longrightarrow f\ x \in B) \wedge (x \notin A \longrightarrow f\ x = d)\}$ (**is** *finite* *?S*)
⟨*proof*⟩

lemma *not-finite-existsD*:
assumes $\neg$ *finite* $\{a.\ P\ a\}$
shows $\exists a.\ P\ a$
⟨*proof*⟩

lemma *finite-converse* [*iff*]: *finite* $(r^{-1}) \longleftrightarrow \mathit{finite}\ r$
⟨*proof*⟩

lemma *finite-Domain*: *finite* $r \Longrightarrow \mathit{finite}\ (Domain\ r)$
⟨*proof*⟩

lemma *finite-Range*: *finite* $r \Longrightarrow \mathit{finite}\ (Range\ r)$
⟨*proof*⟩

lemma *finite-Field*: *finite r* $\Longrightarrow$ *finite* (*Field r*)
⟨*proof*⟩

lemma *finite-Image*[*simp*]: *finite R* $\Longrightarrow$ *finite* (*R* `` *A*)
⟨*proof*⟩

20.3 Further induction rules on finite sets

lemma *finite-ne-induct* [*case-names singleton insert*, *consumes 2*]:
assumes *finite F* **and** $F \neq \{\}$
assumes $\bigwedge x.\ P\ \{x\}$
and $\bigwedge x\ F.$ *finite* $F \Longrightarrow F \neq \{\} \Longrightarrow x \notin F \Longrightarrow P\ F \Longrightarrow P$ (*insert x F*)
shows *P F*
⟨*proof*⟩

lemma *finite-subset-induct* [*consumes 2*, *case-names empty insert*]:
assumes *finite F* **and** $F \subseteq A$
and *empty*: *P* {}
and *insert*: $\bigwedge a\ F.$ *finite* $F \Longrightarrow a \in A \Longrightarrow a \notin F \Longrightarrow P\ F \Longrightarrow P$ (*insert a F*)
shows *P F*
⟨*proof*⟩

lemma *finite-empty-induct*:
assumes *finite A*
and *P A*
and *remove*: $\bigwedge a\ A.$ *finite* $A \Longrightarrow a \in A \Longrightarrow P\ A \Longrightarrow P\ (A - \{a\})$
shows *P* {}
⟨*proof*⟩

lemma *finite-update-induct* [*consumes 1*, *case-names const update*]:
assumes *finite*: *finite* $\{a.\ f\ a \neq c\}$
and *const*: *P* $(\lambda a.\ c)$
and *update*: $\bigwedge a\ b\ f.$ *finite* $\{a.\ f\ a \neq c\} \Longrightarrow f\ a = c \Longrightarrow b \neq c \Longrightarrow P\ f \Longrightarrow P$ $(f(a := b))$
shows *P f*
⟨*proof*⟩

lemma *finite-subset-induct'* [*consumes 2*, *case-names empty insert*]:
assumes *finite F* **and** $F \subseteq A$
and *empty*: *P* {}
and *insert*: $\bigwedge a\ F.\ [\![$*finite F*; $a \in A$; $F \subseteq A$; $a \notin F$; *P F* $]\!] \Longrightarrow P$ (*insert a F*)
shows *P F*
⟨*proof*⟩

20.4 Class *finite*

class *finite* =
assumes *finite-UNIV*: *finite* (*UNIV* :: *'a set*)
begin

lemma *finite* [*simp*]: *finite* (*A* :: *'a set*)
⟨*proof*⟩

lemma *finite-code* [*code*]: *finite* (*A* :: *'a set*) ⟷ *True*
⟨*proof*⟩

end

instance *prod* :: (*finite*, *finite*) *finite*
⟨*proof*⟩

lemma *inj-graph*: *inj* (λ*f*. {(*x*, *y*). *y* = *f x*})
⟨*proof*⟩

instance *fun* :: (*finite*, *finite*) *finite*
⟨*proof*⟩

instance *bool* :: *finite*
⟨*proof*⟩

instance *set* :: (*finite*) *finite*
⟨*proof*⟩

instance *unit* :: *finite*
⟨*proof*⟩

instance *sum* :: (*finite*, *finite*) *finite*
⟨*proof*⟩

20.5 A basic fold functional for finite sets

The intended behaviour is *fold f z* {$x_1, \ldots, x_n$} = $f\ x_1\ (\ldots\ (f\ x_n\ z)\ldots)$ if *f* is "left-commutative". The commutativity requirement is relativised to the carrier set *S*:

locale *comp-fun-commute-on* =
fixes *S* :: *'a set*
fixes *f* :: *'a* ⇒ *'b* ⇒ *'b*
assumes *comp-fun-commute-on*: *x* ∈ *S* ⟹ *y* ∈ *S* ⟹ *f y* ∘ *f x* = *f x* ∘ *f y*
begin

lemma *fun-left-comm*: *x* ∈ *S* ⟹ *y* ∈ *S* ⟹ *f y* (*f x z*) = *f x* (*f y z*)
⟨*proof*⟩

lemma *commute-left-comp*: *x* ∈ *S* ⟹ *y* ∈ *S* ⟹ *f y* ∘ (*f x* ∘ *g*) = *f x* ∘ (*f y* ∘ *g*)
⟨*proof*⟩

end

inductive *fold-graph* :: $('a \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'b \Rightarrow 'a\ set \Rightarrow 'b \Rightarrow bool$
for $f :: 'a \Rightarrow 'b \Rightarrow 'b$ **and** $z :: 'b$
where
emptyI [*intro*]: *fold-graph f z* {} *z*
| *insertI* [*intro*]: $x \notin A \Longrightarrow$ *fold-graph f z A y* $\Longrightarrow$ *fold-graph f z* (*insert x A*) (*f x y*)

inductive-cases *empty-fold-graphE* [*elim*!]: *fold-graph f z* {} *x*

lemma *fold-graph-closed-lemma*:
fold-graph f z A x $\wedge$ $x \in B$
if *fold-graph g z A x*
$\bigwedge a\ b.\ a \in A \Longrightarrow b \in B \Longrightarrow f\ a\ b = g\ a\ b$
$\bigwedge a\ b.\ a \in A \Longrightarrow b \in B \Longrightarrow g\ a\ b \in B$
$z \in B$
⟨*proof*⟩

lemma *fold-graph-closed-eq*:
fold-graph f z A = *fold-graph g z A*
if $\bigwedge a\ b.\ a \in A \Longrightarrow b \in B \Longrightarrow f\ a\ b = g\ a\ b$
$\bigwedge a\ b.\ a \in A \Longrightarrow b \in B \Longrightarrow g\ a\ b \in B$
$z \in B$
⟨*proof*⟩

definition *fold* :: $('a \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'b \Rightarrow 'a\ set \Rightarrow 'b$
where *fold f z A* = (*if finite A then* (*THE y. fold-graph f z A y*) *else z*)

lemma *fold-closed-eq*: *fold f z A* = *fold g z A*
if $\bigwedge a\ b.\ a \in A \Longrightarrow b \in B \Longrightarrow f\ a\ b = g\ a\ b$
$\bigwedge a\ b.\ a \in A \Longrightarrow b \in B \Longrightarrow g\ a\ b \in B$
$z \in B$
⟨*proof*⟩

A tempting alternative for the definition is *if finite A then THE y. fold-graph f z A y else e*. It allows the removal of finiteness assumptions from the theorems *fold-comm*, *fold-reindex* and *fold-distrib*. The proofs become ugly. It is not worth the effort. (???)

lemma *finite-imp-fold-graph*: *finite A* $\Longrightarrow \exists x.$ *fold-graph f z A x*
⟨*proof*⟩

20.5.1 From *fold-graph* to *fold*

context *comp-fun-commute-on*
begin

lemma *fold-graph-finite*:
assumes *fold-graph f z A y*
shows *finite A*
⟨*proof*⟩

lemma *fold-graph-insertE-aux*:
assumes $A \subseteq S$
assumes *fold-graph f z A y* $a \in A$
shows $\exists y'.\ y = f\ a\ y' \wedge$ *fold-graph f z* $(A - \{a\})\ y'$
⟨*proof*⟩

lemma *fold-graph-insertE*:
assumes *insert x A* $\subseteq S$
assumes *fold-graph f z* (*insert x A*) *v* **and** $x \notin A$
obtains *y* **where** $v = f\ x\ y$ **and** *fold-graph f z A y*
⟨*proof*⟩

lemma *fold-graph-determ*:
assumes $A \subseteq S$
assumes *fold-graph f z A x fold-graph f z A y*
shows $y = x$
⟨*proof*⟩

lemma *fold-equality*: $A \subseteq S \Longrightarrow$ *fold-graph f z A y* $\Longrightarrow$ *fold f z A* $= y$
⟨*proof*⟩

lemma *fold-graph-fold*:
assumes $A \subseteq S$
assumes *finite A*
shows *fold-graph f z A* (*fold f z A*)
⟨*proof*⟩

The base case for *fold*:

lemma (**in** −) *fold-infinite* [*simp*]: ¬ *finite A* $\Longrightarrow$ *fold f z A* $= z$
⟨*proof*⟩

lemma (**in** −) *fold-empty* [*simp*]: *fold f z* $\{\} = z$
⟨*proof*⟩

The various recursion equations for *fold*:

lemma *fold-insert* [*simp*]:
assumes *insert x A* $\subseteq S$
assumes *finite A* **and** $x \notin A$
shows *fold f z* (*insert x A*) $= f\ x$ (*fold f z A*)
⟨*proof*⟩

declare (**in** −) *empty-fold-graphE* [*rule del*] *fold-graph.intros* [*rule del*]
— No more proofs involve these.

lemma *fold-fun-left-comm*:
assumes *insert x A* $\subseteq S$ *finite A*
shows $f\ x$ (*fold f z A*) $=$ *fold f* $(f\ x\ z)\ A$
⟨*proof*⟩

lemma *fold-insert2*:
insert x A $\subseteq$ *S* $\Longrightarrow$ *finite A* $\Longrightarrow$ *x* $\notin$ *A* $\Longrightarrow$ *fold f z* (*insert x A*) = *fold f* (*f x z*) *A*
⟨*proof*⟩

lemma *fold-rec*:
assumes *A* $\subseteq$ *S*
assumes *finite A* **and** *x* $\in$ *A*
shows *fold f z A* = *f x* (*fold f z* (*A* − {*x*}))
⟨*proof*⟩

lemma *fold-insert-remove*:
assumes *insert x A* $\subseteq$ *S*
assumes *finite A*
shows *fold f z* (*insert x A*) = *f x* (*fold f z* (*A* − {*x*}))
⟨*proof*⟩

lemma *fold-set-union-disj*:
assumes *A* $\subseteq$ *S B* $\subseteq$ *S*
assumes *finite A finite B A* ∩ *B* = {}
shows *Finite-Set.fold f z* (*A* ∪ *B*) = *Finite-Set.fold f* (*Finite-Set.fold f z A*) *B*
⟨*proof*⟩

end

Other properties of *fold*:

lemma *fold-graph-image*:
assumes *inj-on g A*
shows *fold-graph f z* (*g* ‘ *A*) = *fold-graph* (*f* ∘ *g*) *z A*
⟨*proof*⟩

lemma *fold-image*:
assumes *inj-on g A*
shows *fold f z* (*g* ‘ *A*) = *fold* (*f* ∘ *g*) *z A*
⟨*proof*⟩

lemma *fold-cong*:
assumes *comp-fun-commute-on S f comp-fun-commute-on S g*
and *A* $\subseteq$ *S finite A*
and *cong*: $\bigwedge$*x*. *x* $\in$ *A* $\Longrightarrow$ *f x* = *g x*
and *s* = *t* **and** *A* = *B*
shows *fold f s A* = *fold g t B*
⟨*proof*⟩

A simplified version for idempotent functions:

locale *comp-fun-idem-on* = *comp-fun-commute-on* +
assumes *comp-fun-idem-on*: *x* $\in$ *S* $\Longrightarrow$ *f x* ∘ *f x* = *f x*

begin

lemma *fun-left-idem*: $x \in S \Longrightarrow f\ x\ (f\ x\ z) = f\ x\ z$
⟨*proof*⟩

lemma *fold-insert-idem*:
assumes *insert x A* ⊆ *S*
assumes *fin*: *finite A*
shows *fold f z* (*insert x A*) = *f x* (*fold f z A*)
⟨*proof*⟩

declare *fold-insert* [*simp del*] *fold-insert-idem* [*simp*]

lemma *fold-insert-idem2*: *insert x A* ⊆ *S* ⟹ *finite A* ⟹ *fold f z* (*insert x A*) = *fold f* (*f x z*) *A*
⟨*proof*⟩

end

20.5.2 Liftings to *comp-fun-commute-on* etc.

lemma (**in** *comp-fun-commute-on*) *comp-comp-fun-commute-on*:
range g ⊆ *S* ⟹ *comp-fun-commute-on R* (*f* ∘ *g*)
⟨*proof*⟩

lemma (**in** *comp-fun-idem-on*) *comp-comp-fun-idem-on*:
assumes *range g* ⊆ *S*
shows *comp-fun-idem-on R* (*f* ∘ *g*)
⟨*proof*⟩

lemma (**in** *comp-fun-commute-on*) *comp-fun-commute-on-funpow*:
comp-fun-commute-on S (λ*x*. *f x* ^^ *g x*)
⟨*proof*⟩

20.5.3 *UNIV* as carrier set

locale *comp-fun-commute* =
fixes *f* :: *'a* ⇒ *'b* ⇒ *'b*
assumes *comp-fun-commute*: *f y* ∘ *f x* = *f x* ∘ *f y*
begin

lemma (**in** −) *comp-fun-commute-def'*: *comp-fun-commute f* = *comp-fun-commute-on UNIV f*
⟨*proof*⟩

We abuse the *rewrites* functionality of locales to remove trivial assumptions that result from instantiating the carrier set to *UNIV*.

sublocale *comp-fun-commute-on UNIV f*
rewrites ⋀*X*. (*X* ⊆ *UNIV*) ≡ *True*

and $\bigwedge x.\ x \in UNIV \equiv True$
and $\bigwedge P.\ (True \Longrightarrow P) \equiv Trueprop\ P$
and $\bigwedge P\ Q.\ (True \Longrightarrow PROP\ P \Longrightarrow PROP\ Q) \equiv (PROP\ P \Longrightarrow True \Longrightarrow PROP\ Q)$
⟨*proof*⟩

end

lemma (**in** *comp-fun-commute*) *comp-comp-fun-commute*: *comp-fun-commute* (*f o g*)
⟨*proof*⟩

lemma (**in** *comp-fun-commute*) *comp-fun-commute-funpow*: *comp-fun-commute* (λ*x*. *f x* ^^ *g x*)
⟨*proof*⟩

locale *comp-fun-idem* = *comp-fun-commute* +
assumes *comp-fun-idem*: *f x o f x* = *f x*
begin

lemma (**in** −) *comp-fun-idem-def'*: *comp-fun-idem f* = *comp-fun-idem-on UNIV f*
⟨*proof*⟩

Again, we abuse the *rewrites* functionality of locales to remove trivial assumptions that result from instantiating the carrier set to *UNIV*.

sublocale *comp-fun-idem-on UNIV f*
rewrites $\bigwedge X.\ (X \subseteq UNIV) \equiv True$
and $\bigwedge x.\ x \in UNIV \equiv True$
and $\bigwedge P.\ (True \Longrightarrow P) \equiv Trueprop\ P$
and $\bigwedge P\ Q.\ (True \Longrightarrow PROP\ P \Longrightarrow PROP\ Q) \equiv (PROP\ P \Longrightarrow True \Longrightarrow PROP\ Q)$
⟨*proof*⟩

end

lemma (**in** *comp-fun-idem*) *comp-comp-fun-idem*: *comp-fun-idem* (*f o g*)
⟨*proof*⟩

20.5.4 Expressing set operations via *fold*

lemma *comp-fun-commute-const*: *comp-fun-commute* (λ-. *f*)
⟨*proof*⟩

lemma *comp-fun-idem-insert*: *comp-fun-idem insert*
⟨*proof*⟩

lemma *comp-fun-idem-remove*: *comp-fun-idem Set.remove*
⟨*proof*⟩

lemma (**in** *semilattice-inf*) *comp-fun-idem-inf*: *comp-fun-idem inf*
⟨*proof*⟩

lemma (**in** *semilattice-sup*) *comp-fun-idem-sup*: *comp-fun-idem sup*
⟨*proof*⟩

lemma *union-fold-insert*:
assumes *finite A*
shows $A \cup B = fold\ insert\ B\ A$
⟨*proof*⟩

lemma *minus-fold-remove*:
assumes *finite A*
shows $B - A = fold\ Set.remove\ B\ A$
⟨*proof*⟩

lemma *comp-fun-commute-filter-fold*:
comp-fun-commute $(\lambda x\ A'.\ if\ P\ x\ then\ Set.insert\ x\ A'\ else\ A')$
⟨*proof*⟩

lemma *Set-filter-fold*:
assumes *finite A*
shows $Set.filter\ P\ A = fold\ (\lambda x\ A'.\ if\ P\ x\ then\ Set.insert\ x\ A'\ else\ A')\ \{\}\ A$
⟨*proof*⟩

lemma *inter-Set-filter*:
assumes *finite B*
shows $A \cap B = Set.filter\ (\lambda x.\ x \in A)\ B$
⟨*proof*⟩

lemma *image-fold-insert*:
assumes *finite A*
shows $image\ f\ A = fold\ (\lambda k\ A.\ Set.insert\ (f\ k)\ A)\ \{\}\ A$
⟨*proof*⟩

lemma *Ball-fold*:
assumes *finite A*
shows $Ball\ A\ P = fold\ (\lambda k\ s.\ s \wedge P\ k)\ True\ A$
⟨*proof*⟩

lemma *Bex-fold*:
assumes *finite A*
shows $Bex\ A\ P = fold\ (\lambda k\ s.\ s \vee P\ k)\ False\ A$
⟨*proof*⟩

lemma *comp-fun-commute-Pow-fold*: $comp\text{-}fun\text{-}commute\ (\lambda x\ A.\ A \cup Set.insert\ x\ `\ A)$
⟨*proof*⟩

lemma *Pow-fold*:
assumes *finite A*
shows *Pow A* = *fold* (λ*x A*. *A* ∪ *Set.insert x* ' *A*) {{}} *A*
⟨*proof*⟩

lemma *fold-union-pair*:
assumes *finite B*
shows (⋃ *y*∈*B*. {(*x*, *y*)}) ∪ *A* = *fold* (λ*y*. *Set.insert* (*x*, *y*)) *A B*
⟨*proof*⟩

lemma *comp-fun-commute-product-fold*:
finite B ⟹ *comp-fun-commute* (λ*x z*. *fold* (λ*y*. *Set.insert* (*x*, *y*)) *z B*)
⟨*proof*⟩

lemma *product-fold*:
assumes *finite A finite B*
shows *A* × *B* = *fold* (λ*x z*. *fold* (λ*y*. *Set.insert* (*x*, *y*)) *z B*) {} *A*
⟨*proof*⟩

context *complete-lattice*
begin

lemma *inf-Inf-fold-inf*:
assumes *finite A*
shows *inf* (*Inf A*) *B* = *fold inf B A*
⟨*proof*⟩

lemma *sup-Sup-fold-sup*:
assumes *finite A*
shows *sup* (*Sup A*) *B* = *fold sup B A*
⟨*proof*⟩

lemma *Inf-fold-inf*: *finite A* ⟹ *Inf A* = *fold inf top A*
⟨*proof*⟩

lemma *Sup-fold-sup*: *finite A* ⟹ *Sup A* = *fold sup bot A*
⟨*proof*⟩

lemma *inf-INF-fold-inf*:
assumes *finite A*
shows *inf B* (⨅(*f* ' *A*)) = *fold* (*inf* ∘ *f*) *B A* (**is** *?inf* = *?fold*)
⟨*proof*⟩

lemma *sup-SUP-fold-sup*:
assumes *finite A*
shows *sup B* (⨆(*f* ' *A*)) = *fold* (*sup* ∘ *f*) *B A* (**is** *?sup* = *?fold*)
⟨*proof*⟩

lemma *INF-fold-inf*: *finite A* $\Longrightarrow \prod (f \text{ ` } A) = \mathit{fold}\ (\mathit{inf} \circ f)\ \mathit{top}\ A$
⟨*proof*⟩

lemma *SUP-fold-sup*: *finite A* $\Longrightarrow \bigsqcup (f \text{ ` } A) = \mathit{fold}\ (\mathit{sup} \circ f)\ \mathit{bot}\ A$
⟨*proof*⟩

lemma *finite-Inf-in*:
assumes *finite A* $A \neq \{\}$ **and** *inf*: $\bigwedge x\ y.\ [\![x \in A;\ y \in A]\!] \Longrightarrow \mathit{inf}\ x\ y \in A$
shows *Inf A* $\in A$
⟨*proof*⟩

lemma *finite-Sup-in*:
assumes *finite A* $A \neq \{\}$ **and** *sup*: $\bigwedge x\ y.\ [\![x \in A;\ y \in A]\!] \Longrightarrow \mathit{sup}\ x\ y \in A$
shows *Sup A* $\in A$
⟨*proof*⟩

end

20.5.5 Expressing relation operations via *fold*

lemma *Id-on-fold*:
assumes *finite A*
shows *Id-on A* = *Finite-Set.fold* $(\lambda x.\ \mathit{Set.insert}\ (\mathit{Pair}\ x\ x))\ \{\}\ A$
⟨*proof*⟩

lemma *comp-fun-commute-Image-fold*:
comp-fun-commute $(\lambda(x,y)\ A.\ \mathit{if}\ x \in S\ \mathit{then}\ \mathit{Set.insert}\ y\ A\ \mathit{else}\ A)$
⟨*proof*⟩

lemma *Image-fold*:
assumes *finite R*
shows R `` S = *Finite-Set.fold* $(\lambda(x,y)\ A.\ \mathit{if}\ x \in S\ \mathit{then}\ \mathit{Set.insert}\ y\ A\ \mathit{else}\ A)$ $\{\}\ R$
⟨*proof*⟩

lemma *insert-relcomp-union-fold*:
assumes *finite S*
shows $\{x\}\ O\ S \cup X$ = *Finite-Set.fold* $(\lambda(w,z)\ A'.\ \mathit{if}\ \mathit{snd}\ x = w\ \mathit{then}\ \mathit{Set.insert}\ (\mathit{fst}\ x,z)\ A'\ \mathit{else}\ A')\ X\ S$
⟨*proof*⟩

lemma *insert-relcomp-fold*:
assumes *finite S*
shows *Set.insert x R O S* =
Finite-Set.fold $(\lambda(w,z)\ A'.\ \mathit{if}\ \mathit{snd}\ x = w\ \mathit{then}\ \mathit{Set.insert}\ (\mathit{fst}\ x,z)\ A'\ \mathit{else}\ A')\ (R\ O\ S)\ S$
⟨*proof*⟩

lemma *comp-fun-commute-relcomp-fold*:

assumes *finite S*
shows *comp-fun-commute* (λ(*x*,*y*) *A*.
Finite-Set.fold (λ(*w*,*z*) *A'*. *if y = w then Set.insert* (*x*,*z*) *A'* *else A'*) *A S*)
⟨*proof*⟩

lemma *relcomp-fold*:
assumes *finite R finite S*
shows *R O S = Finite-Set.fold*
(λ(*x*,*y*) *A*. *Finite-Set.fold* (λ(*w*,*z*) *A'*. *if y = w then Set.insert* (*x*,*z*) *A'* *else A'*)
A S) {} *R*
⟨*proof*⟩

20.6 Locales as mini-packages for fold operations

20.6.1 The natural case

locale *folding-on* =
fixes *S* :: *'a set*
fixes *f* :: *'a* ⇒ *'b* ⇒ *'b* **and** *z* :: *'b*
assumes *comp-fun-commute-on*: *x* ∈ *S* ⟹ *y* ∈ *S* ⟹ *f y o f x = f x o f y*
begin

interpretation *fold?*: *comp-fun-commute-on S f*
⟨*proof*⟩

definition *F* :: *'a set* ⇒ *'b*
where *eq-fold*: *F A = Finite-Set.fold f z A*

lemma *empty* [*simp*]: *F* {} = *z*
⟨*proof*⟩

lemma *infinite* [*simp*]: ¬ *finite A* ⟹ *F A = z*
⟨*proof*⟩

lemma *insert* [*simp*]:
assumes *insert x A* ⊆ *S* **and** *finite A* **and** *x* ∉ *A*
shows *F* (*insert x A*) = *f x* (*F A*)
⟨*proof*⟩

lemma *remove*:
assumes *A* ⊆ *S* **and** *finite A* **and** *x* ∈ *A*
shows *F A = f x* (*F* (*A* − {*x*}))
⟨*proof*⟩

lemma *insert-remove*:
assumes *insert x A* ⊆ *S* **and** *finite A*
shows *F* (*insert x A*) = *f x* (*F* (*A* − {*x*}))
⟨*proof*⟩

end

20.6.2 With idempotency

locale *folding-idem-on* = *folding-on* +
  assumes *comp-fun-idem-on*: $x \in S \Longrightarrow y \in S \Longrightarrow f\ x \circ f\ x = f\ x$
begin

declare *insert* [*simp del*]

interpretation *fold?*: *comp-fun-idem-on S f*
  ⟨*proof*⟩

lemma *insert-idem* [*simp*]:
  assumes *insert x A* ⊆ *S* **and** *finite A*
  shows *F* (*insert x A*) = *f x* (*F A*)
⟨*proof*⟩

end

20.6.3 *UNIV* as the carrier set

locale *folding* =
  fixes *f* :: $'a \Rightarrow 'b \Rightarrow 'b$ **and** *z* :: $'b$
  assumes *comp-fun-commute*: *f y* ∘ *f x* = *f x* ∘ *f y*
begin

lemma (**in** −) *folding-def'*: *folding f* = *folding-on UNIV f*
  ⟨*proof*⟩

Again, we abuse the *rewrites* functionality of locales to remove trivial assumptions that result from instantiating the carrier set to *UNIV*.

sublocale *folding-on UNIV f*
  rewrites $\bigwedge X.\ (X \subseteq UNIV) \equiv True$
    and $\bigwedge x.\ x \in UNIV \equiv True$
    and $\bigwedge P.\ (True \Longrightarrow P) \equiv Trueprop\ P$
    and $\bigwedge P\ Q.\ (True \Longrightarrow PROP\ P \Longrightarrow PROP\ Q) \equiv (PROP\ P \Longrightarrow True \Longrightarrow PROP\ Q)$
⟨*proof*⟩

end

locale *folding-idem* = *folding* +
  assumes *comp-fun-idem*: *f x* ∘ *f x* = *f x*
begin

lemma (**in** −) *folding-idem-def'*: *folding-idem f* = *folding-idem-on UNIV f*
  ⟨*proof*⟩

Again, we abuse the *rewrites* functionality of locales to remove trivial assumptions that result from instantiating the carrier set to *UNIV*.

sublocale *folding-idem-on UNIV f*

rewrites $\bigwedge X.\ (X \subseteq UNIV) \equiv True$
and $\bigwedge x.\ x \in UNIV \equiv True$
and $\bigwedge P.\ (True \Longrightarrow P) \equiv Trueprop\ P$
and $\bigwedge P\ Q.\ (True \Longrightarrow PROP\ P \Longrightarrow PROP\ Q) \equiv (PROP\ P \Longrightarrow True \Longrightarrow PROP\ Q)$
⟨*proof*⟩

end

20.7 Finite cardinality

The traditional definition $card\ A \equiv LEAST\ n.\ \exists f.\ A = \{f\ i\ |i.\ i < n\}$ is ugly to work with. But now that we have *fold* things are easy:

global-interpretation *card*: *folding* λ-. *Suc 0*
defines $card = folding\text{-}on.F\ (\lambda\text{-}.\ Suc)\ 0$
⟨*proof*⟩

lemma *card-insert-disjoint*: $finite\ A \Longrightarrow x \notin A \Longrightarrow card\ (insert\ x\ A) = Suc\ (card\ A)$
⟨*proof*⟩

lemma *card-insert-if*: $finite\ A \Longrightarrow card\ (insert\ x\ A) = (if\ x \in A\ then\ card\ A\ else\ Suc\ (card\ A))$
⟨*proof*⟩

lemma *card-ge-0-finite*: $card\ A > 0 \Longrightarrow finite\ A$
⟨*proof*⟩

lemma *card-0-eq* [*simp*]: $finite\ A \Longrightarrow card\ A = 0 \longleftrightarrow A = \{\}$
⟨*proof*⟩

lemma *finite-UNIV-card-ge-0*: $finite\ (UNIV :: 'a\ set) \Longrightarrow card\ (UNIV :: 'a\ set) > 0$
⟨*proof*⟩

lemma *card-eq-0-iff*: $card\ A = 0 \longleftrightarrow A = \{\} \vee \neg\ finite\ A$
⟨*proof*⟩

lemma *card-range-greater-zero*: $finite\ (range\ f) \Longrightarrow card\ (range\ f) > 0$
⟨*proof*⟩

lemma *card-gt-0-iff*: $0 < card\ A \longleftrightarrow A \neq \{\} \wedge finite\ A$
⟨*proof*⟩

lemma *card-Suc-Diff1*:
assumes $finite\ A\ x \in A$ **shows** $Suc\ (card\ (A - \{x\})) = card\ A$
⟨*proof*⟩

lemma *card-insert-le-m1*:

assumes $n > 0$ *card* $y \leq n - 1$ **shows** *card* (*insert* x y) $\leq n$
⟨*proof*⟩

lemma *card-Diff-singleton*:
assumes $x \in A$ **shows** *card* $(A - \{x\})$ = *card* $A - 1$
⟨*proof*⟩

lemma *card-Diff-singleton-if*:
card $(A - \{x\})$ = (*if* $x \in A$ *then card* $A - 1$ *else card* A)
⟨*proof*⟩

lemma *card-Diff-insert*[*simp*]:
assumes $a \in A$ **and** $a \notin B$
shows *card* $(A$ − *insert* a $B)$ = *card* $(A - B) - 1$
⟨*proof*⟩

lemma *card-insert-le*: *card* $A \leq$ *card* (*insert* x A)
⟨*proof*⟩

lemma *card-Collect-less-nat*[*simp*]: *card* $\{i$::*nat*. $i < n\} = n$
⟨*proof*⟩

lemma *card-Collect-le-nat*[*simp*]: *card* $\{i$::*nat*. $i \leq n\}$ = *Suc* n
⟨*proof*⟩

lemma *card-mono*:
assumes *finite* B **and** $A \subseteq B$
shows *card* $A \leq$ *card* B
⟨*proof*⟩

lemma *card-seteq*:
assumes *finite* B **and** A: $A \subseteq B$ *card* $B \leq$ *card* A
shows $A = B$
⟨*proof*⟩

lemma *psubset-card-mono*: *finite* $B \Longrightarrow A < B \Longrightarrow$ *card* $A <$ *card* B
⟨*proof*⟩

lemma *card-Un-Int*:
assumes *finite* A *finite* B
shows *card* A + *card* B = *card* $(A \cup B)$ + *card* $(A \cap B)$
⟨*proof*⟩

lemma *card-Un-disjoint*: *finite* $A \Longrightarrow$ *finite* $B \Longrightarrow A \cap B = \{\} \Longrightarrow$ *card* $(A \cup B)$ = *card* A + *card* B
⟨*proof*⟩

lemma *card-Un-disjnt*: ⟦*finite* A; *finite* B; *disjnt* A B⟧ ⟹ *card* $(A \cup B)$ = *card* A + *card* B

⟨*proof*⟩

lemma *card-Un-le*: *card* $(A \cup B) \leq$ *card* $A +$ *card* B
⟨*proof*⟩

lemma *card-Diff-subset*:
assumes *finite* B
and $B \subseteq A$
shows *card* $(A - B) =$ *card* $A -$ *card* B
⟨*proof*⟩

lemma *card-Diff-subset-Int*:
assumes *finite* $(A \cap B)$
shows *card* $(A - B) =$ *card* $A -$ *card* $(A \cap B)$
⟨*proof*⟩

lemma *diff-card-le-card-Diff*:
assumes *finite* B
shows *card* $A -$ *card* $B \leq$ *card* $(A - B)$
⟨*proof*⟩

lemma *card-le-sym-Diff*:
assumes *finite* A *finite* B *card* $A \leq$ *card* B
shows *card*$(A - B) \leq$ *card*$(B - A)$
⟨*proof*⟩

lemma *card-less-sym-Diff*:
assumes *finite* A *finite* B *card* $A <$ *card* B
shows *card*$(A - B) <$ *card*$(B - A)$
⟨*proof*⟩

lemma *card-Diff1-less-iff*: *card* $(A - \{x\}) <$ *card* $A \longleftrightarrow$ *finite* $A \wedge x \in A$
⟨*proof*⟩

lemma *card-Diff1-less*: *finite* $A \Longrightarrow x \in A \Longrightarrow$ *card* $(A - \{x\}) <$ *card* A
⟨*proof*⟩

lemma *card-Diff2-less*:
assumes *finite* A $x \in A$ $y \in A$ **shows** *card* $(A - \{x\} - \{y\}) <$ *card* A
⟨*proof*⟩

lemma *card-Diff1-le*: *card* $(A - \{x\}) \leq$ *card* A
⟨*proof*⟩

lemma *card-psubset*: *finite* $B \Longrightarrow A \subseteq B \Longrightarrow$ *card* $A <$ *card* $B \Longrightarrow A < B$
⟨*proof*⟩

lemma *card-le-inj*:
assumes *fA*: *finite* A

and *fB*: *finite B*
and *c*: *card A* ≤ *card B*
shows ∃*f*. *f* ' *A* ⊆ *B* ∧ *inj-on f A*
⟨*proof*⟩

lemma *card-subset-eq*:
assumes *fB*: *finite B*
and *AB*: *A* ⊆ *B*
and *c*: *card A* = *card B*
shows *A* = *B*
⟨*proof*⟩

lemma *insert-partition*:
x ∉ *F* ⟹ ∀ *c1* ∈ *insert x F*. ∀ *c2* ∈ *insert x F*. *c1* ≠ *c2* ⟶ *c1* ∩ *c2* = {} ⟹ *x* ∩ ⋃ *F* = {}
⟨*proof*⟩

lemma *finite-psubset-induct* [*consumes 1*, *case-names psubset*]:
assumes *finite*: *finite A*
and *major*: ⋀*A*. *finite A* ⟹ (⋀*B*. *B* ⊂ *A* ⟹ *P B*) ⟹ *P A*
shows *P A*
⟨*proof*⟩

lemma *finite-induct-select* [*consumes 1*, *case-names empty select*]:
assumes *finite S*
and *P* {}
and *select*: ⋀ *T*. *T* ⊂ *S* ⟹ *P T* ⟹ ∃ *s*∈*S* − *T*. *P* (*insert s T*)
shows *P S*
⟨*proof*⟩

lemma *remove-induct* [*case-names empty infinite remove*]:
assumes *empty*: *P* ({} :: *'a set*)
and *infinite*: ¬ *finite B* ⟹ *P B*
and *remove*: ⋀*A*. *finite A* ⟹ *A* ≠ {} ⟹ *A* ⊆ *B* ⟹ (⋀*x*. *x* ∈ *A* ⟹ *P* (*A* − {*x*})) ⟹ *P A*
shows *P B*
⟨*proof*⟩

lemma *finite-remove-induct* [*consumes 1*, *case-names empty remove*]:
fixes *P* :: *'a set* ⇒ *bool*
assumes *finite B*
and *P* {}
and ⋀*A*. *finite A* ⟹ *A* ≠ {} ⟹ *A* ⊆ *B* ⟹ (⋀*x*. *x* ∈ *A* ⟹ *P* (*A* − {*x*})) ⟹ *P A*
defines *B'* ≡ *B*
shows *P B'*
⟨*proof*⟩

Main cardinality theorem.

lemma *card-partition* [*rule-format*]:
finite C $\Longrightarrow$ *finite* $(\bigcup C)$ $\Longrightarrow$ $(\forall c{\in}C.\ card\ c = k) \Longrightarrow$
$(\forall c1 \in C.\ \forall c2 \in C.\ c1 \neq c2 \longrightarrow c1 \cap c2 = \{\}) \Longrightarrow$
$k * card\ C = card\ (\bigcup C)$
⟨*proof*⟩

lemma *card-eq-UNIV-imp-eq-UNIV*:
assumes *fin*: *finite* (*UNIV* :: *'a set*)
and *card*: *card A* = *card* (*UNIV* :: *'a set*)
shows *A* = (*UNIV* :: *'a set*)
⟨*proof*⟩

The form of a finite set of given cardinality

lemma *card-eq-SucD*:
assumes *card A* = *Suc k*
shows $\exists b\ B.\ A = insert\ b\ B \wedge b \notin B \wedge card\ B = k \wedge (k = 0 \longrightarrow B = \{\})$
⟨*proof*⟩

lemma *card-Suc-eq*:
card A = *Suc k* $\longleftrightarrow$
$(\exists b\ B.\ A = insert\ b\ B \wedge b \notin B \wedge card\ B = k \wedge (k = 0 \longrightarrow B = \{\}))$
⟨*proof*⟩

lemma *card-Suc-eq-finite*:
card A = *Suc k* $\longleftrightarrow$ $(\exists b\ B.\ A = insert\ b\ B \wedge b \notin B \wedge card\ B = k \wedge finite\ B)$
⟨*proof*⟩

lemma *card-1-singletonE*:
assumes *card A* = *1*
obtains *x* **where** *A* = $\{x\}$
⟨*proof*⟩

lemma *is-singleton-altdef*: *is-singleton A* $\longleftrightarrow$ *card A* = *1*
⟨*proof*⟩

lemma *card-1-singleton-iff*: *card A* = *Suc 0* $\longleftrightarrow$ $(\exists x.\ A = \{x\})$
⟨*proof*⟩

lemma *card-le-Suc0-iff-eq*:
assumes *finite A*
shows *card A* $\leq$ *Suc 0* $\longleftrightarrow$ $(\forall a1 \in A.\ \forall a2 \in A.\ a1 = a2)$ (**is** *?C* = *?A*)
⟨*proof*⟩

lemma *card-le-Suc-iff*:
Suc n $\leq$ *card A* = $(\exists a\ B.\ A = insert\ a\ B \wedge a \notin B \wedge n \leq card\ B \wedge finite\ B)$
⟨*proof*⟩

lemma *finite-fun-UNIVD2*:
assumes *fin*: *finite* (*UNIV* :: (*'a* ⇒ *'b*) *set*)

shows *finite* (*UNIV* :: *'b set*)
⟨*proof*⟩

lemma *card-UNIV-unit* [*simp*]: *card* (*UNIV* :: *unit set*) = *1*
⟨*proof*⟩

lemma *infinite-arbitrarily-large*:
assumes ¬ *finite A*
shows ∃ *B*. *finite B* ∧ *card B* = *n* ∧ *B* ⊆ *A*
⟨*proof*⟩

Sometimes, to prove that a set is finite, it is convenient to work with finite subsets and to show that their cardinalities are uniformly bounded. This possibility is formalized in the next criterion.

lemma *finite-if-finite-subsets-card-bdd*:
assumes ⋀*G*. *G* ⊆ *F* ⟹ *finite G* ⟹ *card G* ≤ *C*
shows *finite F* ∧ *card F* ≤ *C*
⟨*proof*⟩

lemma *obtain-subset-with-card-n*:
assumes *n* ≤ *card S*
obtains *T* **where** *T* ⊆ *S card T* = *n finite T*
⟨*proof*⟩

lemma *exists-subset-between*:
assumes
card A ≤ *n*
n ≤ *card C*
A ⊆ *C*
finite C
shows ∃ *B*. *A* ⊆ *B* ∧ *B* ⊆ *C* ∧ *card B* = *n*
⟨*proof*⟩

20.7.1 Cardinality of image

lemma *card-image-le*: *finite A* ⟹ *card* (*f* ‘ *A*) ≤ *card A*
⟨*proof*⟩

lemma *card-image*: *inj-on f A* ⟹ *card* (*f* ‘ *A*) = *card A*
⟨*proof*⟩

lemma *bij-betw-same-card*: *bij-betw f A B* ⟹ *card A* = *card B*
⟨*proof*⟩

lemma *endo-inj-surj*: *finite A* ⟹ *f* ‘ *A* ⊆ *A* ⟹ *inj-on f A* ⟹ *f* ‘ *A* = *A*
⟨*proof*⟩

lemma *eq-card-imp-inj-on*:
assumes *finite A card*(*f* ‘ *A*) = *card A*

shows *inj-on f A*
⟨*proof*⟩

lemma *inj-on-iff-eq-card*: *finite A* ⟹ *inj-on f A* ⟷ *card* (*f* ‘ *A*) = *card A*
⟨*proof*⟩

lemma *card-inj-on-le*:
assumes *inj-on f A f* ‘ *A* ⊆ *B finite B*
shows *card A* ≤ *card B*
⟨*proof*⟩

lemma *inj-on-iff-card-le*:
⟦ *finite A*; *finite B* ⟧ ⟹ (∃*f*. *inj-on f A* ∧ *f* ‘ *A* ≤ *B*) = (*card A* ≤ *card B*)
⟨*proof*⟩

lemma *surj-card-le*: *finite A* ⟹ *B* ⊆ *f* ‘ *A* ⟹ *card B* ≤ *card A*
⟨*proof*⟩

lemma *card-bij-eq*:
inj-on f A ⟹ *f* ‘ *A* ⊆ *B* ⟹ *inj-on g B* ⟹ *g* ‘ *B* ⊆ *A* ⟹ *finite A* ⟹ *finite B*
⟹ *card A* = *card B*
⟨*proof*⟩

lemma *bij-betw-finite*: *bij-betw f A B* ⟹ *finite A* ⟷ *finite B*
⟨*proof*⟩

lemma *inj-on-finite*: *inj-on f A* ⟹ *f* ‘ *A* ≤ *B* ⟹ *finite B* ⟹ *finite A*
⟨*proof*⟩

lemma *card-vimage-inj-on-le*:
assumes *inj-on f D finite A*
shows *card* (*f*−‘*A* ∩ *D*) ≤ *card A*
⟨*proof*⟩

lemma *card-vimage-inj*: *inj f* ⟹ *A* ⊆ *range f* ⟹ *card* (*f* −‘ *A*) = *card A*
⟨*proof*⟩

lemma *card-inverse*[*simp*]: *card* (R^{-1}) = *card R*
⟨*proof*⟩

20.7.2 Pigeonhole Principles

lemma *pigeonhole*: *card A* > *card* (*f* ‘ *A*) ⟹ ¬ *inj-on f A*
⟨*proof*⟩

lemma *pigeonhole-infinite*:
assumes ¬ *finite A* **and** *finite* (*f*‘*A*)
shows ∃ *a0*∈*A*. ¬ *finite* {*a*∈*A*. *f a* = *f a0*}
⟨*proof*⟩

lemma *pigeonhole-infinite-rel*:
assumes ¬ *finite A*
and *finite B*
and ∀ *a*∈*A*. ∃ *b*∈*B*. *R a b*
shows ∃ *b*∈*B*. ¬ *finite* {*a*:*A*. *R a b*}
⟨*proof*⟩

20.7.3 Cardinality of sums

lemma *card-Plus*:
assumes *finite A finite B*
shows *card* (*A* <+> *B*) = *card A* + *card B*
⟨*proof*⟩

lemma *card-Plus-conv-if*:
card (*A* <+> *B*) = (*if finite A* ∧ *finite B then card A* + *card B else* 0)
⟨*proof*⟩

Relates to equivalence classes. Based on a theorem of F. Kammüller.

lemma *dvd-partition*:
assumes *f*: *finite* (⋃ *C*)
and ∀ *c*∈*C*. *k dvd card c* ∀ *c1*∈*C*. ∀ *c2*∈*C*. *c1* ≠ *c2* ⟶ *c1* ∩ *c2* = {}
shows *k dvd card* (⋃ *C*)
⟨*proof*⟩

20.8 Minimal and maximal elements of finite sets

context begin

qualified lemma
assumes *finite A* **and** *asymp-on A R* **and** *transp-on A R* **and** ∃ *x* ∈ *A*. *P x*
shows
bex-min-element-with-property: ∃ *x* ∈ *A*. *P x* ∧ (∀ *y* ∈ *A*. *R y x* ⟶ ¬ *P y*) **and**
bex-max-element-with-property: ∃ *x* ∈ *A*. *P x* ∧ (∀ *y* ∈ *A*. *R x y* ⟶ ¬ *P y*)
⟨*proof*⟩ **lemma**
assumes *finite A* **and** *asymp-on A R* **and** *transp-on A R* **and** *A* ≠ {}
shows
bex-min-element: ∃ *m* ∈ *A*. ∀ *x* ∈ *A*. *x* ≠ *m* ⟶ ¬ *R x m* **and**
bex-max-element: ∃ *m* ∈ *A*. ∀ *x* ∈ *A*. *x* ≠ *m* ⟶ ¬ *R m x*
⟨*proof*⟩

end

The following alternative form might sometimes be easier to work with.

lemma *is-min-element-in-set-iff*:
asymp-on A R ⟹ (∀ *y* ∈ *A*. *y* ≠ *x* ⟶ ¬ *R y x*) ⟷ (∀ *y*. *R y x* ⟶ *y* ∉ *A*)
⟨*proof*⟩

lemma *is-max-element-in-set-iff*:
asymp-on A R $\Longrightarrow$ $(\forall\, y \in A.\ y \neq x \longrightarrow \neg\ R\ x\ y) \longleftrightarrow (\forall\, y.\ R\ x\ y \longrightarrow y \notin A)$
⟨*proof*⟩

context begin

qualified lemma
assumes *finite A* **and** $A \neq \{\}$ **and** *transp-on A R* **and** *totalp-on A R*
shows
bex-least-element: $\exists\, l \in A.\ \forall\, x \in A.\ x \neq l \longrightarrow R\ l\ x$ **and**
bex-greatest-element: $\exists\, g \in A.\ \forall\, x \in A.\ x \neq g \longrightarrow R\ x\ g$
⟨*proof*⟩

end

20.8.1 Finite orders

context *order*
begin

lemma *finite-has-maximal*:
assumes *finite A* **and** $A \neq \{\}$
shows $\exists\ m \in A.\ \forall\ b \in A.\ m \leq b \longrightarrow m = b$
⟨*proof*⟩

lemma *finite-has-maximal2*:
$[\![$ *finite A*; $a \in A\ ]\!] \Longrightarrow \exists\ m \in A.\ a \leq m \wedge (\forall\ b \in A.\ m \leq b \longrightarrow m = b)$
⟨*proof*⟩

lemma *finite-has-minimal*:
assumes *finite A* **and** $A \neq \{\}$
shows $\exists\ m \in A.\ \forall\ b \in A.\ b \leq m \longrightarrow m = b$
⟨*proof*⟩

lemma *finite-has-minimal2*:
$[\![$ *finite A*; $a \in A\ ]\!] \Longrightarrow \exists\ m \in A.\ m \leq a \wedge (\forall\ b \in A.\ b \leq m \longrightarrow m = b)$
⟨*proof*⟩

end

20.8.2 Relating injectivity and surjectivity

lemma *finite-surj-inj*:
assumes *finite A* $A \subseteq f\ `\ A$
shows *inj-on f A*
⟨*proof*⟩

lemma *finite-UNIV-surj-inj*: *finite*(*UNIV*:: *'a set*) $\Longrightarrow$ *surj f* $\Longrightarrow$ *inj f*
for *f* :: *'a* $\Rightarrow$ *'a*
⟨*proof*⟩

lemma *finite-UNIV-inj-surj*: *finite*(*UNIV*:: *'a set*) $\Longrightarrow$ *inj f* $\Longrightarrow$ *surj f*
for *f* :: *'a* $\Rightarrow$ *'a*
⟨*proof*⟩

lemma *surjective-iff-injective-gen*:
assumes *fS*: *finite S*
and *fT*: *finite T*
and *c*: *card S* = *card T*
and *ST*: *f* ' *S* $\subseteq$ *T*
shows ($\forall$ *y* $\in$ *T*. $\exists$ *x* $\in$ *S*. *f x* = *y*) $\longleftrightarrow$ *inj-on f S*
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

hide-const (**open**) *Finite-Set.fold*

20.9 Infinite Sets

Some elementary facts about infinite sets, mostly by Stephan Merz. Beware! Because "infinite" merely abbreviates a negation, these lemmas may not work well with *blast*.

abbreviation *infinite* :: *'a set* $\Rightarrow$ *bool*
where *infinite S* $\equiv$ $\neg$ *finite S*

Infinite sets are non-empty, and if we remove some elements from an infinite set, the result is still infinite.

lemma *infinite-UNIV-nat* [*iff*]: *infinite* (*UNIV* :: *nat set*)
⟨*proof*⟩

lemma *infinite-UNIV-char-0*: *infinite* (*UNIV* :: *'a*::*semiring-char-0 set*)
⟨*proof*⟩

lemma *infinite-imp-nonempty*: *infinite S* $\Longrightarrow$ *S* $\neq$ {}
⟨*proof*⟩

lemma *infinite-remove*: *infinite S* $\Longrightarrow$ *infinite* (*S* $-$ {*a*})
⟨*proof*⟩

lemma *Diff-infinite-finite*:
assumes *finite T infinite S*
shows *infinite* (*S* $-$ *T*)
⟨*proof*⟩

lemma *Un-infinite*: *infinite S* $\Longrightarrow$ *infinite* (*S* $\cup$ *T*)
⟨*proof*⟩

lemma *infinite-Un*: *infinite* (*S* $\cup$ *T*) $\longleftrightarrow$ *infinite S* $\vee$ *infinite T*
⟨*proof*⟩

lemma *infinite-super*:
assumes $S \subseteq T$
and *infinite* S
shows *infinite* T
⟨*proof*⟩

proposition *infinite-coinduct* [*consumes 1*, *case-names infinite*]:
assumes X A
and *step*: $\bigwedge A.\ X\ A \Longrightarrow \exists x{\in}A.\ X\ (A - \{x\}) \vee$ *infinite* $(A - \{x\})$
shows *infinite* A
⟨*proof*⟩

For any function with infinite domain and finite range there is some element that is the image of infinitely many domain elements. In particular, any infinite sequence of elements from a finite set contains some element that occurs infinitely often.

lemma *inf-img-fin-dom'*:
assumes *img*: *finite* $(f\ `\ A)$
and *dom*: *infinite* A
shows $\exists\, y \in f\ `\ A.$ *infinite* $(f\ -`\ \{y\} \cap A)$
⟨*proof*⟩

lemma *inf-img-fin-domE'*:
assumes *finite* $(f\ `\ A)$ **and** *infinite* A
obtains y **where** $y \in f`A$ **and** *infinite* $(f\ -`\ \{y\} \cap A)$
⟨*proof*⟩

lemma *inf-img-fin-dom*:
assumes *img*: *finite* $(f`A)$ **and** *dom*: *infinite* A
shows $\exists\, y \in f`A.$ *infinite* $(f\ -`\ \{y\})$
⟨*proof*⟩

lemma *inf-img-fin-domE*:
assumes *finite* $(f`A)$ **and** *infinite* A
obtains y **where** $y \in f`A$ **and** *infinite* $(f\ -`\ \{y\})$
⟨*proof*⟩

proposition *finite-image-absD*: *finite* $(abs\ `\ S) \Longrightarrow$ *finite* S
for S :: *'a*::*linordered-ring set*
⟨*proof*⟩

20.10 The finite powerset operator

definition *Fpow* :: *'a set* ⇒ *'a set set*
where *Fpow* $A \equiv \{X.\ X \subseteq A \wedge$ *finite* $X\}$

lemma *Fpow-mono*: $A \subseteq B \Longrightarrow$ *Fpow* $A \subseteq$ *Fpow* B
⟨*proof*⟩

lemma *empty-in-Fpow*: $\{\} \in Fpow\ A$
⟨*proof*⟩

lemma *Fpow-not-empty*: $Fpow\ A \neq \{\}$
⟨*proof*⟩

lemma *Fpow-subset-Pow*: $Fpow\ A \subseteq Pow\ A$
⟨*proof*⟩

lemma *Fpow-Pow-finite*: $Fpow\ A = Pow\ A\ Int\ \{A.\ finite\ A\}$
⟨*proof*⟩

lemma *inj-on-image-Fpow*:
assumes *inj-on f A*
shows *inj-on* (*image f*) (*Fpow A*)
⟨*proof*⟩

lemma *image-Fpow-mono*:
assumes $f\ `\ A \subseteq B$
shows $(image\ f)\ `\ (Fpow\ A) \subseteq Fpow\ B$
⟨*proof*⟩

end

21 Reflexive and Transitive closure of a relation

theory *Transitive-Closure*
imports *Finite-Set*
abbrevs $\hat{}* = {}^{*}\ {}^{**}$
and $\hat{}+ = {}^{+}\ {}^{++}$
and $\hat{}= = {}^{=}\ {}^{==}$
begin

⟨*ML*⟩

rtrancl is reflexive/transitive closure, *trancl* is transitive closure, *reflcl* is reflexive closure.

These postfix operators have *maximum priority*, forcing their operands to be atomic.

context notes [[*inductive-internals*]]
begin

inductive-set *rtrancl* :: $('a \times 'a)\ set \Rightarrow ('a \times 'a)\ set$ ((-*) [*1000*] *999*)
for *r* :: $('a \times 'a)\ set$
where
rtrancl-refl [*intro*!, *Pure.intro*!, *simp*]: $(a, a) \in r^*$
| *rtrancl-into-rtrancl* [*Pure.intro*]: $(a, b) \in r^* \Longrightarrow (b, c) \in r \Longrightarrow (a, c) \in r^*$

inductive-set *trancl* :: $('a \times 'a)$ *set* $\Rightarrow$ $('a \times 'a)$ *set* $((\text{-}^+)\ [1000]\ 999)$
  for r :: $('a \times 'a)$ *set*
  where
    r-into-trancl [*intro*, *Pure.intro*]: $(a, b) \in r \Longrightarrow (a, b) \in r^+$
  | *trancl-into-trancl* [*Pure.intro*]: $(a, b) \in r^+ \Longrightarrow (b, c) \in r \Longrightarrow (a, c) \in r^+$

notation
  rtranclp $((\text{-}^{**})\ [1000]\ 1000)$ **and**
  tranclp $((\text{-}^{++})\ [1000]\ 1000)$

declare
  rtrancl-def [*nitpick-unfold del*]
  rtranclp-def [*nitpick-unfold del*]
  trancl-def [*nitpick-unfold del*]
  tranclp-def [*nitpick-unfold del*]

end

abbreviation *reflcl* :: $('a \times 'a)$ *set* $\Rightarrow$ $('a \times 'a)$ *set* $((\text{-}^=)\ [1000]\ 999)$
  where $r^= \equiv r \cup Id$

abbreviation *reflclp* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ $((\text{-}^{==})\ [1000]\ 1000)$
  where $r^{==} \equiv sup\ r\ (=)$

notation (*ASCII*)
  rtrancl $((\text{-}\hat{}*)\ [1000]\ 999)$ **and**
  trancl $((\text{-}\hat{}+)\ [1000]\ 999)$ **and**
  reflcl $((\text{-}\hat{}=)\ [1000]\ 999)$ **and**
  rtranclp $((\text{-}\hat{}**)\ [1000]\ 1000)$ **and**
  tranclp $((\text{-}\hat{}++)\ [1000]\ 1000)$ **and**
  reflclp $((\text{-}\hat{}==)\ [1000]\ 1000)$

21.1 Reflexive closure

lemma *reflcl-set-eq* [*pred-set-conv*]: $(sup\ (\lambda x\ y.\ (x, y) \in r)\ (=)) = (\lambda x\ y.\ (x, y) \in r \cup Id)$
  ⟨*proof*⟩

lemma *refl-reflcl*[*simp*]: *refl* $(r^=)$
  ⟨*proof*⟩

lemma *reflp-on-reflclp*[*simp*]: *reflp-on A* $R^{==}$
  ⟨*proof*⟩

lemma *antisym-on-reflcl*[*simp*]: *antisym-on A* $(r^=) \longleftrightarrow$ *antisym-on A r*
  ⟨*proof*⟩

lemma *antisymp-on-reflcp*[*simp*]: *antisymp-on A* $R^{==} \longleftrightarrow$ *antisymp-on A R*

⟨*proof*⟩

lemma *trans-on-reflcl*[*simp*]: *trans-on A r* ⟹ *trans-on A* ($r^=$)
⟨*proof*⟩

lemma *transp-on-reflclp*[*simp*]: *transp-on A R* ⟹ *transp-on A* $R^{==}$
⟨*proof*⟩

lemma *reflclp-idemp* [*simp*]: $(P^{==})^{==} = P^{==}$
⟨*proof*⟩

lemma *reflclp-ident-if-reflp*[*simp*]: *reflp R* ⟹ $R^{==} = R$
⟨*proof*⟩

21.2 Reflexive-transitive closure

lemma *r-into-rtrancl* [*intro*]: ⋀*p*. *p* ∈ *r* ⟹ *p* ∈ r^*
— *rtrancl* of *r* contains *r*
⟨*proof*⟩

lemma *r-into-rtranclp* [*intro*]: *r x y* ⟹ r^{**} *x y*
— *rtrancl* of *r* contains *r*
⟨*proof*⟩

lemma *rtranclp-mono*: *r* ≤ *s* ⟹ $r^{**} \le s^{**}$
— monotonicity of *rtrancl*
⟨*proof*⟩

lemma *mono-rtranclp*[*mono*]: (⋀*a b*. *x a b* ⟶ *y a b*) ⟹ x^{**} *a b* ⟶ y^{**} *a b*
⟨*proof*⟩

lemmas *rtrancl-mono* = *rtranclp-mono* [*to-set*]

theorem *rtranclp-induct* [*consumes 1*, *case-names base step*, *induct set*: *rtranclp*]:
assumes *a*: r^{**} *a b*
and *cases*: *P a* ⋀*y z*. r^{**} *a y* ⟹ *r y z* ⟹ *P y* ⟹ *P z*
shows *P b*
⟨*proof*⟩

lemmas *rtrancl-induct* [*induct set*: *rtrancl*] = *rtranclp-induct* [*to-set*]

lemmas *rtranclp-induct2* =
rtranclp-induct[*of* - (*ax*,*ay*) (*bx*,*by*), *split-rule*, *consumes 1*, *case-names refl step*]

lemmas *rtrancl-induct2* =
rtrancl-induct[*of* (*ax*,*ay*) (*bx*,*by*), *split-format* (*complete*), *consumes 1*, *case-names refl step*]

lemma *refl-rtrancl*: *refl* (r^*)

⟨*proof*⟩

Transitivity of transitive closure.

lemma *trans-rtrancl*: *trans* (r^*)
⟨*proof*⟩

lemmas *rtrancl-trans* = *trans-rtrancl* [*THEN transD*]

lemma *rtranclp-trans*:
assumes r^{**} x y
and r^{**} y z
shows r^{**} x z
⟨*proof*⟩

lemma *rtranclE* [*cases set*: *rtrancl*]:
fixes a b :: $'a$
assumes *major*: $(a, b) \in r^*$
obtains
(*base*) $a = b$
| (*step*) y **where** $(a, y) \in r^*$ **and** $(y, b) \in r$
— elimination of *rtrancl* – by induction on a special formula
⟨*proof*⟩

lemma *rtrancl-Int-subset*: $Id \subseteq s \Longrightarrow (r^* \cap s)\ O\ r \subseteq s \Longrightarrow r^* \subseteq s$
⟨*proof*⟩

lemma *converse-rtranclp-into-rtranclp*: $r\ a\ b \Longrightarrow r^{**}\ b\ c \Longrightarrow r^{**}\ a\ c$
⟨*proof*⟩

lemmas *converse-rtrancl-into-rtrancl* = *converse-rtranclp-into-rtranclp* [*to-set*]

More r^* equations and inclusions.

lemma *rtranclp-idemp* [*simp*]: $(r^{**})^{**} = r^{**}$
⟨*proof*⟩

lemmas *rtrancl-idemp* [*simp*] = *rtranclp-idemp* [*to-set*]

lemma *rtrancl-idemp-self-comp* [*simp*]: $R^*\ O\ R^* = R^*$
⟨*proof*⟩

lemma *rtrancl-subset-rtrancl*: $r \subseteq s^* \Longrightarrow r^* \subseteq s^*$
⟨*proof*⟩

lemma *rtranclp-subset*: $R \leq S \Longrightarrow S \leq R^{**} \Longrightarrow S^{**} = R^{**}$
⟨*proof*⟩

lemmas *rtrancl-subset* = *rtranclp-subset* [*to-set*]

lemma *rtranclp-sup-rtranclp*: $(sup\ (R^{**})\ (S^{**}))^{**} = (sup\ R\ S)^{**}$

⟨*proof*⟩

lemmas *rtrancl-Un-rtrancl* = *rtranclp-sup-rtranclp* [*to-set*]

lemma *rtranclp-reflclp* [*simp*]: $(R^{==})^{**} = R^{**}$
⟨*proof*⟩

lemmas *rtrancl-reflcl* [*simp*] = *rtranclp-reflclp* [*to-set*]

lemma *rtrancl-r-diff-Id*: $(r - Id)^* = r^*$
⟨*proof*⟩

lemma *rtranclp-r-diff-Id*: $(inf\ r\ (\neq))^{**} = r^{**}$
⟨*proof*⟩

theorem *rtranclp-converseD*:
assumes $(r^{-1-1})^{**}\ x\ y$
shows $r^{**}\ y\ x$
⟨*proof*⟩

lemmas *rtrancl-converseD* = *rtranclp-converseD* [*to-set*]

theorem *rtranclp-converseI*:
assumes $r^{**}\ y\ x$
shows $(r^{-1-1})^{**}\ x\ y$
⟨*proof*⟩

lemmas *rtrancl-converseI* = *rtranclp-converseI* [*to-set*]

lemma *rtrancl-converse*: $(r^{-1})^* = (r^*)^{-1}$
⟨*proof*⟩

lemma *sym-rtrancl*: $sym\ r \Longrightarrow sym\ (r^*)$
⟨*proof*⟩

theorem *converse-rtranclp-induct* [*consumes 1*, *case-names base step*]:
assumes *major*: $r^{**}\ a\ b$
and *cases*: $P\ b\ \bigwedge y\ z.\ r\ y\ z \Longrightarrow r^{**}\ z\ b \Longrightarrow P\ z \Longrightarrow P\ y$
shows $P\ a$
⟨*proof*⟩

lemmas *converse-rtrancl-induct* = *converse-rtranclp-induct* [*to-set*]

lemmas *converse-rtranclp-induct2* =
converse-rtranclp-induct [*of* - (*ax*, *ay*) (*bx*, *by*), *split-rule*, *consumes 1*, *case-names refl step*]

lemmas *converse-rtrancl-induct2* =
converse-rtrancl-induct [*of* (*ax*, *ay*) (*bx*, *by*), *split-format* (*complete*),

consumes 1, *case-names refl step*]

lemma *converse-rtranclpE* [*consumes 1*, *case-names base step*]:
assumes *major*: $r^{**}\ x\ z$
and *cases*: $x = z \Longrightarrow P \bigwedge y.\ r\ x\ y \Longrightarrow r^{**}\ y\ z \Longrightarrow P$
shows P
⟨*proof*⟩

lemmas *converse-rtranclE* = *converse-rtranclpE* [*to-set*]

lemmas *converse-rtranclpE2* = *converse-rtranclpE* [*of* - (*xa*,*xb*) (*za*,*zb*), *split-rule*]

lemmas *converse-rtranclE2* = *converse-rtranclE* [*of* (*xa*,*xb*) (*za*,*zb*), *split-rule*]

lemma *r-comp-rtrancl-eq*: $r\ O\ r^* = r^*\ O\ r$
⟨*proof*⟩

lemma *rtrancl-unfold*: $r^* = Id \cup r^*\ O\ r$
⟨*proof*⟩

lemma *rtrancl-Un-separatorE*:
$(a,\ b) \in (P \cup Q)^* \Longrightarrow \forall x\ y.\ (a,\ x) \in P^* \longrightarrow (x,\ y) \in Q \longrightarrow x = y \Longrightarrow (a,\ b) \in P^*$
⟨*proof*⟩

lemma *rtrancl-Un-separator-converseE*:
$(a,\ b) \in (P \cup Q)^* \Longrightarrow \forall x\ y.\ (x,\ b) \in P^* \longrightarrow (y,\ x) \in Q \longrightarrow y = x \Longrightarrow (a,\ b) \in P^*$
⟨*proof*⟩

lemma *Image-closed-trancl*:
assumes $r\ ``\ X \subseteq X$
shows $r^*\ ``\ X = X$
⟨*proof*⟩

21.3 Transitive closure

lemma *totalp-on-tranclp*: *totalp-on A R* $\Longrightarrow$ *totalp-on A* (*tranclp R*)
⟨*proof*⟩

lemma *total-on-trancl*: *total-on A r* $\Longrightarrow$ *total-on A* (*trancl r*)
⟨*proof*⟩

lemma *trancl-mono*:
assumes $p \in r^+\ r \subseteq s$
shows $p \in s^+$
⟨*proof*⟩

lemma *r-into-trancl'*: $\bigwedge p.\ p \in r \Longrightarrow p \in r^+$

⟨*proof*⟩

Conversions between *trancl* and *rtrancl*.

lemma *tranclp-into-rtranclp*: $r^{++}\ a\ b \Longrightarrow r^{**}\ a\ b$
⟨*proof*⟩

lemmas *trancl-into-rtrancl* = *tranclp-into-rtranclp* [*to-set*]

lemma *rtranclp-into-tranclp1*:
assumes $r^{**}\ a\ b$
shows $r\ b\ c \Longrightarrow r^{++}\ a\ c$
⟨*proof*⟩

lemmas *rtrancl-into-trancl1* = *rtranclp-into-tranclp1* [*to-set*]

lemma *rtranclp-into-tranclp2*:
assumes $r\ a\ b\ r^{**}\ b\ c$ **shows** $r^{++}\ a\ c$
— intro rule from r and *rtrancl*
⟨*proof*⟩

lemmas *rtrancl-into-trancl2* = *rtranclp-into-tranclp2* [*to-set*]

Nice induction rule for *trancl*

lemma *tranclp-induct* [*consumes 1*, *case-names base step*, *induct pred*: *tranclp*]:
assumes *a*: $r^{++}\ a\ b$
and *cases*: $\bigwedge y.\ r\ a\ y \Longrightarrow P\ y\ \bigwedge y\ z.\ r^{++}\ a\ y \Longrightarrow r\ y\ z \Longrightarrow P\ y \Longrightarrow P\ z$
shows $P\ b$
⟨*proof*⟩

lemmas *trancl-induct* [*induct set*: *trancl*] = *tranclp-induct* [*to-set*]

lemmas *tranclp-induct2* =
tranclp-induct [*of* - (*ax*, *ay*) (*bx*, *by*), *split-rule*, *consumes 1*, *case-names base step*]

lemmas *trancl-induct2* =
trancl-induct [*of* (*ax*, *ay*) (*bx*, *by*), *split-format* (*complete*),
consumes 1, *case-names base step*]

lemma *tranclp-trans-induct*:
assumes *major*: $r^{++}\ x\ y$
and *cases*: $\bigwedge x\ y.\ r\ x\ y \Longrightarrow P\ x\ y\ \bigwedge x\ y\ z.\ r^{++}\ x\ y \Longrightarrow P\ x\ y \Longrightarrow r^{++}\ y\ z \Longrightarrow P\ y\ z \Longrightarrow P\ x\ z$
shows $P\ x\ y$
— Another induction rule for trancl, incorporating transitivity
⟨*proof*⟩

lemmas *trancl-trans-induct* = *tranclp-trans-induct* [*to-set*]

lemma *tranclE* [*cases set*: *trancl*]:
assumes $(a, b) \in r^+$
obtains
$(base)$ $(a, b) \in r$
| $(step)$ c **where** $(a, c) \in r^+$ **and** $(c, b) \in r$
⟨*proof*⟩

lemma *trancl-Int-subset*: $r \subseteq s \Longrightarrow (r^+ \cap s)\ O\ r \subseteq s \Longrightarrow r^+ \subseteq s$
⟨*proof*⟩

lemma *trancl-unfold*: $r^+ = r \cup r^+\ O\ r$
⟨*proof*⟩

Transitivity of r^+

lemma *trans-trancl* [*simp*]: *trans* (r^+)
⟨*proof*⟩

lemmas *trancl-trans* = *trans-trancl* [*THEN transD*]

lemma *tranclp-trans*:
assumes $r^{++}\ x\ y$
and $r^{++}\ y\ z$
shows $r^{++}\ x\ z$
⟨*proof*⟩

lemma *trancl-id* [*simp*]: *trans* $r \Longrightarrow r^+ = r$
⟨*proof*⟩

lemma *rtranclp-tranclp-tranclp*:
assumes $r^{**}\ x\ y$
shows $\bigwedge z.\ r^{++}\ y\ z \Longrightarrow r^{++}\ x\ z$
⟨*proof*⟩

lemmas *rtrancl-trancl-trancl* = *rtranclp-tranclp-tranclp* [*to-set*]

lemma *tranclp-into-tranclp2*: $r\ a\ b \Longrightarrow r^{++}\ b\ c \Longrightarrow r^{++}\ a\ c$
⟨*proof*⟩

lemmas *trancl-into-trancl2* = *tranclp-into-tranclp2* [*to-set*]

lemma *tranclp-converseI*:
assumes $(r^{++})^{-1-1}\ x\ y$ **shows** $(r^{-1-1})^{++}\ x\ y$
⟨*proof*⟩

lemmas *trancl-converseI* = *tranclp-converseI* [*to-set*]

lemma *tranclp-converseD*:
assumes $(r^{-1-1})^{++}\ x\ y$ **shows** $(r^{++})^{-1-1}\ x\ y$
⟨*proof*⟩

lemmas *trancl-converseD* = *tranclp-converseD* [*to-set*]

lemma *tranclp-converse*: $(r^{-1-1})^{++} = (r^{++})^{-1-1}$
⟨*proof*⟩

lemmas *trancl-converse* = *tranclp-converse* [*to-set*]

lemma *sym-trancl*: *sym r* ⟹ *sym* (r^+)
⟨*proof*⟩

lemma *converse-tranclp-induct* [*consumes 1*, *case-names base step*]:
assumes *major*: r^{++} *a b*
and *cases*: ⋀*y*. *r y b* ⟹ *P y* ⋀*y z*. *r y z* ⟹ r^{++} *z b* ⟹ *P z* ⟹ *P y*
shows *P a*
⟨*proof*⟩

lemmas *converse-trancl-induct* = *converse-tranclp-induct* [*to-set*]

lemma *tranclpD*: R^{++} *x y* ⟹ ∃ *z*. *R x z* ∧ R^{**} *z y*
⟨*proof*⟩

lemmas *tranclD* = *tranclpD* [*to-set*]

lemma *converse-tranclpE*:
assumes *major*: *tranclp r x z*
and *base*: *r x z* ⟹ *P*
and *step*: ⋀*y*. *r x y* ⟹ *tranclp r y z* ⟹ *P*
shows *P*
⟨*proof*⟩

lemmas *converse-tranclE* = *converse-tranclpE* [*to-set*]

lemma *tranclD2*: $(x, y) \in R^+ \Longrightarrow \exists\, z.\ (x, z) \in R^* \wedge (z, y) \in R$
⟨*proof*⟩

lemma *irrefl-tranclI*: $r^{-1} \cap r^* = \{\} \Longrightarrow (x, x) \notin r^+$
⟨*proof*⟩

lemma *irrefl-trancl-rD*: $\forall x.\ (x, x) \notin r^+ \Longrightarrow (x, y) \in r \Longrightarrow x \neq y$
⟨*proof*⟩

lemma *trancl-subset-Sigma-aux*: $(a, b) \in r^* \Longrightarrow r \subseteq A \times A \Longrightarrow a = b \vee a \in A$
⟨*proof*⟩

lemma *trancl-subset-Sigma*:
assumes $r \subseteq A \times A$ **shows** $r^+ \subseteq A \times A$
⟨*proof*⟩

lemma *reflclp-tranclp* [*simp*]: $(r^{++})^{==} = r^{**}$
⟨*proof*⟩

lemmas *reflcl-trancl* [*simp*] = *reflclp-tranclp* [*to-set*]

lemma *trancl-reflcl* [*simp*]: $(r^{=})^{+} = r^{*}$
⟨*proof*⟩

lemma *rtrancl-trancl-reflcl* [*code*]: $r^{*} = (r^{+})^{=}$
⟨*proof*⟩

lemma *trancl-empty* [*simp*]: $\{\}^{+} = \{\}$
⟨*proof*⟩

lemma *rtrancl-empty* [*simp*]: $\{\}^{*} = Id$
⟨*proof*⟩

lemma *rtranclpD*: $R^{**}\ a\ b \Longrightarrow a = b \vee a \neq b \wedge R^{++}\ a\ b$
⟨*proof*⟩

lemmas *rtranclD* = *rtranclpD* [*to-set*]

lemma *rtrancl-eq-or-trancl*: $(x,y) \in R^{*} \longleftrightarrow x = y \vee x \neq y \wedge (x,\ y) \in R^{+}$
⟨*proof*⟩

lemma *trancl-unfold-right*: $r^{+} = r^{*}\ O\ r$
⟨*proof*⟩

lemma *trancl-unfold-left*: $r^{+} = r\ O\ r^{*}$
⟨*proof*⟩

lemma *trancl-insert*: $(insert\ (y,\ x)\ r)^{+} = r^{+} \cup \{(a,\ b).\ (a,\ y) \in r^{*} \wedge (x,\ b) \in r^{*}\}$
— primitive recursion for *trancl* over finite relations
⟨*proof*⟩

lemma *trancl-insert2*:
$(insert\ (a,\ b)\ r)^{+} = r^{+} \cup \{(x,\ y).\ ((x,\ a) \in r^{+} \vee x = a) \wedge ((b,\ y) \in r^{+} \vee y = b)\}$
⟨*proof*⟩

lemma *rtrancl-insert*: $(insert\ (a,b)\ r)^{*} = r^{*} \cup \{(x,\ y).\ (x,\ a) \in r^{*} \wedge (b,\ y) \in r^{*}\}$
⟨*proof*⟩

Simplifying nested closures

lemma *rtrancl-trancl-absorb*[*simp*]: $(R^{*})^{+} = R^{*}$
⟨*proof*⟩

lemma *trancl-rtrancl-absorb*[*simp*]: $(R^{+})^{*} = R^{*}$
⟨*proof*⟩

lemma *rtrancl-reflcl-absorb*[*simp*]: $(R^*)^= = R^*$
⟨*proof*⟩

Domain and *Range*

lemma *Domain-rtrancl* [*simp*]: *Domain* $(R^*) = UNIV$
⟨*proof*⟩

lemma *Range-rtrancl* [*simp*]: *Range* $(R^*) = UNIV$
⟨*proof*⟩

lemma *rtrancl-Un-subset*: $(R^* \cup S^*) \subseteq (R \cup S)^*$
⟨*proof*⟩

lemma *in-rtrancl-UnI*: $x \in R^* \vee x \in S^* \Longrightarrow x \in (R \cup S)^*$
⟨*proof*⟩

lemma *trancl-domain* [*simp*]: *Domain* $(r^+) =$ *Domain* r
⟨*proof*⟩

lemma *trancl-range* [*simp*]: *Range* $(r^+) =$ *Range* r
⟨*proof*⟩

lemma *Not-Domain-rtrancl*:
assumes $x \notin$ *Domain* R **shows** $(x, y) \in R^* \longleftrightarrow x = y$
⟨*proof*⟩

lemma *trancl-subset-Field2*: $r^+ \subseteq$ *Field* $r \times$ *Field* r
⟨*proof*⟩

lemma *finite-trancl*[*simp*]: *finite* $(r^+) =$ *finite* r
⟨*proof*⟩

lemma *finite-rtrancl-Image*[*simp*]: **assumes** *finite* R *finite* A **shows** *finite* $(R^* \text{ `` } A)$
⟨*proof*⟩

More about converse *rtrancl* and *trancl*, should be merged with main body.

lemma *single-valued-confluent*:
assumes *single-valued* r **and** *xy*: $(x, y) \in r^*$ **and** *xz*: $(x, z) \in r^*$
shows $(y, z) \in r^* \vee (z, y) \in r^*$
⟨*proof*⟩

lemma *r-r-into-trancl*: $(a, b) \in R \Longrightarrow (b, c) \in R \Longrightarrow (a, c) \in R^+$
⟨*proof*⟩

lemma *trancl-into-trancl*: $(a, b) \in r^+ \Longrightarrow (b, c) \in r \Longrightarrow (a, c) \in r^+$
⟨*proof*⟩

lemma *tranclp-rtranclp-tranclp*:
  assumes r^{++} *a b* r^{**} *b c* **shows** r^{++} *a c*
⟨*proof*⟩

lemma *rtranclp-conversep*: $r^{-1-1**} = r^{**-1-1}$
  ⟨*proof*⟩

lemmas *symp-rtranclp* = *sym-rtrancl*[*to-pred*]

lemmas *symp-conv-conversep-eq* = *sym-conv-converse-eq*[*to-pred*]

lemmas *rtranclp-tranclp-absorb* [*simp*] = *rtrancl-trancl-absorb*[*to-pred*]
lemmas *tranclp-rtranclp-absorb* [*simp*] = *trancl-rtrancl-absorb*[*to-pred*]
lemmas *rtranclp-reflclp-absorb* [*simp*] = *rtrancl-reflcl-absorb*[*to-pred*]

lemmas *trancl-rtrancl-trancl* = *tranclp-rtranclp-tranclp* [*to-set*]

lemmas *transitive-closure-trans* [*trans*] =
  r-r-into-trancl trancl-trans rtrancl-trans
  trancl.trancl-into-trancl trancl-into-trancl2
  rtrancl.rtrancl-into-rtrancl converse-rtrancl-into-rtrancl
  rtrancl-trancl-trancl trancl-rtrancl-trancl

lemmas *transitive-closurep-trans'* [*trans*] =
  tranclp-trans rtranclp-trans
  tranclp.trancl-into-trancl tranclp-into-tranclp2
  rtranclp.rtrancl-into-rtrancl converse-rtranclp-into-rtranclp
  rtranclp-tranclp-tranclp tranclp-rtranclp-tranclp

declare *trancl-into-rtrancl* [*elim*]

21.4 Symmetric closure

definition *symclp* :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ '*a* ⇒ '*a* ⇒ *bool*
where *symclp r x y* ⟷ *r x y* ∨ *r y x*

lemma *symclpI* [*simp*, *intro?*]:
  shows *symclpI1*: *r x y* ⟹ *symclp r x y*
    and *symclpI2*: *r y x* ⟹ *symclp r x y*
  ⟨*proof*⟩

lemma *symclpE* [*consumes 1*, *cases pred*]:
  assumes *symclp r x y*
  obtains (*base*) *r x y* | (*sym*) *r y x*
  ⟨*proof*⟩

lemma *symclp-pointfree*: *symclp r* = *sup r* r^{-1-1}
  ⟨*proof*⟩

lemma *symclp-greater*: $r \leq symclp\ r$
⟨*proof*⟩

lemma *symclp-conversep* [*simp*]: $symclp\ r^{-1-1} = symclp\ r$
⟨*proof*⟩

lemma *symp-on-symclp* [*simp*]: $symp\text{-}on\ A\ (symclp\ R)$
⟨*proof*⟩

lemma *symp-symclp-eq*: $symp\ r \Longrightarrow symclp\ r = r$
⟨*proof*⟩

lemma *symp-rtranclp-symclp* [*simp*]: $symp\ (symclp\ r)^{**}$
⟨*proof*⟩

lemma *rtranclp-symclp-sym* [*sym*]: $(symclp\ r)^{**}\ x\ y \Longrightarrow (symclp\ r)^{**}\ y\ x$
⟨*proof*⟩

lemma *symclp-idem* [*simp*]: $symclp\ (symclp\ r) = symclp\ r$
⟨*proof*⟩

lemma *reflp-on-rtranclp* [*simp*]: $reflp\text{-}on\ A\ R^{**}$
⟨*proof*⟩

21.5 The power operation on relations

$R\ ^{\frown\frown}\ n = R\ O\ \ldots\ O\ R$, the n-fold composition of R

overloading
$relpow \equiv compow :: nat \Rightarrow ('a \times 'a)\ set \Rightarrow ('a \times 'a)\ set$
$relpowp \equiv compow :: nat \Rightarrow ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool)$
begin

primrec $relpow :: nat \Rightarrow ('a \times 'a)\ set \Rightarrow ('a \times 'a)\ set$
where
$relpow\ 0\ R = Id$
| $relpow\ (Suc\ n)\ R = (R\ ^{\frown\frown}\ n)\ O\ R$

primrec $relpowp :: nat \Rightarrow ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool)$
where
$relpowp\ 0\ R = HOL.eq$
| $relpowp\ (Suc\ n)\ R = (R\ ^{\frown\frown}\ n)\ OO\ R$

end

lemma *relpowp-relpow-eq* [*pred-set-conv*]:
$(\lambda x\ y.\ (x,\ y) \in R)\ ^{\frown\frown}\ n = (\lambda x\ y.\ (x,\ y) \in R\ ^{\frown\frown}\ n)$ **for** $R :: 'a\ rel$
⟨*proof*⟩

For code generation:

definition *relpow* :: *nat* ⇒ (*'a* × *'a*) *set* ⇒ (*'a* × *'a*) *set*
where *relpow-code-def* [*code-abbrev*]: *relpow* = *compow*

definition *relpowp* :: *nat* ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ (*'a* ⇒ *'a* ⇒ *bool*)
where *relpowp-code-def* [*code-abbrev*]: *relpowp* = *compow*

lemma [*code*]:
relpow (*Suc n*) *R* = (*relpow n R*) *O R*
relpow 0 R = *Id*
⟨*proof*⟩

lemma [*code*]:
relpowp (*Suc n*) *R* = (*R* ^^ *n*) *OO R*
relpowp 0 R = *HOL.eq*
⟨*proof*⟩

hide-const (**open**) *relpow*
hide-const (**open**) *relpowp*

lemma *relpow-1* [*simp*]: *R* ^^ *1* = *R*
for *R* :: (*'a* × *'a*) *set*
⟨*proof*⟩

lemma *relpowp-1* [*simp*]: *P* ^^ *1* = *P*
for *P* :: *'a* ⇒ *'a* ⇒ *bool*
⟨*proof*⟩

lemma *relpow-0-I*: (*x*, *x*) ∈ *R* ^^ *0*
⟨*proof*⟩

lemma *relpowp-0-I*: (*P* ^^ *0*) *x x*
⟨*proof*⟩

lemma *relpow-Suc-I*: (*x*, *y*) ∈ *R* ^^ *n* ⟹ (*y*, *z*) ∈ *R* ⟹ (*x*, *z*) ∈ *R* ^^ *Suc n*
⟨*proof*⟩

lemma *relpowp-Suc-I*: (*P* ^^ *n*) *x y* ⟹ *P y z* ⟹ (*P* ^^ *Suc n*) *x z*
⟨*proof*⟩

lemma *relpow-Suc-I2*: (*x*, *y*) ∈ *R* ⟹ (*y*, *z*) ∈ *R* ^^ *n* ⟹ (*x*, *z*) ∈ *R* ^^ *Suc n*
⟨*proof*⟩

lemma *relpowp-Suc-I2*: *P x y* ⟹ (*P* ^^ *n*) *y z* ⟹ (*P* ^^ *Suc n*) *x z*
⟨*proof*⟩

lemma *relpow-0-E*: (*x*, *y*) ∈ *R* ^^ *0* ⟹ (*x* = *y* ⟹ *P*) ⟹ *P*
⟨*proof*⟩

lemma *relpowp-0-E*: (*P* ^^ *0*) *x y* ⟹ (*x* = *y* ⟹ *Q*) ⟹ *Q*

⟨*proof*⟩

lemma *relpow-Suc-E*: $(x, z) \in R \mathbin{^\frown\!^\frown} \mathit{Suc}\ n \Longrightarrow (\bigwedge y.\ (x, y) \in R \mathbin{^\frown\!^\frown} n \Longrightarrow (y, z) \in R \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *relpowp-Suc-E*: $(P \mathbin{^\frown\!^\frown} \mathit{Suc}\ n)\ x\ z \Longrightarrow (\bigwedge y.\ (P \mathbin{^\frown\!^\frown} n)\ x\ y \Longrightarrow P\ y\ z \Longrightarrow Q) \Longrightarrow Q$
⟨*proof*⟩

lemma *relpow-E*:
$(x, z) \in R \mathbin{^\frown\!^\frown} n \Longrightarrow$
$(n = 0 \Longrightarrow x = z \Longrightarrow P) \Longrightarrow$
$(\bigwedge y\ m.\ n = \mathit{Suc}\ m \Longrightarrow (x, y) \in R \mathbin{^\frown\!^\frown} m \Longrightarrow (y, z) \in R \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *relpowp-E*:
$(P \mathbin{^\frown\!^\frown} n)\ x\ z \Longrightarrow$
$(n = 0 \Longrightarrow x = z \Longrightarrow Q) \Longrightarrow$
$(\bigwedge y\ m.\ n = \mathit{Suc}\ m \Longrightarrow (P \mathbin{^\frown\!^\frown} m)\ x\ y \Longrightarrow P\ y\ z \Longrightarrow Q) \Longrightarrow Q$
⟨*proof*⟩

lemma *relpow-Suc-D2*: $(x, z) \in R \mathbin{^\frown\!^\frown} \mathit{Suc}\ n \Longrightarrow (\exists y.\ (x, y) \in R \wedge (y, z) \in R \mathbin{^\frown\!^\frown} n)$
⟨*proof*⟩

lemma *relpowp-Suc-D2*: $(P \mathbin{^\frown\!^\frown} \mathit{Suc}\ n)\ x\ z \Longrightarrow \exists y.\ P\ x\ y \wedge (P \mathbin{^\frown\!^\frown} n)\ y\ z$
⟨*proof*⟩

lemma *relpow-Suc-E2*: $(x, z) \in R \mathbin{^\frown\!^\frown} \mathit{Suc}\ n \Longrightarrow (\bigwedge y.\ (x, y) \in R \Longrightarrow (y, z) \in R \mathbin{^\frown\!^\frown} n \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *relpowp-Suc-E2*: $(P \mathbin{^\frown\!^\frown} \mathit{Suc}\ n)\ x\ z \Longrightarrow (\bigwedge y.\ P\ x\ y \Longrightarrow (P \mathbin{^\frown\!^\frown} n)\ y\ z \Longrightarrow Q) \Longrightarrow Q$
⟨*proof*⟩

lemma *relpow-Suc-D2'*: $\forall x\ y\ z.\ (x, y) \in R \mathbin{^\frown\!^\frown} n \wedge (y, z) \in R \longrightarrow (\exists w.\ (x, w) \in R \wedge (w, z) \in R \mathbin{^\frown\!^\frown} n)$
⟨*proof*⟩

lemma *relpowp-Suc-D2'*: $\forall x\ y\ z.\ (P \mathbin{^\frown\!^\frown} n)\ x\ y \wedge P\ y\ z \longrightarrow (\exists w.\ P\ x\ w \wedge (P \mathbin{^\frown\!^\frown} n)\ w\ z)$
⟨*proof*⟩

lemma *relpow-E2*:
assumes $(x, z) \in R \mathbin{^\frown\!^\frown} n$ $n = 0 \Longrightarrow x = z \Longrightarrow P$
$\bigwedge y\ m.\ n = \mathit{Suc}\ m \Longrightarrow (x, y) \in R \Longrightarrow (y, z) \in R \mathbin{^\frown\!^\frown} m \Longrightarrow P$
shows P

⟨*proof*⟩

lemma *relpowp-E2*:
(*P* ^^ *n*) *x z* ⟹
(*n* = *0* ⟹ *x* = *z* ⟹ *Q*) ⟹
(⋀*y m*. *n* = *Suc m* ⟹ *P x y* ⟹ (*P* ^^ *m*) *y z* ⟹ *Q*) ⟹ *Q*
⟨*proof*⟩

lemma *relpow-add*: *R* ^^ (*m* + *n*) = *R*^^*m* *O* *R*^^*n*
⟨*proof*⟩

lemma *relpowp-add*: *P* ^^ (*m* + *n*) = *P* ^^ *m* *OO* *P* ^^ *n*
⟨*proof*⟩

lemma *relpow-commute*: *R O R* ^^ *n* = *R* ^^ *n O R*
⟨*proof*⟩

lemma *relpowp-commute*: *P OO P* ^^ *n* = *P* ^^ *n OO P*
⟨*proof*⟩

lemma *relpow-empty*: *0* < *n* ⟹ ({} :: ('*a* × '*a*) *set*) ^^ *n* = {}
⟨*proof*⟩

lemma *relpowp-bot*: *0* < *n* ⟹ (⊥ :: '*a* ⇒ '*a* ⇒ *bool*) ^^ *n* = ⊥
⟨*proof*⟩

lemma *rtrancl-imp-UN-relpow*:
assumes *p* ∈ *R**
shows *p* ∈ (⋃ *n*. *R* ^^ *n*)
⟨*proof*⟩

lemma *rtranclp-imp-Sup-relpowp*:
assumes (*P***) *x y*
shows (⨆ *n*. *P* ^^ *n*) *x y*
⟨*proof*⟩

lemma *relpow-imp-rtrancl*:
assumes *p* ∈ *R* ^^ *n*
shows *p* ∈ *R**
⟨*proof*⟩

lemma *relpowp-imp-rtranclp*: (*P* ^^ *n*) *x y* ⟹ (*P***) *x y*
⟨*proof*⟩

lemma *rtrancl-is-UN-relpow*: *R** = (⋃ *n*. *R* ^^ *n*)
⟨*proof*⟩

lemma *rtranclp-is-Sup-relpowp*: *P*** = (⨆ *n*. *P* ^^ *n*)
⟨*proof*⟩

lemma *rtrancl-power*: $p \in R^* \longleftrightarrow (\exists\, n.\ p \in R\ \widehat{\ }\ n)$
⟨*proof*⟩

lemma *rtranclp-power*: $(P^{**})\ x\ y \longleftrightarrow (\exists\, n.\ (P\ \widehat{\ }\ n)\ x\ y)$
⟨*proof*⟩

lemma *trancl-power*: $p \in R^+ \longleftrightarrow (\exists\, n > 0.\ p \in R\ \widehat{\ }\ n)$
⟨*proof*⟩

lemma *tranclp-power*: $(P^{++})\ x\ y \longleftrightarrow (\exists\, n > 0.\ (P\ \widehat{\ }\ n)\ x\ y)$
⟨*proof*⟩

lemma *rtrancl-imp-relpow*: $p \in R^* \Longrightarrow \exists\, n.\ p \in R\ \widehat{\ }\ n$
⟨*proof*⟩

lemma *rtranclp-imp-relpowp*: $(P^{**})\ x\ y \Longrightarrow \exists\, n.\ (P\ \widehat{\ }\ n)\ x\ y$
⟨*proof*⟩

By Sternagel/Thiemann:

lemma *relpow-fun-conv*: $(a,\ b) \in R\ \widehat{\ }\ n \longleftrightarrow (\exists f.\ f\ 0 = a \wedge f\ n = b \wedge (\forall i{<}n.\ (f\ i,\ f\ (Suc\ i)) \in R))$
⟨*proof*⟩

lemma *relpowp-fun-conv*: $(P\ \widehat{\ }\ n)\ x\ y \longleftrightarrow (\exists f.\ f\ 0 = x \wedge f\ n = y \wedge (\forall i{<}n.\ P\ (f\ i)\ (f\ (Suc\ i))))$
⟨*proof*⟩

lemma *relpow-finite-bounded1*:
fixes $R :: ('a \times 'a)\ set$
assumes *finite* R **and** $k > 0$
shows $R\widehat{\ }k \subseteq (\bigcup n{\in}\{n.\ 0 < n \wedge n \leq card\ R\}.\ R\widehat{\ }n)$
(**is** - ⊆ *?r*)
⟨*proof*⟩

lemma *relpow-finite-bounded*:
fixes $R :: ('a \times 'a)\ set$
assumes *finite* R
shows $R\widehat{\ }k \subseteq (\bigcup n{\in}\{n.\ n \leq card\ R\}.\ R\widehat{\ }n)$
⟨*proof*⟩

lemma *rtrancl-finite-eq-relpow*: *finite* $R \Longrightarrow R^* = (\bigcup n{\in}\{n.\ n \leq card\ R\}.\ R\widehat{\ }n)$
⟨*proof*⟩

lemma *trancl-finite-eq-relpow*:
assumes *finite* R **shows** $R^+ = (\bigcup n{\in}\{n.\ 0 < n \wedge n \leq card\ R\}.\ R\widehat{\ }n)$
⟨*proof*⟩

lemma *finite-relcomp*[*simp*,*intro*]:

assumes *finite R* **and** *finite S*
shows *finite* (*R O S*)
⟨*proof*⟩

lemma *finite-relpow* [*simp*, *intro*]:
fixes *R* :: (*'a* × *'a*) *set*
assumes *finite R*
shows *n* > *0* ⟹ *finite* (*R*^^*n*)
⟨*proof*⟩

lemma *single-valued-relpow*:
fixes *R* :: (*'a* × *'a*) *set*
shows *single-valued R* ⟹ *single-valued* (*R* ^^ *n*)
⟨*proof*⟩

21.6 Bounded transitive closure

definition *ntrancl* :: *nat* ⇒ (*'a* × *'a*) *set* ⇒ (*'a* × *'a*) *set*
where *ntrancl n R* = (⋃ *i*∈{*i*. *0* < *i* ∧ *i* ≤ *Suc n*}. *R* ^^ *i*)

lemma *ntrancl-Zero* [*simp*, *code*]: *ntrancl 0 R* = *R*
⟨*proof*⟩

lemma *ntrancl-Suc* [*simp*]: *ntrancl* (*Suc n*) *R* = *ntrancl n R O* (*Id* ∪ *R*)
⟨*proof*⟩

lemma [*code*]: *ntrancl* (*Suc n*) *r* = (*let r'* = *ntrancl n r in r'* ∪ *r' O r*)
⟨*proof*⟩

lemma *finite-trancl-ntranl*: *finite R* ⟹ *trancl R* = *ntrancl* (*card R* − *1*) *R*
⟨*proof*⟩

21.7 Acyclic relations

definition *acyclic* :: (*'a* × *'a*) *set* ⇒ *bool*
where *acyclic r* ⟷ (∀ *x*. (*x*,*x*) ∉ *r*^+)

abbreviation *acyclicP* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool*
where *acyclicP r* ≡ *acyclic* {(*x*, *y*). *r x y*}

lemma *acyclic-irrefl* [*code*]: *acyclic r* ⟷ *irrefl* (*r*^+)
⟨*proof*⟩

lemma *acyclicI*: ∀ *x*. (*x*, *x*) ∉ *r*^+ ⟹ *acyclic r*
⟨*proof*⟩

lemma (**in** *preorder*) *acyclicI-order*:
assumes ∗: ⋀*a b*. (*a*, *b*) ∈ *r* ⟹ *f b* < *f a*
shows *acyclic r*
⟨*proof*⟩

lemma *acyclic-insert* [*iff*]: *acyclic* (*insert* (*y*, *x*) *r*) ⟷ *acyclic r* ∧ (*x*, *y*) ∉ *r**
⟨*proof*⟩

lemma *acyclic-converse* [*iff*]: *acyclic* (r^{-1}) ⟷ *acyclic r*
⟨*proof*⟩

lemmas *acyclicP-converse* [*iff*] = *acyclic-converse* [*to-pred*]

lemma *acyclic-impl-antisym-rtrancl*: *acyclic r* ⟹ *antisym* (r^*)
⟨*proof*⟩

lemma *acyclic-subset*: *acyclic s* ⟹ *r* ⊆ *s* ⟹ *acyclic r*
⟨*proof*⟩

21.8 Setup of transitivity reasoner

⟨*ML*⟩

lemma *transp-rtranclp* [*simp*]: *transp* R^{**}
⟨*proof*⟩

Optional methods.

⟨*ML*⟩

end

22 Well-founded Recursion

theory *Wellfounded*
imports *Transitive-Closure*
begin

22.1 Basic Definitions

definition *wf* :: (′*a* × ′*a*) *set* ⇒ *bool*
where *wf r* ⟷ (∀ *P*. (∀ *x*. (∀ *y*. (*y*, *x*) ∈ *r* ⟶ *P y*) ⟶ *P x*) ⟶ (∀ *x*. *P x*))

definition *wfP* :: (′*a* ⇒ ′*a* ⇒ *bool*) ⇒ *bool*
where *wfP r* ⟷ *wf* {(*x*, *y*). *r x y*}

lemma *wfP-wf-eq* [*pred-set-conv*]: *wfP* (λ*x y*. (*x*, *y*) ∈ *r*) = *wf r*
⟨*proof*⟩

lemma *wfUNIVI*: (⋀*P x*. (∀ *x*. (∀ *y*. (*y*, *x*) ∈ *r* ⟶ *P y*) ⟶ *P x*) ⟹ *P x*) ⟹ *wf r*
⟨*proof*⟩

lemmas *wfPUNIVI* = *wfUNIVI* [*to-pred*]

Restriction to domain A and range B. If r is well-founded over their intersection, then *wf r*.

lemma *wfI*:
assumes $r \subseteq A \times B$
and $\bigwedge x\ P.\ [\![\forall x.\ (\forall y.\ (y, x) \in r \longrightarrow P\ y) \longrightarrow P\ x;\ \ x \in A;\ x \in B]\!] \Longrightarrow P\ x$
shows *wf r*
⟨*proof*⟩

lemma *wf-induct*:
assumes *wf r*
and $\bigwedge x.\ \forall y.\ (y, x) \in r \longrightarrow P\ y \Longrightarrow P\ x$
shows $P\ a$
⟨*proof*⟩

lemmas *wfP-induct* = *wf-induct* [*to-pred*]

lemmas *wf-induct-rule* = *wf-induct* [*rule-format*, *consumes 1*, *case-names less*, *induct set*: *wf*]

lemmas *wfP-induct-rule* = *wf-induct-rule* [*to-pred*, *induct set*: *wfP*]

lemma *wf-not-sym*: *wf r* $\Longrightarrow (a, x) \in r \Longrightarrow (x, a) \notin r$
⟨*proof*⟩

lemma *wf-asym*:
assumes *wf r* $(a, x) \in r$
obtains $(x, a) \notin r$
⟨*proof*⟩

lemma *wf-imp-asym*: *wf r* $\Longrightarrow$ *asym r*
⟨*proof*⟩

lemma *wfP-imp-asymp*: *wfP r* $\Longrightarrow$ *asymp r*
⟨*proof*⟩

lemma *wf-not-refl* [*simp*]: *wf r* $\Longrightarrow (a, a) \notin r$
⟨*proof*⟩

lemma *wf-irrefl*:
assumes *wf r*
obtains $(a, a) \notin r$
⟨*proof*⟩

lemma *wf-imp-irrefl*:
assumes *wf r* **shows** *irrefl r*
⟨*proof*⟩

lemma *wfP-imp-irreflp*: *wfP r* $\Longrightarrow$ *irreflp r*
⟨*proof*⟩

lemma *wf-wellorderI*:
assumes *wf*: *wf* {(*x*::'*a*::*ord*, *y*). *x* < *y*}
and *lin*: *OFCLASS*('*a*::*ord*, *linorder-class*)
shows *OFCLASS*('*a*::*ord*, *wellorder-class*)
⟨*proof*⟩

lemma (**in** *wellorder*) *wf*: *wf* {(*x*, *y*). *x* < *y*}
⟨*proof*⟩

lemma (**in** *wellorder*) *wfP-less*[*simp*]: *wfP* (<)
⟨*proof*⟩

22.2 Basic Results

Point-free characterization of well-foundedness

lemma *wfE-pf*:
assumes *wf*: *wf R*
and *a*: *A* ⊆ *R* `` *A*
shows *A* = {}
⟨*proof*⟩

lemma *wfI-pf*:
assumes *a*: ⋀*A*. *A* ⊆ *R* `` *A* $\Longrightarrow$ *A* = {}
shows *wf R*
⟨*proof*⟩

22.2.1 Minimal-element characterization of well-foundedness

lemma *wfE-min*:
assumes *wf*: *wf R* **and** *Q*: *x* ∈ *Q*
obtains *z* **where** *z* ∈ *Q* ⋀*y*. (*y*, *z*) ∈ *R* $\Longrightarrow$ *y* ∉ *Q*
⟨*proof*⟩

lemma *wfE-min'*:
wf R $\Longrightarrow$ *Q* ≠ {} $\Longrightarrow$ (⋀*z*. *z* ∈ *Q* $\Longrightarrow$ (⋀*y*. (*y*, *z*) ∈ *R* $\Longrightarrow$ *y* ∉ *Q*) $\Longrightarrow$ *thesis*) $\Longrightarrow$ *thesis*
⟨*proof*⟩

lemma *wfI-min*:
assumes *a*: ⋀*x Q*. *x* ∈ *Q* $\Longrightarrow$ ∃ *z*∈*Q*. ∀ *y*. (*y*, *z*) ∈ *R* ⟶ *y* ∉ *Q*
shows *wf R*
⟨*proof*⟩

lemma *wf-eq-minimal*: *wf r* ⟷ (∀ *Q x*. *x* ∈ *Q* ⟶ (∃ *z*∈*Q*. ∀ *y*. (*y*, *z*) ∈ *r* ⟶ *y* ∉ *Q*))

⟨*proof*⟩

lemmas *wfP-eq-minimal* = *wf-eq-minimal* [*to-pred*]

22.2.2 Well-foundedness of transitive closure

lemma *wf-trancl*:
assumes *wf r*
shows *wf* (r^+)
⟨*proof*⟩

lemmas *wfP-trancl* = *wf-trancl* [*to-pred*]

lemma *wf-converse-trancl*: *wf* $(r^{-1}) \Longrightarrow$ *wf* $((r^+)^{-1})$
⟨*proof*⟩

Well-foundedness of subsets

lemma *wf-subset*: *wf r* $\Longrightarrow$ $p \subseteq r \Longrightarrow$ *wf p*
⟨*proof*⟩

lemmas *wfP-subset* = *wf-subset* [*to-pred*]

Well-foundedness of the empty relation

lemma *wf-empty* [*iff*]: *wf* {}
⟨*proof*⟩

lemma *wfP-empty* [*iff*]: *wfP* (λ*x y*. *False*)
⟨*proof*⟩

lemma *wf-Int1*: *wf r* $\Longrightarrow$ *wf* $(r \cap r')$
⟨*proof*⟩

lemma *wf-Int2*: *wf r* $\Longrightarrow$ *wf* $(r' \cap r)$
⟨*proof*⟩

Exponentiation.

lemma *wf-exp*:
assumes *wf* $(R \mathbin{\frown\!\!\frown} n)$
shows *wf R*
⟨*proof*⟩

Well-foundedness of *insert*.

lemma *wf-insert* [*iff*]: *wf* (*insert* (*y*,*x*) *r*) $\longleftrightarrow$ *wf r* $\wedge$ (*x*,*y*) $\notin r^*$ (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

22.2.3 Well-foundedness of image

lemma *wf-map-prod-image-Dom-Ran*:
fixes *r*:: $('a \times 'a)$ *set*

and $f :: 'a \Rightarrow 'b$
assumes *wf-r*: *wf r*
and *inj*: $\bigwedge a\ a'.\ a \in Domain\ r \Longrightarrow a' \in Range\ r \Longrightarrow f\ a = f\ a' \Longrightarrow a = a'$
shows *wf* (*map-prod f f* ‘ *r*)
⟨*proof*⟩

lemma *wf-map-prod-image*: *wf r* ⟹ *inj f* ⟹ *wf* (*map-prod f f* ‘ *r*)
⟨*proof*⟩

22.3 Well-Foundedness Results for Unions

lemma *wf-union-compatible*:
assumes *wf R wf S*
assumes *R O S* ⊆ *R*
shows *wf* (*R* ∪ *S*)
⟨*proof*⟩

Well-foundedness of indexed union with disjoint domains and ranges.

lemma *wf-UN*:
assumes *r*: $\bigwedge i.\ i \in I \Longrightarrow wf\ (r\ i)$
and *disj*: $\bigwedge i\ j.\ [\![i \in I;\ j \in I;\ r\ i \neq r\ j]\!] \Longrightarrow Domain\ (r\ i) \cap Range\ (r\ j) = \{\}$
shows *wf* ($\bigcup i{\in}I.\ r\ i$)
⟨*proof*⟩

lemma *wfP-SUP*:
$\forall i.\ wfP\ (r\ i) \Longrightarrow \forall i\ j.\ r\ i \neq r\ j \longrightarrow inf\ (Domainp\ (r\ i))\ (Rangep\ (r\ j)) = bot$
⟹
wfP ($\bigsqcup$(*range r*))
⟨*proof*⟩

lemma *wf-Union*:
assumes $\forall r{\in}R.\ wf\ r$
and $\forall r{\in}R.\ \forall s{\in}R.\ r \neq s \longrightarrow Domain\ r \cap Range\ s = \{\}$
shows *wf* ($\bigcup R$)
⟨*proof*⟩

Intuition: We find an $R \cup S$-min element of a nonempty subset A by case distinction.

1. There is a step $a\ {-}R{\rightarrow}\ b$ with $a,\ b \in A$. Pick an R-min element z of the (nonempty) set $\{a{\in}A \mid \exists\, b{\in}A.\ a\ {-}R{\rightarrow}\ b\}$. By definition, there is $z' \in A$ s.t. $z\ {-}R{\rightarrow}\ z'$. Because z is R-min in the subset, z' must be R-min in A. Because z' has an R-predecessor, it cannot have an S-successor and is thus S-min in A as well.

2. There is no such step. Pick an S-min element of A. In this case it must be an R-min element of A as well.

lemma *wf-Un*: *wf r* ⟹ *wf s* ⟹ *Domain r* ∩ *Range s* = {} ⟹ *wf* (*r* ∪ *s*)

⟨*proof*⟩

lemma *wf-union-merge*: *wf* $(R \cup S) = wf\ (R\ O\ R \cup S\ O\ R \cup S)$
(**is** *wf ?A* = *wf ?B*)
⟨*proof*⟩

lemma *wf-comp-self*: *wf* $R \longleftrightarrow wf\ (R\ O\ R)$ — special case
⟨*proof*⟩

22.4 Well-Foundedness of Composition

Bachmair and Dershowitz 1986, Lemma 2. [Provided by Tjark Weber]

lemma *qc-wf-relto-iff*:
assumes $R\ O\ S \subseteq (R \cup S)^*\ O\ R$ — R quasi-commutes over S
shows $wf\ (S^*\ O\ R\ O\ S^*) \longleftrightarrow wf\ R$
(**is** *wf ?S* ⟷ -)
⟨*proof*⟩

corollary *wf-relcomp-compatible*:
assumes *wf R* **and** $R\ O\ S \subseteq S\ O\ R$
shows $wf\ (S\ O\ R)$
⟨*proof*⟩

22.5 Acyclic relations

lemma *wf-acyclic*: *wf r* ⟹ *acyclic r*
⟨*proof*⟩

lemmas *wfP-acyclicP* = *wf-acyclic* [*to-pred*]

22.5.1 Wellfoundedness of finite acyclic relations

lemma *finite-acyclic-wf*:
assumes *finite r acyclic r* **shows** *wf r*
⟨*proof*⟩

lemma *finite-acyclic-wf-converse*: *finite r* ⟹ *acyclic r* ⟹ $wf\ (r^{-1})$
⟨*proof*⟩

Observe that the converse of an irreflexive, transitive, and finite relation is again well-founded. Thus, we may employ it for well-founded induction.

lemma *wf-converse*:
assumes *irefl r* **and** *trans r* **and** *finite r*
shows $wf\ (r^{-1})$
⟨*proof*⟩

lemma *wf-iff-acyclic-if-finite*: *finite r* ⟹ *wf r* = *acyclic r*
⟨*proof*⟩

22.6 *nat* is well-founded

lemma *less-nat-rel*: $(<) = (\lambda m\ n.\ n = Suc\ m)^{++}$
⟨*proof*⟩

definition *pred-nat* :: $(nat \times nat)\ set$
where $pred\text{-}nat = \{(m,\ n).\ n = Suc\ m\}$

definition *less-than* :: $(nat \times nat)\ set$
where $less\text{-}than = pred\text{-}nat^{+}$

lemma *less-eq*: $(m,\ n) \in pred\text{-}nat^{+} \longleftrightarrow m < n$
⟨*proof*⟩

lemma *pred-nat-trancl-eq-le*: $(m,\ n) \in pred\text{-}nat^{*} \longleftrightarrow m \leq n$
⟨*proof*⟩

lemma *wf-pred-nat*: *wf pred-nat*
⟨*proof*⟩

lemma *wf-less-than* [*iff*]: *wf less-than*
⟨*proof*⟩

lemma *trans-less-than* [*iff*]: *trans less-than*
⟨*proof*⟩

lemma *less-than-iff* [*iff*]: $((x,y) \in less\text{-}than) = (x<y)$
⟨*proof*⟩

lemma *irrefl-less-than*: *irrefl less-than*
⟨*proof*⟩

lemma *asym-less-than*: *asym less-than*
⟨*proof*⟩

lemma *total-less-than*: *total less-than* **and** *total-on-less-than* [*simp*]: *total-on A less-than*
⟨*proof*⟩

lemma *wf-less*: *wf* $\{(x,\ y{::}nat).\ x < y\}$
⟨*proof*⟩

22.7 Accessible Part

Inductive definition of the accessible part *acc r* of a relation; see also [6].

inductive-set *acc* :: $('a \times 'a)\ set \Rightarrow 'a\ set$ **for** r :: $('a \times 'a)\ set$
where *accI*: $(\bigwedge y.\ (y,\ x) \in r \Longrightarrow y \in acc\ r) \Longrightarrow x \in acc\ r$

abbreviation *termip* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a \Rightarrow bool$

where *termip r* ≡ *accp* (r^{-1-1})

abbreviation *termi* :: (′*a* × ′*a*) *set* ⇒ ′*a set*
where *termi r* ≡ *acc* (r^{-1})

lemmas *accpI* = *accp.accI*

lemma *accp-eq-acc* [*code*]: *accp r* = (λ*x*. *x* ∈ *Wellfounded.acc* {(*x*, *y*). *r x y*})
⟨*proof*⟩

Induction rules

theorem *accp-induct*:
assumes *major*: *accp r a*
assumes *hyp*: ⋀*x*. *accp r x* ⟹ ∀ *y*. *r y x* ⟶ *P y* ⟹ *P x*
shows *P a*
⟨*proof*⟩

lemmas *accp-induct-rule* = *accp-induct* [*rule-format*, *induct set*: *accp*]

theorem *accp-downward*: *accp r b* ⟹ *r a b* ⟹ *accp r a*
⟨*proof*⟩

lemma *not-accp-down*:
assumes *na*: ¬ *accp R x*
obtains *z* **where** *R z x* **and** ¬ *accp R z*
⟨*proof*⟩

lemma *accp-downwards-aux*: r^{**} *b a* ⟹ *accp r a* ⟶ *accp r b*
⟨*proof*⟩

theorem *accp-downwards*: *accp r a* ⟹ r^{**} *b a* ⟹ *accp r b*
⟨*proof*⟩

theorem *accp-wfPI*: ∀ *x*. *accp r x* ⟹ *wfP r*
⟨*proof*⟩

theorem *accp-wfPD*: *wfP r* ⟹ *accp r x*
⟨*proof*⟩

theorem *wfP-accp-iff*: *wfP r* = (∀ *x*. *accp r x*)
⟨*proof*⟩

Smaller relations have bigger accessible parts:

lemma *accp-subset*:
assumes *R1* ≤ *R2*
shows *accp R2* ≤ *accp R1*
⟨*proof*⟩

This is a generalized induction theorem that works on subsets of the acces-

sible part.

lemma *accp-subset-induct*:
assumes *subset*: $D \leq accp\ R$
and *dcl*: $\bigwedge x\ z.\ D\ x \Longrightarrow R\ z\ x \Longrightarrow D\ z$
and $D\ x$
and *istep*: $\bigwedge x.\ D\ x \Longrightarrow (\bigwedge z.\ R\ z\ x \Longrightarrow P\ z) \Longrightarrow P\ x$
shows $P\ x$
⟨*proof*⟩

Set versions of the above theorems

lemmas *acc-induct* = *accp-induct* [*to-set*]
lemmas *acc-induct-rule* = *acc-induct* [*rule-format*, *induct set*: *acc*]
lemmas *acc-downward* = *accp-downward* [*to-set*]
lemmas *not-acc-down* = *not-accp-down* [*to-set*]
lemmas *acc-downwards-aux* = *accp-downwards-aux* [*to-set*]
lemmas *acc-downwards* = *accp-downwards* [*to-set*]
lemmas *acc-wfI* = *accp-wfPI* [*to-set*]
lemmas *acc-wfD* = *accp-wfPD* [*to-set*]
lemmas *wf-acc-iff* = *wfP-accp-iff* [*to-set*]
lemmas *acc-subset* = *accp-subset* [*to-set*]
lemmas *acc-subset-induct* = *accp-subset-induct* [*to-set*]

22.8 Tools for building wellfounded relations

Inverse Image

lemma *wf-inv-image* [*simp*,*intro*!]:
fixes $f :: 'a \Rightarrow 'b$
assumes *wf r*
shows *wf* (*inv-image r f*)
⟨*proof*⟩

22.8.1 Conversion to a known well-founded relation

lemma *wf-if-convertible-to-wf*:
fixes $r :: 'a\ rel$ **and** $s :: 'b\ rel$ **and** $f :: 'a \Rightarrow 'b$
assumes *wf s* **and** *convertible*: $\bigwedge x\ y.\ (x, y) \in r \Longrightarrow (f\ x, f\ y) \in s$
shows *wf r*
⟨*proof*⟩

lemma *wfP-if-convertible-to-wfP*: *wfP S* $\Longrightarrow (\bigwedge x\ y.\ R\ x\ y \Longrightarrow S\ (f\ x)\ (f\ y)) \Longrightarrow$ *wfP R*
⟨*proof*⟩

Converting to *nat* is a very common special case that might be found more easily by Sledgehammer.

lemma *wfP-if-convertible-to-nat*:
fixes $f :: - \Rightarrow nat$
shows $(\bigwedge x\ y.\ R\ x\ y \Longrightarrow f\ x < f\ y) \Longrightarrow$ *wfP R*
⟨*proof*⟩

22.8.2 Measure functions into *nat*

definition *measure* :: $('a \Rightarrow nat) \Rightarrow ('a \times 'a)\ set$
where $measure = inv\text{-}image\ less\text{-}than$

lemma *in-measure*[*simp*, *code-unfold*]: $(x, y) \in measure\ f \longleftrightarrow f\ x < f\ y$
⟨*proof*⟩

lemma *wf-measure* [*iff*]: $wf\ (measure\ f)$
⟨*proof*⟩

lemma *wf-if-measure*: $(\bigwedge x.\ P\ x \Longrightarrow f(g\ x) < f\ x) \Longrightarrow wf\ \{(y,x).\ P\ x \wedge y = g\ x\}$
for $f :: 'a \Rightarrow nat$
⟨*proof*⟩

22.8.3 Lexicographic combinations

definition *lex-prod* :: $('a \times 'a)\ set \Rightarrow ('b \times 'b)\ set \Rightarrow (('a \times 'b) \times ('a \times 'b))\ set$
(**infixr** $<*lex*>$ *80*)
where $ra <*lex*> rb = \{((a, b), (a', b')).\ (a, a') \in ra \vee a = a' \wedge (b, b') \in rb\}$

lemma *in-lex-prod*[*simp*]: $((a, b), (a', b')) \in r <*lex*> s \longleftrightarrow (a, a') \in r \vee a = a' \wedge (b, b') \in s$
⟨*proof*⟩

lemma *wf-lex-prod* [*intro*!]:
assumes *wf ra* *wf rb*
shows $wf\ (ra <*lex*> rb)$
⟨*proof*⟩

lemma *refl-lex-prod*[*simp*]: $refl\ r_B \Longrightarrow refl\ (r_A <*lex*> r_B)$
⟨*proof*⟩

lemma *irrefl-on-lex-prod*[*simp*]:
$irrefl\text{-}on\ A\ r_A \Longrightarrow irrefl\text{-}on\ B\ r_B \Longrightarrow irrefl\text{-}on\ (A \times B)\ (r_A <*lex*> r_B)$
⟨*proof*⟩

lemma *irrefl-lex-prod*[*simp*]: $irrefl\ r_A \Longrightarrow irrefl\ r_B \Longrightarrow irrefl\ (r_A <*lex*> r_B)$
⟨*proof*⟩

lemma *sym-on-lex-prod*[*simp*]:
$sym\text{-}on\ A\ r_A \Longrightarrow sym\text{-}on\ B\ r_B \Longrightarrow sym\text{-}on\ (A \times B)\ (r_A <*lex*> r_B)$
⟨*proof*⟩

lemma *sym-lex-prod*[*simp*]:
$sym\ r_A \Longrightarrow sym\ r_B \Longrightarrow sym\ (r_A <*lex*> r_B)$
⟨*proof*⟩

lemma *asym-on-lex-prod*[*simp*]:
$asym\text{-}on\ A\ r_A \Longrightarrow asym\text{-}on\ B\ r_B \Longrightarrow asym\text{-}on\ (A \times B)\ (r_A <*lex*> r_B)$

⟨*proof*⟩

lemma *asym-lex-prod*[*simp*]:
asym $r_A \Longrightarrow$ *asym* $r_B \Longrightarrow$ *asym* $(r_A <*lex*> r_B)$
⟨*proof*⟩

lemma *trans-on-lex-prod*[*simp*]:
assumes *trans-on* A r_A **and** *trans-on* B r_B
shows *trans-on* $(A \times B)$ $(r_A <*lex*> r_B)$
⟨*proof*⟩

lemma *trans-lex-prod* [*simp*,*intro*!]: *trans* $r_A \Longrightarrow$ *trans* $r_B \Longrightarrow$ *trans* $(r_A <*lex*> r_B)$
⟨*proof*⟩

lemma *total-on-lex-prod*[*simp*]:
total-on A $r_A \Longrightarrow$ *total-on* B $r_B \Longrightarrow$ *total-on* $(A \times B)$ $(r_A <*lex*> r_B)$
⟨*proof*⟩

lemma *total-lex-prod*[*simp*]: *total* $r_A \Longrightarrow$ *total* $r_B \Longrightarrow$ *total* $(r_A <*lex*> r_B)$
⟨*proof*⟩

lexicographic combinations with measure functions

definition *mlex-prod* :: $('a \Rightarrow nat) \Rightarrow ('a \times 'a)\ set \Rightarrow ('a \times 'a)\ set$ (**infixr** $<*mlex*>$ *80*)
where $f <*mlex*> R = inv\text{-}image\ (less\text{-}than <*lex*> R)\ (\lambda x.\ (f\ x,\ x))$

lemma
wf-mlex: *wf* $R \Longrightarrow$ *wf* $(f <*mlex*> R)$ **and**
mlex-less: $f\ x < f\ y \Longrightarrow (x,\ y) \in f <*mlex*> R$ **and**
mlex-leq: $f\ x \leq f\ y \Longrightarrow (x,\ y) \in R \Longrightarrow (x,\ y) \in f <*mlex*> R$ **and**
mlex-iff: $(x,\ y) \in f <*mlex*> R \longleftrightarrow f\ x < f\ y \vee f\ x = f\ y \wedge (x,\ y) \in R$
⟨*proof*⟩

Proper subset relation on finite sets.

definition *finite-psubset* :: $('a\ set \times 'a\ set)\ set$
where *finite-psubset* $= \{(A,\ B).\ A \subset B \wedge finite\ B\}$

lemma *wf-finite-psubset*[*simp*]: *wf finite-psubset*
⟨*proof*⟩

lemma *trans-finite-psubset*: *trans finite-psubset*
⟨*proof*⟩

lemma *in-finite-psubset*[*simp*]: $(A,\ B) \in finite\text{-}psubset \longleftrightarrow A \subset B \wedge finite\ B$
⟨*proof*⟩

max- and min-extension of order to finite sets

inductive-set *max-ext* :: $('a \times 'a)\ set \Rightarrow ('a\ set \times 'a\ set)\ set$

for $R :: ('a \times 'a)\ set$
where *max-extI*[*intro*]:
finite $X \Longrightarrow$ *finite* $Y \Longrightarrow Y \neq \{\} \Longrightarrow (\bigwedge x.\ x \in X \Longrightarrow \exists\, y{\in}Y.\ (x,\ y) \in R) \Longrightarrow (X,\ Y) \in$ *max-ext* R

lemma *max-ext-wf*:
assumes *wf*: *wf* r
shows *wf* (*max-ext* r)
⟨*proof*⟩

lemma *max-ext-additive*: $(A,\ B) \in$ *max-ext* $R \Longrightarrow (C,\ D) \in$ *max-ext* $R \Longrightarrow (A \cup C,\ B \cup D) \in$ *max-ext* R
⟨*proof*⟩

definition *min-ext* $:: ('a \times 'a)\ set \Rightarrow ('a\ set \times 'a\ set)\ set$
where *min-ext* $r = \{(X,\ Y) \mid X\ Y.\ X \neq \{\} \wedge (\forall\, y \in Y.\ (\exists\, x \in X.\ (x,\ y) \in r))\}$

lemma *min-ext-wf*:
assumes *wf* r
shows *wf* (*min-ext* r)
⟨*proof*⟩

22.8.4 Bounded increase must terminate

lemma *wf-bounded-measure*:
fixes $ub :: 'a \Rightarrow nat$
and $f :: 'a \Rightarrow nat$
assumes $\bigwedge a\ b.\ (b,\ a) \in r \Longrightarrow ub\ b \leq ub\ a \wedge ub\ a \geq f\ b \wedge f\ b > f\ a$
shows *wf* r
⟨*proof*⟩

lemma *wf-bounded-set*:
fixes $ub :: 'a \Rightarrow 'b\ set$
and $f :: 'a \Rightarrow 'b\ set$
assumes $\bigwedge a\ b.\ (b,a) \in r \Longrightarrow$ *finite* $(ub\ a) \wedge ub\ b \subseteq ub\ a \wedge ub\ a \supseteq f\ b \wedge f\ b \supset f\ a$
shows *wf* r
⟨*proof*⟩

lemma *finite-subset-wf*:
assumes *finite* A
shows *wf* $\{(X,\ Y).\ X \subset Y \wedge Y \subseteq A\}$
⟨*proof*⟩

hide-const (**open**) *acc accp*

end

23 Well-Founded Recursion Combinator

theory *Wfrec*
imports *Wellfounded*
begin

inductive *wfrec-rel* :: $('a \times 'a)\ set \Rightarrow (('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b)) \Rightarrow 'a \Rightarrow 'b \Rightarrow bool$
for *R F*
where *wfrecI*: $(\bigwedge z.\ (z, x) \in R \Longrightarrow wfrec\text{-}rel\ R\ F\ z\ (g\ z)) \Longrightarrow wfrec\text{-}rel\ R\ F\ x\ (F\ g\ x)$

definition *cut* :: $('a \Rightarrow 'b) \Rightarrow ('a \times 'a)\ set \Rightarrow 'a \Rightarrow 'a \Rightarrow 'b$
where $cut\ f\ R\ x = (\lambda y.\ if\ (y, x) \in R\ then\ f\ y\ else\ undefined)$

definition *adm-wf* :: $('a \times 'a)\ set \Rightarrow (('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b)) \Rightarrow bool$
where $adm\text{-}wf\ R\ F \longleftrightarrow (\forall f\ g\ x.\ (\forall z.\ (z, x) \in R \longrightarrow f\ z = g\ z) \longrightarrow F\ f\ x = F\ g\ x)$

definition *wfrec* :: $('a \times 'a)\ set \Rightarrow (('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b)) \Rightarrow ('a \Rightarrow 'b)$
where $wfrec\ R\ F = (\lambda x.\ THE\ y.\ wfrec\text{-}rel\ R\ (\lambda f\ x.\ F\ (cut\ f\ R\ x)\ x)\ x\ y)$

lemma *cuts-eq*: $(cut\ f\ R\ x = cut\ g\ R\ x) \longleftrightarrow (\forall y.\ (y, x) \in R \longrightarrow f\ y = g\ y)$
⟨*proof*⟩

lemma *cut-apply*: $(x, a) \in R \Longrightarrow cut\ f\ R\ a\ x = f\ x$
⟨*proof*⟩

Inductive characterization of *wfrec* combinator; for details see: John Harrison, "Inductive definitions: automation and application".

lemma *theI-unique*: $\exists!x.\ P\ x \Longrightarrow P\ x \longleftrightarrow x = The\ P$
⟨*proof*⟩

lemma *wfrec-unique*:
assumes *adm-wf R F wf R*
shows $\exists!y.\ wfrec\text{-}rel\ R\ F\ x\ y$
⟨*proof*⟩

lemma *adm-lemma*: $adm\text{-}wf\ R\ (\lambda f\ x.\ F\ (cut\ f\ R\ x)\ x)$
⟨*proof*⟩

lemma *wfrec*: $wf\ R \Longrightarrow wfrec\ R\ F\ a = F\ (cut\ (wfrec\ R\ F)\ R\ a)\ a$
⟨*proof*⟩

This form avoids giant explosions in proofs. NOTE USE OF ≡.

lemma *def-wfrec*: $f \equiv wfrec\ R\ F \Longrightarrow wf\ R \Longrightarrow f\ a = F\ (cut\ f\ R\ a)\ a$
⟨*proof*⟩

23.0.1 Well-founded recursion via genuine fixpoints

lemma *wfrec-fixpoint*:

assumes *wf*: *wf R*
and *adm*: *adm-wf R F*
shows *wfrec R F* = *F* (*wfrec R F*)
⟨*proof*⟩

lemma *wfrec-def-adm*: *f* ≡ *wfrec R F* ⟹ *wf R* ⟹ *adm-wf R F* ⟹ *f* = *F f*
⟨*proof*⟩

23.1 Wellfoundedness of *same-fst*

definition *same-fst* :: ('*a* ⇒ *bool*) ⇒ ('*a* ⇒ ('*b* × '*b*) *set*) ⇒ (('*a* × '*b*) × ('*a* × '*b*)) *set*
where *same-fst P R* = {((*x'*, *y'*), (*x*, *y*)) . *x'* = *x* ∧ *P x* ∧ (*y'*,*y*) ∈ *R x*}
— For *wfrec* declarations where the first n parameters stay unchanged in the recursive call.

lemma *same-fstI* [*intro*!]: *P x* ⟹ (*y'*, *y*) ∈ *R x* ⟹ ((*x*, *y'*), (*x*, *y*)) ∈ *same-fst P R*
⟨*proof*⟩

lemma *wf-same-fst*:
assumes ⋀*x*. *P x* ⟹ *wf* (*R x*)
shows *wf* (*same-fst P R*)
⟨*proof*⟩

end

24 Orders as Relations

theory *Order-Relation*
imports *Wfrec*
begin

24.1 Orders on a set

definition *preorder-on A r* ≡ *refl-on A r* ∧ *trans r*

definition *partial-order-on A r* ≡ *preorder-on A r* ∧ *antisym r*

definition *linear-order-on A r* ≡ *partial-order-on A r* ∧ *total-on A r*

definition *strict-linear-order-on A r* ≡ *trans r* ∧ *irrefl r* ∧ *total-on A r*

definition *well-order-on A r* ≡ *linear-order-on A r* ∧ *wf*(*r* − *Id*)

lemmas *order-on-defs* =
preorder-on-def partial-order-on-def linear-order-on-def
strict-linear-order-on-def well-order-on-def

lemma *partial-order-onD*:
assumes *partial-order-on A r* **shows** *refl-on A r* **and** *trans r* **and** *antisym r*
⟨*proof*⟩

lemma *preorder-on-empty*[*simp*]: *preorder-on* {} {}
⟨*proof*⟩

lemma *partial-order-on-empty*[*simp*]: *partial-order-on* {} {}
⟨*proof*⟩

lemma *lnear-order-on-empty*[*simp*]: *linear-order-on* {} {}
⟨*proof*⟩

lemma *well-order-on-empty*[*simp*]: *well-order-on* {} {}
⟨*proof*⟩

lemma *preorder-on-converse*[*simp*]: *preorder-on A* (r^{-1}) = *preorder-on A r*
⟨*proof*⟩

lemma *partial-order-on-converse*[*simp*]: *partial-order-on A* (r^{-1}) = *partial-order-on A r*
⟨*proof*⟩

lemma *linear-order-on-converse*[*simp*]: *linear-order-on A* (r^{-1}) = *linear-order-on A r*
⟨*proof*⟩

lemma *partial-order-on-acyclic*:
partial-order-on A r ⟹ *acyclic* $(r - Id)$
⟨*proof*⟩

lemma *partial-order-on-well-order-on*:
finite r ⟹ *partial-order-on A r* ⟹ *wf* $(r - Id)$
⟨*proof*⟩

lemma *strict-linear-order-on-diff-Id*: *linear-order-on A r* ⟹ *strict-linear-order-on A* $(r - Id)$
⟨*proof*⟩

lemma *linear-order-on-singleton* [*simp*]: *linear-order-on* {*x*} {(*x*, *x*)}
⟨*proof*⟩

lemma *linear-order-on-acyclic*:
assumes *linear-order-on A r*
shows *acyclic* $(r - Id)$
⟨*proof*⟩

lemma *linear-order-on-well-order-on*:
assumes *finite r*
shows *linear-order-on A r* $\longleftrightarrow$ *well-order-on A r*
⟨*proof*⟩

24.2 Orders on the field

abbreviation *Refl r* ≡ *refl-on* (*Field r*) *r*

abbreviation *Preorder r* ≡ *preorder-on* (*Field r*) *r*

abbreviation *Partial-order r* ≡ *partial-order-on* (*Field r*) *r*

abbreviation *Total r* ≡ *total-on* (*Field r*) *r*

abbreviation *Linear-order r* ≡ *linear-order-on* (*Field r*) *r*

abbreviation *Well-order r* ≡ *well-order-on* (*Field r*) *r*

lemma *subset-Image-Image-iff*:
Preorder r $\Longrightarrow$ *A* ⊆ *Field r* $\Longrightarrow$ *B* ⊆ *Field r* $\Longrightarrow$
r `` *A* ⊆ *r* `` *B* $\longleftrightarrow$ (∀ *a*∈*A*.∃ *b*∈*B*. (*b*, *a*) ∈ *r*)
⟨*proof*⟩

lemma *subset-Image1-Image1-iff*:
Preorder r $\Longrightarrow$ *a* ∈ *Field r* $\Longrightarrow$ *b* ∈ *Field r* $\Longrightarrow$ *r* `` {*a*} ⊆ *r* `` {*b*} $\longleftrightarrow$ (*b*, *a*) ∈ *r*
⟨*proof*⟩

lemma *Refl-antisym-eq-Image1-Image1-iff*:
assumes *Refl r*
and *as*: *antisym r*
and *abf*: *a* ∈ *Field r b* ∈ *Field r*
shows *r* `` {*a*} = *r* `` {*b*} $\longleftrightarrow$ *a* = *b*
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *Partial-order-eq-Image1-Image1-iff*:
Partial-order r $\Longrightarrow$ *a* ∈ *Field r* $\Longrightarrow$ *b* ∈ *Field r* $\Longrightarrow$ *r* `` {*a*} = *r* `` {*b*} $\longleftrightarrow$ *a* = *b*
⟨*proof*⟩

lemma *Total-Id-Field*:
assumes *Total r*
and *not-Id*: ¬ *r* ⊆ *Id*
shows *Field r* = *Field* (*r* − *Id*)
⟨*proof*⟩

24.3 Relations given by a predicate and the field

definition *relation-of* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a\ set \Rightarrow ('a \times 'a)\ set$
where *relation-of P A* $\equiv \{ (a, b) \in A \times A.\ P\ a\ b \}$

lemma *Field-relation-of*:
assumes *refl-on A* (*relation-of P A*) **shows** *Field* (*relation-of P A*) $= A$
⟨*proof*⟩

lemma *partial-order-on-relation-ofI*:
assumes *refl*: $\bigwedge a.\ a \in A \Longrightarrow P\ a\ a$
and *trans*: $\bigwedge a\ b\ c.\ [\![a \in A;\ b \in A;\ c \in A]\!] \Longrightarrow P\ a\ b \Longrightarrow P\ b\ c \Longrightarrow P\ a\ c$
and *antisym*: $\bigwedge a\ b.\ [\![a \in A;\ b \in A]\!] \Longrightarrow P\ a\ b \Longrightarrow P\ b\ a \Longrightarrow a = b$
shows *partial-order-on A* (*relation-of P A*)
⟨*proof*⟩

lemma *Partial-order-relation-ofI*:
assumes *partial-order-on A* (*relation-of P A*) **shows** *Partial-order* (*relation-of P A*)
⟨*proof*⟩

24.4 Orders on a type

abbreviation *strict-linear-order* ≡ *strict-linear-order-on UNIV*

abbreviation *linear-order* ≡ *linear-order-on UNIV*

abbreviation *well-order* ≡ *well-order-on UNIV*

24.5 Order-like relations

In this subsection, we develop basic concepts and results pertaining to order-like relations, i.e., to reflexive and/or transitive and/or symmetric and/or total relations. We also further define upper and lower bounds operators.

24.5.1 Auxiliaries

lemma *refl-on-domain*: *refl-on A r* $\Longrightarrow (a, b) \in r \Longrightarrow a \in A \wedge b \in A$
⟨*proof*⟩

corollary *well-order-on-domain*: *well-order-on A r* $\Longrightarrow (a, b) \in r \Longrightarrow a \in A \wedge b \in A$
⟨*proof*⟩

lemma *well-order-on-Field*: *well-order-on A r* $\Longrightarrow A =$ *Field r*
⟨*proof*⟩

lemma *well-order-on-Well-order*: *well-order-on A r* $\Longrightarrow A =$ *Field r* $\wedge$ *Well-order r*

⟨*proof*⟩

lemma *Total-subset-Id*:
assumes *Total r*
and $r \subseteq Id$
shows $r = \{\} \vee (\exists\, a.\ r = \{(a,\ a)\})$
⟨*proof*⟩

lemma *Linear-order-in-diff-Id*:
assumes *Linear-order r*
and $a \in Field\ r$
and $b \in Field\ r$
shows $(a,\ b) \in r \longleftrightarrow (b,\ a) \notin r - Id$
⟨*proof*⟩

24.5.2 The upper and lower bounds operators

Here we define upper ("above") and lower ("below") bounds operators. We think of r as a *non-strict* relation. The suffix S at the names of some operators indicates that the bounds are strict – e.g., *underS a* is the set of all strict lower bounds of a (w.r.t. r). Capitalization of the first letter in the name reminds that the operator acts on sets, rather than on individual elements.

definition *under* :: *'a rel* ⇒ *'a* ⇒ *'a set*
where $under\ r\ a \equiv \{b.\ (b,\ a) \in r\}$

definition *underS* :: *'a rel* ⇒ *'a* ⇒ *'a set*
where $underS\ r\ a \equiv \{b.\ b \neq a \wedge (b,\ a) \in r\}$

definition *Under* :: *'a rel* ⇒ *'a set* ⇒ *'a set*
where $Under\ r\ A \equiv \{b \in Field\ r.\ \forall\, a \in A.\ (b,\ a) \in r\}$

definition *UnderS* :: *'a rel* ⇒ *'a set* ⇒ *'a set*
where $UnderS\ r\ A \equiv \{b \in Field\ r.\ \forall\, a \in A.\ b \neq a \wedge (b,\ a) \in r\}$

definition *above* :: *'a rel* ⇒ *'a* ⇒ *'a set*
where $above\ r\ a \equiv \{b.\ (a,\ b) \in r\}$

definition *aboveS* :: *'a rel* ⇒ *'a* ⇒ *'a set*
where $aboveS\ r\ a \equiv \{b.\ b \neq a \wedge (a,\ b) \in r\}$

definition *Above* :: *'a rel* ⇒ *'a set* ⇒ *'a set*
where $Above\ r\ A \equiv \{b \in Field\ r.\ \forall\, a \in A.\ (a,\ b) \in r\}$

definition *AboveS* :: *'a rel* ⇒ *'a set* ⇒ *'a set*
where $AboveS\ r\ A \equiv \{b \in Field\ r.\ \forall\, a \in A.\ b \neq a \wedge (a,\ b) \in r\}$

definition *ofilter* :: *'a rel* ⇒ *'a set* ⇒ *bool*

where *ofilter r A* ≡ *A* ⊆ *Field r* ∧ (∀ *a* ∈ *A*. *under r a* ⊆ *A*)

Note: In the definitions of *Above*[*S*] and *Under*[*S*], we bounded comprehension by *Field r* in order to properly cover the case of *A* being empty.

lemma *underS-subset-under*: *underS r a* ⊆ *under r a*
⟨*proof*⟩

lemma *underS-notIn*: *a* ∉ *underS r a*
⟨*proof*⟩

lemma *Refl-under-in*: *Refl r* ⟹ *a* ∈ *Field r* ⟹ *a* ∈ *under r a*
⟨*proof*⟩

lemma *AboveS-disjoint*: *A* ∩ (*AboveS r A*) = {}
⟨*proof*⟩

lemma *in-AboveS-underS*: *a* ∈ *Field r* ⟹ *a* ∈ *AboveS r* (*underS r a*)
⟨*proof*⟩

lemma *Refl-under-underS*: *Refl r* ⟹ *a* ∈ *Field r* ⟹ *under r a* = *underS r a* ∪ {*a*}
⟨*proof*⟩

lemma *underS-empty*: *a* ∉ *Field r* ⟹ *underS r a* = {}
⟨*proof*⟩

lemma *under-Field*: *under r a* ⊆ *Field r*
⟨*proof*⟩

lemma *underS-Field*: *underS r a* ⊆ *Field r*
⟨*proof*⟩

lemma *underS-Field2*: *a* ∈ *Field r* ⟹ *underS r a* ⊂ *Field r*
⟨*proof*⟩

lemma *underS-Field3*: *Field r* ≠ {} ⟹ *underS r a* ⊂ *Field r*
⟨*proof*⟩

lemma *AboveS-Field*: *AboveS r A* ⊆ *Field r*
⟨*proof*⟩

lemma *under-incr*:
assumes *trans r*
and (*a*, *b*) ∈ *r*
shows *under r a* ⊆ *under r b*
⟨*proof*⟩

lemma *underS-incr*:
assumes *trans r*

and *antisym r*
and *ab*: $(a, b) \in r$
shows *underS r a* $\subseteq$ *underS r b*
⟨*proof*⟩

lemma *underS-incl-iff*:
assumes *LO*: *Linear-order r*
and *INa*: $a \in$ *Field r*
and *INb*: $b \in$ *Field r*
shows *underS r a* $\subseteq$ *underS r b* $\longleftrightarrow$ $(a, b) \in r$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *finite-Partial-order-induct*[*consumes 3*, *case-names step*]:
assumes *Partial-order r*
and $x \in$ *Field r*
and *finite r*
and *step*: $\bigwedge x.\ x \in$ *Field r* $\Longrightarrow$ $(\bigwedge y.\ y \in$ *aboveS r x* $\Longrightarrow$ *P y*$) \Longrightarrow$ *P x*
shows *P x*
⟨*proof*⟩

lemma *finite-Linear-order-induct*[*consumes 3*, *case-names step*]:
assumes *Linear-order r*
and $x \in$ *Field r*
and *finite r*
and *step*: $\bigwedge x.\ x \in$ *Field r* $\Longrightarrow$ $(\bigwedge y.\ y \in$ *aboveS r x* $\Longrightarrow$ *P y*$) \Longrightarrow$ *P x*
shows *P x*
⟨*proof*⟩

24.6 Variations on Well-Founded Relations

This subsection contains some variations of the results from *HOL.Wellfounded*:

- means for slightly more direct definitions by well-founded recursion;
- variations of well-founded induction;
- means for proving a linear order to be a well-order.

24.6.1 Characterizations of well-foundedness

A transitive relation is well-founded iff it is "locally" well-founded, i.e., iff its restriction to the lower bounds of of any element is well-founded.

lemma *trans-wf-iff*:
assumes *trans r*
shows *wf r* $\longleftrightarrow$ $(\forall a.$ *wf* $(r \cap (r^{-1}\text{``}\{a\} \times r^{-1}\text{``}\{a\})))$
⟨*proof*⟩

A transitive relation is well-founded if all initial segments are finite.

corollary *wf-finite-segments*:
  assumes *irrefl r* **and** *trans r* **and** $\bigwedge x.$ *finite* $\{y.\ (y,\ x) \in r\}$
  shows *wf r*
⟨*proof*⟩

The next lemma is a variation of *wf-eq-minimal* from Wellfounded, allowing one to assume the set included in the field.

lemma *wf-eq-minimal2*: *wf r* $\longleftrightarrow$ $(\forall A.\ A \subseteq$ *Field r* $\wedge\ A \neq \{\} \longrightarrow (\exists a \in A.\ \forall a' \in A.\ (a',\ a) \notin r))$
⟨*proof*⟩

24.6.2 Characterizations of well-foundedness

The next lemma and its corollary enable one to prove that a linear order is a well-order in a way which is more standard than via well-foundedness of the strict version of the relation.

lemma *Linear-order-wf-diff-Id*:
  assumes *Linear-order r*
  shows *wf* $(r - Id) \longleftrightarrow (\forall A \subseteq$ *Field r*. $A \neq \{\} \longrightarrow (\exists a \in A.\ \forall a' \in A.\ (a,\ a') \in r))$
⟨*proof*⟩

corollary *Linear-order-Well-order-iff*:
  Linear-order r $\Longrightarrow$
    Well-order r $\longleftrightarrow (\forall A \subseteq$ *Field r*. $A \neq \{\} \longrightarrow (\exists a \in A.\ \forall a' \in A.\ (a,\ a') \in r))$
  ⟨*proof*⟩

end

25 Hilbert's Epsilon-Operator and the Axiom of Choice

theory *Hilbert-Choice*
  imports *Wellfounded*
  keywords *specification* :: *thy-goal-defn*
begin

25.1 Hilbert's epsilon

axiomatization *Eps* :: $('a \Rightarrow bool) \Rightarrow 'a$
  where *someI*: $P\ x \Longrightarrow P\ (Eps\ P)$

syntax (*epsilon*)
  -Eps :: *pttrn* $\Rightarrow$ *bool* $\Rightarrow$ $'a$ ((3ϵ -./ -) [0, 10] 10)
syntax (*input*)
  -Eps :: *pttrn* $\Rightarrow$ *bool* $\Rightarrow$ $'a$ (($3@$ -./ -) [0, 10] 10)

syntax
-Eps :: *pttrn* ⇒ *bool* ⇒ *'a* ((*3SOME -./ -*) [*0*, *10*] *10*)
translations
SOME x. P ⇌ *CONST Eps* (λ*x. P*)

⟨*ML*⟩

definition *inv-into* :: *'a set* ⇒ (*'a* ⇒ *'b*) ⇒ (*'b* ⇒ *'a*) **where**
inv-into A f = (λ*x. SOME y. y* ∈ *A* ∧ *f y* = *x*)

lemma *inv-into-def2*: *inv-into A f x* = (*SOME y. y* ∈ *A* ∧ *f y* = *x*)
⟨*proof*⟩

abbreviation *inv* :: (*'a* ⇒ *'b*) ⇒ (*'b* ⇒ *'a*) **where**
inv ≡ *inv-into UNIV*

25.2 Hilbert's Epsilon-operator

lemma *Eps-cong*:
assumes ⋀*x. P x* = *Q x*
shows *Eps P* = *Eps Q*
⟨*proof*⟩

Easier to use than *someI* if the witness comes from an existential formula.

lemma *someI-ex* [*elim?*]: ∃ *x. P x* ⟹ *P* (*SOME x. P x*)
⟨*proof*⟩

lemma *some-eq-imp*:
assumes *Eps P* = *a P b* **shows** *P a*
⟨*proof*⟩

Easier to use than *someI* because the conclusion has only one occurrence of *P*.

lemma *someI2*: *P a* ⟹ (⋀*x. P x* ⟹ *Q x*) ⟹ *Q* (*SOME x. P x*)
⟨*proof*⟩

Easier to use than *someI2* if the witness comes from an existential formula.

lemma *someI2-ex*: ∃ *a. P a* ⟹ (⋀*x. P x* ⟹ *Q x*) ⟹ *Q* (*SOME x. P x*)
⟨*proof*⟩

lemma *someI2-bex*: ∃ *a*∈*A. P a* ⟹ (⋀*x. x* ∈ *A* ∧ *P x* ⟹ *Q x*) ⟹ *Q* (*SOME x. x* ∈ *A* ∧ *P x*)
⟨*proof*⟩

lemma *some-equality* [*intro*]: *P a* ⟹ (⋀*x. P x* ⟹ *x* = *a*) ⟹ (*SOME x. P x*) = *a*
⟨*proof*⟩

lemma *some1-equality*: ∃!*x. P x* ⟹ *P a* ⟹ (*SOME x. P x*) = *a*

⟨*proof*⟩

lemma *some-eq-ex*: $P\ (SOME\ x.\ P\ x) \longleftrightarrow (\exists\, x.\ P\ x)$
⟨*proof*⟩

lemma *some-in-eq*: $(SOME\ x.\ x \in A) \in A \longleftrightarrow A \neq \{\}$
⟨*proof*⟩

lemma *some-eq-trivial* [*simp*]: $(SOME\ y.\ y = x) = x$
⟨*proof*⟩

lemma *some-sym-eq-trivial* [*simp*]: $(SOME\ y.\ x = y) = x$
⟨*proof*⟩

25.3 Axiom of Choice, Proved Using the Description Operator

lemma *choice*: $\forall\, x.\ \exists\, y.\ Q\ x\ y \Longrightarrow \exists f.\ \forall\, x.\ Q\ x\ (f\ x)$
⟨*proof*⟩

lemma *bchoice*: $\forall\, x{\in}S.\ \exists\, y.\ Q\ x\ y \Longrightarrow \exists f.\ \forall\, x{\in}S.\ Q\ x\ (f\ x)$
⟨*proof*⟩

lemma *choice-iff*: $(\forall\, x.\ \exists\, y.\ Q\ x\ y) \longleftrightarrow (\exists f.\ \forall\, x.\ Q\ x\ (f\ x))$
⟨*proof*⟩

lemma *choice-iff'*: $(\forall\, x.\ P\ x \longrightarrow (\exists\, y.\ Q\ x\ y)) \longleftrightarrow (\exists f.\ \forall\, x.\ P\ x \longrightarrow Q\ x\ (f\ x))$
⟨*proof*⟩

lemma *bchoice-iff*: $(\forall\, x{\in}S.\ \exists\, y.\ Q\ x\ y) \longleftrightarrow (\exists f.\ \forall\, x{\in}S.\ Q\ x\ (f\ x))$
⟨*proof*⟩

lemma *bchoice-iff'*: $(\forall\, x{\in}S.\ P\ x \longrightarrow (\exists\, y.\ Q\ x\ y)) \longleftrightarrow (\exists f.\ \forall\, x{\in}S.\ P\ x \longrightarrow Q\ x\ (f\ x))$
⟨*proof*⟩

lemma *dependent-nat-choice*:
assumes *1*: $\exists\, x.\ P\ 0\ x$
and *2*: $\bigwedge x\ n.\ P\ n\ x \Longrightarrow \exists\, y.\ P\ (Suc\ n)\ y \wedge Q\ n\ x\ y$
shows $\exists f.\ \forall\, n.\ P\ n\ (f\ n) \wedge Q\ n\ (f\ n)\ (f\ (Suc\ n))$
⟨*proof*⟩

lemma *finite-subset-Union*:
assumes *finite* A $A \subseteq \bigcup \mathcal{B}$
obtains $\mathcal{F}$ **where** *finite* $\mathcal{F}$ $\mathcal{F} \subseteq \mathcal{B}$ $A \subseteq \bigcup \mathcal{F}$
⟨*proof*⟩

25.4 Function Inverse

lemma *inv-def*: $inv\ f = (\lambda y.\ SOME\ x.\ f\ x = y)$
⟨*proof*⟩

lemma *inv-into-into*: $x \in f\ `\ A \Longrightarrow inv\text{-}into\ A\ f\ x \in A$
⟨*proof*⟩

lemma *inv-identity* [*simp*]: $inv\ (\lambda a.\ a) = (\lambda a.\ a)$
⟨*proof*⟩

lemma *inv-id* [*simp*]: $inv\ id = id$
⟨*proof*⟩

lemma *inv-into-f-f* [*simp*]: $inj\text{-}on\ f\ A \Longrightarrow x \in A \Longrightarrow inv\text{-}into\ A\ f\ (f\ x) = x$
⟨*proof*⟩

lemma *inv-f-f*: $inj\ f \Longrightarrow inv\ f\ (f\ x) = x$
⟨*proof*⟩

lemma *f-inv-into-f*: $y \in f`A \Longrightarrow f\ (inv\text{-}into\ A\ f\ y) = y$
⟨*proof*⟩

lemma *inv-into-f-eq*: $inj\text{-}on\ f\ A \Longrightarrow x \in A \Longrightarrow f\ x = y \Longrightarrow inv\text{-}into\ A\ f\ y = x$
⟨*proof*⟩

lemma *inv-f-eq*: $inj\ f \Longrightarrow f\ x = y \Longrightarrow inv\ f\ y = x$
⟨*proof*⟩

lemma *inj-imp-inv-eq*: $inj\ f \Longrightarrow \forall x.\ f\ (g\ x) = x \Longrightarrow inv\ f = g$
⟨*proof*⟩

But is it useful?

lemma *inj-transfer*:
assumes *inj*: $inj\ f$
and *minor*: $\bigwedge y.\ y \in range\ f \Longrightarrow P\ (inv\ f\ y)$
shows $P\ x$
⟨*proof*⟩

lemma *inj-iff*: $inj\ f \longleftrightarrow inv\ f \circ f = id$
⟨*proof*⟩

lemma *inv-o-cancel*[*simp*]: $inj\ f \Longrightarrow inv\ f \circ f = id$
⟨*proof*⟩

lemma *o-inv-o-cancel*[*simp*]: $inj\ f \Longrightarrow g \circ inv\ f \circ f = g$
⟨*proof*⟩

lemma *inv-into-image-cancel*[*simp*]: $inj\text{-}on\ f\ A \Longrightarrow S \subseteq A \Longrightarrow inv\text{-}into\ A\ f\ `\ f\ `\ S = S$

⟨*proof*⟩

lemma *inj-imp-surj-inv*: *inj f* ⟹ *surj* (*inv f*)
⟨*proof*⟩

lemma *surj-f-inv-f*: *surj f* ⟹ *f* (*inv f y*) = *y*
⟨*proof*⟩

lemma *bij-inv-eq-iff*: *bij p* ⟹ *x* = *inv p y* ⟷ *p x* = *y*
⟨*proof*⟩

lemma *inv-into-injective*:
assumes *eq*: *inv-into A f x* = *inv-into A f y*
and *x*: *x* ∈ *f*‘*A*
and *y*: *y* ∈ *f*‘*A*
shows *x* = *y*
⟨*proof*⟩

lemma *inj-on-inv-into*: *B* ⊆ *f*‘*A* ⟹ *inj-on* (*inv-into A f*) *B*
⟨*proof*⟩

lemma *inj-imp-bij-betw-inv*: *inj f* ⟹ *bij-betw* (*inv f*) (*f* ‘ *M*) *M*
⟨*proof*⟩

lemma *bij-betw-inv-into*: *bij-betw f A B* ⟹ *bij-betw* (*inv-into A f*) *B A*
⟨*proof*⟩

lemma *surj-imp-inj-inv*: *surj f* ⟹ *inj* (*inv f*)
⟨*proof*⟩

lemma *surj-iff*: *surj f* ⟷ *f* ∘ *inv f* = *id*
⟨*proof*⟩

lemma *surj-iff-all*: *surj f* ⟷ (∀ *x*. *f* (*inv f x*) = *x*)
⟨*proof*⟩

lemma *surj-imp-inv-eq*:
assumes *surj f* **and** *gf*: ⋀*x*. *g* (*f x*) = *x*
shows *inv f* = *g*
⟨*proof*⟩

lemma *bij-imp-bij-inv*: *bij f* ⟹ *bij* (*inv f*)
⟨*proof*⟩

lemma *inv-equality*: (⋀*x*. *g* (*f x*) = *x*) ⟹ (⋀*y*. *f* (*g y*) = *y*) ⟹ *inv f* = *g*
⟨*proof*⟩

lemma *inv-inv-eq*: *bij f* ⟹ *inv* (*inv f*) = *f*
⟨*proof*⟩

bij (*inv f*) implies little about *f*. Consider $f :: bool \Rightarrow bool$ such that $f\ True = f\ False = True$. Then it ia consistent with axiom *someI* that *inv f* could be any function at all, including the identity function. If $inv\ f = id$ then *inv f* is a bijection, but *inj f*, *surj f* and $inv\ (inv\ f) = f$ all fail.

lemma *inv-into-comp*:
inj-on f (*g* ‘ *A*) ⟹ *inj-on g A* ⟹ *x* ∈ *f* ‘ *g* ‘ *A* ⟹
inv-into A (*f* ∘ *g*) *x* = (*inv-into A g* ∘ *inv-into* (*g* ‘ *A*) *f*) *x*
⟨*proof*⟩

lemma *o-inv-distrib*: *bij f* ⟹ *bij g* ⟹ *inv* (*f* ∘ *g*) = *inv g* ∘ *inv f*
⟨*proof*⟩

lemma *image-f-inv-f*: *surj f* ⟹ *f* ‘ (*inv f* ‘ *A*) = *A*
⟨*proof*⟩

lemma *image-inv-f-f*: *inj f* ⟹ *inv f* ‘ (*f* ‘ *A*) = *A*
⟨*proof*⟩

lemma *bij-image-Collect-eq*:
assumes *bij f*
shows *f* ‘ *Collect P* = {*y*. *P* (*inv f y*)}
⟨*proof*⟩

lemma *bij-vimage-eq-inv-image*:
assumes *bij f*
shows *f* −‘ *A* = *inv f* ‘ *A*
⟨*proof*⟩

lemma *inv-fn-o-fn-is-id*:
fixes *f*::′*a* ⇒ ′*a*
assumes *bij f*
shows ((*inv f*) ^^ *n*) *o* (*f* ^^ *n*) = (λ*x*. *x*)
⟨*proof*⟩

lemma *fn-o-inv-fn-is-id*:
fixes *f*::′*a* ⇒ ′*a*
assumes *bij f*
shows (*f* ^^ *n*) *o* ((*inv f*) ^^ *n*) = (λ*x*. *x*)
⟨*proof*⟩

lemma *inv-fn*:
fixes *f*::′*a* ⇒ ′*a*
assumes *bij f*
shows *inv* (*f* ^^ *n*) = ((*inv f*) ^^ *n*)
⟨*proof*⟩

lemma *funpow-inj-finite*:
assumes ‹*inj p*› ‹*finite* {*y*. ∃ *n*. *y* = (*p* ^^ *n*) *x*}›
obtains *n* **where** ‹*n* > *0*› ‹(*p* ^^ *n*) *x* = *x*›

⟨*proof*⟩

lemma *mono-inv*:
fixes *f*::*'a*::*linorder* ⇒ *'b*::*linorder*
assumes *mono f bij f*
shows *mono* (*inv f*)
⟨*proof*⟩

lemma *strict-mono-inv-on-range*:
fixes *f* :: *'a*::*linorder* ⇒ *'b*::*order*
assumes *strict-mono f*
shows *strict-mono-on* (*range f*) (*inv f*)
⟨*proof*⟩

lemma *mono-bij-Inf*:
fixes *f* :: *'a*::*complete-linorder* ⇒ *'b*::*complete-linorder*
assumes *mono f bij f*
shows *f* (*Inf A*) = *Inf* (*f*‘*A*)
⟨*proof*⟩

lemma *finite-fun-UNIVD1*:
assumes *fin*: *finite* (*UNIV* :: (*'a* ⇒ *'b*) *set*)
and *card*: *card* (*UNIV* :: *'b set*) ≠ *Suc 0*
shows *finite* (*UNIV* :: *'a set*)
⟨*proof*⟩

Every infinite set contains a countable subset. More precisely we show that a set S is infinite if and only if there exists an injective function from the naturals into S.

The "only if" direction is harder because it requires the construction of a sequence of pairwise different elements of an infinite set S. The idea is to construct a sequence of non-empty and infinite subsets of S obtained by successively removing elements of S.

lemma *infinite-countable-subset*:
assumes *inf*: ¬ *finite S*
shows ∃*f*::*nat* ⇒ *'a*. *inj f* ∧ *range f* ⊆ *S*
— Courtesy of Stephan Merz
⟨*proof*⟩

lemma *infinite-iff-countable-subset*: ¬ *finite S* ⟷ (∃*f*::*nat* ⇒ *'a*. *inj f* ∧ *range f* ⊆ *S*)
— Courtesy of Stephan Merz
⟨*proof*⟩

lemma *image-inv-into-cancel*:
assumes *surj*: *f*‘*A* = *A'*
and *sub*: *B'* ⊆ *A'*

shows f ‘$((\mathit{inv\text{-}into}\ A\ f)$‘$B') = B'$
⟨*proof*⟩

lemma *inv-into-inv-into-eq*:
assumes *bij-betw* $f\ A\ A'$
and a: $a \in A$
shows *inv-into* A' (*inv-into* $A\ f$) $a = f\ a$
⟨*proof*⟩

lemma *inj-on-iff-surj*:
assumes $A \neq \{\}$
shows $(\exists f.\ \mathit{inj\text{-}on}\ f\ A \wedge f$ ‘ $A \subseteq A') \longleftrightarrow (\exists g.\ g$ ‘ $A' = A)$
⟨*proof*⟩

lemma *Ex-inj-on-UNION-Sigma*:
$\exists f.\ (\mathit{inj\text{-}on}\ f\ (\bigcup i \in I.\ A\ i) \wedge f$ ‘ $(\bigcup i \in I.\ A\ i) \subseteq (\mathit{SIGMA}\ i : I.\ A\ i))$
⟨*proof*⟩

lemma *inv-unique-comp*:
assumes *fg*: $f \circ g = id$
and *gf*: $g \circ f = id$
shows *inv* $f = g$
⟨*proof*⟩

lemma *subset-image-inj*:
$S \subseteq f$ ‘ $T \longleftrightarrow (\exists U.\ U \subseteq T \wedge \mathit{inj\text{-}on}\ f\ U \wedge S = f$ ‘ $U)$
⟨*proof*⟩

25.5 Other Consequences of Hilbert's Epsilon

Hilbert's Epsilon and the *split* Operator

Looping simprule!

lemma *split-paired-Eps*: $(\mathit{SOME}\ x.\ P\ x) = (\mathit{SOME}\ (a,\ b).\ P\ (a,\ b))$
⟨*proof*⟩

lemma *Eps-case-prod*: *Eps* (*case-prod* P) = (*SOME xy*. P (*fst xy*) (*snd xy*))
⟨*proof*⟩

lemma *Eps-case-prod-eq* [*simp*]: $(\mathit{SOME}\ (x',\ y').\ x = x' \wedge y = y') = (x,\ y)$
⟨*proof*⟩

A relation is wellfounded iff it has no infinite descending chain.

lemma *wf-iff-no-infinite-down-chain*: *wf* $r \longleftrightarrow (\nexists f.\ \forall i.\ (f\ (\mathit{Suc}\ i),\ f\ i) \in r)$
(**is** - $\longleftrightarrow \neg$ *?ex*)
⟨*proof*⟩

lemma *wf-no-infinite-down-chainE*:
assumes *wf* r

obtains k **where** $(f\ (Suc\ k), f\ k) \notin r$
⟨*proof*⟩

A dynamically-scoped fact for TFL

lemma *tfl-some*: $\forall P\ x.\ P\ x \longrightarrow P\ (Eps\ P)$
⟨*proof*⟩

25.6 An aside: bounded accessible part

Finite monotone eventually stable sequences

lemma *finite-mono-remains-stable-implies-strict-prefix*:
fixes $f :: nat \Rightarrow 'a::order$
assumes S: *finite* (*range* f) *mono* f
and *eq*: $\forall n.\ f\ n = f\ (Suc\ n) \longrightarrow f\ (Suc\ n) = f\ (Suc\ (Suc\ n))$
shows $\exists N.\ (\forall n{\leq}N.\ \forall m{\leq}N.\ m < n \longrightarrow f\ m < f\ n) \wedge (\forall n{\geq}N.\ f\ N = f\ n)$
⟨*proof*⟩

lemma *finite-mono-strict-prefix-implies-finite-fixpoint*:
fixes $f :: nat \Rightarrow 'a\ set$
assumes S: $\bigwedge i.\ f\ i \subseteq S$ *finite* S
and *ex*: $\exists N.\ (\forall n{\leq}N.\ \forall m{\leq}N.\ m < n \longrightarrow f\ m \subset f\ n) \wedge (\forall n{\geq}N.\ f\ N = f\ n)$
shows $f\ (card\ S) = (\bigcup n.\ f\ n)$
⟨*proof*⟩

25.7 More on injections, bijections, and inverses

locale *bijection* =
fixes $f :: 'a \Rightarrow 'a$
assumes *bij*: *bij* f
begin

lemma *bij-inv*: *bij* (*inv* f)
⟨*proof*⟩

lemma *surj* [*simp*]: *surj* f
⟨*proof*⟩

lemma *inj*: *inj* f
⟨*proof*⟩

lemma *surj-inv* [*simp*]: *surj* (*inv* f)
⟨*proof*⟩

lemma *inj-inv*: *inj* (*inv* f)
⟨*proof*⟩

lemma *eqI*: $f\ a = f\ b \Longrightarrow a = b$
⟨*proof*⟩

lemma *eq-iff* [*simp*]: $f\ a = f\ b \longleftrightarrow a = b$
⟨*proof*⟩

lemma *eq-invI*: $inv\ f\ a = inv\ f\ b \Longrightarrow a = b$
⟨*proof*⟩

lemma *eq-inv-iff* [*simp*]: $inv\ f\ a = inv\ f\ b \longleftrightarrow a = b$
⟨*proof*⟩

lemma *inv-left* [*simp*]: $inv\ f\ (f\ a) = a$
⟨*proof*⟩

lemma *inv-comp-left* [*simp*]: $inv\ f \circ f = id$
⟨*proof*⟩

lemma *inv-right* [*simp*]: $f\ (inv\ f\ a) = a$
⟨*proof*⟩

lemma *inv-comp-right* [*simp*]: $f \circ inv\ f = id$
⟨*proof*⟩

lemma *inv-left-eq-iff* [*simp*]: $inv\ f\ a = b \longleftrightarrow f\ b = a$
⟨*proof*⟩

lemma *inv-right-eq-iff* [*simp*]: $b = inv\ f\ a \longleftrightarrow f\ b = a$
⟨*proof*⟩

end

lemma *infinite-imp-bij-betw*:
assumes *infinite*: $\neg$ *finite* A
shows $\exists h.\ bij\text{-}betw\ h\ A\ (A - \{a\})$
⟨*proof*⟩

lemma *infinite-imp-bij-betw2*:
assumes $\neg$ *finite* A
shows $\exists h.\ bij\text{-}betw\ h\ A\ (A \cup \{a\})$
⟨*proof*⟩

lemma *bij-betw-inv-into-left*: $bij\text{-}betw\ f\ A\ A' \Longrightarrow a \in A \Longrightarrow inv\text{-}into\ A\ f\ (f\ a) = a$
⟨*proof*⟩

lemma *bij-betw-inv-into-right*: $bij\text{-}betw\ f\ A\ A' \Longrightarrow a' \in A' \Longrightarrow f\ (inv\text{-}into\ A\ f\ a') = a'$
⟨*proof*⟩

lemma *bij-betw-inv-into-subset*:
$bij\text{-}betw\ f\ A\ A' \Longrightarrow B \subseteq A \Longrightarrow f\ `\ B = B' \Longrightarrow bij\text{-}betw\ (inv\text{-}into\ A\ f)\ B'\ B$
⟨*proof*⟩

25.8 Specification package – Hilbertized version

lemma *exE-some*: $Ex\ P \Longrightarrow c \equiv Eps\ P \Longrightarrow P\ c$
⟨*proof*⟩

⟨*ML*⟩

25.9 Complete Distributive Lattices – Properties depending on Hilbert Choice

context *complete-distrib-lattice*
begin

lemma *Sup-Inf*: $\bigsqcup\ (Inf\ `\ A) = \bigsqcap\ (Sup\ `\ \{f\ `\ A\ |f.\ \forall B{\in}A.\ f\ B\ \in\ B\})$
⟨*proof*⟩

lemma *dual-complete-distrib-lattice*:
class.complete-distrib-lattice Sup Inf sup $(\geq)$ $(>)$ *inf* $\top$ $\bot$
⟨*proof*⟩

lemma *sup-Inf*: $a \sqcup \bigsqcap B = \bigsqcap((\sqcup)\ a\ `\ B)$
⟨*proof*⟩

lemma *inf-Sup*: $a \sqcap \bigsqcup B = \bigsqcup((\sqcap)\ a\ `\ B)$
⟨*proof*⟩

lemma *INF-SUP*: $(\bigsqcap y.\ \bigsqcup x.\ P\ x\ y) = (\bigsqcup f.\ \bigsqcap x.\ P\ (f\ x)\ x)$
⟨*proof*⟩

lemma *INF-SUP-set*: $(\bigsqcap B{\in}A.\ \bigsqcup(g\ `\ B)) = (\bigsqcup B{\in}\{f\ `\ A\ |f.\ \forall C{\in}A.\ f\ C\ \in\ C\}.\ \bigsqcap(g\ `\ B))$
(**is** - = $(\bigsqcup B{\in}?F.$ -))
⟨*proof*⟩

lemma *SUP-INF*: $(\bigsqcup y.\ \bigsqcap x.\ P\ x\ y) = (\bigsqcap x.\ \bigsqcup y.\ P\ (x\ y)\ y)$
⟨*proof*⟩

lemma *SUP-INF-set*: $(\bigsqcup x{\in}A.\ \bigsqcap\ (g\ `\ x)) = (\bigsqcap x{\in}\{f\ `\ A\ |f.\ \forall Y{\in}A.\ f\ Y\ \in\ Y\}.\ \bigsqcup\ (g\ `\ x))$
⟨*proof*⟩

end

context *complete-distrib-lattice*
begin

lemma *sup-INF*: $a \sqcup (\bigsqcap b{\in}B.\ f\ b) = (\bigsqcap b{\in}B.\ a \sqcup f\ b)$
⟨*proof*⟩

lemma *inf-SUP*: $a \sqcap (\bigsqcup b{\in}B.\ f\ b) = (\bigsqcup b{\in}B.\ a \sqcap f\ b)$
⟨*proof*⟩

lemma *Inf-sup*: $\bigsqcap B \sqcup a = (\bigsqcap b{\in}B.\ b \sqcup a)$
⟨*proof*⟩

lemma *Sup-inf*: $\bigsqcup B \sqcap a = (\bigsqcup b{\in}B.\ b \sqcap a)$
⟨*proof*⟩

lemma *INF-sup*: $(\bigsqcap b{\in}B.\ f\ b) \sqcup a = (\bigsqcap b{\in}B.\ f\ b \sqcup a)$
⟨*proof*⟩

lemma *SUP-inf*: $(\bigsqcup b{\in}B.\ f\ b) \sqcap a = (\bigsqcup b{\in}B.\ f\ b \sqcap a)$
⟨*proof*⟩

lemma *Inf-sup-eq-top-iff*: $(\bigsqcap B \sqcup a = \top) \longleftrightarrow (\forall\, b{\in}B.\ b \sqcup a = \top)$
⟨*proof*⟩

lemma *Sup-inf-eq-bot-iff*: $(\bigsqcup B \sqcap a = \bot) \longleftrightarrow (\forall\, b{\in}B.\ b \sqcap a = \bot)$
⟨*proof*⟩

lemma *INF-sup-distrib2*: $(\bigsqcap a{\in}A.\ f\ a) \sqcup (\bigsqcap b{\in}B.\ g\ b) = (\bigsqcap a{\in}A.\ \bigsqcap b{\in}B.\ f\ a \sqcup g\ b)$
⟨*proof*⟩

lemma *SUP-inf-distrib2*: $(\bigsqcup a{\in}A.\ f\ a) \sqcap (\bigsqcup b{\in}B.\ g\ b) = (\bigsqcup a{\in}A.\ \bigsqcup b{\in}B.\ f\ a \sqcap g\ b)$
⟨*proof*⟩

end

instantiation *set* :: (*type*) *complete-distrib-lattice*
begin
instance ⟨*proof*⟩
end

instance *set* :: (*type*) *complete-boolean-algebra* ⟨*proof*⟩

instantiation *fun* :: (*type*, *complete-distrib-lattice*) *complete-distrib-lattice*
begin
instance ⟨*proof*⟩
end

instance *fun* :: (*type*, *complete-boolean-algebra*) *complete-boolean-algebra* ⟨*proof*⟩

context *complete-linorder*
begin

subclass *complete-distrib-lattice*

⟨*proof*⟩
end

end

26 Zorn's Lemma and the Well-ordering Theorem

theory *Zorn*
imports *Order-Relation Hilbert-Choice*
begin

26.1 Zorn's Lemma for the Subset Relation

26.1.1 Results that do not require an order

Let P be a binary predicate on the set A.

locale *pred-on* =
fixes A :: $'a$ *set*
and P :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infix** $\sqsubset$ *50*)
begin

abbreviation *Peq* :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infix** $\sqsubseteq$ *50*)
where $x \sqsubseteq y \equiv P^{==}\ x\ y$

A chain is a totally ordered subset of A.

definition *chain* :: $'a$ *set* $\Rightarrow$ *bool*
where *chain* $C \longleftrightarrow C \subseteq A \wedge (\forall x{\in}C.\ \forall y{\in}C.\ x \sqsubseteq y \vee y \sqsubseteq x)$

We call a chain that is a proper superset of some set X, but not necessarily a chain itself, a superchain of X.

abbreviation *superchain* :: $'a$ *set* $\Rightarrow$ $'a$ *set* $\Rightarrow$ *bool* (**infix** $<c$ *50*)
where $X <c\ C \equiv$ *chain* $C \wedge X \subset C$

A maximal chain is a chain that does not have a superchain.

definition *maxchain* :: $'a$ *set* $\Rightarrow$ *bool*
where *maxchain* $C \longleftrightarrow$ *chain* $C \wedge (\nexists S.\ C <c S)$

We define the successor of a set to be an arbitrary superchain, if such exists, or the set itself, otherwise.

definition *suc* :: $'a$ *set* $\Rightarrow$ $'a$ *set*
where *suc* $C =$ (*if* $\neg$ *chain* $C \vee$ *maxchain* C *then* C *else* (*SOME D.* $C <c D$))

lemma *chainI* [*Pure.intro?*]: $C \subseteq A \Longrightarrow (\bigwedge x\ y.\ x \in C \Longrightarrow y \in C \Longrightarrow x \sqsubseteq y \vee y \sqsubseteq x) \Longrightarrow$ *chain* C
⟨*proof*⟩

lemma *chain-total*: *chain* $C \Longrightarrow x \in C \Longrightarrow y \in C \Longrightarrow x \sqsubseteq y \vee y \sqsubseteq x$

⟨*proof*⟩

lemma *not-chain-suc* [*simp*]: ¬ *chain X* ⟹ *suc X* = *X*
⟨*proof*⟩

lemma *maxchain-suc* [*simp*]: *maxchain X* ⟹ *suc X* = *X*
⟨*proof*⟩

lemma *suc-subset*: *X* ⊆ *suc X*
⟨*proof*⟩

lemma *chain-empty* [*simp*]: *chain* {}
⟨*proof*⟩

lemma *not-maxchain-Some*: *chain C* ⟹ ¬ *maxchain C* ⟹ *C* <*c* (*SOME D*. *C* <*c D*)
⟨*proof*⟩

lemma *suc-not-equals*: *chain C* ⟹ ¬ *maxchain C* ⟹ *suc C* ≠ *C*
⟨*proof*⟩

lemma *subset-suc*:
assumes *X* ⊆ *Y*
shows *X* ⊆ *suc Y*
⟨*proof*⟩

We build a set $\mathcal{C}$ that is closed under applications of *suc* and contains the union of all its subsets.

inductive-set *suc-Union-closed* (*C*)
where
suc: *X* ∈ *C* ⟹ *suc X* ∈ *C*
| *Union* [*unfolded Pow-iff*]: *X* ∈ *Pow C* ⟹ ⋃ *X* ∈ *C*

Since the empty set as well as the set itself is a subset of every set, $\mathcal{C}$ contains at least {} ∈ $\mathcal{C}$ and ⋃ $\mathcal{C}$ ∈ $\mathcal{C}$.

lemma *suc-Union-closed-empty*: {} ∈ *C*
and *suc-Union-closed-Union*: ⋃*C* ∈ *C*
⟨*proof*⟩

Thus closure under *suc* will hit a maximal chain eventually, as is shown below.

lemma *suc-Union-closed-induct* [*consumes 1*, *case-names suc Union*, *induct pred*: *suc-Union-closed*]:
assumes *X* ∈ *C*
and ⋀*X*. *X* ∈ *C* ⟹ *Q X* ⟹ *Q* (*suc X*)
and ⋀*X*. *X* ⊆ *C* ⟹ ∀ *x*∈*X*. *Q x* ⟹ *Q* (⋃ *X*)
shows *Q X*
⟨*proof*⟩

lemma *suc-Union-closed-cases* [*consumes 1*, *case-names suc Union*, *cases pred*: *suc-Union-closed*]:
assumes $X \in \mathcal{C}$
and $\bigwedge Y.\ X = \mathit{suc}\ Y \Longrightarrow Y \in \mathcal{C} \Longrightarrow Q$
and $\bigwedge Y.\ X = \bigcup Y \Longrightarrow Y \subseteq \mathcal{C} \Longrightarrow Q$
shows Q
⟨*proof*⟩

On chains, *suc* yields a chain.

lemma *chain-suc*:
assumes *chain* X
shows *chain* (*suc* X)
⟨*proof*⟩

lemma *chain-sucD*:
assumes *chain* X
shows $\mathit{suc}\ X \subseteq A \wedge \mathit{chain}\ (\mathit{suc}\ X)$
⟨*proof*⟩

lemma *suc-Union-closed-total'*:
assumes $X \in \mathcal{C}$ **and** $Y \in \mathcal{C}$
and $*$: $\bigwedge Z.\ Z \in \mathcal{C} \Longrightarrow Z \subseteq Y \Longrightarrow Z = Y \vee \mathit{suc}\ Z \subseteq Y$
shows $X \subseteq Y \vee \mathit{suc}\ Y \subseteq X$
⟨*proof*⟩

lemma *suc-Union-closed-subsetD*:
assumes $Y \subseteq X$ **and** $X \in \mathcal{C}$ **and** $Y \in \mathcal{C}$
shows $X = Y \vee \mathit{suc}\ Y \subseteq X$
⟨*proof*⟩

The elements of $\mathcal{C}$ are totally ordered by the subset relation.

lemma *suc-Union-closed-total*:
assumes $X \in \mathcal{C}$ **and** $Y \in \mathcal{C}$
shows $X \subseteq Y \vee Y \subseteq X$
⟨*proof*⟩

Once we hit a fixed point w.r.t. *suc*, all other elements of $\mathcal{C}$ are subsets of this fixed point.

lemma *suc-Union-closed-suc*:
assumes $X \in \mathcal{C}$ **and** $Y \in \mathcal{C}$ **and** $\mathit{suc}\ Y = Y$
shows $X \subseteq Y$
⟨*proof*⟩

lemma *eq-suc-Union*:
assumes $X \in \mathcal{C}$
shows $\mathit{suc}\ X = X \longleftrightarrow X = \bigcup \mathcal{C}$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *suc-in-carrier*:
assumes $X \subseteq A$
shows *suc* $X \subseteq A$
⟨*proof*⟩

lemma *suc-Union-closed-in-carrier*:
assumes $X \in \mathcal{C}$
shows $X \subseteq A$
⟨*proof*⟩

All elements of $\mathcal{C}$ are chains.

lemma *suc-Union-closed-chain*:
assumes $X \in \mathcal{C}$
shows *chain* X
⟨*proof*⟩

26.1.2 Hausdorff's Maximum Principle

There exists a maximal totally ordered subset of A. (Note that we do not require A to be partially ordered.)

theorem *Hausdorff*: $\exists\, C.$ *maxchain* C
⟨*proof*⟩

Make notation $\mathcal{C}$ available again.

no-notation *suc-Union-closed* $(\mathcal{C})$

lemma *chain-extend*: *chain* $C \Longrightarrow z \in A \Longrightarrow \forall\, x{\in}C.\ x \sqsubseteq z \Longrightarrow$ *chain* $(\{z\} \cup C)$
⟨*proof*⟩

lemma *maxchain-imp-chain*: *maxchain* $C \Longrightarrow$ *chain* C
⟨*proof*⟩

end

Hide constant *pred-on.suc-Union-closed*, which was just needed for the proof of Hausforff's maximum principle.

hide-const *pred-on.suc-Union-closed*

lemma *chain-mono*:
assumes $\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow P\ x\ y \Longrightarrow Q\ x\ y$
and *pred-on.chain* $A\ P\ C$
shows *pred-on.chain* $A\ Q\ C$
⟨*proof*⟩

26.1.3 Results for the proper subset relation

interpretation *subset*: *pred-on* A $(\subset)$ **for** A ⟨*proof*⟩

lemma *subset-maxchain-max*:
assumes *subset.maxchain A C*
and $X \in A$
and $\bigcup C \subseteq X$
shows $\bigcup C = X$
⟨*proof*⟩

lemma *subset-chain-def*: $\bigwedge \mathcal{A}$. *subset.chain* $\mathcal{A}$ $\mathcal{C}$ = ($\mathcal{C} \subseteq \mathcal{A} \wedge (\forall X{\in}\mathcal{C}.\ \forall Y{\in}\mathcal{C}.\ X \subseteq Y \vee Y \subseteq X)$)
⟨*proof*⟩

lemma *subset-chain-insert*:
subset.chain $\mathcal{A}$ (*insert* B $\mathcal{B}$) $\longleftrightarrow$ $B \in \mathcal{A} \wedge (\forall X{\in}\mathcal{B}.\ X \subseteq B \vee B \subseteq X) \wedge$ *subset.chain* $\mathcal{A}$ $\mathcal{B}$
⟨*proof*⟩

26.1.4 Zorn's lemma

If every chain has an upper bound, then there is a maximal set.

theorem *subset-Zorn*:
assumes $\bigwedge C$. *subset.chain A C* $\Longrightarrow \exists U{\in}A.\ \forall X{\in}C.\ X \subseteq U$
shows $\exists M{\in}A.\ \forall X{\in}A.\ M \subseteq X \longrightarrow X = M$
⟨*proof*⟩

Alternative version of Zorn's lemma for the subset relation.

lemma *subset-Zorn'*:
assumes $\bigwedge C$. *subset.chain A C* $\Longrightarrow \bigcup C \in A$
shows $\exists M{\in}A.\ \forall X{\in}A.\ M \subseteq X \longrightarrow X = M$
⟨*proof*⟩

26.2 Zorn's Lemma for Partial Orders

Relate old to new definitions.

definition *chain-subset* :: *'a set set* ⇒ *bool* (*chain*$_\subseteq$)
where *chain*$_\subseteq$ $C \longleftrightarrow (\forall A{\in}C.\ \forall B{\in}C.\ A \subseteq B \vee B \subseteq A)$

definition *chains* :: *'a set set* ⇒ *'a set set set*
where *chains A* = $\{C.\ C \subseteq A \wedge$ *chain*$_\subseteq$ $C\}$

definition *Chains* :: (*'a* × *'a*) *set* ⇒ *'a set set*
where *Chains r* = $\{C.\ \forall a{\in}C.\ \forall b{\in}C.\ (a, b) \in r \vee (b, a) \in r\}$

lemma *chains-extend*: $c \in$ *chains* $S \Longrightarrow z \in S \Longrightarrow \forall x \in c.\ x \subseteq z \Longrightarrow \{z\} \cup c \in$ *chains* S
for z :: *'a set*
⟨*proof*⟩

lemma *mono-Chains*: $r \subseteq s \Longrightarrow \mathit{Chains}\ r \subseteq \mathit{Chains}\ s$
⟨*proof*⟩

lemma *chain-subset-alt-def*: $\mathit{chain}_{\subseteq}\ C = \mathit{subset.chain}\ \mathit{UNIV}\ C$
⟨*proof*⟩

lemma *chains-alt-def*: $\mathit{chains}\ A = \{C.\ \mathit{subset.chain}\ A\ C\}$
⟨*proof*⟩

lemma *Chains-subset*: $\mathit{Chains}\ r \subseteq \{C.\ \mathit{pred\text{-}on.chain}\ \mathit{UNIV}\ (\lambda x\ y.\ (x,\ y) \in r)\ C\}$
⟨*proof*⟩

lemma *Chains-subset'*:
assumes *refl r*
shows $\{C.\ \mathit{pred\text{-}on.chain}\ \mathit{UNIV}\ (\lambda x\ y.\ (x,\ y) \in r)\ C\} \subseteq \mathit{Chains}\ r$
⟨*proof*⟩

lemma *Chains-alt-def*:
assumes *refl r*
shows $\mathit{Chains}\ r = \{C.\ \mathit{pred\text{-}on.chain}\ \mathit{UNIV}\ (\lambda x\ y.\ (x,\ y) \in r)\ C\}$
⟨*proof*⟩

lemma *Chains-relation-of*:
assumes $C \in \mathit{Chains}\ (\mathit{relation\text{-}of}\ P\ A)$ **shows** $C \subseteq A$
⟨*proof*⟩

lemma *pairwise-chain-Union*:
assumes P: $\bigwedge S.\ S \in \mathcal{C} \Longrightarrow \mathit{pairwise}\ R\ S$ **and** $\mathit{chain}_{\subseteq}\ \mathcal{C}$
shows $\mathit{pairwise}\ R\ (\bigcup \mathcal{C})$
⟨*proof*⟩

lemma *Zorn-Lemma*: $\forall\, C{\in}\mathit{chains}\ A.\ \bigcup C \in A \Longrightarrow \exists\, M{\in}A.\ \forall\, X{\in}A.\ M \subseteq X \longrightarrow X = M$
⟨*proof*⟩

lemma *Zorn-Lemma2*: $\forall\, C{\in}\mathit{chains}\ A.\ \exists\, U{\in}A.\ \forall\, X{\in}C.\ X \subseteq U \Longrightarrow \exists\, M{\in}A.\ \forall\, X{\in}A.\ M \subseteq X \longrightarrow X = M$
⟨*proof*⟩

26.3 Other variants of Zorn's Lemma

lemma *chainsD*: $c \in \mathit{chains}\ S \Longrightarrow x \in c \Longrightarrow y \in c \Longrightarrow x \subseteq y \vee y \subseteq x$
⟨*proof*⟩

lemma *chainsD2*: $c \in \mathit{chains}\ S \Longrightarrow c \subseteq S$
⟨*proof*⟩

lemma *Zorns-po-lemma*:
assumes *po*: *Partial-order r*

and u: $\bigwedge C.\ C \in \mathit{Chains}\ r \Longrightarrow \exists\, u{\in}\mathit{Field}\ r.\ \forall\, a{\in}C.\ (a,\ u) \in r$
shows $\exists\, m{\in}\mathit{Field}\ r.\ \forall\, a{\in}\mathit{Field}\ r.\ (m,\ a) \in r \longrightarrow a = m$
⟨*proof*⟩

lemma *predicate-Zorn*:
assumes *po*: *partial-order-on A* (*relation-of P A*)
and *ch*: $\bigwedge C.\ C \in \mathit{Chains}\ (\mathit{relation\text{-}of}\ P\ A) \Longrightarrow \exists\, u \in A.\ \forall\, a \in C.\ P\ a\ u$
shows $\exists\, m \in A.\ \forall\, a \in A.\ P\ m\ a \longrightarrow a = m$
⟨*proof*⟩

lemma *Union-in-chain*: $[\![\mathit{finite}\ \mathcal{B};\ \mathcal{B} \neq \{\};\ \mathit{subset.chain}\ \mathcal{A}\ \mathcal{B}]\!] \Longrightarrow \bigcup \mathcal{B} \in \mathcal{B}$
⟨*proof*⟩

lemma *Inter-in-chain*: $[\![\mathit{finite}\ \mathcal{B};\ \mathcal{B} \neq \{\};\ \mathit{subset.chain}\ \mathcal{A}\ \mathcal{B}]\!] \Longrightarrow \bigcap \mathcal{B} \in \mathcal{B}$
⟨*proof*⟩

lemma *finite-subset-Union-chain*:
assumes *finite A* $A \subseteq \bigcup \mathcal{B}$ $\mathcal{B} \neq \{\}$ **and** *sub*: *subset.chain* $\mathcal{A}$ $\mathcal{B}$
obtains B **where** $B \in \mathcal{B}$ $A \subseteq B$
⟨*proof*⟩

lemma *subset-Zorn-nonempty*:
assumes $\mathcal{A} \neq \{\}$ **and** *ch*: $\bigwedge \mathcal{C}.\ [\![\mathcal{C}{\neq}\{\};\ \mathit{subset.chain}\ \mathcal{A}\ \mathcal{C}]\!] \Longrightarrow \bigcup \mathcal{C} \in \mathcal{A}$
shows $\exists\, M{\in}\mathcal{A}.\ \forall\, X{\in}\mathcal{A}.\ M \subseteq X \longrightarrow X = M$
⟨*proof*⟩

26.4 The Well Ordering Theorem

definition *init-seg-of* :: $((\prime a \times \prime a)\ \mathit{set} \times (\prime a \times \prime a)\ \mathit{set})\ \mathit{set}$
where *init-seg-of* $= \{(r,\ s).\ r \subseteq s \wedge (\forall\, a\ b\ c.\ (a,\ b) \in s \wedge (b,\ c) \in r \longrightarrow (a,\ b) \in r)\}$

abbreviation *initial-segment-of-syntax* :: $(\prime a \times \prime a)\ \mathit{set} \Rightarrow (\prime a \times \prime a)\ \mathit{set} \Rightarrow \mathit{bool}$
(**infix** *initial′-segment′-of 55*)
where r *initial-segment-of* $s \equiv (r,\ s) \in$ *init-seg-of*

lemma *refl-on-init-seg-of* [*simp*]: r *initial-segment-of* r
⟨*proof*⟩

lemma *trans-init-seg-of*:
r *initial-segment-of* $s \Longrightarrow s$ *initial-segment-of* $t \Longrightarrow r$ *initial-segment-of* t
⟨*proof*⟩

lemma *antisym-init-seg-of*: r *initial-segment-of* $s \Longrightarrow s$ *initial-segment-of* $r \Longrightarrow r = s$
⟨*proof*⟩

lemma *Chains-init-seg-of-Union*: $R \in \mathit{Chains}$ *init-seg-of* $\Longrightarrow r{\in}R \Longrightarrow r$ *initial-segment-of* $\bigcup R$

⟨*proof*⟩

lemma *chain-subset-trans-Union*:
assumes $chain_{\subseteq}\ R\ \forall r{\in}R.\ trans\ r$
shows $trans\ (\bigcup R)$
⟨*proof*⟩

lemma *chain-subset-antisym-Union*:
assumes $chain_{\subseteq}\ R\ \forall r{\in}R.\ antisym\ r$
shows $antisym\ (\bigcup R)$
⟨*proof*⟩

lemma *chain-subset-Total-Union*:
assumes $chain_{\subseteq}\ R$ **and** $\forall r{\in}R.\ Total\ r$
shows $Total\ (\bigcup R)$
⟨*proof*⟩

lemma *wf-Union-wf-init-segs*:
assumes $R \in Chains\ init\text{-}seg\text{-}of$
and $\forall r{\in}R.\ wf\ r$
shows $wf\ (\bigcup R)$
⟨*proof*⟩

lemma *initial-segment-of-Diff*: $p\ initial\text{-}segment\text{-}of\ q \Longrightarrow p - s\ initial\text{-}segment\text{-}of\ q - s$
⟨*proof*⟩

lemma *Chains-inits-DiffI*: $R \in Chains\ init\text{-}seg\text{-}of \Longrightarrow \{r - s\ |r.\ r \in R\} \in Chains\ init\text{-}seg\text{-}of$
⟨*proof*⟩

theorem *well-ordering*: $\exists r{::}'a\ rel.\ Well\text{-}order\ r \wedge Field\ r = UNIV$
⟨*proof*⟩

corollary *well-order-on*: $\exists r{::}'a\ rel.\ well\text{-}order\text{-}on\ A\ r$
⟨*proof*⟩

lemma *dependent-wf-choice*:
fixes $P :: ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b \Rightarrow bool$
assumes $wf\ R$
and *adm*: $\bigwedge f\ g\ x\ r.\ (\bigwedge z.\ (z, x) \in R \Longrightarrow f\ z = g\ z) \Longrightarrow P\ f\ x\ r = P\ g\ x\ r$
and *P*: $\bigwedge x\ f.\ (\bigwedge y.\ (y, x) \in R \Longrightarrow P\ f\ y\ (f\ y)) \Longrightarrow \exists r.\ P\ f\ x\ r$
shows $\exists f.\ \forall x.\ P\ f\ x\ (f\ x)$
⟨*proof*⟩

lemma (**in** *wellorder*) *dependent-wellorder-choice*:
assumes $\bigwedge r\ f\ g\ x.\ (\bigwedge y.\ y < x \Longrightarrow f\ y = g\ y) \Longrightarrow P\ f\ x\ r = P\ g\ x\ r$
and *P*: $\bigwedge x\ f.\ (\bigwedge y.\ y < x \Longrightarrow P\ f\ y\ (f\ y)) \Longrightarrow \exists r.\ P\ f\ x\ r$
shows $\exists f.\ \forall x.\ P\ f\ x\ (f\ x)$

⟨*proof*⟩

end

27 Well-Order Relations as Needed by Bounded Natural Functors

theory *BNF-Wellorder-Relation*
imports *Order-Relation*
begin

In this section, we develop basic concepts and results pertaining to well-order relations. Note that we consider well-order relations as *non-strict relations*, i.e., as containing the diagonals of their fields.

locale *wo-rel* =
fixes $r :: 'a\ rel$
assumes *WELL*: *Well-order r*
begin

The following context encompasses all this section. In other words, for the whole section, we consider a fixed well-order relation r.

abbreviation *under* **where** *under* ≡ *Order-Relation.under r*
abbreviation *underS* **where** *underS* ≡ *Order-Relation.underS r*
abbreviation *Under* **where** *Under* ≡ *Order-Relation.Under r*
abbreviation *UnderS* **where** *UnderS* ≡ *Order-Relation.UnderS r*
abbreviation *above* **where** *above* ≡ *Order-Relation.above r*
abbreviation *aboveS* **where** *aboveS* ≡ *Order-Relation.aboveS r*
abbreviation *Above* **where** *Above* ≡ *Order-Relation.Above r*
abbreviation *AboveS* **where** *AboveS* ≡ *Order-Relation.AboveS r*
abbreviation *ofilter* **where** *ofilter* ≡ *Order-Relation.ofilter r*
lemmas *ofilter-def* = *Order-Relation.ofilter-def*[*of r*]

27.1 Auxiliaries

lemma *REFL*: *Refl r*
⟨*proof*⟩

lemma *TRANS*: *trans r*
⟨*proof*⟩

lemma *ANTISYM*: *antisym r*
⟨*proof*⟩

lemma *TOTAL*: *Total r*
⟨*proof*⟩

lemma *TOTALS*: ∀ *a* ∈ *Field r*. ∀ *b* ∈ *Field r*. (*a*,*b*) ∈ *r* ∨ (*b*,*a*) ∈ *r*
⟨*proof*⟩

lemma *LIN*: *Linear-order r*
⟨*proof*⟩

lemma *WF*: *wf* (*r* − *Id*)
⟨*proof*⟩

lemma *cases-Total*:
⋀ *phi a b*. ⟦{*a*,*b*} <= *Field r*; ((*a*,*b*) ∈ *r* ⟹ *phi a b*); ((*b*,*a*) ∈ *r* ⟹ *phi a b*)⟧
⟹ *phi a b*
⟨*proof*⟩

lemma *cases-Total3*:
⋀ *phi a b*. ⟦{*a*,*b*} ≤ *Field r*; ((*a*,*b*) ∈ *r* − *Id* ∨ (*b*,*a*) ∈ *r* − *Id* ⟹ *phi a b*);
(*a* = *b* ⟹ *phi a b*)⟧ ⟹ *phi a b*
⟨*proof*⟩

27.2 Well-founded induction and recursion adapted to non-strict well-order relations

Here we provide induction and recursion principles specific to *non-strict* well-order relations. Although minor variations of those for well-founded relations, they will be useful for doing away with the tediousness of having to take out the diagonal each time in order to switch to a well-founded relation.

lemma *well-order-induct*:
assumes *IND*: ⋀*x*. ∀ *y*. *y* ≠ *x* ∧ (*y*, *x*) ∈ *r* ⟶ *P y* ⟹ *P x*
shows *P a*
⟨*proof*⟩

definition
worec :: (('*a* ⇒ '*b*) ⇒ '*a* ⇒ '*b*) ⇒ '*a* ⇒ '*b*
where
worec F ≡ *wfrec* (*r* − *Id*) *F*

definition
adm-wo :: (('*a* ⇒ '*b*) ⇒ '*a* ⇒ '*b*) ⇒ *bool*
where
adm-wo H ≡ ∀ *f g x*. (∀ *y* ∈ *underS x*. *f y* = *g y*) ⟶ *H f x* = *H g x*

lemma *worec-fixpoint*:
assumes *ADM*: *adm-wo H*
shows *worec H* = *H* (*worec H*)
⟨*proof*⟩

27.3 The notions of maximum, minimum, supremum, successor and order filter

We define the successor *of a set*, and not of an element (the latter is of course a particular case). Also, we define the maximum *of two elements*, *max2*, and the minimum *of a set*, *minim* – we chose these variants since we consider them the most useful for well-orders. The minimum is defined in terms of the auxiliary relational operator *isMinim*. Then, supremum and successor are defined in terms of minimum as expected. The minimum is only meaningful for non-empty sets, and the successor is only meaningful for sets for which strict upper bounds exist. Order filters for well-orders are also known as "initial segments".

definition $max2 :: 'a \Rightarrow 'a \Rightarrow 'a$
where $max2\ a\ b \equiv if\ (a,b) \in r\ then\ b\ else\ a$

definition $isMinim :: 'a\ set \Rightarrow 'a \Rightarrow bool$
where $isMinim\ A\ b \equiv b \in A \wedge (\forall a \in A.\ (b,a) \in r)$

definition $minim :: 'a\ set \Rightarrow 'a$
where $minim\ A \equiv THE\ b.\ isMinim\ A\ b$

definition $supr :: 'a\ set \Rightarrow 'a$
where $supr\ A \equiv minim\ (Above\ A)$

definition $suc :: 'a\ set \Rightarrow 'a$
where $suc\ A \equiv minim\ (AboveS\ A)$

27.3.1 Properties of max2

lemma *max2-greater-among*:
assumes $a \in Field\ r$ **and** $b \in Field\ r$
shows $(a,\ max2\ a\ b) \in r \wedge (b,\ max2\ a\ b) \in r \wedge max2\ a\ b \in \{a,b\}$
⟨*proof*⟩

lemma *max2-greater*:
assumes $a \in Field\ r$ **and** $b \in Field\ r$
shows $(a,\ max2\ a\ b) \in r \wedge (b,\ max2\ a\ b) \in r$
⟨*proof*⟩

lemma *max2-among*:
assumes $a \in Field\ r$ **and** $b \in Field\ r$
shows $max2\ a\ b \in \{a,\ b\}$
⟨*proof*⟩

lemma *max2-equals1*:
assumes $a \in Field\ r$ **and** $b \in Field\ r$
shows $(max2\ a\ b = a) = ((b,a) \in r)$
⟨*proof*⟩

lemma *max2-equals2*:
assumes $a \in Field\ r$ **and** $b \in Field\ r$
shows $(max2\ a\ b = b) = ((a,b) \in r)$
⟨*proof*⟩

lemma *in-notinI*:
assumes $(j,i) \notin r \vee j = i$ **and** $i \in Field\ r$ **and** $j \in Field\ r$
shows $(i,j) \in r$ ⟨*proof*⟩

27.3.2 Existence and uniqueness for isMinim and well-definedness of minim

lemma *isMinim-unique*:
assumes *isMinim B a isMinim B a'*
shows $a = a'$
⟨*proof*⟩

lemma *Well-order-isMinim-exists*:
assumes *SUB*: $B \leq Field\ r$ **and** *NE*: $B \neq \{\}$
shows $\exists\, b.\ isMinim\ B\ b$
⟨*proof*⟩

lemma *minim-isMinim*:
assumes *SUB*: $B \leq Field\ r$ **and** *NE*: $B \neq \{\}$
shows *isMinim B* (*minim B*)
⟨*proof*⟩

27.3.3 Properties of minim

lemma *minim-in*:
assumes $B \leq Field\ r$ **and** $B \neq \{\}$
shows $minim\ B \in B$
⟨*proof*⟩

lemma *minim-inField*:
assumes $B \leq Field\ r$ **and** $B \neq \{\}$
shows $minim\ B \in Field\ r$
⟨*proof*⟩

lemma *minim-least*:
assumes *SUB*: $B \leq Field\ r$ **and** *IN*: $b \in B$
shows (*minim B*, *b*) ∈ *r*
⟨*proof*⟩

lemma *equals-minim*:
assumes *SUB*: $B \leq Field\ r$ **and** *IN*: $a \in B$ **and**
LEAST: $\bigwedge\ b.\ b \in B \Longrightarrow (a,b) \in r$
shows $a = minim\ B$
⟨*proof*⟩

27.3.4 Properties of successor

lemma *suc-AboveS*:
assumes *SUB*: $B \leq Field\ r$ **and** *ABOVES*: $AboveS\ B \neq \{\}$
shows $suc\ B \in AboveS\ B$
⟨*proof*⟩

lemma *suc-greater*:
assumes *SUB*: $B \leq Field\ r$ **and** *ABOVES*: $AboveS\ B \neq \{\}$ **and** *IN*: $b \in B$
shows $suc\ B \neq b \wedge (b,suc\ B) \in r$
⟨*proof*⟩

lemma *suc-least-AboveS*:
assumes *ABOVES*: $a \in AboveS\ B$
shows $(suc\ B,a) \in r$
⟨*proof*⟩

lemma *suc-inField*:
assumes $B \leq Field\ r$ **and** $AboveS\ B \neq \{\}$
shows $suc\ B \in Field\ r$
⟨*proof*⟩

lemma *equals-suc-AboveS*:
assumes $B \leq Field\ r$ **and** $a \in AboveS\ B$ **and** $\bigwedge a'.\ a' \in AboveS\ B \Longrightarrow (a,a') \in r$
shows $a = suc\ B$
⟨*proof*⟩

lemma *suc-underS*:
assumes *IN*: $a \in Field\ r$
shows $a = suc\ (underS\ a)$
⟨*proof*⟩

27.3.5 Properties of order filters

lemma *under-ofilter*: $ofilter\ (under\ a)$
⟨*proof*⟩

lemma *underS-ofilter*: $ofilter\ (underS\ a)$
⟨*proof*⟩

lemma *Field-ofilter*:
$ofilter\ (Field\ r)$
⟨*proof*⟩

lemma *ofilter-underS-Field*:
$ofilter\ A = ((\exists\, a \in Field\ r.\ A = underS\ a) \vee (A = Field\ r))$
⟨*proof*⟩

lemma *ofilter-UNION*:

$(\bigwedge i.\ i \in I \Longrightarrow ofilter(A\ i)) \Longrightarrow ofilter\ (\bigcup i \in I.\ A\ i)$
⟨*proof*⟩

lemma *ofilter-under-UNION*:
assumes *ofilter A*
shows $A = (\bigcup a \in A.\ under\ a)$
⟨*proof*⟩

27.3.6 Other properties

lemma *ofilter-linord*:
assumes *OF1*: *ofilter A* **and** *OF2*: *ofilter B*
shows $A \leq B \vee B \leq A$
⟨*proof*⟩

lemma *ofilter-AboveS-Field*:
assumes *ofilter A*
shows $A \cup (AboveS\ A) = Field\ r$
⟨*proof*⟩

lemma *suc-ofilter-in*:
assumes *OF*: *ofilter A* **and** *ABOVE-NE*: $AboveS\ A \neq \{\}$ **and**
REL: $(b,suc\ A) \in r$ **and** *DIFF*: $b \neq suc\ A$
shows $b \in A$
⟨*proof*⟩

end

end

28 Well-Order Embeddings as Needed by Bounded Natural Functors

theory *BNF-Wellorder-Embedding*
imports *Hilbert-Choice BNF-Wellorder-Relation*
begin

In this section, we introduce well-order *embeddings* and *isomorphisms* and prove their basic properties. The notion of embedding is considered from the point of view of the theory of ordinals, and therefore requires the source to be injected as an *initial segment* (i.e., *order filter*) of the target. A main result of this section is the existence of embeddings (in one direction or another) between any two well-orders, having as a consequence the fact that, given any two sets on any two types, one is smaller than (i.e., can be injected into) the other.

28.1 Auxiliaries

lemma *UNION-inj-on-ofilter*:
assumes *WELL*: *Well-order r* **and**
OF: $\bigwedge i.\ i \in I \Longrightarrow$ *wo-rel.ofilter r* (*A i*) **and**
INJ: $\bigwedge i.\ i \in I \Longrightarrow$ *inj-on f* (*A i*)
shows *inj-on f* ($\bigcup i \in I.$ *A i*)
⟨*proof*⟩

lemma *under-underS-bij-betw*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'* **and**
IN: *a* ∈ *Field r* **and** *IN'*: *f a* ∈ *Field r'* **and**
BIJ: *bij-betw f* (*underS r a*) (*underS r'* (*f a*))
shows *bij-betw f* (*under r a*) (*under r'* (*f a*))
⟨*proof*⟩

28.2 (Well-order) embeddings, strict embeddings, isomorphisms and order-compatible functions

Standardly, a function is an embedding of a well-order in another if it injectively and order-compatibly maps the former into an order filter of the latter. Here we opt for a more succinct definition (operator *embed*), asking that, for any element in the source, the function should be a bijection between the set of strict lower bounds of that element and the set of strict lower bounds of its image. (Later we prove equivalence with the standard definition – lemma *embed-iff-compat-inj-on-ofilter*.) A *strict embedding* (operator *embedS*) is a non-bijective embedding and an isomorphism (operator *iso*) is a bijective embedding.

definition *embed* :: *'a rel* ⇒ *'a' rel* ⇒ (*'a* ⇒ *'a'*) ⇒ *bool*
where
embed r r' f ≡ ∀ *a* ∈ *Field r*. *bij-betw f* (*under r a*) (*under r'* (*f a*))

lemmas *embed-defs* = *embed-def embed-def*[*abs-def*]

Strict embeddings:

definition *embedS* :: *'a rel* ⇒ *'a' rel* ⇒ (*'a* ⇒ *'a'*) ⇒ *bool*
where
embedS r r' f ≡ *embed r r' f* ∧ ¬ *bij-betw f* (*Field r*) (*Field r'*)

lemmas *embedS-defs* = *embedS-def embedS-def*[*abs-def*]

definition *iso* :: *'a rel* ⇒ *'a' rel* ⇒ (*'a* ⇒ *'a'*) ⇒ *bool*
where
iso r r' f ≡ *embed r r' f* ∧ *bij-betw f* (*Field r*) (*Field r'*)

lemmas *iso-defs* = *iso-def iso-def*[*abs-def*]

definition *compat* :: *'a rel* ⇒ *'a' rel* ⇒ (*'a* ⇒ *'a'*) ⇒ *bool*

where
compat r r′ f ≡ ∀ *a b*. (*a*,*b*) ∈ *r* ⟶ (*f a*, *f b*) ∈ *r′*

lemma *compat-wf*:
assumes *CMP*: *compat r r′ f* **and** *WF*: *wf r′*
shows *wf r*
⟨*proof*⟩

lemma *id-embed*: *embed r r id*
⟨*proof*⟩

lemma *id-iso*: *iso r r id*
⟨*proof*⟩

lemma *embed-compat*:
assumes *EMB*: *embed r r′ f*
shows *compat r r′ f*
⟨*proof*⟩

lemma *embed-in-Field*:
assumes *EMB*: *embed r r′ f* **and** *IN*: *a* ∈ *Field r*
shows *f a* ∈ *Field r′*
⟨*proof*⟩

lemma *comp-embed*:
assumes *EMB*: *embed r r′ f* **and** *EMB′*: *embed r′ r″ f′*
shows *embed r r″* (*f′* ∘ *f*)
⟨*proof*⟩

lemma *comp-iso*:
assumes *EMB*: *iso r r′ f* **and** *EMB′*: *iso r′ r″ f′*
shows *iso r r″* (*f′* ∘ *f*)
⟨*proof*⟩

That *embedS* is also preserved by function composition shall be proved only later.

lemma *embed-Field*: *embed r r′ f* ⟹ *f*‘(*Field r*) ≤ *Field r′*
⟨*proof*⟩

lemma *embed-preserves-ofilter*:
assumes *WELL*: *Well-order r* **and** *WELL′*: *Well-order r′* **and**
EMB: *embed r r′ f* **and** *OF*: *wo-rel.ofilter r A*
shows *wo-rel.ofilter r′* (*f*‘*A*)
⟨*proof*⟩

lemma *embed-Field-ofilter*:
assumes *WELL*: *Well-order r* **and** *WELL′*: *Well-order r′* **and**
EMB: *embed r r′ f*
shows *wo-rel.ofilter r′* (*f*‘(*Field r*))

⟨*proof*⟩

lemma *embed-inj-on*:
assumes *WELL*: *Well-order r* **and** *EMB*: *embed r r′ f*
shows *inj-on f* (*Field r*)
⟨*proof*⟩

lemma *embed-underS*:
assumes *WELL*: *Well-order r* **and**
EMB: *embed r r′ f* **and** *IN*: *a* ∈ *Field r*
shows *bij-betw f* (*underS r a*) (*underS r′* (*f a*))
⟨*proof*⟩

lemma *embed-iff-compat-inj-on-ofilter*:
assumes *WELL*: *Well-order r* **and** *WELL′*: *Well-order r′*
shows *embed r r′ f* = (*compat r r′ f* ∧ *inj-on f* (*Field r*) ∧ *wo-rel.ofilter r′* (*f*‘(*Field r*)))
⟨*proof*⟩

lemma *inv-into-ofilter-embed*:
assumes *WELL*: *Well-order r* **and** *OF*: *wo-rel.ofilter r A* **and**
BIJ: ∀ *b* ∈ *A*. *bij-betw f* (*under r b*) (*under r′* (*f b*)) **and**
IMAGE: *f* ‘ *A* = *Field r′*
shows *embed r′ r* (*inv-into A f*)
⟨*proof*⟩

lemma *inv-into-underS-embed*:
assumes *WELL*: *Well-order r* **and**
BIJ: ∀ *b* ∈ *underS r a*. *bij-betw f* (*under r b*) (*under r′* (*f b*)) **and**
IN: *a* ∈ *Field r* **and**
IMAGE: *f* ‘ (*underS r a*) = *Field r′*
shows *embed r′ r* (*inv-into* (*underS r a*) *f*)
⟨*proof*⟩

lemma *inv-into-Field-embed*:
assumes *WELL*: *Well-order r* **and** *EMB*: *embed r r′ f* **and**
IMAGE: *Field r′* ≤ *f* ‘ (*Field r*)
shows *embed r′ r* (*inv-into* (*Field r*) *f*)
⟨*proof*⟩

lemma *inv-into-Field-embed-bij-betw*:
assumes *EMB*: *embed r r′ f* **and** *BIJ*: *bij-betw f* (*Field r*) (*Field r′*)
shows *embed r′ r* (*inv-into* (*Field r*) *f*)
⟨*proof*⟩

28.3 Given any two well-orders, one can be embedded in the other

Here is an overview of the proof of of this fact, stated in theorem *wellorders-totally-ordered*:

Fix the well-orders $r::'a\ rel$ and $r'::'a'\ rel$. Attempt to define an embedding $f::'a \Rightarrow 'a'$ from r to r' in the natural way by well-order recursion ("hoping" that *Field* r turns out to be smaller than *Field* r'), but also record, at the recursive step, in a function $g::'a \Rightarrow bool$, the extra information of whether *Field* r' gets exhausted or not.

If *Field* r' does not get exhausted, then *Field* r is indeed smaller and f is the desired embedding from r to r' (lemma *wellorders-totally-ordered-aux*). Otherwise, it means that *Field* r' is the smaller one, and the inverse of (the "good" segment of) f is the desired embedding from r' to r (lemma *wellorders-totally-ordered-aux2*).

lemma *wellorders-totally-ordered-aux*:
fixes $r::'a\ rel$ **and** $r'::'a'\ rel$ **and**
$f :: 'a \Rightarrow 'a'$ **and** $a::'a$
assumes *WELL*: *Well-order* r **and** *WELL'*: *Well-order* r' **and** *IN*: $a \in$ *Field* r **and**
IH: $\forall b \in$ *underS* $r\ a$. *bij-betw* f (*under* $r\ b$) (*under* r' ($f\ b$)) **and**
NOT: f ‘ (*underS* $r\ a$) $\neq$ *Field* r' **and** *SUC*: $f\ a =$ *wo-rel.suc* r' (f‘(*underS* $r\ a$))
shows *bij-betw* f (*under* $r\ a$) (*under* r' ($f\ a$))
⟨*proof*⟩

lemma *wellorders-totally-ordered-aux2*:
fixes $r::'a\ rel$ **and** $r'::'a'\ rel$ **and**
$f :: 'a \Rightarrow 'a'$ **and** $g :: 'a \Rightarrow bool$ **and** $a::'a$
assumes *WELL*: *Well-order* r **and** *WELL'*: *Well-order* r' **and**
MAIN1:
$\bigwedge a.$ (*False* $\notin g$‘(*underS* $r\ a$) $\wedge$ f‘(*underS* $r\ a$) $\neq$ *Field* r'
$\longrightarrow f\ a =$ *wo-rel.suc* r' (f‘(*underS* $r\ a$)) $\wedge$ $g\ a =$ *True*)
$\wedge$
(¬(*False* $\notin$ (g‘(*underS* $r\ a$)) $\wedge$ f‘(*underS* $r\ a$) $\neq$ *Field* r')
$\longrightarrow g\ a =$ *False*) **and**
MAIN2: $\bigwedge a.\ a \in$ *Field* $r \wedge$ *False* $\notin g$‘(*under* $r\ a$) $\longrightarrow$
bij-betw f (*under* $r\ a$) (*under* r' ($f\ a$)) **and**
Case: $a \in$ *Field* $r \wedge$ *False* $\in g$‘(*under* $r\ a$)
shows $\exists f'.$ *embed* $r'\ r\ f'$
⟨*proof*⟩

theorem *wellorders-totally-ordered*:
fixes $r::'a\ rel$ **and** $r'::'a'\ rel$
assumes *WELL*: *Well-order* r **and** *WELL'*: *Well-order* r'
shows ($\exists f.$ *embed* $r\ r'\ f$) $\vee$ ($\exists f'.$ *embed* $r'\ r\ f'$)
⟨*proof*⟩

28.4 Uniqueness of embeddings

Here we show a fact complementary to the one from the previous subsection – namely, that between any two well-orders there is *at most* one embed-

ding, and is the one definable by the expected well-order recursive equation. As a consequence, any two embeddings of opposite directions are mutually inverse.

lemma *embed-determined*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'* **and**
EMB: *embed r r' f* **and** *IN*: $a \in Field\ r$
shows $f\ a = wo\text{-}rel.suc\ r'\ (f`(underS\ r\ a))$
⟨*proof*⟩

lemma *embed-unique*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'* **and**
EMBf: *embed r r' f* **and** *EMBg*: *embed r r' g*
shows $a \in Field\ r \longrightarrow f\ a = g\ a$
⟨*proof*⟩

lemma *embed-bothWays-inverse*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'* **and**
EMB: *embed r r' f* **and** *EMB'*: *embed r' r f'*
shows $(\forall a \in Field\ r.\ f'(f\ a) = a) \wedge (\forall a' \in Field\ r'.\ f(f'\ a') = a')$
⟨*proof*⟩

lemma *embed-bothWays-bij-betw*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'* **and**
EMB: *embed r r' f* **and** *EMB'*: *embed r' r g*
shows *bij-betw f* (*Field r*) (*Field r'*)
⟨*proof*⟩

lemma *embed-bothWays-iso*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'* **and**
EMB: *embed r r' f* **and** *EMB'*: *embed r' r g*
shows *iso r r' f*
⟨*proof*⟩

28.5 More properties of embeddings, strict embeddings and isomorphisms

lemma *embed-bothWays-Field-bij-betw*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'* **and**
EMB: *embed r r' f* **and** *EMB'*: *embed r' r f'*
shows *bij-betw f* (*Field r*) (*Field r'*)
⟨*proof*⟩

lemma *embedS-comp-embed*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'*
and *EMB*: *embedS r r' f* **and** *EMB'*: *embed r' r'' f'*
shows *embedS r r''* $(f' \circ f)$
⟨*proof*⟩

lemma *embed-comp-embedS*:

assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'*
and *EMB*: *embed r r' f* **and** *EMB'*: *embedS r' r'' f'*
shows *embedS r r'' (f' ∘ f)*
⟨*proof*⟩

lemma *embed-comp-iso*:
assumes *EMB*: *embed r r' f* **and** *EMB'*: *iso r' r'' f'*
shows *embed r r'' (f' ∘ f)* ⟨*proof*⟩

lemma *iso-comp-embed*:
assumes *EMB*: *iso r r' f* **and** *EMB'*: *embed r' r'' f'*
shows *embed r r'' (f' ∘ f)*
⟨*proof*⟩

lemma *embedS-comp-iso*:
assumes *EMB*: *embedS r r' f* **and** *EMB'*: *iso r' r'' f'*
shows *embedS r r'' (f' ∘ f)*
⟨*proof*⟩

lemma *iso-comp-embedS*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'*
and *EMB*: *iso r r' f* **and** *EMB'*: *embedS r' r'' f'*
shows *embedS r r'' (f' ∘ f)*
⟨*proof*⟩

lemma *embedS-Field*:
assumes *WELL*: *Well-order r* **and** *EMB*: *embedS r r' f*
shows *f ‘ (Field r) < Field r'*
⟨*proof*⟩

lemma *embedS-iff*:
assumes *WELL*: *Well-order r* **and** *ISO*: *embed r r' f*
shows *embedS r r' f = (f ‘ (Field r) < Field r')*
⟨*proof*⟩

lemma *iso-Field*: *iso r r' f ⟹ f ‘ (Field r) = Field r'*
⟨*proof*⟩

lemma *iso-iff*:
assumes *Well-order r*
shows *iso r r' f = (embed r r' f ∧ f ‘ (Field r) = Field r')*
⟨*proof*⟩

lemma *iso-iff2*: *iso r r' f ⟷*
bij-betw f (Field r) (Field r') ∧
(∀ a ∈ Field r. ∀ b ∈ Field r. (a, b) ∈ r ⟷ (f a, f b) ∈ r')
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *iso-iff3*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'*
shows *iso r r' f* = (*bij-betw f* (*Field r*) (*Field r'*) ∧ *compat r r' f*)
⟨*proof*⟩

lemma *iso-imp-inj-on*:
assumes *iso r r' f* **shows** *inj-on f* (*Field r*)
⟨*proof*⟩

lemma *iso-backward-Field*:
assumes *x* ∈ *Field r' iso r r' f*
shows *inv-into* (*Field r*) *f x* ∈ *Field r*
⟨*proof*⟩

lemma *iso-backward*:
assumes (*x*,*y*) ∈ *r'* **and** *iso*: *iso r r' f*
shows (*inv-into* (*Field r*) *f x*, *inv-into* (*Field r*) *f y*) ∈ *r*
⟨*proof*⟩

lemma *iso-forward*:
assumes (*x*,*y*) ∈ *r iso r r' f* **shows** (*f x*,*f y*) ∈ *r'*
⟨*proof*⟩

lemma *iso-trans*:
assumes *trans r* **and** *iso*: *iso r r' f* **shows** *trans r'*
⟨*proof*⟩

lemma *iso-Total*:
assumes *Total r* **and** *iso*: *iso r r' f* **shows** *Total r'*
⟨*proof*⟩

lemma *iso-wf*:
assumes *wf r* **and** *iso*: *iso r r' f* **shows** *wf r'*
⟨*proof*⟩

end

29 Constructions on Wellorders as Needed by Bounded Natural Functors

theory *BNF-Wellorder-Constructions*
imports *BNF-Wellorder-Embedding*
begin

In this section, we study basic constructions on well-orders, such as restriction to a set/order filter, copy via direct images, ordinal-like sum of disjoint well-orders, and bounded square. We also define between well-orders the relations *ordLeq*, of being embedded (abbreviated $\leq o$), *ordLess*, of being

strictly embedded (abbreviated $<o$), and *ordIso*, of being isomorphic (abbreviated $=o$). We study the connections between these relations, order filters, and the aforementioned constructions. A main result of this section is that $<o$ is well-founded.

29.1 Restriction to a set

abbreviation *Restr* :: *'a rel* ⇒ *'a set* ⇒ *'a rel*
where *Restr r A* ≡ *r Int* (*A* × *A*)

lemma *Restr-subset*:
A ≤ *B* ⟹ *Restr* (*Restr r B*) *A* = *Restr r A*
⟨*proof*⟩

lemma *Restr-Field*: *Restr r* (*Field r*) = *r*
⟨*proof*⟩

lemma *Refl-Restr*: *Refl r* ⟹ *Refl*(*Restr r A*)
⟨*proof*⟩

lemma *linear-order-on-Restr*:
linear-order-on A r ⟹ *linear-order-on* (*A* ∩ *above r x*) (*Restr r* (*above r x*))
⟨*proof*⟩

lemma *antisym-Restr*:
antisym r ⟹ *antisym*(*Restr r A*)
⟨*proof*⟩

lemma *Total-Restr*:
Total r ⟹ *Total*(*Restr r A*)
⟨*proof*⟩

lemma *total-on-imp-Total-Restr*: *total-on A r* ⟹ *Total* (*Restr r A*)
⟨*proof*⟩

lemma *trans-Restr*:
trans r ⟹ *trans*(*Restr r A*)
⟨*proof*⟩

lemma *Preorder-Restr*:
Preorder r ⟹ *Preorder*(*Restr r A*)
⟨*proof*⟩

lemma *Partial-order-Restr*:
Partial-order r ⟹ *Partial-order*(*Restr r A*)
⟨*proof*⟩

lemma *Linear-order-Restr*:
Linear-order r ⟹ *Linear-order*(*Restr r A*)

⟨*proof*⟩

lemma *Well-order-Restr*:
assumes *Well-order r*
shows *Well-order*(*Restr r A*)
⟨*proof*⟩

lemma *Field-Restr-subset*: *Field*(*Restr r A*) ≤ *A*
⟨*proof*⟩

lemma *Refl-Field-Restr*:
Refl r ⟹ *Field*(*Restr r A*) = (*Field r*) *Int A*
⟨*proof*⟩

lemma *Refl-Field-Restr2*:
⟦*Refl r*; *A* ≤ *Field r*⟧ ⟹ *Field*(*Restr r A*) = *A*
⟨*proof*⟩

lemma *well-order-on-Restr*:
assumes *WELL*: *Well-order r* **and** *SUB*: *A* ≤ *Field r*
shows *well-order-on A* (*Restr r A*)
⟨*proof*⟩

29.2 Order filters versus restrictions and embeddings

lemma *Field-Restr-ofilter*:
⟦*Well-order r*; *wo-rel.ofilter r A*⟧ ⟹ *Field*(*Restr r A*) = *A*
⟨*proof*⟩

lemma *ofilter-Restr-under*:
assumes *WELL*: *Well-order r* **and** *OF*: *wo-rel.ofilter r A* **and** *IN*: *a* ∈ *A*
shows *under* (*Restr r A*) *a* = *under r a*
⟨*proof*⟩

lemma *ofilter-embed*:
assumes *Well-order r*
shows *wo-rel.ofilter r A* = (*A* ≤ *Field r* ∧ *embed* (*Restr r A*) *r id*)
⟨*proof*⟩

lemma *ofilter-Restr-Int*:
assumes *WELL*: *Well-order r* **and** *OFA*: *wo-rel.ofilter r A*
shows *wo-rel.ofilter* (*Restr r B*) (*A Int B*)
⟨*proof*⟩

lemma *ofilter-Restr-subset*:
assumes *WELL*: *Well-order r* **and** *OFA*: *wo-rel.ofilter r A* **and** *SUB*: *A* ≤ *B*
shows *wo-rel.ofilter* (*Restr r B*) *A*
⟨*proof*⟩

lemma *ofilter-subset-embed*:
assumes *WELL*: *Well-order r* **and**
OFA: *wo-rel.ofilter r A* **and** *OFB*: *wo-rel.ofilter r B*
shows $(A \leq B) = (embed\ (Restr\ r\ A)\ (Restr\ r\ B)\ id)$
⟨*proof*⟩

lemma *ofilter-subset-embedS-iso*:
assumes *WELL*: *Well-order r* **and**
OFA: *wo-rel.ofilter r A* **and** *OFB*: *wo-rel.ofilter r B*
shows $((A < B) = (embedS\ (Restr\ r\ A)\ (Restr\ r\ B)\ id)) \wedge$
$((A = B) = (iso\ (Restr\ r\ A)\ (Restr\ r\ B)\ id))$
⟨*proof*⟩

lemma *ofilter-subset-embedS*:
assumes *WELL*: *Well-order r* **and**
OFA: *wo-rel.ofilter r A* **and** *OFB*: *wo-rel.ofilter r B*
shows $(A < B) = embedS\ (Restr\ r\ A)\ (Restr\ r\ B)\ id$
⟨*proof*⟩

lemma *embed-implies-iso-Restr*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'* **and**
EMB: *embed r' r f*
shows $iso\ r'\ (Restr\ r\ (f\ `\ (Field\ r')))\ f$
⟨*proof*⟩

29.3 The strict inclusion on proper ofilters is well-founded

definition *ofilterIncl* :: *'a rel* ⇒ *'a set rel*
where
$ofilterIncl\ r \equiv \{(A,B).\ wo\text{-}rel.ofilter\ r\ A \wedge A \neq Field\ r \wedge$
$wo\text{-}rel.ofilter\ r\ B \wedge B \neq Field\ r \wedge A < B\}$

lemma *wf-ofilterIncl*:
assumes *WELL*: *Well-order r*
shows $wf(ofilterIncl\ r)$
⟨*proof*⟩

29.4 Ordering the well-orders by existence of embeddings

We define three relations between well-orders:

- *ordLeq*, of being embedded (abbreviated $\leq o$);
- *ordLess*, of being strictly embedded (abbreviated $<o$);
- *ordIso*, of being isomorphic (abbreviated $=o$).

The prefix "ord" and the index "o" in these names stand for "ordinal-like". These relations shall be proved to be inter-connected in a similar fashion as the trio $\leq$, $<$, $=$ associated to a total order on a set.

definition *ordLeq* :: (*'a rel* ∗ *'a' rel*) *set*
where
ordLeq = {(*r*,*r'*). *Well-order r* ∧ *Well-order r'* ∧ (∃*f*. *embed r r' f*)}

abbreviation *ordLeq2* :: *'a rel* ⇒ *'a' rel* ⇒ *bool* (**infix** <=*o* 50)
where *r* <=*o r'* ≡ (*r*,*r'*) ∈ *ordLeq*

abbreviation *ordLeq3* :: *'a rel* ⇒ *'a' rel* ⇒ *bool* (**infix** ≤*o* 50)
where *r* ≤*o r'* ≡ *r* <=*o r'*

definition *ordLess* :: (*'a rel* ∗ *'a' rel*) *set*
where
ordLess = {(*r*,*r'*). *Well-order r* ∧ *Well-order r'* ∧ (∃*f*. *embedS r r' f*)}

abbreviation *ordLess2* :: *'a rel* ⇒ *'a' rel* ⇒ *bool* (**infix** <*o* 50)
where *r* <*o r'* ≡ (*r*,*r'*) ∈ *ordLess*

definition *ordIso* :: (*'a rel* ∗ *'a' rel*) *set*
where
ordIso = {(*r*,*r'*). *Well-order r* ∧ *Well-order r'* ∧ (∃*f*. *iso r r' f*)}

abbreviation *ordIso2* :: *'a rel* ⇒ *'a' rel* ⇒ *bool* (**infix** =*o* 50)
where *r* =*o r'* ≡ (*r*,*r'*) ∈ *ordIso*

lemmas *ordRels-def* = *ordLeq-def ordLess-def ordIso-def*

lemma *ordLeq-Well-order-simp*:
assumes *r* ≤*o r'*
shows *Well-order r* ∧ *Well-order r'*
⟨*proof*⟩

Notice that the relations ≤*o*, <*o*, =*o* connect well-orders on potentially *distinct* types. However, some of the lemmas below, including the next one, restrict implicitly the type of these relations to ((*'a rel*) ∗ (*'a rel*)) *set* , i.e., to *'a rel rel*.

lemma *ordLeq-reflexive*:
Well-order r ⟹ *r* ≤*o r*
⟨*proof*⟩

lemma *ordLeq-transitive*[*trans*]:
assumes *r* ≤*o r'* **and** *r'* ≤*o r''*
shows *r* ≤*o r''*
⟨*proof*⟩

lemma *ordLeq-total*:
⟦*Well-order r*; *Well-order r'*⟧ ⟹ *r* ≤*o r'* ∨ *r'* ≤*o r*
⟨*proof*⟩

lemma *ordIso-reflexive*:

Well-order $r \Longrightarrow r =o\ r$
⟨*proof*⟩

lemma *ordIso-transitive*[*trans*]:
assumes ∗: $r =o\ r'$ **and** ∗∗: $r' =o\ r''$
shows $r =o\ r''$
⟨*proof*⟩

lemma *ordIso-symmetric*:
assumes ∗: $r =o\ r'$
shows $r' =o\ r$
⟨*proof*⟩

lemma *ordLeq-ordLess-trans*[*trans*]:
assumes $r \leq o\ r'$ **and** $r' <o\ r''$
shows $r <o\ r''$
⟨*proof*⟩

lemma *ordLess-ordLeq-trans*[*trans*]:
assumes $r <o\ r'$ **and** $r' \leq o\ r''$
shows $r <o\ r''$
⟨*proof*⟩

lemma *ordLeq-ordIso-trans*[*trans*]:
assumes $r \leq o\ r'$ **and** $r' =o\ r''$
shows $r \leq o\ r''$
⟨*proof*⟩

lemma *ordIso-ordLeq-trans*[*trans*]:
assumes $r =o\ r'$ **and** $r' \leq o\ r''$
shows $r \leq o\ r''$
⟨*proof*⟩

lemma *ordLess-ordIso-trans*[*trans*]:
assumes $r <o\ r'$ **and** $r' =o\ r''$
shows $r <o\ r''$
⟨*proof*⟩

lemma *ordIso-ordLess-trans*[*trans*]:
assumes $r =o\ r'$ **and** $r' <o\ r''$
shows $r <o\ r''$
⟨*proof*⟩

lemma *ordLess-not-embed*:
assumes $r <o\ r'$
shows $\neg(\exists f'.\ embed\ r'\ r\ f')$
⟨*proof*⟩

lemma *ordLess-Field*:

assumes *OL*: *r1* $<o$ *r2* **and** *EMB*: *embed r1 r2 f*
shows $\neg$ (*f*‘(*Field r1*) = *Field r2*)
⟨*proof*⟩

lemma *ordLess-iff*:
$r <o r' = (\textit{Well-order}\ r \wedge \textit{Well-order}\ r' \wedge \neg(\exists f'.\ \textit{embed}\ r'\ r\ f'))$
⟨*proof*⟩

lemma *ordLess-irreflexive*: $\neg\ r <o r$
⟨*proof*⟩

lemma *ordLeq-iff-ordLess-or-ordIso*:
$r \leq o\ r' = (r <o r' \vee r =o r')$
⟨*proof*⟩

lemma *ordIso-iff-ordLeq*:
$(r =o r') = (r \leq o\ r' \wedge r' \leq o\ r)$
⟨*proof*⟩

lemma *not-ordLess-ordLeq*:
$r <o r' \Longrightarrow \neg\ r' \leq o\ r$
⟨*proof*⟩

lemma *not-ordLeq-ordLess*:
$r \leq o\ r' \Longrightarrow \neg\ r' <o r$
⟨*proof*⟩

lemma *ordLess-or-ordLeq*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'*
shows $r <o r' \vee r' \leq o\ r$
⟨*proof*⟩

lemma *not-ordLess-ordIso*:
$r <o r' \Longrightarrow \neg\ r =o r'$
⟨*proof*⟩

lemma *not-ordLeq-iff-ordLess*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'*
shows $(\neg\ r' \leq o\ r) = (r <o r')$
⟨*proof*⟩

lemma *not-ordLess-iff-ordLeq*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'*
shows $(\neg\ r' <o r) = (r \leq o\ r')$
⟨*proof*⟩

lemma *ordLess-transitive*[*trans*]:
$[\![r <o r';\ r' <o r'']\!] \Longrightarrow r <o r''$
⟨*proof*⟩

corollary *ordLess-trans*: *trans ordLess*
⟨*proof*⟩

lemmas *ordIso-equivalence* = *ordIso-transitive ordIso-reflexive ordIso-symmetric*

lemma *ordIso-imp-ordLeq*:
$r =o\ r' \Longrightarrow r \leq o\ r'$
⟨*proof*⟩

lemma *ordLess-imp-ordLeq*:
$r <o\ r' \Longrightarrow r \leq o\ r'$
⟨*proof*⟩

lemma *ofilter-subset-ordLeq*:
assumes *WELL*: *Well-order r* **and**
OFA: *wo-rel.ofilter r A* **and** *OFB*: *wo-rel.ofilter r B*
shows $(A \leq B) = (Restr\ r\ A \leq o\ Restr\ r\ B)$
⟨*proof*⟩

lemma *ofilter-subset-ordLess*:
assumes *WELL*: *Well-order r* **and**
OFA: *wo-rel.ofilter r A* **and** *OFB*: *wo-rel.ofilter r B*
shows $(A < B) = (Restr\ r\ A <o\ Restr\ r\ B)$
⟨*proof*⟩

lemma *ofilter-ordLess*:
$[\![Well\text{-}order\ r;\ wo\text{-}rel.ofilter\ r\ A]\!] \Longrightarrow (A < Field\ r) = (Restr\ r\ A <o\ r)$
⟨*proof*⟩

corollary *underS-Restr-ordLess*:
assumes *Well-order r* **and** $Field\ r \neq \{\}$
shows $Restr\ r\ (underS\ r\ a) <o\ r$
⟨*proof*⟩

lemma *embed-ordLess-ofilterIncl*:
assumes
OL12: *r1* $<o$ *r2* **and** *OL23*: *r2* $<o$ *r3* **and**
EMB13: *embed r1 r3 f13* **and** *EMB23*: *embed r2 r3 f23*
shows $(f13`(Field\ r1),\ f23`(Field\ r2)) \in (ofilterIncl\ r3)$
⟨*proof*⟩

lemma *ordLess-iff-ordIso-Restr*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'*
shows $(r' <o\ r) = (\exists\, a \in Field\ r.\ r' =o\ Restr\ r\ (underS\ r\ a))$
⟨*proof*⟩

lemma *internalize-ordLess*:
$(r' <o\ r) = (\exists\, p.\ Field\ p < Field\ r \wedge r' =o\ p \wedge p <o\ r)$

⟨*proof*⟩

lemma *internalize-ordLeq*:
(*r'* ≤*o* *r*) = (∃ *p*. *Field p* ≤ *Field r* ∧ *r'* =*o* *p* ∧ *p* ≤*o* *r*)
⟨*proof*⟩

lemma *ordLeq-iff-ordLess-Restr*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'*
shows (*r* ≤*o* *r'*) = (∀ *a* ∈ *Field r*. *Restr r* (*underS r a*) <*o* *r'*)
⟨*proof*⟩

lemma *finite-ordLess-infinite*:
assumes *WELL*: *Well-order r* **and** *WELL'*: *Well-order r'* **and**
FIN: *finite*(*Field r*) **and** *INF*: ¬*finite*(*Field r'*)
shows *r* <*o* *r'*
⟨*proof*⟩

lemma *finite-well-order-on-ordIso*:
assumes *FIN*: *finite A* **and**
WELL: *well-order-on A r* **and** *WELL'*: *well-order-on A r'*
shows *r* =*o* *r'*
⟨*proof*⟩

29.5 <*o* is well-founded

Of course, it only makes sense to state that the <*o* is well-founded on the restricted type *'a rel rel*. We prove this by first showing that, for any set of well-orders all embedded in a fixed well-order, the function mapping each well-order in the set to an order filter of the fixed well-order is compatible w.r.t. to <*o* versus *strict inclusion*; and we already know that strict inclusion of order filters is well-founded.

definition *ord-to-filter* :: *'a rel* ⇒ *'a rel* ⇒ *'a set*
where *ord-to-filter r0 r* ≡ (*SOME f*. *embed r r0 f*) ‘ (*Field r*)

lemma *ord-to-filter-compat*:
compat (*ordLess Int* (*ordLess*$^{-1}$ ‘‘{*r0*} × *ordLess*$^{-1}$ ‘‘{*r0*}))
(*ofilterIncl r0*)
(*ord-to-filter r0*)
⟨*proof*⟩

theorem *wf-ordLess*: *wf ordLess*
⟨*proof*⟩

corollary *exists-minim-Well-order*:
assumes *NE*: *R* ≠ {} **and** *WELL*: ∀ *r* ∈ *R*. *Well-order r*
shows ∃ *r* ∈ *R*. ∀ *r'* ∈ *R*. *r* ≤*o* *r'*
⟨*proof*⟩

29.6 Copy via direct images

The direct image operator is the dual of the inverse image operator *inv-image* from *Relation.thy*. It is useful for transporting a well-order between different types.

definition *dir-image* :: *'a rel* ⇒ (*'a* ⇒ *'a'*) ⇒ *'a' rel*
  where
    dir-image r f = {(*f a*, *f b*)| *a b*. (*a*,*b*) ∈ *r*}

lemma *dir-image-Field*:
  Field(*dir-image r f*) = *f* ‘ (*Field r*)
  ⟨*proof*⟩

lemma *dir-image-minus-Id*:
  inj-on f (*Field r*) ⟹ (*dir-image r f*) − *Id* = *dir-image* (*r* − *Id*) *f*
  ⟨*proof*⟩

lemma *Refl-dir-image*:
  assumes *Refl r*
  shows *Refl*(*dir-image r f*)
⟨*proof*⟩

lemma *trans-dir-image*:
  assumes *TRANS*: *trans r* **and** *INJ*: *inj-on f* (*Field r*)
  shows *trans*(*dir-image r f*)
  ⟨*proof*⟩

lemma *Preorder-dir-image*:
  ⟦*Preorder r*; *inj-on f* (*Field r*)⟧ ⟹ *Preorder* (*dir-image r f*)
  ⟨*proof*⟩

lemma *antisym-dir-image*:
  assumes *AN*: *antisym r* **and** *INJ*: *inj-on f* (*Field r*)
  shows *antisym*(*dir-image r f*)
  ⟨*proof*⟩

lemma *Partial-order-dir-image*:
  ⟦*Partial-order r*; *inj-on f* (*Field r*)⟧ ⟹ *Partial-order* (*dir-image r f*)
  ⟨*proof*⟩

lemma *Total-dir-image*:
  assumes *TOT*: *Total r* **and** *INJ*: *inj-on f* (*Field r*)
  shows *Total*(*dir-image r f*)
⟨*proof*⟩

lemma *Linear-order-dir-image*:
  ⟦*Linear-order r*; *inj-on f* (*Field r*)⟧ ⟹ *Linear-order* (*dir-image r f*)
  ⟨*proof*⟩

lemma *wf-dir-image*:
assumes *WF*: *wf r* **and** *INJ*: *inj-on f* (*Field r*)
shows *wf*(*dir-image r f*)
⟨*proof*⟩

lemma *Well-order-dir-image*:
⟦*Well-order r*; *inj-on f* (*Field r*)⟧ ⟹ *Well-order* (*dir-image r f*)
⟨*proof*⟩

lemma *dir-image-bij-betw*:
⟦*inj-on f* (*Field r*)⟧ ⟹ *bij-betw f* (*Field r*) (*Field* (*dir-image r f*))
⟨*proof*⟩

lemma *dir-image-compat*:
compat r (*dir-image r f*) *f*
⟨*proof*⟩

lemma *dir-image-iso*:
⟦*Well-order r*; *inj-on f* (*Field r*)⟧ ⟹ *iso r* (*dir-image r f*) *f*
⟨*proof*⟩

lemma *dir-image-ordIso*:
⟦*Well-order r*; *inj-on f* (*Field r*)⟧ ⟹ *r* =*o dir-image r f*
⟨*proof*⟩

lemma *Well-order-iso-copy*:
assumes *WELL*: *well-order-on A r* **and** *BIJ*: *bij-betw f A A′*
shows ∃ *r′*. *well-order-on A′ r′* ∧ *r* =*o r′*
⟨*proof*⟩

29.7 Bounded square

This construction essentially defines, for an order relation r, a lexicographic order *bsqr r* on (*Field r*) × (*Field r*), applying the following criteria (in this order):

- compare the maximums;
- compare the first components;
- compare the second components.

The only application of this construction that we are aware of is at proving that the square of an infinite set has the same cardinal as that set. The essential property required there (and which is ensured by this construction) is that any proper order filter of the product order is included in a rectangle, i.e., in a product of proper filters on the original relation (assumed to be a well-order).

definition *bsqr* :: *'a rel => ('a * 'a)rel*
where
bsqr r = {((*a1*,*a2*),(*b1*,*b2*)).
{*a1*,*a2*,*b1*,*b2*} ≤ *Field r* ∧
(*a1* = *b1* ∧ *a2* = *b2* ∨
(*wo-rel.max2 r a1 a2*, *wo-rel.max2 r b1 b2*) ∈ *r* − *Id* ∨
wo-rel.max2 r a1 a2 = *wo-rel.max2 r b1 b2* ∧ (*a1*,*b1*) ∈ *r* − *Id* ∨
wo-rel.max2 r a1 a2 = *wo-rel.max2 r b1 b2* ∧ *a1* = *b1* ∧ (*a2*,*b2*) ∈ *r* − *Id*
)}

lemma *Field-bsqr*:
Field (*bsqr r*) = *Field r* × *Field r*
⟨*proof*⟩

lemma *bsqr-Refl*: *Refl*(*bsqr r*)
⟨*proof*⟩

lemma *bsqr-Trans*:
assumes *Well-order r*
shows *trans* (*bsqr r*)
⟨*proof*⟩

lemma *bsqr-antisym*:
assumes *Well-order r*
shows *antisym* (*bsqr r*)
⟨*proof*⟩

lemma *bsqr-Total*:
assumes *Well-order r*
shows *Total*(*bsqr r*)
⟨*proof*⟩

lemma *bsqr-Linear-order*:
assumes *Well-order r*
shows *Linear-order*(*bsqr r*)
⟨*proof*⟩

lemma *bsqr-Well-order*:
assumes *Well-order r*
shows *Well-order*(*bsqr r*)
⟨*proof*⟩

lemma *bsqr-max2*:
assumes *WELL*: *Well-order r* **and** *LEQ*: ((*a1*,*a2*),(*b1*,*b2*)) ∈ *bsqr r*
shows (*wo-rel.max2 r a1 a2*, *wo-rel.max2 r b1 b2*) ∈ *r*
⟨*proof*⟩

lemma *bsqr-ofilter*:

assumes *WELL*: *Well-order r* **and**
OF: *wo-rel.ofilter* (*bsqr r*) *D* **and** *SUB*: $D < Field\ r \times Field\ r$ **and**
NE: $\neg\ (\exists\, a.\ Field\ r = under\ r\ a)$
shows $\exists\, A.$ *wo-rel.ofilter r A* $\wedge$ $A < Field\ r \wedge D \leq A \times A$
⟨*proof*⟩

definition *Func* **where**
$Func\ A\ B = \{f\ .\ (\forall\ a \in A.\ f\ a \in B) \wedge (\forall\ a.\ a \notin A \longrightarrow f\ a = undefined)\}$

lemma *Func-empty*:
$Func\ \{\}\ B = \{\lambda x.\ undefined\}$
⟨*proof*⟩

lemma *Func-elim*:
assumes $g \in Func\ A\ B$ **and** $a \in A$
shows $\exists\ b.\ b \in B \wedge g\ a = b$
⟨*proof*⟩

definition *curr* **where**
$curr\ A\ f \equiv \lambda\ a.\ if\ a \in A\ then\ \lambda b.\ f\ (a,b)\ else\ undefined$

lemma *curr-in*:
assumes *f*: $f \in Func\ (A \times B)\ C$
shows $curr\ A\ f \in Func\ A\ (Func\ B\ C)$
⟨*proof*⟩

lemma *curr-inj*:
assumes $f1 \in Func\ (A \times B)\ C$ **and** $f2 \in Func\ (A \times B)\ C$
shows $curr\ A\ f1 = curr\ A\ f2 \longleftrightarrow f1 = f2$
⟨*proof*⟩

lemma *curr-surj*:
assumes $g \in Func\ A\ (Func\ B\ C)$
shows $\exists\ f \in Func\ (A \times B)\ C.\ curr\ A\ f = g$
⟨*proof*⟩

lemma *bij-betw-curr*:
$bij\text{-}betw\ (curr\ A)\ (Func\ (A \times B)\ C)\ (Func\ A\ (Func\ B\ C))$
⟨*proof*⟩

definition *Func-map* **where**
$Func\text{-}map\ B2\ f1\ f2\ g\ b2 \equiv if\ b2 \in B2\ then\ f1\ (g\ (f2\ b2))\ else\ undefined$

lemma *Func-map*:
assumes *g*: $g \in Func\ A2\ A1$ **and** *f1*: $f1\ `\ A1 \subseteq B1$ **and** *f2*: $f2\ `\ B2 \subseteq A2$
shows $Func\text{-}map\ B2\ f1\ f2\ g \in Func\ B2\ B1$
⟨*proof*⟩

lemma *Func-non-emp*:

assumes $B \neq \{\}$
shows *Func A B* $\neq \{\}$
⟨*proof*⟩

lemma *Func-is-emp*:
Func A B $= \{\} \longleftrightarrow A \neq \{\} \wedge B = \{\}$ (**is** *?L* $\longleftrightarrow$ *?R*)
⟨*proof*⟩

lemma *Func-map-surj*:
assumes *B1*: *f1* ‘ *A1* $=$ *B1* **and** *A2*: *inj-on f2 B2 f2* ‘ *B2* $\subseteq$ *A2*
and *B2A2*: *B2* $= \{\} \Longrightarrow$ *A2* $= \{\}$
shows *Func B2 B1* $=$ *Func-map B2 f1 f2* ‘ *Func A2 A1*
⟨*proof*⟩

end

30 Cardinal-Order Relations as Needed by Bounded Natural Functors

theory *BNF-Cardinal-Order-Relation*
imports *Zorn BNF-Wellorder-Constructions*
begin

In this section, we define cardinal-order relations to be minim well-orders on their field. Then we define the cardinal of a set to be *some* cardinal-order relation on that set, which will be unique up to order isomorphism. Then we study the connection between cardinals and:

- standard set-theoretic constructions: products, sums, unions, lists, powersets, set-of finite sets operator;
- finiteness and infiniteness (in particular, with the numeric cardinal operator for finite sets, *card*, from the theory *Finite-Sets.thy*).

On the way, we define the canonical ω cardinal and finite cardinals. We also define (again, up to order isomorphism) the successor of a cardinal, and show that any cardinal admits a successor.

Main results of this section are the existence of cardinal relations and the facts that, in the presence of infiniteness, most of the standard set-theoretic constructions (except for the powerset) *do not increase cardinality*. In particular, e.g., the set of words/lists over any infinite set has the same cardinality (hence, is in bijection) with that set.

30.1 Cardinal orders

A cardinal order in our setting shall be a well-order *minim* w.r.t. the order-embedding relation, $\leq o$ (which is the same as being *minimal* w.r.t. the strict

order-embedding relation, $<o$), among all the well-orders on its field.

definition *card-order-on* :: *'a set* ⇒ *'a rel* ⇒ *bool*
  where
    card-order-on A r ≡ *well-order-on A r* ∧ (∀ *r'*. *well-order-on A r'* ⟶ *r* ≤*o r'*)

abbreviation *Card-order r* ≡ *card-order-on* (*Field r*) *r*
abbreviation *card-order r* ≡ *card-order-on UNIV r*

lemma *card-order-on-well-order-on*:
  assumes *card-order-on A r*
  shows *well-order-on A r*
  ⟨*proof*⟩

lemma *card-order-on-Card-order*:
  card-order-on A r ⟹ *A* = *Field r* ∧ *Card-order r*
  ⟨*proof*⟩

The existence of a cardinal relation on any given set (which will mean that any set has a cardinal) follows from two facts:

- Zermelo's theorem (proved in *Zorn.thy* as theorem *well-order-on*), which states that on any given set there exists a well-order;
- The well-founded-ness of $<o$, ensuring that then there exists a minimal such well-order, i.e., a cardinal order.

theorem *card-order-on*: ∃ *r*. *card-order-on A r*
⟨*proof*⟩

lemma *card-order-on-ordIso*:
  assumes *CO*: *card-order-on A r* **and** *CO'*: *card-order-on A r'*
  shows *r* =*o r'*
  ⟨*proof*⟩

lemma *Card-order-ordIso*:
  assumes *CO*: *Card-order r* **and** *ISO*: *r'* =*o r*
  shows *Card-order r'*
  ⟨*proof*⟩

lemma *Card-order-ordIso2*:
  assumes *CO*: *Card-order r* **and** *ISO*: *r* =*o r'*
  shows *Card-order r'*
  ⟨*proof*⟩

30.2 Cardinal of a set

We define the cardinal of set to be *some* cardinal order on that set. We shall prove that this notion is unique up to order isomorphism, meaning that order isomorphism shall be the true identity of cardinals.

definition *card-of* :: *'a set* ⇒ *'a rel* (|-|)
where *card-of A* = (*SOME r*. *card-order-on A r*)

lemma *card-of-card-order-on*: *card-order-on A* |*A*|
⟨*proof*⟩

lemma *card-of-well-order-on*: *well-order-on A* |*A*|
⟨*proof*⟩

lemma *Field-card-of*: *Field* |*A*| = *A*
⟨*proof*⟩

lemma *card-of-Card-order*: *Card-order* |*A*|
⟨*proof*⟩

corollary *ordIso-card-of-imp-Card-order*:
r =*o* |*A*| ⟹ *Card-order r*
⟨*proof*⟩

lemma *card-of-Well-order*: *Well-order* |*A*|
⟨*proof*⟩

lemma *card-of-refl*: |*A*| =*o* |*A*|
⟨*proof*⟩

lemma *card-of-least*: *well-order-on A r* ⟹ |*A*| ≤*o r*
⟨*proof*⟩

lemma *card-of-ordIso*:
(∃*f*. *bij-betw f A B*) = (|*A*| =*o* |*B*|)
⟨*proof*⟩

lemma *card-of-ordLeq*:
(∃*f*. *inj-on f A* ∧ *f* ‘ *A* ≤ *B*) = (|*A*| ≤*o* |*B*|)
⟨*proof*⟩

lemma *card-of-ordLeq2*:
A ≠ {} ⟹ (∃ *g*. *g* ‘ *B* = *A*) = (|*A*| ≤*o* |*B*|)
⟨*proof*⟩

lemma *card-of-ordLess*:
(¬(∃*f*. *inj-on f A* ∧ *f* ‘ *A* ≤ *B*)) = (|*B*| <*o* |*A*|)
⟨*proof*⟩

lemma *card-of-ordLess2*:
B ≠ {} ⟹ (¬(∃*f*. *f* ‘ *A* = *B*)) = (|*A*| <*o* |*B*|)
⟨*proof*⟩

lemma *card-of-ordIsoI*:

assumes *bij-betw f A B*
shows $|A| =o |B|$
⟨*proof*⟩

lemma *card-of-ordLeqI*:
assumes *inj-on f A* **and** $\bigwedge a.\ a \in A \Longrightarrow f\ a \in B$
shows $|A| \leq o\ |B|$
⟨*proof*⟩

lemma *card-of-unique*:
card-order-on A r $\Longrightarrow$ $r =o |A|$
⟨*proof*⟩

lemma *card-of-mono1*:
$A \leq B \Longrightarrow |A| \leq o\ |B|$
⟨*proof*⟩

lemma *card-of-mono2*:
assumes $r \leq o\ r'$
shows $|Field\ r| \leq o\ |Field\ r'|$
⟨*proof*⟩

lemma *card-of-cong*: $r =o r' \Longrightarrow |Field\ r| =o |Field\ r'|$
⟨*proof*⟩

lemma *card-of-Field-ordIso*:
assumes *Card-order r*
shows $|Field\ r| =o r$
⟨*proof*⟩

lemma *Card-order-iff-ordIso-card-of*:
Card-order r = $(r =o |Field\ r|\ )$
⟨*proof*⟩

lemma *Card-order-iff-ordLeq-card-of*:
Card-order r = $(r \leq o\ |Field\ r|\ )$
⟨*proof*⟩

lemma *Card-order-iff-Restr-underS*:
assumes *Well-order r*
shows *Card-order r* = $(\forall\, a \in Field\ r.\ Restr\ r\ (underS\ r\ a) <o\ |Field\ r|\ )$
⟨*proof*⟩

lemma *card-of-underS*:
assumes *r*: *Card-order r* **and** *a*: $a \in Field\ r$
shows $|underS\ r\ a| <o\ r$
⟨*proof*⟩

lemma *ordLess-Field*:

assumes $r <o r'$
shows $|Field\ r| <o r'$
⟨*proof*⟩

lemma *internalize-card-of-ordLeq*:
$(\ |A| \leq o\ r) = (\exists B \leq Field\ r.\ |A| =o |B| \wedge |B| \leq o\ r)$
⟨*proof*⟩

lemma *internalize-card-of-ordLeq2*:
$(\ |A| \leq o\ |C|\ ) = (\exists B \leq C.\ |A| =o |B| \wedge |B| \leq o\ |C|\ )$
⟨*proof*⟩

30.3 Cardinals versus set operations on arbitrary sets

Here we embark in a long journey of simple results showing that the standard set-theoretic operations are well-behaved w.r.t. the notion of cardinal – essentially, this means that they preserve the "cardinal identity" $=o$ and are monotonic w.r.t. $\leq o$.

lemma *card-of-empty*: $|\{\}| \leq o\ |A|$
⟨*proof*⟩

lemma *card-of-empty1*:
assumes *Well-order r* $\vee$ *Card-order r*
shows $|\{\}| \leq o\ r$
⟨*proof*⟩

corollary *Card-order-empty*:
Card-order r $\Longrightarrow |\{\}| \leq o\ r$ ⟨*proof*⟩

lemma *card-of-empty2*:
assumes $|A| =o |\{\}|$
shows $A = \{\}$
⟨*proof*⟩

lemma *card-of-empty3*:
assumes $|A| \leq o |\{\}|$
shows $A = \{\}$
⟨*proof*⟩

lemma *card-of-empty-ordIso*:
$|\{\}::'a\ set| =o |\{\}::'b\ set|$
⟨*proof*⟩

lemma *card-of-image*:
$|f\ `\ A| \leq o\ |A|$
⟨*proof*⟩

lemma *surj-imp-ordLeq*:

assumes $B \subseteq f$ ‘ A
shows $|B| \leq o\ |A|$
⟨*proof*⟩

lemma *card-of-singl-ordLeq*:
assumes $A \neq \{\}$
shows $|\{b\}| \leq o\ |A|$
⟨*proof*⟩

corollary *Card-order-singl-ordLeq*:
$[\![Card\text{-}order\ r;\ Field\ r \neq \{\}]\!] \Longrightarrow |\{b\}| \leq o\ r$
⟨*proof*⟩

lemma *card-of-Pow*: $|A| <o\ |Pow\ A|$
⟨*proof*⟩

corollary *Card-order-Pow*:
$Card\text{-}order\ r \Longrightarrow r <o\ |Pow(Field\ r)|$
⟨*proof*⟩

lemma *card-of-Plus1*: $|A| \leq o\ |A <+> B|$ **and** *card-of-Plus2*: $|B| \leq o\ |A <+> B|$
⟨*proof*⟩

corollary *Card-order-Plus1*:
$Card\text{-}order\ r \Longrightarrow r \leq o\ |(Field\ r) <+> B|$
⟨*proof*⟩

corollary *Card-order-Plus2*:
$Card\text{-}order\ r \Longrightarrow r \leq o\ |A <+> (Field\ r)|$
⟨*proof*⟩

lemma *card-of-Plus-empty1*: $|A| =o\ |A <+> \{\}|$
⟨*proof*⟩

lemma *card-of-Plus-empty2*: $|A| =o\ |\{\} <+> A|$
⟨*proof*⟩

lemma *card-of-Plus-commute*: $|A <+> B| =o\ |B <+> A|$
⟨*proof*⟩

lemma *card-of-Plus-assoc*:
fixes $A :: 'a\ set$ **and** $B :: 'b\ set$ **and** $C :: 'c\ set$
shows $|(A <+> B) <+> C| =o\ |A <+> B <+> C|$
⟨*proof*⟩

lemma *card-of-Plus-mono1*:
assumes $|A| \leq o\ |B|$
shows $|A <+> C| \leq o\ |B <+> C|$
⟨*proof*⟩

corollary *ordLeq-Plus-mono1*:
assumes $r \leq o\ r'$
shows $|(Field\ r) <+> C| \leq o\ |(Field\ r') <+> C|$
⟨*proof*⟩

lemma *card-of-Plus-mono2*:
assumes $|A| \leq o\ |B|$
shows $|C <+> A| \leq o\ |C <+> B|$
⟨*proof*⟩

corollary *ordLeq-Plus-mono2*:
assumes $r \leq o\ r'$
shows $|A <+> (Field\ r)| \leq o\ |A <+> (Field\ r')|$
⟨*proof*⟩

lemma *card-of-Plus-mono*:
assumes $|A| \leq o\ |B|$ **and** $|C| \leq o\ |D|$
shows $|A <+> C| \leq o\ |B <+> D|$
⟨*proof*⟩

corollary *ordLeq-Plus-mono*:
assumes $r \leq o\ r'$ **and** $p \leq o\ p'$
shows $|(Field\ r) <+> (Field\ p)| \leq o\ |(Field\ r') <+> (Field\ p')|$
⟨*proof*⟩

lemma *card-of-Plus-cong1*:
assumes $|A| =o\ |B|$
shows $|A <+> C| =o\ |B <+> C|$
⟨*proof*⟩

corollary *ordIso-Plus-cong1*:
assumes $r =o\ r'$
shows $|(Field\ r) <+> C| =o\ |(Field\ r') <+> C|$
⟨*proof*⟩

lemma *card-of-Plus-cong2*:
assumes $|A| =o\ |B|$
shows $|C <+> A| =o\ |C <+> B|$
⟨*proof*⟩

corollary *ordIso-Plus-cong2*:
assumes $r =o\ r'$
shows $|A <+> (Field\ r)| =o\ |A <+> (Field\ r')|$
⟨*proof*⟩

lemma *card-of-Plus-cong*:
assumes $|A| =o\ |B|$ **and** $|C| =o\ |D|$
shows $|A <+> C| =o\ |B <+> D|$

⟨*proof*⟩

corollary *ordIso-Plus-cong*:
assumes $r =o r'$ **and** $p =o p'$
shows $|(Field\ r) <+> (Field\ p)| =o |(Field\ r') <+> (Field\ p')|$
⟨*proof*⟩

lemma *card-of-Un-Plus-ordLeq*:
$|A \cup B| \leq o\ |A <+> B|$
⟨*proof*⟩

lemma *card-of-Times1*:
assumes $A \neq \{\}$
shows $|B| \leq o\ |B \times A|$
⟨*proof*⟩

lemma *card-of-Times-commute*: $|A \times B| =o |B \times A|$
⟨*proof*⟩

lemma *card-of-Times2*:
assumes $A \neq \{\}$ **shows** $|B| \leq o\ |A \times B|$
⟨*proof*⟩

corollary *Card-order-Times1*:
$[\![Card\text{-}order\ r;\ B \neq \{\}]\!] \Longrightarrow r \leq o\ |(Field\ r) \times B|$
⟨*proof*⟩

corollary *Card-order-Times2*:
$[\![Card\text{-}order\ r;\ A \neq \{\}]\!] \Longrightarrow r \leq o\ |A \times (Field\ r)|$
⟨*proof*⟩

lemma *card-of-Times3*: $|A| \leq o\ |A \times A|$
⟨*proof*⟩

lemma *card-of-Plus-Times-bool*: $|A <+> A| =o |A \times (UNIV::bool\ set)|$
⟨*proof*⟩

lemma *card-of-Times-mono1*:
assumes $|A| \leq o\ |B|$
shows $|A \times C| \leq o\ |B \times C|$
⟨*proof*⟩

corollary *ordLeq-Times-mono1*:
assumes $r \leq o\ r'$
shows $|(Field\ r) \times C| \leq o\ |(Field\ r') \times C|$
⟨*proof*⟩

lemma *card-of-Times-mono2*:
assumes $|A| \leq o\ |B|$

shows $|C \times A| \leq o\ |C \times B|$
⟨*proof*⟩

corollary *ordLeq-Times-mono2*:
assumes $r \leq o\ r'$
shows $|A \times (Field\ r)| \leq o\ |A \times (Field\ r')|$
⟨*proof*⟩

lemma *card-of-Sigma-mono1*:
assumes $\forall i \in I.\ |A\ i| \leq o\ |B\ i|$
shows $|SIGMA\ i : I.\ A\ i| \leq o\ |SIGMA\ i : I.\ B\ i|$
⟨*proof*⟩

lemma *card-of-UNION-Sigma*:
$|\bigcup i \in I.\ A\ i| \leq o\ |SIGMA\ i : I.\ A\ i|$
⟨*proof*⟩

lemma *card-of-bool*:
assumes $a1 \neq a2$
shows $|UNIV{::}bool\ set| =o\ |\{a1,a2\}|$
⟨*proof*⟩

lemma *card-of-Plus-Times-aux*:
assumes *A2*: $a1 \neq a2 \wedge \{a1,a2\} \leq A$ **and**
LEQ: $|A| \leq o\ |B|$
shows $|A <+> B| \leq o\ |A \times B|$
⟨*proof*⟩

lemma *card-of-Plus-Times*:
assumes *A2*: $a1 \neq a2 \wedge \{a1,a2\} \leq A$ **and** *B2*: $b1 \neq b2 \wedge \{b1,b2\} \leq B$
shows $|A <+> B| \leq o\ |A \times B|$
⟨*proof*⟩

lemma *card-of-Times-Plus-distrib*:
$|A \times (B <+> C)| =o\ |A \times B <+> A \times C|$ (**is** $|?RHS| =o\ |?LHS|$)
⟨*proof*⟩

lemma *card-of-ordLeq-finite*:
assumes $|A| \leq o\ |B|$ **and** *finite B*
shows *finite A*
⟨*proof*⟩

lemma *card-of-ordLeq-infinite*:
assumes $|A| \leq o\ |B|$ **and** ¬ *finite A*
shows ¬ *finite B*
⟨*proof*⟩

lemma *card-of-ordIso-finite*:
assumes $|A| =o\ |B|$

shows *finite A = finite B*
⟨*proof*⟩

lemma *card-of-ordIso-finite-Field*:
assumes *Card-order r* **and** *r* =*o* |*A*|
shows *finite*(*Field r*) = *finite A*
⟨*proof*⟩

30.4 Cardinals versus set operations involving infinite sets

Here we show that, for infinite sets, most set-theoretic constructions do not increase the cardinality. The cornerstone for this is theorem *Card-order-Times-same-infinite*, which states that self-product does not increase cardinality – the proof of this fact adapts a standard set-theoretic argument, as presented, e.g., in the proof of theorem 1.5.11 at page 47 in [4]. Then everything else follows fairly easily.

lemma *infinite-iff-card-of-nat*:
¬ *finite A* ⟷ (|*UNIV*::*nat set*| ≤*o* |*A*|)
⟨*proof*⟩

The next two results correspond to the ZF fact that all infinite cardinals are limit ordinals:

lemma *Card-order-infinite-not-under*:
assumes *CARD*: *Card-order r* **and** *INF*: ¬*finite* (*Field r*)
shows ¬ (∃ *a*. *Field r* = *under r a*)
⟨*proof*⟩

lemma *infinite-Card-order-limit*:
assumes *r*: *Card-order r* **and** ¬*finite* (*Field r*)
and *a*: *a* ∈ *Field r*
shows ∃ *b* ∈ *Field r*. *a* ≠ *b* ∧ (*a*,*b*) ∈ *r*
⟨*proof*⟩

theorem *Card-order-Times-same-infinite*:
assumes *CO*: *Card-order r* **and** *INF*: ¬*finite*(*Field r*)
shows |*Field r* × *Field r*| ≤*o r*
⟨*proof*⟩

corollary *card-of-Times-same-infinite*:
assumes ¬*finite A*
shows |*A* × *A*| =*o* |*A*|
⟨*proof*⟩

lemma *card-of-Times-infinite*:
assumes *INF*: ¬*finite A* **and** *NE*: *B* ≠ {} **and** *LEQ*: |*B*| ≤*o* |*A*|
shows |*A* × *B*| =*o* |*A*| ∧ |*B* × *A*| =*o* |*A*|
⟨*proof*⟩

corollary *Card-order-Times-infinite*:
assumes *INF*: ¬*finite*(*Field r*) **and** *CARD*: *Card-order r* **and**
NE: *Field p* ≠ {} **and** *LEQ*: *p* ≤*o r*
shows | (*Field r*) × (*Field p*) | =*o r* ∧ | (*Field p*) × (*Field r*) | =*o r*
⟨*proof*⟩

lemma *card-of-Sigma-ordLeq-infinite*:
assumes *INF*: ¬*finite B* **and**
LEQ-I: |*I*| ≤*o* |*B*| **and** *LEQ*: ∀ *i* ∈ *I*. |*A i*| ≤*o* |*B*|
shows |*SIGMA i* : *I*. *A i*| ≤*o* |*B*|
⟨*proof*⟩

lemma *card-of-Sigma-ordLeq-infinite-Field*:
assumes *INF*: ¬*finite* (*Field r*) **and** *r*: *Card-order r* **and**
LEQ-I: |*I*| ≤*o r* **and** *LEQ*: ∀ *i* ∈ *I*. |*A i*| ≤*o r*
shows |*SIGMA i* : *I*. *A i*| ≤*o r*
⟨*proof*⟩

lemma *card-of-Times-ordLeq-infinite-Field*:
⟦¬*finite* (*Field r*); |*A*| ≤*o r*; |*B*| ≤*o r*; *Card-order r*⟧ ⟹ |*A* × *B*| ≤*o r*
⟨*proof*⟩

lemma *card-of-Times-infinite-simps*:
⟦¬*finite A*; *B* ≠ {}; |*B*| ≤*o* |*A*|⟧ ⟹ |*A* × *B*| =*o* |*A*|
⟦¬*finite A*; *B* ≠ {}; |*B*| ≤*o* |*A*|⟧ ⟹ |*A*| =*o* |*A* × *B*|
⟦¬*finite A*; *B* ≠ {}; |*B*| ≤*o* |*A*|⟧ ⟹ |*B* × *A*| =*o* |*A*|
⟦¬*finite A*; *B* ≠ {}; |*B*| ≤*o* |*A*|⟧ ⟹ |*A*| =*o* |*B* × *A*|
⟨*proof*⟩

lemma *card-of-UNION-ordLeq-infinite*:
assumes *INF*: ¬*finite B* **and** *LEQ-I*: |*I*| ≤*o* |*B*| **and** *LEQ*: ∀ *i* ∈ *I*. |*A i*| ≤*o* |*B*|
shows |⋃ *i* ∈ *I*. *A i*| ≤*o* |*B*|
⟨*proof*⟩

corollary *card-of-UNION-ordLeq-infinite-Field*:
assumes *INF*: ¬*finite* (*Field r*) **and** *r*: *Card-order r* **and**
LEQ-I: |*I*| ≤*o r* **and** *LEQ*: ∀ *i* ∈ *I*. |*A i*| ≤*o r*
shows |⋃ *i* ∈ *I*. *A i*| ≤*o r*
⟨*proof*⟩

lemma *card-of-Plus-infinite1*:
assumes *INF*: ¬*finite A* **and** *LEQ*: |*B*| ≤*o* |*A*|
shows |*A* <+> *B*| =*o* |*A*|
⟨*proof*⟩

lemma *card-of-Plus-infinite2*:
assumes *INF*: ¬*finite A* **and** *LEQ*: |*B*| ≤*o* |*A*|
shows |*B* <+> *A*| =*o* |*A*|
⟨*proof*⟩

lemma *card-of-Plus-infinite*:
assumes *INF*: ¬*finite A* **and** *LEQ*: $|B| \leq o\ |A|$
shows $|A <+> B| =o\ |A| \wedge |B <+> A| =o\ |A|$
⟨*proof*⟩

corollary *Card-order-Plus-infinite*:
assumes *INF*: ¬*finite*(*Field r*) **and** *CARD*: *Card-order r* **and**
LEQ: $p \leq o\ r$
shows $|\ (Field\ r) <+> (Field\ p)\ | =o\ r \wedge |\ (Field\ p) <+> (Field\ r)\ | =o\ r$
⟨*proof*⟩

30.5 The cardinal ω and the finite cardinals

The cardinal ω, of natural numbers, shall be the standard non-strict order relation on *nat*, that we abbreviate by *natLeq*. The finite cardinals shall be the restrictions of these relations to the numbers smaller than fixed numbers n, that we abbreviate by *natLeq-on n*.

definition (*natLeq*::(*nat* ∗ *nat*) *set*) ≡ $\{(x,y).\ x \leq y\}$
definition (*natLess*::(*nat* ∗ *nat*) *set*) ≡ $\{(x,y).\ x < y\}$

abbreviation *natLeq-on* :: *nat* ⇒ (*nat* ∗ *nat*) *set*
where *natLeq-on n* ≡ $\{(x,y).\ x < n \wedge y < n \wedge x \leq y\}$

lemma *infinite-cartesian-product*:
assumes ¬*finite A* ¬*finite B*
shows ¬*finite* (*A* × *B*)
⟨*proof*⟩

30.5.1 First as well-orders

lemma *Field-natLeq*: *Field natLeq* = (*UNIV*::*nat set*)
⟨*proof*⟩

lemma *natLeq-Refl*: *Refl natLeq*
⟨*proof*⟩

lemma *natLeq-trans*: *trans natLeq*
⟨*proof*⟩

lemma *natLeq-Preorder*: *Preorder natLeq*
⟨*proof*⟩

lemma *natLeq-antisym*: *antisym natLeq*
⟨*proof*⟩

lemma *natLeq-Partial-order*: *Partial-order natLeq*
⟨*proof*⟩

lemma *natLeq-Total*: *Total natLeq*
⟨*proof*⟩

lemma *natLeq-Linear-order*: *Linear-order natLeq*
⟨*proof*⟩

lemma *natLeq-natLess-Id*: *natLess* = *natLeq* − *Id*
⟨*proof*⟩

lemma *natLeq-Well-order*: *Well-order natLeq*
⟨*proof*⟩

lemma *Field-natLeq-on*: *Field* (*natLeq-on n*) = {*x*. *x* < *n*}
⟨*proof*⟩

lemma *natLeq-underS-less*: *underS natLeq n* = {*x*. *x* < *n*}
⟨*proof*⟩

lemma *Restr-natLeq*: *Restr natLeq* {*x*. *x* < *n*} = *natLeq-on n*
⟨*proof*⟩

lemma *Restr-natLeq2*:
Restr natLeq (*underS natLeq n*) = *natLeq-on n*
⟨*proof*⟩

lemma *natLeq-on-Well-order*: *Well-order*(*natLeq-on n*)
⟨*proof*⟩

corollary *natLeq-on-well-order-on*: *well-order-on* {*x*. *x* < *n*} (*natLeq-on n*)
⟨*proof*⟩

lemma *natLeq-on-wo-rel*: *wo-rel*(*natLeq-on n*)
⟨*proof*⟩

30.5.2 Then as cardinals

lemma *natLeq-Card-order*: *Card-order natLeq*
⟨*proof*⟩

corollary *card-of-Field-natLeq*:
|*Field natLeq*| =*o natLeq*
⟨*proof*⟩

corollary *card-of-nat*:
|*UNIV*::*nat set*| =*o natLeq*
⟨*proof*⟩

corollary *infinite-iff-natLeq-ordLeq*:
¬*finite A* = (*natLeq* ≤*o* |*A*|)

⟨*proof*⟩

corollary *finite-iff-ordLess-natLeq*:
finite A = (|*A*| <*o natLeq*)
⟨*proof*⟩

30.6 The successor of a cardinal

First we define *isCardSuc r r′*, the notion of r' being a successor cardinal of r. Although the definition does not require r to be a cardinal, only this case will be meaningful.

definition *isCardSuc* :: *'a rel* ⇒ *'a set rel* ⇒ *bool*
where
isCardSuc r r′ ≡
Card-order r′ ∧ *r* <*o r′* ∧
(∀ (*r″*::*'a set rel*). *Card-order r″* ∧ *r* <*o r″* ⟶ *r′* ≤*o r″*)

Now we introduce the cardinal-successor operator *cardSuc*, by picking *some* cardinal-order relation fulfilling *isCardSuc*. Again, the picked item shall be proved unique up to order-isomorphism.

definition *cardSuc* :: *'a rel* ⇒ *'a set rel*
where *cardSuc r* ≡ *SOME r′*. *isCardSuc r r′*

lemma *exists-minim-Card-order*:
⟦*R* ≠ {}; ∀ *r* ∈ *R*. *Card-order r*⟧ ⟹ ∃ *r* ∈ *R*. ∀ *r′* ∈ *R*. *r* ≤*o r′*
⟨*proof*⟩

lemma *exists-isCardSuc*:
assumes *Card-order r*
shows ∃ *r′*. *isCardSuc r r′*
⟨*proof*⟩

lemma *cardSuc-isCardSuc*:
assumes *Card-order r*
shows *isCardSuc r* (*cardSuc r*)
⟨*proof*⟩

lemma *cardSuc-Card-order*:
Card-order r ⟹ *Card-order*(*cardSuc r*)
⟨*proof*⟩

lemma *cardSuc-greater*:
Card-order r ⟹ *r* <*o cardSuc r*
⟨*proof*⟩

lemma *cardSuc-ordLeq*:
Card-order r ⟹ *r* ≤*o cardSuc r*
⟨*proof*⟩

The minimality property of *cardSuc* originally present in its definition is local to the type *'a set rel*, i.e., that of *cardSuc r*:

lemma *cardSuc-least-aux*:
⟦*Card-order* (*r*::*'a rel*); *Card-order* (*r'*::*'a set rel*); *r* <*o r'*⟧ ⟹ *cardSuc r* ≤*o r'*
⟨*proof*⟩

But from this we can infer general minimality:

lemma *cardSuc-least*:
assumes *CARD*: *Card-order r* **and** *CARD'*: *Card-order r'* **and** *LESS*: *r* <*o r'*
shows *cardSuc r* ≤*o r'*
⟨*proof*⟩

lemma *cardSuc-ordLess-ordLeq*:
assumes *CARD*: *Card-order r* **and** *CARD'*: *Card-order r'*
shows (*r* <*o r'*) = (*cardSuc r* ≤*o r'*)
⟨*proof*⟩

lemma *cardSuc-ordLeq-ordLess*:
assumes *CARD*: *Card-order r* **and** *CARD'*: *Card-order r'*
shows (*r'* <*o cardSuc r*) = (*r'* ≤*o r*)
⟨*proof*⟩

lemma *cardSuc-mono-ordLeq*:
assumes *CARD*: *Card-order r* **and** *CARD'*: *Card-order r'*
shows (*cardSuc r* ≤*o cardSuc r'*) = (*r* ≤*o r'*)
⟨*proof*⟩

lemma *cardSuc-invar-ordIso*:
assumes *CARD*: *Card-order r* **and** *CARD'*: *Card-order r'*
shows (*cardSuc r* =*o cardSuc r'*) = (*r* =*o r'*)
⟨*proof*⟩

lemma *card-of-cardSuc-finite*:
finite(*Field*(*cardSuc* |*A*|)) = *finite A*
⟨*proof*⟩

lemma *cardSuc-finite*:
assumes *Card-order r*
shows *finite* (*Field* (*cardSuc r*)) = *finite* (*Field r*)
⟨*proof*⟩

lemma *Field-cardSuc-not-empty*:
assumes *Card-order r*
shows *Field* (*cardSuc r*) ≠ {}
⟨*proof*⟩

typedef *'a suc* = *Field* (*cardSuc* |*UNIV* :: *'a set*|)
⟨*proof*⟩

definition *card-suc* :: *'a rel* ⇒ *'a suc rel* **where**
card-suc ≡ λ-. *map-prod Abs-suc Abs-suc* ‘ *cardSuc* |*UNIV* :: *'a set*|

lemma *Field-card-suc*: *Field* (*card-suc r*) = *UNIV*
⟨*proof*⟩

lemma *card-suc-alt*: *card-suc r* = *dir-image* (*cardSuc* |*UNIV* :: *'a set*|) *Abs-suc*
⟨*proof*⟩

lemma *cardSuc-Well-order*: *Card-order r* ⟹ *Well-order*(*cardSuc r*)
⟨*proof*⟩

lemma *cardSuc-ordIso-card-suc*:
assumes *card-order r*
shows *cardSuc r* =*o card-suc* (*r* :: *'a rel*)
⟨*proof*⟩

lemma *Card-order-card-suc*: *card-order r* ⟹ *Card-order* (*card-suc r*)
⟨*proof*⟩

lemma *card-order-card-suc*: *card-order r* ⟹ *card-order* (*card-suc r*)
⟨*proof*⟩

lemma *card-suc-greater*: *card-order r* ⟹ *r* <*o card-suc r*
⟨*proof*⟩

lemma *card-of-Plus-ordLess-infinite*:
assumes *INF*: ¬*finite C* **and** *LESS1*: |*A*| <*o* |*C*| **and** *LESS2*: |*B*| <*o* |*C*|
shows |*A* <+> *B*| <*o* |*C*|
⟨*proof*⟩

lemma *card-of-Plus-ordLess-infinite-Field*:
assumes *INF*: ¬*finite* (*Field r*) **and** *r*: *Card-order r* **and**
LESS1: |*A*| <*o r* **and** *LESS2*: |*B*| <*o r*
shows |*A* <+> *B*| <*o r*
⟨*proof*⟩

lemma *card-of-Plus-ordLeq-infinite-Field*:
assumes *r*: ¬*finite* (*Field r*) **and** *A*: |*A*| ≤*o r* **and** *B*: |*B*| ≤*o r*
and *c*: *Card-order r*
shows |*A* <+> *B*| ≤*o r*
⟨*proof*⟩

lemma *card-of-Un-ordLeq-infinite-Field*:
assumes *C*: ¬*finite* (*Field r*) **and** *A*: |*A*| ≤*o r* **and** *B*: |*B*| ≤*o r*
and *Card-order r*
shows |*A Un B*| ≤*o r*
⟨*proof*⟩

lemma *card-of-Un-ordLess-infinite*:
assumes *INF*: ¬*finite C* **and**
LESS1: |*A*| <*o* |*C*| **and** *LESS2*: |*B*| <*o* |*C*|
shows |*A* ∪ *B*| <*o* |*C*|
⟨*proof*⟩

lemma *card-of-Un-ordLess-infinite-Field*:
assumes *INF*: ¬*finite* (*Field r*) **and** *r*: *Card-order r* **and**
LESS1: |*A*| <*o r* **and** *LESS2*: |*B*| <*o r*
shows |*A Un B*| <*o r*
⟨*proof*⟩

30.7 Regular cardinals

definition *cofinal* **where**
cofinal A r ≡ ∀ *a* ∈ *Field r*. ∃ *b* ∈ *A*. *a* ≠ *b* ∧ (*a*,*b*) ∈ *r*

definition *regularCard* **where**
regularCard r ≡ ∀ *K*. *K* ≤ *Field r* ∧ *cofinal K r* ⟶ |*K*| =*o r*

definition *relChain* **where**
relChain r As ≡ ∀ *i j*. (*i*,*j*) ∈ *r* ⟶ *As i* ≤ *As j*

lemma *regularCard-UNION*:
assumes *r*: *Card-order r* *regularCard r*
and *As*: *relChain r As*
and *Bsub*: *B* ≤ (⋃ *i* ∈ *Field r*. *As i*)
and *cardB*: |*B*| <*o r*
shows ∃ *i* ∈ *Field r*. *B* ≤ *As i*
⟨*proof*⟩

lemma *infinite-cardSuc-regularCard*:
assumes *r-inf*: ¬*finite* (*Field r*) **and** *r-card*: *Card-order r*
shows *regularCard* (*cardSuc r*)
⟨*proof*⟩

lemma *cardSuc-UNION*:
assumes *r*: *Card-order r* **and** ¬*finite* (*Field r*)
and *As*: *relChain* (*cardSuc r*) *As*
and *Bsub*: *B* ≤ (⋃ *i* ∈ *Field* (*cardSuc r*). *As i*)
and *cardB*: |*B*| ≤*o r*
shows ∃ *i* ∈ *Field* (*cardSuc r*). *B* ≤ *As i*
⟨*proof*⟩

30.8 Others

lemma *card-of-Func-Times*:
|*Func* (*A* × *B*) *C*| =*o* |*Func A* (*Func B C*)|
⟨*proof*⟩

lemma *card-of-Pow-Func*:
$|Pow\ A| =o |Func\ A\ (UNIV::bool\ set)|$
⟨*proof*⟩

lemma *card-of-Func-UNIV*:
$|Func\ (UNIV::'a\ set)\ (B::'b\ set)| =o |\{f::'a \Rightarrow 'b.\ range\ f \subseteq B\}|$
⟨*proof*⟩

lemma *Func-Times-Range*:
$|Func\ A\ (B \times C)| =o |Func\ A\ B \times Func\ A\ C|$ (**is** $|?LHS| =o |?RHS|$)
⟨*proof*⟩

30.9 Regular vs. stable cardinals

definition *stable* :: $'a\ rel \Rightarrow bool$
where
$stable\ r \equiv \forall (A::'a\ set)\ (F :: 'a \Rightarrow 'a\ set).$
$|A| <o r \wedge (\forall a \in A.\ |F\ a| <o r)$
$\longrightarrow |SIGMA\ a : A.\ F\ a| <o r$

lemma *regularCard-stable*:
assumes *cr*: *Card-order r* **and** *ir*: ¬*finite* (*Field r*) **and** *reg*: *regularCard r*
shows *stable r*
⟨*proof*⟩

lemma *stable-regularCard*:
assumes *cr*: *Card-order r* **and** *ir*: ¬*finite* (*Field r*) **and** *st*: *stable r*
shows *regularCard r*
⟨*proof*⟩

lemma *internalize-card-of-ordLess*:
$(|A| <o r) = (\exists B < Field\ r.\ |A| =o |B| \wedge |B| <o r)$
⟨*proof*⟩

lemma *card-of-Sigma-cong1*:
assumes $\forall i \in I.\ |A\ i| =o |B\ i|$
shows $|SIGMA\ i : I.\ A\ i| =o |SIGMA\ i : I.\ B\ i|$
⟨*proof*⟩

lemma *card-of-Sigma-cong2*:
assumes *bij-betw f* $(I::'i\ set)\ (J::'j\ set)$
shows $|SIGMA\ i : I.\ (A::'j \Rightarrow 'a\ set)\ (f\ i)| =o |SIGMA\ j : J.\ A\ j|$
⟨*proof*⟩

lemma *card-of-Sigma-cong*:
assumes *BIJ*: *bij-betw f I J* **and** *ISO*: $\forall j \in J.\ |A\ j| =o |B\ j|$
shows $|SIGMA\ i : I.\ A\ (f\ i)| =o |SIGMA\ j : J.\ B\ j|$
⟨*proof*⟩

lemma *stable-elim*:
assumes *ST*: *stable r* **and** *A-LESS*: $|A| <o r$ **and**
F-LESS: $\bigwedge a.\ a \in A \Longrightarrow |F\ a| <o r$
shows $|SIGMA\ a : A.\ F\ a| <o r$
⟨*proof*⟩

lemma *stable-natLeq*: *stable natLeq*
⟨*proof*⟩

corollary *regularCard-natLeq*: *regularCard natLeq*
⟨*proof*⟩

lemma *stable-ordIso1*:
assumes *ST*: *stable r* **and** *ISO*: $r' =o r$
shows *stable r'*
⟨*proof*⟩

lemma *stable-UNION*:
assumes *stable r* **and** $|A| <o r$ **and** $\bigwedge a.\ a \in A \Longrightarrow |F\ a| <o r$
shows $|\bigcup a \in A.\ F\ a| <o r$
⟨*proof*⟩

corollary *card-of-UNION-ordLess-infinite*:
assumes *stable* $|B|$ **and** $|I| <o |B|$ **and** $\forall i \in I.\ |A\ i| <o |B|$
shows $|\bigcup i \in I.\ A\ i| <o |B|$
⟨*proof*⟩

corollary *card-of-UNION-ordLess-infinite-Field*:
assumes *ST*: *stable r* **and** *r*: *Card-order r* **and**
LEQ-I: $|I| <o r$ **and** *LEQ*: $\forall i \in I.\ |A\ i| <o r$
shows $|\bigcup i \in I.\ A\ i| <o r$
⟨*proof*⟩

end

31 Cardinal Arithmetic as Needed by Bounded Natural Functors

theory *BNF-Cardinal-Arithmetic*
imports *BNF-Cardinal-Order-Relation*
begin

lemma *dir-image*: $[\![\bigwedge x\ y.\ (f\ x = f\ y) = (x = y);$ *Card-order r*$]\!] \Longrightarrow r =o$ *dir-image r f*
⟨*proof*⟩

lemma *card-order-dir-image*:

assumes *bij*: *bij f* **and** *co*: *card-order r*
shows *card-order* (*dir-image r f*)
⟨*proof*⟩

lemma *ordIso-refl*: *Card-order r* ⟹ *r* =*o r*
⟨*proof*⟩

lemma *ordLeq-refl*: *Card-order r* ⟹ *r* ≤*o r*
⟨*proof*⟩

lemma *card-of-ordIso-subst*: *A* = *B* ⟹ |*A*| =*o* |*B*|
⟨*proof*⟩

lemma *Field-card-order*: *card-order r* ⟹ *Field r* = *UNIV*
⟨*proof*⟩

31.1 Zero

definition *czero* **where**
czero = *card-of* {}

lemma *czero-ordIso*: *czero* =*o czero*
⟨*proof*⟩

lemma *card-of-ordIso-czero-iff-empty*:
|*A*| =*o* (*czero* :: 'b *rel*) ⟷ *A* = ({} :: 'a *set*)
⟨*proof*⟩

abbreviation *Cnotzero* **where**
Cnotzero (*r* :: 'a *rel*) ≡ ¬(*r* =*o* (*czero* :: 'a *rel*)) ∧ *Card-order r*

lemma *Cnotzero-imp-not-empty*: *Cnotzero r* ⟹ *Field r* ≠ {}
⟨*proof*⟩

lemma *czeroI*:
⟦*Card-order r*; *Field r* = {}⟧ ⟹ *r* =*o czero*
⟨*proof*⟩

lemma *czeroE*:
r =*o czero* ⟹ *Field r* = {}
⟨*proof*⟩

lemma *Cnotzero-mono*:
⟦*Cnotzero r*; *Card-order q*; *r* ≤*o q*⟧ ⟹ *Cnotzero q*
⟨*proof*⟩

31.2 (In)finite cardinals

definition *cinfinite* **where**
cinfinite r ≡ (¬ *finite* (*Field r*))

abbreviation *Cinfinite* **where**
Cinfinite r ≡ *cinfinite r* ∧ *Card-order r*

definition *cfinite* **where**
cfinite r = *finite* (*Field r*)

abbreviation *Cfinite* **where**
Cfinite r ≡ *cfinite r* ∧ *Card-order r*

lemma *Cfinite-ordLess-Cinfinite*: ⟦*Cfinite r*; *Cinfinite s*⟧ ⟹ *r* <*o s*
⟨*proof*⟩

lemmas *natLeq-card-order* = *natLeq-Card-order*[*unfolded Field-natLeq*]

lemma *natLeq-cinfinite*: *cinfinite natLeq*
⟨*proof*⟩

lemma *natLeq-Cinfinite*: *Cinfinite natLeq*
⟨*proof*⟩

lemma *natLeq-ordLeq-cinfinite*:
assumes *inf*: *Cinfinite r*
shows *natLeq* ≤*o r*
⟨*proof*⟩

lemma *cinfinite-not-czero*: *cinfinite r* ⟹ ¬ (*r* =*o* (*czero* :: *'a rel*))
⟨*proof*⟩

lemma *Cinfinite-Cnotzero*: *Cinfinite r* ⟹ *Cnotzero r*
⟨*proof*⟩

lemma *Cinfinite-cong*: ⟦*r1* =*o r2*; *Cinfinite r1*⟧ ⟹ *Cinfinite r2*
⟨*proof*⟩

lemma *cinfinite-mono*: ⟦*r1* ≤*o r2*; *cinfinite r1*⟧ ⟹ *cinfinite r2*
⟨*proof*⟩

lemma *regularCard-ordIso*:
assumes *k* =*o k'* **and** *Cinfinite k* **and** *regularCard k*
shows *regularCard k'*
⟨*proof*⟩

corollary *card-of-UNION-ordLess-infinite-Field-regularCard*:
assumes *regularCard r* **and** *Cinfinite r* **and** |*I*| <*o r* **and** ∀ *i* ∈ *I*. |*A i*| <*o r*
shows |⋃ *i* ∈ *I*. *A i*| <*o r*

⟨*proof*⟩

31.3 Binary sum

definition *csum* (**infixr** +*c* 65)
  where *r1* +*c* *r2* ≡ |*Field r1* <+> *Field r2*|

lemma *Field-csum*: *Field* (*r* +*c* *s*) = *Inl* ‘ *Field r* ∪ *Inr* ‘ *Field s*
⟨*proof*⟩

lemma *Card-order-csum*: *Card-order* (*r1* +*c* *r2*)
⟨*proof*⟩

lemma *csum-Cnotzero1*: *Cnotzero r1* ⟹ *Cnotzero* (*r1* +*c* *r2*)
⟨*proof*⟩

lemma *card-order-csum*:
assumes *card-order r1 card-order r2*
shows *card-order* (*r1* +*c* *r2*)
⟨*proof*⟩

lemma *cinfinite-csum*:
cinfinite r1 ∨ *cinfinite r2* ⟹ *cinfinite* (*r1* +*c* *r2*)
⟨*proof*⟩

lemma *Cinfinite-csum*:
Cinfinite r1 ∨ *Cinfinite r2* ⟹ *Cinfinite* (*r1* +*c* *r2*)
⟨*proof*⟩

lemma *Cinfinite-csum1*: *Cinfinite r1* ⟹ *Cinfinite* (*r1* +*c* *r2*)
⟨*proof*⟩

lemma *Cinfinite-csum-weak*:
⟦*Cinfinite r1*; *Cinfinite r2*⟧ ⟹ *Cinfinite* (*r1* +*c* *r2*)
⟨*proof*⟩

lemma *csum-cong*: ⟦*p1* =*o* *r1*; *p2* =*o* *r2*⟧ ⟹ *p1* +*c* *p2* =*o* *r1* +*c* *r2*
⟨*proof*⟩

lemma *csum-cong1*: *p1* =*o* *r1* ⟹ *p1* +*c* *q* =*o* *r1* +*c* *q*
⟨*proof*⟩

lemma *csum-cong2*: *p2* =*o* *r2* ⟹ *q* +*c* *p2* =*o* *q* +*c* *r2*
⟨*proof*⟩

lemma *csum-mono*: ⟦*p1* ≤*o* *r1*; *p2* ≤*o* *r2*⟧ ⟹ *p1* +*c* *p2* ≤*o* *r1* +*c* *r2*
⟨*proof*⟩

lemma *csum-mono1*: *p1* ≤*o* *r1* ⟹ *p1* +*c* *q* ≤*o* *r1* +*c* *q*

⟨*proof*⟩

lemma *csum-mono2*: *p2* ≤*o r2* ⟹ *q* +*c p2* ≤*o q* +*c r2*
⟨*proof*⟩

lemma *ordLeq-csum1*: *Card-order p1* ⟹ *p1* ≤*o p1* +*c p2*
⟨*proof*⟩

lemma *ordLeq-csum2*: *Card-order p2* ⟹ *p2* ≤*o p1* +*c p2*
⟨*proof*⟩

lemma *csum-com*: *p1* +*c p2* =*o p2* +*c p1*
⟨*proof*⟩

lemma *csum-assoc*: (*p1* +*c p2*) +*c p3* =*o p1* +*c p2* +*c p3*
⟨*proof*⟩

lemma *Cfinite-csum*: ⟦*Cfinite r*; *Cfinite s*⟧ ⟹ *Cfinite* (*r* +*c s*)
⟨*proof*⟩

lemma *csum-csum*: (*r1* +*c r2*) +*c* (*r3* +*c r4*) =*o* (*r1* +*c r3*) +*c* (*r2* +*c r4*)
⟨*proof*⟩

lemma *Plus-csum*: |*A* <+> *B*| =*o* |*A*| +*c* |*B*|
⟨*proof*⟩

lemma *Un-csum*: |*A* ∪ *B*| ≤*o* |*A*| +*c* |*B*|
⟨*proof*⟩

31.4 One

definition *cone* **where**
cone = *card-of* {()}

lemma *Card-order-cone*: *Card-order cone*
⟨*proof*⟩

lemma *Cfinite-cone*: *Cfinite cone*
⟨*proof*⟩

lemma *cone-not-czero*: ¬ (*cone* =*o czero*)
⟨*proof*⟩

lemma *cone-ordLeq-Cnotzero*: *Cnotzero r* ⟹ *cone* ≤*o r*
⟨*proof*⟩

31.5 Two

definition *ctwo* **where**
ctwo = |*UNIV* :: *bool set*|

lemma *Card-order-ctwo*: *Card-order ctwo*
⟨*proof*⟩

lemma *ctwo-not-czero*: ¬ (*ctwo* =*o* *czero*)
⟨*proof*⟩

lemma *ctwo-Cnotzero*: *Cnotzero ctwo*
⟨*proof*⟩

31.6 Family sum

definition *Csum* **where**
Csum r rs ≡ |*SIGMA i* : *Field r*. *Field* (*rs i*)|

syntax -*Csum* ::
pttrn => ('*a* ∗ '*a*) *set* => '*b* ∗ '*b set* => (('*a* ∗ '*b*) ∗ ('*a* ∗ '*b*)) *set*
((*3CSUM* -:-. -) [*0*, *51*, *10*] *10*)

translations
CSUM i:*r*. *rs* == *CONST Csum r* (%*i*. *rs*)

lemma *SIGMA-CSUM*: |*SIGMA i* : *I*. *As i*| = (*CSUM i* : |*I*|. |*As i*|)
⟨*proof*⟩

31.7 Product

definition *cprod* (**infixr** ∗*c* *80*) **where**
r1 ∗*c* *r2* = |*Field r1* × *Field r2*|

lemma *card-order-cprod*:
assumes *card-order r1 card-order r2*
shows *card-order* (*r1* ∗*c* *r2*)
⟨*proof*⟩

lemma *Card-order-cprod*: *Card-order* (*r1* ∗*c* *r2*)
⟨*proof*⟩

lemma *cprod-mono1*: *p1* ≤*o* *r1* ⟹ *p1* ∗*c* *q* ≤*o* *r1* ∗*c* *q*
⟨*proof*⟩

lemma *cprod-mono2*: *p2* ≤*o* *r2* ⟹ *q* ∗*c* *p2* ≤*o* *q* ∗*c* *r2*
⟨*proof*⟩

lemma *cprod-mono*: ⟦*p1* ≤*o* *r1*; *p2* ≤*o* *r2*⟧ ⟹ *p1* ∗*c* *p2* ≤*o* *r1* ∗*c* *r2*
⟨*proof*⟩

lemma *ordLeq-cprod2*: ⟦*Cnotzero p1*; *Card-order p2*⟧ ⟹ *p2* ≤*o* *p1* ∗*c* *p2*
⟨*proof*⟩

lemma *cinfinite-cprod*: $[\![cinfinite\ r1;\ cinfinite\ r2]\!] \Longrightarrow cinfinite\ (r1 *c\ r2)$
⟨*proof*⟩

lemma *cinfinite-cprod2*: $[\![Cnotzero\ r1;\ Cinfinite\ r2]\!] \Longrightarrow cinfinite\ (r1 *c\ r2)$
⟨*proof*⟩

lemma *Cinfinite-cprod2*: $[\![Cnotzero\ r1;\ Cinfinite\ r2]\!] \Longrightarrow Cinfinite\ (r1 *c\ r2)$
⟨*proof*⟩

lemma *cprod-cong*: $[\![p1 =o\ r1;\ p2 =o\ r2]\!] \Longrightarrow p1 *c\ p2 =o\ r1 *c\ r2$
⟨*proof*⟩

lemma *cprod-cong1*: $[\![p1 =o\ r1]\!] \Longrightarrow p1 *c\ p2 =o\ r1 *c\ p2$
⟨*proof*⟩

lemma *cprod-cong2*: $p2 =o\ r2 \Longrightarrow q *c\ p2 =o\ q *c\ r2$
⟨*proof*⟩

lemma *cprod-com*: $p1 *c\ p2 =o\ p2 *c\ p1$
⟨*proof*⟩

lemma *card-of-Csum-Times*:
$\forall i \in I.\ |A\ i| \leq o\ |B| \Longrightarrow (CSUM\ i : |I|.\ |A\ i|\ ) \leq o\ |I| *c\ |B|$
⟨*proof*⟩

lemma *card-of-Csum-Times'*:
assumes *Card-order r* $\forall i \in I.\ |A\ i| \leq o\ r$
shows $(CSUM\ i : |I|.\ |A\ i|\ ) \leq o\ |I| *c\ r$
⟨*proof*⟩

lemma *cprod-csum-distrib1*: $r1 *c\ r2 +c\ r1 *c\ r3 =o\ r1 *c\ (r2 +c\ r3)$
⟨*proof*⟩

lemma *csum-absorb2'*: $[\![Card\text{-}order\ r2;\ r1 \leq o\ r2;\ cinfinite\ r1 \vee cinfinite\ r2]\!] \Longrightarrow$
$r1 +c\ r2 =o\ r2$
⟨*proof*⟩

lemma *csum-absorb1'*:
assumes *card*: *Card-order r2*
and *r12*: $r1 \leq o\ r2$ **and** *cr12*: *cinfinite r1* $\vee$ *cinfinite r2*
shows $r2 +c\ r1 =o\ r2$
⟨*proof*⟩

lemma *csum-absorb1*: $[\![Cinfinite\ r2;\ r1 \leq o\ r2]\!] \Longrightarrow r2 +c\ r1 =o\ r2$
⟨*proof*⟩

lemma *csum-absorb2*: $[\![Cinfinite\ r2\ ;\ r1 \leq o\ r2]\!] \Longrightarrow r1 +c\ r2 =o\ r2$
⟨*proof*⟩

lemma *regularCard-csum*:
assumes *Cinfinite r Cinfinite s regularCard r regularCard s*
shows *regularCard* (*r* +*c* *s*)
⟨*proof*⟩

lemma *csum-mono-strict*:
assumes *Card-order*: *Card-order r Card-order q*
and *Cinfinite*: *Cinfinite r′ Cinfinite q′*
and *less*: *r* <*o* *r′* *q* <*o* *q′*
shows *r* +*c* *q* <*o* *r′* +*c* *q′*
⟨*proof*⟩

31.8 Exponentiation

definition *cexp* (**infixr** ^*c* *90*) **where**
r1 ^*c* *r2* ≡ |*Func* (*Field r2*) (*Field r1*)|

lemma *Card-order-cexp*: *Card-order* (*r1* ^*c* *r2*)
⟨*proof*⟩

lemma *cexp-mono′*:
assumes *1*: *p1* ≤*o* *r1* **and** *2*: *p2* ≤*o* *r2*
and *n*: *Field p2* = {} ⟹ *Field r2* = {}
shows *p1* ^*c* *p2* ≤*o* *r1* ^*c* *r2*
⟨*proof*⟩

lemma *cexp-mono*:
assumes *1*: *p1* ≤*o* *r1* **and** *2*: *p2* ≤*o* *r2*
and *n*: *p2* =*o* *czero* ⟹ *r2* =*o* *czero* **and** *card*: *Card-order p2*
shows *p1* ^*c* *p2* ≤*o* *r1* ^*c* *r2*
⟨*proof*⟩

lemma *cexp-mono1*:
assumes *1*: *p1* ≤*o* *r1* **and** *q*: *Card-order q*
shows *p1* ^*c* *q* ≤*o* *r1* ^*c* *q*
⟨*proof*⟩

lemma *cexp-mono2′*:
assumes *2*: *p2* ≤*o* *r2* **and** *q*: *Card-order q*
and *n*: *Field p2* = {} ⟹ *Field r2* = {}
shows *q* ^*c* *p2* ≤*o* *q* ^*c* *r2*
⟨*proof*⟩

lemma *cexp-mono2*:
assumes *2*: *p2* ≤*o* *r2* **and** *q*: *Card-order q*
and *n*: *p2* =*o* *czero* ⟹ *r2* =*o* *czero* **and** *card*: *Card-order p2*
shows *q* ^*c* *p2* ≤*o* *q* ^*c* *r2*
⟨*proof*⟩

lemma *cexp-mono2-Cnotzero*:
assumes *p2* $\leq o$ *r2* *Card-order q* *Cnotzero p2*
shows *q* $\hat{}c$ *p2* $\leq o$ *q* $\hat{}c$ *r2*
⟨*proof*⟩

lemma *cexp-cong*:
assumes *1*: *p1* $=o$ *r1* **and** *2*: *p2* $=o$ *r2*
and *Cr*: *Card-order r2*
and *Cp*: *Card-order p2*
shows *p1* $\hat{}c$ *p2* $=o$ *r1* $\hat{}c$ *r2*
⟨*proof*⟩

lemma *cexp-cong1*:
assumes *1*: *p1* $=o$ *r1* **and** *q*: *Card-order q*
shows *p1* $\hat{}c$ *q* $=o$ *r1* $\hat{}c$ *q*
⟨*proof*⟩

lemma *cexp-cong2*:
assumes *2*: *p2* $=o$ *r2* **and** *q*: *Card-order q* **and** *p*: *Card-order p2*
shows *q* $\hat{}c$ *p2* $=o$ *q* $\hat{}c$ *r2*
⟨*proof*⟩

lemma *cexp-cone*:
assumes *Card-order r*
shows *r* $\hat{}c$ *cone* $=o$ *r*
⟨*proof*⟩

lemma *cexp-cprod*:
assumes *r1*: *Card-order r1*
shows (*r1* $\hat{}c$ *r2*) $\hat{}c$ *r3* $=o$ *r1* $\hat{}c$ (*r2* $*c$ *r3*) (**is** *?L* $=o$ *?R*)
⟨*proof*⟩

lemma *cprod-infinite1'*: ⟦*Cinfinite r*; *Cnotzero p*; *p* $\leq o$ *r*⟧ $\Longrightarrow$ *r* $*c$ *p* $=o$ *r*
⟨*proof*⟩

lemma *cprod-infinite*: *Cinfinite r* $\Longrightarrow$ *r* $*c$ *r* $=o$ *r*
⟨*proof*⟩

lemma *cexp-cprod-ordLeq*:
assumes *r1*: *Card-order r1* **and** *r2*: *Cinfinite r2*
and *r3*: *Cnotzero r3* *r3* $\leq o$ *r2*
shows (*r1* $\hat{}c$ *r2*) $\hat{}c$ *r3* $=o$ *r1* $\hat{}c$ *r2* (**is** *?L* $=o$ *?R*)
⟨*proof*⟩

lemma *Cnotzero-UNIV*: *Cnotzero* |*UNIV*|
⟨*proof*⟩

lemma *ordLess-ctwo-cexp*:

assumes *Card-order r*
shows $r <o$ *ctwo* $\hat{}c$ *r*
⟨*proof*⟩

lemma *ordLeq-cexp1*:
assumes *Cnotzero r Card-order q*
shows $q \leq o$ *q* $\hat{}c$ *r*
⟨*proof*⟩

lemma *ordLeq-cexp2*:
assumes *ctwo* $\leq o$ *q Card-order r*
shows $r \leq o$ *q* $\hat{}c$ *r*
⟨*proof*⟩

lemma *cinfinite-cexp*: ⟦*ctwo* $\leq o$ *q*; *Cinfinite r*⟧ ⟹ *cinfinite* (*q* $\hat{}c$ *r*)
⟨*proof*⟩

lemma *Cinfinite-cexp*:
⟦*ctwo* $\leq o$ *q*; *Cinfinite r*⟧ ⟹ *Cinfinite* (*q* $\hat{}c$ *r*)
⟨*proof*⟩

lemma *card-order-cexp*:
assumes *card-order r1 card-order r2*
shows *card-order* (*r1* $\hat{}c$ *r2*)
⟨*proof*⟩

lemma *ctwo-ordLess-natLeq*: *ctwo* $<o$ *natLeq*
⟨*proof*⟩

lemma *ctwo-ordLess-Cinfinite*: *Cinfinite r* ⟹ *ctwo* $<o$ *r*
⟨*proof*⟩

lemma *ctwo-ordLeq-Cinfinite*:
assumes *Cinfinite r*
shows *ctwo* $\leq o$ *r*
⟨*proof*⟩

lemma *Un-Cinfinite-bound*: ⟦|*A*| $\leq o$ *r*; |*B*| $\leq o$ *r*; *Cinfinite r*⟧ ⟹ |*A* ∪ *B*| $\leq o$ *r*
⟨*proof*⟩

lemma *Un-Cinfinite-bound-strict*: ⟦|*A*| $<o$ *r*; |*B*| $<o$ *r*; *Cinfinite r*⟧ ⟹ |*A* ∪ *B*| $<o$ *r*
⟨*proof*⟩

lemma *UNION-Cinfinite-bound*: ⟦|*I*| $\leq o$ *r*; ∀ *i* ∈ *I*. |*A i*| $\leq o$ *r*; *Cinfinite r*⟧ ⟹ |⋃ *i* ∈ *I*. *A i*| $\leq o$ *r*
⟨*proof*⟩

lemma *csum-cinfinite-bound*:

assumes $p \leq o\ r\ q \leq o\ r$ *Card-order p Card-order q Cinfinite r*
shows $p +c\ q \leq o\ r$
⟨*proof*⟩

lemma *cprod-cinfinite-bound*:
assumes $p \leq o\ r\ q \leq o\ r$ *Card-order p Card-order q Cinfinite r*
shows $p *c\ q \leq o\ r$
⟨*proof*⟩

lemma *cprod-infinite2'*: $[\![Cnotzero\ r1;\ Cinfinite\ r2;\ r1 \leq o\ r2]\!] \Longrightarrow r1 *c\ r2 =o\ r2$
⟨*proof*⟩

lemma *regularCard-cprod*:
assumes *Cinfinite r Cinfinite s regularCard r regularCard s*
shows *regularCard* $(r *c\ s)$
⟨*proof*⟩

lemma *cprod-csum-cexp*:
$r1 *c\ r2 \leq o\ (r1 +c\ r2)\ ^c\ ctwo$
⟨*proof*⟩

lemma *Cfinite-cprod-Cinfinite*: $[\![Cfinite\ r;\ Cinfinite\ s]\!] \Longrightarrow r *c\ s \leq o\ s$
⟨*proof*⟩

lemma *cprod-cexp*: $(r *c\ s)\ ^c\ t =o\ r\ ^c\ t *c\ s\ ^c\ t$
⟨*proof*⟩

lemma *cprod-cexp-csum-cexp-Cinfinite*:
assumes *t*: *Cinfinite t*
shows $(r *c\ s)\ ^c\ t \leq o\ (r +c\ s)\ ^c\ t$
⟨*proof*⟩

lemma *Cfinite-cexp-Cinfinite*:
assumes *s*: *Cfinite s* **and** *t*: *Cinfinite t*
shows $s\ ^c\ t \leq o\ ctwo\ ^c\ t$
⟨*proof*⟩

lemma *csum-Cfinite-cexp-Cinfinite*:
assumes *r*: *Card-order r* **and** *s*: *Cfinite s* **and** *t*: *Cinfinite t*
shows $(r +c\ s)\ ^c\ t \leq o\ (r +c\ ctwo)\ ^c\ t$
⟨*proof*⟩

lemma *Cinfinite-cardSuc*: *Cinfinite r* $\Longrightarrow$ *Cinfinite* $(cardSuc\ r)$
⟨*proof*⟩

lemma *cardSuc-UNION-Cinfinite*:

assumes *Cinfinite r relChain* (*cardSuc r*) *As B* ≤ (⋃ *i* ∈ *Field* (*cardSuc r*). *As i*) |*B*| <=*o r*
shows ∃ *i* ∈ *Field* (*cardSuc r*). *B* ≤ *As i*
⟨*proof*⟩

lemma *Cinfinite-card-suc*: ⟦ *Cinfinite r* ; *card-order r* ⟧ ⟹ *Cinfinite* (*card-suc r*)
⟨*proof*⟩

lemma *card-suc-least*: ⟦*card-order r*; *Card-order s*; *r* <*o s*⟧ ⟹ *card-suc r* ≤*o s*
⟨*proof*⟩

lemma *regularCard-cardSuc*: *Cinfinite k* ⟹ *regularCard* (*cardSuc k*)
⟨*proof*⟩

lemma *regularCard-card-suc*: *card-order r* ⟹ *Cinfinite r* ⟹ *regularCard* (*card-suc r*)
⟨*proof*⟩

end

32 Function Definition Base

theory *Fun-Def-Base*
imports *Ctr-Sugar Set Wellfounded*
begin

⟨*ML*⟩
named-theorems *termination-simp simplification rules for termination proofs*
⟨*ML*⟩

end

33 Definition of Bounded Natural Functors

theory *BNF-Def*
imports *BNF-Cardinal-Arithmetic Fun-Def-Base*
keywords
print-bnfs :: *diag* **and**
bnf :: *thy-goal-defn*
begin

lemma *Collect-case-prodD*: *x* ∈ *Collect* (*case-prod A*) ⟹ *A* (*fst x*) (*snd x*)
⟨*proof*⟩

inductive
rel-sum :: ('*a* ⇒ '*c* ⇒ *bool*) ⇒ ('*b* ⇒ '*d* ⇒ *bool*) ⇒ '*a* + '*b* ⇒ '*c* + '*d* ⇒ *bool*
for *R1 R2*
where

R1 a c $\Longrightarrow$ *rel-sum R1 R2* (*Inl a*) (*Inl c*)
| *R2 b d* $\Longrightarrow$ *rel-sum R1 R2* (*Inr b*) (*Inr d*)

definition
rel-fun :: ('*a* $\Rightarrow$ '*c* $\Rightarrow$ *bool*) $\Rightarrow$ ('*b* $\Rightarrow$ '*d* $\Rightarrow$ *bool*) $\Rightarrow$ ('*a* $\Rightarrow$ '*b*) $\Rightarrow$ ('*c* $\Rightarrow$ '*d*) $\Rightarrow$ *bool*
where
rel-fun A B = ($\lambda f\ g.\ \forall x\ y.\ A\ x\ y \longrightarrow B\ (f\ x)\ (g\ y)$)

lemma *rel-funI* [*intro*]:
assumes $\bigwedge x\ y.\ A\ x\ y \Longrightarrow B\ (f\ x)\ (g\ y)$
shows *rel-fun A B f g*
⟨*proof*⟩

lemma *rel-funD*:
assumes *rel-fun A B f g* **and** *A x y*
shows *B* (*f x*) (*g y*)
⟨*proof*⟩

lemma *rel-fun-mono*:
⟦ *rel-fun X A f g*; $\bigwedge x\ y.\ Y\ x\ y \longrightarrow X\ x\ y$; $\bigwedge x\ y.\ A\ x\ y \Longrightarrow B\ x\ y$ ⟧ $\Longrightarrow$ *rel-fun Y B f g*
⟨*proof*⟩

lemma *rel-fun-mono'* [*mono*]:
⟦ $\bigwedge x\ y.\ Y\ x\ y \longrightarrow X\ x\ y$; $\bigwedge x\ y.\ A\ x\ y \longrightarrow B\ x\ y$ ⟧ $\Longrightarrow$ *rel-fun X A f g* $\longrightarrow$ *rel-fun Y B f g*
⟨*proof*⟩

definition *rel-set* :: ('*a* $\Rightarrow$ '*b* $\Rightarrow$ *bool*) $\Rightarrow$ '*a set* $\Rightarrow$ '*b set* $\Rightarrow$ *bool*
where *rel-set R* = ($\lambda A\ B.\ (\forall x{\in}A.\ \exists y{\in}B.\ R\ x\ y) \wedge (\forall y{\in}B.\ \exists x{\in}A.\ R\ x\ y)$)

lemma *rel-setI*:
assumes $\bigwedge x.\ x \in A \Longrightarrow \exists y{\in}B.\ R\ x\ y$
assumes $\bigwedge y.\ y \in B \Longrightarrow \exists x{\in}A.\ R\ x\ y$
shows *rel-set R A B*
⟨*proof*⟩

lemma *predicate2-transferD*:
⟦*rel-fun R1* (*rel-fun R2* (=)) *P Q*; $a \in A$; $b \in B$; $A \subseteq \{(x,\ y).\ R1\ x\ y\}$; $B \subseteq \{(x,\ y).\ R2\ x\ y\}$⟧ $\Longrightarrow$
P (*fst a*) (*fst b*) $\longleftrightarrow$ *Q* (*snd a*) (*snd b*)
⟨*proof*⟩

definition *collect* **where**
collect F x = ($\bigcup f \in F.\ f\ x$)

lemma *fstI*: *x* = (*y*, *z*) $\Longrightarrow$ *fst x* = *y*
⟨*proof*⟩

lemma *sndI*: $x = (y,\ z) \Longrightarrow snd\ x = z$
⟨*proof*⟩

lemma *bijI'*: $[\![\bigwedge x\ y.\ (f\ x = f\ y) = (x = y);\ \bigwedge y.\ \exists x.\ y = f\ x]\!] \Longrightarrow bij\ f$
⟨*proof*⟩

definition $Gr\ A\ f = \{(a,\ f\ a)\ |\ a.\ a \in A\}$

definition $Grp\ A\ f = (\lambda a\ b.\ b = f\ a \wedge a \in A)$

definition *vimage2p* **where**
$vimage2p\ f\ g\ R = (\lambda x\ y.\ R\ (f\ x)\ (g\ y))$

lemma *collect-comp*: $collect\ F \circ g = collect\ ((\lambda f.\ f \circ g)\ `\ F)$
⟨*proof*⟩

definition *convol* (⟨(-,/ -)⟩) **where**
$\langle f,\ g\rangle \equiv \lambda a.\ (f\ a,\ g\ a)$

lemma *fst-convol*: $fst \circ \langle f,\ g\rangle = f$
⟨*proof*⟩

lemma *snd-convol*: $snd \circ \langle f,\ g\rangle = g$
⟨*proof*⟩

lemma *convol-mem-GrpI*:
$x \in A \Longrightarrow \langle id,\ g\rangle\ x \in (Collect\ (case\text{-}prod\ (Grp\ A\ g)))$
⟨*proof*⟩

definition *csquare* **where**
$csquare\ A\ f1\ f2\ p1\ p2 \longleftrightarrow (\forall\ a \in A.\ f1\ (p1\ a) = f2\ (p2\ a))$

lemma *eq-alt*: $(=) = Grp\ UNIV\ id$
⟨*proof*⟩

lemma *leq-conversepI*: $R = (=) \Longrightarrow R \leq R^{-1-1}$
⟨*proof*⟩

lemma *leq-OOI*: $R = (=) \Longrightarrow R \leq R\ OO\ R$
⟨*proof*⟩

lemma *OO-Grp-alt*: $(Grp\ A\ f)^{-1-1}\ OO\ Grp\ A\ g = (\lambda x\ y.\ \exists z.\ z \in A \wedge f\ z = x \wedge g\ z = y)$
⟨*proof*⟩

lemma *Grp-UNIV-id*: $f = id \Longrightarrow (Grp\ UNIV\ f)^{-1-1}\ OO\ Grp\ UNIV\ f = Grp\ UNIV\ f$
⟨*proof*⟩

lemma *Grp-UNIV-idI*: $x = y \Longrightarrow Grp\ UNIV\ id\ x\ y$
⟨*proof*⟩

lemma *Grp-mono*: $A \leq B \Longrightarrow Grp\ A\ f \leq Grp\ B\ f$
⟨*proof*⟩

lemma *GrpI*: $[\![f\ x = y;\ x \in A]\!] \Longrightarrow Grp\ A\ f\ x\ y$
⟨*proof*⟩

lemma *GrpE*: $Grp\ A\ f\ x\ y \Longrightarrow ([\![f\ x = y;\ x \in A]\!] \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

lemma *Collect-case-prod-Grp-eqD*: $z \in Collect\ (case\text{-}prod\ (Grp\ A\ f)) \Longrightarrow (f \circ fst)\ z = snd\ z$
⟨*proof*⟩

lemma *Collect-case-prod-Grp-in*: $z \in Collect\ (case\text{-}prod\ (Grp\ A\ f)) \Longrightarrow fst\ z \in A$
⟨*proof*⟩

definition *pick-middlep* $P\ Q\ a\ c = (SOME\ b.\ P\ a\ b \wedge Q\ b\ c)$

lemma *pick-middlep*:
$(P\ OO\ Q)\ a\ c \Longrightarrow P\ a\ (pick\text{-}middlep\ P\ Q\ a\ c) \wedge Q\ (pick\text{-}middlep\ P\ Q\ a\ c)\ c$
⟨*proof*⟩

definition *fstOp* **where**
$fstOp\ P\ Q\ ac = (fst\ ac,\ pick\text{-}middlep\ P\ Q\ (fst\ ac)\ (snd\ ac))$

definition *sndOp* **where**
$sndOp\ P\ Q\ ac = (pick\text{-}middlep\ P\ Q\ (fst\ ac)\ (snd\ ac),\ (snd\ ac))$

lemma *fstOp-in*: $ac \in Collect\ (case\text{-}prod\ (P\ OO\ Q)) \Longrightarrow fstOp\ P\ Q\ ac \in Collect\ (case\text{-}prod\ P)$
⟨*proof*⟩

lemma *fst-fstOp*: $fst\ bc = (fst \circ fstOp\ P\ Q)\ bc$
⟨*proof*⟩

lemma *snd-sndOp*: $snd\ bc = (snd \circ sndOp\ P\ Q)\ bc$
⟨*proof*⟩

lemma *sndOp-in*: $ac \in Collect\ (case\text{-}prod\ (P\ OO\ Q)) \Longrightarrow sndOp\ P\ Q\ ac \in Collect\ (case\text{-}prod\ Q)$
⟨*proof*⟩

lemma *csquare-fstOp-sndOp*:
$csquare\ (Collect\ (f\ (P\ OO\ Q)))\ snd\ fst\ (fstOp\ P\ Q)\ (sndOp\ P\ Q)$
⟨*proof*⟩

lemma *snd-fst-flip*: *snd xy* = (*fst* ∘ (%(*x*, *y*). (*y*, *x*))) *xy*
⟨*proof*⟩

lemma *fst-snd-flip*: *fst xy* = (*snd* ∘ (%(*x*, *y*). (*y*, *x*))) *xy*
⟨*proof*⟩

lemma *flip-pred*: *A* ⊆ *Collect* (*case-prod* (*R* $^{-1-1}$)) ⟹ (%(*x*, *y*). (*y*, *x*)) ‘ *A* ⊆ *Collect* (*case-prod R*)
⟨*proof*⟩

lemma *predicate2-eqD*: *A* = *B* ⟹ *A a b* ⟷ *B a b*
⟨*proof*⟩

lemma *case-sum-o-inj*: *case-sum f g* ∘ *Inl* = *f case-sum f g* ∘ *Inr* = *g*
⟨*proof*⟩

lemma *map-sum-o-inj*: *map-sum f g* ∘ *Inl* = *Inl* ∘ *f map-sum f g* ∘ *Inr* = *Inr* ∘ *g*
⟨*proof*⟩

lemma *card-order-csum-cone-cexp-def*:
card-order r ⟹ (|*A1*| +*c cone*) ^*c r* = |*Func UNIV* (*Inl* ‘ *A1* ∪ {*Inr* ()})|
⟨*proof*⟩

lemma *If-the-inv-into-in-Func*:
⟦*inj-on g C*; *C* ⊆ *B* ∪ {*x*}⟧ ⟹
(λ*i*. *if i* ∈ *g* ‘ *C then the-inv-into C g i else x*) ∈ *Func UNIV* (*B* ∪ {*x*})
⟨*proof*⟩

lemma *If-the-inv-into-f-f*:
⟦*i* ∈ *C*; *inj-on g C*⟧ ⟹ ((λ*i*. *if i* ∈ *g* ‘ *C then the-inv-into C g i else x*) ∘ *g*) *i* = *id i*
⟨*proof*⟩

lemma *the-inv-f-o-f-id*: *inj f* ⟹ (*the-inv f* ∘ *f*) *z* = *id z*
⟨*proof*⟩

lemma *vimage2pI*: *R* (*f x*) (*g y*) ⟹ *vimage2p f g R x y*
⟨*proof*⟩

lemma *rel-fun-iff-leq-vimage2p*: (*rel-fun R S*) *f g* = (*R* ≤ *vimage2p f g S*)
⟨*proof*⟩

lemma *convol-image-vimage2p*: ⟨*f* ∘ *fst*, *g* ∘ *snd*⟩ ‘ *Collect* (*case-prod* (*vimage2p f g R*)) ⊆ *Collect* (*case-prod R*)
⟨*proof*⟩

lemma *vimage2p-Grp*: *vimage2p f g P* = *Grp UNIV f OO P OO* (*Grp UNIV g*)$^{-1-1}$

⟨*proof*⟩

lemma *subst-Pair*: $P\ x\ y \Longrightarrow a = (x,\ y) \Longrightarrow P\ (fst\ a)\ (snd\ a)$
⟨*proof*⟩

lemma *comp-apply-eq*: $f\ (g\ x) = h\ (k\ x) \Longrightarrow (f \circ g)\ x = (h \circ k)\ x$
⟨*proof*⟩

lemma *refl-ge-eq*: $(\bigwedge x.\ R\ x\ x) \Longrightarrow (=) \leq R$
⟨*proof*⟩

lemma *ge-eq-refl*: $(=) \leq R \Longrightarrow R\ x\ x$
⟨*proof*⟩

lemma *reflp-eq*: $reflp\ R = ((=) \leq R)$
⟨*proof*⟩

lemma *transp-relcompp*: $transp\ r \longleftrightarrow r\ OO\ r \leq r$
⟨*proof*⟩

lemma *symp-conversep*: $symp\ R = (R^{-1-1} \leq R)$
⟨*proof*⟩

lemma *diag-imp-eq-le*: $(\bigwedge x.\ x \in A \Longrightarrow R\ x\ x) \Longrightarrow \forall x\ y.\ x \in A \longrightarrow y \in A \longrightarrow x = y \longrightarrow R\ x\ y$
⟨*proof*⟩

definition *eq-onp* :: $('a \Rightarrow bool) \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$
where $eq\text{-}onp\ R = (\lambda x\ y.\ R\ x \wedge x = y)$

lemma *eq-onp-Grp*: $eq\text{-}onp\ P = BNF\text{-}Def.Grp\ (Collect\ P)\ id$
⟨*proof*⟩

lemma *eq-onp-to-eq*: $eq\text{-}onp\ P\ x\ y \Longrightarrow x = y$
⟨*proof*⟩

lemma *eq-onp-top-eq-eq*: $eq\text{-}onp\ top = (=)$
⟨*proof*⟩

lemma *eq-onp-same-args*: $eq\text{-}onp\ P\ x\ x = P\ x$
⟨*proof*⟩

lemma *eq-onp-eqD*: $eq\text{-}onp\ P = Q \Longrightarrow P\ x = Q\ x\ x$
⟨*proof*⟩

lemma *Ball-Collect*: $Ball\ A\ P = (A \subseteq (Collect\ P))$
⟨*proof*⟩

lemma *eq-onp-mono0*: $\forall x{\in}A.\ P\ x \longrightarrow Q\ x \Longrightarrow \forall x{\in}A.\ \forall y{\in}A.\ eq\text{-}onp\ P\ x\ y \longrightarrow$

eq-onp Q x y
⟨*proof*⟩

lemma *eq-onp-True*: *eq-onp* (λ-. *True*) = (=)
⟨*proof*⟩

lemma *Ball-image-comp*: *Ball* (*f* ‘ *A*) *g* = *Ball A* (*g* ∘ *f*)
⟨*proof*⟩

lemma *rel-fun-Collect-case-prodD*:
rel-fun A B f g ⟹ *X* ⊆ *Collect* (*case-prod A*) ⟹ *x* ∈ *X* ⟹ *B* ((*f* ∘ *fst*) *x*) ((*g* ∘ *snd*) *x*)
⟨*proof*⟩

lemma *eq-onp-mono-iff*: *eq-onp P* ≤ *eq-onp Q* ⟷ *P* ≤ *Q*
⟨*proof*⟩

⟨*ML*⟩

end

34 Composition of Bounded Natural Functors

theory *BNF-Composition*
imports *BNF-Def*
begin

lemma *ssubst-mem*: ⟦*t* = *s*; *s* ∈ *X*⟧ ⟹ *t* ∈ *X*
⟨*proof*⟩

lemma *empty-natural*: (λ-. {}) ∘ *f* = *image g* ∘ (λ-. {})
⟨*proof*⟩

lemma *Cinfinite-gt-empty*: *Cinfinite r* ⟹ |{}| <*o r*
⟨*proof*⟩

lemma *Union-natural*: *Union* ∘ *image* (*image f*) = *image f* ∘ *Union*
⟨*proof*⟩

lemma *in-Union-o-assoc*: *x* ∈ (*Union* ∘ *gset* ∘ *gmap*) *A* ⟹ *x* ∈ (*Union* ∘ (*gset* ∘ *gmap*)) *A*
⟨*proof*⟩

lemma *regularCard-UNION-bound*:
assumes *Cinfinite r regularCard r* **and** |*I*| <*o r* ⋀*i*. *i* ∈ *I* ⟹ |*A i*| <*o r*
shows |⋃ *i*∈*I*. *A i*| <*o r*
⟨*proof*⟩

lemma *comp-single-set-bd-strict*:

assumes *fbd*: *Cinfinite fbd regularCard fbd* **and**
gbd: *Cinfinite gbd regularCard gbd* **and**
fset-bd: $\bigwedge x.\ |fset\ x| <o\ fbd$ **and**
gset-bd: $\bigwedge x.\ |gset\ x| <o\ gbd$
shows $|\bigcup (fset\ `\ gset\ x)| <o\ gbd *c\ fbd$
⟨*proof*⟩

lemma *comp-single-set-bd*:
assumes *fbd-Card-order*: *Card-order fbd* **and**
fset-bd: $\bigwedge x.\ |fset\ x| \leq o\ fbd$ **and**
gset-bd: $\bigwedge x.\ |gset\ x| \leq o\ gbd$
shows $|\bigcup (fset\ `\ gset\ x)| \leq o\ gbd *c\ fbd$
⟨*proof*⟩

lemma *csum-dup*: *cinfinite r* $\Longrightarrow$ *Card-order r* $\Longrightarrow$ $p +c\ p' =o\ r +c\ r \Longrightarrow p +c\ p' =o\ r$
⟨*proof*⟩

lemma *cprod-dup*: *cinfinite r* $\Longrightarrow$ *Card-order r* $\Longrightarrow$ $p *c\ p' =o\ r *c\ r \Longrightarrow p *c\ p' =o\ r$
⟨*proof*⟩

lemma *Union-image-insert*: $\bigcup (f\ `\ insert\ a\ B) = f\ a \cup \bigcup (f\ `\ B)$
⟨*proof*⟩

lemma *Union-image-empty*: $A \cup \bigcup (f\ `\ \{\}) = A$
⟨*proof*⟩

lemma *image-o-collect*: $collect\ ((\lambda f.\ image\ g \circ f)\ `\ F) = image\ g \circ collect\ F$
⟨*proof*⟩

lemma *conj-subset-def*: $A \subseteq \{x.\ P\ x \wedge Q\ x\} = (A \subseteq \{x.\ P\ x\} \wedge A \subseteq \{x.\ Q\ x\})$
⟨*proof*⟩

lemma *UN-image-subset*: $\bigcup (f\ `\ g\ x) \subseteq X = (g\ x \subseteq \{x.\ f\ x \subseteq X\})$
⟨*proof*⟩

lemma *comp-set-bd-Union-o-collect*: $|\bigcup (\bigcup ((\lambda f.\ f\ x)\ `\ X))| \leq o\ hbd \Longrightarrow |(Union \circ collect\ X)\ x| \leq o\ hbd$
⟨*proof*⟩

lemma *comp-set-bd-Union-o-collect-strict*: $|\bigcup (\bigcup ((\lambda f.\ f\ x)\ `\ X))| <o\ hbd \Longrightarrow |(Union \circ collect\ X)\ x| <o\ hbd$
⟨*proof*⟩

lemma *Collect-inj*: $Collect\ P = Collect\ Q \Longrightarrow P = Q$
⟨*proof*⟩

lemma *Grp-fst-snd*: $(Grp\ (Collect\ (case\text{-}prod\ R))\ fst)^{-1-1}\ OO\ Grp\ (Collect\ (case\text{-}prod$

$R))$ *snd* $= R$
⟨*proof*⟩

lemma *OO-Grp-cong*: $A = B \Longrightarrow (Grp\ A\ f)^{-1-1}\ OO\ Grp\ A\ g = (Grp\ B\ f)^{-1-1}\ OO\ Grp\ B\ g$
⟨*proof*⟩

lemma *vimage2p-relcompp-mono*: $R\ OO\ S \leq T \Longrightarrow$
vimage2p f g R OO vimage2p g h S $\leq$ *vimage2p f h T*
⟨*proof*⟩

lemma *type-copy-map-cong0*: $M\ (g\ x) = N\ (h\ x) \Longrightarrow (f \circ M \circ g)\ x = (f \circ N \circ h)\ x$
⟨*proof*⟩

lemma *type-copy-set-bd*: $(\bigwedge y.\ |S\ y| <o\ bd) \Longrightarrow |(S \circ Rep)\ x| <o\ bd$
⟨*proof*⟩

lemma *vimage2p-cong*: $R = S \Longrightarrow$ *vimage2p f g R* $=$ *vimage2p f g S*
⟨*proof*⟩

lemma *Ball-comp-iff*: $(\lambda x.\ Ball\ (A\ x)\ f) \circ g = (\lambda x.\ Ball\ ((A \circ g)\ x)\ f)$
⟨*proof*⟩

lemma *conj-comp-iff*: $(\lambda x.\ P\ x \wedge Q\ x) \circ g = (\lambda x.\ (P \circ g)\ x \wedge (Q \circ g)\ x)$
⟨*proof*⟩

context
fixes *Rep Abs*
assumes *type-copy*: *type-definition Rep Abs UNIV*
begin

lemma *type-copy-map-id0*: $M = id \Longrightarrow Abs \circ M \circ Rep = id$
⟨*proof*⟩

lemma *type-copy-map-comp0*: $M = M1 \circ M2 \Longrightarrow f \circ M \circ g = (f \circ M1 \circ Rep) \circ (Abs \circ M2 \circ g)$
⟨*proof*⟩

lemma *type-copy-set-map0*: $S \circ M = image\ f \circ S' \Longrightarrow (S \circ Rep) \circ (Abs \circ M \circ g) = image\ f \circ (S' \circ g)$
⟨*proof*⟩

lemma *type-copy-wit*: $x \in (S \circ Rep)\ (Abs\ y) \Longrightarrow x \in S\ y$
⟨*proof*⟩

lemma *type-copy-vimage2p-Grp-Rep*: *vimage2p f Rep* $(Grp\ (Collect\ P)\ h) =$
$Grp\ (Collect\ (\lambda x.\ P\ (f\ x)))\ (Abs \circ h \circ f)$
⟨*proof*⟩

lemma *type-copy-vimage2p-Grp-Abs*:
⋀*h*. *vimage2p g Abs* (*Grp* (*Collect P*) *h*) = *Grp* (*Collect* (λ*x*. *P* (*g x*))) (*Rep* ∘ *h* ∘ *g*)
⟨*proof*⟩

lemma *type-copy-ex-RepI*: (∃ *b*. *F b*) = (∃ *b*. *F* (*Rep b*))
⟨*proof*⟩

lemma *vimage2p-relcompp-converse*:
vimage2p f g (*R*$^{-1-1}$ *OO S*) = (*vimage2p Rep f R*)$^{-1-1}$ *OO vimage2p Rep g S*
⟨*proof*⟩

end

bnf *DEADID*: *'a*
map: *id* :: *'a* ⇒ *'a*
bd: *natLeq*
rel: (=) :: *'a* ⇒ *'a* ⇒ *bool*
⟨*proof*⟩

definition *id-bnf* :: *'a* ⇒ *'a* **where**
id-bnf ≡ (λ*x*. *x*)

lemma *id-bnf-apply*: *id-bnf x* = *x*
⟨*proof*⟩

bnf *ID*: *'a*
map: *id-bnf* :: (*'a* ⇒ *'b*) ⇒ *'a* ⇒ *'b*
sets: λ*x*. {*x*}
bd: *natLeq*
rel: *id-bnf* :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *'a* ⇒ *'b* ⇒ *bool*
pred: *id-bnf* :: (*'a* ⇒ *bool*) ⇒ *'a* ⇒ *bool*
⟨*proof*⟩

lemma *type-definition-id-bnf-UNIV*: *type-definition id-bnf id-bnf UNIV*
⟨*proof*⟩

⟨*ML*⟩

hide-fact
DEADID.inj-map DEADID.inj-map-strong DEADID.map-comp DEADID.map-cong DEADID.map-cong0
DEADID.map-cong-simp DEADID.map-id DEADID.map-id0 DEADID.map-ident DEADID.map-transfer
DEADID.rel-Grp DEADID.rel-compp DEADID.rel-compp-Grp DEADID.rel-conversep DEADID.rel-eq
DEADID.rel-flip DEADID.rel-map DEADID.rel-mono DEADID.rel-transfer
ID.inj-map ID.inj-map-strong ID.map-comp ID.map-cong ID.map-cong0 ID.map-cong-simp

ID.map-id
ID.map-id0 ID.map-ident ID.map-transfer ID.rel-Grp ID.rel-compp ID.rel-compp-Grp ID.rel-conversep
ID.rel-eq ID.rel-flip ID.rel-map ID.rel-mono ID.rel-transfer ID.set-map ID.set-transfer

end

35 Registration of Basic Types as Bounded Natural Functors

theory *Basic-BNFs*
imports *BNF-Def*
begin

inductive-set *setl* :: $'a + 'b \Rightarrow 'a\ set$ **for** $s :: 'a + 'b$ **where**
$s = Inl\ x \Longrightarrow x \in setl\ s$
inductive-set *setr* :: $'a + 'b \Rightarrow 'b\ set$ **for** $s :: 'a + 'b$ **where**
$s = Inr\ x \Longrightarrow x \in setr\ s$

lemma *sum-set-defs*[*code*]:
$setl = (\lambda x.\ case\ x\ of\ Inl\ z \Rightarrow \{z\} \mid - \Rightarrow \{\})$
$setr = (\lambda x.\ case\ x\ of\ Inr\ z \Rightarrow \{z\} \mid - \Rightarrow \{\})$
⟨*proof*⟩

lemma *rel-sum-simps*[*code*, *simp*]:
$rel\text{-}sum\ R1\ R2\ (Inl\ a1)\ (Inl\ b1) = R1\ a1\ b1$
$rel\text{-}sum\ R1\ R2\ (Inl\ a1)\ (Inr\ b2) = False$
$rel\text{-}sum\ R1\ R2\ (Inr\ a2)\ (Inl\ b1) = False$
$rel\text{-}sum\ R1\ R2\ (Inr\ a2)\ (Inr\ b2) = R2\ a2\ b2$
⟨*proof*⟩

inductive
$pred\text{-}sum :: ('a \Rightarrow bool) \Rightarrow ('b \Rightarrow bool) \Rightarrow 'a + 'b \Rightarrow bool$ **for** *P1 P2*
where
$P1\ a \Longrightarrow pred\text{-}sum\ P1\ P2\ (Inl\ a)$
| $P2\ b \Longrightarrow pred\text{-}sum\ P1\ P2\ (Inr\ b)$

lemma *pred-sum-inject*[*code*, *simp*]:
$pred\text{-}sum\ P1\ P2\ (Inl\ a) \longleftrightarrow P1\ a$
$pred\text{-}sum\ P1\ P2\ (Inr\ b) \longleftrightarrow P2\ b$
⟨*proof*⟩

bnf $'a + 'b$
map: *map-sum*
sets: *setl setr*
bd: *natLeq*
wits: *Inl Inr*
rel: *rel-sum*

pred: *pred-sum*
⟨*proof*⟩

inductive-set *fsts* :: $'a \times 'b \Rightarrow 'a$ *set* **for** $p :: 'a \times 'b$ **where**
fst p ∈ *fsts p*
inductive-set *snds* :: $'a \times 'b \Rightarrow 'b$ *set* **for** $p :: 'a \times 'b$ **where**
snd p ∈ *snds p*

lemma *prod-set-defs*[*code*]: *fsts* = (λ*p*. {*fst p*}) *snds* = (λ*p*. {*snd p*})
⟨*proof*⟩

inductive
rel-prod :: $('a \Rightarrow 'b \Rightarrow$ *bool*$) \Rightarrow ('c \Rightarrow 'd \Rightarrow$ *bool*$) \Rightarrow 'a \times 'c \Rightarrow 'b \times 'd \Rightarrow$ *bool*
for *R1 R2*
where
⟦*R1 a b*; *R2 c d*⟧ ⟹ *rel-prod R1 R2* (*a*, *c*) (*b*, *d*)

inductive
pred-prod :: $('a \Rightarrow$ *bool*$) \Rightarrow ('b \Rightarrow$ *bool*$) \Rightarrow 'a \times 'b \Rightarrow$ *bool* **for** *P1 P2*
where
⟦*P1 a*; *P2 b*⟧ ⟹ *pred-prod P1 P2* (*a*, *b*)

lemma *rel-prod-inject* [*code*, *simp*]:
rel-prod R1 R2 (*a*, *b*) (*c*, *d*) ⟷ *R1 a c* ∧ *R2 b d*
⟨*proof*⟩

lemma *pred-prod-inject* [*code*, *simp*]:
pred-prod P1 P2 (*a*, *b*) ⟷ *P1 a* ∧ *P2 b*
⟨*proof*⟩

lemma *rel-prod-conv*:
rel-prod R1 R2 = (λ(*a*, *b*) (*c*, *d*). *R1 a c* ∧ *R2 b d*)
⟨*proof*⟩

definition
pred-fun :: $('a \Rightarrow$ *bool*$) \Rightarrow ('b \Rightarrow$ *bool*$) \Rightarrow ('a \Rightarrow 'b) \Rightarrow$ *bool*
where
pred-fun A B = (λ*f*. ∀ *x*. *A x* ⟶ *B* (*f x*))

lemma *pred-funI*: (⋀*x*. *A x* ⟹ *B* (*f x*)) ⟹ *pred-fun A B f*
⟨*proof*⟩

bnf $'a \times 'b$
map: *map-prod*
sets: *fsts snds*
bd: *natLeq*
rel: *rel-prod*
pred: *pred-prod*
⟨*proof*⟩

lemma *card-order-bd-fun*: *card-order* (*natLeq* +*c* *card-suc* (|*UNIV*|))
⟨*proof*⟩

lemma *Cinfinite-bd-fun*: *Cinfinite* (*natLeq* +*c* *card-suc* (|*UNIV*|))
⟨*proof*⟩

lemma *regularCard-bd-fun*: *regularCard* (*natLeq* +*c* *card-suc* (|*UNIV*|))
(**is** *regularCard* (- +*c* *card-suc* *?U*))
⟨*proof*⟩

lemma *ordLess-bd-fun*: |*UNIV*::'*a* *set*| <*o* *natLeq* +*c* *card-suc* (|*UNIV*::'*a* *set*|)
(**is** - <*o* (- +*c* *card-suc* (*?U* :: '*a* *rel*)))
⟨*proof*⟩

bnf '*a* ⇒ '*b*
map: (∘)
sets: *range*
bd: *natLeq* +*c* *card-suc* (|*UNIV*::'*a* *set*|)
rel: *rel-fun* (=)
pred: *pred-fun* (λ-. *True*)
⟨*proof*⟩

end

36 Shared Fixpoint Operations on Bounded Natural Functors

theory *BNF-Fixpoint-Base*
imports *BNF-Composition Basic-BNFs*
begin

lemma *conj-imp-eq-imp-imp*: (*P* ∧ *Q* ⟹ *PROP* *R*) ≡ (*P* ⟹ *Q* ⟹ *PROP* *R*)
⟨*proof*⟩

lemma *predicate2D-conj*: *P* ≤ *Q* ∧ *R* ⟹ *R* ∧ (*P* *x* *y* ⟶ *Q* *x* *y*)
⟨*proof*⟩

lemma *eq-sym-Unity-conv*: (*x* = ((() = ()))) = *x*
⟨*proof*⟩

lemma *case-unit-Unity*: (*case* *u* *of* () ⇒ *f*) = *f*
⟨*proof*⟩

lemma *case-prod-Pair-iden*: (*case* *p* *of* (*x*, *y*) ⇒ (*x*, *y*)) = *p*
⟨*proof*⟩

lemma *unit-all-impI*: (*P* () ⟹ *Q* ()) ⟹ ∀*x*. *P* *x* ⟶ *Q* *x*

⟨*proof*⟩

lemma *pointfree-idE*: $f \circ g = id \Longrightarrow f\ (g\ x) = x$
⟨*proof*⟩

lemma *o-bij*:
assumes *gf*: $g \circ f = id$ **and** *fg*: $f \circ g = id$
shows *bij f*
⟨*proof*⟩

lemma *case-sum-step*:
case-sum (*case-sum f′ g′*) *g* (*Inl p*) = *case-sum f′ g′ p*
case-sum f (*case-sum f′ g′*) (*Inr p*) = *case-sum f′ g′ p*
⟨*proof*⟩

lemma *obj-one-pointE*: $\forall x.\ s = x \longrightarrow P \Longrightarrow P$
⟨*proof*⟩

lemma *type-copy-obj-one-point-absE*:
assumes *type-definition Rep Abs UNIV* $\forall x.\ s = Abs\ x \longrightarrow P$ **shows** *P*
⟨*proof*⟩

lemma *obj-sumE-f*:
assumes $\forall x.\ s = f\ (Inl\ x) \longrightarrow P$ $\forall x.\ s = f\ (Inr\ x) \longrightarrow P$
shows $\forall x.\ s = f\ x \longrightarrow P$
⟨*proof*⟩

lemma *case-sum-if*:
case-sum f g (*if p then Inl x else Inr y*) = (*if p then f x else g y*)
⟨*proof*⟩

lemma *prod-set-simps*[*simp*]:
fsts (*x*, *y*) = {*x*}
snds (*x*, *y*) = {*y*}
⟨*proof*⟩

lemma *sum-set-simps*[*simp*]:
setl (*Inl x*) = {*x*}
setl (*Inr x*) = {}
setr (*Inl x*) = {}
setr (*Inr x*) = {*x*}
⟨*proof*⟩

lemma *Inl-Inr-False*: (*Inl x* = *Inr y*) = *False*
⟨*proof*⟩

lemma *Inr-Inl-False*: (*Inr x* = *Inl y*) = *False*
⟨*proof*⟩

lemma *spec2*: $\forall x\ y.\ P\ x\ y \Longrightarrow P\ x\ y$
⟨*proof*⟩

lemma *rewriteR-comp-comp*: $[\![g \circ h = r]\!] \Longrightarrow f \circ g \circ h = f \circ r$
⟨*proof*⟩

lemma *rewriteR-comp-comp2*: $[\![g \circ h = r1 \circ r2;\ f \circ r1 = l]\!] \Longrightarrow f \circ g \circ h = l \circ r2$
⟨*proof*⟩

lemma *rewriteL-comp-comp*: $[\![f \circ g = l]\!] \Longrightarrow f \circ (g \circ h) = l \circ h$
⟨*proof*⟩

lemma *rewriteL-comp-comp2*: $[\![f \circ g = l1 \circ l2;\ l2 \circ h = r]\!] \Longrightarrow f \circ (g \circ h) = l1 \circ r$
⟨*proof*⟩

lemma *convol-o*: $\langle f,\ g\rangle \circ h = \langle f \circ h,\ g \circ h\rangle$
⟨*proof*⟩

lemma *map-prod-o-convol*: *map-prod h1 h2* $\circ\ \langle f,\ g\rangle = \langle h1 \circ f,\ h2 \circ g\rangle$
⟨*proof*⟩

lemma *map-prod-o-convol-id*: $(\mathit{map\text{-}prod}\ f\ \mathit{id} \circ \langle \mathit{id},\ g\rangle)\ x = \langle \mathit{id} \circ f,\ g\rangle\ x$
⟨*proof*⟩

lemma *o-case-sum*: $h \circ \mathit{case\text{-}sum}\ f\ g = \mathit{case\text{-}sum}\ (h \circ f)\ (h \circ g)$
⟨*proof*⟩

lemma *case-sum-o-map-sum*: *case-sum f g* $\circ$ *map-sum h1 h2* = *case-sum* $(f \circ h1)$ $(g \circ h2)$
⟨*proof*⟩

lemma *case-sum-o-map-sum-id*: $(\mathit{case\text{-}sum}\ \mathit{id}\ g \circ \mathit{map\text{-}sum}\ f\ \mathit{id})\ x = \mathit{case\text{-}sum}\ (f \circ \mathit{id})\ g\ x$
⟨*proof*⟩

lemma *rel-fun-def-butlast*:
rel-fun R (*rel-fun S T*) *f g* = $(\forall x\ y.\ R\ x\ y \longrightarrow (\mathit{rel\text{-}fun}\ S\ T)\ (f\ x)\ (g\ y))$
⟨*proof*⟩

lemma *subst-eq-imp*: $(\forall a\ b.\ a = b \longrightarrow P\ a\ b) \equiv (\forall a.\ P\ a\ a)$
⟨*proof*⟩

lemma *eq-subset*: $(=) \leq (\lambda a\ b.\ P\ a\ b \vee a = b)$
⟨*proof*⟩

lemma *eq-le-Grp-id-iff*: $((=) \leq \mathit{Grp}\ (\mathit{Collect}\ R)\ \mathit{id}) = (\mathit{All}\ R)$
⟨*proof*⟩

lemma *Grp-id-mono-subst*: $(\bigwedge x\ y.\ Grp\ P\ id\ x\ y \Longrightarrow Grp\ Q\ id\ (f\ x)\ (f\ y)) \equiv (\bigwedge x.\ x \in P \Longrightarrow f\ x \in Q)$
⟨*proof*⟩

lemma *vimage2p-mono*: $vimage2p\ f\ g\ R\ x\ y \Longrightarrow R \leq S \Longrightarrow vimage2p\ f\ g\ S\ x\ y$
⟨*proof*⟩

lemma *vimage2p-refl*: $(\bigwedge x.\ R\ x\ x) \Longrightarrow vimage2p\ f\ f\ R\ x\ x$
⟨*proof*⟩

lemma
assumes *type-definition Rep Abs UNIV*
shows *type-copy-Rep-o-Abs*: $Rep \circ Abs = id$ **and** *type-copy-Abs-o-Rep*: $Abs \circ Rep = id$
⟨*proof*⟩

lemma *type-copy-map-comp0-undo*:
assumes *type-definition Rep Abs UNIV*
type-definition Rep' Abs' UNIV
type-definition Rep'' Abs'' UNIV
shows $Abs' \circ M \circ Rep'' = (Abs' \circ M1 \circ Rep) \circ (Abs \circ M2 \circ Rep'') \Longrightarrow M1 \circ M2 = M$
⟨*proof*⟩

lemma *vimage2p-id*: $vimage2p\ id\ id\ R = R$
⟨*proof*⟩

lemma *vimage2p-comp*: $vimage2p\ (f1 \circ f2)\ (g1 \circ g2) = vimage2p\ f2\ g2 \circ vimage2p\ f1\ g1$
⟨*proof*⟩

lemma *vimage2p-rel-fun*: $rel\text{-}fun\ (vimage2p\ f\ g\ R)\ R\ f\ g$
⟨*proof*⟩

lemma *fun-cong-unused-0*: $f = (\lambda x.\ g) \Longrightarrow f\ (\lambda x.\ 0) = g$
⟨*proof*⟩

lemma *inj-on-convol-ident*: $inj\text{-}on\ (\lambda x.\ (x, f\ x))\ X$
⟨*proof*⟩

lemma *map-sum-if-distrib-then*:
$\bigwedge f\ g\ e\ x\ y.\ map\text{-}sum\ f\ g\ (if\ e\ then\ Inl\ x\ else\ y) = (if\ e\ then\ Inl\ (f\ x)\ else\ map\text{-}sum\ f\ g\ y)$
$\bigwedge f\ g\ e\ x\ y.\ map\text{-}sum\ f\ g\ (if\ e\ then\ Inr\ x\ else\ y) = (if\ e\ then\ Inr\ (g\ x)\ else\ map\text{-}sum\ f\ g\ y)$
⟨*proof*⟩

lemma *map-sum-if-distrib-else*:

⋀*f g e x y*. *map-sum f g* (*if e then x else Inl y*) = (*if e then map-sum f g x else Inl* (*f y*))
⋀*f g e x y*. *map-sum f g* (*if e then x else Inr y*) = (*if e then map-sum f g x else Inr* (*g y*))
⟨*proof*⟩

lemma *case-prod-app*: *case-prod f x y* = *case-prod* (λ*l r*. *f l r y*) *x*
⟨*proof*⟩

lemma *case-sum-map-sum*: *case-sum l r* (*map-sum f g x*) = *case-sum* (*l* ∘ *f*) (*r* ∘ *g*) *x*
⟨*proof*⟩

lemma *case-sum-transfer*:
rel-fun (*rel-fun R T*) (*rel-fun* (*rel-fun S T*) (*rel-fun* (*rel-sum R S*) *T*)) *case-sum case-sum*
⟨*proof*⟩

lemma *case-prod-map-prod*: *case-prod h* (*map-prod f g x*) = *case-prod* (λ*l r*. *h* (*f l*) (*g r*)) *x*
⟨*proof*⟩

lemma *case-prod-o-map-prod*: *case-prod f* ∘ *map-prod g1 g2* = *case-prod* (λ*l r*. *f* (*g1 l*) (*g2 r*))
⟨*proof*⟩

lemma *case-prod-transfer*:
(*rel-fun* (*rel-fun A* (*rel-fun B C*)) (*rel-fun* (*rel-prod A B*) *C*)) *case-prod case-prod*
⟨*proof*⟩

lemma *eq-ifI*: (*P* ⟶ *t* = *u1*) ⟹ (¬ *P* ⟶ *t* = *u2*) ⟹ *t* = (*if P then u1 else u2*)
⟨*proof*⟩

lemma *comp-transfer*:
rel-fun (*rel-fun B C*) (*rel-fun* (*rel-fun A B*) (*rel-fun A C*)) (∘) (∘)
⟨*proof*⟩

lemma *If-transfer*: *rel-fun* (=) (*rel-fun A* (*rel-fun A A*)) *If If*
⟨*proof*⟩

lemma *Abs-transfer*:
assumes *type-copy1*: *type-definition Rep1 Abs1 UNIV*
assumes *type-copy2*: *type-definition Rep2 Abs2 UNIV*
shows *rel-fun R* (*vimage2p Rep1 Rep2 R*) *Abs1 Abs2*
⟨*proof*⟩

lemma *Inl-transfer*:
rel-fun S (*rel-sum S T*) *Inl Inl*

⟨*proof*⟩

lemma *Inr-transfer*:
rel-fun T (*rel-sum S T*) *Inr Inr*
⟨*proof*⟩

lemma *Pair-transfer*: *rel-fun A* (*rel-fun B* (*rel-prod A B*)) *Pair Pair*
⟨*proof*⟩

lemma *eq-onp-live-step*: $x = y \Longrightarrow$ *eq-onp P a a* $\wedge$ $x \longleftrightarrow$ *P a* $\wedge$ y
⟨*proof*⟩

lemma *top-conj*: *top x* $\wedge$ *P* $\longleftrightarrow$ *P P* $\wedge$ *top x* $\longleftrightarrow$ *P*
⟨*proof*⟩

lemma *fst-convol'*: *fst* (⟨*f*, *g*⟩ *x*) = *f x*
⟨*proof*⟩

lemma *snd-convol'*: *snd* (⟨*f*, *g*⟩ *x*) = *g x*
⟨*proof*⟩

lemma *convol-expand-snd*: *fst* ∘ *f* = *g* ⟹ ⟨*g*, *snd* ∘ *f*⟩ = *f*
⟨*proof*⟩

lemma *convol-expand-snd'*:
assumes (*fst* ∘ *f* = *g*)
shows *h* = *snd* ∘ *f* ⟷ ⟨*g*, *h*⟩ = *f*
⟨*proof*⟩

lemma *case-sum-expand-Inr-pointfree*: *f* ∘ *Inl* = *g* ⟹ *case-sum g* (*f* ∘ *Inr*) = *f*
⟨*proof*⟩

lemma *case-sum-expand-Inr'*: *f* ∘ *Inl* = *g* ⟹ *h* = *f* ∘ *Inr* ⟷ *case-sum g h* = *f*
⟨*proof*⟩

lemma *case-sum-expand-Inr*: *f* ∘ *Inl* = *g* ⟹ *f x* = *case-sum g* (*f* ∘ *Inr*) *x*
⟨*proof*⟩

lemma *id-transfer*: *rel-fun A A id id*
⟨*proof*⟩

lemma *fst-transfer*: *rel-fun* (*rel-prod A B*) *A fst fst*
⟨*proof*⟩

lemma *snd-transfer*: *rel-fun* (*rel-prod A B*) *B snd snd*
⟨*proof*⟩

lemma *convol-transfer*:
rel-fun (*rel-fun R S*) (*rel-fun* (*rel-fun R T*) (*rel-fun R* (*rel-prod S T*))) *BNF-Def.convol*

BNF-Def.convol
⟨*proof*⟩

lemma *Let-const*: *Let x* (λ-. *c*) = *c*
⟨*proof*⟩

⟨*ML*⟩

end

37 Least Fixpoint (Datatype) Operation on Bounded Natural Functors

theory *BNF-Least-Fixpoint*
imports *BNF-Fixpoint-Base*
keywords
datatype :: *thy-defn* **and**
datatype-compat :: *thy-defn*
begin

lemma *subset-emptyI*: (⋀*x*. *x* ∈ *A* ⟹ *False*) ⟹ *A* ⊆ {}
⟨*proof*⟩

lemma *image-Collect-subsetI*: (⋀*x*. *P x* ⟹ *f x* ∈ *B*) ⟹ *f* ' {*x*. *P x*} ⊆ *B*
⟨*proof*⟩

lemma *Collect-restrict*: {*x*. *x* ∈ *X* ∧ *P x*} ⊆ *X*
⟨*proof*⟩

lemma *prop-restrict*: ⟦*x* ∈ *Z*; *Z* ⊆ {*x*. *x* ∈ *X* ∧ *P x*}⟧ ⟹ *P x*
⟨*proof*⟩

lemma *underS-I*: ⟦*i* ≠ *j*; (*i*, *j*) ∈ *R*⟧ ⟹ *i* ∈ *underS R j*
⟨*proof*⟩

lemma *underS-E*: *i* ∈ *underS R j* ⟹ *i* ≠ *j* ∧ (*i*, *j*) ∈ *R*
⟨*proof*⟩

lemma *underS-Field*: *i* ∈ *underS R j* ⟹ *i* ∈ *Field R*
⟨*proof*⟩

lemma *ex-bij-betw*: |*A*| ≤*o* (*r* :: 'b *rel*) ⟹ ∃*f B* :: 'b *set*. *bij-betw f B A*
⟨*proof*⟩

lemma *bij-betwI'*:
⟦⋀*x y*. ⟦*x* ∈ *X*; *y* ∈ *X*⟧ ⟹ (*f x* = *f y*) = (*x* = *y*);
⋀*x*. *x* ∈ *X* ⟹ *f x* ∈ *Y*;
⋀*y*. *y* ∈ *Y* ⟹ ∃*x* ∈ *X*. *y* = *f x*⟧ ⟹ *bij-betw f X Y*

⟨*proof*⟩

lemma *surj-fun-eq*:
assumes *surj-on*: f ' $X = UNIV$ **and** *eq-on*: $\forall x \in X.\ (g1 \circ f)\ x = (g2 \circ f)\ x$
shows $g1 = g2$
⟨*proof*⟩

lemma *Card-order-wo-rel*: $Card\text{-}order\ r \Longrightarrow wo\text{-}rel\ r$
⟨*proof*⟩

lemma *Cinfinite-limit*: $[\![x \in Field\ r;\ Cinfinite\ r]\!] \Longrightarrow \exists y \in Field\ r.\ x \neq y \wedge (x, y) \in r$
⟨*proof*⟩

lemma *Card-order-trans*:
$[\![Card\text{-}order\ r;\ x \neq y;\ (x, y) \in r;\ y \neq z;\ (y, z) \in r]\!] \Longrightarrow x \neq z \wedge (x, z) \in r$
⟨*proof*⟩

lemma *Cinfinite-limit2*:
assumes *x1*: $x1 \in Field\ r$ **and** *x2*: $x2 \in Field\ r$ **and** *r*: $Cinfinite\ r$
shows $\exists y \in Field\ r.\ (x1 \neq y \wedge (x1, y) \in r) \wedge (x2 \neq y \wedge (x2, y) \in r)$
⟨*proof*⟩

lemma *Cinfinite-limit-finite*:
$[\![finite\ X;\ X \subseteq Field\ r;\ Cinfinite\ r]\!] \Longrightarrow \exists y \in Field\ r.\ \forall x \in X.\ (x \neq y \wedge (x, y) \in r)$
⟨*proof*⟩

lemma *insert-subsetI*: $[\![x \in A;\ X \subseteq A]\!] \Longrightarrow insert\ x\ X \subseteq A$
⟨*proof*⟩

lemmas *well-order-induct-imp* = *wo-rel.well-order-induct*[*of r* $\lambda x.\ x \in Field\ r \longrightarrow P\ x$ **for** $r\ P$]

lemma *meta-spec2*:
assumes $(\bigwedge x\ y.\ PROP\ P\ x\ y)$
shows $PROP\ P\ x\ y$
⟨*proof*⟩

lemma *nchotomy-relcomppE*:
assumes $\bigwedge y.\ \exists x.\ y = f\ x\ (r\ OO\ s)\ a\ c \bigwedge b.\ r\ a\ (f\ b) \Longrightarrow s\ (f\ b)\ c \Longrightarrow P$
shows P
⟨*proof*⟩

lemma *predicate2D-vimage2p*: $[\![R \leq vimage2p\ f\ g\ S;\ R\ x\ y]\!] \Longrightarrow S\ (f\ x)\ (g\ y)$
⟨*proof*⟩

lemma *ssubst-Pair-rhs*: $[\![(r, s) \in R;\ s' = s]\!] \Longrightarrow (r, s') \in R$
⟨*proof*⟩

lemma *all-mem-range1*:
$(\bigwedge y.\ y \in range\ f \Longrightarrow P\ y) \equiv (\bigwedge x.\ P\ (f\ x))$
⟨*proof*⟩

lemma *all-mem-range2*:
$(\bigwedge fa\ y.\ fa \in range\ f \Longrightarrow y \in range\ fa \Longrightarrow P\ y) \equiv (\bigwedge x\ xa.\ P\ (f\ x\ xa))$
⟨*proof*⟩

lemma *all-mem-range3*:
$(\bigwedge fa\ fb\ y.\ fa \in range\ f \Longrightarrow fb \in range\ fa \Longrightarrow y \in range\ fb \Longrightarrow P\ y) \equiv (\bigwedge x\ xa\ xb.\ P\ (f\ x\ xa\ xb))$
⟨*proof*⟩

lemma *all-mem-range4*:
$(\bigwedge fa\ fb\ fc\ y.\ fa \in range\ f \Longrightarrow fb \in range\ fa \Longrightarrow fc \in range\ fb \Longrightarrow y \in range\ fc \Longrightarrow P\ y) \equiv$
$(\bigwedge x\ xa\ xb\ xc.\ P\ (f\ x\ xa\ xb\ xc))$
⟨*proof*⟩

lemma *all-mem-range5*:
$(\bigwedge fa\ fb\ fc\ fd\ y.\ fa \in range\ f \Longrightarrow fb \in range\ fa \Longrightarrow fc \in range\ fb \Longrightarrow fd \in range\ fc \Longrightarrow$
$y \in range\ fd \Longrightarrow P\ y) \equiv$
$(\bigwedge x\ xa\ xb\ xc\ xd.\ P\ (f\ x\ xa\ xb\ xc\ xd))$
⟨*proof*⟩

lemma *all-mem-range6*:
$(\bigwedge fa\ fb\ fc\ fd\ fe\ ff\ y.\ fa \in range\ f \Longrightarrow fb \in range\ fa \Longrightarrow fc \in range\ fb \Longrightarrow fd \in range\ fc \Longrightarrow$
$fe \in range\ fd \Longrightarrow ff \in range\ fe \Longrightarrow y \in range\ ff \Longrightarrow P\ y) \equiv$
$(\bigwedge x\ xa\ xb\ xc\ xd\ xe\ xf.\ P\ (f\ x\ xa\ xb\ xc\ xd\ xe\ xf))$
⟨*proof*⟩

lemma *all-mem-range7*:
$(\bigwedge fa\ fb\ fc\ fd\ fe\ ff\ fg\ y.\ fa \in range\ f \Longrightarrow fb \in range\ fa \Longrightarrow fc \in range\ fb \Longrightarrow fd \in range\ fc \Longrightarrow$
$fe \in range\ fd \Longrightarrow ff \in range\ fe \Longrightarrow fg \in range\ ff \Longrightarrow y \in range\ fg \Longrightarrow P\ y) \equiv$
$(\bigwedge x\ xa\ xb\ xc\ xd\ xe\ xf\ xg.\ P\ (f\ x\ xa\ xb\ xc\ xd\ xe\ xf\ xg))$
⟨*proof*⟩

lemma *all-mem-range8*:
$(\bigwedge fa\ fb\ fc\ fd\ fe\ ff\ fg\ fh\ y.\ fa \in range\ f \Longrightarrow fb \in range\ fa \Longrightarrow fc \in range\ fb \Longrightarrow fd \in range\ fc \Longrightarrow$
$fe \in range\ fd \Longrightarrow ff \in range\ fe \Longrightarrow fg \in range\ ff \Longrightarrow fh \in range\ fg \Longrightarrow y \in range\ fh \Longrightarrow P\ y) \equiv$
$(\bigwedge x\ xa\ xb\ xc\ xd\ xe\ xf\ xg\ xh.\ P\ (f\ x\ xa\ xb\ xc\ xd\ xe\ xf\ xg\ xh))$
⟨*proof*⟩

lemmas *all-mem-range = all-mem-range1 all-mem-range2 all-mem-range3 all-mem-range4 all-mem-range5*
all-mem-range6 all-mem-range7 all-mem-range8

lemma *pred-fun-True-id*: *NO-MATCH id p* $\Longrightarrow$ *pred-fun* $(\lambda x.\ True)$ *p f = pred-fun* $(\lambda x.\ True)$ *id* $(p \circ f)$
⟨*proof*⟩

⟨*ML*⟩

end

38 Equivalence Relations in Higher-Order Set Theory

theory *Equiv-Relations*
imports *BNF-Least-Fixpoint*
begin

38.1 Equivalence relations – set version

definition *equiv* :: $'a\ set \Rightarrow ('a \times 'a)\ set \Rightarrow bool$
where *equiv A r* $\longleftrightarrow$ *refl-on A r* $\wedge$ *sym r* $\wedge$ *trans r*

lemma *equivI*: *refl-on A r* $\Longrightarrow$ *sym r* $\Longrightarrow$ *trans r* $\Longrightarrow$ *equiv A r*
⟨*proof*⟩

lemma *equivE*:
assumes *equiv A r*
obtains *refl-on A r* **and** *sym r* **and** *trans r*
⟨*proof*⟩

Suppes, Theorem 70: r is an equiv relation iff $r^{-1}\ O\ r = r$.
First half: *equiv A r* $\Longrightarrow$ $r^{-1}\ O\ r = r$.

lemma *sym-trans-comp-subset*: *sym r* $\Longrightarrow$ *trans r* $\Longrightarrow$ $r^{-1}\ O\ r \subseteq r$
⟨*proof*⟩

lemma *refl-on-comp-subset*: *refl-on A r* $\Longrightarrow$ $r \subseteq r^{-1}\ O\ r$
⟨*proof*⟩

lemma *equiv-comp-eq*: *equiv A r* $\Longrightarrow$ $r^{-1}\ O\ r = r$
⟨*proof*⟩

Second half.

lemma *comp-equivI*:
assumes $r^{-1}\ O\ r = r$ *Domain r = A*
shows *equiv A r*
⟨*proof*⟩

38.2 Equivalence classes

lemma *equiv-class-subset*: *equiv A r* $\Longrightarrow$ $(a, b) \in r \Longrightarrow r``\{a\} \subseteq r``\{b\}$
— lemma for the next result
⟨*proof*⟩

theorem *equiv-class-eq*: *equiv A r* $\Longrightarrow$ $(a, b) \in r \Longrightarrow r``\{a\} = r``\{b\}$
⟨*proof*⟩

lemma *equiv-class-self*: *equiv A r* $\Longrightarrow$ $a \in A \Longrightarrow a \in r``\{a\}$
⟨*proof*⟩

lemma *subset-equiv-class*: *equiv A r* $\Longrightarrow$ $r``\{b\} \subseteq r``\{a\} \Longrightarrow b \in A \Longrightarrow (a, b) \in r$
— lemma for the next result
⟨*proof*⟩

lemma *eq-equiv-class*: $r``\{a\} = r``\{b\} \Longrightarrow$ *equiv A r* $\Longrightarrow b \in A \Longrightarrow (a, b) \in r$
⟨*proof*⟩

lemma *equiv-class-nondisjoint*: *equiv A r* $\Longrightarrow x \in (r``\{a\} \cap r``\{b\}) \Longrightarrow (a, b) \in r$
⟨*proof*⟩

lemma *equiv-type*: *equiv A r* $\Longrightarrow r \subseteq A \times A$
⟨*proof*⟩

lemma *equiv-class-eq-iff*: *equiv A r* $\Longrightarrow (x, y) \in r \longleftrightarrow r``\{x\} = r``\{y\} \wedge x \in A \wedge y \in A$
⟨*proof*⟩

lemma *eq-equiv-class-iff*: *equiv A r* $\Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow r``\{x\} = r``\{y\} \longleftrightarrow (x, y) \in r$
⟨*proof*⟩

lemma *disjnt-equiv-class*: *equiv A r* $\Longrightarrow$ *disjnt* $(r``\{a\})\ (r``\{b\}) \longleftrightarrow (a, b) \notin r$
⟨*proof*⟩

38.3 Quotients

definition *quotient* :: *'a set* ⇒ *('a* × *'a) set* ⇒ *'a set set* (**infixl** *'//' 90*)
where $A//r = (\bigcup x \in A.\ \{r``\{x\}\})$ — set of equiv classes

lemma *quotientI*: $x \in A \Longrightarrow r``\{x\} \in A//r$
⟨*proof*⟩

lemma *quotientE*: $X \in A//r \Longrightarrow (\bigwedge x.\ X = r``\{x\} \Longrightarrow x \in A \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *Union-quotient*: *equiv A r* $\Longrightarrow \bigcup(A//r) = A$
⟨*proof*⟩

lemma *quotient-disj*: *equiv A r* $\Longrightarrow$ $X \in A//r \Longrightarrow Y \in A//r \Longrightarrow X = Y \vee X \cap Y = \{\}$
⟨*proof*⟩

lemma *quotient-eqI*:
assumes *equiv A r* $X \in A//r$ $Y \in A//r$ **and** *xy*: $x \in X$ $y \in Y$ $(x, y) \in r$
shows $X = Y$
⟨*proof*⟩

lemma *quotient-eq-iff*:
assumes *equiv A r* $X \in A//r$ $Y \in A//r$ **and** *xy*: $x \in X$ $y \in Y$
shows $X = Y \longleftrightarrow (x, y) \in r$
⟨*proof*⟩

lemma *eq-equiv-class-iff2*: *equiv A r* $\Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow \{x\}//r = \{y\}//r \longleftrightarrow (x, y) \in r$
⟨*proof*⟩

lemma *quotient-empty* [*simp*]: $\{\}//r = \{\}$
⟨*proof*⟩

lemma *quotient-is-empty* [*iff*]: $A//r = \{\} \longleftrightarrow A = \{\}$
⟨*proof*⟩

lemma *quotient-is-empty2* [*iff*]: $\{\} = A//r \longleftrightarrow A = \{\}$
⟨*proof*⟩

lemma *singleton-quotient*: $\{x\}//r = \{r \text{ `` } \{x\}\}$
⟨*proof*⟩

lemma *quotient-diff1*: *inj-on* $(\lambda a.\ \{a\}//r)$ $A \Longrightarrow a \in A \Longrightarrow (A - \{a\})//r = A//r - \{a\}//r$
⟨*proof*⟩

38.4 Refinement of one equivalence relation WRT another

lemma *refines-equiv-class-eq*: $R \subseteq S \Longrightarrow$ *equiv A R* $\Longrightarrow$ *equiv A S* $\Longrightarrow R``(S``\{a\}) = S``\{a\}$
⟨*proof*⟩

lemma *refines-equiv-class-eq2*: $R \subseteq S \Longrightarrow$ *equiv A R* $\Longrightarrow$ *equiv A S* $\Longrightarrow S``(R``\{a\}) = S``\{a\}$
⟨*proof*⟩

lemma *refines-equiv-image-eq*: $R \subseteq S \Longrightarrow$ *equiv A R* $\Longrightarrow$ *equiv A S* $\Longrightarrow (\lambda X.\ S``X) \text{ ` } (A//R) = A//S$
⟨*proof*⟩

lemma *finite-refines-finite*:

finite $(A//R) \Longrightarrow R \subseteq S \Longrightarrow$ *equiv* $A\ R \Longrightarrow$ *equiv* $A\ S \Longrightarrow$ *finite* $(A//S)$
⟨*proof*⟩

lemma *finite-refines-card-le*:
finite $(A//R) \Longrightarrow R \subseteq S \Longrightarrow$ *equiv* $A\ R \Longrightarrow$ *equiv* $A\ S \Longrightarrow$ *card* $(A//S) \leq$ *card* $(A//R)$
⟨*proof*⟩

38.5 Defining unary operations upon equivalence classes

A congruence-preserving function.

definition *congruent* :: $('a \times 'a)$ *set* $\Rightarrow ('a \Rightarrow 'b) \Rightarrow$ *bool*
where *congruent* $r\ f \longleftrightarrow (\forall (y,\ z) \in r.\ f\ y = f\ z)$

lemma *congruentI*: $(\bigwedge y\ z.\ (y,\ z) \in r \Longrightarrow f\ y = f\ z) \Longrightarrow$ *congruent* $r\ f$
⟨*proof*⟩

lemma *congruentD*: *congruent* $r\ f \Longrightarrow (y,\ z) \in r \Longrightarrow f\ y = f\ z$
⟨*proof*⟩

abbreviation *RESPECTS* :: $('a \Rightarrow 'b) \Rightarrow ('a \times 'a)$ *set* $\Rightarrow$ *bool* (**infixr** *respects* *80*)
where f *respects* $r \equiv$ *congruent* $r\ f$

lemma *UN-constant-eq*: $a \in A \Longrightarrow \forall y \in A.\ f\ y = c \Longrightarrow (\bigcup y \in A.\ f\ y) = c$
— lemma required to prove *UN-equiv-class*
⟨*proof*⟩

lemma *UN-equiv-class*:
assumes *equiv* $A\ r\ f$ *respects* $r\ a \in A$
shows $(\bigcup x \in r``\{a\}.\ f\ x) = f\ a$
— Conversion rule
⟨*proof*⟩

lemma *UN-equiv-class-type*:
assumes r: *equiv* $A\ r\ f$ *respects* r **and** X: $X \in A//r$ **and** AB: $\bigwedge x.\ x \in A \Longrightarrow f\ x \in B$
shows $(\bigcup x \in X.\ f\ x) \in B$
⟨*proof*⟩

Sufficient conditions for injectiveness. Could weaken premises! major premise could be an inclusion; *bcong* could be $\bigwedge y.\ y \in A \Longrightarrow f\ y \in B$.

lemma *UN-equiv-class-inject*:
assumes *equiv* $A\ r\ f$ *respects* r
and *eq*: $(\bigcup x \in X.\ f\ x) = (\bigcup y \in Y.\ f\ y)$
and X: $X \in A//r$ **and** Y: $Y \in A//r$
and *fr*: $\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow f\ x = f\ y \Longrightarrow (x,\ y) \in r$

shows $X = Y$
⟨*proof*⟩

38.6 Defining binary operations upon equivalence classes

A congruence-preserving function of two arguments.

definition *congruent2* :: $('a \times 'a)\ set \Rightarrow ('b \times 'b)\ set \Rightarrow ('a \Rightarrow 'b \Rightarrow 'c) \Rightarrow bool$
where *congruent2 r1 r2 f* $\longleftrightarrow$ $(\forall (y1, z1) \in r1.\ \forall (y2, z2) \in r2.\ f\ y1\ y2 = f\ z1\ z2)$

lemma *congruent2I'*:
assumes $\bigwedge y1\ z1\ y2\ z2.\ (y1, z1) \in r1 \Longrightarrow (y2, z2) \in r2 \Longrightarrow f\ y1\ y2 = f\ z1\ z2$
shows *congruent2 r1 r2 f*
⟨*proof*⟩

lemma *congruent2D*: *congruent2 r1 r2 f* $\Longrightarrow$ $(y1, z1) \in r1 \Longrightarrow (y2, z2) \in r2 \Longrightarrow$ *f y1 y2 = f z1 z2*
⟨*proof*⟩

Abbreviation for the common case where the relations are identical.

abbreviation *RESPECTS2*:: $('a \Rightarrow 'a \Rightarrow 'b) \Rightarrow ('a \times 'a)\ set \Rightarrow bool$ (**infixr** *respects2 80*)
where *f respects2 r* $\equiv$ *congruent2 r r f*

lemma *congruent2-implies-congruent*:
equiv A r1 $\Longrightarrow$ *congruent2 r1 r2 f* $\Longrightarrow$ $a \in A \Longrightarrow$ *congruent r2 (f a)*
⟨*proof*⟩

lemma *congruent2-implies-congruent-UN*:
assumes *equiv A1 r1 equiv A2 r2 congruent2 r1 r2 f a* $\in$ *A2*
shows *congruent r1* $(\lambda x1.\ \bigcup x2 \in r2``\{a\}.\ f\ x1\ x2)$
⟨*proof*⟩

lemma *UN-equiv-class2*:
equiv A1 r1 $\Longrightarrow$ *equiv A2 r2* $\Longrightarrow$ *congruent2 r1 r2 f* $\Longrightarrow$ *a1* $\in$ *A1* $\Longrightarrow$ *a2* $\in$ *A2* $\Longrightarrow$
$(\bigcup x1 \in r1``\{a1\}.\ \bigcup x2 \in r2``\{a2\}.\ f\ x1\ x2) = f\ a1\ a2$
⟨*proof*⟩

lemma *UN-equiv-class-type2*:
equiv A1 r1 $\Longrightarrow$ *equiv A2 r2* $\Longrightarrow$ *congruent2 r1 r2 f*
$\Longrightarrow$ *X1* $\in$ *A1//r1* $\Longrightarrow$ *X2* $\in$ *A2//r2*
$\Longrightarrow$ $(\bigwedge x1\ x2.\ x1 \in A1 \Longrightarrow x2 \in A2 \Longrightarrow f\ x1\ x2 \in B)$
$\Longrightarrow$ $(\bigcup x1 \in X1.\ \bigcup x2 \in X2.\ f\ x1\ x2) \in B$
⟨*proof*⟩

lemma *UN-UN-split-split-eq*:

$(\bigcup(x1,\ x2) \in X.\ \bigcup(y1,\ y2) \in Y.\ A\ x1\ x2\ y1\ y2) =$
$(\bigcup x \in X.\ \bigcup y \in Y.\ (\lambda(x1,\ x2).\ (\lambda(y1,\ y2).\ A\ x1\ x2\ y1\ y2)\ y)\ x)$
— Allows a natural expression of binary operators,
— without explicit calls to *split*
⟨*proof*⟩

lemma *congruent2I*:
$equiv\ A1\ r1 \Longrightarrow equiv\ A2\ r2$
$\Longrightarrow (\bigwedge y\ z\ w.\ w \in A2 \Longrightarrow (y,z) \in r1 \Longrightarrow f\ y\ w = f\ z\ w)$
$\Longrightarrow (\bigwedge y\ z\ w.\ w \in A1 \Longrightarrow (y,z) \in r2 \Longrightarrow f\ w\ y = f\ w\ z)$
$\Longrightarrow congruent2\ r1\ r2\ f$
— Suggested by John Harrison – the two subproofs may be
— *much* simpler than the direct proof.
⟨*proof*⟩

lemma *congruent2-commuteI*:
assumes *equivA*: $equiv\ A\ r$
and *commute*: $\bigwedge y\ z.\ y \in A \Longrightarrow z \in A \Longrightarrow f\ y\ z = f\ z\ y$
and *congt*: $\bigwedge y\ z\ w.\ w \in A \Longrightarrow (y,z) \in r \Longrightarrow f\ w\ y = f\ w\ z$
shows $f\ respects2\ r$
⟨*proof*⟩

38.7 Quotients and finiteness

Suggested by Florian Kammüller

lemma *finite-quotient*:
assumes $finite\ A\ r \subseteq A \times A$
shows $finite\ (A//r)$
— recall $equiv\ ?A\ ?r \Longrightarrow ?r \subseteq ?A \times ?A$
⟨*proof*⟩

lemma *finite-equiv-class*: $finite\ A \Longrightarrow r \subseteq A \times A \Longrightarrow X \in A//r \Longrightarrow finite\ X$
⟨*proof*⟩

lemma *equiv-imp-dvd-card*:
assumes $finite\ A\ equiv\ A\ r\ \bigwedge X.\ X \in A//r \Longrightarrow k\ dvd\ card\ X$
shows $k\ dvd\ card\ A$
⟨*proof*⟩

38.8 Projection

definition *proj* :: $('b \times 'a)\ set \Rightarrow 'b \Rightarrow 'a\ set$
where $proj\ r\ x = r\ ``\ \{x\}$

lemma *proj-preserves*: $x \in A \Longrightarrow proj\ r\ x \in A//r$
⟨*proof*⟩

lemma *proj-in-iff*:
assumes $equiv\ A\ r$

shows *proj r x* ∈ *A*//*r* ⟷ *x* ∈ *A*
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *proj-iff*: *equiv A r* ⟹ {*x*, *y*} ⊆ *A* ⟹ *proj r x* = *proj r y* ⟷ (*x*, *y*) ∈ *r*
⟨*proof*⟩

lemma *proj-image*: *proj r* ‘ *A* = *A*//*r*
⟨*proof*⟩

lemma *in-quotient-imp-non-empty*: *equiv A r* ⟹ *X* ∈ *A*//*r* ⟹ *X* ≠ {}
⟨*proof*⟩

lemma *in-quotient-imp-in-rel*: *equiv A r* ⟹ *X* ∈ *A*//*r* ⟹ {*x*, *y*} ⊆ *X* ⟹ (*x*, *y*) ∈ *r*
⟨*proof*⟩

lemma *in-quotient-imp-closed*: *equiv A r* ⟹ *X* ∈ *A*//*r* ⟹ *x* ∈ *X* ⟹ (*x*, *y*) ∈ *r* ⟹ *y* ∈ *X*
⟨*proof*⟩

lemma *in-quotient-imp-subset*: *equiv A r* ⟹ *X* ∈ *A*//*r* ⟹ *X* ⊆ *A*
⟨*proof*⟩

38.9 Equivalence relations – predicate version

Partial equivalences.

definition *part-equivp* :: (′*a* ⇒ ′*a* ⇒ *bool*) ⇒ *bool*
where *part-equivp R* ⟷ (∃ *x*. *R x x*) ∧ (∀ *x y*. *R x y* ⟷ *R x x* ∧ *R y y* ∧ *R x* = *R y*)
— John-Harrison-style characterization

lemma *part-equivpI*: ∃ *x*. *R x x* ⟹ *symp R* ⟹ *transp R* ⟹ *part-equivp R*
⟨*proof*⟩

lemma *part-equivpE*:
assumes *part-equivp R*
obtains *x* **where** *R x x* **and** *symp R* **and** *transp R*
⟨*proof*⟩

lemma *part-equivp-refl-symp-transp*: *part-equivp R* ⟷ (∃ *x*. *R x x*) ∧ *symp R* ∧ *transp R*
⟨*proof*⟩

lemma *part-equivp-symp*: *part-equivp R* ⟹ *R x y* ⟹ *R y x*
⟨*proof*⟩

lemma *part-equivp-transp*: *part-equivp R* $\Longrightarrow$ *R x y* $\Longrightarrow$ *R y z* $\Longrightarrow$ *R x z*
⟨*proof*⟩

lemma *part-equivp-typedef*: *part-equivp R* $\Longrightarrow$ $\exists\, d.\ d \in \{c.\ \exists\, x.\ R\ x\ x \wedge c = Collect\ (R\ x)\}$
⟨*proof*⟩

Total equivalences.

definition *equivp* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$
where *equivp R* $\longleftrightarrow$ $(\forall\, x\ y.\ R\ x\ y = (R\ x = R\ y))$ — John-Harrison-style characterization

lemma *equivpI*: *reflp R* $\Longrightarrow$ *symp R* $\Longrightarrow$ *transp R* $\Longrightarrow$ *equivp R*
⟨*proof*⟩

lemma *equivpE*:
assumes *equivp R*
obtains *reflp R* **and** *symp R* **and** *transp R*
⟨*proof*⟩

lemma *equivp-implies-part-equivp*: *equivp R* $\Longrightarrow$ *part-equivp R*
⟨*proof*⟩

lemma *equivp-equiv*: *equiv UNIV A* $\longleftrightarrow$ *equivp* $(\lambda x\ y.\ (x,\ y) \in A)$
⟨*proof*⟩

lemma *equivp-reflp-symp-transp*: *equivp R* $\longleftrightarrow$ *reflp R* $\wedge$ *symp R* $\wedge$ *transp R*
⟨*proof*⟩

lemma *identity-equivp*: *equivp* (=)
⟨*proof*⟩

lemma *equivp-reflp*: *equivp R* $\Longrightarrow$ *R x x*
⟨*proof*⟩

lemma *equivp-symp*: *equivp R* $\Longrightarrow$ *R x y* $\Longrightarrow$ *R y x*
⟨*proof*⟩

lemma *equivp-transp*: *equivp R* $\Longrightarrow$ *R x y* $\Longrightarrow$ *R y z* $\Longrightarrow$ *R x z*
⟨*proof*⟩

lemma *equivp-rtranclp*: *symp r* $\Longrightarrow$ *equivp* r^{**}
⟨*proof*⟩

lemmas *equivp-rtranclp-symclp* [*simp*] = *equivp-rtranclp*[*OF symp-on-symclp*]

lemma *equivp-vimage2p*: *equivp R* $\Longrightarrow$ *equivp* (*vimage2p f f R*)
⟨*proof*⟩

lemma *equivp-imp-transp*: *equivp R* $\Longrightarrow$ *transp R*
⟨*proof*⟩

38.10 Equivalence closure

definition *equivclp* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ **where**
equivclp r = $(symclp\ r)^{**}$

lemma *transp-equivclp* [*simp*]: *transp* (*equivclp r*)
⟨*proof*⟩

lemma *reflp-equivclp* [*simp*]: *reflp* (*equivclp r*)
⟨*proof*⟩

lemma *symp-equivclp* [*simp*]: *symp* (*equivclp r*)
⟨*proof*⟩

lemma *equivp-evquivclp* [*simp*]: *equivp* (*equivclp r*)
⟨*proof*⟩

lemma *tranclp-equivclp* [*simp*]: $(equivclp\ r)^{++}$ = *equivclp r*
⟨*proof*⟩

lemma *rtranclp-equivclp* [*simp*]: $(equivclp\ r)^{**}$ = *equivclp r*
⟨*proof*⟩

lemma *symclp-equivclp* [*simp*]: *symclp* (*equivclp r*) = *equivclp r*
⟨*proof*⟩

lemma *equivclp-symclp* [*simp*]: *equivclp* (*symclp r*) = *equivclp r*
⟨*proof*⟩

lemma *equivclp-conversep* [*simp*]: *equivclp* (*conversep r*) = *equivclp r*
⟨*proof*⟩

lemma *equivclp-sym* [*sym*]: *equivclp r x y* $\Longrightarrow$ *equivclp r y x*
⟨*proof*⟩

lemma *equivclp-OO-equivclp-le-equivclp*: *equivclp r OO equivclp r* $\leq$ *equivclp r*
⟨*proof*⟩

lemma *rtranlcp-le-equivclp*: $r^{**} \leq$ *equivclp r*
⟨*proof*⟩

lemma *rtranclp-conversep-le-equivclp*: $r^{-1-1**} \leq$ *equivclp r*
⟨*proof*⟩

lemma *symclp-rtranclp-le-equivclp*: *symclp* $r^{**} \leq$ *equivclp r*
⟨*proof*⟩

lemma *r-OO-conversep-into-equivclp*:
$r^{**}\ OO\ r^{-1-1**} \leq equivclp\ r$
⟨*proof*⟩

lemma *equivclp-induct* [*consumes 1*, *case-names base step*, *induct pred*: *equivclp*]:
assumes *a*: $equivclp\ r\ a\ b$
and *cases*: $P\ a\ \bigwedge y\ z.\ equivclp\ r\ a\ y \Longrightarrow r\ y\ z \vee r\ z\ y \Longrightarrow P\ y \Longrightarrow P\ z$
shows $P\ b$
⟨*proof*⟩

lemma *converse-equivclp-induct* [*consumes 1*, *case-names base step*]:
assumes *major*: $equivclp\ r\ a\ b$
and *cases*: $P\ b\ \bigwedge y\ z.\ r\ y\ z \vee r\ z\ y \Longrightarrow equivclp\ r\ z\ b \Longrightarrow P\ z \Longrightarrow P\ y$
shows $P\ a$
⟨*proof*⟩

lemma *equivclp-refl* [*simp*]: $equivclp\ r\ x\ x$
⟨*proof*⟩

lemma *r-into-equivclp* [*intro*]: $r\ x\ y \Longrightarrow equivclp\ r\ x\ y$
⟨*proof*⟩

lemma *converse-r-into-equivclp* [*intro*]: $r\ y\ x \Longrightarrow equivclp\ r\ x\ y$
⟨*proof*⟩

lemma *rtranclp-into-equivclp*: $r^{**}\ x\ y \Longrightarrow equivclp\ r\ x\ y$
⟨*proof*⟩

lemma *converse-rtranclp-into-equivclp*: $r^{**}\ y\ x \Longrightarrow equivclp\ r\ x\ y$
⟨*proof*⟩

lemma *equivclp-into-equivclp*: $[\![\ equivclp\ r\ a\ b;\ r\ b\ c \vee r\ c\ b\ ]\!] \Longrightarrow equivclp\ r\ a\ c$
⟨*proof*⟩

lemma *equivclp-trans* [*trans*]: $[\![\ equivclp\ r\ a\ b;\ equivclp\ r\ b\ c\ ]\!] \Longrightarrow equivclp\ r\ a\ c$
⟨*proof*⟩

hide-const (**open**) *proj*

end

theory *Basic-BNF-LFPs*
imports *BNF-Least-Fixpoint*
begin

definition *xtor* :: $'a \Rightarrow 'a$ **where**
$xtor\ x = x$

lemma *xtor-map*: $f\ (xtor\ x) = xtor\ (f\ x)$
⟨*proof*⟩

lemma *xtor-map-unique*: $u \circ xtor = xtor \circ f \Longrightarrow u = f$
⟨*proof*⟩

lemma *xtor-set*: $f\ (xtor\ x) = f\ x$
⟨*proof*⟩

lemma *xtor-rel*: $R\ (xtor\ x)\ (xtor\ y) = R\ x\ y$
⟨*proof*⟩

lemma *xtor-induct*: $(\bigwedge x.\ P\ (xtor\ x)) \Longrightarrow P\ z$
⟨*proof*⟩

lemma *xtor-xtor*: $xtor\ (xtor\ x) = x$
⟨*proof*⟩

lemmas *xtor-inject* = *xtor-rel*[*of* (=)]

lemma *xtor-rel-induct*: $(\bigwedge x\ y.\ vimage2p\ id\text{-}bnf\ id\text{-}bnf\ R\ x\ y \Longrightarrow IR\ (xtor\ x)\ (xtor\ y)) \Longrightarrow R \leq IR$
⟨*proof*⟩

lemma *xtor-iff-xtor*: $u = xtor\ w \longleftrightarrow xtor\ u = w$
⟨*proof*⟩

lemma *Inl-def-alt*: $Inl \equiv (\lambda a.\ xtor\ (id\text{-}bnf\ (Inl\ a)))$
⟨*proof*⟩

lemma *Inr-def-alt*: $Inr \equiv (\lambda a.\ xtor\ (id\text{-}bnf\ (Inr\ a)))$
⟨*proof*⟩

lemma *Pair-def-alt*: $Pair \equiv (\lambda a\ b.\ xtor\ (id\text{-}bnf\ (a,\ b)))$
⟨*proof*⟩

definition *ctor-rec* :: $'a \Rightarrow 'a$ **where**
ctor-rec x = *x*

lemma *ctor-rec*: $g = id \Longrightarrow ctor\text{-}rec\ f\ (xtor\ x) = f\ ((id\text{-}bnf \circ g \circ id\text{-}bnf)\ x)$
⟨*proof*⟩

lemma *ctor-rec-unique*: $g = id \Longrightarrow f \circ xtor = s \circ (id\text{-}bnf \circ g \circ id\text{-}bnf) \Longrightarrow f = ctor\text{-}rec\ s$
⟨*proof*⟩

lemma *ctor-rec-def-alt*: $f = ctor\text{-}rec\ (f \circ id\text{-}bnf)$
⟨*proof*⟩

lemma *ctor-rec-o-map*: *ctor-rec f ∘ g = ctor-rec (f ∘ (id-bnf ∘ g ∘ id-bnf))*
⟨*proof*⟩

lemma *ctor-rec-transfer*: *rel-fun (rel-fun (vimage2p id-bnf id-bnf R) S) (rel-fun R S) ctor-rec ctor-rec*
⟨*proof*⟩

lemma *eq-fst-iff*: *a = fst p ⟷ (∃ b. p = (a, b))*
⟨*proof*⟩

lemma *eq-snd-iff*: *b = snd p ⟷ (∃ a. p = (a, b))*
⟨*proof*⟩

lemma *ex-neg-all-pos*: *((∃ x. P x) ⟹ Q) ≡ (⋀x. P x ⟹ Q)*
⟨*proof*⟩

lemma *hypsubst-in-prems*: *(⋀x. y = x ⟹ z = f x ⟹ P) ≡ (z = f y ⟹ P)*
⟨*proof*⟩

lemma *isl-map-sum*:
isl (map-sum f g s) = isl s
⟨*proof*⟩

lemma *map-sum-sel*:
isl s ⟹ projl (map-sum f g s) = f (projl s)
¬ isl s ⟹ projr (map-sum f g s) = g (projr s)
⟨*proof*⟩

lemma *set-sum-sel*:
isl s ⟹ projl s ∈ setl s
¬ isl s ⟹ projr s ∈ setr s
⟨*proof*⟩

lemma *rel-sum-sel*: *rel-sum R1 R2 a b = (isl a = isl b ∧*
(isl a ⟶ isl b ⟶ R1 (projl a) (projl b)) ∧
(¬ isl a ⟶ ¬ isl b ⟶ R2 (projr a) (projr b)))
⟨*proof*⟩

lemma *isl-transfer*: *rel-fun (rel-sum A B) (=) isl isl*
⟨*proof*⟩

lemma *rel-prod-sel*: *rel-prod R1 R2 p q = (R1 (fst p) (fst q) ∧ R2 (snd p) (snd q))*
⟨*proof*⟩

⟨*ML*⟩

declare *prod.size* [*no-atp*]

hide-const (**open**) *xtor ctor-rec*

hide-fact (**open**)
xtor-def xtor-map xtor-set xtor-rel xtor-induct xtor-xtor xtor-inject ctor-rec-def ctor-rec
ctor-rec-def-alt ctor-rec-o-map xtor-rel-induct Inl-def-alt Inr-def-alt Pair-def-alt

end

39 MESON Proof Method

theory *Meson*
imports *Nat*
begin

39.1 Negation Normal Form

de Morgan laws

lemma *not-conjD*: $\neg(P \wedge Q) \Longrightarrow \neg P \vee \neg Q$
and *not-disjD*: $\neg(P \vee Q) \Longrightarrow \neg P \wedge \neg Q$
and *not-notD*: $\neg\neg P \Longrightarrow P$
and *not-allD*: $\bigwedge P.\ \neg(\forall x.\ P(x)) \Longrightarrow \exists x.\ \neg P(x)$
and *not-exD*: $\bigwedge P.\ \neg(\exists x.\ P(x)) \Longrightarrow \forall x.\ \neg P(x)$
⟨*proof*⟩

Removal of $\longrightarrow$ and $\longleftrightarrow$ (positive and negative occurrences)

lemma *imp-to-disjD*: $P \longrightarrow Q \Longrightarrow \neg P \vee Q$
and *not-impD*: $\neg(P \longrightarrow Q) \Longrightarrow P \wedge \neg Q$
and *iff-to-disjD*: $P = Q \Longrightarrow (\neg P \vee Q) \wedge (\neg Q \vee P)$
and *not-iffD*: $\neg(P = Q) \Longrightarrow (P \vee Q) \wedge (\neg P \vee \neg Q)$
— Much more efficient than $P \wedge \neg Q \vee Q \wedge \neg P$ for computing CNF
and *not-refl-disj-D*: $x \neq x \vee P \Longrightarrow P$
⟨*proof*⟩

39.2 Pulling out the existential quantifiers

Conjunction

lemma *conj-exD1*: $\bigwedge P\ Q.\ (\exists x.\ P(x)) \wedge Q \Longrightarrow \exists x.\ P(x) \wedge Q$
and *conj-exD2*: $\bigwedge P\ Q.\ P \wedge (\exists x.\ Q(x)) \Longrightarrow \exists x.\ P \wedge Q(x)$
⟨*proof*⟩

Disjunction

lemma *disj-exD*: $\bigwedge P\ Q.\ (\exists x.\ P(x)) \vee (\exists x.\ Q(x)) \Longrightarrow \exists x.\ P(x) \vee Q(x)$
— DO NOT USE with forall-Skolemization: makes fewer schematic variables!!
— With ex-Skolemization, makes fewer Skolem constants
and *disj-exD1*: $\bigwedge P\ Q.\ (\exists x.\ P(x)) \vee Q \Longrightarrow \exists x.\ P(x) \vee Q$

and *disj-exD2*: $\bigwedge P\ Q.\ P \vee (\exists x.\ Q(x)) \Longrightarrow \exists x.\ P \vee Q(x)$
⟨*proof*⟩

lemma *disj-assoc*: $(P \vee Q) \vee R \Longrightarrow P \vee (Q \vee R)$
and *disj-comm*: $P \vee Q \Longrightarrow Q \vee P$
and *disj-FalseD1*: $False \vee P \Longrightarrow P$
and *disj-FalseD2*: $P \vee False \Longrightarrow P$
⟨*proof*⟩

Generation of contrapositives

Inserts negated disjunct after removing the negation; P is a literal. Model elimination requires assuming the negation of every attempted subgoal, hence the negated disjuncts.

lemma *make-neg-rule*: $\neg P \vee Q \Longrightarrow ((\neg P \Longrightarrow P) \Longrightarrow Q)$
⟨*proof*⟩

Version for Plaisted's "Postive refinement" of the Meson procedure

lemma *make-refined-neg-rule*: $\neg P \vee Q \Longrightarrow (P \Longrightarrow Q)$
⟨*proof*⟩

P should be a literal

lemma *make-pos-rule*: $P \vee Q \Longrightarrow ((P \Longrightarrow \neg P) \Longrightarrow Q)$
⟨*proof*⟩

Versions of *make-neg-rule* and *make-pos-rule* that don't insert new assumptions, for ordinary resolution.

lemmas *make-neg-rule'* = *make-refined-neg-rule*

lemma *make-pos-rule'*: $[\![P \vee Q;\ \neg P]\!] \Longrightarrow Q$
⟨*proof*⟩

Generation of a goal clause – put away the final literal

lemma *make-neg-goal*: $\neg P \Longrightarrow ((\neg P \Longrightarrow P) \Longrightarrow False)$
⟨*proof*⟩

lemma *make-pos-goal*: $P \Longrightarrow ((P \Longrightarrow \neg P) \Longrightarrow False)$
⟨*proof*⟩

39.3 Lemmas for Forward Proof

There is a similarity to congruence rules. They are also useful in ordinary proofs.

lemma *conj-forward*: $[\![P' \wedge Q';\ \ P' \Longrightarrow P;\ \ Q' \Longrightarrow Q\ ]\!] \Longrightarrow P \wedge Q$
⟨*proof*⟩

lemma *disj-forward*: $[\![P' \vee Q';\ \ P' \Longrightarrow P;\ \ Q' \Longrightarrow Q\ ]\!] \Longrightarrow P \vee Q$

⟨*proof*⟩

lemma *imp-forward*: ⟦$P' \longrightarrow Q'$; $P \Longrightarrow P'$; $Q' \Longrightarrow Q$ ⟧ ⟹ $P \longrightarrow Q$
⟨*proof*⟩

lemma *imp-forward2*: ⟦$P' \longrightarrow Q'$; $P \Longrightarrow P'$; $P' \Longrightarrow Q' \Longrightarrow Q$ ⟧ ⟹ $P \longrightarrow Q$
⟨*proof*⟩

lemma *disj-forward2*: ⟦ $P' \vee Q'$; $P' \Longrightarrow P$; ⟦Q'; $P \Longrightarrow False$⟧ ⟹ Q⟧ ⟹ $P \vee Q$
⟨*proof*⟩

lemma *all-forward*: [| $\forall x.\ P'(x)$; !!$x.\ P'(x)$ ==> $P(x)$ |] ==> $\forall x.\ P(x)$
⟨*proof*⟩

lemma *ex-forward*: [| $\exists x.\ P'(x)$; !!$x.\ P'(x)$ ==> $P(x)$ |] ==> $\exists x.\ P(x)$
⟨*proof*⟩

39.4 Clausification helper

lemma *TruepropI*: $P \equiv Q \Longrightarrow Trueprop\ P \equiv Trueprop\ Q$
⟨*proof*⟩

lemma *ext-cong-neq*: $F\ g \neq F\ h \Longrightarrow F\ g \neq F\ h \wedge (\exists x.\ g\ x \neq h\ x)$
⟨*proof*⟩

Combinator translation helpers

definition *COMBI* :: $'a \Rightarrow 'a$ **where**
$COMBI\ P = P$

definition *COMBK* :: $'a \Rightarrow 'b \Rightarrow 'a$ **where**
$COMBK\ P\ Q = P$

definition *COMBB* :: $('b => 'c) \Rightarrow ('a => 'b) \Rightarrow 'a \Rightarrow 'c$ **where**
$COMBB\ P\ Q\ R = P\ (Q\ R)$

definition *COMBC* :: $('a \Rightarrow 'b \Rightarrow 'c) \Rightarrow 'b \Rightarrow 'a \Rightarrow 'c$ **where**
$COMBC\ P\ Q\ R = P\ R\ Q$

definition *COMBS* :: $('a \Rightarrow 'b \Rightarrow 'c) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'c$ **where**
$COMBS\ P\ Q\ R = P\ R\ (Q\ R)$

lemma *abs-S*: $\lambda x.\ (f\ x)\ (g\ x) \equiv COMBS\ f\ g$
⟨*proof*⟩

lemma *abs-I*: $\lambda x.\ x \equiv COMBI$
⟨*proof*⟩

lemma *abs-K*: $\lambda x.\ y \equiv COMBK\ y$

⟨*proof*⟩

lemma *abs-B*: $\lambda x.\ a\ (g\ x) \equiv COMBB\ a\ g$
⟨*proof*⟩

lemma *abs-C*: $\lambda x.\ (f\ x)\ b \equiv COMBC\ f\ b$
⟨*proof*⟩

39.5 Skolemization helpers

definition *skolem* :: $'a \Rightarrow 'a$ **where**
$skolem = (\lambda x.\ x)$

lemma *skolem-COMBK-iff*: $P \longleftrightarrow skolem\ (COMBK\ P\ (i{::}nat))$
⟨*proof*⟩

lemmas *skolem-COMBK-I* = *iffD1* [*OF skolem-COMBK-iff*]

39.6 Meson package

⟨*ML*⟩

hide-const (**open**) *COMBI COMBK COMBB COMBC COMBS skolem*
hide-fact (**open**) *not-conjD not-disjD not-notD not-allD not-exD imp-to-disjD not-impD iff-to-disjD not-iffD not-refl-disj-D conj-exD1 conj-exD2 disj-exD disj-exD1 disj-exD2 disj-assoc disj-comm disj-FalseD1 disj-FalseD2 TrueпropI ext-cong-neq COMBI-def COMBK-def COMBB-def COMBC-def COMBS-def abs-I abs-K abs-B abs-C abs-S skolem-def skolem-COMBK-iff skolem-COMBK-I*
end

40 Automatic Theorem Provers (ATPs)

theory *ATP*
imports *Meson Hilbert-Choice*
begin

40.1 ATP problems and proofs

⟨*ML*⟩

40.2 Higher-order reasoning helpers

definition *fFalse* :: *bool* **where**
$fFalse \longleftrightarrow False$

definition *fTrue* :: *bool* **where**
$fTrue \longleftrightarrow True$

definition *fNot* :: *bool* ⇒ *bool* **where**
fNot P ⟷ ¬ *P*

definition *fComp* :: (*'a* ⇒ *bool*) ⇒ *'a* ⇒ *bool* **where**
fComp P = (λ*x*. ¬ *P x*)

definition *fconj* :: *bool* ⇒ *bool* ⇒ *bool* **where**
fconj P Q ⟷ *P* ∧ *Q*

definition *fdisj* :: *bool* ⇒ *bool* ⇒ *bool* **where**
fdisj P Q ⟷ *P* ∨ *Q*

definition *fimplies* :: *bool* ⇒ *bool* ⇒ *bool* **where**
fimplies P Q ⟷ (*P* ⟶ *Q*)

definition *fAll* :: (*'a* ⇒ *bool*) ⇒ *bool* **where**
fAll P ⟷ *All P*

definition *fEx* :: (*'a* ⇒ *bool*) ⇒ *bool* **where**
fEx P ⟷ *Ex P*

definition *fequal* :: *'a* ⇒ *'a* ⇒ *bool* **where**
fequal x y ⟷ (*x* = *y*)

definition *fChoice* :: (*'a* ⇒ *bool*) ⇒ *'a* **where**
fChoice ≡ *Hilbert-Choice.Eps*

lemma *fTrue-ne-fFalse*: *fFalse* ≠ *fTrue*
⟨*proof*⟩

lemma *fNot-table*:
fNot fFalse = *fTrue*
fNot fTrue = *fFalse*
⟨*proof*⟩

lemma *fconj-table*:
fconj fFalse P = *fFalse*
fconj P fFalse = *fFalse*
fconj fTrue fTrue = *fTrue*
⟨*proof*⟩

lemma *fdisj-table*:
fdisj fTrue P = *fTrue*
fdisj P fTrue = *fTrue*
fdisj fFalse fFalse = *fFalse*
⟨*proof*⟩

lemma *fimplies-table*:
fimplies P fTrue = *fTrue*

fimplies fFalse P = fTrue
fimplies fTrue fFalse = fFalse
⟨*proof*⟩

lemma *fAll-table*:
Ex (fComp P) ∨ fAll P = fTrue
All P ∨ fAll P = fFalse
⟨*proof*⟩

lemma *fEx-table*:
All (fComp P) ∨ fEx P = fTrue
Ex P ∨ fEx P = fFalse
⟨*proof*⟩

lemma *fequal-table*:
fequal x x = fTrue
x = y ∨ fequal x y = fFalse
⟨*proof*⟩

lemma *fNot-law*:
fNot P ≠ P
⟨*proof*⟩

lemma *fComp-law*:
fComp P x ⟷ ¬ P x
⟨*proof*⟩

lemma *fconj-laws*:
fconj P P ⟷ P
fconj P Q ⟷ fconj Q P
fNot (fconj P Q) ⟷ fdisj (fNot P) (fNot Q)
⟨*proof*⟩

lemma *fdisj-laws*:
fdisj P P ⟷ P
fdisj P Q ⟷ fdisj Q P
fNot (fdisj P Q) ⟷ fconj (fNot P) (fNot Q)
⟨*proof*⟩

lemma *fimplies-laws*:
fimplies P Q ⟷ fdisj (¬ P) Q
fNot (fimplies P Q) ⟷ fconj P (fNot Q)
⟨*proof*⟩

lemma *fAll-law*:
fNot (fAll R) ⟷ fEx (fComp R)
⟨*proof*⟩

lemma *fEx-law*:

fNot (*fEx R*) ⟷ *fAll* (*fComp R*)
⟨*proof*⟩

lemma *fequal-laws*:
fequal x y = *fequal y x*
fequal x y = *fFalse* ∨ *fequal y z* = *fFalse* ∨ *fequal x z* = *fTrue*
fequal x y = *fFalse* ∨ *fequal* (*f x*) (*f y*) = *fTrue*
⟨*proof*⟩

lemma *fChoice-iff-Ex*: *P* (*fChoice P*) ⟷ *HOL.Ex P*
⟨*proof*⟩

We use the *Ex* constant on the right-hand side of *fChoice-iff-Ex* because we want to use the TPTP-native version if fChoice is introduced in a logic that supports FOOL. In logics that don't support it, it gets replaced by fEx during processing. Notice that we cannot use ∃ *x*. *P x*, as existentials are not skolimized by the metis proof method but *Ex P* is eta-expanded if FOOL is supported.

40.3 Basic connection between ATPs and HOL

⟨*ML*⟩

end

41 Metis Proof Method

theory *Metis*
imports *ATP*
begin

⟨*ML*⟩

41.1 Literal selection and lambda-lifting helpers

definition *select* :: *'a* ⇒ *'a* **where**
select = (λ*x*. *x*)

lemma *not-atomize*: (¬ *A* ⟹ *False*) ≡ *Trueprop A*
⟨*proof*⟩

lemma *atomize-not-select*: (*A* ⟹ *select False*) ≡ *Trueprop* (¬ *A*)
⟨*proof*⟩

lemma *not-atomize-select*: (¬ *A* ⟹ *select False*) ≡ *Trueprop A*
⟨*proof*⟩

lemma *select-FalseI*: *False* ⟹ *select False*

⟨*proof*⟩

definition *lambda* :: *'a* ⇒ *'a* **where**
lambda = (λ*x*. *x*)

lemma *eq-lambdaI*: *x* ≡ *y* ⟹ *x* ≡ *lambda y*
⟨*proof*⟩

41.2 Metis package

⟨*ML*⟩

hide-const (**open**) *select fFalse fTrue fNot fComp fconj fdisj fimplies fAll fEx fequal lambda*
hide-fact (**open**) *select-def not-atomize atomize-not-select not-atomize-select select-FalseI*
fFalse-def fTrue-def fNot-def fconj-def fdisj-def fimplies-def fAll-def fEx-def fequal-def
fTrue-ne-fFalse fNot-table fconj-table fdisj-table fimplies-table fAll-table fEx-table
fequal-table fAll-table fEx-table fNot-law fComp-law fconj-laws fdisj-laws fimplies-laws
fequal-laws fAll-law fEx-law lambda-def eq-lambdaI

end

42 Generic theorem transfer using relations

theory *Transfer*
imports *Basic-BNF-LFPs Hilbert-Choice Metis*
begin

42.1 Relator for function space

bundle *lifting-syntax*
begin
notation *rel-fun* (**infixr** ===> *55*)
notation *map-fun* (**infixr** −−−> *55*)
end

context includes *lifting-syntax*
begin

lemma *rel-funD2*:
assumes *rel-fun A B f g* **and** *A x x*
shows *B* (*f x*) (*g x*)
⟨*proof*⟩

lemma *rel-funE*:
assumes *rel-fun A B f g* **and** *A x y*

obtains $B\ (f\ x)\ (g\ y)$
⟨*proof*⟩

lemmas *rel-fun-eq* = *fun.rel-eq*

lemma *rel-fun-eq-rel*:
shows *rel-fun* $(=)\ R = (\lambda f\ g.\ \forall x.\ R\ (f\ x)\ (g\ x))$
⟨*proof*⟩

42.2 Transfer method

Explicit tag for relation membership allows for backward proof methods.

definition *Rel* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow 'a \Rightarrow 'b \Rightarrow bool$
where $Rel\ r \equiv r$

Handling of equality relations

definition *is-equality* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$
where *is-equality* $R \longleftrightarrow R = (=)$

lemma *is-equality-eq*: *is-equality* $(=)$
⟨*proof*⟩

Reverse implication for monotonicity rules

definition *rev-implies* **where**
rev-implies $x\ y \longleftrightarrow (y \longrightarrow x)$

Handling of meta-logic connectives

definition *transfer-forall* **where**
transfer-forall $\equiv$ *All*

definition *transfer-implies* **where**
transfer-implies $\equiv (\longrightarrow)$

definition *transfer-bforall* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow bool) \Rightarrow bool$
where *transfer-bforall* $\equiv (\lambda P\ Q.\ \forall x.\ P\ x \longrightarrow Q\ x)$

lemma *transfer-forall-eq*: $(\bigwedge x.\ P\ x) \equiv$ *Trueprop* (*transfer-forall* $(\lambda x.\ P\ x)$)
⟨*proof*⟩

lemma *transfer-implies-eq*: $(A \Longrightarrow B) \equiv$ *Trueprop* (*transfer-implies* $A\ B$)
⟨*proof*⟩

lemma *transfer-bforall-unfold*:
Trueprop (*transfer-bforall* $P\ (\lambda x.\ Q\ x)) \equiv (\bigwedge x.\ P\ x \Longrightarrow Q\ x)$
⟨*proof*⟩

lemma *transfer-start*: $[\![P;\ Rel\ (=)\ P\ Q]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma *transfer-start'*: $[\![P;\ Rel\ (\longrightarrow)\ P\ Q]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma *transfer-prover-start*: $[\![x = x';\ Rel\ R\ x'\ y]\!] \Longrightarrow Rel\ R\ x\ y$
⟨*proof*⟩

lemma *untransfer-start*: $[\![Q;\ Rel\ (=)\ P\ Q]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *Rel-eq-refl*: $Rel\ (=)\ x\ x$
⟨*proof*⟩

lemma *Rel-app*:
assumes $Rel\ (A\ ===>\ B)\ f\ g$ **and** $Rel\ A\ x\ y$
shows $Rel\ B\ (f\ x)\ (g\ y)$
⟨*proof*⟩

lemma *Rel-abs*:
assumes $\bigwedge x\ y.\ Rel\ A\ x\ y \Longrightarrow Rel\ B\ (f\ x)\ (g\ y)$
shows $Rel\ (A\ ===>\ B)\ (\lambda x.\ f\ x)\ (\lambda y.\ g\ y)$
⟨*proof*⟩

42.3 Predicates on relations, i.e. "class constraints"

definition *left-total* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$
where $left\text{-}total\ R \longleftrightarrow (\forall x.\ \exists y.\ R\ x\ y)$

definition *left-unique* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$
where $left\text{-}unique\ R \longleftrightarrow (\forall x\ y\ z.\ R\ x\ z \longrightarrow R\ y\ z \longrightarrow x = y)$

definition *right-total* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$
where $right\text{-}total\ R \longleftrightarrow (\forall y.\ \exists x.\ R\ x\ y)$

definition *right-unique* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$
where $right\text{-}unique\ R \longleftrightarrow (\forall x\ y\ z.\ R\ x\ y \longrightarrow R\ x\ z \longrightarrow y = z)$

definition *bi-total* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$
where $bi\text{-}total\ R \longleftrightarrow (\forall x.\ \exists y.\ R\ x\ y) \wedge (\forall y.\ \exists x.\ R\ x\ y)$

definition *bi-unique* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$
where $bi\text{-}unique\ R \longleftrightarrow$
$(\forall x\ y\ z.\ R\ x\ y \longrightarrow R\ x\ z \longrightarrow y = z) \wedge$
$(\forall x\ y\ z.\ R\ x\ z \longrightarrow R\ y\ z \longrightarrow x = y)$

lemma *left-unique-iff*: $left\text{-}unique\ R \longleftrightarrow (\forall z.\ \exists_{\leq 1} x.\ R\ x\ z)$
⟨*proof*⟩

lemma *left-uniqueI*: $(\bigwedge x\ y\ z.\ [\![\ A\ x\ z;\ A\ y\ z\ ]\!] \Longrightarrow x = y) \Longrightarrow left\text{-}unique\ A$

⟨*proof*⟩

lemma *left-uniqueD*: ⟦ *left-unique A*; *A x z*; *A y z* ⟧ ⟹ $x = y$
⟨*proof*⟩

lemma *left-totalI*:
$(\bigwedge x.\ \exists\, y.\ R\ x\ y) \Longrightarrow$ *left-total R*
⟨*proof*⟩

lemma *left-totalE*:
assumes *left-total R*
obtains $(\bigwedge x.\ \exists\, y.\ R\ x\ y)$
⟨*proof*⟩

lemma *bi-uniqueDr*: ⟦ *bi-unique A*; *A x y*; *A x z* ⟧ ⟹ $y = z$
⟨*proof*⟩

lemma *bi-uniqueDl*: ⟦ *bi-unique A*; *A x y*; *A z y* ⟧ ⟹ $x = z$
⟨*proof*⟩

lemma *bi-unique-iff*: *bi-unique R* ⟷ $(\forall\, z.\ \exists_{\le 1} x.\ R\ x\ z) \wedge (\forall\, z.\ \exists_{\le 1} x.\ R\ z\ x)$
⟨*proof*⟩

lemma *right-unique-iff*: *right-unique R* ⟷ $(\forall\, z.\ \exists_{\le 1} x.\ R\ z\ x)$
⟨*proof*⟩

lemma *right-uniqueI*: $(\bigwedge x\ y\ z.$ ⟦ *A x y*; *A x z* ⟧ ⟹ $y = z) \Longrightarrow$ *right-unique A*
⟨*proof*⟩

lemma *right-uniqueD*: ⟦ *right-unique A*; *A x y*; *A x z* ⟧ ⟹ $y = z$
⟨*proof*⟩

lemma *right-totalI*: $(\bigwedge y.\ \exists\, x.\ A\ x\ y) \Longrightarrow$ *right-total A*
⟨*proof*⟩

lemma *right-totalE*:
assumes *right-total A*
obtains *x* **where** *A x y*
⟨*proof*⟩

lemma *right-total-alt-def2*:
right-total R ⟷ ((*R* ===> (⟶)) ===> (⟶)) *All All* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *right-unique-alt-def2*:
right-unique R ⟷ (*R* ===> *R* ===> (⟶)) (=) (=)
⟨*proof*⟩

lemma *bi-total-alt-def2*:

bi-total R $\longleftrightarrow$ ((*R* ===> (=)) ===> (=)) *All All* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *bi-unique-alt-def2*:
bi-unique R $\longleftrightarrow$ (*R* ===> *R* ===> (=)) (=) (=)
⟨*proof*⟩

lemma [*simp*]:
shows *left-unique-conversep*: *left-unique* A^{-1-1} $\longleftrightarrow$ *right-unique A*
and *right-unique-conversep*: *right-unique* A^{-1-1} $\longleftrightarrow$ *left-unique A*
⟨*proof*⟩

lemma [*simp*]:
shows *left-total-conversep*: *left-total* A^{-1-1} $\longleftrightarrow$ *right-total A*
and *right-total-conversep*: *right-total* A^{-1-1} $\longleftrightarrow$ *left-total A*
⟨*proof*⟩

lemma *bi-unique-conversep* [*simp*]: *bi-unique* R^{-1-1} = *bi-unique R*
⟨*proof*⟩

lemma *bi-total-conversep* [*simp*]: *bi-total* R^{-1-1} = *bi-total R*
⟨*proof*⟩

lemma *right-unique-alt-def*: *right-unique R* = (*conversep R OO R* ≤ (=)) ⟨*proof*⟩
lemma *left-unique-alt-def*: *left-unique R* = (*R OO* (*conversep R*) ≤ (=)) ⟨*proof*⟩

lemma *right-total-alt-def*: *right-total R* = (*conversep R OO R* ≥ (=)) ⟨*proof*⟩
lemma *left-total-alt-def*: *left-total R* = (*R OO conversep R* ≥ (=)) ⟨*proof*⟩

lemma *bi-total-alt-def*: *bi-total A* = (*left-total A* ∧ *right-total A*)
⟨*proof*⟩

lemma *bi-unique-alt-def*: *bi-unique A* = (*left-unique A* ∧ *right-unique A*)
⟨*proof*⟩

lemma *bi-totalI*: *left-total R* ⟹ *right-total R* ⟹ *bi-total R*
⟨*proof*⟩

lemma *bi-uniqueI*: *left-unique R* ⟹ *right-unique R* ⟹ *bi-unique R*
⟨*proof*⟩

end

lemma *is-equality-lemma*: (⋀*R*. *is-equality R* ⟹ *PROP* (*P R*)) ≡ *PROP* (*P* (=))
⟨*proof*⟩

lemma *Domainp-lemma*: (⋀*R*. *Domainp T* = *R* ⟹ *PROP* (*P R*)) ≡ *PROP* (*P* (*Domainp T*))

⟨*proof*⟩

⟨*ML*⟩
declare *refl* [*transfer-rule*]

hide-const (**open**) *Rel*

context includes *lifting-syntax*
begin

Handling of domains

lemma *Domainp-iff*: *Domainp T x* ⟷ (∃ *y*. *T x y*)
⟨*proof*⟩

lemma *Domainp-refl*[*transfer-domain-rule*]:
Domainp T = *Domainp T* ⟨*proof*⟩

lemma *Domain-eq-top*[*transfer-domain-rule*]: *Domainp* (=) = *top* ⟨*proof*⟩

lemma *Domainp-pred-fun-eq*[*relator-domain*]:
assumes *left-unique T*
shows *Domainp* (*T* ===> *S*) = *pred-fun* (*Domainp T*) (*Domainp S*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

Properties are preserved by relation composition.

lemma *OO-def*: *R OO S* = (λ*x z*. ∃ *y*. *R x y* ∧ *S y z*)
⟨*proof*⟩

lemma *bi-total-OO*: ⟦*bi-total A*; *bi-total B*⟧ ⟹ *bi-total* (*A OO B*)
⟨*proof*⟩

lemma *bi-unique-OO*: ⟦*bi-unique A*; *bi-unique B*⟧ ⟹ *bi-unique* (*A OO B*)
⟨*proof*⟩

lemma *right-total-OO*:
⟦*right-total A*; *right-total B*⟧ ⟹ *right-total* (*A OO B*)
⟨*proof*⟩

lemma *right-unique-OO*:
⟦*right-unique A*; *right-unique B*⟧ ⟹ *right-unique* (*A OO B*)
⟨*proof*⟩

lemma *left-total-OO*: *left-total R* ⟹ *left-total S* ⟹ *left-total* (*R OO S*)
⟨*proof*⟩

lemma *left-unique-OO*: *left-unique R* ⟹ *left-unique S* ⟹ *left-unique* (*R OO S*)
⟨*proof*⟩

42.4 Properties of relators

lemma *left-total-eq*[*transfer-rule*]: *left-total* (=)
⟨*proof*⟩

lemma *left-unique-eq*[*transfer-rule*]: *left-unique* (=)
⟨*proof*⟩

lemma *right-total-eq* [*transfer-rule*]: *right-total* (=)
⟨*proof*⟩

lemma *right-unique-eq* [*transfer-rule*]: *right-unique* (=)
⟨*proof*⟩

lemma *bi-total-eq*[*transfer-rule*]: *bi-total* (=)
⟨*proof*⟩

lemma *bi-unique-eq*[*transfer-rule*]: *bi-unique* (=)
⟨*proof*⟩

lemma *left-total-fun*[*transfer-rule*]:
assumes *left-unique A left-total B*
shows *left-total* (*A* ===> *B*)
⟨*proof*⟩

lemma *left-unique-fun*[*transfer-rule*]:
⟦*left-total A*; *left-unique B*⟧ ⟹ *left-unique* (*A* ===> *B*)
⟨*proof*⟩

lemma *right-total-fun* [*transfer-rule*]:
assumes *right-unique A right-total B*
shows *right-total* (*A* ===> *B*)
⟨*proof*⟩

lemma *right-unique-fun* [*transfer-rule*]:
⟦*right-total A*; *right-unique B*⟧ ⟹ *right-unique* (*A* ===> *B*)
⟨*proof*⟩

lemma *bi-total-fun*[*transfer-rule*]:
⟦*bi-unique A*; *bi-total B*⟧ ⟹ *bi-total* (*A* ===> *B*)
⟨*proof*⟩

lemma *bi-unique-fun*[*transfer-rule*]:
⟦*bi-total A*; *bi-unique B*⟧ ⟹ *bi-unique* (*A* ===> *B*)
⟨*proof*⟩

end

lemma *if-conn*:
(*if P* ∧ *Q then t else e*) = (*if P then if Q then t else e else e*)

(*if* $P \vee Q$ *then* t *else* e) = (*if* P *then* t *else if* Q *then* t *else* e)
(*if* $P \longrightarrow Q$ *then* t *else* e) = (*if* P *then if* Q *then* t *else* e *else* t)
(*if* $\neg P$ *then* t *else* e) = (*if* P *then* e *else* t)
⟨*proof*⟩

⟨*ML*⟩

declare *pred-fun-def* [*simp*]
declare *rel-fun-eq* [*relator-eq*]

declare *fun.Domainp-rel*[*relator-domain del*]

42.5 Transfer rules

context includes *lifting-syntax*
begin

lemma *Domainp-forall-transfer* [*transfer-rule*]:
assumes *right-total A*
shows ((*A* ===> (=)) ===> (=))
(*transfer-bforall* (*Domainp A*)) *transfer-forall*
⟨*proof*⟩

Transfer rules using implication instead of equality on booleans.

lemma *transfer-forall-transfer* [*transfer-rule*]:
bi-total A ⟹ ((*A* ===> (=)) ===> (=)) *transfer-forall transfer-forall*
right-total A ⟹ ((*A* ===> (=)) ===> *implies*) *transfer-forall transfer-forall*
right-total A ⟹ ((*A* ===> *implies*) ===> *implies*) *transfer-forall transfer-forall*
bi-total A ⟹ ((*A* ===> (=)) ===> *rev-implies*) *transfer-forall transfer-forall*
bi-total A ⟹ ((*A* ===> *rev-implies*) ===> *rev-implies*) *transfer-forall transfer-forall*
⟨*proof*⟩

lemma *transfer-implies-transfer* [*transfer-rule*]:
((=) ===> (=) ===> (=)) *transfer-implies transfer-implies*
(*rev-implies* ===> *implies* ===> *implies*) *transfer-implies transfer-implies*
(*rev-implies* ===> (=) ===> *implies*) *transfer-implies transfer-implies*
((=) ===> *implies* ===> *implies*) *transfer-implies transfer-implies*
((=) ===> (=) ===> *implies*) *transfer-implies transfer-implies*
(*implies* ===> *rev-implies* ===> *rev-implies*) *transfer-implies transfer-implies*
(*implies* ===> (=) ===> *rev-implies*) *transfer-implies transfer-implies*
((=) ===> *rev-implies* ===> *rev-implies*) *transfer-implies transfer-implies*
((=) ===> (=) ===> *rev-implies*) *transfer-implies transfer-implies*
⟨*proof*⟩

lemma *eq-imp-transfer* [*transfer-rule*]:
right-unique A ⟹ (*A* ===> *A* ===> (⟶)) (=) (=)

⟨*proof*⟩

Transfer rules using equality.

lemma *left-unique-transfer* [*transfer-rule*]:
assumes *right-total A*
assumes *right-total B*
assumes *bi-unique A*
shows ((*A* ===> *B* ===> (=)) ===> *implies*) *left-unique left-unique*
⟨*proof*⟩

lemma *eq-transfer* [*transfer-rule*]:
assumes *bi-unique A*
shows (*A* ===> *A* ===> (=)) (=) (=)
⟨*proof*⟩

lemma *right-total-Ex-transfer*[*transfer-rule*]:
assumes *right-total A*
shows ((*A* ===> (=)) ===> (=)) (*Bex* (*Collect* (*Domainp A*))) *Ex*
⟨*proof*⟩

lemma *right-total-All-transfer*[*transfer-rule*]:
assumes *right-total A*
shows ((*A* ===> (=)) ===> (=)) (*Ball* (*Collect* (*Domainp A*))) *All*
⟨*proof*⟩

context
includes *lifting-syntax*
begin

lemma *right-total-fun-eq-transfer*:
assumes [*transfer-rule*]: *right-total A bi-unique B*
shows ((*A* ===> *B*) ===> (*A* ===> *B*) ===> (=)) (λ*f g*. ∀ *x*∈*Collect*(*Domainp A*). *f x* = *g x*) (=)
⟨*proof*⟩

end

lemma *All-transfer* [*transfer-rule*]:
assumes *bi-total A*
shows ((*A* ===> (=)) ===> (=)) *All All*
⟨*proof*⟩

lemma *Ex-transfer* [*transfer-rule*]:
assumes *bi-total A*
shows ((*A* ===> (=)) ===> (=)) *Ex Ex*
⟨*proof*⟩

lemma *Ex1-parametric* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A bi-total A*

shows $((A \Longrightarrow (=)) \Longrightarrow (=))$ *Ex1 Ex1*
⟨*proof*⟩

declare *If-transfer* [*transfer-rule*]

lemma *Let-transfer* [*transfer-rule*]: $(A \Longrightarrow (A \Longrightarrow B) \Longrightarrow B)$ *Let Let*
⟨*proof*⟩

declare *id-transfer* [*transfer-rule*]

declare *comp-transfer* [*transfer-rule*]

lemma *curry-transfer* [*transfer-rule*]:
$((\textit{rel-prod}\ A\ B \Longrightarrow C) \Longrightarrow A \Longrightarrow B \Longrightarrow C)$ *curry curry*
⟨*proof*⟩

lemma *fun-upd-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows $((A \Longrightarrow B) \Longrightarrow A \Longrightarrow B \Longrightarrow A \Longrightarrow B)$ *fun-upd fun-upd*
⟨*proof*⟩

lemma *case-nat-transfer* [*transfer-rule*]:
$(A \Longrightarrow ((=) \Longrightarrow A) \Longrightarrow (=) \Longrightarrow A)$ *case-nat case-nat*
⟨*proof*⟩

lemma *rec-nat-transfer* [*transfer-rule*]:
$(A \Longrightarrow ((=) \Longrightarrow A \Longrightarrow A) \Longrightarrow (=) \Longrightarrow A)$ *rec-nat rec-nat*
⟨*proof*⟩

lemma *funpow-transfer* [*transfer-rule*]:
$((=) \Longrightarrow (A \Longrightarrow A) \Longrightarrow (A \Longrightarrow A))$ *compow compow*
⟨*proof*⟩

lemma *mono-transfer*[*transfer-rule*]:
assumes [*transfer-rule*]: *bi-total A*
assumes [*transfer-rule*]: $(A \Longrightarrow A \Longrightarrow (=))\ (\leq)\ (\leq)$
assumes [*transfer-rule*]: $(B \Longrightarrow B \Longrightarrow (=))\ (\leq)\ (\leq)$
shows $((A \Longrightarrow B) \Longrightarrow (=))$ *mono mono*
⟨*proof*⟩

lemma *right-total-relcompp-transfer*[*transfer-rule*]:
assumes [*transfer-rule*]: *right-total B*
shows $((A \Longrightarrow B \Longrightarrow (=)) \Longrightarrow (B \Longrightarrow C \Longrightarrow (=)) \Longrightarrow A \Longrightarrow C \Longrightarrow (=))$
$(\lambda R\ S\ x\ z.\ \exists y{\in}\textit{Collect}\ (\textit{Domainp}\ B).\ R\ x\ y \wedge S\ y\ z)\ (OO)$
⟨*proof*⟩

lemma *relcompp-transfer*[*transfer-rule*]:

assumes [*transfer-rule*]: *bi-total B*
shows ((*A* ===> *B* ===> (=)) ===> (*B* ===> *C* ===> (=)) ===> *A* ===> *C* ===> (=)) (*OO*) (*OO*)
⟨*proof*⟩

lemma *right-total-Domainp-transfer*[*transfer-rule*]:
assumes [*transfer-rule*]: *right-total B*
shows ((*A* ===> *B* ===> (=)) ===> *A* ===> (=)) (λ*T x*. ∃ *y*∈*Collect*(*Domainp B*). *T x y*) *Domainp*
⟨*proof*⟩

lemma *Domainp-transfer*[*transfer-rule*]:
assumes [*transfer-rule*]: *bi-total B*
shows ((*A* ===> *B* ===> (=)) ===> *A* ===> (=)) *Domainp Domainp*
⟨*proof*⟩

lemma *reflp-transfer*[*transfer-rule*]:
bi-total A ⟹ ((*A* ===> *A* ===> (=)) ===> (=)) *reflp reflp*
right-total A ⟹ ((*A* ===> *A* ===> *implies*) ===> *implies*) *reflp reflp*
right-total A ⟹ ((*A* ===> *A* ===> (=)) ===> *implies*) *reflp reflp*
bi-total A ⟹ ((*A* ===> *A* ===> *rev-implies*) ===> *rev-implies*) *reflp reflp*
bi-total A ⟹ ((*A* ===> *A* ===> (=)) ===> *rev-implies*) *reflp reflp*
⟨*proof*⟩

lemma *right-unique-transfer* [*transfer-rule*]:
⟦ *right-total A*; *right-total B*; *bi-unique B* ⟧
⟹ ((*A* ===> *B* ===> (=)) ===> *implies*) *right-unique right-unique*
⟨*proof*⟩

lemma *left-total-parametric* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-total A bi-total B*
shows ((*A* ===> *B* ===> (=)) ===> (=)) *left-total left-total*
⟨*proof*⟩

lemma *right-total-parametric* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-total A bi-total B*
shows ((*A* ===> *B* ===> (=)) ===> (=)) *right-total right-total*
⟨*proof*⟩

lemma *left-unique-parametric* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A bi-total A bi-total B*
shows ((*A* ===> *B* ===> (=)) ===> (=)) *left-unique left-unique*
⟨*proof*⟩

lemma *prod-pred-parametric* [*transfer-rule*]:
((*A* ===> (=)) ===> (*B* ===> (=)) ===> *rel-prod A B* ===> (=)) *pred-prod pred-prod*
⟨*proof*⟩

lemma *apfst-parametric* [*transfer-rule*]:
((A ===> B) ===> *rel-prod* A C ===> *rel-prod* B C) *apfst apfst*
⟨*proof*⟩

lemma *rel-fun-eq-eq-onp*: ((=) ===> *eq-onp* P) = *eq-onp* ($\lambda f.\ \forall x.\ P(f\ x)$)
⟨*proof*⟩

lemma *rel-fun-eq-onp-rel*:
shows ((*eq-onp* R) ===> S) = ($\lambda f\ g.\ \forall x.\ R\ x \longrightarrow S\ (f\ x)\ (g\ x)$)
⟨*proof*⟩

lemma *eq-onp-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique* A
shows ((A ===> (=)) ===> A ===> A ===> (=)) *eq-onp eq-onp*
⟨*proof*⟩

lemma *rtranclp-parametric* [*transfer-rule*]:
assumes *bi-unique* A *bi-total* A
shows ((A ===> A ===> (=)) ===> A ===> A ===> (=)) *rtranclp rtranclp*
⟨*proof*⟩

lemma *right-unique-parametric* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-total* A *bi-unique* B *bi-total* B
shows ((A ===> B ===> (=)) ===> (=)) *right-unique right-unique*
⟨*proof*⟩

lemma *map-fun-parametric* [*transfer-rule*]:
((A ===> B) ===> (C ===> D) ===> (B ===> C) ===> A ===> D) *map-fun map-fun*
⟨*proof*⟩

end

42.6 *of-bool* and *of-nat*

context
includes *lifting-syntax*
begin

lemma *transfer-rule-of-bool*:
‹((⟷) ===> (≅)) *of-bool of-bool*›
if [*transfer-rule*]: ‹*0* ≅ *0*› ‹*1* ≅ *1*›
for R :: ‹'*a*::*zero-neq-one* ⇒ '*b*::*zero-neq-one* ⇒ *bool*› (**infix** ‹≅› *50*)
⟨*proof*⟩

lemma *transfer-rule-of-nat*:
((=) ===> (≅)) *of-nat of-nat*
if [*transfer-rule*]: ‹*0* ≅ *0*› ‹*1* ≅ *1*›

‹((≅) ===> (≅) ===> (≅)) (+) (+)›
for *R* :: ‹'*a*::*semiring-1* ⇒ '*b*::*semiring-1* ⇒ *bool*› (**infix** ‹≅› *50*)
⟨*proof*⟩

end

end

43 Lifting package

theory *Lifting*
imports *Equiv-Relations Transfer*
keywords
parametric **and**
print-quot-maps print-quotients :: *diag* **and**
lift-definition :: *thy-goal-defn* **and**
setup-lifting lifting-forget lifting-update :: *thy-decl*
begin

43.1 Function map

context includes *lifting-syntax*
begin

lemma *map-fun-id*:
(*id* −−−> *id*) = *id*
⟨*proof*⟩

43.2 Quotient Predicate

definition
Quotient R Abs Rep T ⟷
(∀ *a*. *Abs* (*Rep a*) = *a*) ∧
(∀ *a*. *R* (*Rep a*) (*Rep a*)) ∧
(∀ *r s*. *R r s* ⟷ *R r r* ∧ *R s s* ∧ *Abs r* = *Abs s*) ∧
T = (λ*x y*. *R x x* ∧ *Abs x* = *y*)

lemma *QuotientI*:
assumes ⋀*a*. *Abs* (*Rep a*) = *a*
and ⋀*a*. *R* (*Rep a*) (*Rep a*)
and ⋀*r s*. *R r s* ⟷ *R r r* ∧ *R s s* ∧ *Abs r* = *Abs s*
and *T* = (λ*x y*. *R x x* ∧ *Abs x* = *y*)
shows *Quotient R Abs Rep T*
⟨*proof*⟩

context
fixes *R Abs Rep T*
assumes *a*: *Quotient R Abs Rep T*
begin

lemma *Quotient-abs-rep*: $Abs\ (Rep\ a) = a$
⟨*proof*⟩

lemma *Quotient-rep-reflp*: $R\ (Rep\ a)\ (Rep\ a)$
⟨*proof*⟩

lemma *Quotient-rel*:
$R\ r\ r \wedge R\ s\ s \wedge Abs\ r = Abs\ s \longleftrightarrow R\ r\ s$ — orientation does not loop on rewriting
⟨*proof*⟩

lemma *Quotient-cr-rel*: $T = (\lambda x\ y.\ R\ x\ x \wedge Abs\ x = y)$
⟨*proof*⟩

lemma *Quotient-refl1*: $R\ r\ s \Longrightarrow R\ r\ r$
⟨*proof*⟩

lemma *Quotient-refl2*: $R\ r\ s \Longrightarrow R\ s\ s$
⟨*proof*⟩

lemma *Quotient-rel-rep*: $R\ (Rep\ a)\ (Rep\ b) \longleftrightarrow a = b$
⟨*proof*⟩

lemma *Quotient-rep-abs*: $R\ r\ r \Longrightarrow R\ (Rep\ (Abs\ r))\ r$
⟨*proof*⟩

lemma *Quotient-rep-abs-eq*: $R\ t\ t \Longrightarrow R \leq (=) \Longrightarrow Rep\ (Abs\ t) = t$
⟨*proof*⟩

lemma *Quotient-rep-abs-fold-unmap*:
assumes $x' \equiv Abs\ x$ **and** $R\ x\ x$ **and** $Rep\ x' \equiv Rep'\ x'$
shows $R\ (Rep'\ x')\ x$
⟨*proof*⟩

lemma *Quotient-Rep-eq*:
assumes $x' \equiv Abs\ x$
shows $Rep\ x' \equiv Rep\ x'$
⟨*proof*⟩

lemma *Quotient-rel-abs*: $R\ r\ s \Longrightarrow Abs\ r = Abs\ s$
⟨*proof*⟩

lemma *Quotient-rel-abs2*:
assumes $R\ (Rep\ x)\ y$
shows $x = Abs\ y$
⟨*proof*⟩

lemma *Quotient-symp*: $symp\ R$
⟨*proof*⟩

lemma *Quotient-transp*: *transp R*
⟨*proof*⟩

lemma *Quotient-part-equivp*: *part-equivp R*
⟨*proof*⟩

end

lemma *identity-quotient*: *Quotient* (=) *id id* (=)
⟨*proof*⟩

TODO: Use one of these alternatives as the real definition.

lemma *Quotient-alt-def*:
Quotient R Abs Rep T ⟷
(∀ *a b*. *T a b* ⟶ *Abs a* = *b*) ∧
(∀ *b*. *T* (*Rep b*) *b*) ∧
(∀ *x y*. *R x y* ⟷ *T x* (*Abs x*) ∧ *T y* (*Abs y*) ∧ *Abs x* = *Abs y*)
⟨*proof*⟩

lemma *Quotient-alt-def2*:
Quotient R Abs Rep T ⟷
(∀ *a b*. *T a b* ⟶ *Abs a* = *b*) ∧
(∀ *b*. *T* (*Rep b*) *b*) ∧
(∀ *x y*. *R x y* ⟷ *T x* (*Abs y*) ∧ *T y* (*Abs x*))
⟨*proof*⟩

lemma *Quotient-alt-def3*:
Quotient R Abs Rep T ⟷
(∀ *a b*. *T a b* ⟶ *Abs a* = *b*) ∧ (∀ *b*. *T* (*Rep b*) *b*) ∧
(∀ *x y*. *R x y* ⟷ (∃ *z*. *T x z* ∧ *T y z*))
⟨*proof*⟩

lemma *Quotient-alt-def4*:
Quotient R Abs Rep T ⟷
(∀ *a b*. *T a b* ⟶ *Abs a* = *b*) ∧ (∀ *b*. *T* (*Rep b*) *b*) ∧ *R* = *T OO conversep T*
⟨*proof*⟩

lemma *Quotient-alt-def5*:
Quotient R Abs Rep T ⟷
T ≤ *BNF-Def.Grp UNIV Abs* ∧ *BNF-Def.Grp UNIV Rep* ≤ T^{-1-1} ∧ *R* = *T OO* T^{-1-1}
⟨*proof*⟩

lemma *fun-quotient*:
assumes *1*: *Quotient R1 abs1 rep1 T1*
assumes *2*: *Quotient R2 abs2 rep2 T2*
shows *Quotient* (*R1* ===> *R2*) (*rep1* ---> *abs2*) (*abs1* ---> *rep2*) (*T1* ===> *T2*)

⟨*proof*⟩

lemma *apply-rsp*:
fixes *f g*::*'a* ⇒ *'c*
assumes *q*: *Quotient R1 Abs1 Rep1 T1*
and *a*: (*R1* ===> *R2*) *f g R1 x y*
shows *R2* (*f x*) (*g y*)
⟨*proof*⟩

lemma *apply-rsp'*:
assumes *a*: (*R1* ===> *R2*) *f g R1 x y*
shows *R2* (*f x*) (*g y*)
⟨*proof*⟩

lemma *apply-rsp''*:
assumes *Quotient R Abs Rep T*
and (*R* ===> *S*) *f f*
shows *S* (*f* (*Rep x*)) (*f* (*Rep x*))
⟨*proof*⟩

43.3 Quotient composition

lemma *Quotient-compose*:
assumes *1*: *Quotient R1 Abs1 Rep1 T1*
assumes *2*: *Quotient R2 Abs2 Rep2 T2*
shows *Quotient* (*T1 OO R2 OO conversep T1*) (*Abs2* ∘ *Abs1*) (*Rep1* ∘ *Rep2*) (*T1 OO T2*)
⟨*proof*⟩

lemma *equivp-reflp2*:
equivp R ⟹ *reflp R*
⟨*proof*⟩

43.4 Respects predicate

definition *Respects* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *'a set*
where *Respects R* = {*x*. *R x x*}

lemma *in-respects*: *x* ∈ *Respects R* ⟷ *R x x*
⟨*proof*⟩

lemma *UNIV-typedef-to-Quotient*:
assumes *type-definition Rep Abs UNIV*
and *T-def*: *T* ≡ (λ*x y*. *x* = *Rep y*)
shows *Quotient* (=) *Abs Rep T*
⟨*proof*⟩

lemma *UNIV-typedef-to-equivp*:
fixes *Abs* :: *'a* ⇒ *'b*
and *Rep* :: *'b* ⇒ *'a*

assumes *type-definition Rep Abs* (*UNIV*::'*a set*)
shows *equivp* ((=) ::'*a*⇒'*a*⇒*bool*)
⟨*proof*⟩

lemma *typedef-to-Quotient*:
assumes *type-definition Rep Abs S*
and *T-def*: *T* ≡ (λ*x y*. *x* = *Rep y*)
shows *Quotient* (*eq-onp* (λ*x*. *x* ∈ *S*)) *Abs Rep T*
⟨*proof*⟩

lemma *typedef-to-part-equivp*:
assumes *type-definition Rep Abs S*
shows *part-equivp* (*eq-onp* (λ*x*. *x* ∈ *S*))
⟨*proof*⟩

lemma *open-typedef-to-Quotient*:
assumes *type-definition Rep Abs* {*x*. *P x*}
and *T-def*: *T* ≡ (λ*x y*. *x* = *Rep y*)
shows *Quotient* (*eq-onp P*) *Abs Rep T*
⟨*proof*⟩

lemma *open-typedef-to-part-equivp*:
assumes *type-definition Rep Abs* {*x*. *P x*}
shows *part-equivp* (*eq-onp P*)
⟨*proof*⟩

lemma *type-definition-Quotient-not-empty*: *Quotient* (*eq-onp P*) *Abs Rep T* ⟹ ∃ *x*. *P x*
⟨*proof*⟩

lemma *type-definition-Quotient-not-empty-witness*: *Quotient* (*eq-onp P*) *Abs Rep T* ⟹ *P* (*Rep undefined*)
⟨*proof*⟩

Generating transfer rules for quotients.

context
fixes *R Abs Rep T*
assumes *1*: *Quotient R Abs Rep T*
begin

lemma *Quotient-right-unique*: *right-unique T*
⟨*proof*⟩

lemma *Quotient-right-total*: *right-total T*
⟨*proof*⟩

lemma *Quotient-rel-eq-transfer*: (*T* ===> *T* ===> (=)) *R* (=)
⟨*proof*⟩

lemma *Quotient-abs-induct*:
assumes $\bigwedge y.\ R\ y\ y \Longrightarrow P\ (Abs\ y)$ **shows** $P\ x$
⟨*proof*⟩

end

Generating transfer rules for total quotients.

context
fixes *R Abs Rep T*
assumes *1*: *Quotient R Abs Rep T* **and** *2*: *reflp R*
begin

lemma *Quotient-left-total*: *left-total T*
⟨*proof*⟩

lemma *Quotient-bi-total*: *bi-total T*
⟨*proof*⟩

lemma *Quotient-id-abs-transfer*: $((=) ===> T)\ (\lambda x.\ x)\ Abs$
⟨*proof*⟩

lemma *Quotient-total-abs-induct*: $(\bigwedge y.\ P\ (Abs\ y)) \Longrightarrow P\ x$
⟨*proof*⟩

lemma *Quotient-total-abs-eq-iff*: $Abs\ x = Abs\ y \longleftrightarrow R\ x\ y$
⟨*proof*⟩

end

Generating transfer rules for a type defined with *typedef*.

context
fixes *Rep Abs A T*
assumes *type*: *type-definition Rep Abs A*
assumes *T-def*: $T \equiv (\lambda(x::'a)\ (y::'b).\ x = Rep\ y)$
begin

lemma *typedef-left-unique*: *left-unique T*
⟨*proof*⟩

lemma *typedef-bi-unique*: *bi-unique T*
⟨*proof*⟩

lemma *typedef-right-unique*: *right-unique T*
⟨*proof*⟩

lemma *typedef-right-total*: *right-total T*
⟨*proof*⟩

lemma *typedef-rep-transfer*: $(T ===> (=))\ (\lambda x.\ x)\ Rep$
⟨*proof*⟩

end

Generating the correspondence rule for a constant defined with *lift-definition*.

lemma *Quotient-to-transfer*:
assumes *Quotient R Abs Rep T* **and** $R\ c\ c$ **and** $c' \equiv Abs\ c$
shows $T\ c\ c'$
⟨*proof*⟩

Proving reflexivity

lemma *Quotient-to-left-total*:
assumes *q*: *Quotient R Abs Rep T*
and *r-R*: *reflp R*
shows *left-total T*
⟨*proof*⟩

lemma *Quotient-composition-ge-eq*:
assumes *left-total T*
assumes $R \geq (=)$
shows $(T\ OO\ R\ OO\ T^{-1-1}) \geq (=)$
⟨*proof*⟩

lemma *Quotient-composition-le-eq*:
assumes *left-unique T*
assumes $R \leq (=)$
shows $(T\ OO\ R\ OO\ T^{-1-1}) \leq (=)$
⟨*proof*⟩

lemma *eq-onp-le-eq*:
eq-onp P $\leq (=)$ ⟨*proof*⟩

lemma *reflp-ge-eq*:
reflp R $\Longrightarrow R \geq (=)$ ⟨*proof*⟩

Proving a parametrized correspondence relation

definition $POS :: ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$ **where**
$POS\ A\ B \equiv A \leq B$

definition $NEG :: ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow bool$ **where**
$NEG\ A\ B \equiv B \leq A$

lemma *pos-OO-eq*:
shows $POS\ (A\ OO\ (=))\ A$
⟨*proof*⟩

lemma *pos-eq-OO*:

shows *POS* ((=) *OO A*) *A*
⟨*proof*⟩

lemma *neg-OO-eq*:
shows *NEG* (*A OO* (=)) *A*
⟨*proof*⟩

lemma *neg-eq-OO*:
shows *NEG* ((=) *OO A*) *A*
⟨*proof*⟩

lemma *POS-trans*:
assumes *POS A B*
assumes *POS B C*
shows *POS A C*
⟨*proof*⟩

lemma *NEG-trans*:
assumes *NEG A B*
assumes *NEG B C*
shows *NEG A C*
⟨*proof*⟩

lemma *POS-NEG*:
POS A B ≡ *NEG B A*
⟨*proof*⟩

lemma *NEG-POS*:
NEG A B ≡ *POS B A*
⟨*proof*⟩

lemma *POS-pcr-rule*:
assumes *POS* (*A OO B*) *C*
shows *POS* (*A OO B OO X*) (*C OO X*)
⟨*proof*⟩

lemma *NEG-pcr-rule*:
assumes *NEG* (*A OO B*) *C*
shows *NEG* (*A OO B OO X*) (*C OO X*)
⟨*proof*⟩

lemma *POS-apply*:
assumes *POS R R′*
assumes *R f g*
shows *R′ f g*
⟨*proof*⟩

Proving a parametrized correspondence relation

lemma *fun-mono*:

assumes $A \geq C$
assumes $B \leq D$
shows $(A \mathrel{===>} B) \leq (C \mathrel{===>} D)$
⟨*proof*⟩

lemma *pos-fun-distr*: $((R \mathrel{===>} S)\ OO\ (R' \mathrel{===>} S')) \leq ((R\ OO\ R') \mathrel{===>} (S\ OO\ S'))$
⟨*proof*⟩

lemma *functional-relation*: *right-unique* $R \Longrightarrow$ *left-total* $R \Longrightarrow \forall x.\ \exists!y.\ R\ x\ y$
⟨*proof*⟩

lemma *functional-converse-relation*: *left-unique* $R \Longrightarrow$ *right-total* $R \Longrightarrow \forall y.\ \exists!x.\ R\ x\ y$
⟨*proof*⟩

lemma *neg-fun-distr1*:
assumes *1*: *left-unique* R *right-total* R
assumes *2*: *right-unique* R' *left-total* R'
shows $(R\ OO\ R' \mathrel{===>} S\ OO\ S') \leq ((R \mathrel{===>} S)\ OO\ (R' \mathrel{===>} S'))$
⟨*proof*⟩

lemma *neg-fun-distr2*:
assumes *1*: *right-unique* R' *left-total* R'
assumes *2*: *left-unique* S' *right-total* S'
shows $(R\ OO\ R' \mathrel{===>} S\ OO\ S') \leq ((R \mathrel{===>} S)\ OO\ (R' \mathrel{===>} S'))$
⟨*proof*⟩

43.5 Domains

lemma *composed-equiv-rel-eq-onp*:
assumes *left-unique* R
assumes $(R \mathrel{===>} (=))\ P\ P'$
assumes *Domainp* $R = P''$
shows $(R\ OO\ eq\text{-}onp\ P'\ OO\ R^{-1-1}) = eq\text{-}onp\ (inf\ P''\ P)$
⟨*proof*⟩

lemma *composed-equiv-rel-eq-eq-onp*:
assumes *left-unique* R
assumes *Domainp* $R = P$
shows $(R\ OO\ (=)\ OO\ R^{-1-1}) = eq\text{-}onp\ P$
⟨*proof*⟩

lemma *pcr-Domainp-par-left-total*:
assumes *Domainp* $B = P$
assumes *left-total* A
assumes $(A \mathrel{===>} (=))\ P'\ P$
shows *Domainp* $(A\ OO\ B) = P'$
⟨*proof*⟩

lemma *pcr-Domainp-par*:
assumes *Domainp B = P2*
assumes *Domainp A = P1*
assumes *(A ===> (=)) P2' P2*
shows *Domainp (A OO B) = (inf P1 P2')*
⟨*proof*⟩

definition *rel-pred-comp* :: *('a => 'b => bool) => ('b => bool) => 'a => bool*
where *rel-pred-comp R P ≡ λx. ∃ y. R x y ∧ P y*

lemma *pcr-Domainp*:
assumes *Domainp B = P*
shows *Domainp (A OO B) = (λx. ∃ y. A x y ∧ P y)*
⟨*proof*⟩

lemma *pcr-Domainp-total*:
assumes *left-total B*
assumes *Domainp A = P*
shows *Domainp (A OO B) = P*
⟨*proof*⟩

lemma *Quotient-to-Domainp*:
assumes *Quotient R Abs Rep T*
shows *Domainp T = (λx. R x x)*
⟨*proof*⟩

lemma *eq-onp-to-Domainp*:
assumes *Quotient (eq-onp P) Abs Rep T*
shows *Domainp T = P*
⟨*proof*⟩

end

lemma *right-total-UNIV-transfer*:
assumes *right-total A*
shows *(rel-set A) (Collect (Domainp A)) UNIV*
⟨*proof*⟩

43.6 ML setup

⟨*ML*⟩

named-theorems *relator-eq-onp*
theorems that a relator of an eq-onp is an eq-onp of the corresponding predicate
⟨*ML*⟩

declare *fun-quotient*[*quot-map*]
declare *fun-mono*[*relator-mono*]
lemmas [*relator-distr*] = *pos-fun-distr neg-fun-distr1 neg-fun-distr2*

⟨*ML*⟩

lemma *pred-prod-beta*: *pred-prod P Q xy* ⟷ *P* (*fst xy*) ∧ *Q* (*snd xy*)
⟨*proof*⟩

lemma *pred-prod-split*: *P* (*pred-prod Q R xy*) ⟷ (∀ *x y*. *xy* = (*x*, *y*) ⟶ *P* (*Q x* ∧ *R y*))
⟨*proof*⟩

hide-const (**open**) *POS NEG*

end

44 Definition of Quotient Types

theory *Quotient*
imports *Lifting*
keywords
print-quotmapsQ3 print-quotientsQ3 print-quotconsts :: *diag* **and**
quotient-type :: *thy-goal-defn* **and** / **and**
quotient-definition :: *thy-goal-defn* **and**
copy-bnf :: *thy-defn* **and**
lift-bnf :: *thy-goal-defn*
begin

Basic definition for equivalence relations that are represented by predicates.

Composition of Relations

abbreviation
rel-conj :: ('*a* ⇒ '*b* ⇒ *bool*) ⇒ ('*b* ⇒ '*a* ⇒ *bool*) ⇒ '*a* ⇒ '*b* ⇒ *bool* (**infixr** *OOO* *75*)
where
r1 OOO r2 ≡ *r1 OO r2 OO r1*

lemma *eq-comp-r*:
shows ((=) *OOO R*) = *R*
⟨*proof*⟩

context includes *lifting-syntax*
begin

44.1 Quotient Predicate

definition
Quotient3 R Abs Rep ⟷

$(\forall\, a.\ Abs\ (Rep\ a) = a) \wedge (\forall\, a.\ R\ (Rep\ a)\ (Rep\ a)) \wedge$
$(\forall\, r\ s.\ R\ r\ s \longleftrightarrow R\ r\ r \wedge R\ s\ s \wedge Abs\ r = Abs\ s)$

lemma *Quotient3I*:
assumes $\bigwedge a.\ Abs\ (Rep\ a) = a$
and $\bigwedge a.\ R\ (Rep\ a)\ (Rep\ a)$
and $\bigwedge r\ s.\ R\ r\ s \longleftrightarrow R\ r\ r \wedge R\ s\ s \wedge Abs\ r = Abs\ s$
shows *Quotient3 R Abs Rep*
⟨*proof*⟩

context
fixes *R Abs Rep*
assumes *a*: *Quotient3 R Abs Rep*
begin

lemma *Quotient3-abs-rep*:
$Abs\ (Rep\ a) = a$
⟨*proof*⟩

lemma *Quotient3-rep-reflp*:
$R\ (Rep\ a)\ (Rep\ a)$
⟨*proof*⟩

lemma *Quotient3-rel*:
$R\ r\ r \wedge R\ s\ s \wedge Abs\ r = Abs\ s \longleftrightarrow R\ r\ s$ — orientation does not loop on rewriting
⟨*proof*⟩

lemma *Quotient3-refl1*:
$R\ r\ s \Longrightarrow R\ r\ r$
⟨*proof*⟩

lemma *Quotient3-refl2*:
$R\ r\ s \Longrightarrow R\ s\ s$
⟨*proof*⟩

lemma *Quotient3-rel-rep*:
$R\ (Rep\ a)\ (Rep\ b) \longleftrightarrow a = b$
⟨*proof*⟩

lemma *Quotient3-rep-abs*:
$R\ r\ r \Longrightarrow R\ (Rep\ (Abs\ r))\ r$
⟨*proof*⟩

lemma *Quotient3-rel-abs*:
$R\ r\ s \Longrightarrow Abs\ r = Abs\ s$
⟨*proof*⟩

lemma *Quotient3-symp*:
symp R

⟨*proof*⟩

lemma *Quotient3-transp*:
transp R
⟨*proof*⟩

lemma *Quotient3-part-equivp*:
part-equivp R
⟨*proof*⟩

lemma *abs-o-rep*:
Abs ∘ *Rep* = *id*
⟨*proof*⟩

lemma *equals-rsp*:
assumes *b*: *R xa xb R ya yb*
shows *R xa ya* = *R xb yb*
⟨*proof*⟩

lemma *rep-abs-rsp*:
assumes *b*: *R x1 x2*
shows *R x1* (*Rep* (*Abs x2*))
⟨*proof*⟩

lemma *rep-abs-rsp-left*:
assumes *b*: *R x1 x2*
shows *R* (*Rep* (*Abs x1*)) *x2*
⟨*proof*⟩

end

lemma *identity-quotient3*:
Quotient3 (=) *id id*
⟨*proof*⟩

lemma *fun-quotient3*:
assumes *q1*: *Quotient3 R1 abs1 rep1*
and *q2*: *Quotient3 R2 abs2 rep2*
shows *Quotient3* (*R1* ===> *R2*) (*rep1* ---> *abs2*) (*abs1* ---> *rep2*)
⟨*proof*⟩

lemma *lambda-prs*:
assumes *q1*: *Quotient3 R1 Abs1 Rep1*
and *q2*: *Quotient3 R2 Abs2 Rep2*
shows (*Rep1* ---> *Abs2*) (λ*x*. *Rep2* (*f* (*Abs1 x*))) = (λ*x*. *f x*)
⟨*proof*⟩

lemma *lambda-prs1*:
assumes *q1*: *Quotient3 R1 Abs1 Rep1*

and *q2*: *Quotient3 R2 Abs2 Rep2*
shows (*Rep1* −−−> *Abs2*) (λ*x*. (*Abs1* −−−> *Rep2*) *f x*) = (λ*x*. *f x*)
⟨*proof*⟩

In the following theorem R1 can be instantiated with anything, but we know some of the types of the Rep and Abs functions; so by solving Quotient assumptions we can get a unique R1 that will be provable; which is why we need to use *apply-rsp* and not the primed version

lemma *apply-rspQ3*:
fixes *f g*::'*a* ⇒ '*c*
assumes *q*: *Quotient3 R1 Abs1 Rep1*
and *a*: (*R1* ===> *R2*) *f g R1 x y*
shows *R2* (*f x*) (*g y*)
⟨*proof*⟩

lemma *apply-rspQ3''*:
assumes *Quotient3 R Abs Rep*
and (*R* ===> *S*) *f f*
shows *S* (*f* (*Rep x*)) (*f* (*Rep x*))
⟨*proof*⟩

44.2 lemmas for regularisation of ball and bex

lemma *ball-reg-eqv*:
fixes *P* :: '*a* ⇒ *bool*
assumes *a*: *equivp R*
shows *Ball* (*Respects R*) *P* = (*All P*)
⟨*proof*⟩

lemma *bex-reg-eqv*:
fixes *P* :: '*a* ⇒ *bool*
assumes *a*: *equivp R*
shows *Bex* (*Respects R*) *P* = (*Ex P*)
⟨*proof*⟩

lemma *ball-reg-right*:
assumes *a*: ⋀*x*. *x* ∈ *R* ⟹ *P x* ⟶ *Q x*
shows *All P* ⟶ *Ball R Q*
⟨*proof*⟩

lemma *bex-reg-left*:
assumes *a*: ⋀*x*. *x* ∈ *R* ⟹ *Q x* ⟶ *P x*
shows *Bex R Q* ⟶ *Ex P*
⟨*proof*⟩

lemma *ball-reg-left*:
assumes *a*: *equivp R*
shows (⋀*x*. (*Q x* ⟶ *P x*)) ⟹ *Ball* (*Respects R*) *Q* ⟶ *All P*
⟨*proof*⟩

lemma *bex-reg-right*:
assumes *a*: *equivp R*
shows $(\bigwedge x.\ (Q\ x \longrightarrow P\ x)) \Longrightarrow Ex\ Q \longrightarrow Bex\ (Respects\ R)\ P$
⟨*proof*⟩

lemma *ball-reg-eqv-range*:
fixes $P::'a \Rightarrow bool$
and $x::'a$
assumes *a*: *equivp R2*
shows $(Ball\ (Respects\ (R1 ===> R2))\ (\lambda f.\ P\ (f\ x)) = All\ (\lambda f.\ P\ (f\ x)))$
⟨*proof*⟩

lemma *bex-reg-eqv-range*:
assumes *a*: *equivp R2*
shows $(Bex\ (Respects\ (R1 ===> R2))\ (\lambda f.\ P\ (f\ x)) = Ex\ (\lambda f.\ P\ (f\ x)))$
⟨*proof*⟩

lemma *all-reg*:
assumes *a*: $\forall x :: 'a.\ (P\ x \longrightarrow Q\ x)$
and *b*: *All P*
shows *All Q*
⟨*proof*⟩

lemma *ex-reg*:
assumes *a*: $\forall x :: 'a.\ (P\ x \longrightarrow Q\ x)$
and *b*: *Ex P*
shows *Ex Q*
⟨*proof*⟩

lemma *ball-reg*:
assumes *a*: $\forall x :: 'a.\ (x \in R \longrightarrow P\ x \longrightarrow Q\ x)$
and *b*: *Ball R P*
shows *Ball R Q*
⟨*proof*⟩

lemma *bex-reg*:
assumes *a*: $\forall x :: 'a.\ (x \in R \longrightarrow P\ x \longrightarrow Q\ x)$
and *b*: *Bex R P*
shows *Bex R Q*
⟨*proof*⟩

lemma *ball-all-comm*:
assumes $\bigwedge y.\ (\forall x{\in}P.\ A\ x\ y) \longrightarrow (\forall x.\ B\ x\ y)$
shows $(\forall x{\in}P.\ \forall y.\ A\ x\ y) \longrightarrow (\forall x.\ \forall y.\ B\ x\ y)$
⟨*proof*⟩

lemma *bex-ex-comm*:
assumes $(\exists\, y.\ \exists\, x.\ A\ x\ y) \longrightarrow (\exists\, y.\ \exists\, x{\in}P.\ B\ x\ y)$
shows $(\exists\, x.\ \exists\, y.\ A\ x\ y) \longrightarrow (\exists\, x{\in}P.\ \exists\, y.\ B\ x\ y)$
⟨*proof*⟩

44.3 Bounded abstraction

definition
$Babs :: 'a\ set \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b$
where
$x \in p \Longrightarrow Babs\ p\ m\ x = m\ x$

lemma *babs-rsp*:
assumes *q*: *Quotient3 R1 Abs1 Rep1*
and *a*: *(R1 ===> R2) f g*
shows *(R1 ===> R2) (Babs (Respects R1) f) (Babs (Respects R1) g)*
⟨*proof*⟩

lemma *babs-prs*:
assumes *q1*: *Quotient3 R1 Abs1 Rep1*
and *q2*: *Quotient3 R2 Abs2 Rep2*
shows *((Rep1 −−−> Abs2) (Babs (Respects R1) ((Abs1 −−−> Rep2) f))) = f*
⟨*proof*⟩

lemma *babs-simp*:
assumes *q*: *Quotient3 R1 Abs Rep*
shows *((R1 ===> R2) (Babs (Respects R1) f) (Babs (Respects R1) g)) = ((R1 ===> R2) f g)*
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

If a user proves that a particular functional relation is an equivalence, this may be useful in regularising

lemma *babs-reg-eqv*:
shows *equivp R ⟹ Babs (Respects R) P = P*
⟨*proof*⟩

lemma *ball-rsp*:
assumes *a*: *(R ===> (=)) f g*
shows *Ball (Respects R) f = Ball (Respects R) g*
⟨*proof*⟩

lemma *bex-rsp*:
assumes *a*: *(R ===> (=)) f g*
shows *(Bex (Respects R) f = Bex (Respects R) g)*
⟨*proof*⟩

lemma *bex1-rsp*:
assumes *a*: $(R ===> (=))\ f\ g$
shows *Ex1* $(\lambda x.\ x \in Respects\ R \wedge f\ x) = Ex1\ (\lambda x.\ x \in Respects\ R \wedge g\ x)$
⟨*proof*⟩

Two lemmas needed for cleaning of quantifiers

lemma *all-prs*:
assumes *a*: *Quotient3 R absf repf*
shows *Ball* (*Respects R*) ((*absf* −−−> *id*) *f*) = *All f*
⟨*proof*⟩

lemma *ex-prs*:
assumes *a*: *Quotient3 R absf repf*
shows *Bex* (*Respects R*) ((*absf* −−−> *id*) *f*) = *Ex f*
⟨*proof*⟩

44.4 *Bex1-rel* quantifier

definition
Bex1-rel :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('a \Rightarrow bool) \Rightarrow bool$
where
$Bex1\text{-}rel\ R\ P \longleftrightarrow (\exists x \in Respects\ R.\ P\ x) \wedge (\forall x \in Respects\ R.\ \forall y \in Respects\ R.\ ((P\ x \wedge P\ y) \longrightarrow (R\ x\ y)))$

lemma *bex1-rel-aux*:
$[\![\forall xa\ ya.\ R\ xa\ ya \longrightarrow x\ xa = y\ ya;\ Bex1\text{-}rel\ R\ x]\!] \Longrightarrow Bex1\text{-}rel\ R\ y$
⟨*proof*⟩

lemma *bex1-rel-aux2*:
$[\![\forall xa\ ya.\ R\ xa\ ya \longrightarrow x\ xa = y\ ya;\ Bex1\text{-}rel\ R\ y]\!] \Longrightarrow Bex1\text{-}rel\ R\ x$
⟨*proof*⟩

lemma *bex1-rel-rsp*:
assumes *a*: *Quotient3 R absf repf*
shows $((R ===> (=)) ===> (=))\ (Bex1\text{-}rel\ R)\ (Bex1\text{-}rel\ R)$
⟨*proof*⟩

lemma *ex1-prs*:
assumes *Quotient3 R absf repf*
shows ((*absf* −−−> *id*) −−−> *id*) (*Bex1-rel R*) *f* = *Ex1 f*
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *bex1-bexeq-reg*:
shows $(\exists!x{\in}Respects\ R.\ P\ x) \longrightarrow (Bex1\text{-}rel\ R\ (\lambda x.\ P\ x))$
⟨*proof*⟩

lemma *bex1-bexeq-reg-eqv*:
assumes *a*: *equivp R*

shows $(\exists!x.\ P\ x) \longrightarrow$ *Bex1-rel R P*
⟨*proof*⟩

44.5 Various respects and preserve lemmas

lemma *quot-rel-rsp*:
assumes *a*: *Quotient3 R Abs Rep*
shows $(R$ ===> R ===> $(=))\ R\ R$
⟨*proof*⟩

lemma *o-prs*:
assumes *q1*: *Quotient3 R1 Abs1 Rep1*
and *q2*: *Quotient3 R2 Abs2 Rep2*
and *q3*: *Quotient3 R3 Abs3 Rep3*
shows $((Abs2$ ---> $Rep3)$ ---> $(Abs1$ ---> $Rep2)$ ---> $(Rep1$ ---> *Abs3*)) $(\circ) = (\circ)$
and $(id$ ---> $(Abs1$ ---> $id)$ ---> $Rep1$ ---> $id)\ (\circ) = (\circ)$
⟨*proof*⟩

lemma *o-rsp*:
$((R2$ ===> $R3)$ ===> $(R1$ ===> $R2)$ ===> $(R1$ ===> $R3))\ (\circ)\ (\circ)$
$((=)$ ===> $(R1$ ===> $(=))$ ===> $R1$ ===> $(=))\ (\circ)\ (\circ)$
⟨*proof*⟩

lemma *cond-prs*:
assumes *a*: *Quotient3 R absf repf*
shows *absf* $(if\ a\ then\ repf\ b\ else\ repf\ c) = (if\ a\ then\ b\ else\ c)$
⟨*proof*⟩

lemma *if-prs*:
assumes *q*: *Quotient3 R Abs Rep*
shows $(id$ ---> Rep ---> Rep ---> $Abs)\ If = If$
⟨*proof*⟩

lemma *if-rsp*:
assumes *q*: *Quotient3 R Abs Rep*
shows $((=)$ ===> R ===> R ===> $R)\ If\ If$
⟨*proof*⟩

lemma *let-prs*:
assumes *q1*: *Quotient3 R1 Abs1 Rep1*
and *q2*: *Quotient3 R2 Abs2 Rep2*
shows $(Rep2$ ---> $(Abs2$ ---> $Rep1)$ ---> $Abs1)\ Let = Let$
⟨*proof*⟩

lemma *let-rsp*:
shows $(R1$ ===> $(R1$ ===> $R2)$ ===> $R2)\ Let\ Let$
⟨*proof*⟩

lemma *id-rsp*:
shows $(R ===> R)\ id\ id$
⟨*proof*⟩

lemma *id-prs*:
assumes *a*: *Quotient3 R Abs Rep*
shows $(Rep ---> Abs)\ id = id$
⟨*proof*⟩

end

locale *quot-type* =
fixes $R :: 'a \Rightarrow 'a \Rightarrow bool$
and $Abs :: 'a\ set \Rightarrow 'b$
and $Rep :: 'b \Rightarrow 'a\ set$
assumes *equivp*: *part-equivp R*
and *rep-prop*: $\bigwedge y.\ \exists x.\ R\ x\ x \wedge Rep\ y = Collect\ (R\ x)$
and *rep-inverse*: $\bigwedge x.\ Abs\ (Rep\ x) = x$
and *abs-inverse*: $\bigwedge c.\ (\exists x.\ ((R\ x\ x) \wedge (c = Collect\ (R\ x)))) \Longrightarrow (Rep\ (Abs\ c)) = c$
and *rep-inject*: $\bigwedge x\ y.\ (Rep\ x = Rep\ y) = (x = y)$
begin

definition
$abs :: 'a \Rightarrow 'b$
where
$abs\ x = Abs\ (Collect\ (R\ x))$

definition
$rep :: 'b \Rightarrow 'a$
where
$rep\ a = (SOME\ x.\ x \in Rep\ a)$

lemma *some-collect*:
assumes *R r r*
shows $R\ (SOME\ x.\ x \in Collect\ (R\ r)) = R\ r$
⟨*proof*⟩

lemma *Quotient*:
shows *Quotient3 R abs rep*
⟨*proof*⟩

end

44.6 Quotient composition

lemma *OOO-quotient3*:
fixes $R1 :: 'a \Rightarrow 'a \Rightarrow bool$
fixes $Abs1 :: 'a \Rightarrow 'b$ **and** $Rep1 :: 'b \Rightarrow 'a$

fixes *Abs2* :: *'b* ⇒ *'c* **and** *Rep2* :: *'c* ⇒ *'b*
fixes *R2'* :: *'a* ⇒ *'a* ⇒ *bool*
fixes *R2* :: *'b* ⇒ *'b* ⇒ *bool*
assumes *R1*: *Quotient3 R1 Abs1 Rep1*
assumes *R2*: *Quotient3 R2 Abs2 Rep2*
assumes *Abs1*: ⋀*x y*. *R2' x y* ⟹ *R1 x x* ⟹ *R1 y y* ⟹ *R2* (*Abs1 x*) (*Abs1 y*)
assumes *Rep1*: ⋀*x y*. *R2 x y* ⟹ *R2'* (*Rep1 x*) (*Rep1 y*)
shows *Quotient3* (*R1 OO R2' OO R1*) (*Abs2* ∘ *Abs1*) (*Rep1* ∘ *Rep2*)
⟨*proof*⟩

lemma *OOO-eq-quotient3*:
fixes *R1* :: *'a* ⇒ *'a* ⇒ *bool*
fixes *Abs1* :: *'a* ⇒ *'b* **and** *Rep1* :: *'b* ⇒ *'a*
fixes *Abs2* :: *'b* ⇒ *'c* **and** *Rep2* :: *'c* ⇒ *'b*
assumes *R1*: *Quotient3 R1 Abs1 Rep1*
assumes *R2*: *Quotient3* (=) *Abs2 Rep2*
shows *Quotient3* (*R1 OOO* (=)) (*Abs2* ∘ *Abs1*) (*Rep1* ∘ *Rep2*)
⟨*proof*⟩

44.7 Quotient3 to Quotient

lemma *Quotient3-to-Quotient*:
assumes *Quotient3 R Abs Rep*
and *T* ≡ λ*x y*. *R x x* ∧ *Abs x* = *y*
shows *Quotient R Abs Rep T*
⟨*proof*⟩

lemma *Quotient3-to-Quotient-equivp*:
assumes *q*: *Quotient3 R Abs Rep*
and *T-def*: *T* ≡ λ*x y*. *Abs x* = *y*
and *eR*: *equivp R*
shows *Quotient R Abs Rep T*
⟨*proof*⟩

44.8 ML setup

Auxiliary data for the quotient package

named-theorems *quot-equiv equivalence relation theorems*
and *quot-respect respectfulness theorems*
and *quot-preserve preservation theorems*
and *id-simps identity simp rules for maps*
and *quot-thm quotient theorems*
⟨*ML*⟩

declare [[*mapQ3 fun* = (*rel-fun*, *fun-quotient3*)]]

lemmas [*quot-thm*] = *fun-quotient3*
lemmas [*quot-respect*] = *quot-rel-rsp if-rsp o-rsp let-rsp id-rsp*
lemmas [*quot-preserve*] = *if-prs o-prs let-prs id-prs*

lemmas [*quot-equiv*] = *identity-equivp*

Lemmas about simplifying id's.

lemmas [*id-simps*] =
id-def[*symmetric*]
map-fun-id
id-apply
id-o
o-id
eq-comp-r
vimage-id

Translation functions for the lifting process.

⟨*ML*⟩

Definitions of the quotient types.

⟨*ML*⟩

Definitions for quotient constants.

⟨*ML*⟩

An auxiliary constant for recording some information about the lifted theorem in a tactic.

definition
Quot-True :: *'a* ⇒ *bool*
where
Quot-True x ⟷ *True*

lemma
shows *QT-all*: *Quot-True* (*All P*) ⟹ *Quot-True P*
and *QT-ex*: *Quot-True* (*Ex P*) ⟹ *Quot-True P*
and *QT-ex1*: *Quot-True* (*Ex1 P*) ⟹ *Quot-True P*
and *QT-lam*: *Quot-True* (λ*x*. *P x*) ⟹ (⋀*x*. *Quot-True* (*P x*))
and *QT-ext*: (⋀*x*. *Quot-True* (*a x*) ⟹ *f x* = *g x*) ⟹ (*Quot-True a* ⟹ *f* = *g*)
⟨*proof*⟩

lemma *QT-imp*: *Quot-True a* ≡ *Quot-True b*
⟨*proof*⟩

context includes *lifting-syntax*
begin

Tactics for proving the lifted theorems

⟨*ML*⟩

end

44.9 Methods / Interface

⟨*ML*⟩

no-notation
rel-conj (**infixr** *OOO* 75)

45 Lifting of BNFs

lemma *sum-insert-Inl-unit*: $x \in A \Longrightarrow (\bigwedge y.\ x = Inr\ y \Longrightarrow Inr\ y \in B) \Longrightarrow x \in insert\ (Inl\ ())\ B$
⟨*proof*⟩

lemma *lift-sum-unit-vimage-commute*:
$insert\ (Inl\ ())\ (Inr\ `\ f\ -`\ A) = map\text{-}sum\ id\ f\ -`\ insert\ (Inl\ ())\ (Inr\ `\ A)$
⟨*proof*⟩

lemma *insert-Inl-int-map-sum-unit*: $insert\ (Inl\ ())\ A \cap range\ (map\text{-}sum\ id\ f) \neq \{\}$
⟨*proof*⟩

lemma *image-map-sum-unit-subset*:
$A \subseteq insert\ (Inl\ ())\ (Inr\ `\ B) \Longrightarrow map\text{-}sum\ id\ f\ `\ A \subseteq insert\ (Inl\ ())\ (Inr\ `\ f\ `\ B)$
⟨*proof*⟩

lemma *subset-lift-sum-unitD*: $A \subseteq insert\ (Inl\ ())\ (Inr\ `\ B) \Longrightarrow Inr\ x \in A \Longrightarrow x \in B$
⟨*proof*⟩

lemma *UNIV-sum-unit-conv*: $insert\ (Inl\ ())\ (range\ Inr) = UNIV$
⟨*proof*⟩

lemma *subset-vimage-image-subset*: $A \subseteq f\ -`\ B \Longrightarrow f\ `\ A \subseteq B$
⟨*proof*⟩

lemma *relcompp-mem-Grp-neq-bot*:
$A \cap range\ f \neq \{\} \Longrightarrow (\lambda x\ y.\ x \in A \wedge y \in A)\ OO\ (Grp\ UNIV\ f)^{-1-1} \neq bot$
⟨*proof*⟩

lemma *comp-projr-Inr*: $projr \circ Inr = id$
⟨*proof*⟩

lemma *in-rel-sum-in-image-projr*:
$B \subseteq \{(x,y).\ rel\text{-}sum\ ((=) :: unit \Rightarrow unit \Rightarrow bool)\ A\ x\ y\} \Longrightarrow$
$Inr\ `\ C = fst\ `\ B \Longrightarrow snd\ `\ B = Inr\ `\ D \Longrightarrow map\text{-}prod\ projr\ projr\ `\ B \subseteq \{(x,y).\ A\ x\ y\}$
⟨*proof*⟩

lemma *subset-rel-sumI*: $B \subseteq \{(x,y).\ A\ x\ y\} \Longrightarrow rel\text{-}sum\ ((=) :: unit => unit =>$

bool) *A*
(*if* $x \in B$ *then Inr* (*fst x*) *else Inl* ())
(*if* $x \in B$ *then Inr* (*snd x*) *else Inl* ())
⟨*proof*⟩

lemma *relcompp-eq-Grp-neq-bot*: $(=)$ *OO* $(Grp\ UNIV\ f)^{-1-1} \neq bot$
⟨*proof*⟩

lemma *rel-fun-rel-OO1*: (*rel-fun Q* (*rel-fun R* $(=)$)) *A B* $\Longrightarrow$ *conversep Q OO A OO R* $\leq$ *B*
⟨*proof*⟩

lemma *rel-fun-rel-OO2*: (*rel-fun Q* (*rel-fun R* $(=)$)) *A B* $\Longrightarrow$ *Q OO B OO conversep R* $\leq$ *A*
⟨*proof*⟩

lemma *rel-sum-eq2-nonempty*: *rel-sum* $(=)$ *A OO rel-sum* $(=)$ *B* $\neq$ *bot*
⟨*proof*⟩

lemma *rel-sum-eq3-nonempty*: *rel-sum* $(=)$ *A OO* (*rel-sum* $(=)$ *B OO rel-sum* $(=)$ *C*) $\neq$ *bot*
⟨*proof*⟩

lemma *hypsubst*: $A = B \Longrightarrow x \in B \Longrightarrow (x \in A \Longrightarrow P) \Longrightarrow P$ ⟨*proof*⟩

lemma *Quotient-crel-quotient*: *Quotient R Abs Rep T* $\Longrightarrow$ *equivp R* $\Longrightarrow T \equiv (\lambda x\ y.\ Abs\ x = y)$
⟨*proof*⟩

lemma *Quotient-crel-typedef*: *Quotient* (*eq-onp P*) *Abs Rep T* $\Longrightarrow T \equiv (\lambda x\ y.\ x = Rep\ y)$
⟨*proof*⟩

lemma *Quotient-crel-typecopy*: *Quotient* $(=)$ *Abs Rep T* $\Longrightarrow T \equiv (\lambda x\ y.\ x = Rep\ y)$
⟨*proof*⟩

lemma *equivp-add-relconj*:
assumes *equiv*: *equivp R equivp R′* **and** *le*: *S OO T OO U* $\leq$ *R OO STU OO R′*
shows *R OO S OO T OO U OO R′* $\leq$ *R OO STU OO R′*
⟨*proof*⟩

lemma *Grp-conversep-eq-onp*: $((BNF\text{-}Def.Grp\ UNIV\ f)^{-1-1}\ OO\ BNF\text{-}Def.Grp\ UNIV\ f) = eq\text{-}onp\ (\lambda x.\ x \in range\ f)$
⟨*proof*⟩

lemma *Grp-conversep-nonempty*: $(BNF\text{-}Def.Grp\ UNIV\ f)^{-1-1}\ OO\ BNF\text{-}Def.Grp\ UNIV\ f \neq bot$
⟨*proof*⟩

lemma *relcomppI2*: *r a b* ⟹ *s b c* ⟹ *t c d* ⟹ (*r OO s OO t*) *a d*
⟨*proof*⟩

lemma *rel-conj-eq-onp*: *equivp R* ⟹ *rel-conj R* (*eq-onp P*) ≤ *R*
⟨*proof*⟩

lemma *Quotient-Quotient3*: *Quotient R Abs Rep T* ⟹ *Quotient3 R Abs Rep*
⟨*proof*⟩

lemma *Quotient-reflp-imp-equivp*: *Quotient R Abs Rep T* ⟹ *reflp R* ⟹ *equivp R*
⟨*proof*⟩

lemma *Quotient-eq-onp-typedef*:
Quotient (*eq-onp P*) *Abs Rep cr* ⟹ *type-definition Rep Abs* {*x*. *P x*}
⟨*proof*⟩

lemma *Quotient-eq-onp-type-copy*:
Quotient (=) *Abs Rep cr* ⟹ *type-definition Rep Abs UNIV*
⟨*proof*⟩

⟨*ML*⟩

hide-fact
sum-insert-Inl-unit lift-sum-unit-vimage-commute insert-Inl-int-map-sum-unit image-map-sum-unit-subset subset-lift-sum-unitD UNIV-sum-unit-conv subset-vimage-image-subset relcompp-mem-Grp-neq-bot comp-projr-Inr in-rel-sum-in-image-projr subset-rel-sumI relcompp-eq-Grp-neq-bot rel-fun-rel-OO1 rel-fun-rel-OO2 rel-sum-eq2-nonempty rel-sum-eq3-nonempty hypsubst equivp-add-relconj Grp-conversep-eq-onp Grp-conversep-nonempty rel-comppI2 rel-conj-eq-onp Quotient-reflp-imp-equivp Quotient-Quotient3

end

46 Binary Numerals

theory *Num*
imports *BNF-Least-Fixpoint Transfer*
begin

46.1 The *num* type

datatype *num* = *One* | *Bit0 num* | *Bit1 num*

Increment function for type *num*

primrec *inc* :: *num* ⇒ *num*
where

inc One = Bit0 One
| *inc (Bit0 x) = Bit1 x*
| *inc (Bit1 x) = Bit0 (inc x)*

Converting between type *num* and type *nat*

primrec *nat-of-num* :: *num* ⇒ *nat*
where
nat-of-num One = Suc 0
| *nat-of-num (Bit0 x) = nat-of-num x + nat-of-num x*
| *nat-of-num (Bit1 x) = Suc (nat-of-num x + nat-of-num x)*

primrec *num-of-nat* :: *nat* ⇒ *num*
where
num-of-nat 0 = One
| *num-of-nat (Suc n) = (if 0 < n then inc (num-of-nat n) else One)*

lemma *nat-of-num-pos*: *0 < nat-of-num x*
⟨*proof*⟩

lemma *nat-of-num-neq-0*: *nat-of-num x ≠ 0*
⟨*proof*⟩

lemma *nat-of-num-inc*: *nat-of-num (inc x) = Suc (nat-of-num x)*
⟨*proof*⟩

lemma *num-of-nat-double*: *0 < n ⟹ num-of-nat (n + n) = Bit0 (num-of-nat n)*
⟨*proof*⟩

Type *num* is isomorphic to the strictly positive natural numbers.

lemma *nat-of-num-inverse*: *num-of-nat (nat-of-num x) = x*
⟨*proof*⟩

lemma *num-of-nat-inverse*: *0 < n ⟹ nat-of-num (num-of-nat n) = n*
⟨*proof*⟩

lemma *num-eq-iff*: *x = y ⟷ nat-of-num x = nat-of-num y*
⟨*proof*⟩

lemma *num-induct* [*case-names One inc*]:
fixes *P* :: *num* ⇒ *bool*
assumes *One*: *P One*
and *inc*: ⋀*x*. *P x ⟹ P (inc x)*
shows *P x*
⟨*proof*⟩

From now on, there are two possible models for *num*: as positive naturals (rule *num-induct*) and as digit representation (rules *num.induct*, *num.cases*).

46.2 Numeral operations

instantiation *num* :: {*plus*,*times*,*linorder*}
begin

definition [*code del*]: $m + n = \textit{num-of-nat}\ (\textit{nat-of-num}\ m + \textit{nat-of-num}\ n)$

definition [*code del*]: $m * n = \textit{num-of-nat}\ (\textit{nat-of-num}\ m * \textit{nat-of-num}\ n)$

definition [*code del*]: $m \leq n \longleftrightarrow \textit{nat-of-num}\ m \leq \textit{nat-of-num}\ n$

definition [*code del*]: $m < n \longleftrightarrow \textit{nat-of-num}\ m < \textit{nat-of-num}\ n$

instance
⟨*proof*⟩

end

lemma *nat-of-num-add*: $\textit{nat-of-num}\ (x + y) = \textit{nat-of-num}\ x + \textit{nat-of-num}\ y$
⟨*proof*⟩

lemma *nat-of-num-mult*: $\textit{nat-of-num}\ (x * y) = \textit{nat-of-num}\ x * \textit{nat-of-num}\ y$
⟨*proof*⟩

lemma *add-num-simps* [*simp*, *code*]:
$\textit{One} + \textit{One} = \textit{Bit0}\ \textit{One}$
$\textit{One} + \textit{Bit0}\ n = \textit{Bit1}\ n$
$\textit{One} + \textit{Bit1}\ n = \textit{Bit0}\ (n + \textit{One})$
$\textit{Bit0}\ m + \textit{One} = \textit{Bit1}\ m$
$\textit{Bit0}\ m + \textit{Bit0}\ n = \textit{Bit0}\ (m + n)$
$\textit{Bit0}\ m + \textit{Bit1}\ n = \textit{Bit1}\ (m + n)$
$\textit{Bit1}\ m + \textit{One} = \textit{Bit0}\ (m + \textit{One})$
$\textit{Bit1}\ m + \textit{Bit0}\ n = \textit{Bit1}\ (m + n)$
$\textit{Bit1}\ m + \textit{Bit1}\ n = \textit{Bit0}\ (m + n + \textit{One})$
⟨*proof*⟩

lemma *mult-num-simps* [*simp*, *code*]:
$m * \textit{One} = m$
$\textit{One} * n = n$
$\textit{Bit0}\ m * \textit{Bit0}\ n = \textit{Bit0}\ (\textit{Bit0}\ (m * n))$
$\textit{Bit0}\ m * \textit{Bit1}\ n = \textit{Bit0}\ (m * \textit{Bit1}\ n)$
$\textit{Bit1}\ m * \textit{Bit0}\ n = \textit{Bit0}\ (\textit{Bit1}\ m * n)$
$\textit{Bit1}\ m * \textit{Bit1}\ n = \textit{Bit1}\ (m + n + \textit{Bit0}\ (m * n))$
⟨*proof*⟩

lemma *eq-num-simps*:
$\textit{One} = \textit{One} \longleftrightarrow \textit{True}$
$\textit{One} = \textit{Bit0}\ n \longleftrightarrow \textit{False}$
$\textit{One} = \textit{Bit1}\ n \longleftrightarrow \textit{False}$
$\textit{Bit0}\ m = \textit{One} \longleftrightarrow \textit{False}$

$Bit1\ m = One \longleftrightarrow False$
$Bit0\ m = Bit0\ n \longleftrightarrow m = n$
$Bit0\ m = Bit1\ n \longleftrightarrow False$
$Bit1\ m = Bit0\ n \longleftrightarrow False$
$Bit1\ m = Bit1\ n \longleftrightarrow m = n$
⟨*proof*⟩

lemma *le-num-simps* [*simp*, *code*]:
$One \leq n \longleftrightarrow True$
$Bit0\ m \leq One \longleftrightarrow False$
$Bit1\ m \leq One \longleftrightarrow False$
$Bit0\ m \leq Bit0\ n \longleftrightarrow m \leq n$
$Bit0\ m \leq Bit1\ n \longleftrightarrow m \leq n$
$Bit1\ m \leq Bit1\ n \longleftrightarrow m \leq n$
$Bit1\ m \leq Bit0\ n \longleftrightarrow m < n$
⟨*proof*⟩

lemma *less-num-simps* [*simp*, *code*]:
$m < One \longleftrightarrow False$
$One < Bit0\ n \longleftrightarrow True$
$One < Bit1\ n \longleftrightarrow True$
$Bit0\ m < Bit0\ n \longleftrightarrow m < n$
$Bit0\ m < Bit1\ n \longleftrightarrow m \leq n$
$Bit1\ m < Bit1\ n \longleftrightarrow m < n$
$Bit1\ m < Bit0\ n \longleftrightarrow m < n$
⟨*proof*⟩

lemma *le-num-One-iff*: $x \leq num.One \longleftrightarrow x = num.One$
⟨*proof*⟩

Rules using *One* and *inc* as constructors.

lemma *add-One*: $x + One = inc\ x$
⟨*proof*⟩

lemma *add-One-commute*: $One + n = n + One$
⟨*proof*⟩

lemma *add-inc*: $x + inc\ y = inc\ (x + y)$
⟨*proof*⟩

lemma *mult-inc*: $x * inc\ y = x * y + x$
⟨*proof*⟩

The *num-of-nat* conversion.

lemma *num-of-nat-One*: $n \leq 1 \Longrightarrow num\text{-}of\text{-}nat\ n = One$
⟨*proof*⟩

lemma *num-of-nat-plus-distrib*:
$0 < m \Longrightarrow 0 < n \Longrightarrow num\text{-}of\text{-}nat\ (m + n) = num\text{-}of\text{-}nat\ m + num\text{-}of\text{-}nat\ n$

⟨*proof*⟩

A double-and-decrement function.

primrec *BitM* :: *num* ⇒ *num*
where
BitM One = *One*
| *BitM* (*Bit0 n*) = *Bit1* (*BitM n*)
| *BitM* (*Bit1 n*) = *Bit1* (*Bit0 n*)

lemma *BitM-plus-one*: *BitM n* + *One* = *Bit0 n*
⟨*proof*⟩

lemma *one-plus-BitM*: *One* + *BitM n* = *Bit0 n*
⟨*proof*⟩

lemma *BitM-inc-eq*:
‹*Num.BitM* (*Num.inc n*) = *Num.Bit1 n*›
⟨*proof*⟩

lemma *inc-BitM-eq*:
‹*Num.inc* (*Num.BitM n*) = *num.Bit0 n*›
⟨*proof*⟩

Squaring and exponentiation.

primrec *sqr* :: *num* ⇒ *num*
where
sqr One = *One*
| *sqr* (*Bit0 n*) = *Bit0* (*Bit0* (*sqr n*))
| *sqr* (*Bit1 n*) = *Bit1* (*Bit0* (*sqr n* + *n*))

primrec *pow* :: *num* ⇒ *num* ⇒ *num*
where
pow x One = *x*
| *pow x* (*Bit0 y*) = *sqr* (*pow x y*)
| *pow x* (*Bit1 y*) = *sqr* (*pow x y*) ∗ *x*

lemma *nat-of-num-sqr*: *nat-of-num* (*sqr x*) = *nat-of-num x* ∗ *nat-of-num x*
⟨*proof*⟩

lemma *sqr-conv-mult*: *sqr x* = *x* ∗ *x*
⟨*proof*⟩

lemma *num-double* [*simp*]:
num.Bit0 num.One ∗ *n* = *num.Bit0 n*
⟨*proof*⟩

46.3 Binary numerals

We embed binary representations into a generic algebraic structure using *numeral*.

class *numeral* = *one* + *semigroup-add*
begin

primrec *numeral* :: *num* ⇒ *'a*
where
numeral-One: *numeral One* = *1*
| *numeral-Bit0*: *numeral* (*Bit0 n*) = *numeral n* + *numeral n*
| *numeral-Bit1*: *numeral* (*Bit1 n*) = *numeral n* + *numeral n* + *1*

lemma *numeral-code* [*code*]:
numeral One = *1*
numeral (*Bit0 n*) = (*let m* = *numeral n in m* + *m*)
numeral (*Bit1 n*) = (*let m* = *numeral n in m* + *m* + *1*)
⟨*proof*⟩

lemma *one-plus-numeral-commute*: *1* + *numeral x* = *numeral x* + *1*
⟨*proof*⟩

lemma *numeral-inc*: *numeral* (*inc x*) = *numeral x* + *1*
⟨*proof*⟩

declare *numeral.simps* [*simp del*]

abbreviation *Numeral1* ≡ *numeral One*

declare *numeral-One* [*code-post*]

end

Numeral syntax.

syntax
-Numeral :: *num-const* ⇒ *'a* (-)

⟨*ML*⟩

46.4 Class-specific numeral rules

numeral is a morphism.

46.4.1 Structures with addition: class *numeral*

context *numeral*
begin

lemma *numeral-add*: *numeral* $(m + n) = numeral\ m + numeral\ n$
⟨*proof*⟩

lemma *numeral-plus-numeral*: $numeral\ m + numeral\ n = numeral\ (m + n)$
⟨*proof*⟩

lemma *numeral-plus-one*: $numeral\ n + 1 = numeral\ (n + One)$
⟨*proof*⟩

lemma *one-plus-numeral*: $1 + numeral\ n = numeral\ (One + n)$
⟨*proof*⟩

lemma *one-add-one*: $1 + 1 = 2$
⟨*proof*⟩

lemmas *add-numeral-special* =
numeral-plus-one one-plus-numeral one-add-one

end

46.4.2 Structures with negation: class *neg-numeral*

class *neg-numeral* = *numeral* + *group-add*
begin

lemma *uminus-numeral-One*: $- Numeral1 = - 1$
⟨*proof*⟩

Numerals form an abelian subgroup.

inductive *is-num* :: $'a \Rightarrow bool$
where
is-num 1
| $is\text{-}num\ x \Longrightarrow is\text{-}num\ (- x)$
| $is\text{-}num\ x \Longrightarrow is\text{-}num\ y \Longrightarrow is\text{-}num\ (x + y)$

lemma *is-num-numeral*: $is\text{-}num\ (numeral\ k)$
⟨*proof*⟩

lemma *is-num-add-commute*: $is\text{-}num\ x \Longrightarrow is\text{-}num\ y \Longrightarrow x + y = y + x$
⟨*proof*⟩

lemma *is-num-add-left-commute*: $is\text{-}num\ x \Longrightarrow is\text{-}num\ y \Longrightarrow x + (y + z) = y + (x + z)$
⟨*proof*⟩

lemmas *is-num-normalize* =
add.assoc is-num-add-commute is-num-add-left-commute
is-num.intros is-num-numeral
minus-add

definition *dbl* :: *'a* ⇒ *'a*
where *dbl x* = *x* + *x*

definition *dbl-inc* :: *'a* ⇒ *'a*
where *dbl-inc x* = *x* + *x* + *1*

definition *dbl-dec* :: *'a* ⇒ *'a*
where *dbl-dec x* = *x* + *x* − *1*

definition *sub* :: *num* ⇒ *num* ⇒ *'a*
where *sub k l* = *numeral k* − *numeral l*

lemma *numeral-BitM*: *numeral* (*BitM n*) = *numeral* (*Bit0 n*) − *1*
⟨*proof*⟩

lemma *sub-inc-One-eq*:
‹*Num.sub* (*Num.inc n*) *num.One* = *numeral n*›
⟨*proof*⟩

lemma *dbl-simps* [*simp*]:
dbl (− *numeral k*) = − *dbl* (*numeral k*)
dbl 0 = *0*
dbl 1 = *2*
dbl (− *1*) = − *2*
dbl (*numeral k*) = *numeral* (*Bit0 k*)
⟨*proof*⟩

lemma *dbl-inc-simps* [*simp*]:
dbl-inc (− *numeral k*) = − *dbl-dec* (*numeral k*)
dbl-inc 0 = *1*
dbl-inc 1 = *3*
dbl-inc (− *1*) = − *1*
dbl-inc (*numeral k*) = *numeral* (*Bit1 k*)
⟨*proof*⟩

lemma *dbl-dec-simps* [*simp*]:
dbl-dec (− *numeral k*) = − *dbl-inc* (*numeral k*)
dbl-dec 0 = − *1*
dbl-dec 1 = *1*
dbl-dec (− *1*) = − *3*
dbl-dec (*numeral k*) = *numeral* (*BitM k*)
⟨*proof*⟩

lemma *sub-num-simps* [*simp*]:
sub One One = *0*
sub One (*Bit0 l*) = − *numeral* (*BitM l*)
sub One (*Bit1 l*) = − *numeral* (*Bit0 l*)
sub (*Bit0 k*) *One* = *numeral* (*BitM k*)

sub (Bit1 k) One = numeral (Bit0 k)
sub (Bit0 k) (Bit0 l) = dbl (sub k l)
sub (Bit0 k) (Bit1 l) = dbl-dec (sub k l)
sub (Bit1 k) (Bit0 l) = dbl-inc (sub k l)
sub (Bit1 k) (Bit1 l) = dbl (sub k l)
⟨*proof*⟩

lemma *add-neg-numeral-simps*:
numeral m + - numeral n = sub m n
- numeral m + numeral n = sub n m
- numeral m + - numeral n = - (numeral m + numeral n)
⟨*proof*⟩

lemma *add-neg-numeral-special*:
1 + - numeral m = sub One m
- numeral m + 1 = sub One m
numeral m + - 1 = sub m One
- 1 + numeral n = sub n One
- 1 + - numeral n = - numeral (inc n)
- numeral m + - 1 = - numeral (inc m)
1 + - 1 = 0
- 1 + 1 = 0
- 1 + - 1 = - 2
⟨*proof*⟩

lemma *diff-numeral-simps*:
numeral m - numeral n = sub m n
numeral m - - numeral n = numeral (m + n)
- numeral m - numeral n = - numeral (m + n)
- numeral m - - numeral n = sub n m
⟨*proof*⟩

lemma *diff-numeral-special*:
1 - numeral n = sub One n
numeral m - 1 = sub m One
1 - - numeral n = numeral (One + n)
- numeral m - 1 = - numeral (m + One)
- 1 - numeral n = - numeral (inc n)
numeral m - - 1 = numeral (inc m)
- 1 - - numeral n = sub n One
- numeral m - - 1 = sub One m
1 - 1 = 0
- 1 - 1 = - 2
1 - - 1 = 2
- 1 - - 1 = 0
⟨*proof*⟩

end

46.4.3 Structures with multiplication: class *semiring-numeral*

class *semiring-numeral* = *semiring* + *monoid-mult*
begin

subclass *numeral* ⟨*proof*⟩

lemma *numeral-mult*: *numeral* (*m* ∗ *n*) = *numeral m* ∗ *numeral n*
⟨*proof*⟩

lemma *numeral-times-numeral*: *numeral m* ∗ *numeral n* = *numeral* (*m* ∗ *n*)
⟨*proof*⟩

lemma *mult-2*: *2* ∗ *z* = *z* + *z*
⟨*proof*⟩

lemma *mult-2-right*: *z* ∗ *2* = *z* + *z*
⟨*proof*⟩

lemma *left-add-twice*:
a + (*a* + *b*) = *2* ∗ *a* + *b*
⟨*proof*⟩

end

46.4.4 Structures with a zero: class *semiring-1*

context *semiring-1*
begin

subclass *semiring-numeral* ⟨*proof*⟩

lemma *of-nat-numeral* [*simp*]: *of-nat* (*numeral n*) = *numeral n*
⟨*proof*⟩

end

lemma *nat-of-num-numeral* [*code-abbrev*]: *nat-of-num* = *numeral*
⟨*proof*⟩

lemma *nat-of-num-code* [*code*]:
nat-of-num One = *1*
nat-of-num (*Bit0 n*) = (*let m* = *nat-of-num n in m* + *m*)
nat-of-num (*Bit1 n*) = (*let m* = *nat-of-num n in Suc* (*m* + *m*))
⟨*proof*⟩

46.4.5 Equality: class *semiring-char-0*

context *semiring-char-0*
begin

lemma *numeral-eq-iff*: *numeral m = numeral n ⟷ m = n*
⟨*proof*⟩

lemma *numeral-eq-one-iff*: *numeral n = 1 ⟷ n = One*
⟨*proof*⟩

lemma *one-eq-numeral-iff*: *1 = numeral n ⟷ One = n*
⟨*proof*⟩

lemma *numeral-neq-zero*: *numeral n ≠ 0*
⟨*proof*⟩

lemma *zero-neq-numeral*: *0 ≠ numeral n*
⟨*proof*⟩

lemmas *eq-numeral-simps* [*simp*] =
numeral-eq-iff
numeral-eq-one-iff
one-eq-numeral-iff
numeral-neq-zero
zero-neq-numeral

end

46.4.6 Comparisons: class *linordered-nonzero-semiring*

context *linordered-nonzero-semiring*
begin

lemma *numeral-le-iff*: *numeral m ≤ numeral n ⟷ m ≤ n*
⟨*proof*⟩

lemma *one-le-numeral*: *1 ≤ numeral n*
⟨*proof*⟩

lemma *numeral-le-one-iff*: *numeral n ≤ 1 ⟷ n ≤ num.One*
⟨*proof*⟩

lemma *numeral-less-iff*: *numeral m < numeral n ⟷ m < n*
⟨*proof*⟩

lemma *not-numeral-less-one*: *¬ numeral n < 1*
⟨*proof*⟩

lemma *one-less-numeral-iff*: *1 < numeral n ⟷ num.One < n*
⟨*proof*⟩

lemma *zero-le-numeral*: *0 ≤ numeral n*

⟨*proof*⟩

lemma *zero-less-numeral*: *0 < numeral n*
⟨*proof*⟩

lemma *not-numeral-le-zero*: *¬ numeral n ≤ 0*
⟨*proof*⟩

lemma *not-numeral-less-zero*: *¬ numeral n < 0*
⟨*proof*⟩

lemmas *le-numeral-extra* =
zero-le-one not-one-le-zero
order-refl [*of 0*] *order-refl* [*of 1*]

lemmas *less-numeral-extra* =
zero-less-one not-one-less-zero
less-irrefl [*of 0*] *less-irrefl* [*of 1*]

lemmas *le-numeral-simps* [*simp*] =
numeral-le-iff
one-le-numeral
numeral-le-one-iff
zero-le-numeral
not-numeral-le-zero

lemmas *less-numeral-simps* [*simp*] =
numeral-less-iff
one-less-numeral-iff
not-numeral-less-one
zero-less-numeral
not-numeral-less-zero

lemma *min-0-1* [*simp*]:
fixes *min′* :: *'a ⇒ 'a ⇒ 'a*
defines *min′ ≡ min*
shows
min′ 0 1 = 0
min′ 1 0 = 0
min′ 0 (*numeral x*) *= 0*
min′ (*numeral x*) *0 = 0*
min′ 1 (*numeral x*) *= 1*
min′ (*numeral x*) *1 = 1*
⟨*proof*⟩

lemma *max-0-1* [*simp*]:
fixes *max′* :: *'a ⇒ 'a ⇒ 'a*
defines *max′ ≡ max*
shows

max′ 0 1 = 1
max′ 1 0 = 1
max′ 0 (numeral x) = numeral x
max′ (numeral x) 0 = numeral x
max′ 1 (numeral x) = numeral x
max′ (numeral x) 1 = numeral x
⟨*proof*⟩

end

Unfold *min* and *max* on numerals.

lemmas *max-number-of* [*simp*] =
max-def [*of numeral u numeral v*]
max-def [*of numeral u − numeral v*]
max-def [*of − numeral u numeral v*]
max-def [*of − numeral u − numeral v*] **for** *u v*

lemmas *min-number-of* [*simp*] =
min-def [*of numeral u numeral v*]
min-def [*of numeral u − numeral v*]
min-def [*of − numeral u numeral v*]
min-def [*of − numeral u − numeral v*] **for** *u v*

46.4.7 Multiplication and negation: class *ring-1*

context *ring-1*
begin

subclass *neg-numeral* ⟨*proof*⟩

lemma *mult-neg-numeral-simps*:
− numeral m ∗ − numeral n = numeral (m ∗ n)
− numeral m ∗ numeral n = − numeral (m ∗ n)
numeral m ∗ − numeral n = − numeral (m ∗ n)
⟨*proof*⟩

lemma *mult-minus1* [*simp*]: *− 1 ∗ z = − z*
⟨*proof*⟩

lemma *mult-minus1-right* [*simp*]: *z ∗ − 1 = − z*
⟨*proof*⟩

lemma *minus-sub-one-diff-one* [*simp*]:
‹*− sub m One − 1 = − numeral m*›
⟨*proof*⟩

end

46.4.8 Equality using *iszero* for rings with non-zero characteristic

context *ring-1*
begin

definition *iszero* :: $'a \Rightarrow bool$
  where $iszero\ z \longleftrightarrow z = 0$

lemma *iszero-0* [*simp*]: $iszero\ 0$
  ⟨*proof*⟩

lemma *not-iszero-1* [*simp*]: $\neg\ iszero\ 1$
  ⟨*proof*⟩

lemma *not-iszero-Numeral1*: $\neg\ iszero\ Numeral1$
  ⟨*proof*⟩

lemma *not-iszero-neg-1* [*simp*]: $\neg\ iszero\ (-\ 1)$
  ⟨*proof*⟩

lemma *not-iszero-neg-Numeral1*: $\neg\ iszero\ (-\ Numeral1)$
  ⟨*proof*⟩

lemma *iszero-neg-numeral* [*simp*]: $iszero\ (-\ numeral\ w) \longleftrightarrow iszero\ (numeral\ w)$
  ⟨*proof*⟩

lemma *eq-iff-iszero-diff*: $x = y \longleftrightarrow iszero\ (x - y)$
  ⟨*proof*⟩

The *eq-numeral-iff-iszero* lemmas are not declared [*simp*] by default, because for rings of characteristic zero, better simp rules are possible. For a type like integers mod n, type-instantiated versions of these rules should be added to the simplifier, along with a type-specific rule for deciding propositions of the form $iszero\ (numeral\ w)$.

bh: Maybe it would not be so bad to just declare these as simp rules anyway? I should test whether these rules take precedence over the *ring-char-0* rules in the simplifier.

lemma *eq-numeral-iff-iszero*:
  $numeral\ x = numeral\ y \longleftrightarrow iszero\ (sub\ x\ y)$
  $numeral\ x = -\ numeral\ y \longleftrightarrow iszero\ (numeral\ (x + y))$
  $-\ numeral\ x = numeral\ y \longleftrightarrow iszero\ (numeral\ (x + y))$
  $-\ numeral\ x = -\ numeral\ y \longleftrightarrow iszero\ (sub\ y\ x)$
  $numeral\ x = 1 \longleftrightarrow iszero\ (sub\ x\ One)$
  $1 = numeral\ y \longleftrightarrow iszero\ (sub\ One\ y)$
  $-\ numeral\ x = 1 \longleftrightarrow iszero\ (numeral\ (x + One))$
  $1 = -\ numeral\ y \longleftrightarrow iszero\ (numeral\ (One + y))$
  $numeral\ x = 0 \longleftrightarrow iszero\ (numeral\ x)$
  $0 = numeral\ y \longleftrightarrow iszero\ (numeral\ y)$
  $-\ numeral\ x = 0 \longleftrightarrow iszero\ (numeral\ x)$

$0 = - \; numeral\ y \longleftrightarrow iszero\ (numeral\ y)$
⟨*proof*⟩

end

46.4.9 Equality and negation: class *ring-char-0*

context *ring-char-0*
begin

lemma *not-iszero-numeral* [*simp*]: $\neg\ iszero\ (numeral\ w)$
⟨*proof*⟩

lemma *neg-numeral-eq-iff*: $-\ numeral\ m = -\ numeral\ n \longleftrightarrow m = n$
⟨*proof*⟩

lemma *numeral-neq-neg-numeral*: $numeral\ m \neq -\ numeral\ n$
⟨*proof*⟩

lemma *neg-numeral-neq-numeral*: $-\ numeral\ m \neq numeral\ n$
⟨*proof*⟩

lemma *zero-neq-neg-numeral*: $0 \neq -\ numeral\ n$
⟨*proof*⟩

lemma *neg-numeral-neq-zero*: $-\ numeral\ n \neq 0$
⟨*proof*⟩

lemma *one-neq-neg-numeral*: $1 \neq -\ numeral\ n$
⟨*proof*⟩

lemma *neg-numeral-neq-one*: $-\ numeral\ n \neq 1$
⟨*proof*⟩

lemma *neg-one-neq-numeral*: $-\ 1 \neq numeral\ n$
⟨*proof*⟩

lemma *numeral-neq-neg-one*: $numeral\ n \neq -\ 1$
⟨*proof*⟩

lemma *neg-one-eq-numeral-iff*: $-\ 1 = -\ numeral\ n \longleftrightarrow n = One$
⟨*proof*⟩

lemma *numeral-eq-neg-one-iff*: $-\ numeral\ n = -\ 1 \longleftrightarrow n = One$
⟨*proof*⟩

lemma *neg-one-neq-zero*: $-\ 1 \neq 0$
⟨*proof*⟩

lemma *zero-neq-neg-one*: $0 \neq -\ 1$
⟨*proof*⟩

lemma *neg-one-neq-one*: $-\ 1 \neq 1$
⟨*proof*⟩

lemma *one-neq-neg-one*: $1 \neq -\ 1$
⟨*proof*⟩

lemmas *eq-neg-numeral-simps* [*simp*] =
neg-numeral-eq-iff
numeral-neq-neg-numeral neg-numeral-neq-numeral
one-neq-neg-numeral neg-numeral-neq-one
zero-neq-neg-numeral neg-numeral-neq-zero
neg-one-neq-numeral numeral-neq-neg-one
neg-one-eq-numeral-iff numeral-eq-neg-one-iff
neg-one-neq-zero zero-neq-neg-one
neg-one-neq-one one-neq-neg-one

end

46.4.10 Structures with negation and order: class *linordered-idom*

context *linordered-idom*
begin

subclass *ring-char-0* ⟨*proof*⟩

lemma *neg-numeral-le-iff*: $-\ \mathit{numeral}\ m \leq -\ \mathit{numeral}\ n \longleftrightarrow n \leq m$
⟨*proof*⟩

lemma *neg-numeral-less-iff*: $-\ \mathit{numeral}\ m < -\ \mathit{numeral}\ n \longleftrightarrow n < m$
⟨*proof*⟩

lemma *neg-numeral-less-zero*: $-\ \mathit{numeral}\ n < 0$
⟨*proof*⟩

lemma *neg-numeral-le-zero*: $-\ \mathit{numeral}\ n \leq 0$
⟨*proof*⟩

lemma *not-zero-less-neg-numeral*: $\neg\ 0 < -\ \mathit{numeral}\ n$
⟨*proof*⟩

lemma *not-zero-le-neg-numeral*: $\neg\ 0 \leq -\ \mathit{numeral}\ n$
⟨*proof*⟩

lemma *neg-numeral-less-numeral*: $-\ \mathit{numeral}\ m < \mathit{numeral}\ n$
⟨*proof*⟩

lemma *neg-numeral-le-numeral*: $-\ numeral\ m \leq numeral\ n$
⟨*proof*⟩

lemma *not-numeral-less-neg-numeral*: $\neg\ numeral\ m < -\ numeral\ n$
⟨*proof*⟩

lemma *not-numeral-le-neg-numeral*: $\neg\ numeral\ m \leq -\ numeral\ n$
⟨*proof*⟩

lemma *neg-numeral-less-one*: $-\ numeral\ m < 1$
⟨*proof*⟩

lemma *neg-numeral-le-one*: $-\ numeral\ m \leq 1$
⟨*proof*⟩

lemma *not-one-less-neg-numeral*: $\neg\ 1 < -\ numeral\ m$
⟨*proof*⟩

lemma *not-one-le-neg-numeral*: $\neg\ 1 \leq -\ numeral\ m$
⟨*proof*⟩

lemma *not-numeral-less-neg-one*: $\neg\ numeral\ m < -\ 1$
⟨*proof*⟩

lemma *not-numeral-le-neg-one*: $\neg\ numeral\ m \leq -\ 1$
⟨*proof*⟩

lemma *neg-one-less-numeral*: $-\ 1 < numeral\ m$
⟨*proof*⟩

lemma *neg-one-le-numeral*: $-\ 1 \leq numeral\ m$
⟨*proof*⟩

lemma *neg-numeral-less-neg-one-iff*: $-\ numeral\ m < -\ 1 \longleftrightarrow m \neq One$
⟨*proof*⟩

lemma *neg-numeral-le-neg-one*: $-\ numeral\ m \leq -\ 1$
⟨*proof*⟩

lemma *not-neg-one-less-neg-numeral*: $\neg\ -\ 1 < -\ numeral\ m$
⟨*proof*⟩

lemma *not-neg-one-le-neg-numeral-iff*: $\neg\ -\ 1 \leq -\ numeral\ m \longleftrightarrow m \neq One$
⟨*proof*⟩

lemma *sub-non-negative*: $sub\ n\ m \geq 0 \longleftrightarrow n \geq m$
⟨*proof*⟩

lemma *sub-positive*: $sub\ n\ m > 0 \longleftrightarrow n > m$

⟨*proof*⟩

lemma *sub-non-positive*: *sub n m* ≤ *0* ⟷ *n* ≤ *m*
⟨*proof*⟩

lemma *sub-negative*: *sub n m* < *0* ⟷ *n* < *m*
⟨*proof*⟩

lemmas *le-neg-numeral-simps* [*simp*] =
neg-numeral-le-iff
neg-numeral-le-numeral not-numeral-le-neg-numeral
neg-numeral-le-zero not-zero-le-neg-numeral
neg-numeral-le-one not-one-le-neg-numeral
neg-one-le-numeral not-numeral-le-neg-one
neg-numeral-le-neg-one not-neg-one-le-neg-numeral-iff

lemma *le-minus-one-simps* [*simp*]:
− *1* ≤ *0*
− *1* ≤ *1*
¬ *0* ≤ − *1*
¬ *1* ≤ − *1*
⟨*proof*⟩

lemmas *less-neg-numeral-simps* [*simp*] =
neg-numeral-less-iff
neg-numeral-less-numeral not-numeral-less-neg-numeral
neg-numeral-less-zero not-zero-less-neg-numeral
neg-numeral-less-one not-one-less-neg-numeral
neg-one-less-numeral not-numeral-less-neg-one
neg-numeral-less-neg-one-iff not-neg-one-less-neg-numeral

lemma *less-minus-one-simps* [*simp*]:
− *1* < *0*
− *1* < *1*
¬ *0* < − *1*
¬ *1* < − *1*
⟨*proof*⟩

lemma *abs-numeral* [*simp*]: |*numeral n*| = *numeral n*
⟨*proof*⟩

lemma *abs-neg-numeral* [*simp*]: |− *numeral n*| = *numeral n*
⟨*proof*⟩

lemma *abs-neg-one* [*simp*]: |− *1*| = *1*
⟨*proof*⟩

end

46.4.11 Natural numbers

lemma *numeral-num-of-nat*:
numeral (*num-of-nat n*) = *n* **if** *n* > *0*
⟨*proof*⟩

lemma *Suc-1* [*simp*]: *Suc 1* = *2*
⟨*proof*⟩

lemma *Suc-numeral* [*simp*]: *Suc* (*numeral n*) = *numeral* (*n* + *One*)
⟨*proof*⟩

definition *pred-numeral* :: *num* ⇒ *nat*
where *pred-numeral k* = *numeral k* − *1*

declare [[*code drop*: *pred-numeral*]]

lemma *numeral-eq-Suc*: *numeral k* = *Suc* (*pred-numeral k*)
⟨*proof*⟩

lemma *eval-nat-numeral*:
numeral One = *Suc 0*
numeral (*Bit0 n*) = *Suc* (*numeral* (*BitM n*))
numeral (*Bit1 n*) = *Suc* (*numeral* (*Bit0 n*))
⟨*proof*⟩

lemma *pred-numeral-simps* [*simp*]:
pred-numeral One = *0*
pred-numeral (*Bit0 k*) = *numeral* (*BitM k*)
pred-numeral (*Bit1 k*) = *numeral* (*Bit0 k*)
⟨*proof*⟩

lemma *pred-numeral-inc* [*simp*]:
pred-numeral (*Num.inc k*) = *numeral k*
⟨*proof*⟩

lemma *numeral-2-eq-2*: *2* = *Suc* (*Suc 0*)
⟨*proof*⟩

lemma *numeral-3-eq-3*: *3* = *Suc* (*Suc* (*Suc 0*))
⟨*proof*⟩

lemma *numeral-1-eq-Suc-0*: *Numeral1* = *Suc 0*
⟨*proof*⟩

lemma *Suc-nat-number-of-add*: *Suc* (*numeral v* + *n*) = *numeral* (*v* + *One*) + *n*
⟨*proof*⟩

lemma *numerals*: *Numeral1* = (*1*::*nat*) *2* = *Suc* (*Suc 0*)
⟨*proof*⟩

lemmas *numeral-nat = eval-nat-numeral BitM.simps One-nat-def*

Comparisons involving *Suc*.

lemma *eq-numeral-Suc* [*simp*]: *numeral k = Suc n ⟷ pred-numeral k = n*
⟨*proof*⟩

lemma *Suc-eq-numeral* [*simp*]: *Suc n = numeral k ⟷ n = pred-numeral k*
⟨*proof*⟩

lemma *less-numeral-Suc* [*simp*]: *numeral k < Suc n ⟷ pred-numeral k < n*
⟨*proof*⟩

lemma *less-Suc-numeral* [*simp*]: *Suc n < numeral k ⟷ n < pred-numeral k*
⟨*proof*⟩

lemma *le-numeral-Suc* [*simp*]: *numeral k ≤ Suc n ⟷ pred-numeral k ≤ n*
⟨*proof*⟩

lemma *le-Suc-numeral* [*simp*]: *Suc n ≤ numeral k ⟷ n ≤ pred-numeral k*
⟨*proof*⟩

lemma *diff-Suc-numeral* [*simp*]: *Suc n − numeral k = n − pred-numeral k*
⟨*proof*⟩

lemma *diff-numeral-Suc* [*simp*]: *numeral k − Suc n = pred-numeral k − n*
⟨*proof*⟩

lemma *max-Suc-numeral* [*simp*]: *max (Suc n) (numeral k) = Suc (max n (pred-numeral k))*
⟨*proof*⟩

lemma *max-numeral-Suc* [*simp*]: *max (numeral k) (Suc n) = Suc (max (pred-numeral k) n)*
⟨*proof*⟩

lemma *min-Suc-numeral* [*simp*]: *min (Suc n) (numeral k) = Suc (min n (pred-numeral k))*
⟨*proof*⟩

lemma *min-numeral-Suc* [*simp*]: *min (numeral k) (Suc n) = Suc (min (pred-numeral k) n)*
⟨*proof*⟩

For *case-nat* and *rec-nat*.

lemma *case-nat-numeral* [*simp*]: *case-nat a f (numeral v) = (let pv = pred-numeral v in f pv)*
⟨*proof*⟩

lemma *case-nat-add-eq-if* [*simp*]:
case-nat a f ((*numeral v*) + *n*) = (*let pv* = *pred-numeral v in f* (*pv* + *n*))
⟨*proof*⟩

lemma *rec-nat-numeral* [*simp*]:
rec-nat a f (*numeral v*) = (*let pv* = *pred-numeral v in f pv* (*rec-nat a f pv*))
⟨*proof*⟩

lemma *rec-nat-add-eq-if* [*simp*]:
rec-nat a f (*numeral v* + *n*) = (*let pv* = *pred-numeral v in f* (*pv* + *n*) (*rec-nat a f* (*pv* + *n*)))
⟨*proof*⟩

Case analysis on $n < (2::'a)$.

lemma *less-2-cases*: $n < 2 \Longrightarrow n = 0 \vee n = Suc\ 0$
⟨*proof*⟩

lemma *less-2-cases-iff*: $n < 2 \longleftrightarrow n = 0 \vee n = Suc\ 0$
⟨*proof*⟩

Removal of Small Numerals: 0, 1 and (in additive positions) 2.

bh: Are these rules really a good idea? LCP: well, it already happens for 0 and 1!

lemma *add-2-eq-Suc* [*simp*]: $2 + n = Suc\ (Suc\ n)$
⟨*proof*⟩

lemma *add-2-eq-Suc'* [*simp*]: $n + 2 = Suc\ (Suc\ n)$
⟨*proof*⟩

Can be used to eliminate long strings of Sucs, but not by default.

lemma *Suc3-eq-add-3*: $Suc\ (Suc\ (Suc\ n)) = 3 + n$
⟨*proof*⟩

lemmas *nat-1-add-1* = *one-add-one* [**where** *'a*=*nat*]

context *semiring-numeral*
begin

lemma *numeral-add-unfold-funpow*:
‹*numeral k* + *a* = ((+) *1* ^^ *numeral k*) *a*›
⟨*proof*⟩

end

context *semiring-1*
begin

lemma *numeral-unfold-funpow*:

‹*numeral k* = ((+) *1* ^^ *numeral k*) *0*›
⟨*proof*⟩

end

context
includes *lifting-syntax*
begin

lemma *transfer-rule-numeral*:
‹((=) ===> *R*) *numeral numeral*›
if [*transfer-rule*]: ‹*R 0 0*› ‹*R 1 1*›
‹(*R* ===> *R* ===> *R*) (+) (+)›
for *R* :: ‹'*a*::{*semiring-numeral*,*monoid-add*} ⇒ '*b*::{*semiring-numeral*,*monoid-add*}
⇒ *bool*›
⟨*proof*⟩

end

46.5 Particular lemmas concerning *2*::'*a*

context *linordered-field*
begin

subclass *field-char-0* ⟨*proof*⟩

lemma *half-gt-zero-iff*: *0* < *a* / *2* ⟷ *0* < *a*
⟨*proof*⟩

lemma *half-gt-zero* [*simp*]: *0* < *a* ⟹ *0* < *a* / *2*
⟨*proof*⟩

end

46.6 Numeral equations as default simplification rules

declare (**in** *numeral*) *numeral-One* [*simp*]
declare (**in** *numeral*) *numeral-plus-numeral* [*simp*]
declare (**in** *numeral*) *add-numeral-special* [*simp*]
declare (**in** *neg-numeral*) *add-neg-numeral-simps* [*simp*]
declare (**in** *neg-numeral*) *add-neg-numeral-special* [*simp*]
declare (**in** *neg-numeral*) *diff-numeral-simps* [*simp*]
declare (**in** *neg-numeral*) *diff-numeral-special* [*simp*]
declare (**in** *semiring-numeral*) *numeral-times-numeral* [*simp*]
declare (**in** *ring-1*) *mult-neg-numeral-simps* [*simp*]

46.6.1 Special Simplification for Constants

These distributive laws move literals inside sums and differences.

lemmas *distrib-right-numeral* [*simp*] = *distrib-right* [*of* - - *numeral v*] **for** *v*
lemmas *distrib-left-numeral* [*simp*] = *distrib-left* [*of numeral v*] **for** *v*
lemmas *left-diff-distrib-numeral* [*simp*] = *left-diff-distrib* [*of* - - *numeral v*] **for** *v*
lemmas *right-diff-distrib-numeral* [*simp*] = *right-diff-distrib* [*of numeral v*] **for** *v*

These are actually for fields, like real

lemmas *zero-less-divide-iff-numeral* [*simp*, *no-atp*] = *zero-less-divide-iff* [*of numeral w*] **for** *w*
lemmas *divide-less-0-iff-numeral* [*simp*, *no-atp*] = *divide-less-0-iff* [*of numeral w*] **for** *w*
lemmas *zero-le-divide-iff-numeral* [*simp*, *no-atp*] = *zero-le-divide-iff* [*of numeral w*] **for** *w*
lemmas *divide-le-0-iff-numeral* [*simp*, *no-atp*] = *divide-le-0-iff* [*of numeral w*] **for** *w*

Replaces *inverse #nn* by *1/#nn*. It looks strange, but then other simprocs simplify the quotient.

lemmas *inverse-eq-divide-numeral* [*simp*] =
inverse-eq-divide [*of numeral w*] **for** *w*

lemmas *inverse-eq-divide-neg-numeral* [*simp*] =
inverse-eq-divide [*of* − *numeral w*] **for** *w*

These laws simplify inequalities, moving unary minus from a term into the literal.

lemmas *equation-minus-iff-numeral* [*no-atp*] =
equation-minus-iff [*of numeral v*] **for** *v*

lemmas *minus-equation-iff-numeral* [*no-atp*] =
minus-equation-iff [*of* - *numeral v*] **for** *v*

lemmas *le-minus-iff-numeral* [*no-atp*] =
le-minus-iff [*of numeral v*] **for** *v*

lemmas *minus-le-iff-numeral* [*no-atp*] =
minus-le-iff [*of* - *numeral v*] **for** *v*

lemmas *less-minus-iff-numeral* [*no-atp*] =
less-minus-iff [*of numeral v*] **for** *v*

lemmas *minus-less-iff-numeral* [*no-atp*] =
minus-less-iff [*of* - *numeral v*] **for** *v*

Cancellation of constant factors in comparisons ($<$ and $\leq$)

lemmas *mult-less-cancel-left-numeral* [*simp*, *no-atp*] = *mult-less-cancel-left* [*of numeral v*] **for** *v*
lemmas *mult-less-cancel-right-numeral* [*simp*, *no-atp*] = *mult-less-cancel-right* [*of* - *numeral v*] **for** *v*

lemmas *mult-le-cancel-left-numeral* [*simp*, *no-atp*] = *mult-le-cancel-left* [*of numeral v*] **for** *v*
lemmas *mult-le-cancel-right-numeral* [*simp*, *no-atp*] = *mult-le-cancel-right* [*of - numeral v*] **for** *v*

Multiplying out constant divisors in comparisons ($<$, $\leq$ and $=$)

named-theorems *divide-const-simps simplification rules to simplify comparisons involving constant divisors*

lemmas *le-divide-eq-numeral1* [*simp*,*divide-const-simps*] =
pos-le-divide-eq [*of numeral w*, *OF zero-less-numeral*]
neg-le-divide-eq [*of − numeral w*, *OF neg-numeral-less-zero*] **for** *w*

lemmas *divide-le-eq-numeral1* [*simp*,*divide-const-simps*] =
pos-divide-le-eq [*of numeral w*, *OF zero-less-numeral*]
neg-divide-le-eq [*of − numeral w*, *OF neg-numeral-less-zero*] **for** *w*

lemmas *less-divide-eq-numeral1* [*simp*,*divide-const-simps*] =
pos-less-divide-eq [*of numeral w*, *OF zero-less-numeral*]
neg-less-divide-eq [*of − numeral w*, *OF neg-numeral-less-zero*] **for** *w*

lemmas *divide-less-eq-numeral1* [*simp*,*divide-const-simps*] =
pos-divide-less-eq [*of numeral w*, *OF zero-less-numeral*]
neg-divide-less-eq [*of − numeral w*, *OF neg-numeral-less-zero*] **for** *w*

lemmas *eq-divide-eq-numeral1* [*simp*,*divide-const-simps*] =
eq-divide-eq [*of - - numeral w*]
eq-divide-eq [*of - - − numeral w*] **for** *w*

lemmas *divide-eq-eq-numeral1* [*simp*,*divide-const-simps*] =
divide-eq-eq [*of - numeral w*]
divide-eq-eq [*of - − numeral w*] **for** *w*

46.6.2 Optional Simplification Rules Involving Constants

Simplify quotients that are compared with a literal constant.

lemmas *le-divide-eq-numeral* [*divide-const-simps*] =
le-divide-eq [*of numeral w*]
le-divide-eq [*of − numeral w*] **for** *w*

lemmas *divide-le-eq-numeral* [*divide-const-simps*] =
divide-le-eq [*of - - numeral w*]
divide-le-eq [*of - - − numeral w*] **for** *w*

lemmas *less-divide-eq-numeral* [*divide-const-simps*] =
less-divide-eq [*of numeral w*]
less-divide-eq [*of − numeral w*] **for** *w*

lemmas *divide-less-eq-numeral* [*divide-const-simps*] =

divide-less-eq [*of* - - *numeral w*]
divide-less-eq [*of* - - − *numeral w*] **for** *w*

lemmas *eq-divide-eq-numeral* [*divide-const-simps*] =
eq-divide-eq [*of numeral w*]
eq-divide-eq [*of* − *numeral w*] **for** *w*

lemmas *divide-eq-eq-numeral* [*divide-const-simps*] =
divide-eq-eq [*of* - - *numeral w*]
divide-eq-eq [*of* - - − *numeral w*] **for** *w*

Not good as automatic simprules because they cause case splits.

lemmas [*divide-const-simps*] =
le-divide-eq-1 divide-le-eq-1 less-divide-eq-1 divide-less-eq-1

46.7 Setting up simprocs

lemma *mult-numeral-1*: *Numeral1* ∗ *a* = *a*
for *a* :: *'a*::*semiring-numeral*
⟨*proof*⟩

lemma *mult-numeral-1-right*: *a* ∗ *Numeral1* = *a*
for *a* :: *'a*::*semiring-numeral*
⟨*proof*⟩

lemma *divide-numeral-1*: *a* / *Numeral1* = *a*
for *a* :: *'a*::*field*
⟨*proof*⟩

lemma *inverse-numeral-1*: *inverse Numeral1* = (*Numeral1*::*'a*::*division-ring*)
⟨*proof*⟩

Theorem lists for the cancellation simprocs. The use of a binary numeral for 1 reduces the number of special cases.

lemma *mult-1s-semiring-numeral*:
Numeral1 ∗ *a* = *a*
a ∗ *Numeral1* = *a*
for *a* :: *'a*::*semiring-numeral*
⟨*proof*⟩

lemma *mult-1s-ring-1*:
− *Numeral1* ∗ *b* = − *b*
b ∗ − *Numeral1* = − *b*
for *b* :: *'a*::*ring-1*
⟨*proof*⟩

lemmas *mult-1s* = *mult-1s-semiring-numeral mult-1s-ring-1*

⟨*ML*⟩

46.7.1 Simplification of arithmetic operations on integer constants

lemmas *arith-special* =
add-numeral-special add-neg-numeral-special
diff-numeral-special

lemmas *arith-extra-simps* =
numeral-plus-numeral add-neg-numeral-simps add-0-left add-0-right
minus-zero
diff-numeral-simps diff-0 diff-0-right
numeral-times-numeral mult-neg-numeral-simps
mult-zero-left mult-zero-right
abs-numeral abs-neg-numeral

For making a minimal simpset, one must include these default simprules. Also include *simp-thms*.

lemmas *arith-simps* =
add-num-simps mult-num-simps sub-num-simps
BitM.simps dbl-simps dbl-inc-simps dbl-dec-simps
abs-zero abs-one arith-extra-simps

lemmas *more-arith-simps* =
neg-le-iff-le
minus-zero left-minus right-minus
mult-1-left mult-1-right
mult-minus-left mult-minus-right
minus-add-distrib minus-minus mult.assoc

lemmas *of-nat-simps* =
of-nat-0 of-nat-1 of-nat-Suc of-nat-add of-nat-mult

Simplification of relational operations.

lemmas *eq-numeral-extra* =
zero-neq-one one-neq-zero

lemmas *rel-simps* =
le-num-simps less-num-simps eq-num-simps
le-numeral-simps le-neg-numeral-simps le-minus-one-simps le-numeral-extra
less-numeral-simps less-neg-numeral-simps less-minus-one-simps less-numeral-extra
eq-numeral-simps eq-neg-numeral-simps eq-numeral-extra

lemma *Let-numeral* [*simp*]: *Let* (*numeral v*) *f* = *f* (*numeral v*)
— Unfold all *let*s involving constants
⟨*proof*⟩

lemma *Let-neg-numeral* [*simp*]: *Let* (− *numeral v*) *f* = *f* (− *numeral v*)
— Unfold all *let*s involving constants
⟨*proof*⟩

⟨*ML*⟩

46.7.2 Simplification of arithmetic when nested to the right

lemma *add-numeral-left* [*simp*]: $numeral\ v + (numeral\ w + z) = (numeral(v + w) + z)$
⟨*proof*⟩

lemma *add-neg-numeral-left* [*simp*]:
$numeral\ v + (-\ numeral\ w + y) = (sub\ v\ w + y)$
$-\ numeral\ v + (numeral\ w + y) = (sub\ w\ v + y)$
$-\ numeral\ v + (-\ numeral\ w + y) = (-\ numeral(v + w) + y)$
⟨*proof*⟩

lemma *mult-numeral-left-semiring-numeral*:
$numeral\ v * (numeral\ w * z) = (numeral(v * w) * z :: 'a{::}semiring\text{-}numeral)$
⟨*proof*⟩

lemma *mult-numeral-left-ring-1*:
$-\ numeral\ v * (numeral\ w * y) = (-\ numeral(v * w) * y :: 'a{::}ring\text{-}1)$
$numeral\ v * (-\ numeral\ w * y) = (-\ numeral(v * w) * y :: 'a{::}ring\text{-}1)$
$-\ numeral\ v * (-\ numeral\ w * y) = (numeral(v * w) * y :: 'a{::}ring\text{-}1)$
⟨*proof*⟩

lemmas *mult-numeral-left* [*simp*] =
mult-numeral-left-semiring-numeral
mult-numeral-left-ring-1

hide-const (**open**) *One Bit0 Bit1 BitM inc pow sqr sub dbl dbl-inc dbl-dec*

46.8 Code module namespace

code-identifier
code-module *Num* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

46.9 Printing of evaluated natural numbers as numerals

lemma [*code-post*]:
$Suc\ 0 = 1$
$Suc\ 1 = 2$
$Suc\ (numeral\ n) = numeral\ (Num.inc\ n)$
⟨*proof*⟩

lemmas [*code-post*] = *Num.inc.simps*

46.10 More on auxiliary conversion

context *semiring-1*
begin

lemma *numeral-num-of-nat-unfold*:
‹*numeral* (*num-of-nat n*) = (*if n = 0 then 1 else of-nat n*)›
⟨*proof*⟩

lemma *num-of-nat-numeral-eq* [*simp*]:
‹*num-of-nat* (*numeral q*) = *q*›
⟨*proof*⟩

end

end

47 Exponentiation

theory *Power*
imports *Num*
begin

47.1 Powers for Arbitrary Monoids

class *power* = *one* + *times*
begin

primrec *power* :: *'a* ⇒ *nat* ⇒ *'a* (**infixr** ^ *80*)
where
power-0: *a* ^ *0* = *1*
| *power-Suc*: *a* ^ *Suc n* = *a* ∗ *a* ^ *n*

notation (*latex* **output**)
power ((-⁻) [*1000*] *1000*)

Special syntax for squares.

abbreviation *power2* :: *'a* ⇒ *'a* ((-²) [*1000*] *999*)
where $x^2 \equiv x \mathbin{\hat{}} 2$

end

context
includes *lifting-syntax*
begin

lemma *power-transfer* [*transfer-rule*]:
‹(*R* ===> (=) ===> *R*) (^) (^)›
if [*transfer-rule*]: ‹*R 1 1*›
‹(*R* ===> *R* ===> *R*) (∗) (∗)›
for *R* :: ‹*'a*::*power* ⇒ *'b*::*power* ⇒ *bool*›
⟨*proof*⟩

end

context *monoid-mult*
begin

subclass *power* ⟨*proof*⟩

lemma *power-one* [*simp*]: $1 \mathbin{\hat{}} n = 1$
⟨*proof*⟩

lemma *power-one-right* [*simp*]: $a \mathbin{\hat{}} 1 = a$
⟨*proof*⟩

lemma *power-Suc0-right* [*simp*]: $a \mathbin{\hat{}} Suc\ 0 = a$
⟨*proof*⟩

lemma *power-commutes*: $a \mathbin{\hat{}} n * a = a * a \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *power-Suc2*: $a \mathbin{\hat{}} Suc\ n = a \mathbin{\hat{}} n * a$
⟨*proof*⟩

lemma *power-add*: $a \mathbin{\hat{}} (m + n) = a \mathbin{\hat{}} m * a \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *power-mult*: $a \mathbin{\hat{}} (m * n) = (a \mathbin{\hat{}} m) \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *power-even-eq*: $a \mathbin{\hat{}} (2 * n) = (a \mathbin{\hat{}} n)^2$
⟨*proof*⟩

lemma *power-odd-eq*: $a \mathbin{\hat{}} Suc\ (2*n) = a * (a \mathbin{\hat{}} n)^2$
⟨*proof*⟩

lemma *power-numeral-even*: $z \mathbin{\hat{}} numeral\ (Num.Bit0\ w) = (let\ w = z \mathbin{\hat{}} (numeral\ w)\ in\ w * w)$
⟨*proof*⟩

lemma *power-numeral-odd*: $z \mathbin{\hat{}} numeral\ (Num.Bit1\ w) = (let\ w = z \mathbin{\hat{}} (numeral\ w)\ in\ z * w * w)$
⟨*proof*⟩

lemma *power2-eq-square*: $a^2 = a * a$
⟨*proof*⟩

lemma *power3-eq-cube*: $a \mathbin{\hat{}} 3 = a * a * a$
⟨*proof*⟩

lemma *power4-eq-xxxx*: $x\hat{}4 = x * x * x * x$

⟨*proof*⟩

lemma *power-numeral-reduce*: $x \mathbin{\hat{}} \mathit{numeral}\ n = x * x \mathbin{\hat{}} \mathit{pred\text{-}numeral}\ n$
⟨*proof*⟩

lemma *funpow-times-power*: $(\mathit{times}\ x \mathbin{\hat{}\hat{}} f\ x) = \mathit{times}\ (x \mathbin{\hat{}} f\ x)$
⟨*proof*⟩

lemma *power-commuting-commutes*:
assumes $x * y = y * x$
shows $x \mathbin{\hat{}} n * y = y * x \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *power-minus-mult*: $0 < n \Longrightarrow a \mathbin{\hat{}} (n - 1) * a = a \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *left-right-inverse-power*:
assumes $x * y = 1$
shows $x \mathbin{\hat{}} n * y \mathbin{\hat{}} n = 1$
⟨*proof*⟩

end

context *comm-monoid-mult*
begin

lemma *power-mult-distrib* [*algebra-simps*, *algebra-split-simps*, *field-simps*, *field-split-simps*, *divide-simps*]:
$(a * b) \mathbin{\hat{}} n = (a \mathbin{\hat{}} n) * (b \mathbin{\hat{}} n)$
⟨*proof*⟩

end

Extract constant factors from powers.

declare *power-mult-distrib* [**where** $a = \mathit{numeral}\ w$ **for** w, *simp*]
declare *power-mult-distrib* [**where** $b = \mathit{numeral}\ w$ **for** w, *simp*]

lemma *power-add-numeral* [*simp*]: $a\hat{}\mathit{numeral}\ m * a\hat{}\mathit{numeral}\ n = a\hat{}\mathit{numeral}\ (m + n)$
for $a :: 'a$::*monoid-mult*
⟨*proof*⟩

lemma *power-add-numeral2* [*simp*]: $a\hat{}\mathit{numeral}\ m * (a\hat{}\mathit{numeral}\ n * b) = a\hat{}\mathit{numeral}\ (m + n) * b$
for $a :: 'a$::*monoid-mult*
⟨*proof*⟩

lemma *power-mult-numeral* [*simp*]: $(a\hat{}\mathit{numeral}\ m)\hat{}\mathit{numeral}\ n = a\hat{}\mathit{numeral}\ (m * n)$

for $a :: 'a$::*monoid-mult*
⟨*proof*⟩

context *semiring-numeral*
begin

lemma *numeral-sqr*: *numeral* (*Num.sqr k*) = *numeral k* ∗ *numeral k*
⟨*proof*⟩

lemma *numeral-pow*: *numeral* (*Num.pow k l*) = *numeral k* ^ *numeral l*
⟨*proof*⟩

lemma *power-numeral* [*simp*]: *numeral k* ^ *numeral l* = *numeral* (*Num.pow k l*)
⟨*proof*⟩

end

context *semiring-1*
begin

lemma *of-nat-power* [*simp*]: *of-nat* (*m* ^ *n*) = *of-nat m* ^ *n*
⟨*proof*⟩

lemma *zero-power*: *0* < *n* ⟹ *0* ^ *n* = *0*
⟨*proof*⟩

lemma *power-zero-numeral* [*simp*]: *0* ^ *numeral k* = *0*
⟨*proof*⟩

lemma *zero-power2*: $0^2 = 0$
⟨*proof*⟩

lemma *one-power2*: $1^2 = 1$
⟨*proof*⟩

lemma *power-0-Suc* [*simp*]: *0* ^ *Suc n* = *0*
⟨*proof*⟩

It looks plausible as a simprule, but its effect can be strange.

lemma *power-0-left*: *0* ^ *n* = (*if n* = *0 then 1 else 0*)
⟨*proof*⟩

end

context *semiring-char-0* **begin**

lemma *numeral-power-eq-of-nat-cancel-iff* [*simp*]:
numeral x ^ *n* = *of-nat y* ⟷ *numeral x* ^ *n* = *y*
⟨*proof*⟩

lemma *real-of-nat-eq-numeral-power-cancel-iff* [*simp*]:
of-nat y = numeral x ^ n ⟷ y = numeral x ^ n
⟨*proof*⟩

lemma *of-nat-eq-of-nat-power-cancel-iff*[*simp*]: *(of-nat b) ^ w = of-nat x ⟷ b ^ w = x*
⟨*proof*⟩

lemma *of-nat-power-eq-of-nat-cancel-iff*[*simp*]: *of-nat x = (of-nat b) ^ w ⟷ x = b ^ w*
⟨*proof*⟩

end

context *comm-semiring-1*
begin

The divides relation.

lemma *le-imp-power-dvd*:
assumes *m ≤ n*
shows *a ^ m dvd a ^ n*
⟨*proof*⟩

lemma *power-le-dvd*: *a ^ n dvd b ⟹ m ≤ n ⟹ a ^ m dvd b*
⟨*proof*⟩

lemma *dvd-power-same*: *x dvd y ⟹ x ^ n dvd y ^ n*
⟨*proof*⟩

lemma *dvd-power-le*: *x dvd y ⟹ m ≥ n ⟹ x ^ n dvd y ^ m*
⟨*proof*⟩

lemma *dvd-power* [*simp*]:
fixes *n* :: *nat*
assumes *n > 0 ∨ x = 1*
shows *x dvd (x ^ n)*
⟨*proof*⟩

end

context *semiring-1-no-zero-divisors*
begin

subclass *power* ⟨*proof*⟩

lemma *power-eq-0-iff* [*simp*]: *a ^ n = 0 ⟷ a = 0 ∧ n > 0*
⟨*proof*⟩

lemma *power-not-zero*: $a \neq 0 \Longrightarrow a \mathbin{\hat{}} n \neq 0$
⟨*proof*⟩

lemma *zero-eq-power2* [*simp*]: $a^2 = 0 \longleftrightarrow a = 0$
⟨*proof*⟩

end

context *ring-1*
begin

lemma *power-minus*: $(-a) \mathbin{\hat{}} n = (-1) \mathbin{\hat{}} n * a \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *power-minus'*: *NO-MATCH 1 x* $\Longrightarrow (-x) \mathbin{\hat{}} n = (-1) \mathbin{\hat{}} n * x \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *power-minus-Bit0*: $(-x) \mathbin{\hat{}} \mathit{numeral}\ (\mathit{Num.Bit0}\ k) = x \mathbin{\hat{}} \mathit{numeral}\ (\mathit{Num.Bit0}\ k)$
⟨*proof*⟩

lemma *power-minus-Bit1*: $(-x) \mathbin{\hat{}} \mathit{numeral}\ (\mathit{Num.Bit1}\ k) = -\ (x \mathbin{\hat{}} \mathit{numeral}\ (\mathit{Num.Bit1}\ k))$
⟨*proof*⟩

lemma *power2-minus* [*simp*]: $(-a)^2 = a^2$
⟨*proof*⟩

lemma *power-minus1-even* [*simp*]: $(-1) \mathbin{\hat{}} (2*n) = 1$
⟨*proof*⟩

lemma *power-minus1-odd*: $(-1) \mathbin{\hat{}} \mathit{Suc}\ (2*n) = -1$
⟨*proof*⟩

lemma *power-minus-even* [*simp*]: $(-a) \mathbin{\hat{}} (2*n) = a \mathbin{\hat{}} (2*n)$
⟨*proof*⟩

end

context *ring-1-no-zero-divisors*
begin

lemma *power2-eq-1-iff*: $a^2 = 1 \longleftrightarrow a = 1 \vee a = -1$
⟨*proof*⟩

end

context *idom*
begin

lemma *power2-eq-iff*: $x^2 = y^2 \longleftrightarrow x = y \vee x = - y$
⟨*proof*⟩

end

context *semidom-divide*
begin

lemma *power-diff*:
$a \mathbin{\hat{}} (m - n) = (a \mathbin{\hat{}} m)\ div\ (a \mathbin{\hat{}} n)$ **if** $a \neq 0$ **and** $n \leq m$
⟨*proof*⟩

end

context *algebraic-semidom*
begin

lemma *div-power*: $b\ dvd\ a \Longrightarrow (a\ div\ b) \mathbin{\hat{}} n = a \mathbin{\hat{}} n\ div\ b \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *is-unit-power-iff*: $is\text{-}unit\ (a \mathbin{\hat{}} n) \longleftrightarrow is\text{-}unit\ a \vee n = 0$
⟨*proof*⟩

lemma *dvd-power-iff*:
assumes $x \neq 0$
shows $x \mathbin{\hat{}} m\ dvd\ x \mathbin{\hat{}} n \longleftrightarrow is\text{-}unit\ x \vee m \leq n$
⟨*proof*⟩

end

context *normalization-semidom-multiplicative*
begin

lemma *normalize-power*: $normalize\ (a \mathbin{\hat{}} n) = normalize\ a \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *unit-factor-power*: $unit\text{-}factor\ (a \mathbin{\hat{}} n) = unit\text{-}factor\ a \mathbin{\hat{}} n$
⟨*proof*⟩

end

context *division-ring*
begin

Perhaps these should be simprules.

lemma *power-inverse* [*field-simps*, *field-split-simps*, *divide-simps*]: $inverse\ a \mathbin{\hat{}} n = inverse\ (a \mathbin{\hat{}} n)$

⟨*proof*⟩

lemma *power-one-over* [*field-simps*, *field-split-simps*, *divide-simps*]: $(1 \mathbin{/} a) \mathbin{\hat{}} n = 1 \mathbin{/} a \mathbin{\hat{}} n$
⟨*proof*⟩

end

context *field*
begin

lemma *power-divide* [*field-simps*, *field-split-simps*, *divide-simps*]: $(a \mathbin{/} b) \mathbin{\hat{}} n = a \mathbin{\hat{}} n \mathbin{/} b \mathbin{\hat{}} n$
⟨*proof*⟩

end

47.2 Exponentiation on ordered types

context *linordered-semidom*
begin

lemma *zero-less-power* [*simp*]: $0 < a \Longrightarrow 0 < a \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *zero-le-power* [*simp*]: $0 \leq a \Longrightarrow 0 \leq a \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *power-mono*: $a \leq b \Longrightarrow 0 \leq a \Longrightarrow a \mathbin{\hat{}} n \leq b \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *one-le-power* [*simp*]: $1 \leq a \Longrightarrow 1 \leq a \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *power-le-one*: $0 \leq a \Longrightarrow a \leq 1 \Longrightarrow a \mathbin{\hat{}} n \leq 1$
⟨*proof*⟩

lemma *power-gt1-lemma*:
assumes *gt1*: $1 < a$
shows $1 < a * a \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *power-gt1*: $1 < a \Longrightarrow 1 < a \mathbin{\hat{}} Suc\ n$
⟨*proof*⟩

lemma *one-less-power* [*simp*]: $1 < a \Longrightarrow 0 < n \Longrightarrow 1 < a \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *power-le-imp-le-exp*:

assumes *gt1*: $1 < a$
shows $a \mathbin{\hat{}} m \leq a \mathbin{\hat{}} n \Longrightarrow m \leq n$
⟨*proof*⟩

lemma *of-nat-zero-less-power-iff* [*simp*]: *of-nat* $x \mathbin{\hat{}} n > 0 \longleftrightarrow x > 0 \vee n = 0$
⟨*proof*⟩

Surely we can strengthen this? It holds for $0<a<1$ too.

lemma *power-inject-exp* [*simp*]:
$‹a \mathbin{\hat{}} m = a \mathbin{\hat{}} n \longleftrightarrow m = n›$ **if** $‹1 < a›$
⟨*proof*⟩

Can relax the first premise to $(0::'a) < a$ in the case of the natural numbers.

lemma *power-less-imp-less-exp*: $1 < a \Longrightarrow a \mathbin{\hat{}} m < a \mathbin{\hat{}} n \Longrightarrow m < n$
⟨*proof*⟩

lemma *power-strict-mono*: $a < b \Longrightarrow 0 \leq a \Longrightarrow 0 < n \Longrightarrow a \mathbin{\hat{}} n < b \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *power-mono-iff* [*simp*]:
shows $[\![a \geq 0;\ b \geq 0;\ n>0]\!] \Longrightarrow a \mathbin{\hat{}} n \leq b \mathbin{\hat{}} n \longleftrightarrow a \leq b$
⟨*proof*⟩

Lemma for *power-strict-decreasing*

lemma *power-Suc-less*: $0 < a \Longrightarrow a < 1 \Longrightarrow a * a \mathbin{\hat{}} n < a \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *power-strict-decreasing*: $n < N \Longrightarrow 0 < a \Longrightarrow a < 1 \Longrightarrow a \mathbin{\hat{}} N < a \mathbin{\hat{}} n$
⟨*proof*⟩

Proof resembles that of *power-strict-decreasing*.

lemma *power-decreasing*: $n \leq N \Longrightarrow 0 \leq a \Longrightarrow a \leq 1 \Longrightarrow a \mathbin{\hat{}} N \leq a \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *power-decreasing-iff* [*simp*]: $[\![0 < b;\ b < 1]\!] \Longrightarrow b \mathbin{\hat{}} m \leq b \mathbin{\hat{}} n \longleftrightarrow n \leq m$
⟨*proof*⟩

lemma *power-strict-decreasing-iff* [*simp*]: $[\![0 < b;\ b < 1]\!] \Longrightarrow b \mathbin{\hat{}} m < b \mathbin{\hat{}} n \longleftrightarrow n < m$
⟨*proof*⟩

lemma *power-Suc-less-one*: $0 < a \Longrightarrow a < 1 \Longrightarrow a \mathbin{\hat{}} Suc\ n < 1$
⟨*proof*⟩

Proof again resembles that of *power-strict-decreasing*.

lemma *power-increasing*: $n \leq N \Longrightarrow 1 \leq a \Longrightarrow a \mathbin{\hat{}} n \leq a \mathbin{\hat{}} N$
⟨*proof*⟩

Lemma for *power-strict-increasing*.

lemma *power-less-power-Suc*: $1 < a \Longrightarrow a \mathbin{\hat{}} n < a * a \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *power-strict-increasing*: $n < N \Longrightarrow 1 < a \Longrightarrow a \mathbin{\hat{}} n < a \mathbin{\hat{}} N$
⟨*proof*⟩

lemma *power-increasing-iff* [*simp*]: $1 < b \Longrightarrow b \mathbin{\hat{}} x \leq b \mathbin{\hat{}} y \longleftrightarrow x \leq y$
⟨*proof*⟩

lemma *power-strict-increasing-iff* [*simp*]: $1 < b \Longrightarrow b \mathbin{\hat{}} x < b \mathbin{\hat{}} y \longleftrightarrow x < y$
⟨*proof*⟩

lemma *power-le-imp-le-base*:
assumes *le*: $a \mathbin{\hat{}} Suc\ n \leq b \mathbin{\hat{}} Suc\ n$
and $0 \leq b$
shows $a \leq b$
⟨*proof*⟩

lemma *power-less-imp-less-base*:
assumes *less*: $a \mathbin{\hat{}} n < b \mathbin{\hat{}} n$
assumes *nonneg*: $0 \leq b$
shows $a < b$
⟨*proof*⟩

lemma *power-inject-base*: $a \mathbin{\hat{}} Suc\ n = b \mathbin{\hat{}} Suc\ n \Longrightarrow 0 \leq a \Longrightarrow 0 \leq b \Longrightarrow a = b$
⟨*proof*⟩

lemma *power-eq-imp-eq-base*: $a \mathbin{\hat{}} n = b \mathbin{\hat{}} n \Longrightarrow 0 \leq a \Longrightarrow 0 \leq b \Longrightarrow 0 < n \Longrightarrow a = b$
⟨*proof*⟩

lemma *power-eq-iff-eq-base*: $0 < n \Longrightarrow 0 \leq a \Longrightarrow 0 \leq b \Longrightarrow a \mathbin{\hat{}} n = b \mathbin{\hat{}} n \longleftrightarrow a = b$
⟨*proof*⟩

lemma *power2-le-imp-le*: $x^2 \leq y^2 \Longrightarrow 0 \leq y \Longrightarrow x \leq y$
⟨*proof*⟩

lemma *power2-less-imp-less*: $x^2 < y^2 \Longrightarrow 0 \leq y \Longrightarrow x < y$
⟨*proof*⟩

lemma *power2-eq-imp-eq*: $x^2 = y^2 \Longrightarrow 0 \leq x \Longrightarrow 0 \leq y \Longrightarrow x = y$
⟨*proof*⟩

lemma *power-Suc-le-self*: $0 \leq a \Longrightarrow a \leq 1 \Longrightarrow a \mathbin{\hat{}} Suc\ n \leq a$
⟨*proof*⟩

lemma *power2-eq-iff-nonneg* [*simp*]:

assumes $0 \leq x\ 0 \leq y$
shows $(x \mathbin{\hat{}} 2 = y \mathbin{\hat{}} 2) \longleftrightarrow x = y$
⟨*proof*⟩

lemma *of-nat-less-numeral-power-cancel-iff*[*simp*]:
$of\text{-}nat\ x < numeral\ i \mathbin{\hat{}} n \longleftrightarrow x < numeral\ i \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *of-nat-le-numeral-power-cancel-iff*[*simp*]:
$of\text{-}nat\ x \leq numeral\ i \mathbin{\hat{}} n \longleftrightarrow x \leq numeral\ i \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *numeral-power-less-of-nat-cancel-iff*[*simp*]:
$numeral\ i \mathbin{\hat{}} n < of\text{-}nat\ x \longleftrightarrow numeral\ i \mathbin{\hat{}} n < x$
⟨*proof*⟩

lemma *numeral-power-le-of-nat-cancel-iff*[*simp*]:
$numeral\ i \mathbin{\hat{}} n \leq of\text{-}nat\ x \longleftrightarrow numeral\ i \mathbin{\hat{}} n \leq x$
⟨*proof*⟩

lemma *of-nat-le-of-nat-power-cancel-iff*[*simp*]: $(of\text{-}nat\ b) \mathbin{\hat{}} w \leq of\text{-}nat\ x \longleftrightarrow b \mathbin{\hat{}} w \leq x$
⟨*proof*⟩

lemma *of-nat-power-le-of-nat-cancel-iff*[*simp*]: $of\text{-}nat\ x \leq (of\text{-}nat\ b) \mathbin{\hat{}} w \longleftrightarrow x \leq b \mathbin{\hat{}} w$
⟨*proof*⟩

lemma *of-nat-less-of-nat-power-cancel-iff*[*simp*]: $(of\text{-}nat\ b) \mathbin{\hat{}} w < of\text{-}nat\ x \longleftrightarrow b \mathbin{\hat{}} w < x$
⟨*proof*⟩

lemma *of-nat-power-less-of-nat-cancel-iff*[*simp*]: $of\text{-}nat\ x < (of\text{-}nat\ b) \mathbin{\hat{}} w \longleftrightarrow x < b \mathbin{\hat{}} w$
⟨*proof*⟩

lemma *power2-nonneg-ge-1-iff*:
assumes $x \geq 0$
shows $x \mathbin{\hat{}} 2 \geq 1 \longleftrightarrow x \geq 1$
⟨*proof*⟩

lemma *power2-nonneg-gt-1-iff*:
assumes $x \geq 0$
shows $x \mathbin{\hat{}} 2 > 1 \longleftrightarrow x > 1$
⟨*proof*⟩

end

Some *nat*-specific lemmas:

lemma *mono-ge2-power-minus-self*:
assumes $k \geq 2$ **shows** *mono* $(\lambda m.\ k \mathbin{\hat{}} m - m)$
⟨*proof*⟩

lemma *self-le-ge2-pow*[*simp*]:
assumes $k \geq 2$ **shows** $m \leq k \mathbin{\hat{}} m$
⟨*proof*⟩

lemma *diff-le-diff-pow*[*simp*]:
assumes $k \geq 2$ **shows** $m - n \leq k \mathbin{\hat{}} m - k \mathbin{\hat{}} n$
⟨*proof*⟩

context *linordered-ring-strict*
begin

lemma *sum-squares-eq-zero-iff*: $x * x + y * y = 0 \longleftrightarrow x = 0 \wedge y = 0$
⟨*proof*⟩

lemma *sum-squares-le-zero-iff*: $x * x + y * y \leq 0 \longleftrightarrow x = 0 \wedge y = 0$
⟨*proof*⟩

lemma *sum-squares-gt-zero-iff*: $0 < x * x + y * y \longleftrightarrow x \neq 0 \vee y \neq 0$
⟨*proof*⟩

end

context *linordered-idom*
begin

lemma *zero-le-power2* [*simp*]: $0 \leq a^2$
⟨*proof*⟩

lemma *zero-less-power2* [*simp*]: $0 < a^2 \longleftrightarrow a \neq 0$
⟨*proof*⟩

lemma *power2-less-0* [*simp*]: $\neg\ a^2 < 0$
⟨*proof*⟩

lemma *power-abs*: $|a \mathbin{\hat{}} n| = |a| \mathbin{\hat{}} n$ — FIXME simp?
⟨*proof*⟩

lemma *power-sgn* [*simp*]: *sgn* $(a \mathbin{\hat{}} n) =$ *sgn* $a \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *abs-power-minus* [*simp*]: $|(-\ a) \mathbin{\hat{}} n| = |a \mathbin{\hat{}} n|$
⟨*proof*⟩

lemma *zero-less-power-abs-iff* [*simp*]: $0 < |a| \mathbin{\hat{}} n \longleftrightarrow a \neq 0 \vee n = 0$

⟨*proof*⟩

lemma *zero-le-power-abs* [*simp*]: $0 \leq |a| \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *power2-less-eq-zero-iff* [*simp*]: $a^2 \leq 0 \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *abs-power2* [*simp*]: $|a^2| = a^2$
⟨*proof*⟩

lemma *power2-abs* [*simp*]: $|a|^2 = a^2$
⟨*proof*⟩

lemma *odd-power-less-zero*: $a < 0 \Longrightarrow a \mathbin{\hat{}} Suc\ (2 * n) < 0$
⟨*proof*⟩

lemma *odd-0-le-power-imp-0-le*: $0 \leq a \mathbin{\hat{}} Suc\ (2 * n) \Longrightarrow 0 \leq a$
⟨*proof*⟩

lemma *zero-le-even-power'*[*simp*]: $0 \leq a \mathbin{\hat{}} (2 * n)$
⟨*proof*⟩

lemma *sum-power2-ge-zero*: $0 \leq x^2 + y^2$
⟨*proof*⟩

lemma *not-sum-power2-lt-zero*: $\neg\ x^2 + y^2 < 0$
⟨*proof*⟩

lemma *sum-power2-eq-zero-iff*: $x^2 + y^2 = 0 \longleftrightarrow x = 0 \wedge y = 0$
⟨*proof*⟩

lemma *sum-power2-le-zero-iff*: $x^2 + y^2 \leq 0 \longleftrightarrow x = 0 \wedge y = 0$
⟨*proof*⟩

lemma *sum-power2-gt-zero-iff*: $0 < x^2 + y^2 \longleftrightarrow x \neq 0 \vee y \neq 0$
⟨*proof*⟩

lemma *abs-le-square-iff*: $|x| \leq |y| \longleftrightarrow x^2 \leq y^2$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *power2-le-iff-abs-le*:
$y \geq 0 \Longrightarrow x^2 \leq y^2 \longleftrightarrow |x| \leq y$
⟨*proof*⟩

lemma *abs-square-le-1*:$x^2 \leq 1 \longleftrightarrow |x| \leq 1$
⟨*proof*⟩

lemma *abs-square-eq-1*: $x^2 = 1 \longleftrightarrow |x| = 1$
⟨*proof*⟩

lemma *abs-square-less-1*: $x^2 < 1 \longleftrightarrow |x| < 1$
⟨*proof*⟩

lemma *square-le-1*:
assumes $-1 \leq x$ $x \leq 1$
shows $x^2 \leq 1$
⟨*proof*⟩

end

47.3 Miscellaneous rules

lemma (**in** *linordered-semidom*) *self-le-power*: $1 \leq a \Longrightarrow 0 < n \Longrightarrow a \leq a \hat{} n$
⟨*proof*⟩

lemma *power2-ge-1-iff*: $x \hat{} 2 \geq 1 \longleftrightarrow x \geq 1 \vee x \leq (-1 :: 'a :: linordered\text{-}idom)$
⟨*proof*⟩

lemma (**in** *power*) *power-eq-if*: $p \hat{} m = (if\ m{=}0\ then\ 1\ else\ p * (p \hat{} (m - 1)))$
⟨*proof*⟩

lemma (**in** *comm-semiring-1*) *power2-sum*: $(x + y)^2 = x^2 + y^2 + 2 * x * y$
⟨*proof*⟩

context *comm-ring-1*
begin

lemma *power2-diff*: $(x - y)^2 = x^2 + y^2 - 2 * x * y$
⟨*proof*⟩

lemma *power2-commute*: $(x - y)^2 = (y - x)^2$
⟨*proof*⟩

lemma *minus-power-mult-self*: $(-a) \hat{} n * (-a) \hat{} n = a \hat{} (2 * n)$
⟨*proof*⟩

lemma *minus-one-mult-self* [*simp*]: $(-1) \hat{} n * (-1) \hat{} n = 1$
⟨*proof*⟩

lemma *left-minus-one-mult-self* [*simp*]: $(-1) \hat{} n * ((-1) \hat{} n * a) = a$
⟨*proof*⟩

end

Simprules for comparisons where common factors can be cancelled.

lemmas *zero-compare-simps* =

add-strict-increasing add-strict-increasing2 add-increasing
zero-le-mult-iff zero-le-divide-iff
zero-less-mult-iff zero-less-divide-iff
mult-le-0-iff divide-le-0-iff
mult-less-0-iff divide-less-0-iff
zero-le-power2 power2-less-0

47.4 Exponentiation for the Natural Numbers

lemma *nat-one-le-power* [*simp*]: $Suc\ 0 \leq i \Longrightarrow Suc\ 0 \leq i \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *nat-zero-less-power-iff* [*simp*]: $x \mathbin{\hat{}} n > 0 \longleftrightarrow x > 0 \vee n = 0$
for $x :: nat$
⟨*proof*⟩

lemma *nat-power-eq-Suc-0-iff* [*simp*]: $x \mathbin{\hat{}} m = Suc\ 0 \longleftrightarrow m = 0 \vee x = Suc\ 0$
⟨*proof*⟩

lemma *power-Suc-0* [*simp*]: $Suc\ 0 \mathbin{\hat{}} n = Suc\ 0$
⟨*proof*⟩

Valid for the naturals, but what if $0 < i < 1$? Premises cannot be weakened: consider the case where $i = 0$, $m = 1$ and $n = 0$.

lemma *nat-power-less-imp-less*:
fixes $i :: nat$
assumes *nonneg*: $0 < i$
assumes *less*: $i \mathbin{\hat{}} m < i \mathbin{\hat{}} n$
shows $m < n$
⟨*proof*⟩

lemma *power-gt-expt*: $n > Suc\ 0 \Longrightarrow n \hat{} k > k$
⟨*proof*⟩

lemma *less-exp* [*simp*]:
‹$n < 2 \mathbin{\hat{}} n$›
⟨*proof*⟩

lemma *power-dvd-imp-le*:
fixes $i :: nat$
assumes $i \mathbin{\hat{}} m\ dvd\ i \mathbin{\hat{}} n\ 1 < i$
shows $m \leq n$
⟨*proof*⟩

lemma *dvd-power-iff-le*:
fixes $k{::}nat$
shows $2 \leq k \Longrightarrow ((k \mathbin{\hat{}} m)\ dvd\ (k \mathbin{\hat{}} n) \longleftrightarrow m \leq n)$
⟨*proof*⟩

lemma *power2-nat-le-eq-le*: $m^2 \leq n^2 \longleftrightarrow m \leq n$
for m n :: *nat*
⟨*proof*⟩

lemma *power2-nat-le-imp-le*:
fixes m n :: *nat*
assumes $m^2 \leq n$
shows $m \leq n$
⟨*proof*⟩

lemma *ex-power-ivl1*: **fixes** b k :: *nat* **assumes** $b \geq 2$
shows $k \geq 1 \Longrightarrow \exists n.\ b\hat{}n \leq k \wedge k < b\hat{}(n+1)$ (**is** - $\Longrightarrow \exists n.$ *?P k n*)
⟨*proof*⟩

lemma *ex-power-ivl2*: **fixes** b k :: *nat* **assumes** $b \geq 2$ $k \geq 2$
shows $\exists n.\ b\hat{}n < k \wedge k \leq b\hat{}(n+1)$
⟨*proof*⟩

47.4.1 Cardinality of the Powerset

lemma *card-UNIV-bool* [*simp*]: *card* (*UNIV* :: *bool set*) = *2*
⟨*proof*⟩

lemma *card-Pow*: *finite A* $\Longrightarrow$ *card* (*Pow A*) = *2* ^ *card A*
⟨*proof*⟩

47.5 Code generator tweak

code-identifier
code-module *Power* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

end

48 Big sum and product over finite (non-empty) sets

theory *Groups-Big*
imports *Power Equiv-Relations*
begin

48.1 Generic monoid operation over a set

locale *comm-monoid-set* = *comm-monoid*
begin

48.1.1 Standard sum or product indexed by a finite set

interpretation *comp-fun-commute f*
⟨*proof*⟩

interpretation *comp?*: *comp-fun-commute* $f \circ g$
⟨*proof*⟩

definition $F :: ('b \Rightarrow 'a) \Rightarrow 'b\ set \Rightarrow 'a$
where *eq-fold*: $F\ g\ A = Finite\text{-}Set.fold\ (f \circ g)\ \mathbf{1}\ A$

lemma *infinite* [*simp*]: $\neg\ finite\ A \Longrightarrow F\ g\ A = \mathbf{1}$
⟨*proof*⟩

lemma *empty* [*simp*]: $F\ g\ \{\} = \mathbf{1}$
⟨*proof*⟩

lemma *insert* [*simp*]: $finite\ A \Longrightarrow x \notin A \Longrightarrow F\ g\ (insert\ x\ A) = g\ x * F\ g\ A$
⟨*proof*⟩

lemma *remove*:
assumes *finite* A **and** $x \in A$
shows $F\ g\ A = g\ x * F\ g\ (A - \{x\})$
⟨*proof*⟩

lemma *insert-remove*: $finite\ A \Longrightarrow F\ g\ (insert\ x\ A) = g\ x * F\ g\ (A - \{x\})$
⟨*proof*⟩

lemma *insert-if*: $finite\ A \Longrightarrow F\ g\ (insert\ x\ A) = (if\ x \in A\ then\ F\ g\ A\ else\ g\ x * F\ g\ A)$
⟨*proof*⟩

lemma *neutral*: $\forall x \in A.\ g\ x = \mathbf{1} \Longrightarrow F\ g\ A = \mathbf{1}$
⟨*proof*⟩

lemma *neutral-const* [*simp*]: $F\ (\lambda\text{-}.\ \mathbf{1})\ A = \mathbf{1}$
⟨*proof*⟩

lemma *union-inter*:
assumes *finite* A **and** *finite* B
shows $F\ g\ (A \cup B) * F\ g\ (A \cap B) = F\ g\ A * F\ g\ B$
— The reversed orientation looks more natural, but LOOPS as a simprule!
⟨*proof*⟩

corollary *union-inter-neutral*:
assumes *finite* A **and** *finite* B
and $\forall x \in A \cap B.\ g\ x = \mathbf{1}$
shows $F\ g\ (A \cup B) = F\ g\ A * F\ g\ B$
⟨*proof*⟩

corollary *union-disjoint*:
assumes *finite* A **and** *finite* B
assumes $A \cap B = \{\}$

shows $F\ g\ (A \cup B) = F\ g\ A * F\ g\ B$
⟨*proof*⟩

lemma *union-diff2*:
assumes *finite* A **and** *finite* B
shows $F\ g\ (A \cup B) = F\ g\ (A - B) * F\ g\ (B - A) * F\ g\ (A \cap B)$
⟨*proof*⟩

lemma *subset-diff*:
assumes $B \subseteq A$ **and** *finite* A
shows $F\ g\ A = F\ g\ (A - B) * F\ g\ B$
⟨*proof*⟩

lemma *Int-Diff*:
assumes *finite* A
shows $F\ g\ A = F\ g\ (A \cap B) * F\ g\ (A - B)$
⟨*proof*⟩

lemma *setdiff-irrelevant*:
assumes *finite* A
shows $F\ g\ (A - \{x.\ g\ x = z\}) = F\ g\ A$
⟨*proof*⟩

lemma *not-neutral-contains-not-neutral*:
assumes $F\ g\ A \neq \mathbf{1}$
obtains a **where** $a \in A$ **and** $g\ a \neq \mathbf{1}$
⟨*proof*⟩

lemma *reindex*:
assumes *inj-on* $h\ A$
shows $F\ g\ (h\ `\ A) = F\ (g \circ h)\ A$
⟨*proof*⟩

lemma *cong* [*fundef-cong*]:
assumes $A = B$
assumes *g-h*: $\bigwedge x.\ x \in B \Longrightarrow g\ x = h\ x$
shows $F\ g\ A = F\ h\ B$
⟨*proof*⟩

lemma *cong-simp* [*cong*]:
$[\![A = B;\ \bigwedge x.\ x \in B =simp=> g\ x = h\ x]\!] \Longrightarrow F\ (\lambda x.\ g\ x)\ A = F\ (\lambda x.\ h\ x)\ B$
⟨*proof*⟩

lemma *reindex-cong*:
assumes *inj-on* $l\ B$
assumes $A = l\ `\ B$
assumes $\bigwedge x.\ x \in B \Longrightarrow g\ (l\ x) = h\ x$
shows $F\ g\ A = F\ h\ B$
⟨*proof*⟩

lemma *image-eq*:
assumes *inj-on g A*
shows $F\ (\lambda x.\ x)\ (g\ `\ A) = F\ g\ A$
⟨*proof*⟩

lemma *UNION-disjoint*:
assumes *finite I* **and** $\forall\, i{\in}I.$ *finite* $(A\ i)$
and $\forall\, i{\in}I.\ \forall\, j{\in}I.\ i \neq j \longrightarrow A\ i \cap A\ j = \{\}$
shows $F\ g\ (\bigcup (A\ `\ I)) = F\ (\lambda x.\ F\ g\ (A\ x))\ I$
⟨*proof*⟩

lemma *Union-disjoint*:
assumes $\forall\, A{\in}C.$ *finite* $A\ \forall\, A{\in}C.\ \forall\, B{\in}C.\ A \neq B \longrightarrow A \cap B = \{\}$
shows $F\ g\ (\bigcup C) = (F \circ F)\ g\ C$
⟨*proof*⟩

lemma *distrib*: $F\ (\lambda x.\ g\ x * h\ x)\ A = F\ g\ A * F\ h\ A$
⟨*proof*⟩

lemma *Sigma*:
assumes *finite* $A\ \forall\, x{\in}A.$ *finite* $(B\ x)$
shows $F\ (\lambda x.\ F\ (g\ x)\ (B\ x))\ A = F\ (\mathit{case\text{-}prod}\ g)\ (\mathit{SIGMA}\ x{:}A.\ B\ x)$
⟨*proof*⟩

lemma *related*:
assumes *Re*: $R\ \mathbf{1}\ \mathbf{1}$
and *Rop*: $\forall\, x1\ y1\ x2\ y2.\ R\ x1\ x2 \wedge R\ y1\ y2 \longrightarrow R\ (x1 * y1)\ (x2 * y2)$
and *fin*: *finite S*
and *R-h-g*: $\forall\, x{\in}S.\ R\ (h\ x)\ (g\ x)$
shows $R\ (F\ h\ S)\ (F\ g\ S)$
⟨*proof*⟩

lemma *mono-neutral-cong-left*:
assumes *finite T*
and $S \subseteq T$
and $\forall\, i \in T - S.\ h\ i = \mathbf{1}$
and $\bigwedge x.\ x \in S \Longrightarrow g\ x = h\ x$
shows $F\ g\ S = F\ h\ T$
⟨*proof*⟩

lemma *mono-neutral-cong-right*:
finite $T \Longrightarrow S \subseteq T \Longrightarrow \forall\, i \in T - S.\ g\ i = \mathbf{1} \Longrightarrow (\bigwedge x.\ x \in S \Longrightarrow g\ x = h\ x)$
$\Longrightarrow$
$F\ g\ T = F\ h\ S$
⟨*proof*⟩

lemma *mono-neutral-left*: *finite* $T \Longrightarrow S \subseteq T \Longrightarrow \forall\, i \in T - S.\ g\ i = \mathbf{1} \Longrightarrow F\ g\ S = F\ g\ T$

⟨*proof*⟩

lemma *mono-neutral-right*: *finite* $T \Longrightarrow S \subseteq T \Longrightarrow \forall i \in T - S.\ g\ i = \mathbf{1} \Longrightarrow F\ g\ T = F\ g\ S$
⟨*proof*⟩

lemma *mono-neutral-cong*:
assumes [*simp*]: *finite* T *finite* S
and $*$: $\bigwedge i.\ i \in T - S \Longrightarrow h\ i = \mathbf{1}$ $\bigwedge i.\ i \in S - T \Longrightarrow g\ i = \mathbf{1}$
and *gh*: $\bigwedge x.\ x \in S \cap T \Longrightarrow g\ x = h\ x$
shows $F\ g\ S = F\ h\ T$
⟨*proof*⟩

lemma *reindex-bij-betw*: *bij-betw* $h\ S\ T \Longrightarrow F\ (\lambda x.\ g\ (h\ x))\ S = F\ g\ T$
⟨*proof*⟩

lemma *reindex-bij-witness*:
assumes *witness*:
$\bigwedge a.\ a \in S \Longrightarrow i\ (j\ a) = a$
$\bigwedge a.\ a \in S \Longrightarrow j\ a \in T$
$\bigwedge b.\ b \in T \Longrightarrow j\ (i\ b) = b$
$\bigwedge b.\ b \in T \Longrightarrow i\ b \in S$
assumes *eq*:
$\bigwedge a.\ a \in S \Longrightarrow h\ (j\ a) = g\ a$
shows $F\ g\ S = F\ h\ T$
⟨*proof*⟩

lemma *reindex-bij-betw-not-neutral*:
assumes *fin*: *finite* S' *finite* T'
assumes *bij*: *bij-betw* $h\ (S - S')\ (T - T')$
assumes *nn*:
$\bigwedge a.\ a \in S' \Longrightarrow g\ (h\ a) = z$
$\bigwedge b.\ b \in T' \Longrightarrow g\ b = z$
shows $F\ (\lambda x.\ g\ (h\ x))\ S = F\ g\ T$
⟨*proof*⟩

lemma *reindex-nontrivial*:
assumes *finite* A
and *nz*: $\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow x \neq y \Longrightarrow h\ x = h\ y \Longrightarrow g\ (h\ x) = \mathbf{1}$
shows $F\ g\ (h\ `\ A) = F\ (g \circ h)\ A$
⟨*proof*⟩

lemma *reindex-bij-witness-not-neutral*:
assumes *fin*: *finite* S' *finite* T'
assumes *witness*:
$\bigwedge a.\ a \in S - S' \Longrightarrow i\ (j\ a) = a$
$\bigwedge a.\ a \in S - S' \Longrightarrow j\ a \in T - T'$
$\bigwedge b.\ b \in T - T' \Longrightarrow j\ (i\ b) = b$
$\bigwedge b.\ b \in T - T' \Longrightarrow i\ b \in S - S'$

assumes *nn*:
$\bigwedge a.\ a \in S' \Longrightarrow g\ a = z$
$\bigwedge b.\ b \in T' \Longrightarrow h\ b = z$
assumes *eq*:
$\bigwedge a.\ a \in S \Longrightarrow h\ (j\ a) = g\ a$
shows $F\ g\ S = F\ h\ T$
⟨*proof*⟩

lemma *delta-remove*:
assumes *fS*: *finite S*
shows $F\ (\lambda k.\ \mathit{if}\ k = a\ \mathit{then}\ b\ k\ \mathit{else}\ c\ k)\ S = (\mathit{if}\ a \in S\ \mathit{then}\ b\ a * F\ c\ (S - \{a\})\ \mathit{else}\ F\ c\ (S - \{a\}))$
⟨*proof*⟩

lemma *delta* [*simp*]:
assumes *fS*: *finite S*
shows $F\ (\lambda k.\ \mathit{if}\ k = a\ \mathit{then}\ b\ k\ \mathit{else}\ \mathbf{1})\ S = (\mathit{if}\ a \in S\ \mathit{then}\ b\ a\ \mathit{else}\ \mathbf{1})$
⟨*proof*⟩

lemma *delta'* [*simp*]:
assumes *fin*: *finite S*
shows $F\ (\lambda k.\ \mathit{if}\ a = k\ \mathit{then}\ b\ k\ \mathit{else}\ \mathbf{1})\ S = (\mathit{if}\ a \in S\ \mathit{then}\ b\ a\ \mathit{else}\ \mathbf{1})$
⟨*proof*⟩

lemma *If-cases*:
fixes $P :: 'b \Rightarrow \mathit{bool}$ **and** $g\ h :: 'b \Rightarrow 'a$
assumes *fin*: *finite A*
shows $F\ (\lambda x.\ \mathit{if}\ P\ x\ \mathit{then}\ h\ x\ \mathit{else}\ g\ x)\ A = F\ h\ (A \cap \{x.\ P\ x\}) * F\ g\ (A \cap - \{x.\ P\ x\})$
⟨*proof*⟩

lemma *cartesian-product*: $F\ (\lambda x.\ F\ (g\ x)\ B)\ A = F\ (\mathit{case\text{-}prod}\ g)\ (A \times B)$
⟨*proof*⟩

lemma *inter-restrict*:
assumes *finite A*
shows $F\ g\ (A \cap B) = F\ (\lambda x.\ \mathit{if}\ x \in B\ \mathit{then}\ g\ x\ \mathit{else}\ \mathbf{1})\ A$
⟨*proof*⟩

lemma *inter-filter*:
finite $A \Longrightarrow F\ g\ \{x \in A.\ P\ x\} = F\ (\lambda x.\ \mathit{if}\ P\ x\ \mathit{then}\ g\ x\ \mathit{else}\ \mathbf{1})\ A$
⟨*proof*⟩

lemma *Union-comp*:
assumes $\forall A \in B.\ \mathit{finite}\ A$
and $\bigwedge A1\ A2\ x.\ A1 \in B \Longrightarrow A2 \in B \Longrightarrow A1 \neq A2 \Longrightarrow x \in A1 \Longrightarrow x \in A2 \Longrightarrow g\ x = \mathbf{1}$
shows $F\ g\ (\bigcup B) = (F \circ F)\ g\ B$
⟨*proof*⟩

lemma *swap*: $F\ (\lambda i.\ F\ (g\ i)\ B)\ A = F\ (\lambda j.\ F\ (\lambda i.\ g\ i\ j)\ A)\ B$
⟨*proof*⟩

lemma *swap-restrict*:
finite $A \Longrightarrow$ *finite* $B \Longrightarrow$
$F\ (\lambda x.\ F\ (g\ x)\ \{y.\ y \in B \wedge R\ x\ y\})\ A = F\ (\lambda y.\ F\ (\lambda x.\ g\ x\ y)\ \{x.\ x \in A \wedge R\ x\ y\})\ B$
⟨*proof*⟩

lemma *image-gen*:
assumes *fin*: *finite* S
shows $F\ h\ S = F\ (\lambda y.\ F\ h\ \{x.\ x \in S \wedge g\ x = y\})\ (g\ `\ S)$
⟨*proof*⟩

lemma *group*:
assumes *fS*: *finite* S **and** *fT*: *finite* T **and** *fST*: $g\ `\ S \subseteq T$
shows $F\ (\lambda y.\ F\ h\ \{x.\ x \in S \wedge g\ x = y\})\ T = F\ h\ S$
⟨*proof*⟩

lemma *Plus*:
fixes $A :: 'b\ set$ **and** $B :: 'c\ set$
assumes *fin*: *finite* A *finite* B
shows $F\ g\ (A <+> B) = F\ (g \circ Inl)\ A * F\ (g \circ Inr)\ B$
⟨*proof*⟩

lemma *same-carrier*:
assumes *finite* C
assumes *subset*: $A \subseteq C\ B \subseteq C$
assumes *trivial*: $\bigwedge a.\ a \in C - A \Longrightarrow g\ a = \mathbf{1}\ \bigwedge b.\ b \in C - B \Longrightarrow h\ b = \mathbf{1}$
shows $F\ g\ A = F\ h\ B \longleftrightarrow F\ g\ C = F\ h\ C$
⟨*proof*⟩

lemma *same-carrierI*:
assumes *finite* C
assumes *subset*: $A \subseteq C\ B \subseteq C$
assumes *trivial*: $\bigwedge a.\ a \in C - A \Longrightarrow g\ a = \mathbf{1}\ \bigwedge b.\ b \in C - B \Longrightarrow h\ b = \mathbf{1}$
assumes $F\ g\ C = F\ h\ C$
shows $F\ g\ A = F\ h\ B$
⟨*proof*⟩

lemma *eq-general*:
assumes *B*: $\bigwedge y.\ y \in B \Longrightarrow \exists!x.\ x \in A \wedge h\ x = y$ **and** *A*: $\bigwedge x.\ x \in A \Longrightarrow h\ x \in B \wedge \gamma(h\ x) = \varphi\ x$
shows $F\ \varphi\ A = F\ \gamma\ B$
⟨*proof*⟩

lemma *eq-general-inverses*:
assumes *B*: $\bigwedge y.\ y \in B \Longrightarrow k\ y \in A \wedge h(k\ y) = y$ **and** *A*: $\bigwedge x.\ x \in A \Longrightarrow h\ x$

$\in B \wedge k(h\ x) = x \wedge \gamma(h\ x) = \varphi\ x$
shows $F\ \varphi\ A = F\ \gamma\ B$
⟨*proof*⟩

48.1.2 HOL Light variant: sum/product indexed by the non-neutral subset

NB only a subset of the properties above are proved

definition $G :: ['b \Rightarrow 'a, 'b\ set] \Rightarrow 'a$
where $G\ p\ I \equiv$ *if finite* $\{x \in I.\ p\ x \neq \mathbf{1}\}$ *then* $F\ p\ \{x \in I.\ p\ x \neq \mathbf{1}\}$ *else* $\mathbf{1}$

lemma *finite-Collect-op*:
shows $[\![$*finite* $\{i \in I.\ x\ i \neq \mathbf{1}\}$; *finite* $\{i \in I.\ y\ i \neq \mathbf{1}\}]\!] \Longrightarrow$ *finite* $\{i \in I.\ x\ i * y\ i \neq \mathbf{1}\}$
⟨*proof*⟩

lemma *empty'* [*simp*]: $G\ p\ \{\} = \mathbf{1}$
⟨*proof*⟩

lemma *eq-sum* [*simp*]: *finite* $I \Longrightarrow G\ p\ I = F\ p\ I$
⟨*proof*⟩

lemma *insert'* [*simp*]:
assumes *finite* $\{x \in I.\ p\ x \neq \mathbf{1}\}$
shows $G\ p\ (insert\ i\ I) =$ (*if* $i \in I$ *then* $G\ p\ I$ *else* $p\ i * G\ p\ I$)
⟨*proof*⟩

lemma *distrib-triv'*:
assumes *finite* I
shows $G\ (\lambda i.\ g\ i * h\ i)\ I = G\ g\ I * G\ h\ I$
⟨*proof*⟩

lemma *non-neutral'*: $G\ g\ \{x \in I.\ g\ x \neq \mathbf{1}\} = G\ g\ I$
⟨*proof*⟩

lemma *distrib'*:
assumes *finite* $\{x \in I.\ g\ x \neq \mathbf{1}\}$ *finite* $\{x \in I.\ h\ x \neq \mathbf{1}\}$
shows $G\ (\lambda i.\ g\ i * h\ i)\ I = G\ g\ I * G\ h\ I$
⟨*proof*⟩

lemma *cong'*:
assumes $A = B$
assumes *g-h*: $\bigwedge x.\ x \in B \Longrightarrow g\ x = h\ x$
shows $G\ g\ A = G\ h\ B$
⟨*proof*⟩

lemma *mono-neutral-cong-left'*:
assumes $S \subseteq T$

and $\bigwedge i.\ i \in T - S \Longrightarrow h\ i = \mathbf{1}$
and $\bigwedge x.\ x \in S \Longrightarrow g\ x = h\ x$
shows $G\ g\ S = G\ h\ T$
⟨*proof*⟩

lemma *mono-neutral-cong-right'*:
$S \subseteq T \Longrightarrow \forall i \in T - S.\ g\ i = \mathbf{1} \Longrightarrow (\bigwedge x.\ x \in S \Longrightarrow g\ x = h\ x) \Longrightarrow$
$G\ g\ T = G\ h\ S$
⟨*proof*⟩

lemma *mono-neutral-left'*: $S \subseteq T \Longrightarrow \forall i \in T - S.\ g\ i = \mathbf{1} \Longrightarrow G\ g\ S = G\ g\ T$
⟨*proof*⟩

lemma *mono-neutral-right'*: $S \subseteq T \Longrightarrow \forall i \in T - S.\ g\ i = \mathbf{1} \Longrightarrow G\ g\ T = G\ g\ S$
⟨*proof*⟩

end

48.2 Generalized summation over a set

context *comm-monoid-add*
begin

sublocale *sum*: *comm-monoid-set plus 0*
defines *sum* = *sum.F* **and** *sum'* = *sum.G* ⟨*proof*⟩

abbreviation *Sum* ($\sum$)
where $\sum \equiv sum\ (\lambda x.\ x)$

end

Now: lots of fancy syntax. First, *sum* $(\lambda x.\ e)\ A$ is written $\sum x{\in}A.\ e$.

syntax (*ASCII*)
-sum :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b*::*comm-monoid-add* ((*3SUM* (-/:-)./ -) [*0*, *51*, *10*] *10*)
syntax
-sum :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b*::*comm-monoid-add* ((*2*$\sum$(-/∈-)./ -) [*0*, *51*, *10*] *10*)
translations — Beware of argument permutation!
$\sum i{\in}A.\ b \rightleftharpoons CONST\ sum\ (\lambda i.\ b)\ A$

Instead of $\sum x{\in}\{x.\ P\}.\ e$ we introduce the shorter $\sum x|P.\ e$.

syntax (*ASCII*)
-qsum :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a* ((*3SUM* - |/ -./ -) [*0*, *0*, *10*] *10*)
syntax
-qsum :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a* ((*2*$\sum$ - | (-)./ -) [*0*, *0*, *10*] *10*)
translations
$\sum x|P.\ t$ => $CONST\ sum\ (\lambda x.\ t)\ \{x.\ P\}$

⟨*ML*⟩

48.2.1 Properties in more restricted classes of structures

lemma *sum-Un*:
finite A ⟹ *finite B* ⟹ *sum f* (*A* ∪ *B*) = *sum f A* + *sum f B* − *sum f* (*A* ∩ *B*)
for *f* :: *'b* ⇒ *'a*::*ab-group-add*
⟨*proof*⟩

lemma *sum-Un2*:
assumes *finite* (*A* ∪ *B*)
shows *sum f* (*A* ∪ *B*) = *sum f* (*A* − *B*) + *sum f* (*B* − *A*) + *sum f* (*A* ∩ *B*)
⟨*proof*⟩

lemma *sum-diff*:
fixes *f* :: *'b* ⇒ *'a*::*ab-group-add*
assumes *finite A B* ⊆ *A*
shows *sum f* (*A* − *B*) = *sum f A* − *sum f B*
⟨*proof*⟩

lemma *sum-diff1*:
fixes *f* :: *'b* ⇒ *'a*::*ab-group-add*
assumes *finite A*
shows *sum f* (*A* − {*a*}) = (*if a* ∈ *A then sum f A* − *f a else sum f A*)
⟨*proof*⟩

lemma *sum-diff1'-aux*:
fixes *f* :: *'a* ⇒ *'b*::*ab-group-add*
assumes *finite F* {*i* ∈ *I*. *f i* ≠ *0*} ⊆ *F*
shows *sum' f* (*I* − {*i*}) = (*if i* ∈ *I then sum' f I* − *f i else sum' f I*)
⟨*proof*⟩

lemma *sum-diff1'*:
fixes *f* :: *'a* ⇒ *'b*::*ab-group-add*
assumes *finite* {*i* ∈ *I*. *f i* ≠ *0*}
shows *sum' f* (*I* − {*i*}) = (*if i* ∈ *I then sum' f I* − *f i else sum' f I*)
⟨*proof*⟩

lemma (**in** *ordered-comm-monoid-add*) *sum-mono*:
(⋀*i*. *i*∈*K* ⟹ *f i* ≤ *g i*) ⟹ (∑ *i*∈*K*. *f i*) ≤ (∑ *i*∈*K*. *g i*)
⟨*proof*⟩

lemma (**in** *strict-ordered-comm-monoid-add*) *sum-strict-mono*:
assumes *finite A A* ≠ {}
and ⋀*x*. *x* ∈ *A* ⟹ *f x* < *g x*
shows *sum f A* < *sum g A*
⟨*proof*⟩

lemma *sum-strict-mono-ex1*:
fixes $f\ g :: 'i \Rightarrow 'a$::*ordered-cancel-comm-monoid-add*
assumes *finite A*
and $\forall x{\in}A.\ f\ x \leq g\ x$
and $\exists a{\in}A.\ f\ a < g\ a$
shows *sum f A* < *sum g A*
⟨*proof*⟩

lemma *sum-mono-inv*:
fixes $f\ g :: 'i \Rightarrow 'a$ :: *ordered-cancel-comm-monoid-add*
assumes *eq*: *sum f I* = *sum g I*
assumes *le*: $\bigwedge i.\ i \in I \Longrightarrow f\ i \leq g\ i$
assumes *i*: $i \in I$
assumes *I*: *finite I*
shows $f\ i = g\ i$
⟨*proof*⟩

lemma *member-le-sum*:
fixes f :: - ⇒ $'b$::{*semiring-1*, *ordered-comm-monoid-add*}
assumes $i \in A$
and *le*: $\bigwedge x.\ x \in A - \{i\} \Longrightarrow 0 \leq f\ x$
and *finite A*
shows $f\ i \leq$ *sum f A*
⟨*proof*⟩

lemma *sum-negf*: $(\sum x{\in}A.\ -\ f\ x) = -\ (\sum x{\in}A.\ f\ x)$
for $f :: 'b \Rightarrow 'a$::*ab-group-add*
⟨*proof*⟩

lemma *sum-subtractf*: $(\sum x{\in}A.\ f\ x\ -\ g\ x) = (\sum x{\in}A.\ f\ x)\ -\ (\sum x{\in}A.\ g\ x)$
for $f\ g :: 'b \Rightarrow 'a$::*ab-group-add*
⟨*proof*⟩

lemma *sum-subtractf-nat*:
$(\bigwedge x.\ x \in A \Longrightarrow g\ x \leq f\ x) \Longrightarrow (\sum x{\in}A.\ f\ x\ -\ g\ x) = (\sum x{\in}A.\ f\ x)\ -\ (\sum x{\in}A.\ g\ x)$
for $f\ g :: 'a \Rightarrow$ *nat*
⟨*proof*⟩

context *ordered-comm-monoid-add*
begin

lemma *sum-nonneg*: $(\bigwedge x.\ x \in A \Longrightarrow 0 \leq f\ x) \Longrightarrow 0 \leq$ *sum f A*
⟨*proof*⟩

lemma *sum-nonpos*: $(\bigwedge x.\ x \in A \Longrightarrow f\ x \leq 0) \Longrightarrow$ *sum f A* ≤ 0
⟨*proof*⟩

lemma *sum-nonneg-eq-0-iff*:

finite A $\Longrightarrow$ $(\bigwedge x.\ x \in A \Longrightarrow 0 \leq f\ x) \Longrightarrow$ *sum f A* $= 0 \longleftrightarrow (\forall x{\in}A.\ f\ x = 0)$
⟨*proof*⟩

lemma *sum-nonneg-0*:
finite s $\Longrightarrow$ $(\bigwedge i.\ i \in s \Longrightarrow f\ i \geq 0) \Longrightarrow (\sum\ i \in s.\ f\ i) = 0 \Longrightarrow i \in s \Longrightarrow f\ i = 0$
⟨*proof*⟩

lemma *sum-nonneg-leq-bound*:
assumes *finite s* $\bigwedge i.\ i \in s \Longrightarrow f\ i \geq 0$ $(\sum i \in s.\ f\ i) = B$ $i \in s$
shows $f\ i \leq B$
⟨*proof*⟩

lemma *sum-mono2*:
assumes *fin*: *finite B*
and *sub*: $A \subseteq B$
and *nn*: $\bigwedge b.\ b \in B{-}A \Longrightarrow 0 \leq f\ b$
shows *sum f A* $\leq$ *sum f B*
⟨*proof*⟩

lemma *sum-le-included*:
assumes *finite s finite t*
and $\forall y{\in}t.\ 0 \leq g\ y\ (\forall x{\in}s.\ \exists y{\in}t.\ i\ y = x \wedge f\ x \leq g\ y)$
shows *sum f s* $\leq$ *sum g t*
⟨*proof*⟩

end

lemma (**in** *canonically-ordered-monoid-add*) *sum-eq-0-iff* [*simp*]:
finite F $\Longrightarrow$ $(sum\ f\ F = 0) = (\forall a{\in}F.\ f\ a = 0)$
⟨*proof*⟩

context *semiring-0*
begin

lemma *sum-distrib-left*: $r * sum\ f\ A = (\sum n{\in}A.\ r * f\ n)$
⟨*proof*⟩

lemma *sum-distrib-right*: $sum\ f\ A * r = (\sum n{\in}A.\ f\ n * r)$
⟨*proof*⟩

end

lemma *sum-divide-distrib*: $sum\ f\ A\ /\ r = (\sum n{\in}A.\ f\ n\ /\ r)$
for $r :: 'a$::*field*
⟨*proof*⟩

lemma *sum-abs*[*iff*]: $|sum\ f\ A| \leq sum\ (\lambda i.\ |f\ i|)\ A$
for $f :: 'a \Rightarrow 'b$::*ordered-ab-group-add-abs*
⟨*proof*⟩

lemma *sum-abs-ge-zero*[*iff*]: $0 \leq sum\ (\lambda i.\ |f\ i|)\ A$
for $f :: 'a \Rightarrow 'b$::*ordered-ab-group-add-abs*
⟨*proof*⟩

lemma *abs-sum-abs*[*simp*]: $|\sum a{\in}A.\ |f\ a|| = (\sum a{\in}A.\ |f\ a|)$
for $f :: 'a \Rightarrow 'b$::*ordered-ab-group-add-abs*
⟨*proof*⟩

lemma *sum-product*:
fixes $f :: 'a \Rightarrow 'b$::*semiring-0*
shows $sum\ f\ A * sum\ g\ B = (\sum i{\in}A.\ \sum j{\in}B.\ f\ i * g\ j)$
⟨*proof*⟩

lemma *sum-mult-sum-if-inj*:
fixes $f :: 'a \Rightarrow 'b$::*semiring-0*
shows *inj-on* $(\lambda(a,\ b).\ f\ a * g\ b)\ (A \times B) \Longrightarrow$
$sum\ f\ A * sum\ g\ B = sum\ id\ \{f\ a * g\ b\ |a\ b.\ a \in A \wedge b \in B\}$
⟨*proof*⟩

lemma *sum-SucD*: $sum\ f\ A = Suc\ n \Longrightarrow \exists\, a{\in}A.\ 0 < f\ a$
⟨*proof*⟩

lemma *sum-eq-Suc0-iff*:
finite $A \Longrightarrow sum\ f\ A = Suc\ 0 \longleftrightarrow (\exists\, a{\in}A.\ f\ a = Suc\ 0 \wedge (\forall\, b{\in}A.\ a \neq b \longrightarrow f\ b = 0))$
⟨*proof*⟩

lemmas *sum-eq-1-iff* = *sum-eq-Suc0-iff*[*simplified One-nat-def*[*symmetric*]]

lemma *sum-Un-nat*:
finite $A \Longrightarrow$ *finite* $B \Longrightarrow sum\ f\ (A \cup B) = sum\ f\ A + sum\ f\ B - sum\ f\ (A \cap B)$
for $f :: 'a \Rightarrow nat$
— For the natural numbers, we have subtraction.
⟨*proof*⟩

lemma *sum-diff1-nat*: $sum\ f\ (A - \{a\}) = (if\ a \in A\ then\ sum\ f\ A - f\ a\ else\ sum\ f\ A)$
for $f :: 'a \Rightarrow nat$
⟨*proof*⟩

lemma *sum-diff-nat*:
fixes $f :: 'a \Rightarrow nat$
assumes *finite* B **and** $B \subseteq A$
shows $sum\ f\ (A - B) = sum\ f\ A - sum\ f\ B$
⟨*proof*⟩

lemma *sum-comp-morphism*:
$h\ 0 = 0 \Longrightarrow (\bigwedge x\ y.\ h\ (x + y) = h\ x + h\ y) \Longrightarrow sum\ (h \circ g)\ A = h\ (sum\ g\ A)$

⟨*proof*⟩

lemma (**in** *comm-semiring-1*) *dvd-sum*: $(\bigwedge a.\ a \in A \Longrightarrow d\ dvd\ f\ a) \Longrightarrow d\ dvd\ sum\ f\ A$
⟨*proof*⟩

lemma (**in** *ordered-comm-monoid-add*) *sum-pos*:
$finite\ I \Longrightarrow I \neq \{\} \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow 0 < f\ i) \Longrightarrow 0 < sum\ f\ I$
⟨*proof*⟩

lemma (**in** *ordered-comm-monoid-add*) *sum-pos2*:
assumes *I*: $finite\ I\ i \in I\ 0 < f\ i \bigwedge i.\ i \in I \Longrightarrow 0 \leq f\ i$
shows $0 < sum\ f\ I$
⟨*proof*⟩

lemma *sum-strict-mono2*:
fixes $f :: 'a \Rightarrow 'b$::*ordered-cancel-comm-monoid-add*
assumes $finite\ B\ A \subseteq B\ b \in B - A\ f\ b > 0$ **and** $\bigwedge x.\ x \in B \Longrightarrow f\ x \geq 0$
shows $sum\ f\ A < sum\ f\ B$
⟨*proof*⟩

lemma *sum-cong-Suc*:
assumes $0 \notin A \bigwedge x.\ Suc\ x \in A \Longrightarrow f\ (Suc\ x) = g\ (Suc\ x)$
shows $sum\ f\ A = sum\ g\ A$
⟨*proof*⟩

48.2.2 Cardinality as special case of *sum*

lemma *card-eq-sum*: $card\ A = sum\ (\lambda x.\ 1)\ A$
⟨*proof*⟩

context *semiring-1*
begin

lemma *sum-constant* [*simp*]:
$(\sum x \in A.\ y) = of\text{-}nat\ (card\ A) * y$
⟨*proof*⟩

context
fixes *A*
assumes ‹*finite A*›
begin

lemma *sum-of-bool-eq* [*simp*]:
‹$(\sum x \in A.\ of\text{-}bool\ (P\ x)) = of\text{-}nat\ (card\ (A \cap \{x.\ P\ x\}))$› **if** ‹*finite A*›
⟨*proof*⟩

lemma *sum-mult-of-bool-eq* [*simp*]:
‹$(\sum x \in A.\ f\ x * of\text{-}bool\ (P\ x)) = (\sum x \in (A \cap \{x.\ P\ x\}).\ f\ x)$›

⟨*proof*⟩

lemma *sum-of-bool-mult-eq* [*simp*]:
‹$(\sum x \in A.\ \textit{of-bool}\ (P\ x) * f\ x) = (\sum x \in (A \cap \{x.\ P\ x\}).\ f\ x)$›
⟨*proof*⟩

end

end

lemma *sum-Suc*: $\textit{sum}\ (\lambda x.\ \textit{Suc}(f\ x))\ A = \textit{sum}\ f\ A + \textit{card}\ A$
⟨*proof*⟩

lemma *sum-bounded-above*:
fixes $K :: 'a::\{\textit{semiring-1},\textit{ordered-comm-monoid-add}\}$
assumes *le*: $\bigwedge i.\ i \in A \Longrightarrow f\ i \leq K$
shows $\textit{sum}\ f\ A \leq \textit{of-nat}\ (\textit{card}\ A) * K$
⟨*proof*⟩

lemma *sum-bounded-above-divide*:
fixes $K :: 'a::\textit{linordered-field}$
assumes *le*: $\bigwedge i.\ i \in A \Longrightarrow f\ i \leq K\ /\ \textit{of-nat}\ (\textit{card}\ A)$ **and** *fin*: $\textit{finite}\ A\ A \neq \{\}$
shows $\textit{sum}\ f\ A \leq K$
⟨*proof*⟩

lemma *sum-bounded-above-strict*:
fixes $K :: 'a::\{\textit{ordered-cancel-comm-monoid-add},\textit{semiring-1}\}$
assumes $\bigwedge i.\ i \in A \Longrightarrow f\ i < K\ \textit{card}\ A > 0$
shows $\textit{sum}\ f\ A < \textit{of-nat}\ (\textit{card}\ A) * K$
⟨*proof*⟩

lemma *sum-bounded-below*:
fixes $K :: 'a::\{\textit{semiring-1},\textit{ordered-comm-monoid-add}\}$
assumes *le*: $\bigwedge i.\ i \in A \Longrightarrow K \leq f\ i$
shows $\textit{of-nat}\ (\textit{card}\ A) * K \leq \textit{sum}\ f\ A$
⟨*proof*⟩

lemma *convex-sum-bound-le*:
fixes $x :: 'a \Rightarrow 'b::\textit{linordered-idom}$
assumes *0*: $\bigwedge i.\ i \in I \Longrightarrow 0 \leq x\ i$ **and** *1*: $\textit{sum}\ x\ I = 1$
and δ: $\bigwedge i.\ i \in I \Longrightarrow |a\ i - b| \leq \delta$
shows $|(\sum i \in I.\ a\ i * x\ i) - b| \leq \delta$
⟨*proof*⟩

lemma *card-UN-disjoint*:
assumes $\textit{finite}\ I$ **and** $\forall i \in I.\ \textit{finite}\ (A\ i)$
and $\forall i \in I.\ \forall j \in I.\ i \neq j \longrightarrow A\ i \cap A\ j = \{\}$
shows $\textit{card}\ (\bigcup (A\ `\ I)) = (\sum i \in I.\ \textit{card}(A\ i))$
⟨*proof*⟩

lemma *card-Union-disjoint*:
assumes *pairwise disjnt C* **and** *fin*: $\bigwedge A.\ A \in C \Longrightarrow$ *finite A*
shows *card* $(\bigcup C)$ = *sum card C*
⟨*proof*⟩

lemma *card-Union-le-sum-card-weak*:
fixes $U :: 'a\ set\ set$
assumes $\forall\, u \in U.$ *finite u*
shows *card* $(\bigcup U) \leq$ *sum card U*
⟨*proof*⟩

lemma *card-Union-le-sum-card*:
fixes $U :: 'a\ set\ set$
shows *card* $(\bigcup U) \leq$ *sum card U*
⟨*proof*⟩

lemma *card-UN-le*:
assumes *finite I*
shows *card*$(\bigcup i{\in}I.\ A\ i) \leq (\sum i{\in}I.\ card(A\ i))$
⟨*proof*⟩

lemma *card-quotient-disjoint*:
assumes *finite A inj-on* $(\lambda x.\ \{x\}\ //\ r)\ A$
shows *card* $(A//r)$ = *card A*
⟨*proof*⟩

lemma *sum-multicount-gen*:
assumes *finite s finite t* $\forall\, j{\in}t.$ (*card* $\{i{\in}s.\ R\ i\ j\} = k\ j$)
shows *sum* $(\lambda i.\ (card\ \{j{\in}t.\ R\ i\ j\}))\ s$ = *sum k t*
(**is** *?l* = *?r*)
⟨*proof*⟩

lemma *sum-multicount*:
assumes *finite S finite T* $\forall\, j{\in}T.$ (*card* $\{i{\in}S.\ R\ i\ j\} = k$)
shows *sum* $(\lambda i.\ card\ \{j{\in}T.\ R\ i\ j\})\ S = k * card\ T$ (**is** *?l* = *?r*)
⟨*proof*⟩

lemma *sum-card-image*:
assumes *finite A*
assumes *pairwise* $(\lambda s\ t.\ disjnt\ (f\ s)\ (f\ t))\ A$
shows *sum card* $(f\ `\ A)$ = *sum* $(\lambda a.\ card\ (f\ a))\ A$
⟨*proof*⟩

By Jakub Kdzioka:

lemma *sum-fun-comp*:
assumes *finite S finite R g* $`\ S \subseteq R$
shows $(\sum x \in S.\ f\ (g\ x)) = (\sum y \in R.\ of\text{-}nat\ (card\ \{x \in S.\ g\ x = y\}) * f\ y)$
⟨*proof*⟩

48.2.3 Cardinality of products

lemma *card-SigmaI* [*simp*]:
finite A $\Longrightarrow \forall a{\in}A.$ *finite* (*B a*) $\Longrightarrow$ *card* (*SIGMA x*: *A*. *B x*) = ($\sum a{\in}A.$ *card* (*B a*))
⟨*proof*⟩

lemma *card-cartesian-product*: *card* (*A* × *B*) = *card A* ∗ *card B*
⟨*proof*⟩

lemma *card-cartesian-product-singleton*: *card* ({*x*} × *A*) = *card A*
⟨*proof*⟩

48.3 Generalized product over a set

context *comm-monoid-mult*
begin

sublocale *prod*: *comm-monoid-set times 1*
defines *prod* = *prod.F* **and** *prod′* = *prod.G* ⟨*proof*⟩

abbreviation *Prod* ($\prod$ - [*1000*] *999*)
where $\prod A \equiv$ *prod* (λ*x*. *x*) *A*

end

syntax (*ASCII*)
-prod :: *pttrn* => *'a set* => *'b* => *'b*::*comm-monoid-mult* ((*4PROD* (-/:-)./ -) [*0*, *51*, *10*] *10*)
syntax
-prod :: *pttrn* => *'a set* => *'b* => *'b*::*comm-monoid-mult* ((*2*$\prod$(-/∈-)./ -) [*0*, *51*, *10*] *10*)
translations — Beware of argument permutation!
$\prod i{\in}A.$ *b* == *CONST prod* (λ*i*. *b*) *A*

Instead of $\prod x{\in}\{x.\ P\}.\ e$ we introduce the shorter $\prod x|P.\ e$.

syntax (*ASCII*)
-qprod :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a* ((*4PROD* - |/ -./ -) [*0*, *0*, *10*] *10*)
syntax
-qprod :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a* ((*2*$\prod$ - | (-)./ -) [*0*, *0*, *10*] *10*)
translations
$\prod x|P.$ *t* => *CONST prod* (λ*x*. *t*) {*x*. *P*}

context *comm-monoid-mult*
begin

lemma *prod-dvd-prod*: ($\bigwedge a.$ *a* ∈ *A* ⟹ *f a dvd g a*) ⟹ *prod f A dvd prod g A*
⟨*proof*⟩

lemma *prod-dvd-prod-subset*: *finite* $B \Longrightarrow A \subseteq B \Longrightarrow$ *prod* f A *dvd prod* f B
⟨*proof*⟩

end

48.3.1 Properties in more restricted classes of structures

context *linordered-nonzero-semiring*
begin

lemma *prod-ge-1*: $(\bigwedge x.\ x \in A \Longrightarrow 1 \leq f\ x) \Longrightarrow 1 \leq$ *prod* f A
⟨*proof*⟩

lemma *prod-le-1*:
fixes $f :: 'b \Rightarrow 'a$
assumes $\bigwedge x.\ x \in A \Longrightarrow 0 \leq f\ x \wedge f\ x \leq 1$
shows *prod* f $A \leq 1$
⟨*proof*⟩

end

context *comm-semiring-1*
begin

lemma *dvd-prod-eqI* [*intro*]:
assumes *finite* A **and** $a \in A$ **and** $b = f\ a$
shows b *dvd prod* f A
⟨*proof*⟩

lemma *dvd-prodI* [*intro*]: *finite* $A \Longrightarrow a \in A \Longrightarrow f\ a$ *dvd prod* f A
⟨*proof*⟩

lemma *prod-zero*:
assumes *finite* A **and** $\exists\, a{\in}A.\ f\ a = 0$
shows *prod* f $A = 0$
⟨*proof*⟩

lemma *prod-dvd-prod-subset2*:
assumes *finite* B **and** $A \subseteq B$ **and** $\bigwedge a.\ a \in A \Longrightarrow f\ a$ *dvd* $g\ a$
shows *prod* f A *dvd prod* g B
⟨*proof*⟩

end

lemma (**in** *semidom*) *prod-zero-iff* [*simp*]:
fixes $f :: 'b \Rightarrow 'a$
assumes *finite* A
shows *prod* f $A = 0 \longleftrightarrow (\exists\, a{\in}A.\ f\ a = 0)$

⟨*proof*⟩

lemma (**in** *semidom-divide*) *prod-diff1*:
assumes *finite A* **and** *f a* ≠ *0*
shows *prod f* (*A* − {*a*}) = (*if a* ∈ *A then prod f A div f a else prod f A*)
⟨*proof*⟩

lemma *sum-zero-power* [*simp*]: (∑ *i*∈*A*. *c i* ∗ *0*^*i*) = (*if finite A* ∧ *0* ∈ *A then c 0 else 0*)
for *c* :: *nat* ⇒ *'a*::*division-ring*
⟨*proof*⟩

lemma *sum-zero-power'* [*simp*]:
(∑ *i*∈*A*. *c i* ∗ *0*^*i* / *d i*) = (*if finite A* ∧ *0* ∈ *A then c 0* / *d 0 else 0*)
for *c* :: *nat* ⇒ *'a*::*field*
⟨*proof*⟩

lemma (**in** *field*) *prod-inversef*: *prod* (*inverse* ∘ *f*) *A* = *inverse* (*prod f A*)
⟨*proof*⟩

lemma (**in** *field*) *prod-dividef*: (∏ *x*∈*A*. *f x* / *g x*) = *prod f A* / *prod g A*
⟨*proof*⟩

lemma *prod-Un*:
fixes *f* :: *'b* ⇒ *'a* :: *field*
assumes *finite A* **and** *finite B*
and ∀ *x*∈*A* ∩ *B*. *f x* ≠ *0*
shows *prod f* (*A* ∪ *B*) = *prod f A* ∗ *prod f B* / *prod f* (*A* ∩ *B*)
⟨*proof*⟩

context *linordered-semidom*
begin

lemma *prod-nonneg*: (∀ *a*∈*A*. *0* ≤ *f a*) ⟹ *0* ≤ *prod f A*
⟨*proof*⟩

lemma *prod-pos*: (∀ *a*∈*A*. *0* < *f a*) ⟹ *0* < *prod f A*
⟨*proof*⟩

lemma *prod-mono*:
(⋀*i*. *i* ∈ *A* ⟹ *0* ≤ *f i* ∧ *f i* ≤ *g i*) ⟹ *prod f A* ≤ *prod g A*
⟨*proof*⟩

lemma *prod-mono-strict*:
assumes *finite A* ⋀*i*. *i* ∈ *A* ⟹ *0* ≤ *f i* ∧ *f i* < *g i A* ≠ {}
shows *prod f A* < *prod g A*
⟨*proof*⟩

lemma *prod-le-power*:

assumes A: $\bigwedge i.\ i \in A \Longrightarrow 0 \leq f\ i \wedge f\ i \leq n$ *card* $A \leq k$ **and** $n \geq 1$
shows *prod* $f\ A \leq n \mathbin{\hat{}} k$
⟨*proof*⟩

end

lemma *prod-mono2*:
fixes $f :: 'a \Rightarrow 'b :: linordered\text{-}idom$
assumes *fin*: *finite* B
and *sub*: $A \subseteq B$
and *nn*: $\bigwedge b.\ b \in B - A \Longrightarrow 1 \leq f\ b$
and A: $\bigwedge a.\ a \in A \Longrightarrow 0 \leq f\ a$
shows *prod* $f\ A \leq$ *prod* $f\ B$
⟨*proof*⟩

lemma *less-1-prod*:
fixes $f :: 'a \Rightarrow 'b :: linordered\text{-}idom$
shows *finite* $I \Longrightarrow I \neq \{\} \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow 1 < f\ i) \Longrightarrow 1 <$ *prod* $f\ I$
⟨*proof*⟩

lemma *less-1-prod2*:
fixes $f :: 'a \Rightarrow 'b :: linordered\text{-}idom$
assumes I: *finite* $I\ i \in I\ 1 < f\ i\ \bigwedge i.\ i \in I \Longrightarrow 1 \leq f\ i$
shows $1 <$ *prod* $f\ I$
⟨*proof*⟩

lemma (**in** *linordered-field*) *abs-prod*: $|$*prod* $f\ A| = (\prod x \in A.\ |f\ x|)$
⟨*proof*⟩

lemma *prod-eq-1-iff* [*simp*]: *finite* $A \Longrightarrow$ *prod* $f\ A = 1 \longleftrightarrow (\forall a \in A.\ f\ a = 1)$
for $f :: 'a \Rightarrow nat$
⟨*proof*⟩

lemma *prod-pos-nat-iff* [*simp*]: *finite* $A \Longrightarrow$ *prod* $f\ A > 0 \longleftrightarrow (\forall a \in A.\ f\ a > 0)$
for $f :: 'a \Rightarrow nat$
⟨*proof*⟩

lemma *prod-constant* [*simp*]: $(\prod x \in A.\ y) = y \mathbin{\hat{}}$ *card* A
for $y :: 'a :: comm\text{-}monoid\text{-}mult$
⟨*proof*⟩

lemma *prod-power-distrib*: *prod* $f\ A \mathbin{\hat{}} n =$ *prod* $(\lambda x.\ (f\ x) \mathbin{\hat{}} n)\ A$
for $f :: 'a \Rightarrow 'b :: comm\text{-}semiring\text{-}1$
⟨*proof*⟩

lemma *power-sum*: $c \mathbin{\hat{}} (\sum a \in A.\ f\ a) = (\prod a \in A.\ c \mathbin{\hat{}} f\ a)$
⟨*proof*⟩

lemma *prod-gen-delta*:

fixes $b :: 'b \Rightarrow 'a$::*comm-monoid-mult*
assumes *fin*: *finite S*
shows *prod* $(\lambda k.$ *if* $k = a$ *then* $b\ k$ *else* $c)\ S =$
$($*if* $a \in S$ *then* $b\ a * c\ \widehat{}\ (card\ S - 1)$ *else* $c\ \widehat{}\ card\ S)$
⟨*proof*⟩

lemma *sum-image-le*:
fixes $g :: 'a \Rightarrow 'b$::*ordered-comm-monoid-add*
assumes *finite* I $\bigwedge i.\ i \in I \Longrightarrow 0 \leq g(f\ i)$
shows $sum\ g\ (f\ `\ I) \leq sum\ (g \circ f)\ I$
⟨*proof*⟩

end

49 Chain-complete partial orders and their fixpoints

theory *Complete-Partial-Order*
imports *Product-Type*
begin

49.1 Chains

A chain is a totally-ordered set. Chains are parameterized over the order for maximal flexibility, since type classes are not enough.

definition $chain :: ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a\ set \Rightarrow bool$
where $chain\ ord\ S \longleftrightarrow (\forall x{\in}S.\ \forall y{\in}S.\ ord\ x\ y \vee ord\ y\ x)$

lemma *chainI*:
assumes $\bigwedge x\ y.\ x \in S \Longrightarrow y \in S \Longrightarrow ord\ x\ y \vee ord\ y\ x$
shows $chain\ ord\ S$
⟨*proof*⟩

lemma *chainD*:
assumes $chain\ ord\ S$ **and** $x \in S$ **and** $y \in S$
shows $ord\ x\ y \vee ord\ y\ x$
⟨*proof*⟩

lemma *chainE*:
assumes $chain\ ord\ S$ **and** $x \in S$ **and** $y \in S$
obtains $ord\ x\ y \mid ord\ y\ x$
⟨*proof*⟩

lemma *chain-empty*: $chain\ ord\ \{\}$
⟨*proof*⟩

lemma *chain-equality*: $chain\ (=)\ A \longleftrightarrow (\forall x{\in}A.\ \forall y{\in}A.\ x = y)$
⟨*proof*⟩

lemma *chain-subset*: *chain ord A* $\Longrightarrow$ *B* $\subseteq$ *A* $\Longrightarrow$ *chain ord B*
⟨*proof*⟩

lemma *chain-imageI*:
assumes *chain*: *chain le-a Y*
and *mono*: $\bigwedge x\ y.\ x \in Y \Longrightarrow y \in Y \Longrightarrow$ *le-a x y* $\Longrightarrow$ *le-b* (*f x*) (*f y*)
shows *chain le-b* (*f* ‘ *Y*)
⟨*proof*⟩

49.2 Chain-complete partial orders

A *ccpo* has a least upper bound for any chain. In particular, the empty set is a chain, so every *ccpo* must have a bottom element.

class *ccpo* = *order* + *Sup* +
assumes *ccpo-Sup-upper*: *chain* ($\leq$) *A* $\Longrightarrow$ *x* $\in$ *A* $\Longrightarrow$ *x* $\leq$ *Sup A*
assumes *ccpo-Sup-least*: *chain* ($\leq$) *A* $\Longrightarrow$ ($\bigwedge x.\ x \in A \Longrightarrow x \leq z$) $\Longrightarrow$ *Sup A* $\leq$ *z*
begin

lemma *chain-singleton*: *Complete-Partial-Order.chain* ($\leq$) {*x*}
⟨*proof*⟩

lemma *ccpo-Sup-singleton* [*simp*]: $\bigsqcup\{x\} = x$
⟨*proof*⟩

49.3 Transfinite iteration of a function

context notes [[*inductive-internals*]]
begin

inductive-set *iterates* :: (′*a* ⇒ ′*a*) ⇒ ′*a set*
for *f* :: ′*a* ⇒ ′*a*
where
step: *x* $\in$ *iterates f* $\Longrightarrow$ *f x* $\in$ *iterates f*
| *Sup*: *chain* ($\leq$) *M* $\Longrightarrow$ $\forall x{\in}M.\ x \in$ *iterates f* $\Longrightarrow$ *Sup M* $\in$ *iterates f*

end

lemma *iterates-le-f*: *x* $\in$ *iterates f* $\Longrightarrow$ *monotone* ($\leq$) ($\leq$) *f* $\Longrightarrow$ *x* $\leq$ *f x*
⟨*proof*⟩

lemma *chain-iterates*:
assumes *f*: *monotone* ($\leq$) ($\leq$) *f*
shows *chain* ($\leq$) (*iterates f*) (**is** *chain - ?C*)
⟨*proof*⟩

lemma *bot-in-iterates*: *Sup* {} $\in$ *iterates f*
⟨*proof*⟩

49.4 Fixpoint combinator

definition *fixp* :: $('a \Rightarrow 'a) \Rightarrow 'a$
  where $fixp\ f = Sup\ (iterates\ f)$

lemma *iterates-fixp*:
  assumes f: $monotone\ (\leq)\ (\leq)\ f$
  shows $fixp\ f \in iterates\ f$
  ⟨*proof*⟩

lemma *fixp-unfold*:
  assumes f: $monotone\ (\leq)\ (\leq)\ f$
  shows $fixp\ f = f\ (fixp\ f)$
⟨*proof*⟩

lemma *fixp-lowerbound*:
  assumes f: $monotone\ (\leq)\ (\leq)\ f$
    and z: $f\ z \leq z$
  shows $fixp\ f \leq z$
  ⟨*proof*⟩

end

49.5 Fixpoint induction

⟨*ML*⟩

definition *admissible* :: $('a\ set \Rightarrow 'a) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('a \Rightarrow bool) \Rightarrow bool$
  where $admissible\ lub\ ord\ P \longleftrightarrow (\forall A.\ chain\ ord\ A \longrightarrow A \neq \{\} \longrightarrow (\forall x{\in}A.\ P\ x) \longrightarrow P\ (lub\ A))$

lemma *admissibleI*:
  assumes $\bigwedge A.\ chain\ ord\ A \Longrightarrow A \neq \{\} \Longrightarrow \forall x{\in}A.\ P\ x \Longrightarrow P\ (lub\ A)$
  shows *ccpo.admissible lub ord P*
  ⟨*proof*⟩

lemma *admissibleD*:
  assumes *ccpo.admissible lub ord P*
  assumes *chain ord A*
  assumes $A \neq \{\}$
  assumes $\bigwedge x.\ x \in A \Longrightarrow P\ x$
  shows $P\ (lub\ A)$
  ⟨*proof*⟩

⟨*ML*⟩

lemma (**in** *ccpo*) *fixp-induct*:
  assumes *adm*: *ccpo.admissible Sup* $(\leq)$ P
  assumes *mono*: $monotone\ (\leq)\ (\leq)\ f$
  assumes *bot*: $P\ (Sup\ \{\})$

assumes *step*: $\bigwedge x.\ P\ x \Longrightarrow P\ (f\ x)$
shows P (*fixp f*)
⟨*proof*⟩

lemma *admissible-True*: *ccpo.admissible lub ord* ($\lambda x.$ *True*)
⟨*proof*⟩

lemma *admissible-const*: *ccpo.admissible lub ord* ($\lambda x.\ t$)
⟨*proof*⟩

lemma *admissible-conj*:
assumes *ccpo.admissible lub ord* ($\lambda x.\ P\ x$)
assumes *ccpo.admissible lub ord* ($\lambda x.\ Q\ x$)
shows *ccpo.admissible lub ord* ($\lambda x.\ P\ x \wedge Q\ x$)
⟨*proof*⟩

lemma *admissible-all*:
assumes $\bigwedge y.$ *ccpo.admissible lub ord* ($\lambda x.\ P\ x\ y$)
shows *ccpo.admissible lub ord* ($\lambda x.\ \forall y.\ P\ x\ y$)
⟨*proof*⟩

lemma *admissible-ball*:
assumes $\bigwedge y.\ y \in A \Longrightarrow$ *ccpo.admissible lub ord* ($\lambda x.\ P\ x\ y$)
shows *ccpo.admissible lub ord* ($\lambda x.\ \forall y{\in}A.\ P\ x\ y$)
⟨*proof*⟩

lemma *chain-compr*: *chain ord A* $\Longrightarrow$ *chain ord* $\{x \in A.\ P\ x\}$
⟨*proof*⟩

context *ccpo*
begin

lemma *admissible-disj*:
fixes $P\ Q :: {'a} \Rightarrow bool$
assumes P: *ccpo.admissible Sup* ($\leq$) ($\lambda x.\ P\ x$)
assumes Q: *ccpo.admissible Sup* ($\leq$) ($\lambda x.\ Q\ x$)
shows *ccpo.admissible Sup* ($\leq$) ($\lambda x.\ P\ x \vee Q\ x$)
⟨*proof*⟩

end

instance *complete-lattice* $\subseteq$ *ccpo*
⟨*proof*⟩

lemma *lfp-eq-fixp*:
assumes *mono*: *mono f*
shows *lfp f* = *fixp f*
⟨*proof*⟩

hide-const (**open**) *iterates fixp*

end

50 Datatype option

theory *Option*
imports *Lifting*
begin

datatype *'a option* =
None
| *Some* (*the*: *'a*)

datatype-compat *option*

lemma [*case-names None Some*, *cases type*: *option*]:
— for backward compatibility – names of variables differ
$(y = None \Longrightarrow P) \Longrightarrow (\bigwedge a.\ y = Some\ a \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma [*case-names None Some*, *induct type*: *option*]:
— for backward compatibility – names of variables differ
$P\ None \Longrightarrow (\bigwedge option.\ P\ (Some\ option)) \Longrightarrow P\ option$
⟨*proof*⟩

Compatibility:

⟨*ML*⟩
lemmas *inducts* = *option.induct*
lemmas *cases* = *option.case*
⟨*ML*⟩

lemma *not-None-eq* [*iff*]: $x \neq None \longleftrightarrow (\exists\, y.\ x = Some\ y)$
⟨*proof*⟩

lemma *not-Some-eq* [*iff*]: $(\forall\, y.\ x \neq Some\ y) \longleftrightarrow x = None$
⟨*proof*⟩

lemma *comp-the-Some*[*simp*]: *the o Some* = *id*
⟨*proof*⟩

Although it may appear that both of these equalities are helpful only when applied to assumptions, in practice it seems better to give them the uniform iff attribute.

lemma *inj-Some* [*simp*]: *inj-on Some A*
⟨*proof*⟩

lemma *case-optionE*:
assumes *c*: (*case x of None ⇒ P | Some y ⇒ Q y*)
obtains
(*None*) *x = None* **and** *P*
| (*Some*) *y* **where** *x = Some y* **and** *Q y*
⟨*proof*⟩

lemma *split-option-all*: (∀ *x*. *P x*) ⟷ *P None* ∧ (∀ *x*. *P* (*Some x*))
⟨*proof*⟩

lemma *split-option-ex*: (∃ *x*. *P x*) ⟷ *P None* ∨ (∃ *x*. *P* (*Some x*))
⟨*proof*⟩

lemma *UNIV-option-conv*: *UNIV = insert None* (*range Some*)
⟨*proof*⟩

lemma *rel-option-None1* [*simp*]: *rel-option P None x* ⟷ *x = None*
⟨*proof*⟩

lemma *rel-option-None2* [*simp*]: *rel-option P x None* ⟷ *x = None*
⟨*proof*⟩

lemma *option-rel-Some1*: *rel-option A* (*Some x*) *y* ⟷ (∃ *y'*. *y = Some y'* ∧ *A x y'*)
⟨*proof*⟩

lemma *option-rel-Some2*: *rel-option A x* (*Some y*) ⟷ (∃ *x'*. *x = Some x'* ∧ *A x' y*)
⟨*proof*⟩

lemma *rel-option-inf*: *inf* (*rel-option A*) (*rel-option B*) = *rel-option* (*inf A B*)
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *rel-option-reflI*:
(⋀*x*. *x* ∈ *set-option y* ⟹ *P x x*) ⟹ *rel-option P y y*
⟨*proof*⟩

50.0.1 Operations

lemma *ospec* [*dest*]: (∀ *x*∈*set-option A*. *P x*) ⟹ *A = Some x* ⟹ *P x*
⟨*proof*⟩

⟨*ML*⟩

lemma *elem-set* [*iff*]: (*x* ∈ *set-option xo*) = (*xo = Some x*)
⟨*proof*⟩

lemma *set-empty-eq* [*simp*]: (*set-option xo* = {}) = (*xo = None*)

⟨*proof*⟩

lemma *map-option-case*: *map-option f y* = (*case y of None* ⇒ *None* | *Some x* ⇒ *Some* (*f x*))
⟨*proof*⟩

lemma *map-option-is-None* [*iff*]: (*map-option f opt* = *None*) = (*opt* = *None*)
⟨*proof*⟩

lemma *None-eq-map-option-iff* [*iff*]: *None* = *map-option f x* ⟷ *x* = *None*
⟨*proof*⟩

lemma *map-option-eq-Some* [*iff*]: (*map-option f xo* = *Some y*) = (∃ *z*. *xo* = *Some z* ∧ *f z* = *y*)
⟨*proof*⟩

lemma *map-option-o-case-sum* [*simp*]:
map-option f ∘ *case-sum g h* = *case-sum* (*map-option f* ∘ *g*) (*map-option f* ∘ *h*)
⟨*proof*⟩

lemma *map-option-cong*: *x* = *y* ⟹ (⋀*a*. *y* = *Some a* ⟹ *f a* = *g a*) ⟹ *map-option f x* = *map-option g y*
⟨*proof*⟩

lemma *map-option-idI*: (⋀*y*. *y* ∈ *set-option x* ⟹ *f y* = *y*) ⟹ *map-option f x* = *x*
⟨*proof*⟩

functor *map-option*: *map-option*
⟨*proof*⟩

lemma *case-map-option* [*simp*]: *case-option g h* (*map-option f x*) = *case-option g* (*h* ∘ *f*) *x*
⟨*proof*⟩

lemma *None-notin-image-Some* [*simp*]: *None* ∉ *Some* ‘ *A*
⟨*proof*⟩

lemma *notin-range-Some*: *x* ∉ *range Some* ⟷ *x* = *None*
⟨*proof*⟩

lemma *rel-option-iff*:
rel-option R x y = (*case* (*x*, *y*) *of* (*None*, *None*) ⇒ *True*
| (*Some x*, *Some y*) ⇒ *R x y*
| - ⇒ *False*)
⟨*proof*⟩

definition *combine-options* :: $('a \Rightarrow 'a \Rightarrow 'a) \Rightarrow 'a\ option \Rightarrow 'a\ option \Rightarrow 'a\ option$
where *combine-options f x y* =
(*case x of None* ⇒ *y* | *Some x* ⇒ (*case y of None* ⇒ *Some x* | *Some y* ⇒ *Some* (*f x y*)))

lemma *combine-options-simps* [*simp*]:
combine-options f None y = *y*
combine-options f x None = *x*
combine-options f (*Some a*) (*Some b*) = *Some* (*f a b*)
⟨*proof*⟩

lemma *combine-options-cases* [*case-names None1 None2 Some*]:
$(x = None \Longrightarrow P\ x\ y) \Longrightarrow (y = None \Longrightarrow P\ x\ y) \Longrightarrow$
$(\bigwedge a\ b.\ x = Some\ a \Longrightarrow y = Some\ b \Longrightarrow P\ x\ y) \Longrightarrow P\ x\ y$
⟨*proof*⟩

lemma *combine-options-commute*:
$(\bigwedge x\ y.\ f\ x\ y = f\ y\ x) \Longrightarrow$ *combine-options f x y* = *combine-options f y x*
⟨*proof*⟩

lemma *combine-options-assoc*:
$(\bigwedge x\ y\ z.\ f\ (f\ x\ y)\ z = f\ x\ (f\ y\ z)) \Longrightarrow$
combine-options f (*combine-options f x y*) *z* =
combine-options f x (*combine-options f y z*)
⟨*proof*⟩

lemma *combine-options-left-commute*:
$(\bigwedge x\ y.\ f\ x\ y = f\ y\ x) \Longrightarrow (\bigwedge x\ y\ z.\ f\ (f\ x\ y)\ z = f\ x\ (f\ y\ z)) \Longrightarrow$
combine-options f y (*combine-options f x z*) =
combine-options f x (*combine-options f y z*)
⟨*proof*⟩

lemmas *combine-options-ac* =
combine-options-commute combine-options-assoc combine-options-left-commute

context
begin

qualified definition *is-none* :: *'a option* ⇒ *bool*
where [*code-post*]: *is-none x* ⟷ *x* = *None*

lemma *is-none-simps* [*simp*]:
is-none None
¬ *is-none* (*Some x*)
⟨*proof*⟩

lemma *is-none-code* [*code*]:
is-none None = *True*

is-none (*Some x*) = *False*
⟨*proof*⟩

lemma *rel-option-unfold*:
rel-option R x y ⟷
(*is-none x* ⟷ *is-none y*) ∧ (¬ *is-none x* ⟶ ¬ *is-none y* ⟶ *R* (*the x*) (*the y*))
⟨*proof*⟩

lemma *rel-optionI*:
⟦ *is-none x* ⟷ *is-none y*; ⟦ ¬ *is-none x*; ¬ *is-none y* ⟧ ⟹ *P* (*the x*) (*the y*) ⟧
⟹ *rel-option P x y*
⟨*proof*⟩

lemma *is-none-map-option* [*simp*]: *is-none* (*map-option f x*) ⟷ *is-none x*
⟨*proof*⟩

lemma *the-map-option*: ¬ *is-none x* ⟹ *the* (*map-option f x*) = *f* (*the x*)
⟨*proof*⟩ **primrec** *bind* :: *'a option* ⇒ (*'a* ⇒ *'b option*) ⇒ *'b option*
where
bind-lzero: *bind None f* = *None*
| *bind-lunit*: *bind* (*Some x*) *f* = *f x*

lemma *is-none-bind*: *is-none* (*bind f g*) ⟷ *is-none f* ∨ *is-none* (*g* (*the f*))
⟨*proof*⟩

lemma *bind-runit*[*simp*]: *bind x Some* = *x*
⟨*proof*⟩

lemma *bind-assoc*[*simp*]: *bind* (*bind x f*) *g* = *bind x* (λ*y*. *bind* (*f y*) *g*)
⟨*proof*⟩

lemma *bind-rzero*[*simp*]: *bind x* (λ*x*. *None*) = *None*
⟨*proof*⟩ **lemma** *bind-cong*: *x* = *y* ⟹ (⋀*a*. *y* = *Some a* ⟹ *f a* = *g a*) ⟹ *bind x f* = *bind y g*
⟨*proof*⟩

lemma *bind-split*: *P* (*bind m f*) ⟷ (*m* = *None* ⟶ *P None*) ∧ (∀ *v*. *m* = *Some v* ⟶ *P* (*f v*))
⟨*proof*⟩

lemma *bind-split-asm*: *P* (*bind m f*) ⟷ ¬ (*m* = *None* ∧ ¬ *P None* ∨ (∃ *x*. *m* = *Some x* ∧ ¬ *P* (*f x*)))
⟨*proof*⟩

lemmas *bind-splits* = *bind-split bind-split-asm*

lemma *bind-eq-Some-conv*: *bind f g* = *Some x* ⟷ (∃ *y*. *f* = *Some y* ∧ *g y* = *Some x*)

⟨*proof*⟩

lemma *bind-eq-None-conv*: *Option.bind a f* = *None* ⟷ *a* = *None* ∨ *f* (*the a*) = *None*
⟨*proof*⟩

lemma *map-option-bind*: *map-option f* (*bind x g*) = *bind x* (*map-option f* ∘ *g*)
⟨*proof*⟩

lemma *bind-option-cong*:
⟦ *x* = *y*; ⋀*z*. *z* ∈ *set-option y* ⟹ *f z* = *g z* ⟧ ⟹ *bind x f* = *bind y g*
⟨*proof*⟩

lemma *bind-option-cong-simp*:
⟦ *x* = *y*; ⋀*z*. *z* ∈ *set-option y* =*simp*=> *f z* = *g z* ⟧ ⟹ *bind x f* = *bind y g*
⟨*proof*⟩

lemma *bind-option-cong-code*: *x* = *y* ⟹ *bind x f* = *bind y f*
⟨*proof*⟩

lemma *bind-map-option*: *bind* (*map-option f x*) *g* = *bind x* (*g* ∘ *f*)
⟨*proof*⟩

lemma *set-bind-option* [*simp*]: *set-option* (*bind x f*) = (⋃((*set-option* ∘ *f*) ‘ *set-option x*))
⟨*proof*⟩

lemma *map-conv-bind-option*: *map-option f x* = *Option.bind x* (*Some* ∘ *f*)
⟨*proof*⟩

end

⟨*ML*⟩

context
begin

qualified definition *these* :: *'a option set* ⇒ *'a set*
where *these A* = *the* ‘ {*x* ∈ *A*. *x* ≠ *None*}

lemma *these-empty* [*simp*]: *these* {} = {}
⟨*proof*⟩

lemma *these-insert-None* [*simp*]: *these* (*insert None A*) = *these A*
⟨*proof*⟩

lemma *these-insert-Some* [*simp*]: *these* (*insert* (*Some x*) *A*) = *insert x* (*these A*)
⟨*proof*⟩

lemma *in-these-eq*: $x \in$ *these* $A \longleftrightarrow$ *Some* $x \in A$
⟨*proof*⟩

lemma *these-image-Some-eq* [*simp*]: *these* (*Some* ‘ A) $= A$
⟨*proof*⟩

lemma *Some-image-these-eq*: *Some* ‘ *these* $A = \{x{\in}A.\ x \neq None\}$
⟨*proof*⟩

lemma *these-empty-eq*: *these* $B = \{\} \longleftrightarrow B = \{\} \vee B = \{None\}$
⟨*proof*⟩

lemma *these-not-empty-eq*: *these* $B \neq \{\} \longleftrightarrow B \neq \{\} \wedge B \neq \{None\}$
⟨*proof*⟩

end

lemma *finite-range-Some*: *finite* (*range* (*Some* :: $'a \Rightarrow 'a$ *option*)) = *finite* (*UNIV* :: $'a$ *set*)
⟨*proof*⟩

50.1 Transfer rules for the Transfer package

context includes *lifting-syntax*
begin

lemma *option-bind-transfer* [*transfer-rule*]:
(*rel-option A* ===> (*A* ===> *rel-option B*) ===> *rel-option B*)
Option.bind Option.bind
⟨*proof*⟩

lemma *pred-option-parametric* [*transfer-rule*]:
((*A* ===> (=)) ===> *rel-option A* ===> (=)) *pred-option pred-option*
⟨*proof*⟩

end

50.1.1 Interaction with finite sets

lemma *finite-option-UNIV* [*simp*]:
finite (*UNIV* :: $'a$ *option set*) = *finite* (*UNIV* :: $'a$ *set*)
⟨*proof*⟩

instance *option* :: (*finite*) *finite*
⟨*proof*⟩

50.1.2 Code generator setup

lemma *equal-None-code-unfold* [*code-unfold*]:

HOL.equal x None ⟷ *Option.is-none x*
HOL.equal None = *Option.is-none*
⟨*proof*⟩

code-printing
type-constructor *option* ⇀
(*SML*) - *option*
and (*OCaml*) - *option*
and (*Haskell*) *Maybe* -
and (*Scala*) !*Option*[(-)]
| **constant** *None* ⇀
(*SML*) *NONE*
and (*OCaml*) *None*
and (*Haskell*) *Nothing*
and (*Scala*) !*None*
| **constant** *Some* ⇀
(*SML*) *SOME*
and (*OCaml*) *Some* -
and (*Haskell*) *Just*
and (*Scala*) *Some*
| **class-instance** *option* :: *equal* ⇀
(*Haskell*) –
| **constant** *HOL.equal* :: *'a option* ⇒ *'a option* ⇒ *bool* ⇀
(*Haskell*) **infix** *4* ==

code-reserved *SML*
option NONE SOME

code-reserved *OCaml*
option None Some

code-reserved *Scala*
Option None Some

end

51 Partial Function Definitions

theory *Partial-Function*
imports *Complete-Partial-Order Option*
keywords *partial-function* :: *thy-defn*
begin

named-theorems *partial-function-mono monotonicity rules for partial function definitions*
⟨*ML*⟩

lemma (**in** *ccpo*) *in-chain-finite*:
assumes *Complete-Partial-Order.chain* (≤) *A finite A A* ≠ {}

shows $\bigsqcup A \in A$
⟨*proof*⟩

lemma (**in** *ccpo*) *admissible-chfin*:
$(\forall S.\ \textit{Complete-Partial-Order.chain}\ (\leq)\ S \longrightarrow \textit{finite}\ S)$
$\Longrightarrow$ *ccpo.admissible Sup* $(\leq)$ *P*
⟨*proof*⟩

51.1 Axiomatic setup

This techical locale constains the requirements for function definitions with ccpo fixed points.

definition *fun-ord ord f g* $\longleftrightarrow$ $(\forall x.\ ord\ (f\ x)\ (g\ x))$
definition *fun-lub L A* $= (\lambda x.\ L\ \{y.\ \exists f{\in}A.\ y = f\ x\})$
definition *img-ord f ord* $= (\lambda x\ y.\ ord\ (f\ x)\ (f\ y))$
definition *img-lub f g Lub* $= (\lambda A.\ g\ (Lub\ (f\ `\ A)))$

lemma *chain-fun*:
assumes *A*: *chain* (*fun-ord ord*) *A*
shows *chain ord* $\{y.\ \exists f{\in}A.\ y = f\ a\}$ (**is** *chain ord ?C*)
⟨*proof*⟩

lemma *call-mono*[*partial-function-mono*]: *monotone* (*fun-ord ord*) *ord* $(\lambda f.\ f\ t)$
⟨*proof*⟩

lemma *let-mono*[*partial-function-mono*]:
$(\bigwedge x.\ monotone\ orda\ ordb\ (\lambda f.\ b\ f\ x))$
$\Longrightarrow$ *monotone orda ordb* $(\lambda f.\ Let\ t\ (b\ f))$
⟨*proof*⟩

lemma *if-mono*[*partial-function-mono*]: *monotone orda ordb F*
$\Longrightarrow$ *monotone orda ordb G*
$\Longrightarrow$ *monotone orda ordb* $(\lambda f.\ if\ c\ then\ F\ f\ else\ G\ f)$
⟨*proof*⟩

definition *mk-less R* $= (\lambda x\ y.\ R\ x\ y \wedge \neg\ R\ y\ x)$

locale *partial-function-definitions* =
fixes *leq* :: $'a \Rightarrow 'a \Rightarrow bool$
fixes *lub* :: $'a\ set \Rightarrow 'a$
assumes *leq-refl*: *leq x x*
assumes *leq-trans*: *leq x y* $\Longrightarrow$ *leq y z* $\Longrightarrow$ *leq x z*
assumes *leq-antisym*: *leq x y* $\Longrightarrow$ *leq y x* $\Longrightarrow$ $x = y$
assumes *lub-upper*: *chain leq A* $\Longrightarrow$ $x \in A$ $\Longrightarrow$ *leq x* (*lub A*)
assumes *lub-least*: *chain leq A* $\Longrightarrow$ $(\bigwedge x.\ x \in A \Longrightarrow leq\ x\ z) \Longrightarrow leq\ (lub\ A)\ z$

lemma *partial-function-lift*:
assumes *partial-function-definitions ord lb*
shows *partial-function-definitions* (*fun-ord ord*) (*fun-lub lb*) (**is** *partial-function-definitions*

?ordf ?lubf)
⟨*proof*⟩

lemma *ccpo*: **assumes** *partial-function-definitions ord lb*
shows *class.ccpo lb ord* (*mk-less ord*)
⟨*proof*⟩

lemma *partial-function-image*:
assumes *partial-function-definitions ord Lub*
assumes *inj*: $\bigwedge x\ y.\ f\ x = f\ y \Longrightarrow x = y$
assumes *inv*: $\bigwedge x.\ f\ (g\ x) = x$
shows *partial-function-definitions* (*img-ord f ord*) (*img-lub f g Lub*)
⟨*proof*⟩

context *partial-function-definitions*
begin

abbreviation *le-fun* ≡ *fun-ord leq*
abbreviation *lub-fun* ≡ *fun-lub lub*
abbreviation *fixp-fun* ≡ *ccpo.fixp lub-fun le-fun*
abbreviation *mono-body* ≡ *monotone le-fun leq*
abbreviation *admissible* ≡ *ccpo.admissible lub-fun le-fun*

Interpret manually, to avoid flooding everything with facts about orders

lemma *ccpo*: *class.ccpo lub-fun le-fun* (*mk-less le-fun*)
⟨*proof*⟩

The crucial fixed-point theorem

lemma *mono-body-fixp*:
$(\bigwedge x.$ *mono-body* $(\lambda f.\ F\ f\ x)) \Longrightarrow$ *fixp-fun* $F = F$ (*fixp-fun* F)
⟨*proof*⟩

Version with curry/uncurry combinators, to be used by package

lemma *fixp-rule-uc*:
fixes $F :: 'c \Rightarrow 'c$ **and**
$U :: 'c \Rightarrow 'b \Rightarrow 'a$ **and**
$C :: ('b \Rightarrow 'a) \Rightarrow 'c$
assumes *mono*: $\bigwedge x.$ *mono-body* $(\lambda f.\ U\ (F\ (C\ f))\ x)$
assumes *eq*: $f \equiv C$ (*fixp-fun* $(\lambda f.\ U\ (F\ (C\ f))))$
assumes *inverse*: $\bigwedge f.\ C\ (U\ f) = f$
shows $f = F\ f$
⟨*proof*⟩

Fixpoint induction rule

lemma *fixp-induct-uc*:
fixes $F :: 'c \Rightarrow 'c$
and $U :: 'c \Rightarrow 'b \Rightarrow 'a$
and $C :: ('b \Rightarrow 'a) \Rightarrow 'c$

and $P :: ('b \Rightarrow 'a) \Rightarrow bool$
assumes *mono*: $\bigwedge x.\ mono\text{-}body\ (\lambda f.\ U\ (F\ (C\ f))\ x)$
and *eq*: $f \equiv C\ (fixp\text{-}fun\ (\lambda f.\ U\ (F\ (C\ f))))$
and *inverse*: $\bigwedge f.\ U\ (C\ f) = f$
and *adm*: *ccpo.admissible lub-fun le-fun P*
and *bot*: $P\ (\lambda\text{-}.\ lub\ \{\})$
and *step*: $\bigwedge f.\ P\ (U\ f) \Longrightarrow P\ (U\ (F\ f))$
shows $P\ (U\ f)$
⟨*proof*⟩

Rules for *monotone le-fun leq*:

lemma *const-mono*[*partial-function-mono*]: *monotone ord leq* $(\lambda f.\ c)$
⟨*proof*⟩

end

51.2 Flat interpretation: tailrec and option

definition
$flat\text{-}ord\ b\ x\ y \longleftrightarrow x = b \vee x = y$

definition
$flat\text{-}lub\ b\ A = (if\ A \subseteq \{b\}\ then\ b\ else\ (THE\ x.\ x \in A - \{b\}))$

lemma *flat-interpretation*:
partial-function-definitions (*flat-ord b*) (*flat-lub b*)
⟨*proof*⟩

lemma *flat-ordI*: $(x \neq a \Longrightarrow x = y) \Longrightarrow flat\text{-}ord\ a\ x\ y$
⟨*proof*⟩

lemma *flat-ord-antisym*: $[\![\ flat\text{-}ord\ a\ x\ y;\ flat\text{-}ord\ a\ y\ x\ ]\!] \Longrightarrow x = y$
⟨*proof*⟩

lemma *antisymp-flat-ord*: *antisymp* (*flat-ord a*)
⟨*proof*⟩

interpretation *tailrec*:
partial-function-definitions flat-ord undefined flat-lub undefined
rewrites *flat-lub undefined* $\{\} \equiv undefined$
⟨*proof*⟩

interpretation *option*:
partial-function-definitions flat-ord None flat-lub None
rewrites *flat-lub None* $\{\} \equiv None$
⟨*proof*⟩

abbreviation *tailrec-ord* $\equiv$ *flat-ord undefined*

abbreviation *mono-tailrec* ≡ *monotone* (*fun-ord tailrec-ord*) *tailrec-ord*

lemma *tailrec-admissible*:
ccpo.admissible (*fun-lub* (*flat-lub c*)) (*fun-ord* (*flat-ord c*))
(λ*a*. ∀ *x*. *a x* ≠ *c* ⟶ *P x* (*a x*))
⟨*proof*⟩

lemma *fixp-induct-tailrec*:
fixes *F* :: *'c* ⇒ *'c* **and**
U :: *'c* ⇒ *'b* ⇒ *'a* **and**
C :: (*'b* ⇒ *'a*) ⇒ *'c* **and**
P :: *'b* ⇒ *'a* ⇒ *bool* **and**
x :: *'b*
assumes *mono*: ⋀*x*. *monotone* (*fun-ord* (*flat-ord c*)) (*flat-ord c*) (λ*f*. *U* (*F* (*C f*)) *x*)
assumes *eq*: *f* ≡ *C* (*ccpo.fixp* (*fun-lub* (*flat-lub c*)) (*fun-ord* (*flat-ord c*)) (λ*f*. *U* (*F* (*C f*))))
assumes *inverse2*: ⋀*f*. *U* (*C f*) = *f*
assumes *step*: ⋀*f x y*. (⋀*x y*. *U f x* = *y* ⟹ *y* ≠ *c* ⟹ *P x y*) ⟹ *U* (*F f*) *x* = *y* ⟹ *y* ≠ *c* ⟹ *P x y*
assumes *result*: *U f x* = *y*
assumes *defined*: *y* ≠ *c*
shows *P x y*
⟨*proof*⟩

lemma *admissible-image*:
assumes *pfun*: *partial-function-definitions le lub*
assumes *adm*: *ccpo.admissible lub le* (*P* ∘ *g*)
assumes *inj*: ⋀*x y*. *f x* = *f y* ⟹ *x* = *y*
assumes *inv*: ⋀*x*. *f* (*g x*) = *x*
shows *ccpo.admissible* (*img-lub f g lub*) (*img-ord f le*) *P*
⟨*proof*⟩

lemma *admissible-fun*:
assumes *pfun*: *partial-function-definitions le lub*
assumes *adm*: ⋀*x*. *ccpo.admissible lub le* (*Q x*)
shows *ccpo.admissible* (*fun-lub lub*) (*fun-ord le*) (λ*f*. ∀ *x*. *Q x* (*f x*))
⟨*proof*⟩

abbreviation *option-ord* ≡ *flat-ord None*
abbreviation *mono-option* ≡ *monotone* (*fun-ord option-ord*) *option-ord*

lemma *bind-mono*[*partial-function-mono*]:
assumes *mf*: *mono-option B* **and** *mg*: ⋀*y*. *mono-option* (λ*f*. *C y f*)
shows *mono-option* (λ*f*. *Option.bind* (*B f*) (λ*y*. *C y f*))
⟨*proof*⟩

lemma *flat-lub-in-chain*:

assumes *ch*: *chain* (*flat-ord b*) *A*
assumes *lub*: *flat-lub b A* = *a*
shows *a* = *b* ∨ *a* ∈ *A*
⟨*proof*⟩

lemma *option-admissible*: *option.admissible* (%(*f*::'*a* ⇒ '*b option*).
(∀ *x y*. *f x* = *Some y* ⟶ *P x y*))
⟨*proof*⟩

lemma *fixp-induct-option*:
fixes *F* :: '*c* ⇒ '*c* **and**
U :: '*c* ⇒ '*b* ⇒ '*a option* **and**
C :: ('*b* ⇒ '*a option*) ⇒ '*c* **and**
P :: '*b* ⇒ '*a* ⇒ *bool*
assumes *mono*: ⋀*x*. *mono-option* (λ*f*. *U* (*F* (*C f*)) *x*)
assumes *eq*: *f* ≡ *C* (*ccpo.fixp* (*fun-lub* (*flat-lub None*)) (*fun-ord option-ord*) (λ*f*. *U* (*F* (*C f*))))
assumes *inverse2*: ⋀*f*. *U* (*C f*) = *f*
assumes *step*: ⋀*f x y*. (⋀*x y*. *U f x* = *Some y* ⟹ *P x y*) ⟹ *U* (*F f*) *x* = *Some y* ⟹ *P x y*
assumes *defined*: *U f x* = *Some y*
shows *P x y*
⟨*proof*⟩

⟨*ML*⟩

hide-const (**open**) *chain*

end

theory *Argo*
imports *HOL*
begin

⟨*ML*⟩

end

52 Reconstructing external resolution proofs for propositional logic

theory *SAT*
imports *Argo*
begin

⟨*ML*⟩

end

53 Function Definitions and Termination Proofs

theory *Fun-Def*
imports *Basic-BNF-LFPs Partial-Function SAT*
keywords
function termination :: *thy-goal-defn* **and**
fun fun-cases :: *thy-defn*
begin

53.1 Definitions with default value

definition *THE-default* :: $'a \Rightarrow ('a \Rightarrow bool) \Rightarrow 'a$
where *THE-default d P* = (*if* $(\exists!x.\ P\ x)$ *then* (*THE x. P x*) *else d*)

lemma *THE-defaultI'*: $\exists!x.\ P\ x \Longrightarrow P\ (THE\text{-}default\ d\ P)$
⟨*proof*⟩

lemma *THE-default1-equality*: $\exists!x.\ P\ x \Longrightarrow P\ a \Longrightarrow THE\text{-}default\ d\ P = a$
⟨*proof*⟩

lemma *THE-default-none*: $\neg\ (\exists!x.\ P\ x) \Longrightarrow THE\text{-}default\ d\ P = d$
⟨*proof*⟩

lemma *fundef-ex1-existence*:
assumes *f-def*: $f \equiv (\lambda x{::}'a.\ THE\text{-}default\ (d\ x)\ (\lambda y.\ G\ x\ y))$
assumes *ex1*: $\exists!y.\ G\ x\ y$
shows $G\ x\ (f\ x)$
⟨*proof*⟩

lemma *fundef-ex1-uniqueness*:
assumes *f-def*: $f \equiv (\lambda x{::}'a.\ THE\text{-}default\ (d\ x)\ (\lambda y.\ G\ x\ y))$
assumes *ex1*: $\exists!y.\ G\ x\ y$
assumes *elm*: $G\ x\ (h\ x)$
shows $h\ x = f\ x$
⟨*proof*⟩

lemma *fundef-ex1-iff*:
assumes *f-def*: $f \equiv (\lambda x{::}'a.\ THE\text{-}default\ (d\ x)\ (\lambda y.\ G\ x\ y))$
assumes *ex1*: $\exists!y.\ G\ x\ y$
shows $(G\ x\ y) = (f\ x = y)$
⟨*proof*⟩

lemma *fundef-default-value*:
assumes *f-def*: $f \equiv (\lambda x{::}'a.\ THE\text{-}default\ (d\ x)\ (\lambda y.\ G\ x\ y))$
assumes *graph*: $\bigwedge x\ y.\ G\ x\ y \Longrightarrow D\ x$

assumes $\neg\ D\ x$
shows $f\ x = d\ x$
⟨*proof*⟩

definition *in-rel-def*[*simp*]: *in-rel R x y* ≡ $(x,\ y) \in R$

lemma *wf-in-rel*: *wf R* ⟹ *wfP* (*in-rel R*)
⟨*proof*⟩

⟨*ML*⟩

53.2 Measure functions

inductive *is-measure* :: $('a \Rightarrow nat) \Rightarrow bool$
where *is-measure-trivial*: *is-measure f*

named-theorems *measure-function rules that guide the heuristic generation of measure functions*
⟨*ML*⟩

lemma *measure-size*[*measure-function*]: *is-measure size*
⟨*proof*⟩

lemma *measure-fst*[*measure-function*]: *is-measure f* ⟹ *is-measure* (λ*p*. *f* (*fst p*))
⟨*proof*⟩

lemma *measure-snd*[*measure-function*]: *is-measure f* ⟹ *is-measure* (λ*p*. *f* (*snd p*))
⟨*proof*⟩

⟨*ML*⟩

53.3 Congruence rules

lemma *let-cong* [*fundef-cong*]: *M* = *N* ⟹ (⋀*x*. *x* = *N* ⟹ *f x* = *g x*) ⟹ *Let M f* = *Let N g*
⟨*proof*⟩

lemmas [*fundef-cong*] =
if-cong image-cong
bex-cong ball-cong imp-cong map-option-cong Option.bind-cong

lemma *split-cong* [*fundef-cong*]:
(⋀*x y*. $(x,\ y)$ = *q* ⟹ *f x y* = *g x y*) ⟹ *p* = *q* ⟹ *case-prod f p* = *case-prod g q*
⟨*proof*⟩

lemma *comp-cong* [*fundef-cong*]: *f* (*g x*) = *f'* (*g' x'*) ⟹ (*f* ∘ *g*) *x* = (*f'* ∘ *g'*) *x'*
⟨*proof*⟩

53.4 Simp rules for termination proofs

declare
trans-less-add1[*termination-simp*]
trans-less-add2[*termination-simp*]
trans-le-add1[*termination-simp*]
trans-le-add2[*termination-simp*]
less-imp-le-nat[*termination-simp*]
le-imp-less-Suc[*termination-simp*]

lemma *size-prod-simp*[*termination-simp*]: *size-prod f g p* = *f* (*fst p*) + *g* (*snd p*) + *Suc 0*
⟨*proof*⟩

53.5 Decomposition

lemma *less-by-empty*: $A = \{\} \Longrightarrow A \subseteq B$
and *union-comp-emptyL*: $A\ O\ C = \{\} \Longrightarrow B\ O\ C = \{\} \Longrightarrow (A \cup B)\ O\ C = \{\}$
and *union-comp-emptyR*: $A\ O\ B = \{\} \Longrightarrow A\ O\ C = \{\} \Longrightarrow A\ O\ (B \cup C) = \{\}$
and *wf-no-loop*: $R\ O\ R = \{\} \Longrightarrow wf\ R$
⟨*proof*⟩

53.6 Reduction pairs

definition *reduction-pair P* ⟷ *wf* (*fst P*) ∧ *fst P O snd P* ⊆ *fst P*

lemma *reduction-pairI*[*intro*]: *wf R* ⟹ *R O S* ⊆ *R* ⟹ *reduction-pair* (*R*, *S*)
⟨*proof*⟩

lemma *reduction-pair-lemma*:
assumes *rp*: *reduction-pair P*
assumes *R* ⊆ *fst P*
assumes *S* ⊆ *snd P*
assumes *wf S*
shows *wf* (*R* ∪ *S*)
⟨*proof*⟩

definition *rp-inv-image* = (λ(*R*,*S*) *f*. (*inv-image R f*, *inv-image S f*))

lemma *rp-inv-image-rp*: *reduction-pair P* ⟹ *reduction-pair* (*rp-inv-image P f*)
⟨*proof*⟩

53.7 Concrete orders for SCNP termination proofs

definition *pair-less* = *less-than* <∗*lex*∗> *less-than*
definition *pair-leq* = *pair-less*$^{=}$
definition *max-strict* = *max-ext pair-less*
definition *max-weak* = *max-ext pair-leq* ∪ {({}, {})}
definition *min-strict* = *min-ext pair-less*
definition *min-weak* = *min-ext pair-leq* ∪ {({}, {})}

lemma *wf-pair-less*[*simp*]: *wf pair-less*
⟨*proof*⟩

lemma *total-pair-less* [*iff*]: *total-on A pair-less* **and** *trans-pair-less* [*iff*]: *trans pair-less*
⟨*proof*⟩

Introduction rules for *pair-less*/*pair-leq*

lemma *pair-leqI1*: $a < b \Longrightarrow ((a, s), (b, t)) \in$ *pair-leq*
and *pair-leqI2*: $a \leq b \Longrightarrow s \leq t \Longrightarrow ((a, s), (b, t)) \in$ *pair-leq*
and *pair-lessI1*: $a < b \Longrightarrow ((a, s), (b, t)) \in$ *pair-less*
and *pair-lessI2*: $a \leq b \Longrightarrow s < t \Longrightarrow ((a, s), (b, t)) \in$ *pair-less*
⟨*proof*⟩

lemma *pair-less-iff1* [*simp*]: $((x,y), (x,z)) \in$ *pair-less* $\longleftrightarrow$ $y<z$
⟨*proof*⟩

Introduction rules for max

lemma *smax-emptyI*: *finite* $Y \Longrightarrow Y \neq \{\} \Longrightarrow (\{\}, Y) \in$ *max-strict*
and *smax-insertI*:
$y \in Y \Longrightarrow (x, y) \in$ *pair-less* $\Longrightarrow (X, Y) \in$ *max-strict* $\Longrightarrow$ (*insert x X*, Y) $\in$ *max-strict*
and *wmax-emptyI*: *finite* $X \Longrightarrow (\{\}, X) \in$ *max-weak*
and *wmax-insertI*:
$y \in YS \Longrightarrow (x, y) \in$ *pair-leq* $\Longrightarrow (XS, YS) \in$ *max-weak* $\Longrightarrow$ (*insert x XS*, *YS*) $\in$ *max-weak*
⟨*proof*⟩

Introduction rules for min

lemma *smin-emptyI*: $X \neq \{\} \Longrightarrow (X, \{\}) \in$ *min-strict*
and *smin-insertI*:
$x \in XS \Longrightarrow (x, y) \in$ *pair-less* $\Longrightarrow (XS, YS) \in$ *min-strict* $\Longrightarrow$ (XS, *insert y YS*) $\in$ *min-strict*
and *wmin-emptyI*: $(X, \{\}) \in$ *min-weak*
and *wmin-insertI*:
$x \in XS \Longrightarrow (x, y) \in$ *pair-leq* $\Longrightarrow (XS, YS) \in$ *min-weak* $\Longrightarrow$ (XS, *insert y YS*) $\in$ *min-weak*
⟨*proof*⟩

Reduction Pairs.

lemma *max-ext-compat*:
assumes $R\ O\ S \subseteq R$
shows *max-ext R O* (*max-ext S* $\cup$ $\{(\{\}, \{\})\}$) $\subseteq$ *max-ext R*
⟨*proof*⟩

lemma *max-rpair-set*: *reduction-pair* (*max-strict*, *max-weak*)
⟨*proof*⟩

lemma *min-ext-compat*:
assumes $R\ O\ S \subseteq R$
shows *min-ext* $R\ O$ (*min-ext* $S \cup \{(\{\},\{\})\}) \subseteq$ *min-ext* R
⟨*proof*⟩

lemma *min-rpair-set*: *reduction-pair* (*min-strict*, *min-weak*)
⟨*proof*⟩

53.8 Yet another induction principle on the natural numbers

lemma *nat-descend-induct* [*case-names base descend*]:
fixes P :: *nat* ⇒ *bool*
assumes *H1*: $\bigwedge k.\ k > n \Longrightarrow P\ k$
assumes *H2*: $\bigwedge k.\ k \leq n \Longrightarrow (\bigwedge i.\ i > k \Longrightarrow P\ i) \Longrightarrow P\ k$
shows $P\ m$
⟨*proof*⟩

53.9 Tool setup

⟨*ML*⟩

end

54 The Integers as Equivalence Classes over Pairs of Natural Numbers

theory *Int*
imports *Quotient Groups-Big Fun-Def*
begin

54.1 Definition of integers as a quotient type

definition *intrel* :: (*nat* × *nat*) ⇒ (*nat* × *nat*) ⇒ *bool*
where *intrel* = (λ(x, y) (u, v). $x + v = u + y$)

lemma *intrel-iff* [*simp*]: *intrel* (x, y) (u, v) ⟷ $x + v = u + y$
⟨*proof*⟩

quotient-type *int* = *nat* × *nat* / *intrel*
morphisms *Rep-Integ Abs-Integ*
⟨*proof*⟩

lemma *eq-Abs-Integ* [*case-names Abs-Integ*, *cases type*: *int*]:
($\bigwedge x\ y.\ z =$ *Abs-Integ* (x, y) ⟹ P) ⟹ P
⟨*proof*⟩

54.2 Integers form a commutative ring

instantiation *int* :: *comm-ring-1*
begin

lift-definition *zero-int* :: *int* **is** (*0*, *0*) ⟨*proof*⟩

lift-definition *one-int* :: *int* **is** (*1*, *0*) ⟨*proof*⟩

lift-definition *plus-int* :: *int* ⇒ *int* ⇒ *int*
  is λ(*x*, *y*) (*u*, *v*). (*x* + *u*, *y* + *v*)
  ⟨*proof*⟩

lift-definition *uminus-int* :: *int* ⇒ *int*
  is λ(*x*, *y*). (*y*, *x*)
  ⟨*proof*⟩

lift-definition *minus-int* :: *int* ⇒ *int* ⇒ *int*
  is λ(*x*, *y*) (*u*, *v*). (*x* + *v*, *y* + *u*)
  ⟨*proof*⟩

lift-definition *times-int* :: *int* ⇒ *int* ⇒ *int*
  is λ(*x*, *y*) (*u*, *v*). (*x*∗*u* + *y*∗*v*, *x*∗*v* + *y*∗*u*)
⟨*proof*⟩

instance
  ⟨*proof*⟩

end

abbreviation *int* :: *nat* ⇒ *int*
  where *int* ≡ *of-nat*

lemma *int-def*: *int n* = *Abs-Integ* (*n*, *0*)
  ⟨*proof*⟩

lemma *int-transfer* [*transfer-rule*]:
  includes *lifting-syntax*
  shows *rel-fun* (=) *pcr-int* (λ*n*. (*n*, *0*)) *int*
  ⟨*proof*⟩

lemma *int-diff-cases*: **obtains** (*diff*) *m n* **where** *z* = *int m* − *int n*
  ⟨*proof*⟩

54.3 Integers are totally ordered

instantiation *int* :: *linorder*
begin

lift-definition *less-eq-int* :: *int* ⇒ *int* ⇒ *bool*

is $\lambda(x,\ y)\ (u,\ v).\ x + v \leq u + y$
⟨*proof*⟩

lift-definition *less-int* :: *int* ⇒ *int* ⇒ *bool*
is $\lambda(x,\ y)\ (u,\ v).\ x + v < u + y$
⟨*proof*⟩

instance
⟨*proof*⟩

end

instantiation *int* :: *distrib-lattice*
begin

definition (*inf* :: *int* ⇒ *int* ⇒ *int*) = *min*

definition (*sup* :: *int* ⇒ *int* ⇒ *int*) = *max*

instance
⟨*proof*⟩

end

54.4 Ordering properties of arithmetic operations

instance *int* :: *ordered-cancel-ab-semigroup-add*
⟨*proof*⟩

Strict Monotonicity of Multiplication.

Strict, in 1st argument; proof is by induction on $k > 0$.

lemma *zmult-zless-mono2-lemma*: $i < j \Longrightarrow 0 < k \Longrightarrow int\ k * i < int\ k * j$
for *i j* :: *int*
⟨*proof*⟩

lemma *zero-le-imp-eq-int*:
assumes $k \geq (0::int)$ **shows** $\exists n.\ k = int\ n$
⟨*proof*⟩

lemma *zero-less-imp-eq-int*:
assumes $k > (0::int)$ **shows** $\exists n{>}0.\ k = int\ n$
⟨*proof*⟩

lemma *zmult-zless-mono2*: $i < j \Longrightarrow 0 < k \Longrightarrow k * i < k * j$
for *i j k* :: *int*
⟨*proof*⟩

The integers form an ordered integral domain.

instantiation *int* :: *linordered-idom*

begin

definition *zabs-def*: $|i::int| = (if\ i < 0\ then - i\ else\ i)$

definition *zsgn-def*: $sgn\ (i::int) = (if\ i = 0\ then\ 0\ else\ if\ 0 < i\ then\ 1\ else - 1)$

instance
⟨*proof*⟩

end

lemma *zless-imp-add1-zle*: $w < z \Longrightarrow w + 1 \leq z$
for $w\ z :: int$
⟨*proof*⟩

lemma *zless-iff-Suc-zadd*: $w < z \longleftrightarrow (\exists n.\ z = w + int\ (Suc\ n))$
for $w\ z :: int$
⟨*proof*⟩

lemma *zabs-less-one-iff* [*simp*]: $|z| < 1 \longleftrightarrow z = 0$ (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
for $z :: int$
⟨*proof*⟩

54.5 Embedding of the Integers into any *ring-1*: *of-int*

context *ring-1*
begin

lift-definition *of-int* :: $int \Rightarrow 'a$
is $\lambda(i, j).\ of\text{-}nat\ i - of\text{-}nat\ j$
⟨*proof*⟩

lemma *of-int-0* [*simp*]: *of-int 0 = 0*
⟨*proof*⟩

lemma *of-int-1* [*simp*]: *of-int 1 = 1*
⟨*proof*⟩

lemma *of-int-add* [*simp*]: *of-int (w + z) = of-int w + of-int z*
⟨*proof*⟩

lemma *of-int-minus* [*simp*]: *of-int (− z) = − (of-int z)*
⟨*proof*⟩

lemma *of-int-diff* [*simp*]: *of-int (w − z) = of-int w − of-int z*
⟨*proof*⟩

lemma *of-int-mult* [*simp*]: *of-int (w∗z) = of-int w ∗ of-int z*
⟨*proof*⟩

lemma *mult-of-int-commute*: *of-int x * y = y * of-int x*
⟨*proof*⟩

Collapse nested embeddings.

lemma *of-int-of-nat-eq* [*simp*]: *of-int (int n) = of-nat n*
⟨*proof*⟩

lemma *of-int-numeral* [*simp*, *code-post*]: *of-int (numeral k) = numeral k*
⟨*proof*⟩

lemma *of-int-neg-numeral* [*code-post*]: *of-int (- numeral k) = - numeral k*
⟨*proof*⟩

lemma *of-int-power* [*simp*]: *of-int (z ^ n) = of-int z ^ n*
⟨*proof*⟩

lemma *of-int-of-bool* [*simp*]:
of-int (of-bool P) = of-bool P
⟨*proof*⟩

end

context *ring-char-0*
begin

lemma *of-int-eq-iff* [*simp*]: *of-int w = of-int z ⟷ w = z*
⟨*proof*⟩

Special cases where either operand is zero.

lemma *of-int-eq-0-iff* [*simp*]: *of-int z = 0 ⟷ z = 0*
⟨*proof*⟩

lemma *of-int-0-eq-iff* [*simp*]: *0 = of-int z ⟷ z = 0*
⟨*proof*⟩

lemma *of-int-eq-1-iff* [*iff*]: *of-int z = 1 ⟷ z = 1*
⟨*proof*⟩

lemma *numeral-power-eq-of-int-cancel-iff* [*simp*]:
numeral x ^ n = of-int y ⟷ numeral x ^ n = y
⟨*proof*⟩

lemma *of-int-eq-numeral-power-cancel-iff* [*simp*]:
of-int y = numeral x ^ n ⟷ y = numeral x ^ n
⟨*proof*⟩

lemma *neg-numeral-power-eq-of-int-cancel-iff* [*simp*]:
(- numeral x) ^ n = of-int y ⟷ (- numeral x) ^ n = y

⟨*proof*⟩

lemma *of-int-eq-neg-numeral-power-cancel-iff* [*simp*]:
of-int y = (− *numeral x*) ^ *n* ⟷ *y* = (− *numeral x*) ^ *n*
⟨*proof*⟩

lemma *of-int-eq-of-int-power-cancel-iff*[*simp*]: (*of-int b*) ^ *w* = *of-int x* ⟷ *b* ^ *w* = *x*
⟨*proof*⟩

lemma *of-int-power-eq-of-int-cancel-iff*[*simp*]: *of-int x* = (*of-int b*) ^ *w* ⟷ *x* = *b* ^ *w*
⟨*proof*⟩

end

context *linordered-idom*
begin

Every *linordered-idom* has characteristic zero.

subclass *ring-char-0* ⟨*proof*⟩

lemma *of-int-le-iff* [*simp*]: *of-int w* ≤ *of-int z* ⟷ *w* ≤ *z*
⟨*proof*⟩

lemma *of-int-less-iff* [*simp*]: *of-int w* < *of-int z* ⟷ *w* < *z*
⟨*proof*⟩

lemma *of-int-0-le-iff* [*simp*]: *0* ≤ *of-int z* ⟷ *0* ≤ *z*
⟨*proof*⟩

lemma *of-int-le-0-iff* [*simp*]: *of-int z* ≤ *0* ⟷ *z* ≤ *0*
⟨*proof*⟩

lemma *of-int-0-less-iff* [*simp*]: *0* < *of-int z* ⟷ *0* < *z*
⟨*proof*⟩

lemma *of-int-less-0-iff* [*simp*]: *of-int z* < *0* ⟷ *z* < *0*
⟨*proof*⟩

lemma *of-int-1-le-iff* [*simp*]: *1* ≤ *of-int z* ⟷ *1* ≤ *z*
⟨*proof*⟩

lemma *of-int-le-1-iff* [*simp*]: *of-int z* ≤ *1* ⟷ *z* ≤ *1*
⟨*proof*⟩

lemma *of-int-1-less-iff* [*simp*]: *1* < *of-int z* ⟷ *1* < *z*
⟨*proof*⟩

lemma *of-int-less-1-iff* [*simp*]: *of-int z < 1 ⟷ z < 1*
⟨*proof*⟩

lemma *of-int-pos*: *z > 0 ⟹ of-int z > 0*
⟨*proof*⟩

lemma *of-int-nonneg*: *z ≥ 0 ⟹ of-int z ≥ 0*
⟨*proof*⟩

lemma *of-int-abs* [*simp*]: *of-int |x| = |of-int x|*
⟨*proof*⟩

lemma *of-int-lessD*:
assumes *|of-int n| < x*
shows *n = 0 ∨ x > 1*
⟨*proof*⟩

lemma *of-int-leD*:
assumes *|of-int n| ≤ x*
shows *n = 0 ∨ 1 ≤ x*
⟨*proof*⟩

lemma *numeral-power-le-of-int-cancel-iff* [*simp*]:
numeral x ^ n ≤ of-int a ⟷ numeral x ^ n ≤ a
⟨*proof*⟩

lemma *of-int-le-numeral-power-cancel-iff* [*simp*]:
of-int a ≤ numeral x ^ n ⟷ a ≤ numeral x ^ n
⟨*proof*⟩

lemma *numeral-power-less-of-int-cancel-iff* [*simp*]:
numeral x ^ n < of-int a ⟷ numeral x ^ n < a
⟨*proof*⟩

lemma *of-int-less-numeral-power-cancel-iff* [*simp*]:
of-int a < numeral x ^ n ⟷ a < numeral x ^ n
⟨*proof*⟩

lemma *neg-numeral-power-le-of-int-cancel-iff* [*simp*]:
(− numeral x) ^ n ≤ of-int a ⟷ (− numeral x) ^ n ≤ a
⟨*proof*⟩

lemma *of-int-le-neg-numeral-power-cancel-iff* [*simp*]:
of-int a ≤ (− numeral x) ^ n ⟷ a ≤ (− numeral x) ^ n
⟨*proof*⟩

lemma *neg-numeral-power-less-of-int-cancel-iff* [*simp*]:
(− numeral x) ^ n < of-int a ⟷ (− numeral x) ^ n < a
⟨*proof*⟩

lemma *of-int-less-neg-numeral-power-cancel-iff* [*simp*]:
of-int a < (− *numeral x*) ^ *n* ⟷ *a* < (− *numeral x*::*int*) ^ *n*
⟨*proof*⟩

lemma *of-int-le-of-int-power-cancel-iff*[*simp*]: (*of-int b*) ^ *w* ≤ *of-int x* ⟷ *b* ^ *w* ≤ *x*
⟨*proof*⟩

lemma *of-int-power-le-of-int-cancel-iff*[*simp*]: *of-int x* ≤ (*of-int b*) ^ *w*⟷ *x* ≤ *b* ^ *w*
⟨*proof*⟩

lemma *of-int-less-of-int-power-cancel-iff*[*simp*]: (*of-int b*) ^ *w* < *of-int x* ⟷ *b* ^ *w* < *x*
⟨*proof*⟩

lemma *of-int-power-less-of-int-cancel-iff*[*simp*]: *of-int x* < (*of-int b*) ^ *w*⟷ *x* < *b* ^ *w*
⟨*proof*⟩

lemma *of-int-max*: *of-int* (*max x y*) = *max* (*of-int x*) (*of-int y*)
⟨*proof*⟩

lemma *of-int-min*: *of-int* (*min x y*) = *min* (*of-int x*) (*of-int y*)
⟨*proof*⟩

end

context *division-ring*
begin

lemmas *mult-inverse-of-int-commute* =
mult-commute-imp-mult-inverse-commute[*OF mult-of-int-commute*]

end

Comparisons involving *of-int*.

lemma *of-int-eq-numeral-iff* [*iff*]: *of-int z* = (*numeral n* :: 'a::*ring-char-0*) ⟷ *z* = *numeral n*
⟨*proof*⟩

lemma *of-int-le-numeral-iff* [*simp*]:
of-int z ≤ (*numeral n* :: 'a::*linordered-idom*) ⟷ *z* ≤ *numeral n*
⟨*proof*⟩

lemma *of-int-numeral-le-iff* [*simp*]:
(*numeral n* :: 'a::*linordered-idom*) ≤ *of-int z* ⟷ *numeral n* ≤ *z*
⟨*proof*⟩

lemma *of-int-less-numeral-iff* [*simp*]:
of-int z < (*numeral n* :: *'a*::*linordered-idom*) ⟷ *z* < *numeral n*
⟨*proof*⟩

lemma *of-int-numeral-less-iff* [*simp*]:
(*numeral n* :: *'a*::*linordered-idom*) < *of-int z* ⟷ *numeral n* < *z*
⟨*proof*⟩

lemma *of-nat-less-of-int-iff*: (*of-nat n*::*'a*::*linordered-idom*) < *of-int x* ⟷ *int n* < *x*
⟨*proof*⟩

lemma *of-int-eq-id* [*simp*]: *of-int* = *id*
⟨*proof*⟩

instance *int* :: *no-top*
⟨*proof*⟩

instance *int* :: *no-bot*
⟨*proof*⟩

54.6 Magnitude of an Integer, as a Natural Number: *nat*

lift-definition *nat* :: *int* ⇒ *nat* **is** λ(*x*, *y*). *x* − *y*
⟨*proof*⟩

lemma *nat-int* [*simp*]: *nat* (*int n*) = *n*
⟨*proof*⟩

lemma *int-nat-eq* [*simp*]: *int* (*nat z*) = (*if* 0 ≤ *z then z else* 0)
⟨*proof*⟩

lemma *nat-0-le*: 0 ≤ *z* ⟹ *int* (*nat z*) = *z*
⟨*proof*⟩

lemma *nat-le-0* [*simp*]: *z* ≤ 0 ⟹ *nat z* = 0
⟨*proof*⟩

lemma *nat-le-eq-zle*: 0 < *w* ∨ 0 ≤ *z* ⟹ *nat w* ≤ *nat z* ⟷ *w* ≤ *z*
⟨*proof*⟩

An alternative condition is (0::*'a*) ≤ *w*.

lemma *nat-mono-iff*: 0 < *z* ⟹ *nat w* < *nat z* ⟷ *w* < *z*
⟨*proof*⟩

lemma *nat-less-eq-zless*: 0 ≤ *w* ⟹ *nat w* < *nat z* ⟷ *w* < *z*
⟨*proof*⟩

lemma *zless-nat-conj* [*simp*]: *nat w* < *nat z* ⟷ *0* < *z* ∧ *w* < *z*
⟨*proof*⟩

lemma *nonneg-int-cases*:
assumes *0* ≤ *k*
obtains *n* **where** *k* = *int n*
⟨*proof*⟩

lemma *pos-int-cases*:
assumes *0* < *k*
obtains *n* **where** *k* = *int n* **and** *n* > *0*
⟨*proof*⟩

lemma *nonpos-int-cases*:
assumes *k* ≤ *0*
obtains *n* **where** *k* = − *int n*
⟨*proof*⟩

lemma *neg-int-cases*:
assumes *k* < *0*
obtains *n* **where** *k* = − *int n* **and** *n* > *0*
⟨*proof*⟩

lemma *nat-eq-iff*: *nat w* = *m* ⟷ (*if 0* ≤ *w then w* = *int m else m* = *0*)
⟨*proof*⟩

lemma *nat-eq-iff2*: *m* = *nat w* ⟷ (*if 0* ≤ *w then w* = *int m else m* = *0*)
⟨*proof*⟩

lemma *nat-0* [*simp*]: *nat 0* = *0*
⟨*proof*⟩

lemma *nat-1* [*simp*]: *nat 1* = *Suc 0*
⟨*proof*⟩

lemma *nat-numeral* [*simp*]: *nat* (*numeral k*) = *numeral k*
⟨*proof*⟩

lemma *nat-neg-numeral* [*simp*]: *nat* (− *numeral k*) = *0*
⟨*proof*⟩

lemma *nat-2*: *nat 2* = *Suc* (*Suc 0*)
⟨*proof*⟩

lemma *nat-less-iff*: *0* ≤ *w* ⟹ *nat w* < *m* ⟷ *w* < *of-nat m*
⟨*proof*⟩

lemma *nat-le-iff*: *nat x* ≤ *n* ⟷ *x* ≤ *int n*
⟨*proof*⟩

lemma *nat-mono*: $x \leq y \Longrightarrow nat\ x \leq nat\ y$
⟨*proof*⟩

lemma *nat-0-iff* [*simp*]: $nat\ i = 0 \longleftrightarrow i \leq 0$
for $i :: int$
⟨*proof*⟩

lemma *int-eq-iff*: $of\text{-}nat\ m = z \longleftrightarrow m = nat\ z \wedge 0 \leq z$
⟨*proof*⟩

lemma *zero-less-nat-eq* [*simp*]: $0 < nat\ z \longleftrightarrow 0 < z$
⟨*proof*⟩

lemma *nat-add-distrib*: $0 \leq z \Longrightarrow 0 \leq z' \Longrightarrow nat\ (z + z') = nat\ z + nat\ z'$
⟨*proof*⟩

lemma *nat-diff-distrib'*: $0 \leq x \Longrightarrow 0 \leq y \Longrightarrow nat\ (x - y) = nat\ x - nat\ y$
⟨*proof*⟩

lemma *nat-diff-distrib*: $0 \leq z' \Longrightarrow z' \leq z \Longrightarrow nat\ (z - z') = nat\ z - nat\ z'$
⟨*proof*⟩

lemma *nat-zminus-int* [*simp*]: $nat\ (-\ int\ n) = 0$
⟨*proof*⟩

lemma *le-nat-iff*: $k \geq 0 \Longrightarrow n \leq nat\ k \longleftrightarrow int\ n \leq k$
⟨*proof*⟩

lemma *zless-nat-eq-int-zless*: $m < nat\ z \longleftrightarrow int\ m < z$
⟨*proof*⟩

lemma (**in** *ring-1*) *of-nat-nat* [*simp*]: $0 \leq z \Longrightarrow of\text{-}nat\ (nat\ z) = of\text{-}int\ z$
⟨*proof*⟩

lemma *diff-nat-numeral* [*simp*]: $(numeral\ v :: nat) - numeral\ v' = nat\ (numeral\ v - numeral\ v')$
⟨*proof*⟩

lemma *nat-abs-triangle-ineq*:
$nat\ |k + l| \leq nat\ |k| + nat\ |l|$
⟨*proof*⟩

lemma *nat-of-bool* [*simp*]:
$nat\ (of\text{-}bool\ P) = of\text{-}bool\ P$
⟨*proof*⟩

lemma *split-nat* [*linarith-split*]: $P\ (nat\ i) \longleftrightarrow ((\forall n.\ i = int\ n \longrightarrow P\ n) \wedge (i < 0 \longrightarrow P\ 0))$

(**is** *?P* = (*?L* ∧ *?R*))
for *i* :: *int*
⟨*proof*⟩

lemma *all-nat*: (∀ *x*. *P x*) ⟷ (∀ *x*≥*0*. *P* (*nat x*))
⟨*proof*⟩

lemma *ex-nat*: (∃ *x*. *P x*) ⟷ (∃ *x*. *0* ≤ *x* ∧ *P* (*nat x*))
⟨*proof*⟩

For termination proofs:

lemma *measure-function-int*[*measure-function*]: *is-measure* (*nat* ∘ *abs*) ⟨*proof*⟩

54.7 Lemmas about the Function *of-nat* and Orderings

lemma *negative-zless-0*: − (*int* (*Suc n*)) < (*0* :: *int*)
⟨*proof*⟩

lemma *negative-zless* [*iff*]: − (*int* (*Suc n*)) < *int m*
⟨*proof*⟩

lemma *negative-zle-0*: − *int n* ≤ *0*
⟨*proof*⟩

lemma *negative-zle* [*iff*]: − *int n* ≤ *int m*
⟨*proof*⟩

lemma *not-zle-0-negative* [*simp*]: ¬ *0* ≤ − *int* (*Suc n*)
⟨*proof*⟩

lemma *int-zle-neg*: *int n* ≤ − *int m* ⟷ *n* = *0* ∧ *m* = *0*
⟨*proof*⟩

lemma *not-int-zless-negative* [*simp*]: ¬ *int n* < − *int m*
⟨*proof*⟩

lemma *negative-eq-positive* [*simp*]: − *int n* = *of-nat m* ⟷ *n* = *0* ∧ *m* = *0*
⟨*proof*⟩

lemma *zle-iff-zadd*: *w* ≤ *z* ⟷ (∃ *n*. *z* = *w* + *int n*)
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *zadd-int-left*: *int m* + (*int n* + *z*) = *int* (*m* + *n*) + *z*
⟨*proof*⟩

lemma *negD*:
assumes *x* < *0* **shows** ∃ *n*. *x* = − (*int* (*Suc n*))
⟨*proof*⟩

54.8 Cases and induction

Now we replace the case analysis rule by a more conventional one: whether an integer is negative or not.

This version is symmetric in the two subgoals.

lemma *int-cases2* [*case-names nonneg nonpos*, *cases type*: *int*]:
$(\bigwedge n.\ z = int\ n \Longrightarrow P) \Longrightarrow (\bigwedge n.\ z = -\ (int\ n) \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

This is the default, with a negative case.

lemma *int-cases* [*case-names nonneg neg*, *cases type*: *int*]:
assumes *pos*: $\bigwedge n.\ z = int\ n \Longrightarrow P$ **and** *neg*: $\bigwedge n.\ z = -\ (int\ (Suc\ n)) \Longrightarrow P$
shows P
⟨*proof*⟩

lemma *int-cases3* [*case-names zero pos neg*]:
fixes $k :: int$
assumes $k = 0 \Longrightarrow P$ **and** $\bigwedge n.\ k = int\ n \Longrightarrow n > 0 \Longrightarrow P$
and $\bigwedge n.\ k = -\ int\ n \Longrightarrow n > 0 \Longrightarrow P$
shows P
⟨*proof*⟩

lemma *int-of-nat-induct* [*case-names nonneg neg*, *induct type*: *int*]:
$(\bigwedge n.\ P\ (int\ n)) \Longrightarrow (\bigwedge n.\ P\ (-\ (int\ (Suc\ n)))) \Longrightarrow P\ z$
⟨*proof*⟩

lemma *sgn-mult-dvd-iff* [*simp*]:
$sgn\ r * l\ dvd\ k \longleftrightarrow l\ dvd\ k \wedge (r = 0 \longrightarrow k = 0)$ **for** $k\ l\ r :: int$
⟨*proof*⟩

lemma *mult-sgn-dvd-iff* [*simp*]:
$l * sgn\ r\ dvd\ k \longleftrightarrow l\ dvd\ k \wedge (r = 0 \longrightarrow k = 0)$ **for** $k\ l\ r :: int$
⟨*proof*⟩

lemma *dvd-sgn-mult-iff* [*simp*]:
$l\ dvd\ sgn\ r * k \longleftrightarrow l\ dvd\ k \vee r = 0$ **for** $k\ l\ r :: int$
⟨*proof*⟩

lemma *dvd-mult-sgn-iff* [*simp*]:
$l\ dvd\ k * sgn\ r \longleftrightarrow l\ dvd\ k \vee r = 0$ **for** $k\ l\ r :: int$
⟨*proof*⟩

lemma *int-sgnE*:
fixes $k :: int$
obtains n **and** l **where** $k = sgn\ l * int\ n$
⟨*proof*⟩

54.8.1 Binary comparisons

Preliminaries

lemma *le-imp-0-less*:
  fixes z :: *int*
  assumes *le*: $0 \leq z$
  shows $0 < 1 + z$
⟨*proof*⟩

lemma *odd-less-0-iff*: $1 + z + z < 0 \longleftrightarrow z < 0$
  for z :: *int*
⟨*proof*⟩

54.8.2 Comparisons, for Ordered Rings

lemma *odd-nonzero*: $1 + z + z \neq 0$
  for z :: *int*
⟨*proof*⟩

54.9 The Set of Integers

context *ring-1*
begin

definition *Ints* :: *'a set* ($\mathbb{Z}$)
  where $\mathbb{Z} = range\ of\text{-}int$

lemma *Ints-of-int* [*simp*]: *of-int* $z \in \mathbb{Z}$
  ⟨*proof*⟩

lemma *Ints-of-nat* [*simp*]: *of-nat* $n \in \mathbb{Z}$
  ⟨*proof*⟩

lemma *Ints-0* [*simp*]: $0 \in \mathbb{Z}$
  ⟨*proof*⟩

lemma *Ints-1* [*simp*]: $1 \in \mathbb{Z}$
  ⟨*proof*⟩

lemma *Ints-numeral* [*simp*]: *numeral* $n \in \mathbb{Z}$
  ⟨*proof*⟩

lemma *Ints-add* [*simp*]: $a \in \mathbb{Z} \Longrightarrow b \in \mathbb{Z} \Longrightarrow a + b \in \mathbb{Z}$
  ⟨*proof*⟩

lemma *Ints-minus* [*simp*]: $a \in \mathbb{Z} \Longrightarrow -a \in \mathbb{Z}$
  ⟨*proof*⟩

lemma *minus-in-Ints-iff*: $-x \in \mathbb{Z} \longleftrightarrow x \in \mathbb{Z}$

⟨*proof*⟩

lemma *Ints-diff* [*simp*]: $a \in \mathbb{Z} \Longrightarrow b \in \mathbb{Z} \Longrightarrow a - b \in \mathbb{Z}$
⟨*proof*⟩

lemma *Ints-mult* [*simp*]: $a \in \mathbb{Z} \Longrightarrow b \in \mathbb{Z} \Longrightarrow a * b \in \mathbb{Z}$
⟨*proof*⟩

lemma *Ints-power* [*simp*]: $a \in \mathbb{Z} \Longrightarrow a \mathbin{\hat{}} n \in \mathbb{Z}$
⟨*proof*⟩

lemma *Ints-cases* [*cases set*: *Ints*]:
assumes $q \in \mathbb{Z}$
obtains (*of-int*) z **where** $q = \textit{of-int}\ z$
⟨*proof*⟩

lemma *Ints-induct* [*case-names of-int*, *induct set*: *Ints*]:
$q \in \mathbb{Z} \Longrightarrow (\bigwedge z.\ P\ (\textit{of-int}\ z)) \Longrightarrow P\ q$
⟨*proof*⟩

lemma *Nats-subset-Ints*: $\mathbb{N} \subseteq \mathbb{Z}$
⟨*proof*⟩

lemma *Nats-altdef1*: $\mathbb{N} = \{\textit{of-int}\ n \mid n.\ n \geq 0\}$
⟨*proof*⟩

end

lemma *Ints-sum* [*intro*]: $(\bigwedge x.\ x \in A \Longrightarrow f\ x \in \mathbb{Z}) \Longrightarrow \textit{sum}\ f\ A \in \mathbb{Z}$
⟨*proof*⟩

lemma *Ints-prod* [*intro*]: $(\bigwedge x.\ x \in A \Longrightarrow f\ x \in \mathbb{Z}) \Longrightarrow \textit{prod}\ f\ A \in \mathbb{Z}$
⟨*proof*⟩

lemma (**in** *linordered-idom*) *Ints-abs* [*simp*]:
shows $a \in \mathbb{Z} \Longrightarrow \textit{abs}\ a \in \mathbb{Z}$
⟨*proof*⟩

lemma (**in** *linordered-idom*) *Nats-altdef2*: $\mathbb{N} = \{n \in \mathbb{Z}.\ n \geq 0\}$
⟨*proof*⟩

lemma (**in** *idom-divide*) *of-int-divide-in-Ints*:
$\textit{of-int}\ a\ \textit{div}\ \textit{of-int}\ b \in \mathbb{Z}$ **if** $b\ \textit{dvd}\ a$
⟨*proof*⟩

The premise involving $\mathbb{Z}$ prevents $a = (1::'a)\ /\ (2::'a)$.

lemma *Ints-double-eq-0-iff*:
fixes $a :: 'a::\textit{ring-char-0}$
assumes *in-Ints*: $a \in \mathbb{Z}$

shows $a + a = 0 \longleftrightarrow a = 0$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *Ints-odd-nonzero*:
fixes a :: $'a$::*ring-char-0*
assumes *in-Ints*: $a \in \mathbb{Z}$
shows $1 + a + a \neq 0$
⟨*proof*⟩

lemma *Nats-numeral* [*simp*]: *numeral* $w \in \mathbb{N}$
⟨*proof*⟩

lemma *Ints-odd-less-0*:
fixes a :: $'a$::*linordered-idom*
assumes *in-Ints*: $a \in \mathbb{Z}$
shows $1 + a + a < 0 \longleftrightarrow a < 0$
⟨*proof*⟩

54.10 *sum* and *prod*

context *semiring-1*
begin

lemma *of-nat-sum* [*simp*]:
of-nat (*sum f A*) = ($\sum x{\in}A.$ *of-nat* (*f x*))
⟨*proof*⟩

end

context *ring-1*
begin

lemma *of-int-sum* [*simp*]:
of-int (*sum f A*) = ($\sum x{\in}A.$ *of-int* (*f x*))
⟨*proof*⟩

end

context *comm-semiring-1*
begin

lemma *of-nat-prod* [*simp*]:
of-nat (*prod f A*) = ($\prod x{\in}A.$ *of-nat* (*f x*))
⟨*proof*⟩

end

context *comm-ring-1*

begin

lemma *of-int-prod* [*simp*]:
of-int (*prod f A*) = ($\prod x{\in}A.$ *of-int* (*f x*))
⟨*proof*⟩

end

54.11 Setting up simplification procedures

⟨*ML*⟩

54.12 More Inequality Reasoning

lemma *zless-add1-eq*: $w < z + 1 \longleftrightarrow w < z \vee w = z$
for *w z* :: *int*
⟨*proof*⟩

lemma *add1-zle-eq*: $w + 1 \leq z \longleftrightarrow w < z$
for *w z* :: *int*
⟨*proof*⟩

lemma *zle-diff1-eq* [*simp*]: $w \leq z - 1 \longleftrightarrow w < z$
for *w z* :: *int*
⟨*proof*⟩

lemma *zle-add1-eq-le* [*simp*]: $w < z + 1 \longleftrightarrow w \leq z$
for *w z* :: *int*
⟨*proof*⟩

lemma *int-one-le-iff-zero-less*: $1 \leq z \longleftrightarrow 0 < z$
for *z* :: *int*
⟨*proof*⟩

lemma *Ints-nonzero-abs-ge1*:
fixes *x*:: *'a* :: *linordered-idom*
assumes $x \in Ints\ x \neq 0$
shows $1 \leq abs\ x$
⟨*proof*⟩

lemma *Ints-nonzero-abs-less1*:
fixes *x*:: *'a* :: *linordered-idom*
shows $[\![x \in Ints;\ abs\ x < 1]\!] \Longrightarrow x = 0$
⟨*proof*⟩

lemma *Ints-eq-abs-less1*:
fixes *x*:: *'a* :: *linordered-idom*
shows $[\![x \in Ints;\ y \in Ints]\!] \Longrightarrow x = y \longleftrightarrow abs\ (x-y) < 1$
⟨*proof*⟩

54.13 The functions *nat* and *int*

Simplify the term $w + - z$.

lemma *one-less-nat-eq* [*simp*]: *Suc 0 < nat z ⟷ 1 < z*
⟨*proof*⟩

lemma *int-eq-iff-numeral* [*simp*]:
int m = numeral v ⟷ m = numeral v
⟨*proof*⟩

lemma *nat-abs-int-diff*:
nat |int a − int b| = (if a ≤ b then b − a else a − b)
⟨*proof*⟩

lemma *nat-int-add*: *nat (int a + int b) = a + b*
⟨*proof*⟩

context *ring-1*
begin

lemma *of-int-of-nat* [*nitpick-simp*]:
of-int k = (if k < 0 then − of-nat (nat (− k)) else of-nat (nat k))
⟨*proof*⟩

end

lemma *transfer-rule-of-int*:
includes *lifting-syntax*
fixes *R* :: *'a*::*ring-1* ⇒ *'b*::*ring-1* ⇒ *bool*
assumes [*transfer-rule*]: *R 0 0 R 1 1*
(R ===> R ===> R) (+) (+)
(R ===> R) uminus uminus
shows *((=) ===> R) of-int of-int*
⟨*proof*⟩

lemma *nat-mult-distrib*:
fixes *z z′* :: *int*
assumes *0 ≤ z*
shows *nat (z ∗ z′) = nat z ∗ nat z′*
⟨*proof*⟩

lemma *nat-mult-distrib-neg*:
assumes *z ≤ (0*::*int)* **shows** *nat (z ∗ z′) = nat (− z) ∗ nat (− z′)* (**is** *?L = ?R*)
⟨*proof*⟩

lemma *nat-abs-mult-distrib*: *nat |w ∗ z| = nat |w| ∗ nat |z|*
⟨*proof*⟩

lemma *int-in-range-abs* [*simp*]: *int n ∈ range abs*

⟨*proof*⟩

lemma *range-abs-Nats* [*simp*]: *range abs* = (ℕ :: *int set*)
⟨*proof*⟩

lemma *Suc-nat-eq-nat-zadd1*: *0* ≤ *z* ⟹ *Suc* (*nat z*) = *nat* (*1* + *z*)
for *z* :: *int*
⟨*proof*⟩

lemma *diff-nat-eq-if*:
nat z − *nat z'* =
(*if z'* < *0 then nat z*
else
let d = *z* − *z'*
in if d < *0 then 0 else nat d*)
⟨*proof*⟩

lemma *nat-numeral-diff-1* [*simp*]: *numeral v* − (*1*::*nat*) = *nat* (*numeral v* − *1*)
⟨*proof*⟩

54.14 Induction principles for int

Well-founded segments of the integers.

definition *int-ge-less-than* :: *int* ⇒ (*int* × *int*) *set*
where *int-ge-less-than d* = {(*z'*, *z*). *d* ≤ *z'* ∧ *z'* < *z*}

lemma *wf-int-ge-less-than*: *wf* (*int-ge-less-than d*)
⟨*proof*⟩

This variant looks odd, but is typical of the relations suggested by RankFinder.

definition *int-ge-less-than2* :: *int* ⇒ (*int* × *int*) *set*
where *int-ge-less-than2 d* = {(*z'*,*z*). *d* ≤ *z* ∧ *z'* < *z*}

lemma *wf-int-ge-less-than2*: *wf* (*int-ge-less-than2 d*)
⟨*proof*⟩

theorem *int-ge-induct* [*case-names base step*, *induct set*: *int*]:
fixes *i* :: *int*
assumes *ge*: *k* ≤ *i*
and *base*: *P k*
and *step*: ⋀*i*. *k* ≤ *i* ⟹ *P i* ⟹ *P* (*i* + *1*)
shows *P i*
⟨*proof*⟩

theorem *int-gr-induct* [*case-names base step*, *induct set*: *int*]:

fixes $i\ k :: int$
assumes $k < i\ P\ (k + 1)\ \bigwedge i.\ k < i \Longrightarrow P\ i \Longrightarrow P\ (i + 1)$
shows $P\ i$
⟨*proof*⟩

theorem *int-le-induct* [*consumes 1*, *case-names base step*]:
fixes $i\ k :: int$
assumes *le*: $i \leq k$
and *base*: $P\ k$
and *step*: $\bigwedge i.\ i \leq k \Longrightarrow P\ i \Longrightarrow P\ (i - 1)$
shows $P\ i$
⟨*proof*⟩

theorem *int-less-induct* [*consumes 1*, *case-names base step*]:
fixes $i\ k :: int$
assumes $i < k\ P\ (k - 1)\ \bigwedge i.\ i < k \Longrightarrow P\ i \Longrightarrow P\ (i - 1)$
shows $P\ i$
⟨*proof*⟩

theorem *int-induct* [*case-names base step1 step2*]:
fixes $k :: int$
assumes *base*: $P\ k$
and *step1*: $\bigwedge i.\ k \leq i \Longrightarrow P\ i \Longrightarrow P\ (i + 1)$
and *step2*: $\bigwedge i.\ k \geq i \Longrightarrow P\ i \Longrightarrow P\ (i - 1)$
shows $P\ i$
⟨*proof*⟩

54.15 Intermediate value theorems

lemma *nat-ivt-aux*:
$[\![\forall i{<}n.\ |f\ (Suc\ i) - f\ i| \leq 1; f\ 0 \leq k; k \leq f\ n]\!] \Longrightarrow \exists i \leq n.\ f\ i = k$
for $m\ n :: nat$ **and** $k :: int$
⟨*proof*⟩

lemma *nat-intermed-int-val*:
fixes $m\ n :: nat$ **and** $k :: int$
assumes $\forall i.\ m \leq i \wedge i < n \longrightarrow |f\ (Suc\ i) - f\ i| \leq 1\ m \leq n\ f\ m \leq k\ k \leq f\ n$
shows $\exists i.\ m \leq i \wedge i \leq n \wedge f\ i = k$
⟨*proof*⟩

lemma *nat0-intermed-int-val*:
$\exists i{\leq}n.\ f\ i = k$
if $\forall i{<}n.\ |f\ (i + 1) - f\ i| \leq 1\ f\ 0 \leq k\ k \leq f\ n$
for $n :: nat$ **and** $k :: int$
⟨*proof*⟩

54.16 Products and 1, by T. M. Rasmussen

lemma *abs-zmult-eq-1*:
fixes $m\ n :: int$

assumes *mn*: $|m * n| = 1$
shows $|m| = 1$
⟨*proof*⟩

lemma *pos-zmult-eq-1-iff-lemma*: $m * n = 1 \Longrightarrow m = 1 \vee m = -1$
for $m\ n :: int$
⟨*proof*⟩

lemma *pos-zmult-eq-1-iff*:
fixes $m\ n :: int$
assumes $0 < m$
shows $m * n = 1 \longleftrightarrow m = 1 \wedge n = 1$
⟨*proof*⟩

lemma *zmult-eq-1-iff*: $m * n = 1 \longleftrightarrow (m = 1 \wedge n = 1) \vee (m = -1 \wedge n = -1)$ (**is** *?L = ?R*)
for $m\ n :: int$
⟨*proof*⟩

lemma *infinite-UNIV-int* [*simp*]: ¬ *finite* (*UNIV*::*int set*)
⟨*proof*⟩

54.17 The divides relation

lemma *zdvd-antisym-nonneg*: $0 \leq m \Longrightarrow 0 \leq n \Longrightarrow m\ dvd\ n \Longrightarrow n\ dvd\ m \Longrightarrow m = n$
for $m\ n :: int$
⟨*proof*⟩

lemma *zdvd-antisym-abs*:
fixes $a\ b :: int$
assumes $a\ dvd\ b$ **and** $b\ dvd\ a$
shows $|a| = |b|$
⟨*proof*⟩

lemma *zdvd-zdiffD*: $k\ dvd\ m - n \Longrightarrow k\ dvd\ n \Longrightarrow k\ dvd\ m$
for $k\ m\ n :: int$
⟨*proof*⟩

lemma *zdvd-reduce*: $k\ dvd\ n + k * m \longleftrightarrow k\ dvd\ n$
for $k\ m\ n :: int$
⟨*proof*⟩

lemma *dvd-imp-le-int*:
fixes $d\ i :: int$
assumes $i \neq 0$ **and** $d\ dvd\ i$
shows $|d| \leq |i|$
⟨*proof*⟩

lemma *zdvd-not-zless*:
fixes *m n* :: *int*
assumes *0 < m* **and** *m < n*
shows ¬ *n dvd m*
⟨*proof*⟩

lemma *zdvd-mult-cancel*:
fixes *k m n* :: *int*
assumes *d*: *k * m dvd k * n*
and *k ≠ 0*
shows *m dvd n*
⟨*proof*⟩

lemma *int-dvd-int-iff* [*simp*]:
int m dvd int n ⟷ m dvd n
⟨*proof*⟩

lemma *dvd-nat-abs-iff* [*simp*]:
n dvd nat |k| ⟷ int n dvd k
⟨*proof*⟩

lemma *nat-abs-dvd-iff* [*simp*]:
nat |k| dvd n ⟷ k dvd int n
⟨*proof*⟩

lemma *zdvd1-eq* [*simp*]: *x dvd 1 ⟷ |x| = 1* (**is** *?lhs ⟷ ?rhs*)
for *x* :: *int*
⟨*proof*⟩

lemma *zdvd-mult-cancel1*:
fixes *m* :: *int*
assumes *mp*: *m ≠ 0*
shows *m * n dvd m ⟷ |n| = 1*
(**is** *?lhs ⟷ ?rhs*)
⟨*proof*⟩

lemma *nat-dvd-iff*: *nat z dvd m ⟷ (if 0 ≤ z then z dvd int m else m = 0)*
⟨*proof*⟩

lemma *eq-nat-nat-iff*: *0 ≤ z ⟹ 0 ≤ z' ⟹ nat z = nat z' ⟷ z = z'*
⟨*proof*⟩

lemma *nat-power-eq*: *0 ≤ z ⟹ nat (z ^ n) = nat z ^ n*
⟨*proof*⟩

lemma *numeral-power-eq-nat-cancel-iff* [*simp*]:
numeral x ^ n = nat y ⟷ numeral x ^ n = y
⟨*proof*⟩

lemma *nat-eq-numeral-power-cancel-iff* [*simp*]:
nat y = numeral x ^ n ⟷ y = numeral x ^ n
⟨*proof*⟩

lemma *numeral-power-le-nat-cancel-iff* [*simp*]:
numeral x ^ n ≤ nat a ⟷ numeral x ^ n ≤ a
⟨*proof*⟩

lemma *nat-le-numeral-power-cancel-iff* [*simp*]:
nat a ≤ numeral x ^ n ⟷ a ≤ numeral x ^ n
⟨*proof*⟩

lemma *numeral-power-less-nat-cancel-iff* [*simp*]:
numeral x ^ n < nat a ⟷ numeral x ^ n < a
⟨*proof*⟩

lemma *nat-less-numeral-power-cancel-iff* [*simp*]:
nat a < numeral x ^ n ⟷ a < numeral x ^ n
⟨*proof*⟩

lemma *zdvd-imp-le*: *z ≤ n* **if** *z dvd n 0 < n* **for** *n z :: int*
⟨*proof*⟩

lemma *zdvd-period*:
fixes *a d :: int*
assumes *a dvd d*
shows *a dvd (x + t) ⟷ a dvd ((x + c * d) + t)*
(**is** *?lhs ⟷ ?rhs*)
⟨*proof*⟩

54.18 Powers with integer exponents

The following allows writing powers with an integer exponent. While the type signature is very generic, most theorems will assume that the underlying type is a division ring or a field.

The notation 'powi' is inspired by the 'powr' notation for real/complex exponentiation.

definition *power-int :: 'a :: {inverse, power} ⇒ int ⇒ 'a* (**infixr** *powi 80*) **where**
power-int x n = (if n ≥ 0 then x ^ nat n else inverse x ^ (nat (−n)))

lemma *power-int-0-right* [*simp*]: *power-int x 0 = 1*
and *power-int-1-right* [*simp*]:
power-int (y :: 'a :: {power, inverse, monoid-mult}) 1 = y
and *power-int-minus1-right* [*simp*]:
power-int (y :: 'a :: {power, inverse, monoid-mult}) (−1) = inverse y
⟨*proof*⟩

lemma *power-int-of-nat* [*simp*]: *power-int x (int n) = x ^ n*

⟨*proof*⟩

lemma *power-int-numeral* [*simp*]: *power-int x* (*numeral n*) = *x* ^ *numeral n*
⟨*proof*⟩

lemma *int-cases4* [*case-names nonneg neg*]:
fixes *m* :: *int*
obtains *n* **where** *m* = *int n* | *n* **where** *n* > *0 m* = −*int n*
⟨*proof*⟩

context
assumes *SORT-CONSTRAINT*('*a*::*division-ring*)
begin

lemma *power-int-minus*: *power-int* (*x*::'*a*) (−*n*) = *inverse* (*power-int x n*)
⟨*proof*⟩

lemma *power-int-minus-divide*: *power-int* (*x*::'*a*) (−*n*) = *1* / (*power-int x n*)
⟨*proof*⟩

lemma *power-int-eq-0-iff* [*simp*]: *power-int* (*x*::'*a*) *n* = *0* ⟷ *x* = *0* ∧ *n* ≠ *0*
⟨*proof*⟩

lemma *power-int-0-left-If*: *power-int* (*0* :: '*a*) *m* = (*if m* = *0 then 1 else 0*)
⟨*proof*⟩

lemma *power-int-0-left* [*simp*]: *m* ≠ *0* ⟹ *power-int* (*0* :: '*a*) *m* = *0*
⟨*proof*⟩

lemma *power-int-1-left* [*simp*]: *power-int 1 n* = (*1* :: '*a* :: *division-ring*)
⟨*proof*⟩

lemma *power-diff-conv-inverse*: *x* ≠ *0* ⟹ *m* ≤ *n* ⟹ (*x* :: '*a*) ^ (*n* − *m*) = *x* ^ *n* ∗ *inverse x* ^ *m*
⟨*proof*⟩

lemma *power-mult-inverse-distrib*: *x* ^ *m* ∗ *inverse* (*x* :: '*a*) = *inverse x* ∗ *x* ^ *m*
⟨*proof*⟩

lemma *power-mult-power-inverse-commute*:
x ^ *m* ∗ *inverse* (*x* :: '*a*) ^ *n* = *inverse x* ^ *n* ∗ *x* ^ *m*
⟨*proof*⟩

lemma *power-int-add*:
assumes *x* ≠ *0* ∨ *m* + *n* ≠ *0*
shows *power-int* (*x*::'*a*) (*m* + *n*) = *power-int x m* ∗ *power-int x n*
⟨*proof*⟩

lemma *power-int-add-1*:
assumes $x \neq 0 \vee m \neq -1$
shows *power-int* $(x::'a)$ $(m + 1)$ = *power-int* x $m * x$
⟨*proof*⟩

lemma *power-int-add-1'*:
assumes $x \neq 0 \vee m \neq -1$
shows *power-int* $(x::'a)$ $(m + 1) = x *$ *power-int* x m
⟨*proof*⟩

lemma *power-int-commutes*: *power-int* $(x :: 'a)$ $n * x = x *$ *power-int* x n
⟨*proof*⟩

lemma *power-int-inverse* [*field-simps*, *field-split-simps*, *divide-simps*]:
power-int (*inverse* $(x :: 'a)$) n = *inverse* (*power-int* x n)
⟨*proof*⟩

lemma *power-int-mult*: *power-int* $(x :: 'a)$ $(m * n)$ = *power-int* (*power-int* x m) n
⟨*proof*⟩

end

context
assumes *SORT-CONSTRAINT*('*a*::*field*)
begin

lemma *power-int-diff*:
assumes $x \neq 0 \vee m \neq n$
shows *power-int* $(x::'a)$ $(m - n)$ = *power-int* x m / *power-int* x n
⟨*proof*⟩

lemma *power-int-minus-mult*: $x \neq 0 \vee n \neq 0 \Longrightarrow$ *power-int* $(x :: 'a)$ $(n - 1) *$ x = *power-int* x n
⟨*proof*⟩

lemma *power-int-mult-distrib*: *power-int* $(x * y :: 'a)$ m = *power-int* x $m *$ *power-int* y m
⟨*proof*⟩

lemmas *power-int-mult-distrib-numeral1* = *power-int-mult-distrib* [**where** x = *numeral* w **for** w, *simp*]
lemmas *power-int-mult-distrib-numeral2* = *power-int-mult-distrib* [**where** y = *numeral* w **for** w, *simp*]

lemma *power-int-divide-distrib*: *power-int* $(x$ / $y :: 'a)$ m = *power-int* x m / *power-int* y m
⟨*proof*⟩

end

lemma *power-int-add-numeral* [*simp*]:
power-int x (*numeral m*) ∗ *power-int x* (*numeral n*) = *power-int x* (*numeral* (*m* + *n*))
for *x* :: *'a* :: *division-ring*
⟨*proof*⟩

lemma *power-int-add-numeral2* [*simp*]:
power-int x (*numeral m*) ∗ (*power-int x* (*numeral n*) ∗ *b*) = *power-int x* (*numeral* (*m* + *n*)) ∗ *b*
for *x* :: *'a* :: *division-ring*
⟨*proof*⟩

lemma *power-int-mult-numeral* [*simp*]:
power-int (*power-int x* (*numeral m*)) (*numeral n*) = *power-int x* (*numeral* (*m* ∗ *n*))
for *x* :: *'a* :: *division-ring*
⟨*proof*⟩

lemma *power-int-not-zero*: (*x* :: *'a* :: *division-ring*) ≠ *0* ∨ *n* = *0* ⟹ *power-int x n* ≠ *0*
⟨*proof*⟩

lemma *power-int-one-over* [*field-simps*, *field-split-simps*, *divide-simps*]:
power-int (*1* / *x* :: *'a* :: *division-ring*) *n* = *1* / *power-int x n*
⟨*proof*⟩

context
assumes *SORT-CONSTRAINT*(*'a* :: *linordered-field*)
begin

lemma *power-int-numeral-neg-numeral* [*simp*]:
power-int (*numeral m*) (−*numeral n*) = (*inverse* (*numeral* (*Num.pow m n*)) :: *'a*)
⟨*proof*⟩

lemma *zero-less-power-int* [*simp*]: *0* < (*x* :: *'a*) ⟹ *0* < *power-int x n*
⟨*proof*⟩

lemma *zero-le-power-int* [*simp*]: *0* ≤ (*x* :: *'a*) ⟹ *0* ≤ *power-int x n*
⟨*proof*⟩

lemma *power-int-mono*: (*x* :: *'a*) ≤ *y* ⟹ *n* ≥ *0* ⟹ *0* ≤ *x* ⟹ *power-int x n* ≤ *power-int y n*
⟨*proof*⟩

lemma *one-le-power-int* [*simp*]: *1* ≤ (*x* :: *'a*) ⟹ *n* ≥ *0* ⟹ *1* ≤ *power-int x n*

⟨*proof*⟩

lemma *power-int-le-one*: $0 \leq (x :: 'a) \Longrightarrow n \geq 0 \Longrightarrow x \leq 1 \Longrightarrow$ *power-int* $x\ n \leq 1$
⟨*proof*⟩

lemma *power-int-le-imp-le-exp*:
assumes *gt1*: $1 < (x :: 'a :: linordered\text{-}field)$
assumes *power-int* $x\ m \leq$ *power-int* $x\ n\ n \geq 0$
shows $m \leq n$
⟨*proof*⟩

lemma *power-int-le-imp-less-exp*:
assumes *gt1*: $1 < (x :: 'a :: linordered\text{-}field)$
assumes *power-int* $x\ m <$ *power-int* $x\ n\ n \geq 0$
shows $m < n$
⟨*proof*⟩

lemma *power-int-strict-mono*:
$(a :: 'a :: linordered\text{-}field) < b \Longrightarrow 0 \leq a \Longrightarrow 0 < n \Longrightarrow$ *power-int* $a\ n <$ *power-int* $b\ n$
⟨*proof*⟩

lemma *power-int-mono-iff* [*simp*]:
fixes $a\ b :: 'a :: linordered\text{-}field$
shows $[\![a \geq 0; b \geq 0; n > 0]\!] \Longrightarrow$ *power-int* $a\ n \leq$ *power-int* $b\ n \longleftrightarrow a \leq b$
⟨*proof*⟩

lemma *power-int-strict-increasing*:
fixes $a :: 'a :: linordered\text{-}field$
assumes $n < N\ 1 < a$
shows *power-int* $a\ N >$ *power-int* $a\ n$
⟨*proof*⟩

lemma *power-int-increasing*:
fixes $a :: 'a :: linordered\text{-}field$
assumes $n \leq N\ a \geq 1$
shows *power-int* $a\ N \geq$ *power-int* $a\ n$
⟨*proof*⟩

lemma *power-int-strict-decreasing*:
fixes $a :: 'a :: linordered\text{-}field$
assumes $n < N\ 0 < a\ a < 1$
shows *power-int* $a\ N <$ *power-int* $a\ n$
⟨*proof*⟩

lemma *power-int-decreasing*:
fixes $a :: 'a :: linordered\text{-}field$
assumes $n \leq N\ 0 \leq a\ a \leq 1\ a \neq 0 \vee N \neq 0 \vee n = 0$

shows *power-int a N ≤ power-int a n*
⟨*proof*⟩

lemma *one-less-power-int*: *1 < (a :: 'a) ⟹ 0 < n ⟹ 1 < power-int a n*
⟨*proof*⟩

lemma *power-int-abs*: *|power-int a n :: 'a| = power-int |a| n*
⟨*proof*⟩

lemma *power-int-sgn* [*simp*]: *sgn (power-int a n :: 'a) = power-int (sgn a) n*
⟨*proof*⟩

lemma *abs-power-int-minus* [*simp*]: *|power-int (− a) n :: 'a| = |power-int a n|*
⟨*proof*⟩

lemma *power-int-strict-antimono*:
assumes *(a :: 'a :: linordered-field) < b 0 < a n < 0*
shows *power-int a n > power-int b n*
⟨*proof*⟩

lemma *power-int-antimono*:
assumes *(a :: 'a :: linordered-field) ≤ b 0 < a n < 0*
shows *power-int a n ≥ power-int b n*
⟨*proof*⟩

end

54.19 Finiteness of intervals

lemma *finite-interval-int1* [*iff*]: *finite {i :: int. a ≤ i ∧ i ≤ b}*
⟨*proof*⟩

lemma *finite-interval-int2* [*iff*]: *finite {i :: int. a ≤ i ∧ i < b}*
⟨*proof*⟩

lemma *finite-interval-int3* [*iff*]: *finite {i :: int. a < i ∧ i ≤ b}*
⟨*proof*⟩

lemma *finite-interval-int4* [*iff*]: *finite {i :: int. a < i ∧ i < b}*
⟨*proof*⟩

54.20 Configuration of the code generator

Constructors

definition *Pos :: num ⇒ int*
where [*simp*, *code-abbrev*]: *Pos = numeral*

definition *Neg :: num ⇒ int*
where [*simp*, *code-abbrev*]: *Neg n = − (Pos n)*

code-datatype *0*::*int Pos Neg*

Auxiliary operations.

definition *dup* :: *int* ⇒ *int*
  where [*simp*]: *dup k* = *k* + *k*

lemma *dup-code* [*code*]:
  dup 0 = *0*
  dup (*Pos n*) = *Pos* (*Num.Bit0 n*)
  dup (*Neg n*) = *Neg* (*Num.Bit0 n*)
  ⟨*proof*⟩

definition *sub* :: *num* ⇒ *num* ⇒ *int*
  where [*simp*]: *sub m n* = *numeral m* − *numeral n*

lemma *sub-code* [*code*]:
  sub Num.One Num.One = *0*
  sub (*Num.Bit0 m*) *Num.One* = *Pos* (*Num.BitM m*)
  sub (*Num.Bit1 m*) *Num.One* = *Pos* (*Num.Bit0 m*)
  sub Num.One (*Num.Bit0 n*) = *Neg* (*Num.BitM n*)
  sub Num.One (*Num.Bit1 n*) = *Neg* (*Num.Bit0 n*)
  sub (*Num.Bit0 m*) (*Num.Bit0 n*) = *dup* (*sub m n*)
  sub (*Num.Bit1 m*) (*Num.Bit1 n*) = *dup* (*sub m n*)
  sub (*Num.Bit1 m*) (*Num.Bit0 n*) = *dup* (*sub m n*) + *1*
  sub (*Num.Bit0 m*) (*Num.Bit1 n*) = *dup* (*sub m n*) − *1*
  ⟨*proof*⟩

lemma *sub-BitM-One-eq*:
  ‹(*Num.sub* (*Num.BitM n*) *num.One*) = *2* ∗ (*Num.sub n Num.One* :: *int*)›
  ⟨*proof*⟩

Implementations.

lemma *one-int-code* [*code*]: *1* = *Pos Num.One*
  ⟨*proof*⟩

lemma *plus-int-code* [*code*]:
  k + *0* = *k*
  0 + *l* = *l*
  Pos m + *Pos n* = *Pos* (*m* + *n*)
  Pos m + *Neg n* = *sub m n*
  Neg m + *Pos n* = *sub n m*
  Neg m + *Neg n* = *Neg* (*m* + *n*)
  for *k l* :: *int*
  ⟨*proof*⟩

lemma *uminus-int-code* [*code*]:
  uminus 0 = (*0*::*int*)
  uminus (*Pos m*) = *Neg m*

uminus (*Neg m*) = *Pos m*
⟨*proof*⟩

lemma *minus-int-code* [*code*]:
k − *0* = *k*
0 − *l* = *uminus l*
Pos m − *Pos n* = *sub m n*
Pos m − *Neg n* = *Pos* (*m* + *n*)
Neg m − *Pos n* = *Neg* (*m* + *n*)
Neg m − *Neg n* = *sub n m*
for *k l* :: *int*
⟨*proof*⟩

lemma *times-int-code* [*code*]:
k ∗ *0* = *0*
0 ∗ *l* = *0*
Pos m ∗ *Pos n* = *Pos* (*m* ∗ *n*)
Pos m ∗ *Neg n* = *Neg* (*m* ∗ *n*)
Neg m ∗ *Pos n* = *Neg* (*m* ∗ *n*)
Neg m ∗ *Neg n* = *Pos* (*m* ∗ *n*)
for *k l* :: *int*
⟨*proof*⟩

instantiation *int* :: *equal*
begin

definition *HOL.equal k l* ⟷ *k* = (*l*::*int*)

instance
⟨*proof*⟩

end

lemma *equal-int-code* [*code*]:
HOL.equal 0 (*0*::*int*) ⟷ *True*
HOL.equal 0 (*Pos l*) ⟷ *False*
HOL.equal 0 (*Neg l*) ⟷ *False*
HOL.equal (*Pos k*) *0* ⟷ *False*
HOL.equal (*Pos k*) (*Pos l*) ⟷ *HOL.equal k l*
HOL.equal (*Pos k*) (*Neg l*) ⟷ *False*
HOL.equal (*Neg k*) *0* ⟷ *False*
HOL.equal (*Neg k*) (*Pos l*) ⟷ *False*
HOL.equal (*Neg k*) (*Neg l*) ⟷ *HOL.equal k l*
⟨*proof*⟩

lemma *equal-int-refl* [*code nbe*]: *HOL.equal k k* ⟷ *True*
for *k* :: *int*
⟨*proof*⟩

lemma *less-eq-int-code* [*code*]:
$0 \leq (0::int) \longleftrightarrow True$
$0 \leq Pos\ l \longleftrightarrow True$
$0 \leq Neg\ l \longleftrightarrow False$
$Pos\ k \leq 0 \longleftrightarrow False$
$Pos\ k \leq Pos\ l \longleftrightarrow k \leq l$
$Pos\ k \leq Neg\ l \longleftrightarrow False$
$Neg\ k \leq 0 \longleftrightarrow True$
$Neg\ k \leq Pos\ l \longleftrightarrow True$
$Neg\ k \leq Neg\ l \longleftrightarrow l \leq k$
⟨*proof*⟩

lemma *less-int-code* [*code*]:
$0 < (0::int) \longleftrightarrow False$
$0 < Pos\ l \longleftrightarrow True$
$0 < Neg\ l \longleftrightarrow False$
$Pos\ k < 0 \longleftrightarrow False$
$Pos\ k < Pos\ l \longleftrightarrow k < l$
$Pos\ k < Neg\ l \longleftrightarrow False$
$Neg\ k < 0 \longleftrightarrow True$
$Neg\ k < Pos\ l \longleftrightarrow True$
$Neg\ k < Neg\ l \longleftrightarrow l < k$
⟨*proof*⟩

lemma *nat-code* [*code*]:
$nat\ (Int.Neg\ k) = 0$
$nat\ 0 = 0$
$nat\ (Int.Pos\ k) = nat\text{-}of\text{-}num\ k$
⟨*proof*⟩

lemma (**in** *ring-1*) *of-int-code* [*code*]:
$of\text{-}int\ (Int.Neg\ k) = -\ numeral\ k$
$of\text{-}int\ 0 = 0$
$of\text{-}int\ (Int.Pos\ k) = numeral\ k$
⟨*proof*⟩

Serializer setup.

code-identifier
code-module *Int* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

quickcheck-params [*default-type* = *int*]

hide-const (**open**) *Pos Neg sub dup*

De-register *int* as a quotient type:

lifting-update *int.lifting*
lifting-forget *int.lifting*

54.21 Duplicates

lemmas *int-sum* = *of-nat-sum* [**where** *'a=int*]
lemmas *int-prod* = *of-nat-prod* [**where** *'a=int*]
lemmas *zle-int* = *of-nat-le-iff* [**where** *'a=int*]
lemmas *int-int-eq* = *of-nat-eq-iff* [**where** *'a=int*]
lemmas *nonneg-eq-int* = *nonneg-int-cases*
lemmas *double-eq-0-iff* = *double-zero*

lemmas *int-distrib* =
distrib-right [*of z1 z2 w*]
distrib-left [*of w z1 z2*]
left-diff-distrib [*of z1 z2 w*]
right-diff-distrib [*of w z1 z2*]
for *z1 z2 w* :: *int*

end

55 Big infimum (minimum) and supremum (maximum) over finite (non-empty) sets

theory *Lattices-Big*
imports *Groups-Big Option*
begin

55.1 Generic lattice operations over a set

55.1.1 Without neutral element

locale *semilattice-set* = *semilattice*
begin

interpretation *comp-fun-idem f*
⟨*proof*⟩

definition *F* :: *'a set* ⇒ *'a*
where
eq-fold': *F A* = *the* (*Finite-Set.fold* (λ*x y*. *Some* (*case y of None* ⇒ *x* | *Some z* ⇒ *f x z*)) *None A*)

lemma *eq-fold*:
assumes *finite A*
shows *F* (*insert x A*) = *Finite-Set.fold f x A*
⟨*proof*⟩

lemma *singleton* [*simp*]:
F {*x*} = *x*
⟨*proof*⟩

lemma *insert-not-elem*:
assumes *finite* A **and** $x \notin A$ **and** $A \neq \{\}$
shows $F\ (insert\ x\ A) = x * F\ A$
⟨*proof*⟩

lemma *in-idem*:
assumes *finite* A **and** $x \in A$
shows $x * F\ A = F\ A$
⟨*proof*⟩

lemma *insert* [*simp*]:
assumes *finite* A **and** $A \neq \{\}$
shows $F\ (insert\ x\ A) = x * F\ A$
⟨*proof*⟩

lemma *union*:
assumes *finite* A $A \neq \{\}$ **and** *finite* B $B \neq \{\}$
shows $F\ (A \cup B) = F\ A * F\ B$
⟨*proof*⟩

lemma *remove*:
assumes *finite* A **and** $x \in A$
shows $F\ A = (if\ A - \{x\} = \{\}\ then\ x\ else\ x * F\ (A - \{x\}))$
⟨*proof*⟩

lemma *insert-remove*:
assumes *finite* A
shows $F\ (insert\ x\ A) = (if\ A - \{x\} = \{\}\ then\ x\ else\ x * F\ (A - \{x\}))$
⟨*proof*⟩

lemma *subset*:
assumes *finite* A $B \neq \{\}$ **and** $B \subseteq A$
shows $F\ B * F\ A = F\ A$
⟨*proof*⟩

lemma *closed*:
assumes *finite* A $A \neq \{\}$ **and** *elem*: $\bigwedge x\ y.\ x * y \in \{x, y\}$
shows $F\ A \in A$
⟨*proof*⟩

lemma *hom-commute*:
assumes *hom*: $\bigwedge x\ y.\ h\ (x * y) = h\ x * h\ y$
and N: *finite* N $N \neq \{\}$
shows $h\ (F\ N) = F\ (h\ `\ N)$
⟨*proof*⟩

lemma *infinite*: $\neg$ *finite* $A \Longrightarrow F\ A = the\ None$
⟨*proof*⟩

end

locale *semilattice-order-set* = *binary?*: *semilattice-order* + *semilattice-set*
begin

lemma *bounded-iff*:
  assumes *finite* A **and** $A \neq \{\}$
  shows $x \leq F\ A \longleftrightarrow (\forall\, a{\in}A.\ x \leq a)$
  ⟨*proof*⟩

lemma *boundedI*:
  assumes *finite* A
  assumes $A \neq \{\}$
  assumes $\bigwedge a.\ a \in A \Longrightarrow x \leq a$
  shows $x \leq F\ A$
  ⟨*proof*⟩

lemma *boundedE*:
  assumes *finite* A **and** $A \neq \{\}$ **and** $x \leq F\ A$
  obtains $\bigwedge a.\ a \in A \Longrightarrow x \leq a$
  ⟨*proof*⟩

lemma *coboundedI*:
  assumes *finite* A
    and $a \in A$
  shows $F\ A \leq a$
⟨*proof*⟩

lemma *subset-imp*:
  assumes $A \subseteq B$ **and** $A \neq \{\}$ **and** *finite* B
  shows $F\ B \leq F\ A$
⟨*proof*⟩

end

55.1.2 With neutral element

locale *semilattice-neutr-set* = *semilattice-neutr*
begin

interpretation *comp-fun-idem* f
  ⟨*proof*⟩

definition $F :: 'a\ set \Rightarrow 'a$
where
  eq-fold: $F\ A = Finite\text{-}Set.fold\ f\ \mathbf{1}\ A$

lemma *infinite* [*simp*]:
  $\neg$ *finite* $A \Longrightarrow F\ A = \mathbf{1}$

⟨*proof*⟩

lemma *empty* [*simp*]:
$F\ \{\} = \mathbf{1}$
⟨*proof*⟩

lemma *insert* [*simp*]:
assumes *finite* A
shows $F\ (\mathit{insert}\ x\ A) = x * F\ A$
⟨*proof*⟩

lemma *in-idem*:
assumes *finite* A **and** $x \in A$
shows $x * F\ A = F\ A$
⟨*proof*⟩

lemma *union*:
assumes *finite* A **and** *finite* B
shows $F\ (A \cup B) = F\ A * F\ B$
⟨*proof*⟩

lemma *remove*:
assumes *finite* A **and** $x \in A$
shows $F\ A = x * F\ (A - \{x\})$
⟨*proof*⟩

lemma *insert-remove*:
assumes *finite* A
shows $F\ (\mathit{insert}\ x\ A) = x * F\ (A - \{x\})$
⟨*proof*⟩

lemma *subset*:
assumes *finite* A **and** $B \subseteq A$
shows $F\ B * F\ A = F\ A$
⟨*proof*⟩

lemma *closed*:
assumes *finite* A $A \neq \{\}$ **and** *elem*: $\bigwedge x\ y.\ x * y \in \{x, y\}$
shows $F\ A \in A$
⟨*proof*⟩

end

locale *semilattice-order-neutr-set* = *binary?*: *semilattice-neutr-order* + *semilattice-neutr-set*
begin

lemma *bounded-iff*:
assumes *finite* A

shows $x \leq F\ A \longleftrightarrow (\forall\, a{\in}A.\ x \leq a)$
⟨*proof*⟩

lemma *boundedI*:
assumes *finite A*
assumes $\bigwedge a.\ a \in A \Longrightarrow x \leq a$
shows $x \leq F\ A$
⟨*proof*⟩

lemma *boundedE*:
assumes *finite A* **and** $x \leq F\ A$
obtains $\bigwedge a.\ a \in A \Longrightarrow x \leq a$
⟨*proof*⟩

lemma *coboundedI*:
assumes *finite A*
and $a \in A$
shows $F\ A \leq a$
⟨*proof*⟩

lemma *subset-imp*:
assumes $A \subseteq B$ **and** *finite B*
shows $F\ B \leq F\ A$
⟨*proof*⟩

end

55.2 Lattice operations on finite sets

context *semilattice-inf*
begin

sublocale *Inf-fin*: *semilattice-order-set inf less-eq less*
defines
Inf-fin ($\bigsqcap_{fin}$ - [*900*] *900*) = *Inf-fin.F* ⟨*proof*⟩

end

context *semilattice-sup*
begin

sublocale *Sup-fin*: *semilattice-order-set sup greater-eq greater*
defines
Sup-fin ($\bigsqcup_{fin}$ - [*900*] *900*) = *Sup-fin.F* ⟨*proof*⟩

end

55.3 Infimum and Supremum over non-empty sets

context *lattice*

begin

lemma *Inf-fin-le-Sup-fin* [*simp*]:
assumes *finite A* **and** $A \neq \{\}$
shows $\bigsqcap_{fin} A \leq \bigsqcup_{fin} A$
⟨*proof*⟩

lemma *sup-Inf-absorb* [*simp*]:
finite A $\Longrightarrow$ $a \in A \Longrightarrow \bigsqcap_{fin} A \sqcup a = a$
⟨*proof*⟩

lemma *inf-Sup-absorb* [*simp*]:
finite A $\Longrightarrow$ $a \in A \Longrightarrow a \sqcap \bigsqcup_{fin} A = a$
⟨*proof*⟩

end

context *distrib-lattice*
begin

lemma *sup-Inf1-distrib*:
assumes *finite A*
and $A \neq \{\}$
shows *sup x* $(\bigsqcap_{fin} A) = \bigsqcap_{fin} \{sup\ x\ a | a.\ a \in A\}$
⟨*proof*⟩

lemma *sup-Inf2-distrib*:
assumes *A*: *finite A* $A \neq \{\}$ **and** *B*: *finite B* $B \neq \{\}$
shows *sup* $(\bigsqcap_{fin} A)$ $(\bigsqcap_{fin} B) = \bigsqcap_{fin} \{sup\ a\ b | a\ b.\ a \in A \wedge b \in B\}$
⟨*proof*⟩

lemma *inf-Sup1-distrib*:
assumes *finite A* **and** $A \neq \{\}$
shows *inf x* $(\bigsqcup_{fin} A) = \bigsqcup_{fin} \{inf\ x\ a | a.\ a \in A\}$
⟨*proof*⟩

lemma *inf-Sup2-distrib*:
assumes *A*: *finite A* $A \neq \{\}$ **and** *B*: *finite B* $B \neq \{\}$
shows *inf* $(\bigsqcup_{fin} A)$ $(\bigsqcup_{fin} B) = \bigsqcup_{fin} \{inf\ a\ b | a\ b.\ a \in A \wedge b \in B\}$
⟨*proof*⟩

end

context *complete-lattice*
begin

lemma *Inf-fin-Inf*:
assumes *finite A* **and** $A \neq \{\}$
shows $\bigsqcap_{fin} A = \bigsqcap A$

⟨*proof*⟩

lemma *Sup-fin-Sup*:
assumes *finite A* **and** $A \neq \{\}$
shows $\bigsqcup_{fin} A = \bigsqcup A$
⟨*proof*⟩

end

55.4 Minimum and Maximum over non-empty sets

context *linorder*
begin

sublocale *Min*: *semilattice-order-set min less-eq less*
+ *Max*: *semilattice-order-set max greater-eq greater*
defines
Min = *Min.F* **and** *Max* = *Max.F* ⟨*proof*⟩

end

syntax
-MIN1 :: *pttrns* ⇒ *'b* ⇒ *'b* (*(3MIN -./ -) [0, 10] 10*)
-MIN :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* (*(3MIN -∈-./ -) [0, 0, 10] 10*)
-MAX1 :: *pttrns* ⇒ *'b* ⇒ *'b* (*(3MAX -./ -) [0, 10] 10*)
-MAX :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* (*(3MAX -∈-./ -) [0, 0, 10] 10*)

translations
MIN x y. f ⇌ *MIN x. MIN y. f*
MIN x. f ⇌ *CONST Min* (*CONST range* (λ*x. f*))
MIN x∈A. f ⇌ *CONST Min* ((λ*x. f*) ‘ *A*)
MAX x y. f ⇌ *MAX x. MAX y. f*
MAX x. f ⇌ *CONST Max* (*CONST range* (λ*x. f*))
MAX x∈A. f ⇌ *CONST Max* ((λ*x. f*) ‘ *A*)

An aside: *Min*/*Max* on linear orders as special case of *Inf-fin*/*Sup-fin*

lemma *Inf-fin-Min*:
Inf-fin = (*Min* :: *'a*::{*semilattice-inf*, *linorder*} *set* ⇒ *'a*)
⟨*proof*⟩

lemma *Sup-fin-Max*:
Sup-fin = (*Max* :: *'a*::{*semilattice-sup*, *linorder*} *set* ⇒ *'a*)
⟨*proof*⟩

context *linorder*
begin

lemma *dual-min*:
ord.min greater-eq = *max*

⟨*proof*⟩

lemma *dual-max*:
ord.max greater-eq = min
⟨*proof*⟩

lemma *dual-Min*:
linorder.Min greater-eq = Max
⟨*proof*⟩

lemma *dual-Max*:
linorder.Max greater-eq = Min
⟨*proof*⟩

lemmas *Min-singleton = Min.singleton*
lemmas *Max-singleton = Max.singleton*
lemmas *Min-insert = Min.insert*
lemmas *Max-insert = Max.insert*
lemmas *Min-Un = Min.union*
lemmas *Max-Un = Max.union*
lemmas *hom-Min-commute = Min.hom-commute*
lemmas *hom-Max-commute = Max.hom-commute*

lemma *Min-in* [*simp*]:
assumes *finite A* **and** $A \neq \{\}$
shows *Min A* $\in A$
⟨*proof*⟩

lemma *Max-in* [*simp*]:
assumes *finite A* **and** $A \neq \{\}$
shows *Max A* $\in A$
⟨*proof*⟩

lemma *Min-insert2*:
assumes *finite A* **and** *min*: $\bigwedge b.\ b \in A \Longrightarrow a \leq b$
shows *Min* (*insert a A*) = *a*
⟨*proof*⟩

lemma *Max-insert2*:
assumes *finite A* **and** *max*: $\bigwedge b.\ b \in A \Longrightarrow b \leq a$
shows *Max* (*insert a A*) = *a*
⟨*proof*⟩

lemma *Max-const*[*simp*]: ⟦ *finite A*; $A \neq \{\}$ ⟧ ⟹ *Max* ((λ-. *c*) ‘ *A*) = *c*
⟨*proof*⟩

lemma *Min-const*[*simp*]: ⟦ *finite A*; $A \neq \{\}$ ⟧ ⟹ *Min* ((λ-. *c*) ‘ *A*) = *c*
⟨*proof*⟩

lemma *Min-le* [*simp*]:
assumes *finite* A **and** $x \in A$
shows *Min* $A \leq x$
⟨*proof*⟩

lemma *Max-ge* [*simp*]:
assumes *finite* A **and** $x \in A$
shows $x \leq$ *Max* A
⟨*proof*⟩

lemma *Min-eqI*:
assumes *finite* A
assumes $\bigwedge y.\ y \in A \Longrightarrow y \geq x$
and $x \in A$
shows *Min* $A = x$
⟨*proof*⟩

lemma *Max-eqI*:
assumes *finite* A
assumes $\bigwedge y.\ y \in A \Longrightarrow y \leq x$
and $x \in A$
shows *Max* $A = x$
⟨*proof*⟩

lemma *eq-Min-iff*:
$[\![$ *finite* A; $A \neq \{\}$ $]\!] \Longrightarrow m =$ *Min* $A \longleftrightarrow m \in A \wedge (\forall a \in A.\ m \leq a)$
⟨*proof*⟩

lemma *Min-eq-iff*:
$[\![$ *finite* A; $A \neq \{\}$ $]\!] \Longrightarrow$ *Min* $A = m \longleftrightarrow m \in A \wedge (\forall a \in A.\ m \leq a)$
⟨*proof*⟩

lemma *eq-Max-iff*:
$[\![$ *finite* A; $A \neq \{\}$ $]\!] \Longrightarrow m =$ *Max* $A \longleftrightarrow m \in A \wedge (\forall a \in A.\ a \leq m)$
⟨*proof*⟩

lemma *Max-eq-iff*:
$[\![$ *finite* A; $A \neq \{\}$ $]\!] \Longrightarrow$ *Max* $A = m \longleftrightarrow m \in A \wedge (\forall a \in A.\ a \leq m)$
⟨*proof*⟩

context
fixes $A :: 'a$ *set*
assumes *fin-nonempty*: *finite* A $A \neq \{\}$
begin

lemma *Min-ge-iff* [*simp*]:
$x \leq$ *Min* $A \longleftrightarrow (\forall a{\in}A.\ x \leq a)$
⟨*proof*⟩

lemma *Max-le-iff* [*simp*]:
$Max\ A \leq x \longleftrightarrow (\forall\, a{\in}A.\ a \leq x)$
⟨*proof*⟩

lemma *Min-gr-iff* [*simp*]:
$x < Min\ A \longleftrightarrow (\forall\, a{\in}A.\ x < a)$
⟨*proof*⟩

lemma *Max-less-iff* [*simp*]:
$Max\ A < x \longleftrightarrow (\forall\, a{\in}A.\ a < x)$
⟨*proof*⟩

lemma *Min-le-iff*:
$Min\ A \leq x \longleftrightarrow (\exists\, a{\in}A.\ a \leq x)$
⟨*proof*⟩

lemma *Max-ge-iff*:
$x \leq Max\ A \longleftrightarrow (\exists\, a{\in}A.\ x \leq a)$
⟨*proof*⟩

lemma *Min-less-iff*:
$Min\ A < x \longleftrightarrow (\exists\, a{\in}A.\ a < x)$
⟨*proof*⟩

lemma *Max-gr-iff*:
$x < Max\ A \longleftrightarrow (\exists\, a{\in}A.\ x < a)$
⟨*proof*⟩

end

lemma *Max-eq-if*:
assumes *finite A* *finite B* $\forall\, a{\in}A.\ \exists\, b{\in}B.\ a \leq b$ $\forall\, b{\in}B.\ \exists\, a{\in}A.\ b \leq a$
shows $Max\ A = Max\ B$
⟨*proof*⟩

lemma *Min-antimono*:
assumes $M \subseteq N$ **and** $M \neq \{\}$ **and** *finite N*
shows $Min\ N \leq Min\ M$
⟨*proof*⟩

lemma *Max-mono*:
assumes $M \subseteq N$ **and** $M \neq \{\}$ **and** *finite N*
shows $Max\ M \leq Max\ N$
⟨*proof*⟩

lemma *mono-Min-commute*:
assumes *mono f*
assumes *finite A* **and** $A \neq \{\}$
shows $f\ (Min\ A) = Min\ (f\ `\ A)$

⟨*proof*⟩

lemma *mono-Max-commute*:
assumes *mono f*
assumes *finite A* **and** $A \neq \{\}$
shows $f\ (Max\ A) = Max\ (f\ `\ A)$
⟨*proof*⟩

lemma *finite-linorder-max-induct* [*consumes 1*, *case-names empty insert*]:
assumes *fin*: *finite A*
and *empty*: $P\ \{\}$
and *insert*: $\bigwedge b\ A.$ *finite* $A \Longrightarrow \forall\, a{\in}A.\ a < b \Longrightarrow P\ A \Longrightarrow P\ (insert\ b\ A)$
shows $P\ A$
⟨*proof*⟩

lemma *finite-linorder-min-induct* [*consumes 1*, *case-names empty insert*]:
$[\![$*finite A*; $P\ \{\}$; $\bigwedge b\ A.\ [\![$*finite A*; $\forall\, a{\in}A.\ b < a$; $P\ A]\!] \Longrightarrow P\ (insert\ b\ A)]\!] \Longrightarrow P\ A$
⟨*proof*⟩

lemma *finite-ranking-induct*[*consumes 1*, *case-names empty insert*]:
fixes $f :: 'b \Rightarrow 'a$
assumes *finite S*
assumes $P\ \{\}$
assumes $\bigwedge x\ S.$ *finite* $S \Longrightarrow (\bigwedge y.\ y \in S \Longrightarrow f\ y \leq f\ x) \Longrightarrow P\ S \Longrightarrow P\ (insert\ x\ S)$
shows $P\ S$
⟨*proof*⟩

lemma *Least-Min*:
assumes *finite* $\{a.\ P\ a\}$ **and** $\exists\, a.\ P\ a$
shows $(LEAST\ a.\ P\ a) = Min\ \{a.\ P\ a\}$
⟨*proof*⟩

lemma *infinite-growing*:
assumes $X \neq \{\}$
assumes $*$: $\bigwedge x.\ x \in X \Longrightarrow \exists\, y{\in}X.\ y > x$
shows $\neg$ *finite X*
⟨*proof*⟩

end

lemma *sum-le-card-Max*: *finite* $A \Longrightarrow sum\ f\ A \leq card\ A * Max\ (f\ `\ A)$
⟨*proof*⟩

lemma *card-Min-le-sum*: *finite* $A \Longrightarrow card\ A * Min\ (f\ `\ A) \leq sum\ f\ A$
⟨*proof*⟩

context *linordered-ab-semigroup-add*

begin

lemma *Min-add-commute*:
fixes k
assumes *finite* S **and** $S \neq \{\}$
shows $Min\ ((\lambda x.\ f\ x + k)\ `\ S) = Min(f\ `\ S) + k$
⟨*proof*⟩

lemma *Max-add-commute*:
fixes k
assumes *finite* S **and** $S \neq \{\}$
shows $Max\ ((\lambda x.\ f\ x + k)\ `\ S) = Max(f\ `\ S) + k$
⟨*proof*⟩

end

context *linordered-ab-group-add*
begin

lemma *minus-Max-eq-Min* [*simp*]:
$finite\ S \Longrightarrow S \neq \{\} \Longrightarrow -\ Max\ S = Min\ (uminus\ `\ S)$
⟨*proof*⟩

lemma *minus-Min-eq-Max* [*simp*]:
$finite\ S \Longrightarrow S \neq \{\} \Longrightarrow -\ Min\ S = Max\ (uminus\ `\ S)$
⟨*proof*⟩

end

context *complete-linorder*
begin

lemma *Min-Inf*:
assumes *finite* A **and** $A \neq \{\}$
shows $Min\ A = Inf\ A$
⟨*proof*⟩

lemma *Max-Sup*:
assumes *finite* A **and** $A \neq \{\}$
shows $Max\ A = Sup\ A$
⟨*proof*⟩

end

lemma *disjnt-ge-max*:
‹*disjnt* $X\ Y$› **if** ‹*finite* Y› ‹$\bigwedge x.\ x \in X \Longrightarrow x > Max\ Y$›
⟨*proof*⟩

55.5 Arg Min

context *ord*
begin

definition *is-arg-min* :: $('b \Rightarrow 'a) \Rightarrow ('b \Rightarrow bool) \Rightarrow 'b \Rightarrow bool$ **where**
$is\text{-}arg\text{-}min\ f\ P\ x = (P\ x \wedge \neg(\exists\, y.\ P\ y \wedge f\ y < f\ x))$

definition *arg-min* :: $('b \Rightarrow 'a) \Rightarrow ('b \Rightarrow bool) \Rightarrow 'b$ **where**
$arg\text{-}min\ f\ P = (SOME\ x.\ is\text{-}arg\text{-}min\ f\ P\ x)$

definition *arg-min-on* :: $('b \Rightarrow 'a) \Rightarrow 'b\ set \Rightarrow 'b$ **where**
$arg\text{-}min\text{-}on\ f\ S = arg\text{-}min\ f\ (\lambda x.\ x \in S)$

end

syntax
-arg-min :: $('b \Rightarrow 'a) \Rightarrow pttrn \Rightarrow bool \Rightarrow 'b$
$((3ARG'\text{-}MIN\ \text{-}\ \text{-}./\ \text{-})\ [1000,\ 0,\ 10]\ 10)$
translations
$ARG\text{-}MIN\ f\ x.\ P \rightleftharpoons CONST\ arg\text{-}min\ f\ (\lambda x.\ P)$

lemma *is-arg-min-linorder*: **fixes** $f :: 'a \Rightarrow 'b :: linorder$
shows $is\text{-}arg\text{-}min\ f\ P\ x = (P\ x \wedge (\forall\, y.\ P\ y \longrightarrow f\ x \leq f\ y))$
⟨*proof*⟩

lemma *is-arg-min-antimono*: **fixes** $f :: 'a \Rightarrow ('b::order)$
shows $[\![\ is\text{-}arg\text{-}min\ f\ P\ x;\ f\ y \leq f\ x;\ P\ y\ ]\!] \Longrightarrow is\text{-}arg\text{-}min\ f\ P\ y$
⟨*proof*⟩

lemma *arg-minI*:
$[\![\ P\ x;$
$\bigwedge y.\ P\ y \Longrightarrow \neg\ f\ y < f\ x;$
$\bigwedge x.\ [\![\ P\ x;\ \forall\, y.\ P\ y \longrightarrow \neg\ f\ y < f\ x\ ]\!] \Longrightarrow Q\ x\ ]\!]$
$\Longrightarrow Q\ (arg\text{-}min\ f\ P)$
⟨*proof*⟩

lemma *arg-min-equality*:
$[\![\ P\ k;\ \bigwedge x.\ P\ x \Longrightarrow f\ k \leq f\ x\ ]\!] \Longrightarrow f\ (arg\text{-}min\ f\ P) = f\ k$
for $f :: \text{-} \Rightarrow 'a::order$
⟨*proof*⟩

lemma *wf-linord-ex-has-least*:
$[\![\ wf\ r;\ \forall\, x\ y.\ (x,\ y) \in r^+ \longleftrightarrow (y,\ x) \notin r^*;\ P\ k\ ]\!]$
$\Longrightarrow \exists\, x.\ P\ x \wedge (\forall\, y.\ P\ y \longrightarrow (m\ x,\ m\ y) \in r^*)$
⟨*proof*⟩

lemma *ex-has-least-nat*: $P\ k \Longrightarrow \exists\, x.\ P\ x \wedge (\forall\, y.\ P\ y \longrightarrow m\ x \leq m\ y)$
for $m :: 'a \Rightarrow nat$
⟨*proof*⟩

lemma *arg-min-nat-lemma*:
  $P\ k \Longrightarrow P(\textit{arg-min}\ m\ P) \wedge (\forall\, y.\ P\ y \longrightarrow m\ (\textit{arg-min}\ m\ P) \leq m\ y)$
  for $m :: 'a \Rightarrow \textit{nat}$
⟨*proof*⟩

lemmas *arg-min-natI* = *arg-min-nat-lemma* [*THEN conjunct1*]

lemma *is-arg-min-arg-min-nat*: **fixes** $m :: 'a \Rightarrow \textit{nat}$
shows $P\ x \Longrightarrow \textit{is-arg-min}\ m\ P\ (\textit{arg-min}\ m\ P)$
⟨*proof*⟩

lemma *arg-min-nat-le*: $P\ x \Longrightarrow m\ (\textit{arg-min}\ m\ P) \leq m\ x$
  for $m :: 'a \Rightarrow \textit{nat}$
⟨*proof*⟩

lemma *ex-min-if-finite*:
  $[\![\ \textit{finite}\ S;\ S \neq \{\}\ ]\!] \Longrightarrow \exists\, m{\in}S.\ \neg(\exists\, x{\in}S.\ x < (m{::}'a{::}\textit{order}))$
⟨*proof*⟩

lemma *ex-is-arg-min-if-finite*: **fixes** $f :: 'a \Rightarrow 'b :: \textit{order}$
shows $[\![\ \textit{finite}\ S;\ S \neq \{\}\ ]\!] \Longrightarrow \exists\, x.\ \textit{is-arg-min}\ f\ (\lambda x.\ x \in S)\ x$
⟨*proof*⟩

lemma *arg-min-SOME-Min*:
  $\textit{finite}\ S \Longrightarrow \textit{arg-min-on}\ f\ S = (\textit{SOME}\ y.\ y \in S \wedge f\ y = \textit{Min}(f\ `\ S))$
⟨*proof*⟩

lemma *arg-min-if-finite*: **fixes** $f :: 'a \Rightarrow 'b :: \textit{order}$
assumes *finite S* $S \neq \{\}$
shows *arg-min-on f S* $\in S$ **and** $\neg(\exists\, x{\in}S.\ f\ x < f\ (\textit{arg-min-on}\ f\ S))$
⟨*proof*⟩

lemma *arg-min-least*: **fixes** $f :: 'a \Rightarrow 'b :: \textit{linorder}$
shows $[\![\ \textit{finite}\ S;\ \ S \neq \{\};\ \ y \in S\ ]\!] \Longrightarrow f(\textit{arg-min-on}\ f\ S) \leq f\ y$
⟨*proof*⟩

lemma *arg-min-inj-eq*: **fixes** $f :: 'a \Rightarrow 'b :: \textit{order}$
shows $[\![\ \textit{inj-on}\ f\ \{x.\ P\ x\};\ P\ a;\ \forall\, y.\ P\ y \longrightarrow f\ a \leq f\ y\ ]\!] \Longrightarrow \textit{arg-min}\ f\ P = a$
⟨*proof*⟩

55.6 Arg Max

context *ord*
begin

definition *is-arg-max* :: $('b \Rightarrow 'a) \Rightarrow ('b \Rightarrow \textit{bool}) \Rightarrow 'b \Rightarrow \textit{bool}$ **where**
$\textit{is-arg-max}\ f\ P\ x = (P\ x \wedge \neg(\exists\, y.\ P\ y \wedge f\ y > f\ x))$

definition *arg-max* :: $('b \Rightarrow 'a) \Rightarrow ('b \Rightarrow bool) \Rightarrow 'b$ **where**
arg-max f P = (*SOME x. is-arg-max f P x*)

definition *arg-max-on* :: $('b \Rightarrow 'a) \Rightarrow 'b\ set \Rightarrow 'b$ **where**
arg-max-on f S = *arg-max f* $(\lambda x.\ x \in S)$

end

syntax
-arg-max :: $('b \Rightarrow 'a) \Rightarrow pttrn \Rightarrow bool \Rightarrow 'a$
((*3ARG'-MAX - -./ -*) [*1000*, *0*, *10*] *10*)
translations
ARG-MAX f x. P $\rightleftharpoons$ *CONST arg-max f* $(\lambda x.\ P)$

lemma *is-arg-max-linorder*: **fixes** $f :: 'a \Rightarrow 'b :: linorder$
shows *is-arg-max f P x* = $(P\ x \wedge (\forall y.\ P\ y \longrightarrow f\ x \geq f\ y))$
⟨*proof*⟩

lemma *arg-maxI*:
$P\ x \Longrightarrow$
$(\bigwedge y.\ P\ y \Longrightarrow \neg\ f\ y > f\ x) \Longrightarrow$
$(\bigwedge x.\ P\ x \Longrightarrow \forall y.\ P\ y \longrightarrow \neg\ f\ y > f\ x \Longrightarrow Q\ x) \Longrightarrow$
Q (*arg-max f P*)
⟨*proof*⟩

lemma *arg-max-equality*:
$[\![\ P\ k;\ \bigwedge x.\ P\ x \Longrightarrow f\ x \leq f\ k\ ]\!] \Longrightarrow f$ (*arg-max f P*) $= f\ k$
for $f :: - \Rightarrow 'a$::*order*
⟨*proof*⟩

lemma *ex-has-greatest-nat-lemma*:
$P\ k \Longrightarrow \forall x.\ P\ x \longrightarrow (\exists y.\ P\ y \wedge \neg\ f\ y \leq f\ x) \Longrightarrow \exists y.\ P\ y \wedge \neg\ f\ y < f\ k + n$
for $f :: 'a \Rightarrow nat$
⟨*proof*⟩

lemma *ex-has-greatest-nat*:
assumes $P\ k$
and $\forall y.\ P\ y \longrightarrow (f :: 'a \Rightarrow nat)\ y < b$
shows $\exists x.\ P\ x \wedge (\forall y.\ P\ y \longrightarrow f\ y \leq f\ x)$
⟨*proof*⟩

lemma *arg-max-nat-lemma*:
$[\![\ P\ k;\ \forall y.\ P\ y \longrightarrow f\ y < b\ ]\!]$
$\Longrightarrow P$ (*arg-max f P*) $\wedge$ $(\forall y.\ P\ y \longrightarrow f\ y \leq f$ (*arg-max f P*))
for $f :: 'a \Rightarrow nat$
⟨*proof*⟩

lemmas *arg-max-natI* = *arg-max-nat-lemma* [*THEN conjunct1*]

lemma *arg-max-nat-le*: $P\ x \Longrightarrow \forall\, y.\ P\ y \longrightarrow f\ y < b \Longrightarrow f\ x \leq f\ (\mathit{arg\text{-}max}\ f\ P)$
for $f :: 'a \Rightarrow \mathit{nat}$
⟨*proof*⟩

end

56 Division in euclidean (semi)rings

theory *Euclidean-Rings*
imports *Int Lattices-Big*
begin

56.1 Euclidean (semi)rings with explicit division and remainder

class *euclidean-semiring* = *semidom-modulo* +
fixes *euclidean-size* :: $'a \Rightarrow \mathit{nat}$
assumes *size-0* [*simp*]: *euclidean-size* 0 = 0
assumes *mod-size-less*:
$b \neq 0 \Longrightarrow \mathit{euclidean\text{-}size}\ (a\ \mathit{mod}\ b) < \mathit{euclidean\text{-}size}\ b$
assumes *size-mult-mono*:
$b \neq 0 \Longrightarrow \mathit{euclidean\text{-}size}\ a \leq \mathit{euclidean\text{-}size}\ (a * b)$
begin

lemma *euclidean-size-eq-0-iff* [*simp*]:
$\mathit{euclidean\text{-}size}\ b = 0 \longleftrightarrow b = 0$
⟨*proof*⟩

lemma *euclidean-size-greater-0-iff* [*simp*]:
$\mathit{euclidean\text{-}size}\ b > 0 \longleftrightarrow b \neq 0$
⟨*proof*⟩

lemma *size-mult-mono'*: $b \neq 0 \Longrightarrow \mathit{euclidean\text{-}size}\ a \leq \mathit{euclidean\text{-}size}\ (b * a)$
⟨*proof*⟩

lemma *dvd-euclidean-size-eq-imp-dvd*:
assumes $a \neq 0$ **and** $\mathit{euclidean\text{-}size}\ a = \mathit{euclidean\text{-}size}\ b$
and *b dvd a*
shows *a dvd b*
⟨*proof*⟩

lemma *euclidean-size-times-unit*:
assumes *is-unit a*
shows $\mathit{euclidean\text{-}size}\ (a * b) = \mathit{euclidean\text{-}size}\ b$
⟨*proof*⟩

lemma *euclidean-size-unit*:
$\mathit{is\text{-}unit}\ a \Longrightarrow \mathit{euclidean\text{-}size}\ a = \mathit{euclidean\text{-}size}\ 1$
⟨*proof*⟩

lemma *unit-iff-euclidean-size*:
is-unit a $\longleftrightarrow$ *euclidean-size a* = *euclidean-size 1* $\wedge$ *a* $\neq$ *0*
⟨*proof*⟩

lemma *euclidean-size-times-nonunit*:
assumes *a* $\neq$ *0 b* $\neq$ *0* $\neg$ *is-unit a*
shows *euclidean-size b* < *euclidean-size* (*a* * *b*)
⟨*proof*⟩

lemma *dvd-imp-size-le*:
assumes *a dvd b b* $\neq$ *0*
shows *euclidean-size a* $\leq$ *euclidean-size b*
⟨*proof*⟩

lemma *dvd-proper-imp-size-less*:
assumes *a dvd b* $\neg$ *b dvd a b* $\neq$ *0*
shows *euclidean-size a* < *euclidean-size b*
⟨*proof*⟩

lemma *unit-imp-mod-eq-0*:
a mod b = *0* **if** *is-unit b*
⟨*proof*⟩

lemma *mod-eq-self-iff-div-eq-0*:
a mod b = *a* $\longleftrightarrow$ *a div b* = *0* (**is** *?P* $\longleftrightarrow$ *?Q*)
⟨*proof*⟩

lemma *coprime-mod-left-iff* [*simp*]:
coprime (*a mod b*) *b* $\longleftrightarrow$ *coprime a b* **if** *b* $\neq$ *0*
⟨*proof*⟩

lemma *coprime-mod-right-iff* [*simp*]:
coprime a (*b mod a*) $\longleftrightarrow$ *coprime a b* **if** *a* $\neq$ *0*
⟨*proof*⟩

end

class *euclidean-ring* = *idom-modulo* + *euclidean-semiring*
begin

lemma *dvd-diff-commute* [*ac-simps*]:
a dvd c − *b* $\longleftrightarrow$ *a dvd b* − *c*
⟨*proof*⟩

end

56.2 Euclidean (semi)rings with cancel rules

class *euclidean-semiring-cancel* = *euclidean-semiring* +
  assumes *div-mult-self1* [*simp*]: $b \neq 0 \Longrightarrow (a + c * b)\ div\ b = c + a\ div\ b$
  and *div-mult-mult1* [*simp*]: $c \neq 0 \Longrightarrow (c * a)\ div\ (c * b) = a\ div\ b$
begin

lemma *div-mult-self2* [*simp*]:
  assumes $b \neq 0$
  shows $(a + b * c)\ div\ b = c + a\ div\ b$
  ⟨*proof*⟩

lemma *div-mult-self3* [*simp*]:
  assumes $b \neq 0$
  shows $(c * b + a)\ div\ b = c + a\ div\ b$
  ⟨*proof*⟩

lemma *div-mult-self4* [*simp*]:
  assumes $b \neq 0$
  shows $(b * c + a)\ div\ b = c + a\ div\ b$
  ⟨*proof*⟩

lemma *mod-mult-self1* [*simp*]: $(a + c * b)\ mod\ b = a\ mod\ b$
⟨*proof*⟩

lemma *mod-mult-self2* [*simp*]:
  $(a + b * c)\ mod\ b = a\ mod\ b$
  ⟨*proof*⟩

lemma *mod-mult-self3* [*simp*]:
  $(c * b + a)\ mod\ b = a\ mod\ b$
  ⟨*proof*⟩

lemma *mod-mult-self4* [*simp*]:
  $(b * c + a)\ mod\ b = a\ mod\ b$
  ⟨*proof*⟩

lemma *mod-mult-self1-is-0* [*simp*]:
  $b * a\ mod\ b = 0$
  ⟨*proof*⟩

lemma *mod-mult-self2-is-0* [*simp*]:
  $a * b\ mod\ b = 0$
  ⟨*proof*⟩

lemma *div-add-self1*:
  assumes $b \neq 0$
  shows $(b + a)\ div\ b = a\ div\ b + 1$
  ⟨*proof*⟩

lemma *div-add-self2*:
assumes $b \neq 0$
shows $(a + b)\ div\ b = a\ div\ b + 1$
⟨*proof*⟩

lemma *mod-add-self1* [*simp*]:
$(b + a)\ mod\ b = a\ mod\ b$
⟨*proof*⟩

lemma *mod-add-self2* [*simp*]:
$(a + b)\ mod\ b = a\ mod\ b$
⟨*proof*⟩

lemma *mod-div-trivial* [*simp*]:
$a\ mod\ b\ div\ b = 0$
⟨*proof*⟩

lemma *mod-mod-trivial* [*simp*]:
$a\ mod\ b\ mod\ b = a\ mod\ b$
⟨*proof*⟩

lemma *mod-mod-cancel*:
assumes $c\ dvd\ b$
shows $a\ mod\ b\ mod\ c = a\ mod\ c$
⟨*proof*⟩

lemma *div-mult-mult2* [*simp*]:
$c \neq 0 \Longrightarrow (a * c)\ div\ (b * c) = a\ div\ b$
⟨*proof*⟩

lemma *div-mult-mult1-if* [*simp*]:
$(c * a)\ div\ (c * b) = (if\ c = 0\ then\ 0\ else\ a\ div\ b)$
⟨*proof*⟩

lemma *mod-mult-mult1*:
$(c * a)\ mod\ (c * b) = c * (a\ mod\ b)$
⟨*proof*⟩

lemma *mod-mult-mult2*:
$(a * c)\ mod\ (b * c) = (a\ mod\ b) * c$
⟨*proof*⟩

lemma *mult-mod-left*: $(a\ mod\ b) * c = (a * c)\ mod\ (b * c)$
⟨*proof*⟩

lemma *mult-mod-right*: $c * (a\ mod\ b) = (c * a)\ mod\ (c * b)$
⟨*proof*⟩

lemma *dvd-mod*: $k\ dvd\ m \Longrightarrow k\ dvd\ n \Longrightarrow k\ dvd\ (m\ mod\ n)$

⟨*proof*⟩

lemma *div-plus-div-distrib-dvd-left*:
c dvd a ⟹ (*a* + *b*) *div c* = *a div c* + *b div c*
⟨*proof*⟩

lemma *div-plus-div-distrib-dvd-right*:
c dvd b ⟹ (*a* + *b*) *div c* = *a div c* + *b div c*
⟨*proof*⟩

lemma *sum-div-partition*:
‹(∑ *a*∈*A*. *f a*) *div b* = (∑ *a*∈*A* ∩ {*a*. *b dvd f a*}. *f a div b*) + (∑ *a*∈*A* ∩ {*a*. ¬ *b dvd f a*}. *f a*) *div b*›
if ‹*finite A*›
⟨*proof*⟩

named-theorems *mod-simps*

Addition respects modular equivalence.

lemma *mod-add-left-eq* [*mod-simps*]:
(*a mod c* + *b*) *mod c* = (*a* + *b*) *mod c*
⟨*proof*⟩

lemma *mod-add-right-eq* [*mod-simps*]:
(*a* + *b mod c*) *mod c* = (*a* + *b*) *mod c*
⟨*proof*⟩

lemma *mod-add-eq*:
(*a mod c* + *b mod c*) *mod c* = (*a* + *b*) *mod c*
⟨*proof*⟩

lemma *mod-sum-eq* [*mod-simps*]:
(∑ *i*∈*A*. *f i mod a*) *mod a* = *sum f A mod a*
⟨*proof*⟩

lemma *mod-add-cong*:
assumes *a mod c* = *a′ mod c*
assumes *b mod c* = *b′ mod c*
shows (*a* + *b*) *mod c* = (*a′* + *b′*) *mod c*
⟨*proof*⟩

Multiplication respects modular equivalence.

lemma *mod-mult-left-eq* [*mod-simps*]:
((*a mod c*) ∗ *b*) *mod c* = (*a* ∗ *b*) *mod c*
⟨*proof*⟩

lemma *mod-mult-right-eq* [*mod-simps*]:
(*a* ∗ (*b mod c*)) *mod c* = (*a* ∗ *b*) *mod c*
⟨*proof*⟩

lemma *mod-mult-eq*:
$((a \ mod \ c) * (b \ mod \ c)) \ mod \ c = (a * b) \ mod \ c$
⟨*proof*⟩

lemma *mod-prod-eq* [*mod-simps*]:
$(\prod i{\in}A.\ f\ i \ mod \ a) \ mod \ a = prod\ f\ A \ mod \ a$
⟨*proof*⟩

lemma *mod-mult-cong*:
assumes $a \ mod \ c = a' \ mod \ c$
assumes $b \ mod \ c = b' \ mod \ c$
shows $(a * b) \ mod \ c = (a' * b') \ mod \ c$
⟨*proof*⟩

Exponentiation respects modular equivalence.

lemma *power-mod* [*mod-simps*]:
$((a \ mod \ b) \ \hat{} \ n) \ mod \ b = (a \ \hat{} \ n) \ mod \ b$
⟨*proof*⟩

lemma *power-diff-power-eq*:
$‹a \ \hat{} \ m \ div \ a \ \hat{} \ n = (if\ n \leq m\ then\ a \ \hat{} \ (m - n)\ else\ 1 \ div \ a \ \hat{} \ (n - m))›$
if $‹a \neq 0›$
⟨*proof*⟩

end

class *euclidean-ring-cancel* = *euclidean-ring* + *euclidean-semiring-cancel*
begin

subclass *idom-divide* ⟨*proof*⟩

lemma *div-minus-minus* [*simp*]: $(-a) \ div \ (-b) = a \ div \ b$
⟨*proof*⟩

lemma *mod-minus-minus* [*simp*]: $(-a) \ mod \ (-b) = -(a \ mod \ b)$
⟨*proof*⟩

lemma *div-minus-right*: $a \ div \ (-b) = (-a) \ div \ b$
⟨*proof*⟩

lemma *mod-minus-right*: $a \ mod \ (-b) = -((-a) \ mod \ b)$
⟨*proof*⟩

lemma *div-minus1-right* [*simp*]: $a \ div \ (-1) = -a$
⟨*proof*⟩

lemma *mod-minus1-right* [*simp*]: $a \ mod \ (-1) = 0$

⟨*proof*⟩

Negation respects modular equivalence.

lemma *mod-minus-eq* [*mod-simps*]:
(− (*a mod b*)) *mod b* = (− *a*) *mod b*
⟨*proof*⟩

lemma *mod-minus-cong*:
assumes *a mod b* = *a′ mod b*
shows (− *a*) *mod b* = (− *a′*) *mod b*
⟨*proof*⟩

Subtraction respects modular equivalence.

lemma *mod-diff-left-eq* [*mod-simps*]:
(*a mod c* − *b*) *mod c* = (*a* − *b*) *mod c*
⟨*proof*⟩

lemma *mod-diff-right-eq* [*mod-simps*]:
(*a* − *b mod c*) *mod c* = (*a* − *b*) *mod c*
⟨*proof*⟩

lemma *mod-diff-eq*:
(*a mod c* − *b mod c*) *mod c* = (*a* − *b*) *mod c*
⟨*proof*⟩

lemma *mod-diff-cong*:
assumes *a mod c* = *a′ mod c*
assumes *b mod c* = *b′ mod c*
shows (*a* − *b*) *mod c* = (*a′* − *b′*) *mod c*
⟨*proof*⟩

lemma *minus-mod-self2* [*simp*]:
(*a* − *b*) *mod b* = *a mod b*
⟨*proof*⟩

lemma *minus-mod-self1* [*simp*]:
(*b* − *a*) *mod b* = − *a mod b*
⟨*proof*⟩

lemma *mod-eq-dvd-iff*:
a mod c = *b mod c* ⟷ *c dvd a* − *b* (**is** *?P* ⟷ *?Q*)
⟨*proof*⟩

lemma *mod-eqE*:
assumes *a mod c* = *b mod c*
obtains *d* **where** *b* = *a* + *c* ∗ *d*
⟨*proof*⟩

lemma *invertible-coprime*:

coprime a c **if** *a * b mod c = 1*
⟨*proof*⟩

end

56.3 Uniquely determined division

class *unique-euclidean-semiring = euclidean-semiring +*
assumes *euclidean-size-mult*: ‹*euclidean-size (a * b) = euclidean-size a * euclidean-size b*›
fixes *division-segment* :: ‹*'a ⇒ 'a*›
assumes *is-unit-division-segment* [*simp*]: ‹*is-unit (division-segment a)*›
and *division-segment-mult*:
‹*a ≠ 0 ⟹ b ≠ 0 ⟹ division-segment (a * b) = division-segment a * division-segment b*›
and *division-segment-mod*:
‹*b ≠ 0 ⟹ ¬ b dvd a ⟹ division-segment (a mod b) = division-segment b*›
assumes *div-bounded*:
‹*b ≠ 0 ⟹ division-segment r = division-segment b*
⟹ euclidean-size r < euclidean-size b
*⟹ (q * b + r) div b = q*›
begin

lemma *division-segment-not-0* [*simp*]:
‹*division-segment a ≠ 0*›
⟨*proof*⟩

lemma *euclidean-relationI* [*case-names by0 divides euclidean-relation*]:
‹*(a div b, a mod b) = (q, r)*›
if *by0*: ‹*b = 0 ⟹ q = 0 ∧ r = a*›
and *divides*: ‹*b ≠ 0 ⟹ b dvd a ⟹ r = 0 ∧ a = q * b*›
and *euclidean-relation*: ‹*b ≠ 0 ⟹ ¬ b dvd a ⟹ division-segment r = division-segment b*
*∧ euclidean-size r < euclidean-size b ∧ a = q * b + r*›
⟨*proof*⟩

subclass *euclidean-semiring-cancel*
⟨*proof*⟩

lemma *div-eq-0-iff*:
‹*a div b = 0 ⟷ euclidean-size a < euclidean-size b ∨ b = 0*› (**is** *- ⟷ ?P*)
if ‹*division-segment a = division-segment b*›
⟨*proof*⟩

lemma *div-mult1-eq*:
‹*(a * b) div c = a * (b div c) + a * (b mod c) div c*›
⟨*proof*⟩

lemma *div-add1-eq*:

‹$(a + b)$ *div* $c = a$ *div* $c + b$ *div* $c + (a$ *mod* $c + b$ *mod* $c)$ *div* c›
⟨*proof*⟩

end

class *unique-euclidean-ring* = *euclidean-ring* + *unique-euclidean-semiring*
begin

subclass *euclidean-ring-cancel* ⟨*proof*⟩

end

56.4 Division on *nat*

instantiation *nat* :: *normalization-semidom*
begin

definition *normalize-nat* :: ‹*nat* ⇒ *nat*›
where [*simp*]: ‹*normalize* = (*id* :: *nat* ⇒ *nat*)›

definition *unit-factor-nat* :: ‹*nat* ⇒ *nat*›
where ‹*unit-factor n* = *of-bool* (*n* > *0*)› **for** *n* :: *nat*

lemma *unit-factor-simps* [*simp*]:
‹*unit-factor 0* = (*0*::*nat*)›
‹*unit-factor* (*Suc n*) = *1*›
⟨*proof*⟩

definition *divide-nat* :: ‹*nat* ⇒ *nat* ⇒ *nat*›
where ‹*m div n* = (*if n* = *0 then 0 else Max* {*k*. *k* ∗ *n* ≤ *m*})› **for** *m n* :: *nat*

instance
⟨*proof*⟩

end

lemma *coprime-Suc-0-left* [*simp*]:
coprime (*Suc 0*) *n*
⟨*proof*⟩

lemma *coprime-Suc-0-right* [*simp*]:
coprime n (*Suc 0*)
⟨*proof*⟩

lemma *coprime-common-divisor-nat*: *coprime a b* ⟹ *x dvd a* ⟹ *x dvd b* ⟹ *x* = *1*
for *a b* :: *nat*
⟨*proof*⟩

instantiation *nat* :: *unique-euclidean-semiring*
begin

definition *euclidean-size-nat* :: ‹*nat* ⇒ *nat*›
  where [*simp*]: ‹*euclidean-size-nat* = *id*›

definition *division-segment-nat* :: ‹*nat* ⇒ *nat*›
  where [*simp*]: ‹*division-segment n* = *1*› **for** *n* :: *nat*

definition *modulo-nat* :: ‹*nat* ⇒ *nat* ⇒ *nat*›
  where ‹*m mod n* = *m* − (*m div n* ∗ *n*)› **for** *m n* :: *nat*

instance ⟨*proof*⟩

end

lemma *euclidean-relation-natI* [*case-names by0 divides euclidean-relation*]:
  ‹(*m div n*, *m mod n*) = (*q*, *r*)›
    if *by0*: ‹*n* = *0* ⟹ *q* = *0* ∧ *r* = *m*›
    and *divides*: ‹*n* > *0* ⟹ *n dvd m* ⟹ *r* = *0* ∧ *m* = *q* ∗ *n*›
    and *euclidean-relation*: ‹*n* > *0* ⟹ ¬ *n dvd m* ⟹ *r* < *n* ∧ *m* = *q* ∗ *n* + *r*›
for *m n q r* :: *nat*
  ⟨*proof*⟩

lemma *div-nat-eqI*:
  ‹*m div n* = *q*› **if** ‹*n* ∗ *q* ≤ *m*› **and** ‹*m* < *n* ∗ *Suc q*› **for** *m n q* :: *nat*
⟨*proof*⟩

lemma *mod-nat-eqI*:
  ‹*m mod n* = *r*› **if** ‹*r* < *n*› **and** ‹*r* ≤ *m*› **and** ‹*n dvd m* − *r*› **for** *m n r* :: *nat*
⟨*proof*⟩

Tool support

⟨*ML*⟩

lemma *div-mult-self-is-m* [*simp*]:
  m ∗ *n div n* = *m* **if** *n* > *0* **for** *m n* :: *nat*
  ⟨*proof*⟩

lemma *div-mult-self1-is-m* [*simp*]:
  n ∗ *m div n* = *m* **if** *n* > *0* **for** *m n* :: *nat*
  ⟨*proof*⟩

lemma *mod-less-divisor* [*simp*]:
  m mod n < *n* **if** *n* > *0* **for** *m n* :: *nat*
  ⟨*proof*⟩

lemma *mod-le-divisor* [*simp*]:
  m mod n ≤ *n* **if** *n* > *0* **for** *m n* :: *nat*

⟨*proof*⟩

lemma *div-times-less-eq-dividend* [*simp*]:
$m\ div\ n * n \leq m$ **for** $m\ n :: nat$
⟨*proof*⟩

lemma *times-div-less-eq-dividend* [*simp*]:
$n * (m\ div\ n) \leq m$ **for** $m\ n :: nat$
⟨*proof*⟩

lemma *dividend-less-div-times*:
$m < n + (m\ div\ n) * n$ **if** $0 < n$ **for** $m\ n :: nat$
⟨*proof*⟩

lemma *dividend-less-times-div*:
$m < n + n * (m\ div\ n)$ **if** $0 < n$ **for** $m\ n :: nat$
⟨*proof*⟩

lemma *mod-Suc-le-divisor* [*simp*]:
$m\ mod\ Suc\ n \leq n$
⟨*proof*⟩

lemma *mod-less-eq-dividend* [*simp*]:
$m\ mod\ n \leq m$ **for** $m\ n :: nat$
⟨*proof*⟩

lemma
div-less [*simp*]: $m\ div\ n = 0$
and *mod-less* [*simp*]: $m\ mod\ n = m$
if $m < n$ **for** $m\ n :: nat$
⟨*proof*⟩

lemma *split-div*:
$‹P\ (m\ div\ n) \longleftrightarrow$
$(n = 0 \longrightarrow P\ 0) \wedge$
$(n \neq 0 \longrightarrow (\forall i\ j.\ j < n \wedge m = n * i + j \longrightarrow P\ i))›$ (**is** *?div*)
and *split-mod*:
$‹Q\ (m\ mod\ n) \longleftrightarrow$
$(n = 0 \longrightarrow Q\ m) \wedge$
$(n \neq 0 \longrightarrow (\forall i\ j.\ j < n \wedge m = n * i + j \longrightarrow Q\ j))›$ (**is** *?mod*)
for $m\ n :: nat$
⟨*proof*⟩

declare *split-div* [*of* - - ‹*numeral n*›, *linarith-split*] **for** *n*
declare *split-mod* [*of* - - ‹*numeral n*›, *linarith-split*] **for** *n*

lemma *split-div'*:
$P\ (m\ div\ n) \longleftrightarrow n = 0 \wedge P\ 0 \vee (\exists q.\ (n * q \leq m \wedge m < n * Suc\ q) \wedge P\ q)$
⟨*proof*⟩

lemma *le-div-geq*:
$m\ div\ n = Suc\ ((m - n)\ div\ n)$ **if** $0 < n$ **and** $n \leq m$ **for** $m\ n :: nat$
⟨*proof*⟩

lemma *le-mod-geq*:
$m\ mod\ n = (m - n)\ mod\ n$ **if** $n \leq m$ **for** $m\ n :: nat$
⟨*proof*⟩

lemma *div-if*:
$m\ div\ n = (if\ m < n \vee n = 0\ then\ 0\ else\ Suc\ ((m - n)\ div\ n))$
⟨*proof*⟩

lemma *mod-if*:
$m\ mod\ n = (if\ m < n\ then\ m\ else\ (m - n)\ mod\ n)$ **for** $m\ n :: nat$
⟨*proof*⟩

lemma *div-eq-0-iff*:
$m\ div\ n = 0 \longleftrightarrow m < n \vee n = 0$ **for** $m\ n :: nat$
⟨*proof*⟩

lemma *div-greater-zero-iff*:
$m\ div\ n > 0 \longleftrightarrow n \leq m \wedge n > 0$ **for** $m\ n :: nat$
⟨*proof*⟩

lemma *mod-greater-zero-iff-not-dvd*:
$m\ mod\ n > 0 \longleftrightarrow \neg\ n\ dvd\ m$ **for** $m\ n :: nat$
⟨*proof*⟩

lemma *div-by-Suc-0* [*simp*]:
$m\ div\ Suc\ 0 = m$
⟨*proof*⟩

lemma *mod-by-Suc-0* [*simp*]:
$m\ mod\ Suc\ 0 = 0$
⟨*proof*⟩

lemma *div2-Suc-Suc* [*simp*]:
$Suc\ (Suc\ m)\ div\ 2 = Suc\ (m\ div\ 2)$
⟨*proof*⟩

lemma *Suc-n-div-2-gt-zero* [*simp*]:
$0 < Suc\ n\ div\ 2$ **if** $n > 0$ **for** $n :: nat$
⟨*proof*⟩

lemma *div-2-gt-zero* [*simp*]:
$0 < n\ div\ 2$ **if** $Suc\ 0 < n$ **for** $n :: nat$
⟨*proof*⟩

lemma *mod2-Suc-Suc* [*simp*]:
Suc (*Suc m*) *mod* 2 = *m mod* 2
⟨*proof*⟩

lemma *add-self-div-2* [*simp*]:
(*m* + *m*) *div* 2 = *m* **for** *m* :: *nat*
⟨*proof*⟩

lemma *add-self-mod-2* [*simp*]:
(*m* + *m*) *mod* 2 = 0 **for** *m* :: *nat*
⟨*proof*⟩

lemma *mod2-gr-0* [*simp*]:
0 < *m mod* 2 ⟷ *m mod* 2 = 1 **for** *m* :: *nat*
⟨*proof*⟩

lemma *mod-Suc-eq* [*mod-simps*]:
Suc (*m mod n*) *mod n* = *Suc m mod n*
⟨*proof*⟩

lemma *mod-Suc-Suc-eq* [*mod-simps*]:
Suc (*Suc* (*m mod n*)) *mod n* = *Suc* (*Suc m*) *mod n*
⟨*proof*⟩

lemma
Suc-mod-mult-self1 [*simp*]: *Suc* (*m* + *k* ∗ *n*) *mod n* = *Suc m mod n*
and *Suc-mod-mult-self2* [*simp*]: *Suc* (*m* + *n* ∗ *k*) *mod n* = *Suc m mod n*
and *Suc-mod-mult-self3* [*simp*]: *Suc* (*k* ∗ *n* + *m*) *mod n* = *Suc m mod n*
and *Suc-mod-mult-self4* [*simp*]: *Suc* (*n* ∗ *k* + *m*) *mod n* = *Suc m mod n*
⟨*proof*⟩

lemma *Suc-0-mod-eq* [*simp*]:
Suc 0 *mod n* = *of-bool* (*n* ≠ *Suc* 0)
⟨*proof*⟩

lemma *div-mult2-eq*:
‹*m div* (*n* ∗ *q*) = (*m div n*) *div q*› (**is** *?Q*)
and *mod-mult2-eq*:
‹*m mod* (*n* ∗ *q*) = *n* ∗ (*m div n mod q*) + *m mod n*› (**is** *?R*)
for *m n q* :: *nat*
⟨*proof*⟩

lemma *div-le-mono*:
m div k ≤ *n div k* **if** *m* ≤ *n* **for** *m n k* :: *nat*
⟨*proof*⟩

Antimonotonicity of (*div*) in second argument

lemma *div-le-mono2*:
k div n ≤ *k div m* **if** 0 < *m* **and** *m* ≤ *n* **for** *m n k* :: *nat*

⟨*proof*⟩

lemma *div-le-dividend* [*simp*]:
m div n ≤ *m* **for** *m n* :: *nat*
⟨*proof*⟩

lemma *div-less-dividend* [*simp*]:
m div n < *m* **if** *1* < *n* **and** *0* < *m* **for** *m n* :: *nat*
⟨*proof*⟩

lemma *div-eq-dividend-iff*:
m div n = *m* ⟷ *n* = *1* **if** *m* > *0* **for** *m n* :: *nat*
⟨*proof*⟩

lemma *less-mult-imp-div-less*:
m div n < *i* **if** *m* < *i* ∗ *n* **for** *m n i* :: *nat*
⟨*proof*⟩

lemma *div-less-iff-less-mult*:
‹*m div q* < *n* ⟷ *m* < *n* ∗ *q*› (**is** ‹*?P* ⟷ *?Q*›)
if ‹*q* > *0*› **for** *m n q* :: *nat*
⟨*proof*⟩

lemma *less-eq-div-iff-mult-less-eq*:
‹*m* ≤ *n div q* ⟷ *m* ∗ *q* ≤ *n*› **if** ‹*q* > *0*› **for** *m n q* :: *nat*
⟨*proof*⟩

lemma *div-Suc*:
‹*Suc m div n* = (*if Suc m mod n* = *0 then Suc* (*m div n*) *else m div n*)›
⟨*proof*⟩

lemma *mod-Suc*:
‹*Suc m mod n* = (*if Suc* (*m mod n*) = *n then 0 else Suc* (*m mod n*))›
⟨*proof*⟩

lemma *Suc-times-mod-eq*:
Suc (*m* ∗ *n*) *mod m* = *1* **if** *Suc 0* < *m*
⟨*proof*⟩

lemma *Suc-times-numeral-mod-eq* [*simp*]:
Suc (*numeral k* ∗ *n*) *mod numeral k* = *1* **if** *numeral k* ≠ (*1*::*nat*)
⟨*proof*⟩

lemma *Suc-div-le-mono* [*simp*]:
m div n ≤ *Suc m div n*
⟨*proof*⟩

These lemmas collapse some needless occurrences of Suc: at least three Sucs, since two and fewer are rewritten back to Suc again! We already have some

rules to simplify operands smaller than 3.

lemma *div-Suc-eq-div-add3* [*simp*]:
m div Suc (Suc (Suc n)) = m div (3 + n)
⟨*proof*⟩

lemma *mod-Suc-eq-mod-add3* [*simp*]:
m mod Suc (Suc (Suc n)) = m mod (3 + n)
⟨*proof*⟩

lemma *Suc-div-eq-add3-div*:
Suc (Suc (Suc m)) div n = (3 + m) div n
⟨*proof*⟩

lemma *Suc-mod-eq-add3-mod*:
Suc (Suc (Suc m)) mod n = (3 + m) mod n
⟨*proof*⟩

lemmas *Suc-div-eq-add3-div-numeral* [*simp*] =
Suc-div-eq-add3-div [*of - numeral v*] **for** *v*

lemmas *Suc-mod-eq-add3-mod-numeral* [*simp*] =
Suc-mod-eq-add3-mod [*of - numeral v*] **for** *v*

lemma (**in** *field-char-0*) *of-nat-div*:
of-nat (m div n) = ((of-nat m − of-nat (m mod n)) / of-nat n)
⟨*proof*⟩

An "induction" law for modulus arithmetic.

lemma *mod-induct* [*consumes 3*, *case-names step*]:
P m **if** *P n* **and** *n < p* **and** *m < p*
and *step*: ⋀*n*. *n < p* ⟹ *P n* ⟹ *P (Suc n mod p)*
⟨*proof*⟩

lemma *funpow-mod-eq*:
‹(*f* ^^ (*m mod n*)) *x* = (*f* ^^ *m*) *x*› **if** ‹(*f* ^^ *n*) *x* = *x*›
⟨*proof*⟩

lemma *mod-eq-dvd-iff-nat*:
‹*m mod q* = *n mod q* ⟷ *q dvd m − n*› (**is** ‹*?P* ⟷ *?Q*›)
if ‹*m* ≥ *n*› **for** *m n q* :: *nat*
⟨*proof*⟩

lemma *mod-eq-iff-dvd-symdiff-nat*:
‹*m mod q* = *n mod q* ⟷ *q dvd nat* |*int m − int n*|›
⟨*proof*⟩

lemma *mod-eq-nat1E*:
fixes *m n q* :: *nat*
assumes *m mod q* = *n mod q* **and** *m* ≥ *n*

obtains s **where** $m = n + q * s$
⟨*proof*⟩

lemma *mod-eq-nat2E*:
fixes $m\ n\ q :: nat$
assumes $m\ mod\ q = n\ mod\ q$ **and** $n \geq m$
obtains s **where** $n = m + q * s$
⟨*proof*⟩

lemma *nat-mod-eq-iff*:
$(x::nat)\ mod\ n = y\ mod\ n \longleftrightarrow (\exists\, q1\ q2.\ x + n * q1 = y + n * q2)$ (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

56.5 Division on *int*

The following specification of integer division rounds towards minus infinity and is advocated by Donald Knuth. See [5] for an overview and terminology of different possibilities to specify integer division; there division rounding towards minus infinitiy is named "F-division".

56.5.1 Basic instantiation

instantiation *int* :: {*normalization-semidom*, *idom-modulo*}
begin

definition *normalize-int* :: ‹*int* ⇒ *int*›
where [*simp*]: ‹*normalize* = (*abs* :: *int* ⇒ *int*)›

definition *unit-factor-int* :: ‹*int* ⇒ *int*›
where [*simp*]: ‹*unit-factor* = (*sgn* :: *int* ⇒ *int*)›

definition *divide-int* :: ‹*int* ⇒ *int* ⇒ *int*›
where ‹*k div l* = (*sgn k* * *sgn l* * *int* (*nat* |*k*| *div nat* |*l*|)
− *of-bool* (*l* ≠ *0* ∧ *sgn k* ≠ *sgn l* ∧ ¬ *l dvd k*))›

lemma *divide-int-unfold*:
‹(*sgn k* * *int m*) *div* (*sgn l* * *int n*) = (*sgn k* * *sgn l* * *int* (*m div n*)
− *of-bool* ((*k* = *0* ⟷ *m* = *0*) ∧ *l* ≠ *0* ∧ *n* ≠ *0* ∧ *sgn k* ≠ *sgn l* ∧ ¬ *n dvd m*))›
⟨*proof*⟩

definition *modulo-int* :: ‹*int* ⇒ *int* ⇒ *int*›
where ‹*k mod l* = *sgn k* * *int* (*nat* |*k*| *mod nat* |*l*|) + *l* * *of-bool* (*sgn k* ≠ *sgn l* ∧ ¬ *l dvd k*)›

lemma *modulo-int-unfold*:
‹(*sgn k* * *int m*) *mod* (*sgn l* * *int n*) =

*sgn k * int (m mod (of-bool (l ≠ 0) * n)) + (sgn l * int n) * of-bool ((k = 0 ⟷ m = 0) ∧ sgn k ≠ sgn l ∧ ¬ n dvd m)*›
⟨*proof*⟩

instance ⟨*proof*⟩

end

56.5.2 Algebraic foundations

lemma *coprime-int-iff* [*simp*]:
coprime (int m) (int n) ⟷ coprime m n (**is** *?P ⟷ ?Q*)
⟨*proof*⟩

lemma *coprime-abs-left-iff* [*simp*]:
coprime |k| l ⟷ coprime k l **for** *k l* :: *int*
⟨*proof*⟩

lemma *coprime-abs-right-iff* [*simp*]:
coprime k |l| ⟷ coprime k l **for** *k l* :: *int*
⟨*proof*⟩

lemma *coprime-nat-abs-left-iff* [*simp*]:
coprime (nat |k|) n ⟷ coprime k (int n)
⟨*proof*⟩

lemma *coprime-nat-abs-right-iff* [*simp*]:
coprime n (nat |k|) ⟷ coprime (int n) k
⟨*proof*⟩

lemma *coprime-common-divisor-int*: *coprime a b ⟹ x dvd a ⟹ x dvd b ⟹ |x| = 1*
for *a b* :: *int*
⟨*proof*⟩

56.5.3 Basic conversions

lemma *div-abs-eq-div-nat*:
|k| div |l| = int (nat |k| div nat |l|)
⟨*proof*⟩

lemma *div-eq-div-abs*:
*k div l = sgn k * sgn l * (|k| div |l|)*
− of-bool (l ≠ 0 ∧ sgn k ≠ sgn l ∧ ¬ l dvd k)›
for *k l* :: *int*
⟨*proof*⟩

lemma *div-abs-eq*:
*|k| div |l| = sgn k * sgn l * (k div l + of-bool (sgn k ≠ sgn l ∧ ¬ l dvd k))*›
for *k l* :: *int*

⟨*proof*⟩

lemma *mod-abs-eq-div-nat*:
|*k*| *mod* |*l*| = *int* (*nat* |*k*| *mod nat* |*l*|)
⟨*proof*⟩

lemma *mod-eq-mod-abs*:
‹*k mod l* = *sgn k* ∗ (|*k*| *mod* |*l*|) + *l* ∗ *of-bool* (*sgn k* ≠ *sgn l* ∧ ¬ *l dvd k*)›
for *k l* :: *int*
⟨*proof*⟩

lemma *mod-abs-eq*:
‹|*k*| *mod* |*l*| = *sgn k* ∗ (*k mod l* − *l* ∗ *of-bool* (*sgn k* ≠ *sgn l* ∧ ¬ *l dvd k*))›
for *k l* :: *int*
⟨*proof*⟩

lemma *div-sgn-abs-cancel*:
fixes *k l v* :: *int*
assumes *v* ≠ *0*
shows (*sgn v* ∗ |*k*|) *div* (*sgn v* ∗ |*l*|) = |*k*| *div* |*l*|
⟨*proof*⟩

lemma *div-eq-sgn-abs*:
fixes *k l v* :: *int*
assumes *sgn k* = *sgn l*
shows *k div l* = |*k*| *div* |*l*|
⟨*proof*⟩

lemma *div-dvd-sgn-abs*:
fixes *k l* :: *int*
assumes *l dvd k*
shows *k div l* = (*sgn k* ∗ *sgn l*) ∗ (|*k*| *div* |*l*|)
⟨*proof*⟩

lemma *div-noneq-sgn-abs*:
fixes *k l* :: *int*
assumes *l* ≠ *0*
assumes *sgn k* ≠ *sgn l*
shows *k div l* = − (|*k*| *div* |*l*|) − *of-bool* (¬ *l dvd k*)
⟨*proof*⟩

56.5.4 Euclidean division

instantiation *int* :: *unique-euclidean-ring*
begin

definition *euclidean-size-int* :: *int* ⇒ *nat*
where [*simp*]: *euclidean-size-int* = (*nat* ∘ *abs* :: *int* ⇒ *nat*)

definition *division-segment-int* :: *int* ⇒ *int*
where *division-segment-int k* = (*if k* ≥ *0 then 1 else* − *1*)

lemma *division-segment-eq-sgn*:
division-segment k = *sgn k* **if** *k* ≠ *0* **for** *k* :: *int*
⟨*proof*⟩

lemma *abs-division-segment* [*simp*]:
|*division-segment k*| = *1* **for** *k* :: *int*
⟨*proof*⟩

lemma *abs-mod-less*:
|*k mod l*| < |*l*| **if** *l* ≠ *0* **for** *k l* :: *int*
⟨*proof*⟩

lemma *sgn-mod*:
sgn (*k mod l*) = *sgn l* **if** *l* ≠ *0* ¬ *l dvd k* **for** *k l* :: *int*
⟨*proof*⟩

instance ⟨*proof*⟩

end

lemma *euclidean-relation-intI* [*case-names by0 divides euclidean-relation*]:
‹(*k div l*, *k mod l*) = (*q*, *r*)›
if *by0′*: ‹*l* = *0* ⟹ *q* = *0* ∧ *r* = *k*›
and *divides′*: ‹*l* ≠ *0* ⟹ *l dvd k* ⟹ *r* = *0* ∧ *k* = *q* ∗ *l*›
and *euclidean-relation′*: ‹*l* ≠ *0* ⟹ ¬ *l dvd k* ⟹ *sgn r* = *sgn l*
∧ |*r*| < |*l*| ∧ *k* = *q* ∗ *l* + *r*› **for** *k l* :: *int*
⟨*proof*⟩

56.5.5 Trivial reduction steps

lemma *div-pos-pos-trivial* [*simp*]:
k div l = *0* **if** *k* ≥ *0* **and** *k* < *l* **for** *k l* :: *int*
⟨*proof*⟩

lemma *mod-pos-pos-trivial* [*simp*]:
k mod l = *k* **if** *k* ≥ *0* **and** *k* < *l* **for** *k l* :: *int*
⟨*proof*⟩

lemma *div-neg-neg-trivial* [*simp*]:
k div l = *0* **if** *k* ≤ *0* **and** *l* < *k* **for** *k l* :: *int*
⟨*proof*⟩

lemma *mod-neg-neg-trivial* [*simp*]:
k mod l = *k* **if** *k* ≤ *0* **and** *l* < *k* **for** *k l* :: *int*
⟨*proof*⟩

lemma
div-pos-neg-trivial: ‹*k div l = − 1*› (**is** *?Q*)
and *mod-pos-neg-trivial*: ‹*k mod l = k + l*› (**is** *?R*)
if ‹*0 < k*› **and** ‹*k + l ≤ 0*› **for** *k l* :: *int*
⟨*proof*⟩

There is neither *div-neg-pos-trivial* nor *mod-neg-pos-trivial* because (*0*::*'a*) *div l* = (*0*::*'a*) would supersede it.

56.5.6 More uniqueness rules

lemma
fixes *a b q r* :: *int*
assumes ‹*a = b * q + r*› ‹*0 ≤ r*› ‹*r < b*›
shows *int-div-pos-eq*:
‹*a div b = q*› (**is** *?Q*)
and *int-mod-pos-eq*:
‹*a mod b = r*› (**is** *?R*)
⟨*proof*⟩

lemma *int-div-neg-eq*:
‹*a div b = q*› **if** ‹*a = b * q + r*› ‹*r ≤ 0*› ‹*b < r*› **for** *a b q r* :: *int*
⟨*proof*⟩

lemma *int-mod-neg-eq*:
‹*a mod b = r*› **if** ‹*a = b * q + r*› ‹*r ≤ 0*› ‹*b < r*› **for** *a b q r* :: *int*
⟨*proof*⟩

56.5.7 Laws for unary minus

lemma *zmod-zminus1-not-zero*:
fixes *k l* :: *int*
shows *− k mod l ≠ 0 ⟹ k mod l ≠ 0*
⟨*proof*⟩

lemma *zmod-zminus2-not-zero*:
fixes *k l* :: *int*
shows *k mod − l ≠ 0 ⟹ k mod l ≠ 0*
⟨*proof*⟩

lemma *zdiv-zminus1-eq-if*:
‹*(− a) div b = (if a mod b = 0 then − (a div b) else − (a div b) − 1)*›
if ‹*b ≠ 0*› **for** *a b* :: *int*
⟨*proof*⟩

lemma *zdiv-zminus2-eq-if*:
‹*a div (− b) = (if a mod b = 0 then − (a div b) else − (a div b) − 1)*›
if ‹*b ≠ 0*› **for** *a b* :: *int*
⟨*proof*⟩

lemma *zmod-zminus1-eq-if*:
‹(− *a*) *mod b* = (*if a mod b* = *0 then 0 else b* − (*a mod b*))›
for *a b* :: *int*
⟨*proof*⟩

lemma *zmod-zminus2-eq-if*:
‹*a mod* (− *b*) = (*if a mod b* = *0 then 0 else* (*a mod b*) − *b*)›
for *a b* :: *int*
⟨*proof*⟩

56.5.8 Borders

lemma *pos-mod-bound* [*simp*]:
k mod l < *l* **if** *l* > *0* **for** *k l* :: *int*
⟨*proof*⟩

lemma *neg-mod-bound* [*simp*]:
l < *k mod l* **if** *l* < *0* **for** *k l* :: *int*
⟨*proof*⟩

lemma *pos-mod-sign* [*simp*]:
0 ≤ *k mod l* **if** *l* > *0* **for** *k l* :: *int*
⟨*proof*⟩

lemma *neg-mod-sign* [*simp*]:
k mod l ≤ *0* **if** *l* < *0* **for** *k l* :: *int*
⟨*proof*⟩

56.5.9 Splitting Rules for div and mod

lemma *split-zdiv*:
‹*P* (*n div k*) ⟷
(*k* = *0* ⟶ *P 0*) ∧
(*0* < *k* ⟶ (∀ *i j*. *0* ≤ *j* ∧ *j* < *k* ∧ *n* = *k* ∗ *i* + *j* ⟶ *P i*)) ∧
(*k* < *0* ⟶ (∀ *i j*. *k* < *j* ∧ *j* ≤ *0* ∧ *n* = *k* ∗ *i* + *j* ⟶ *P i*))› (**is** *?div*)
and *split-zmod*:
‹*Q* (*n mod k*) ⟷
(*k* = *0* ⟶ *Q n*) ∧
(*0* < *k* ⟶ (∀ *i j*. *0* ≤ *j* ∧ *j* < *k* ∧ *n* = *k* ∗ *i* + *j* ⟶ *Q j*)) ∧
(*k* < *0* ⟶ (∀ *i j*. *k* < *j* ∧ *j* ≤ *0* ∧ *n* = *k* ∗ *i* + *j* ⟶ *Q j*))› (**is** *?mod*)
for *n k* :: *int*
⟨*proof*⟩

Enable (lin)arith to deal with (*div*) and (*mod*) when these are applied to some constant that is of the form *numeral k*:

declare *split-zdiv* [*of* - - ‹*numeral n*›, *linarith-split*] **for** *n*
declare *split-zdiv* [*of* - - ‹− *numeral n*›, *linarith-split*] **for** *n*
declare *split-zmod* [*of* - - ‹*numeral n*›, *linarith-split*] **for** *n*

declare *split-zmod* [*of - - ‹– numeral n›*, *linarith-split*] **for** *n*

lemma *zdiv-eq-0-iff*:
i div k = 0 ⟷ k = 0 ∨ 0 ≤ i ∧ i < k ∨ i ≤ 0 ∧ k < i (**is** *?L = ?R*)
for *i k :: int*
⟨*proof*⟩

lemma *zmod-trivial-iff*:
fixes *i k :: int*
shows *i mod k = i ⟷ k = 0 ∨ 0 ≤ i ∧ i < k ∨ i ≤ 0 ∧ k < i*
⟨*proof*⟩

56.5.10 Algebraic rewrites

lemma *zdiv-zmult2-eq*: *‹a div (b * c) = (a div b) div c›* (**is** *?Q*)
and *zmod-zmult2-eq*: *‹a mod (b * c) = b * (a div b mod c) + a mod b›* (**is** *?P*)
if *‹c ≥ 0›* **for** *a b c :: int*
⟨*proof*⟩

lemma *zdiv-zmult2-eq'*:
*‹k div (l * j) = ((sgn j * k) div l) div |j|›* **for** *k l j :: int*
⟨*proof*⟩

lemma *half-nonnegative-int-iff* [*simp*]:
‹k div 2 ≥ 0 ⟷ k ≥ 0› **for** *k :: int*
⟨*proof*⟩

lemma *half-negative-int-iff* [*simp*]:
‹k div 2 < 0 ⟷ k < 0› **for** *k :: int*
⟨*proof*⟩

56.5.11 Distributive laws for conversions.

lemma *zdiv-int*:
‹int (m div n) = int m div int n›
⟨*proof*⟩

lemma *zmod-int*:
‹int (m mod n) = int m mod int n›
⟨*proof*⟩

lemma *nat-div-distrib*:
‹nat (x div y) = nat x div nat y› **if** *‹0 ≤ x›*
⟨*proof*⟩

lemma *nat-div-distrib'*:
‹nat (x div y) = nat x div nat y› **if** *‹0 ≤ y›*
⟨*proof*⟩

lemma *nat-mod-distrib*: — Fails if y<0: the LHS collapses to (nat z) but the RHS doesn't
‹*nat* (*x mod y*) = *nat x mod nat y*› **if** ‹*0* ≤ *x*› ‹*0* ≤ *y*›
⟨*proof*⟩

56.5.12 Monotonicity in the First Argument (Dividend)

lemma *zdiv-mono1*:
‹*a div b* ≤ *a'* *div b*›
if ‹*a* ≤ *a'*› ‹*0* < *b*›
for *a b b'* :: *int*
⟨*proof*⟩

lemma *zdiv-mono1-neg*:
‹*a'* *div b* ≤ *a div b*›
if ‹*a* ≤ *a'*› ‹*b* < *0*›
for *a a' b* :: *int*
⟨*proof*⟩

56.5.13 Monotonicity in the Second Argument (Divisor)

lemma *zdiv-mono2*:
‹*a div b* ≤ *a div b'*› **if** ‹*0* ≤ *a*› ‹*0* < *b'*› ‹*b'* ≤ *b*› **for** *a b b'* :: *int*
⟨*proof*⟩

lemma *zdiv-mono2-neg*:
‹*a div b'* ≤ *a div b*› **if** ‹*a* < *0*› ‹*0* < *b'*› ‹*b'* ≤ *b*› **for** *a b b'* :: *int*
⟨*proof*⟩

56.5.14 Quotients of Signs

lemma *div-eq-minus1*:
‹*0* < *b* ⟹ − *1 div b* = − *1*› **for** *b* :: *int*
⟨*proof*⟩

lemma *zmod-minus1*:
‹*0* < *b* ⟹ − *1 mod b* = *b* − *1*› **for** *b* :: *int*
⟨*proof*⟩

lemma *minus-mod-int-eq*:
‹− *k mod l* = *l* − *1* − (*k* − *1*) *mod l*› **if** ‹*l* ≥ *0*› **for** *k l* :: *int*
⟨*proof*⟩

lemma *div-neg-pos-less0*:
‹*a div b* < *0*› **if** ‹*a* < *0*› ‹*0* < *b*› **for** *a b* :: *int*
⟨*proof*⟩

lemma *div-nonneg-neg-le0*:
‹*a div b* ≤ *0*› **if** ‹*0* ≤ *a*› ‹*b* < *0*› **for** *a b* :: *int*
⟨*proof*⟩

lemma *div-nonpos-pos-le0*:
‹*a div b* ≤ *0*› **if** ‹*a* ≤ *0*› ‹*0* < *b*› **for** *a b* :: *int*
⟨*proof*⟩

Now for some equivalences of the form *a div b* >=< *0* ⟷ ... conditional upon the sign of *a* or *b*. There are many more. They should all be simp rules unless that causes too much search.

lemma *pos-imp-zdiv-nonneg-iff*:
‹*0* ≤ *a div b* ⟷ *0* ≤ *a*›
if ‹*0* < *b*› **for** *a b* :: *int*
⟨*proof*⟩

lemma *neg-imp-zdiv-nonneg-iff*:
‹*0* ≤ *a div b* ⟷ *a* ≤ *0*› **if** ‹*b* < *0*› **for** *a b* :: *int*
⟨*proof*⟩

lemma *pos-imp-zdiv-pos-iff*:
‹*0* < (*i*::*int*) *div k* ⟷ *k* ≤ *i*› **if** ‹*0* < *k*› **for** *i k* :: *int*
⟨*proof*⟩

lemma *pos-imp-zdiv-neg-iff*:
‹*a div b* < *0* ⟷ *a* < *0*› **if** ‹*0* < *b*› **for** *a b* :: *int*
— But not (*a div b* ≤ *0*) = (*a* ≤ *0*); consider *a* = *1*, *b* = *2* when *a div b* = *0*.
⟨*proof*⟩

lemma *neg-imp-zdiv-neg-iff*:
— But not (*a div b* ≤ *0*) = (*0* ≤ *a*); consider *a* = − *1*, *b* = − *2* when *a div b* = *0*.
‹*a div b* < *0* ⟷ *0* < *a*› **if** ‹*b* < *0*› **for** *a b* :: *int*
⟨*proof*⟩

lemma *nonneg1-imp-zdiv-pos-iff*:
‹*a div b* > *0* ⟷ *a* ≥ *b* ∧ *b* > *0*› **if** ‹*0* ≤ *a*› **for** *a b* :: *int*
⟨*proof*⟩

lemma *zmod-le-nonneg-dividend*:
‹*m mod k* ≤ *m*› **if** ‹(*m*::*int*) ≥ *0*› **for** *m k* :: *int*
⟨*proof*⟩

lemma *sgn-div-eq-sgn-mult*:
‹*sgn* (*k div l*) = *of-bool* (*k div l* ≠ *0*) ∗ *sgn* (*k* ∗ *l*)›
for *k l* :: *int*
⟨*proof*⟩

56.5.15 Further properties

lemma *div-int-pos-iff*:
k div l ≥ *0* ⟷ *k* = *0* ∨ *l* = *0* ∨ *k* ≥ *0* ∧ *l* ≥ *0*

$\vee\ k < 0 \wedge l < 0$
for $k\ l :: int$
⟨*proof*⟩

lemma *mod-int-pos-iff*:
‹$k\ mod\ l \geq 0 \longleftrightarrow l\ dvd\ k \vee l = 0 \wedge k \geq 0 \vee l > 0$›
for $k\ l :: int$
⟨*proof*⟩

lemma *abs-div*:
‹$|x\ div\ y| = |x|\ div\ |y|$› **if** ‹$y\ dvd\ x$› **for** $x\ y :: int$
⟨*proof*⟩

lemma *int-power-div-base*:
‹$k\ \hat{}\ m\ div\ k = k\ \hat{}\ (m - Suc\ 0)$› **if** ‹$0 < m$› ‹$0 < k$› **for** $k :: int$
⟨*proof*⟩

lemma *int-div-less-self*:
‹$x\ div\ k < x$› **if** ‹$0 < x$› ‹$1 < k$› **for** $x\ k :: int$
⟨*proof*⟩

56.5.16 Computing *div* and *mod* by shifting

lemma *div-pos-geq*:
‹$k\ div\ l = (k - l)\ div\ l + 1$› **if** ‹$0 < l$› ‹$l \leq k$› **for** $k\ l :: int$
⟨*proof*⟩

lemma *mod-pos-geq*:
‹$k\ mod\ l = (k - l)\ mod\ l$› **if** ‹$0 < l$› ‹$l \leq k$› **for** $k\ l :: int$
⟨*proof*⟩

lemma *pos-zdiv-mult-2*: ‹$(1 + 2 * b)\ div\ (2 * a) = b\ div\ a$› (**is** *?Q*)
and *pos-zmod-mult-2*: ‹$(1 + 2 * b)\ mod\ (2 * a) = 1 + 2 * (b\ mod\ a)$› (**is** *?R*)
if ‹$0 \leq a$› **for** $a\ b :: int$
⟨*proof*⟩

lemma *neg-zdiv-mult-2*: ‹$(1 + 2 * b)\ div\ (2 * a) = (b + 1)\ div\ a$› (**is** *?Q*)
and *neg-zmod-mult-2*: ‹$(1 + 2 * b)\ mod\ (2 * a) = 2 * ((b + 1)\ mod\ a) - 1$›
(**is** *?R*)
if ‹$a \leq 0$› **for** $a\ b :: int$
⟨*proof*⟩

lemma *zdiv-numeral-Bit0* [*simp*]:
‹$numeral\ (Num.Bit0\ v)\ div\ numeral\ (Num.Bit0\ w) =$
$numeral\ v\ div\ (numeral\ w :: int)$›
⟨*proof*⟩

lemma *zdiv-numeral-Bit1* [*simp*]:
‹$numeral\ (Num.Bit1\ v)\ div\ numeral\ (Num.Bit0\ w) =$

(*numeral v div* (*numeral w* :: *int*))›
⟨*proof*⟩

lemma *zmod-numeral-Bit0* [*simp*]:
‹*numeral* (*Num.Bit0 v*) *mod numeral* (*Num.Bit0 w*) =
(*2*::*int*) ∗ (*numeral v mod numeral w*)›
⟨*proof*⟩

lemma *zmod-numeral-Bit1* [*simp*]:
‹*numeral* (*Num.Bit1 v*) *mod numeral* (*Num.Bit0 w*) =
2 ∗ (*numeral v mod numeral w*) + (*1*::*int*)›
⟨*proof*⟩

56.6 Code generation

context
begin

qualified definition *divmod-nat* :: *nat* ⇒ *nat* ⇒ *nat* × *nat*
where *divmod-nat m n* = (*m div n*, *m mod n*)

qualified lemma *divmod-nat-if* [*code*]:
divmod-nat m n = (*if n* = *0* ∨ *m* < *n then* (*0*, *m*) *else*
let (*q*, *r*) = *divmod-nat* (*m* − *n*) *n in* (*Suc q*, *r*))
⟨*proof*⟩ **lemma** [*code*]:
m div n = *fst* (*divmod-nat m n*)
m mod n = *snd* (*divmod-nat m n*)
⟨*proof*⟩

end

code-identifier
code-module *Euclidean-Rings* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

end

57 Parity in rings and semirings

theory *Parity*
imports *Euclidean-Rings*
begin

57.1 Ring structures with parity and *even*/*odd* predicates

class *semiring-parity* = *comm-semiring-1* + *semiring-modulo* +
assumes *even-iff-mod-2-eq-zero*: *2 dvd a* ⟷ *a mod 2* = *0*
and *odd-iff-mod-2-eq-one*: ¬ *2 dvd a* ⟷ *a mod 2* = *1*
and *odd-one* [*simp*]: ¬ *2 dvd 1*

begin

abbreviation *even* :: *'a* ⇒ *bool*
  where *even a* ≡ *2 dvd a*

abbreviation *odd* :: *'a* ⇒ *bool*
  where *odd a* ≡ ¬ *2 dvd a*

end

class *ring-parity* = *ring* + *semiring-parity*
begin

subclass *comm-ring-1* ⟨*proof*⟩

end

instance *nat* :: *semiring-parity*
  ⟨*proof*⟩

instance *int* :: *ring-parity*
  ⟨*proof*⟩

context *semiring-parity*
begin

lemma *parity-cases* [*case-names even odd*]:
  assumes *even a* ⟹ *a mod 2 = 0* ⟹ *P*
  assumes *odd a* ⟹ *a mod 2 = 1* ⟹ *P*
  shows *P*
  ⟨*proof*⟩

lemma *odd-of-bool-self* [*simp*]:
  ‹*odd* (*of-bool p*) ⟷ *p*›
  ⟨*proof*⟩

lemma *not-mod-2-eq-0-eq-1* [*simp*]:
  a mod 2 ≠ *0* ⟷ *a mod 2 = 1*
  ⟨*proof*⟩

lemma *not-mod-2-eq-1-eq-0* [*simp*]:
  a mod 2 ≠ *1* ⟷ *a mod 2 = 0*
  ⟨*proof*⟩

lemma *evenE* [*elim?*]:
  assumes *even a*
  obtains *b* **where** *a = 2 * b*
  ⟨*proof*⟩

lemma *oddE* [*elim?*]:
assumes *odd a*
obtains *b* **where** *a = 2 * b + 1*
⟨*proof*⟩

lemma *mod-2-eq-odd*:
a mod 2 = of-bool (odd a)
⟨*proof*⟩

lemma *of-bool-odd-eq-mod-2*:
of-bool (odd a) = a mod 2
⟨*proof*⟩

lemma *even-mod-2-iff* [*simp*]:
‹*even (a mod 2) ⟷ even a*›
⟨*proof*⟩

lemma *mod2-eq-if*:
a mod 2 = (if even a then 0 else 1)
⟨*proof*⟩

lemma *even-zero* [*simp*]:
even 0
⟨*proof*⟩

lemma *odd-even-add*:
even (a + b) **if** *odd a* **and** *odd b*
⟨*proof*⟩

lemma *even-add* [*simp*]:
even (a + b) ⟷ (even a ⟷ even b)
⟨*proof*⟩

lemma *odd-add* [*simp*]:
odd (a + b) ⟷ ¬ (odd a ⟷ odd b)
⟨*proof*⟩

lemma *even-plus-one-iff* [*simp*]:
even (a + 1) ⟷ odd a
⟨*proof*⟩

lemma *even-mult-iff* [*simp*]:
*even (a * b) ⟷ even a ∨ even b* (**is** *?P ⟷ ?Q*)
⟨*proof*⟩

lemma *even-numeral* [*simp*]: *even (numeral (Num.Bit0 n))*
⟨*proof*⟩

lemma *odd-numeral* [*simp*]: *odd (numeral (Num.Bit1 n))*

⟨*proof*⟩

lemma *odd-numeral-BitM* [*simp*]:
‹*odd* (*numeral* (*Num.BitM w*))›
⟨*proof*⟩

lemma *even-power* [*simp*]: *even* (*a* ^ *n*) ⟷ *even a* ∧ *n* > *0*
⟨*proof*⟩

lemma *even-prod-iff*:
‹*even* (*prod f A*) ⟷ (∃ *a*∈*A*. *even* (*f a*))› **if** ‹*finite A*›
⟨*proof*⟩

lemma *mask-eq-sum-exp*:
‹*2* ^ *n* − *1* = (∑ *m*∈{*q*. *q* < *n*}. *2* ^ *m*)›
⟨*proof*⟩

lemma (**in** −) *mask-eq-sum-exp-nat*:
‹*2* ^ *n* − *Suc 0* = (∑ *m*∈{*q*. *q* < *n*}. *2* ^ *m*)›
⟨*proof*⟩

end

context *ring-parity*
begin

lemma *even-minus*:
even (− *a*) ⟷ *even a*
⟨*proof*⟩

lemma *even-diff* [*simp*]:
even (*a* − *b*) ⟷ *even* (*a* + *b*)
⟨*proof*⟩

end

57.2 Instance for *nat*

lemma *even-Suc-Suc-iff* [*simp*]:
even (*Suc* (*Suc n*)) ⟷ *even n*
⟨*proof*⟩

lemma *even-Suc* [*simp*]: *even* (*Suc n*) ⟷ *odd n*
⟨*proof*⟩

lemma *even-diff-nat* [*simp*]:
even (*m* − *n*) ⟷ *m* < *n* ∨ *even* (*m* + *n*) **for** *m n* :: *nat*
⟨*proof*⟩

lemma *odd-pos*:
odd n $\Longrightarrow$ *0 < n* **for** *n* :: *nat*
⟨*proof*⟩

lemma *Suc-double-not-eq-double*:
*Suc (2 * m)* $\neq$ *2 * n*
⟨*proof*⟩

lemma *double-not-eq-Suc-double*:
*2 * m* $\neq$ *Suc (2 * n)*
⟨*proof*⟩

lemma *odd-Suc-minus-one* [*simp*]: *odd n* $\Longrightarrow$ *Suc (n − Suc 0) = n*
⟨*proof*⟩

lemma *even-Suc-div-two* [*simp*]:
even n $\Longrightarrow$ *Suc n div 2 = n div 2*
⟨*proof*⟩

lemma *odd-Suc-div-two* [*simp*]:
odd n $\Longrightarrow$ *Suc n div 2 = Suc (n div 2)*
⟨*proof*⟩

lemma *odd-two-times-div-two-nat* [*simp*]:
assumes *odd n*
shows *2 * (n div 2) = n − (1 :: nat)*
⟨*proof*⟩

lemma *not-mod2-eq-Suc-0-eq-0* [*simp*]:
n mod 2 $\neq$ *Suc 0* $\longleftrightarrow$ *n mod 2 = 0*
⟨*proof*⟩

lemma *odd-card-imp-not-empty*:
‹*A* $\neq$ {}› **if** ‹*odd (card A)*›
⟨*proof*⟩

lemma *nat-induct2* [*case-names 0 1 step*]:
assumes *P 0 P 1* **and** *step*: $\bigwedge$*n*::*nat*. *P n* $\Longrightarrow$ *P (n + 2)*
shows *P n*
⟨*proof*⟩

context *semiring-parity*
begin

lemma *even-sum-iff*:
‹*even (sum f A)* $\longleftrightarrow$ *even (card* {*a*∈*A*. *odd (f a)*})› **if** ‹*finite A*›
⟨*proof*⟩

lemma *even-mask-iff* [*simp*]:

‹*even* (2 ^ *n* − 1) ⟷ *n* = 0›
⟨*proof*⟩

lemma *even-of-nat-iff* [*simp*]:
even (*of-nat n*) ⟷ *even n*
⟨*proof*⟩

end

57.3 Parity and powers

context *ring-1*
begin

lemma *power-minus-even* [*simp*]: *even n* ⟹ (− *a*) ^ *n* = *a* ^ *n*
⟨*proof*⟩

lemma *power-minus-odd* [*simp*]: *odd n* ⟹ (− *a*) ^ *n* = − (*a* ^ *n*)
⟨*proof*⟩

lemma *uminus-power-if*:
(− *a*) ^ *n* = (*if even n then a* ^ *n else* − (*a* ^ *n*))
⟨*proof*⟩

lemma *neg-one-even-power* [*simp*]: *even n* ⟹ (− 1) ^ *n* = 1
⟨*proof*⟩

lemma *neg-one-odd-power* [*simp*]: *odd n* ⟹ (− 1) ^ *n* = − 1
⟨*proof*⟩

lemma *neg-one-power-add-eq-neg-one-power-diff*: *k* ≤ *n* ⟹ (− 1) ^ (*n* + *k*) = (− 1) ^ (*n* − *k*)
⟨*proof*⟩

lemma *minus-one-power-iff*: (− 1) ^ *n* = (*if even n then* 1 *else* − 1)
⟨*proof*⟩

end

context *linordered-idom*
begin

lemma *zero-le-even-power*: *even n* ⟹ 0 ≤ *a* ^ *n*
⟨*proof*⟩

lemma *zero-le-odd-power*: *odd n* ⟹ 0 ≤ *a* ^ *n* ⟷ 0 ≤ *a*
⟨*proof*⟩

lemma *zero-le-power-eq*: 0 ≤ *a* ^ *n* ⟷ *even n* ∨ *odd n* ∧ 0 ≤ *a*

⟨*proof*⟩

lemma *zero-less-power-eq*: $0 < a \mathbin{\hat{}} n \longleftrightarrow n = 0 \vee even\ n \wedge a \neq 0 \vee odd\ n \wedge 0 < a$
⟨*proof*⟩

lemma *power-less-zero-eq* [*simp*]: $a \mathbin{\hat{}} n < 0 \longleftrightarrow odd\ n \wedge a < 0$
⟨*proof*⟩

lemma *power-le-zero-eq*: $a \mathbin{\hat{}} n \leq 0 \longleftrightarrow n > 0 \wedge (odd\ n \wedge a \leq 0 \vee even\ n \wedge a = 0)$
⟨*proof*⟩

lemma *power-even-abs*: $even\ n \Longrightarrow |a| \mathbin{\hat{}} n = a \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *power-mono-even*:
assumes *even n* **and** $|a| \leq |b|$
shows $a \mathbin{\hat{}} n \leq b \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *power-mono-odd*:
assumes *odd n* **and** $a \leq b$
shows $a \mathbin{\hat{}} n \leq b \mathbin{\hat{}} n$
⟨*proof*⟩

Simplify, when the exponent is a numeral

lemma *zero-le-power-eq-numeral* [*simp*]:
$0 \leq a \mathbin{\hat{}} numeral\ w \longleftrightarrow even\ (numeral\ w :: nat) \vee odd\ (numeral\ w :: nat) \wedge 0 \leq a$
⟨*proof*⟩

lemma *zero-less-power-eq-numeral* [*simp*]:
$0 < a \mathbin{\hat{}} numeral\ w \longleftrightarrow$
$numeral\ w = (0 :: nat) \vee$
$even\ (numeral\ w :: nat) \wedge a \neq 0 \vee$
$odd\ (numeral\ w :: nat) \wedge 0 < a$
⟨*proof*⟩

lemma *power-le-zero-eq-numeral* [*simp*]:
$a \mathbin{\hat{}} numeral\ w \leq 0 \longleftrightarrow$
$(0 :: nat) < numeral\ w \wedge$
$(odd\ (numeral\ w :: nat) \wedge a \leq 0 \vee even\ (numeral\ w :: nat) \wedge a = 0)$
⟨*proof*⟩

lemma *power-less-zero-eq-numeral* [*simp*]:
$a \mathbin{\hat{}} numeral\ w < 0 \longleftrightarrow odd\ (numeral\ w :: nat) \wedge a < 0$
⟨*proof*⟩

lemma *power-even-abs-numeral* [*simp*]:
even (*numeral w* :: *nat*) ⟹ |*a*| ^ *numeral w* = *a* ^ *numeral w*
⟨*proof*⟩

end

57.4 Instance for *int*

lemma *even-diff-iff*:
even (*k* − *l*) ⟷ *even* (*k* + *l*) **for** *k l* :: *int*
⟨*proof*⟩

lemma *even-abs-add-iff*:
even (|*k*| + *l*) ⟷ *even* (*k* + *l*) **for** *k l* :: *int*
⟨*proof*⟩

lemma *even-add-abs-iff*:
even (*k* + |*l*|) ⟷ *even* (*k* + *l*) **for** *k l* :: *int*
⟨*proof*⟩

lemma *even-nat-iff*: 0 ≤ *k* ⟹ *even* (*nat k*) ⟷ *even k*
⟨*proof*⟩

context
assumes *SORT-CONSTRAINT*('*a*::*division-ring*)
begin

lemma *power-int-minus-left*:
power-int (−*a* :: '*a*) *n* = (*if even n then power-int a n else* −*power-int a n*)
⟨*proof*⟩

lemma *power-int-minus-left-even* [*simp*]: *even n* ⟹ *power-int* (−*a* :: '*a*) *n* = *power-int a n*
⟨*proof*⟩

lemma *power-int-minus-left-odd* [*simp*]: *odd n* ⟹ *power-int* (−*a* :: '*a*) *n* = −*power-int a n*
⟨*proof*⟩

lemma *power-int-minus-left-distrib*:
NO-MATCH (−1) *x* ⟹ *power-int* (−*a* :: '*a*) *n* = *power-int* (−1) *n* ∗ *power-int a n*
⟨*proof*⟩

lemma *power-int-minus-one-minus*: *power-int* (−1 :: '*a*) (−*n*) = *power-int* (−1) *n*
⟨*proof*⟩

lemma *power-int-minus-one-diff-commute*: *power-int* (−1 :: '*a*) (*a* − *b*) = *power-int*

$(-1)\ (b - a)$
⟨*proof*⟩

lemma *power-int-minus-one-mult-self* [*simp*]:
$power\text{-}int\ (-1 :: 'a)\ m * power\text{-}int\ (-1)\ m = 1$
⟨*proof*⟩

lemma *power-int-minus-one-mult-self'* [*simp*]:
$power\text{-}int\ (-1 :: 'a)\ m * (power\text{-}int\ (-1)\ m * b) = b$
⟨*proof*⟩

end

57.5 Special case: euclidean rings containing the natural numbers

class *unique-euclidean-semiring-with-nat* = *semidom* + *semiring-char-0* + *unique-euclidean-semiring* +
assumes *of-nat-div*: $of\text{-}nat\ (m\ div\ n) = of\text{-}nat\ m\ div\ of\text{-}nat\ n$
and *division-segment-of-nat* [*simp*]: $division\text{-}segment\ (of\text{-}nat\ n) = 1$
and *division-segment-euclidean-size* [*simp*]: $division\text{-}segment\ a * of\text{-}nat\ (euclidean\text{-}size\ a) = a$
begin

lemma *division-segment-eq-iff*:
$a = b$ **if** $division\text{-}segment\ a = division\text{-}segment\ b$
and $euclidean\text{-}size\ a = euclidean\text{-}size\ b$
⟨*proof*⟩

lemma *euclidean-size-of-nat* [*simp*]:
$euclidean\text{-}size\ (of\text{-}nat\ n) = n$
⟨*proof*⟩

lemma *of-nat-euclidean-size*:
$of\text{-}nat\ (euclidean\text{-}size\ a) = a\ div\ division\text{-}segment\ a$
⟨*proof*⟩

lemma *division-segment-1* [*simp*]:
$division\text{-}segment\ 1 = 1$
⟨*proof*⟩

lemma *division-segment-numeral* [*simp*]:
$division\text{-}segment\ (numeral\ k) = 1$
⟨*proof*⟩

lemma *euclidean-size-1* [*simp*]:
$euclidean\text{-}size\ 1 = 1$
⟨*proof*⟩

lemma *euclidean-size-numeral* [*simp*]:
euclidean-size (*numeral k*) = *numeral k*
⟨*proof*⟩

lemma *of-nat-dvd-iff*:
of-nat m dvd of-nat n ⟷ *m dvd n* (**is** *?P* ⟷ *?Q*)
⟨*proof*⟩

lemma *of-nat-mod*:
of-nat (*m mod n*) = *of-nat m mod of-nat n*
⟨*proof*⟩

lemma *one-div-two-eq-zero* [*simp*]:
1 div 2 = *0*
⟨*proof*⟩

lemma *one-mod-two-eq-one* [*simp*]:
1 mod 2 = *1*
⟨*proof*⟩

lemma *one-mod-2-pow-eq* [*simp*]:
1 mod (*2* ^ *n*) = *of-bool* (*n* > *0*)
⟨*proof*⟩

lemma *one-div-2-pow-eq* [*simp*]:
1 div (*2* ^ *n*) = *of-bool* (*n* = *0*)
⟨*proof*⟩

lemma *div-mult2-eq′*:
‹*a div* (*of-nat m* ∗ *of-nat n*) = *a div of-nat m div of-nat n*›
⟨*proof*⟩

lemma *mod-mult2-eq′*:
a mod (*of-nat m* ∗ *of-nat n*) = *of-nat m* ∗ (*a div of-nat m mod of-nat n*) + *a mod of-nat m*
⟨*proof*⟩

lemma *div-mult2-numeral-eq*:
a div numeral k div numeral l = *a div numeral* (*k* ∗ *l*) (**is** *?A* = *?B*)
⟨*proof*⟩

lemma *numeral-Bit0-div-2*:
numeral (*num.Bit0 n*) *div 2* = *numeral n*
⟨*proof*⟩

lemma *numeral-Bit1-div-2*:
numeral (*num.Bit1 n*) *div 2* = *numeral n*
⟨*proof*⟩

lemma *exp-mod-exp*:
‹$2 \mathbin{\hat{}} m \bmod 2 \mathbin{\hat{}} n = \textit{of-bool}\ (m < n) * 2 \mathbin{\hat{}} m$›
⟨*proof*⟩

lemma *mask-mod-exp*:
‹$(2 \mathbin{\hat{}} n - 1) \bmod 2 \mathbin{\hat{}} m = 2 \mathbin{\hat{}} \textit{min}\ m\ n - 1$›
⟨*proof*⟩

lemma *of-bool-half-eq-0* [*simp*]:
‹$\textit{of-bool}\ b\ \textit{div}\ 2 = 0$›
⟨*proof*⟩

end

class *unique-euclidean-ring-with-nat* = *ring* + *unique-euclidean-semiring-with-nat*

instance *nat* :: *unique-euclidean-semiring-with-nat*
⟨*proof*⟩

instance *int* :: *unique-euclidean-ring-with-nat*
⟨*proof*⟩

context *unique-euclidean-semiring-with-nat*
begin

subclass *semiring-parity*
⟨*proof*⟩

lemma *even-succ-div-two* [*simp*]:
$\textit{even}\ a \Longrightarrow (a + 1)\ \textit{div}\ 2 = a\ \textit{div}\ 2$
⟨*proof*⟩

lemma *odd-succ-div-two* [*simp*]:
$\textit{odd}\ a \Longrightarrow (a + 1)\ \textit{div}\ 2 = a\ \textit{div}\ 2 + 1$
⟨*proof*⟩

lemma *even-two-times-div-two*:
$\textit{even}\ a \Longrightarrow 2 * (a\ \textit{div}\ 2) = a$
⟨*proof*⟩

lemma *odd-two-times-div-two-succ* [*simp*]:
$\textit{odd}\ a \Longrightarrow 2 * (a\ \textit{div}\ 2) + 1 = a$
⟨*proof*⟩

lemma *coprime-left-2-iff-odd* [*simp*]:
$\textit{coprime}\ 2\ a \longleftrightarrow \textit{odd}\ a$
⟨*proof*⟩

lemma *coprime-right-2-iff-odd* [*simp*]:
coprime a 2 ⟷ *odd a*
⟨*proof*⟩

end

context *unique-euclidean-ring-with-nat*
begin

subclass *ring-parity* ⟨*proof*⟩

lemma *minus-1-mod-2-eq* [*simp*]:
− 1 mod 2 = 1
⟨*proof*⟩

lemma *minus-1-div-2-eq* [*simp*]:
− 1 div 2 = − 1
⟨*proof*⟩

end

context *unique-euclidean-semiring-with-nat*
begin

lemma *even-mask-div-iff′*:
‹*even* ((*2* ^ *m − 1*) *div 2* ^ *n*) ⟷ *m* ≤ *n*›
⟨*proof*⟩

end

57.6 Generic symbolic computations

The following type class contains everything necessary to formulate a division algorithm in ring structures with numerals, restricted to its positive segments.

class *unique-euclidean-semiring-with-nat-division* = *unique-euclidean-semiring-with-nat* +
fixes *divmod* :: ‹*num* ⇒ *num* ⇒ *'a* × *'a*›
and *divmod-step* :: ‹*'a* ⇒ *'a* × *'a* ⇒ *'a* × *'a*› — These are conceptually definitions but force generated code to be monomorphic wrt. particular instances of this class which yields a significant speedup.
assumes *divmod-def*: ‹*divmod m n* = (*numeral m div numeral n*, *numeral m mod numeral n*)›
and *divmod-step-def* [*simp*]: ‹*divmod-step l* (*q*, *r*) =
(*if euclidean-size l* ≤ *euclidean-size r then* (*2* ∗ *q* + *1*, *r* − *l*)
else (*2* ∗ *q*, *r*))› — This is a formulation of one step (referring to one digit position) in school-method division: compare the dividend at the current digit position with the remainder from previous division steps and evaluate accordingly.
begin

lemma *fst-divmod*:
‹*fst* (*divmod m n*) = *numeral m div numeral n*›
⟨*proof*⟩

lemma *snd-divmod*:
‹*snd* (*divmod m n*) = *numeral m mod numeral n*›
⟨*proof*⟩

Following a formulation of school-method division. If the divisor is smaller than the dividend, terminate. If not, shift the dividend to the right until termination occurs and then reiterate single division steps in the opposite direction.

lemma *divmod-divmod-step*:
‹*divmod m n* = (*if m* < *n then* (*0*, *numeral m*)
else divmod-step (*numeral n*) (*divmod m* (*Num.Bit0 n*)))›
⟨*proof*⟩

The division rewrite proper – first, trivial results involving *1*

lemma *divmod-trivial* [*simp*]:
divmod m Num.One = (*numeral m*, *0*)
divmod num.One (*num.Bit0 n*) = (*0*, *Numeral1*)
divmod num.One (*num.Bit1 n*) = (*0*, *Numeral1*)
⟨*proof*⟩

Division by an even number is a right-shift

lemma *divmod-cancel* [*simp*]:
‹*divmod* (*Num.Bit0 m*) (*Num.Bit0 n*) = (*case divmod m n of* (*q*, *r*) ⇒ (*q*, *2* ∗ *r*))› (**is** *?P*)
‹*divmod* (*Num.Bit1 m*) (*Num.Bit0 n*) = (*case divmod m n of* (*q*, *r*) ⇒ (*q*, *2* ∗ *r* + *1*))› (**is** *?Q*)
⟨*proof*⟩

The really hard work

lemma *divmod-steps* [*simp*]:
divmod (*num.Bit0 m*) (*num.Bit1 n*) =
(*if m* ≤ *n then* (*0*, *numeral* (*num.Bit0 m*))
else divmod-step (*numeral* (*num.Bit1 n*))
(*divmod* (*num.Bit0 m*)
(*num.Bit0* (*num.Bit1 n*))))
divmod (*num.Bit1 m*) (*num.Bit1 n*) =
(*if m* < *n then* (*0*, *numeral* (*num.Bit1 m*))
else divmod-step (*numeral* (*num.Bit1 n*))
(*divmod* (*num.Bit1 m*)
(*num.Bit0* (*num.Bit1 n*))))
⟨*proof*⟩

lemmas *divmod-algorithm-code* = *divmod-trivial divmod-cancel divmod-steps*

Special case: divisibility

definition *divides-aux* :: *'a* × *'a* ⇒ *bool*
where
divides-aux qr ⟷ *snd qr* = *0*

lemma *divides-aux-eq* [*simp*]:
divides-aux (*q*, *r*) ⟷ *r* = *0*
⟨*proof*⟩

lemma *dvd-numeral-simp* [*simp*]:
numeral m dvd numeral n ⟷ *divides-aux* (*divmod n m*)
⟨*proof*⟩

Generic computation of quotient and remainder

lemma *numeral-div-numeral* [*simp*]:
numeral k div numeral l = *fst* (*divmod k l*)
⟨*proof*⟩

lemma *numeral-mod-numeral* [*simp*]:
numeral k mod numeral l = *snd* (*divmod k l*)
⟨*proof*⟩

lemma *one-div-numeral* [*simp*]:
1 div numeral n = *fst* (*divmod num.One n*)
⟨*proof*⟩

lemma *one-mod-numeral* [*simp*]:
1 mod numeral n = *snd* (*divmod num.One n*)
⟨*proof*⟩

end

instantiation *nat* :: *unique-euclidean-semiring-with-nat-division*
begin

definition *divmod-nat* :: *num* ⇒ *num* ⇒ *nat* × *nat*
where
divmod'-nat-def: *divmod-nat m n* = (*numeral m div numeral n*, *numeral m mod numeral n*)

definition *divmod-step-nat* :: *nat* ⇒ *nat* × *nat* ⇒ *nat* × *nat*
where
divmod-step-nat l qr = (*let* (*q*, *r*) = *qr*
in if r ≥ *l then* (*2* ∗ *q* + *1*, *r* − *l*)
else (*2* ∗ *q*, *r*))

instance
⟨*proof*⟩

end

declare *divmod-algorithm-code* [**where** *?'a = nat*, *code*]

lemma *Suc-0-div-numeral* [*simp*]:
‹*Suc 0 div numeral Num.One = 1*›
‹*Suc 0 div numeral (Num.Bit0 n) = 0*›
‹*Suc 0 div numeral (Num.Bit1 n) = 0*›
⟨*proof*⟩

lemma *Suc-0-mod-numeral* [*simp*]:
‹*Suc 0 mod numeral Num.One = 0*›
‹*Suc 0 mod numeral (Num.Bit0 n) = 1*›
‹*Suc 0 mod numeral (Num.Bit1 n) = 1*›
⟨*proof*⟩

instantiation *int* :: *unique-euclidean-semiring-with-nat-division*
begin

definition *divmod-int* :: *num* ⇒ *num* ⇒ *int* × *int*
where
divmod-int m n = (*numeral m div numeral n*, *numeral m mod numeral n*)

definition *divmod-step-int* :: *int* ⇒ *int* × *int* ⇒ *int* × *int*
where
divmod-step-int l qr = (*let* (*q*, *r*) = *qr*
in if |*l*| ≤ |*r*| *then* (2 ∗ *q* + 1, *r* − *l*)
else (2 ∗ *q*, *r*))

instance
⟨*proof*⟩

end

declare *divmod-algorithm-code* [**where** *?'a = int*, *code*]

context
begin

qualified definition *adjust-div* :: *int* × *int* ⇒ *int*
where
adjust-div qr = (*let* (*q*, *r*) = *qr in q* + *of-bool* (*r* ≠ 0))

qualified lemma *adjust-div-eq* [*simp*, *code*]:
adjust-div (*q*, *r*) = *q* + *of-bool* (*r* ≠ 0)
⟨*proof*⟩ **definition** *adjust-mod* :: *num* ⇒ *int* ⇒ *int*
where
[*simp*]: *adjust-mod l r* = (*if r* = 0 *then* 0 *else numeral l* − *r*)

lemma *minus-numeral-div-numeral* [*simp*]:
− numeral m div numeral n = − (adjust-div (divmod m n) :: int)
⟨*proof*⟩

lemma *minus-numeral-mod-numeral* [*simp*]:
− numeral m mod numeral n = adjust-mod n (snd (divmod m n) :: int)
⟨*proof*⟩

lemma *numeral-div-minus-numeral* [*simp*]:
numeral m div − numeral n = − (adjust-div (divmod m n) :: int)
⟨*proof*⟩

lemma *numeral-mod-minus-numeral* [*simp*]:
numeral m mod − numeral n = − adjust-mod n (snd (divmod m n) :: int)
⟨*proof*⟩

lemma *minus-one-div-numeral* [*simp*]:
− 1 div numeral n = − (adjust-div (divmod Num.One n) :: int)
⟨*proof*⟩

lemma *minus-one-mod-numeral* [*simp*]:
− 1 mod numeral n = adjust-mod n (snd (divmod Num.One n) :: int)
⟨*proof*⟩

lemma *one-div-minus-numeral* [*simp*]:
1 div − numeral n = − (adjust-div (divmod Num.One n) :: int)
⟨*proof*⟩

lemma *one-mod-minus-numeral* [*simp*]:
1 mod − numeral n = − adjust-mod n (snd (divmod Num.One n) :: int)
⟨*proof*⟩

lemma [*code*]:
fixes *k :: int*
shows
k div 0 = 0
k mod 0 = k
0 div k = 0
0 mod k = 0
k div Int.Pos Num.One = k
k mod Int.Pos Num.One = 0
k div Int.Neg Num.One = − k
k mod Int.Neg Num.One = 0
Int.Pos m div Int.Pos n = (fst (divmod m n) :: int)
Int.Pos m mod Int.Pos n = (snd (divmod m n) :: int)
Int.Neg m div Int.Pos n = − (adjust-div (divmod m n) :: int)
Int.Neg m mod Int.Pos n = adjust-mod n (snd (divmod m n) :: int)
Int.Pos m div Int.Neg n = − (adjust-div (divmod m n) :: int)
Int.Pos m mod Int.Neg n = − adjust-mod n (snd (divmod m n) :: int)

Int.Neg m div Int.Neg n = (fst (divmod m n) :: int)
Int.Neg m mod Int.Neg n = − (snd (divmod m n) :: int)
⟨*proof*⟩

end

lemma *divmod-BitM-2-eq* [*simp*]:
‹*divmod (Num.BitM m) (Num.Bit0 Num.One) = (numeral m − 1, (1 :: int))*›
⟨*proof*⟩

57.6.1 Computation by simplification

lemma *euclidean-size-nat-less-eq-iff*:
‹*euclidean-size m ≤ euclidean-size n ⟷ m ≤ n*› **for** *m n :: nat*
⟨*proof*⟩

lemma *euclidean-size-int-less-eq-iff*:
‹*euclidean-size k ≤ euclidean-size l ⟷ |k| ≤ |l|*› **for** *k l :: int*
⟨*proof*⟩

⟨*ML*⟩

57.7 Computing congruences modulo $2 \mathbin{\widehat{}} q$

context *unique-euclidean-semiring-with-nat-division*
begin

lemma *cong-exp-iff-simps*:
numeral n mod numeral Num.One = 0
⟷ True
numeral (Num.Bit0 n) mod numeral (Num.Bit0 q) = 0
⟷ numeral n mod numeral q = 0
numeral (Num.Bit1 n) mod numeral (Num.Bit0 q) = 0
⟷ False
numeral m mod numeral Num.One = (numeral n mod numeral Num.One)
⟷ True
numeral Num.One mod numeral (Num.Bit0 q) = (numeral Num.One mod numeral (Num.Bit0 q))
⟷ True
numeral Num.One mod numeral (Num.Bit0 q) = (numeral (Num.Bit0 n) mod numeral (Num.Bit0 q))
⟷ False
numeral Num.One mod numeral (Num.Bit0 q) = (numeral (Num.Bit1 n) mod numeral (Num.Bit0 q))
⟷ (numeral n mod numeral q) = 0
numeral (Num.Bit0 m) mod numeral (Num.Bit0 q) = (numeral Num.One mod numeral (Num.Bit0 q))
⟷ False
numeral (Num.Bit0 m) mod numeral (Num.Bit0 q) = (numeral (Num.Bit0 n) mod numeral (Num.Bit0 q))

$\longleftrightarrow$ *numeral m mod numeral q = (numeral n mod numeral q)*
numeral (Num.Bit0 m) mod numeral (Num.Bit0 q) = (numeral (Num.Bit1 n) mod numeral (Num.Bit0 q))
$\longleftrightarrow$ *False*
numeral (Num.Bit1 m) mod numeral (Num.Bit0 q) = (numeral Num.One mod numeral (Num.Bit0 q))
$\longleftrightarrow$ *(numeral m mod numeral q) = 0*
numeral (Num.Bit1 m) mod numeral (Num.Bit0 q) = (numeral (Num.Bit0 n) mod numeral (Num.Bit0 q))
$\longleftrightarrow$ *False*
numeral (Num.Bit1 m) mod numeral (Num.Bit0 q) = (numeral (Num.Bit1 n) mod numeral (Num.Bit0 q))
$\longleftrightarrow$ *numeral m mod numeral q = (numeral n mod numeral q)*
⟨*proof*⟩

end

code-identifier
code-module *Parity* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

lemmas *even-of-nat = even-of-nat-iff*

end

58 Combination and Cancellation Simprocs for Numeral Expressions

theory *Numeral-Simprocs*
imports *Parity*
begin

⟨*ML*⟩

lemmas *semiring-norm =*
Let-def arith-simps diff-nat-numeral rel-simps
if-False if-True
add-Suc add-numeral-left
add-neg-numeral-left mult-numeral-left
numeral-One [*symmetric*] *uminus-numeral-One* [*symmetric*] *Suc-eq-plus1*
eq-numeral-iff-iszero not-iszero-Numeral1

For *combine-numerals*

lemma *left-add-mult-distrib*: *i*u + (j*u + k) = (i+j)*u + (k::nat)*
⟨*proof*⟩

For *cancel-numerals*

lemma *nat-diff-add-eq1*:

$j <= (i::nat) ==> ((i*u + m) - (j*u + n)) = (((i-j)*u + m) - n)$
⟨*proof*⟩

lemma *nat-diff-add-eq2*:
$i <= (j::nat) ==> ((i*u + m) - (j*u + n)) = (m - ((j-i)*u + n))$
⟨*proof*⟩

lemma *nat-eq-add-iff1*:
$j <= (i::nat) ==> (i*u + m = j*u + n) = ((i-j)*u + m = n)$
⟨*proof*⟩

lemma *nat-eq-add-iff2*:
$i <= (j::nat) ==> (i*u + m = j*u + n) = (m = (j-i)*u + n)$
⟨*proof*⟩

lemma *nat-less-add-iff1*:
$j <= (i::nat) ==> (i*u + m < j*u + n) = ((i-j)*u + m < n)$
⟨*proof*⟩

lemma *nat-less-add-iff2*:
$i <= (j::nat) ==> (i*u + m < j*u + n) = (m < (j-i)*u + n)$
⟨*proof*⟩

lemma *nat-le-add-iff1*:
$j <= (i::nat) ==> (i*u + m <= j*u + n) = ((i-j)*u + m <= n)$
⟨*proof*⟩

lemma *nat-le-add-iff2*:
$i <= (j::nat) ==> (i*u + m <= j*u + n) = (m <= (j-i)*u + n)$
⟨*proof*⟩

For *cancel-numeral-factors*

lemma *nat-mult-le-cancel1*: $(0::nat) < k ==> (k*m <= k*n) = (m<=n)$
⟨*proof*⟩

lemma *nat-mult-less-cancel1*: $(0::nat) < k ==> (k*m < k*n) = (m<n)$
⟨*proof*⟩

lemma *nat-mult-eq-cancel1*: $(0::nat) < k ==> (k*m = k*n) = (m=n)$
⟨*proof*⟩

lemma *nat-mult-div-cancel1*: $(0::nat) < k ==> (k*m)\ div\ (k*n) = (m\ div\ n)$
⟨*proof*⟩

lemma *nat-mult-dvd-cancel-disj*[*simp*]:
$(k*m)\ dvd\ (k*n) = (k=0 \vee m\ dvd\ (n::nat))$
⟨*proof*⟩

lemma *nat-mult-dvd-cancel1*: $0 < k \Longrightarrow (k*m)\ dvd\ (k*n::nat) = (m\ dvd\ n)$

⟨*proof*⟩

For *cancel-factor*

lemmas *nat-mult-le-cancel-disj* = *mult-le-cancel1*

lemmas *nat-mult-less-cancel-disj* = *mult-less-cancel1*

lemma *nat-mult-eq-cancel-disj*:
fixes *k m n* :: *nat*
shows $k * m = k * n \longleftrightarrow k = 0 \vee m = n$
⟨*proof*⟩

lemma *nat-mult-div-cancel-disj*:
fixes *k m n* :: *nat*
shows $(k * m)\ div\ (k * n) = (if\ k = 0\ then\ 0\ else\ m\ div\ n)$
⟨*proof*⟩

lemma *numeral-times-minus-swap*:
fixes *x*:: *'a*::*comm-ring-1* **shows** $numeral\ w * -x = x * -\ numeral\ w$
⟨*proof*⟩

⟨*ML*⟩

end

59 Semiring normalization

theory *Semiring-Normalization*
imports *Numeral-Simprocs*
begin

Prelude

class *comm-semiring-1-cancel-crossproduct* = *comm-semiring-1-cancel* +
assumes *crossproduct-eq*: $w * y + x * z = w * z + x * y \longleftrightarrow w = x \vee y = z$
begin

lemma *crossproduct-noteq*:
$a \neq b \wedge c \neq d \longleftrightarrow a * c + b * d \neq a * d + b * c$
⟨*proof*⟩

lemma *add-scale-eq-noteq*:
$r \neq 0 \Longrightarrow a = b \wedge c \neq d \Longrightarrow a + r * c \neq b + r * d$
⟨*proof*⟩

lemma *add-0-iff*:
$b = b + a \longleftrightarrow a = 0$
⟨*proof*⟩

end

subclass (**in** *idom*) *comm-semiring-1-cancel-crossproduct*
⟨*proof*⟩

instance *nat* :: *comm-semiring-1-cancel-crossproduct*
⟨*proof*⟩

Semiring normalization proper

⟨*ML*⟩

context *comm-semiring-1*
begin

lemma *semiring-normalization-rules* [*no-atp*]:
$(a * m) + (b * m) = (a + b) * m$
$(a * m) + m = (a + 1) * m$
$m + (a * m) = (a + 1) * m$
$m + m = (1 + 1) * m$
$0 + a = a$
$a + 0 = a$
$a * b = b * a$
$(a + b) * c = (a * c) + (b * c)$
$0 * a = 0$
$a * 0 = 0$
$1 * a = a$
$a * 1 = a$
$(lx * ly) * (rx * ry) = (lx * rx) * (ly * ry)$
$(lx * ly) * (rx * ry) = lx * (ly * (rx * ry))$
$(lx * ly) * (rx * ry) = rx * ((lx * ly) * ry)$
$(lx * ly) * rx = (lx * rx) * ly$
$(lx * ly) * rx = lx * (ly * rx)$
$lx * (rx * ry) = (lx * rx) * ry$
$lx * (rx * ry) = rx * (lx * ry)$
$(a + b) + (c + d) = (a + c) + (b + d)$
$(a + b) + c = a + (b + c)$
$a + (c + d) = c + (a + d)$
$(a + b) + c = (a + c) + b$
$a + c = c + a$
$a + (c + d) = (a + c) + d$
$(x \hat{} p) * (x \hat{} q) = x \hat{} (p + q)$
$x * (x \hat{} q) = x \hat{} (Suc\ q)$
$(x \hat{} q) * x = x \hat{} (Suc\ q)$
$x * x = x^2$
$(x * y) \hat{} q = (x \hat{} q) * (y \hat{} q)$
$(x \hat{} p) \hat{} q = x \hat{} (p * q)$
$x \hat{} 0 = 1$
$x \hat{} 1 = x$
$x * (y + z) = (x * y) + (x * z)$

$x \mathbin{\hat{}} (Suc\ q) = x * (x \mathbin{\hat{}} q)$
$x \mathbin{\hat{}} (2{*}n) = (x \mathbin{\hat{}} n) * (x \mathbin{\hat{}} n)$
⟨*proof*⟩

⟨*ML*⟩

end

context *comm-ring-1*
begin

lemma *ring-normalization-rules* [*no-atp*]:
$- x = (- 1) * x$
$x - y = x + (- y)$
⟨*proof*⟩

⟨*ML*⟩

end

context *comm-semiring-1-cancel-crossproduct*
begin

⟨*ML*⟩

end

context *idom*
begin

⟨*ML*⟩

end

context *field*
begin

⟨*ML*⟩

end

code-identifier
code-module *Semiring-Normalization* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

end

60 Groebner bases

theory *Groebner-Basis*
imports *Semiring-Normalization Parity*
begin

60.1 Groebner Bases

lemmas *bool-simps* = *simp-thms*(*1*−*34*) — FIXME move to *HOL.HOL*

lemma *nnf-simps*: — FIXME shadows fact binding in *HOL.HOL*
$(\neg(P \wedge Q)) = (\neg P \vee \neg Q)$ $(\neg(P \vee Q)) = (\neg P \wedge \neg Q)$
$(P \longrightarrow Q) = (\neg P \vee Q)$
$(P = Q) = ((P \wedge Q) \vee (\neg P \wedge \neg Q))$ $(\neg \neg(P)) = P$
⟨*proof*⟩

lemma *dnf*:
$(P \wedge (Q \vee R)) = ((P \wedge Q) \vee (P \wedge R))$
$((Q \vee R) \wedge P) = ((Q \wedge P) \vee (R \wedge P))$
$(P \wedge Q) = (Q \wedge P)$
$(P \vee Q) = (Q \vee P)$
⟨*proof*⟩

lemmas *weak-dnf-simps* = *dnf bool-simps*

lemma *PFalse*:
$P \equiv False \Longrightarrow \neg P$
$\neg P \Longrightarrow (P \equiv False)$
⟨*proof*⟩

named-theorems *algebra pre*−*simplification rules for algebraic methods*
⟨*ML*⟩

declare *dvd-def*[*algebra*]
declare *mod-eq-0-iff-dvd*[*algebra*]
declare *mod-div-trivial*[*algebra*]
declare *mod-mod-trivial*[*algebra*]
declare *div-by-0*[*algebra*]
declare *mod-by-0*[*algebra*]
declare *mult-div-mod-eq*[*algebra*]
declare *div-minus-minus*[*algebra*]
declare *mod-minus-minus*[*algebra*]
declare *div-minus-right*[*algebra*]
declare *mod-minus-right*[*algebra*]
declare *div-0*[*algebra*]
declare *mod-0*[*algebra*]
declare *mod-by-1*[*algebra*]
declare *div-by-1*[*algebra*]
declare *mod-minus1-right*[*algebra*]
declare *div-minus1-right*[*algebra*]

declare *mod-mult-self2-is-0*[*algebra*]
declare *mod-mult-self1-is-0*[*algebra*]

lemma *zmod-eq-0-iff* [*algebra*]:
‹*m mod d* = *0* ⟷ (∃ *q*. *m* = *d* ∗ *q*)› **for** *m d* :: *int*
⟨*proof*⟩

declare *dvd-0-left-iff*[*algebra*]
declare *zdvd1-eq*[*algebra*]
declare *mod-eq-dvd-iff*[*algebra*]
declare *nat-mod-eq-iff*[*algebra*]

context *semiring-parity*
begin

declare *even-mult-iff* [*algebra*]
declare *even-power* [*algebra*]

end

context *ring-parity*
begin

declare *even-minus* [*algebra*]

end

declare *even-Suc* [*algebra*]
declare *even-diff-nat* [*algebra*]

end

61 Set intervals

theory *Set-Interval*
imports *Parity*
begin

lemma *card-2-iff*: *card S* = *2* ⟷ (∃ *x y*. *S* = {*x*,*y*} ∧ *x* ≠ *y*)
⟨*proof*⟩

lemma *card-2-iff'*: *card S* = *2* ⟷ (∃ *x*∈*S*. ∃ *y*∈*S*. *x* ≠ *y* ∧ (∀ *z*∈*S*. *z* = *x* ∨ *z* = *y*))
⟨*proof*⟩

lemma *card-3-iff*: *card S* = *3* ⟷ (∃ *x y z*. *S* = {*x*,*y*,*z*} ∧ *x* ≠ *y* ∧ *y* ≠ *z* ∧ *x* ≠ *z*)
⟨*proof*⟩

context *ord*
begin

definition
lessThan :: *'a => 'a set* ((*1*{..<-})) **where**
{..<*u*} == {*x*. *x* < *u*}

definition
atMost :: *'a => 'a set* ((*1*{..-})) **where**
{..*u*} == {*x*. *x* ≤ *u*}

definition
greaterThan :: *'a => 'a set* ((*1*{-<..})) **where**
{*l*<..} == {*x*. *l*<*x*}

definition
atLeast :: *'a => 'a set* ((*1*{-..})) **where**
{*l*..} == {*x*. *l*≤*x*}

definition
greaterThanLessThan :: *'a => 'a => 'a set* ((*1*{-<..<-})) **where**
{*l*<..<*u*} == {*l*<..} *Int* {..<*u*}

definition
atLeastLessThan :: *'a => 'a => 'a set* ((*1*{-..<-})) **where**
{*l*..<*u*} == {*l*..} *Int* {..<*u*}

definition
greaterThanAtMost :: *'a => 'a => 'a set* ((*1*{-<..-})) **where**
{*l*<..*u*} == {*l*<..} *Int* {..*u*}

definition
atLeastAtMost :: *'a => 'a => 'a set* ((*1*{-..-})) **where**
{*l*..*u*} == {*l*..} *Int* {..*u*}

end

A note of warning when using {..<*n*} on type *nat*: it is equivalent to {*0*..<*n*} but some lemmas involving {*m*..<*n*} may not exist in {..<*n*}-form as well.

syntax (*ASCII*)
-UNION-le :: *'a => 'a => 'b set => 'b set* ((*3UN -<=-./ -*) [*0*, *0*, *10*] *10*)
-UNION-less :: *'a => 'a => 'b set => 'b set* ((*3UN -<-./ -*) [*0*, *0*, *10*] *10*)
-INTER-le :: *'a => 'a => 'b set => 'b set* ((*3INT -<=-./ -*) [*0*, *0*, *10*] *10*)
-INTER-less :: *'a => 'a => 'b set => 'b set* ((*3INT -<-./ -*) [*0*, *0*, *10*] *10*)

syntax (*latex* **output**)

-UNION-le :: *'a* ⇒ *'a* => *'b set* => *'b set* ((*3*⋃(‹*unbreakable*›- ≤ -)/ -) [*0*, *0*, *10*] *10*)
-UNION-less :: *'a* ⇒ *'a* => *'b set* => *'b set* ((*3*⋃(‹*unbreakable*›- < -)/ -) [*0*, *0*, *10*] *10*)
-INTER-le :: *'a* ⇒ *'a* => *'b set* => *'b set* ((*3*⋂(‹*unbreakable*›- ≤ -)/ -) [*0*, *0*, *10*] *10*)
-INTER-less :: *'a* ⇒ *'a* => *'b set* => *'b set* ((*3*⋂(‹*unbreakable*›- < -)/ -) [*0*, *0*, *10*] *10*)

syntax
-UNION-le :: *'a* => *'a* => *'b set* => *'b set* ((*3*⋃-≤-./ -) [*0*, *0*, *10*] *10*)
-UNION-less :: *'a* => *'a* => *'b set* => *'b set* ((*3*⋃-<-./ -) [*0*, *0*, *10*] *10*)
-INTER-le :: *'a* => *'a* => *'b set* => *'b set* ((*3*⋂-≤-./ -) [*0*, *0*, *10*] *10*)
-INTER-less :: *'a* => *'a* => *'b set* => *'b set* ((*3*⋂-<-./ -) [*0*, *0*, *10*] *10*)

translations
⋃*i*≤*n*. *A* ⇌ ⋃*i*∈{..*n*}. *A*
⋃*i*<*n*. *A* ⇌ ⋃*i*∈{..<*n*}. *A*
⋂*i*≤*n*. *A* ⇌ ⋂*i*∈{..*n*}. *A*
⋂*i*<*n*. *A* ⇌ ⋂*i*∈{..<*n*}. *A*

61.1 Various equivalences

lemma (**in** *ord*) *lessThan-iff* [*iff*]: (*i* ∈ *lessThan k*) = (*i*<*k*)
⟨*proof*⟩

lemma *Compl-lessThan* [*simp*]:
!!*k*:: *'a*::*linorder*. −*lessThan k* = *atLeast k*
⟨*proof*⟩

lemma *single-Diff-lessThan* [*simp*]: !!*k*:: *'a*::*preorder*. {*k*} − *lessThan k* = {*k*}
⟨*proof*⟩

lemma (**in** *ord*) *greaterThan-iff* [*iff*]: (*i* ∈ *greaterThan k*) = (*k*<*i*)
⟨*proof*⟩

lemma *Compl-greaterThan* [*simp*]:
!!*k*:: *'a*::*linorder*. −*greaterThan k* = *atMost k*
⟨*proof*⟩

lemma *Compl-atMost* [*simp*]: !!*k*:: *'a*::*linorder*. −*atMost k* = *greaterThan k*
⟨*proof*⟩

lemma (**in** *ord*) *atLeast-iff* [*iff*]: (*i* ∈ *atLeast k*) = (*k*<=*i*)
⟨*proof*⟩

lemma *Compl-atLeast* [*simp*]: !!*k*:: *'a*::*linorder*. −*atLeast k* = *lessThan k*
⟨*proof*⟩

lemma (**in** *ord*) *atMost-iff* [*iff*]: $(i \in atMost\ k) = (i <= k)$
⟨*proof*⟩

lemma *atMost-Int-atLeast*: !!n:: $'a$::*order*. $atMost\ n\ Int\ atLeast\ n = \{n\}$
⟨*proof*⟩

lemma (**in** *linorder*) *lessThan-Int-lessThan*: $\{\ a <..\} \cap \{\ b <..\} = \{\ max\ a\ b <..\}$
⟨*proof*⟩

lemma (**in** *linorder*) *greaterThan-Int-greaterThan*: $\{..< a\} \cap \{..< b\} = \{..< min\ a\ b\}$
⟨*proof*⟩

61.2 Logical Equivalences for Set Inclusion and Equality

lemma *atLeast-empty-triv* [*simp*]: $\{\{\}..\} = UNIV$
⟨*proof*⟩

lemma *atMost-UNIV-triv* [*simp*]: $\{..UNIV\} = UNIV$
⟨*proof*⟩

lemma *atLeast-subset-iff* [*iff*]:
$(atLeast\ x \subseteq atLeast\ y) = (y \leq (x::'a::preorder))$
⟨*proof*⟩

lemma *atLeast-eq-iff* [*iff*]:
$(atLeast\ x = atLeast\ y) = (x = (y::'a::order))$
⟨*proof*⟩

lemma *greaterThan-subset-iff* [*iff*]:
$(greaterThan\ x \subseteq greaterThan\ y) = (y \leq (x::'a::linorder))$
⟨*proof*⟩

lemma *greaterThan-eq-iff* [*iff*]:
$(greaterThan\ x = greaterThan\ y) = (x = (y::'a::linorder))$
⟨*proof*⟩

lemma *atMost-subset-iff* [*iff*]: $(atMost\ x \subseteq atMost\ y) = (x \leq (y::'a::preorder))$
⟨*proof*⟩

lemma *atMost-eq-iff* [*iff*]: $(atMost\ x = atMost\ y) = (x = (y::'a::order))$
⟨*proof*⟩

lemma *lessThan-subset-iff* [*iff*]:
$(lessThan\ x \subseteq lessThan\ y) = (x \leq (y::'a::linorder))$
⟨*proof*⟩

lemma *lessThan-eq-iff* [*iff*]:
$(lessThan\ x = lessThan\ y) = (x = (y::'a::linorder))$

⟨*proof*⟩

lemma *lessThan-strict-subset-iff*:
fixes $m\ n :: 'a::linorder$
shows $\{..<m\} < \{..<n\} \longleftrightarrow m < n$
⟨*proof*⟩

lemma (**in** *linorder*) *Ici-subset-Ioi-iff*: $\{a\ ..\} \subseteq \{b\ <..\} \longleftrightarrow b < a$
⟨*proof*⟩

lemma (**in** *linorder*) *Iic-subset-Iio-iff*: $\{..\ a\} \subseteq \{..<\ b\} \longleftrightarrow a < b$
⟨*proof*⟩

lemma (**in** *preorder*) *Ioi-le-Ico*: $\{a\ <..\} \subseteq \{a\ ..\}$
⟨*proof*⟩

61.3 Two-sided intervals

context *ord*
begin

lemma *greaterThanLessThan-iff* [*simp*]: $(i \in \{l<..<u\}) = (l < i \wedge i < u)$
⟨*proof*⟩

lemma *atLeastLessThan-iff* [*simp*]: $(i \in \{l..<u\}) = (l \leq i \wedge i < u)$
⟨*proof*⟩

lemma *greaterThanAtMost-iff* [*simp*]: $(i \in \{l<..u\}) = (l < i \wedge i \leq u)$
⟨*proof*⟩

lemma *atLeastAtMost-iff* [*simp*]: $(i \in \{l..u\}) = (l \leq i \wedge i \leq u)$
⟨*proof*⟩

The above four lemmas could be declared as iffs. Unfortunately this breaks many proofs. Since it only helps blast, it is better to leave them alone.

lemma *greaterThanLessThan-eq*: $\{\ a\ <..<\ b\} = \{\ a\ <..\} \cap \{..<\ b\ \}$
⟨*proof*⟩

lemma (**in** *order*) *atLeastLessThan-eq-atLeastAtMost-diff*:
$\{a..<b\} = \{a..b\} - \{b\}$
⟨*proof*⟩

lemma (**in** *order*) *greaterThanAtMost-eq-atLeastAtMost-diff*:
$\{a<..b\} = \{a..b\} - \{a\}$
⟨*proof*⟩

end

61.3.1 Emptyness, singletons, subset

context *preorder*
begin

lemma *atLeastatMost-empty-iff*[*simp*]:
$\{a..b\} = \{\} \longleftrightarrow (\neg\ a \leq b)$
⟨*proof*⟩

lemma *atLeastatMost-empty-iff2*[*simp*]:
$\{\} = \{a..b\} \longleftrightarrow (\neg\ a \leq b)$
⟨*proof*⟩

lemma *atLeastLessThan-empty-iff*[*simp*]:
$\{a..<b\} = \{\} \longleftrightarrow (\neg\ a < b)$
⟨*proof*⟩

lemma *atLeastLessThan-empty-iff2*[*simp*]:
$\{\} = \{a..<b\} \longleftrightarrow (\neg\ a < b)$
⟨*proof*⟩

lemma *greaterThanAtMost-empty-iff*[*simp*]: $\{k<..l\} = \{\} \longleftrightarrow \neg\ k < l$
⟨*proof*⟩

lemma *greaterThanAtMost-empty-iff2*[*simp*]: $\{\} = \{k<..l\} \longleftrightarrow \neg\ k < l$
⟨*proof*⟩

lemma *atLeastatMost-subset-iff*[*simp*]:
$\{a..b\} \leq \{c..d\} \longleftrightarrow (\neg\ a \leq b) \vee c \leq a \wedge b \leq d$
⟨*proof*⟩

lemma *atLeastatMost-psubset-iff*:
$\{a..b\} < \{c..d\} \longleftrightarrow$
$((\neg\ a \leq b) \vee c \leq a \wedge b \leq d \wedge (c < a \vee b < d)) \wedge c \leq d$
⟨*proof*⟩

lemma *atLeastAtMost-subseteq-atLeastLessThan-iff*:
$\{a..b\} \subseteq \{c\ ..<\ d\} \longleftrightarrow (a \leq b \longrightarrow c \leq a \wedge b < d)$
⟨*proof*⟩

lemma *Icc-subset-Ici-iff*[*simp*]:
$\{l..h\} \subseteq \{l'..\} = (\neg\ l{\leq}h \vee l{\geq}l')$
⟨*proof*⟩

lemma *Icc-subset-Iic-iff*[*simp*]:
$\{l..h\} \subseteq \{..h'\} = (\neg\ l{\leq}h \vee h{\leq}h')$
⟨*proof*⟩

lemma *not-Ici-eq-empty*[*simp*]: $\{l..\} \neq \{\}$
⟨*proof*⟩

lemma *not-Iic-eq-empty*[*simp*]: $\{..h\} \neq \{\}$
⟨*proof*⟩

lemmas *not-empty-eq-Ici-eq-empty*[*simp*] = *not-Ici-eq-empty*[*symmetric*]
lemmas *not-empty-eq-Iic-eq-empty*[*simp*] = *not-Iic-eq-empty*[*symmetric*]

end

context *order*
begin

lemma *atLeastatMost-empty*[*simp*]: $b < a \Longrightarrow \{a..b\} = \{\}$
and *atLeastatMost-empty'*[*simp*]: $\neg\ a \leq b \Longrightarrow \{a..b\} = \{\}$
⟨*proof*⟩

lemma *atLeastLessThan-empty*[*simp*]:
$b \leq a \Longrightarrow \{a..<b\} = \{\}$
⟨*proof*⟩

lemma *greaterThanAtMost-empty*[*simp*]: $l \leq k \Longrightarrow \{k<..l\} = \{\}$
⟨*proof*⟩

lemma *greaterThanLessThan-empty*[*simp*]:$l \leq k \Longrightarrow \{k<..<l\} = \{\}$
⟨*proof*⟩

lemma *atLeastAtMost-singleton* [*simp*]: $\{a..a\} = \{a\}$
⟨*proof*⟩

lemma *atLeastAtMost-singleton'*: $a = b \Longrightarrow \{a\ ..\ b\} = \{a\}$ ⟨*proof*⟩

lemma *Icc-eq-Icc*[*simp*]:
$\{l..h\} = \{l'..h'\} = (l{=}l' \wedge h{=}h' \vee \neg\ l{\leq}h \wedge \neg\ l'{\leq}h')$
⟨*proof*⟩

lemma (**in** *linorder*) *Ico-eq-Ico*:
$\{l..<h\} = \{l'..<h'\} = (l{=}l' \wedge h{=}h' \vee \neg\ l{<}h \wedge \neg\ l'{<}h')$
⟨*proof*⟩

lemma *atLeastAtMost-singleton-iff*[*simp*]:
$\{a\ ..\ b\} = \{c\} \longleftrightarrow a = b \wedge b = c$
⟨*proof*⟩

end

context *no-top*
begin

lemma *not-UNIV-le-Icc*[*simp*]: ¬ *UNIV* ⊆ {*l*..*h*}
⟨*proof*⟩

lemma *not-UNIV-le-Iic*[*simp*]: ¬ *UNIV* ⊆ {..*h*}
⟨*proof*⟩

lemma *not-Ici-le-Icc*[*simp*]: ¬ {*l*..} ⊆ {*l'*..*h'*}
⟨*proof*⟩

lemma *not-Ici-le-Iic*[*simp*]: ¬ {*l*..} ⊆ {..*h'*}
⟨*proof*⟩

end

context *no-bot*
begin

lemma *not-UNIV-le-Ici*[*simp*]: ¬ *UNIV* ⊆ {*l*..}
⟨*proof*⟩

lemma *not-Iic-le-Icc*[*simp*]: ¬ {..*h*} ⊆ {*l'*..*h'*}
⟨*proof*⟩

lemma *not-Iic-le-Ici*[*simp*]: ¬ {..*h*} ⊆ {*l'*..}
⟨*proof*⟩

end

context *no-top*
begin

lemma *not-UNIV-eq-Icc*[*simp*]: ¬ *UNIV* = {*l'*..*h'*}
⟨*proof*⟩

lemmas *not-Icc-eq-UNIV*[*simp*] = *not-UNIV-eq-Icc*[*symmetric*]

lemma *not-UNIV-eq-Iic*[*simp*]: ¬ *UNIV* = {..*h'*}
⟨*proof*⟩

lemmas *not-Iic-eq-UNIV*[*simp*] = *not-UNIV-eq-Iic*[*symmetric*]

lemma *not-Icc-eq-Ici*[*simp*]: ¬ {*l*..*h*} = {*l'*..}
⟨*proof*⟩

lemmas *not-Ici-eq-Icc*[*simp*] = *not-Icc-eq-Ici*[*symmetric*]

lemma *not-Iic-eq-Ici*[*simp*]: $\neg\ \{..h\} = \{l'..\}$
⟨*proof*⟩

lemmas *not-Ici-eq-Iic*[*simp*] = *not-Iic-eq-Ici*[*symmetric*]

end

context *no-bot*
begin

lemma *not-UNIV-eq-Ici*[*simp*]: $\neg\ UNIV = \{l'..\}$
⟨*proof*⟩

lemmas *not-Ici-eq-UNIV*[*simp*] = *not-UNIV-eq-Ici*[*symmetric*]

lemma *not-Icc-eq-Iic*[*simp*]: $\neg\ \{l..h\} = \{..h'\}$
⟨*proof*⟩

lemmas *not-Iic-eq-Icc*[*simp*] = *not-Icc-eq-Iic*[*symmetric*]

end

context *dense-linorder*
begin

lemma *greaterThanLessThan-empty-iff*[*simp*]:
$\{\ a <..< b\ \} = \{\} \longleftrightarrow b \leq a$
⟨*proof*⟩

lemma *greaterThanLessThan-empty-iff2*[*simp*]:
$\{\} = \{\ a <..< b\ \} \longleftrightarrow b \leq a$
⟨*proof*⟩

lemma *atLeastLessThan-subseteq-atLeastAtMost-iff*:
$\{a ..< b\} \subseteq \{\ c .. d\ \} \longleftrightarrow (a < b \longrightarrow c \leq a \wedge b \leq d)$
⟨*proof*⟩

lemma *greaterThanAtMost-subseteq-atLeastAtMost-iff*:
$\{a <.. b\} \subseteq \{\ c .. d\ \} \longleftrightarrow (a < b \longrightarrow c \leq a \wedge b \leq d)$
⟨*proof*⟩

lemma *greaterThanLessThan-subseteq-atLeastAtMost-iff*:
$\{a <..< b\} \subseteq \{\ c .. d\ \} \longleftrightarrow (a < b \longrightarrow c \leq a \wedge b \leq d)$
⟨*proof*⟩

lemma *greaterThanLessThan-subseteq-greaterThanLessThan*:
$\{a <..< b\} \subseteq \{c <..< d\} \longleftrightarrow (a < b \longrightarrow a \geq c \wedge b \leq d)$
⟨*proof*⟩

lemma *greaterThanAtMost-subseteq-atLeastLessThan-iff*:
$\{a <.. b\} \subseteq \{\ c ..< d\ \} \longleftrightarrow (a < b \longrightarrow c \leq a \wedge b < d)$
⟨*proof*⟩

lemma *greaterThanLessThan-subseteq-atLeastLessThan-iff*:
$\{a <..< b\} \subseteq \{\ c ..< d\ \} \longleftrightarrow (a < b \longrightarrow c \leq a \wedge b \leq d)$
⟨*proof*⟩

lemma *greaterThanLessThan-subseteq-greaterThanAtMost-iff*:
$\{a <..< b\} \subseteq \{\ c <.. d\ \} \longleftrightarrow (a < b \longrightarrow c \leq a \wedge b \leq d)$
⟨*proof*⟩

end

context *no-top*
begin

lemma *greaterThan-non-empty*[*simp*]: $\{x <..\} \neq \{\}$
⟨*proof*⟩

end

context *no-bot*
begin

lemma *lessThan-non-empty*[*simp*]: $\{..< x\} \neq \{\}$
⟨*proof*⟩

end

lemma (**in** *linorder*) *atLeastLessThan-subset-iff*:
$\{a..<b\} \subseteq \{c..<d\} \Longrightarrow b \leq a \vee c \leq a \wedge b \leq d$
⟨*proof*⟩

lemma *atLeastLessThan-inj*:
fixes $a\ b\ c\ d :: 'a::linorder$
assumes *eq*: $\{a ..< b\} = \{c ..< d\}$ **and** $a < b\ c < d$
shows $a = c\ b = d$
⟨*proof*⟩

lemma *atLeastLessThan-eq-iff*:
fixes $a\ b\ c\ d :: 'a::linorder$
assumes $a < b\ c < d$
shows $\{a ..< b\} = \{c ..< d\} \longleftrightarrow a = c \wedge b = d$
⟨*proof*⟩

lemma (**in** *linorder*) *Ioc-inj*:
‹$\{a <.. b\} = \{c <.. d\} \longleftrightarrow (b \leq a \wedge d \leq c) \vee a = c \wedge b = d$› (**is** ‹*?P* ⟷ *?Q*›)

⟨*proof*⟩

lemma (**in** *order*) *Iio-Int-singleton*: $\{..<k\} \cap \{x\} = (\mathit{if}\ x < k\ \mathit{then}\ \{x\}\ \mathit{else}\ \{\})$
⟨*proof*⟩

lemma (**in** *linorder*) *Ioc-subset-iff*: $\{a<..b\} \subseteq \{c<..d\} \longleftrightarrow (b \leq a \vee c \leq a \wedge b \leq d)$
⟨*proof*⟩

lemma (**in** *order-bot*) *atLeast-eq-UNIV-iff*: $\{x..\} = \mathit{UNIV} \longleftrightarrow x = \mathit{bot}$
⟨*proof*⟩

lemma (**in** *order-top*) *atMost-eq-UNIV-iff*: $\{..x\} = \mathit{UNIV} \longleftrightarrow x = \mathit{top}$
⟨*proof*⟩

lemma (**in** *bounded-lattice*) *atLeastAtMost-eq-UNIV-iff*:
$\{x..y\} = \mathit{UNIV} \longleftrightarrow (x = \mathit{bot} \wedge y = \mathit{top})$
⟨*proof*⟩

lemma *Iio-eq-empty-iff*: $\{..< n::'a::\{\mathit{linorder}, \mathit{order\text{-}bot}\}\} = \{\} \longleftrightarrow n = \mathit{bot}$
⟨*proof*⟩

lemma *lessThan-empty-iff*: $\{..< n::\mathit{nat}\} = \{\} \longleftrightarrow n = 0$
⟨*proof*⟩

lemma *mono-image-least*:
assumes *f-mono*: *mono f* **and** *f-img*: $f\ `\ \{m\ ..<\ n\} = \{m'\ ..<\ n'\}$ $m < n$
shows $f\ m = m'$
⟨*proof*⟩

61.4 Infinite intervals

context *dense-linorder*
begin

lemma *infinite-Ioo*:
assumes $a < b$
shows ¬ *finite* $\{a<..<b\}$
⟨*proof*⟩

lemma *infinite-Icc*: $a < b \Longrightarrow$ ¬ *finite* $\{a\ ..\ b\}$
⟨*proof*⟩

lemma *infinite-Ico*: $a < b \Longrightarrow$ ¬ *finite* $\{a\ ..<\ b\}$
⟨*proof*⟩

lemma *infinite-Ioc*: $a < b \Longrightarrow$ ¬ *finite* $\{a\ <..\ b\}$
⟨*proof*⟩

lemma *infinite-Ioo-iff* [*simp*]: *infinite* $\{a<..<b\} \longleftrightarrow a < b$
⟨*proof*⟩

lemma *infinite-Icc-iff* [*simp*]: *infinite* $\{a\ ..\ b\} \longleftrightarrow a < b$
⟨*proof*⟩

lemma *infinite-Ico-iff* [*simp*]: *infinite* $\{a..<b\} \longleftrightarrow a < b$
⟨*proof*⟩

lemma *infinite-Ioc-iff* [*simp*]: *infinite* $\{a<..b\} \longleftrightarrow a < b$
⟨*proof*⟩

end

lemma *infinite-Iio*: ¬ *finite* $\{..<\ a :: 'a :: \{no\text{-}bot,\ linorder\}\}$
⟨*proof*⟩

lemma *infinite-Iic*: ¬ *finite* $\{..\ a :: 'a :: \{no\text{-}bot,\ linorder\}\}$
⟨*proof*⟩

lemma *infinite-Ioi*: ¬ *finite* $\{a :: 'a :: \{no\text{-}top,\ linorder\}\ <..\}$
⟨*proof*⟩

lemma *infinite-Ici*: ¬ *finite* $\{a :: 'a :: \{no\text{-}top,\ linorder\}\ ..\}$
⟨*proof*⟩

61.4.1 Intersection

context *linorder*
begin

lemma *Int-atLeastAtMost*[*simp*]: $\{a..b\}\ Int\ \{c..d\} = \{max\ a\ c\ ..\ min\ b\ d\}$
⟨*proof*⟩

lemma *Int-atLeastAtMostR1*[*simp*]: $\{..b\}\ Int\ \{c..d\} = \{c\ ..\ min\ b\ d\}$
⟨*proof*⟩

lemma *Int-atLeastAtMostR2*[*simp*]: $\{a..\}\ Int\ \{c..d\} = \{max\ a\ c\ ..\ d\}$
⟨*proof*⟩

lemma *Int-atLeastAtMostL1*[*simp*]: $\{a..b\}\ Int\ \{..d\} = \{a\ ..\ min\ b\ d\}$
⟨*proof*⟩

lemma *Int-atLeastAtMostL2*[*simp*]: $\{a..b\}\ Int\ \{c..\} = \{max\ a\ c\ ..\ b\}$
⟨*proof*⟩

lemma *Int-atLeastLessThan*[*simp*]: $\{a..<b\}\ Int\ \{c..<d\} = \{max\ a\ c\ ..<\ min\ b\ d\}$
⟨*proof*⟩

lemma *Int-greaterThanAtMost*[*simp*]: $\{a<..b\}$ *Int* $\{c<..d\} = \{max\ a\ c <..\ min\ b\ d\}$
⟨*proof*⟩

lemma *Int-greaterThanLessThan*[*simp*]: $\{a<..<b\}$ *Int* $\{c<..<d\} = \{max\ a\ c <..< min\ b\ d\}$
⟨*proof*⟩

lemma *Int-atMost*[*simp*]: $\{..a\} \cap \{..b\} = \{..\ min\ a\ b\}$
⟨*proof*⟩

lemma *Ioc-disjoint*: $\{a<..b\} \cap \{c<..d\} = \{\} \longleftrightarrow b \leq a \vee d \leq c \vee b \leq c \vee d \leq a$
⟨*proof*⟩

end

context *complete-lattice*
begin

lemma
shows *Sup-atLeast*[*simp*]: $Sup\ \{x\ ..\} = top$
and *Sup-greaterThanAtLeast*[*simp*]: $x < top \Longrightarrow Sup\ \{x <..\} = top$
and *Sup-atMost*[*simp*]: $Sup\ \{..\ y\} = y$
and *Sup-atLeastAtMost*[*simp*]: $x \leq y \Longrightarrow Sup\ \{\ x\ ..\ y\} = y$
and *Sup-greaterThanAtMost*[*simp*]: $x < y \Longrightarrow Sup\ \{\ x <..\ y\} = y$
⟨*proof*⟩

lemma
shows *Inf-atMost*[*simp*]: $Inf\ \{..\ x\} = bot$
and *Inf-atMostLessThan*[*simp*]: $top < x \Longrightarrow Inf\ \{..< x\} = bot$
and *Inf-atLeast*[*simp*]: $Inf\ \{x\ ..\} = x$
and *Inf-atLeastAtMost*[*simp*]: $x \leq y \Longrightarrow Inf\ \{\ x\ ..\ y\} = x$
and *Inf-atLeastLessThan*[*simp*]: $x < y \Longrightarrow Inf\ \{\ x\ ..< y\} = x$
⟨*proof*⟩

end

lemma
fixes $x\ y :: 'a :: \{complete\text{-}lattice,\ dense\text{-}linorder\}$
shows *Sup-lessThan*[*simp*]: $Sup\ \{..< y\} = y$
and *Sup-atLeastLessThan*[*simp*]: $x < y \Longrightarrow Sup\ \{\ x\ ..< y\} = y$
and *Sup-greaterThanLessThan*[*simp*]: $x < y \Longrightarrow Sup\ \{\ x <..< y\} = y$
and *Inf-greaterThan*[*simp*]: $Inf\ \{x <..\} = x$
and *Inf-greaterThanAtMost*[*simp*]: $x < y \Longrightarrow Inf\ \{\ x <..\ y\} = x$
and *Inf-greaterThanLessThan*[*simp*]: $x < y \Longrightarrow Inf\ \{\ x <..< y\} = x$
⟨*proof*⟩

61.5 Intervals of natural numbers

61.5.1 The Constant *lessThan*

lemma *lessThan-0* [*simp*]: *lessThan* (*0*::*nat*) = {}
⟨*proof*⟩

lemma *lessThan-Suc*: *lessThan* (*Suc k*) = *insert k* (*lessThan k*)
⟨*proof*⟩

The following proof is convenient in induction proofs where new elements get indices at the beginning. So it is used to transform {..<*Suc n*} to *0* and {..<*n*}.

lemma *zero-notin-Suc-image* [*simp*]: *0* ∉ *Suc* ‘ *A*
⟨*proof*⟩

lemma *lessThan-Suc-eq-insert-0*: {..<*Suc n*} = *insert 0* (*Suc* ‘ {..<*n*})
⟨*proof*⟩

lemma *lessThan-Suc-atMost*: *lessThan* (*Suc k*) = *atMost k*
⟨*proof*⟩

lemma *atMost-Suc-eq-insert-0*: {.. *Suc n*} = *insert 0* (*Suc* ‘ {.. *n*})
⟨*proof*⟩

lemma *UN-lessThan-UNIV*: (⋃ *m*::*nat*. *lessThan m*) = *UNIV*
⟨*proof*⟩

61.5.2 The Constant *greaterThan*

lemma *greaterThan-0*: *greaterThan 0* = *range Suc*
⟨*proof*⟩

lemma *greaterThan-Suc*: *greaterThan* (*Suc k*) = *greaterThan k* − {*Suc k*}
⟨*proof*⟩

lemma *INT-greaterThan-UNIV*: (⋂ *m*::*nat*. *greaterThan m*) = {}
⟨*proof*⟩

61.5.3 The Constant *atLeast*

lemma *atLeast-0* [*simp*]: *atLeast* (*0*::*nat*) = *UNIV*
⟨*proof*⟩

lemma *atLeast-Suc*: *atLeast* (*Suc k*) = *atLeast k* − {*k*}
⟨*proof*⟩

lemma *atLeast-Suc-greaterThan*: *atLeast* (*Suc k*) = *greaterThan k*
⟨*proof*⟩

lemma *UN-atLeast-UNIV*: $(\bigcup m{::}nat.\ atLeast\ m) = UNIV$
⟨*proof*⟩

61.5.4 The Constant *atMost*

lemma *atMost-0* [*simp*]: $atMost\ (0{::}nat) = \{0\}$
⟨*proof*⟩

lemma *atMost-Suc*: $atMost\ (Suc\ k) = insert\ (Suc\ k)\ (atMost\ k)$
⟨*proof*⟩

lemma *UN-atMost-UNIV*: $(\bigcup m{::}nat.\ atMost\ m) = UNIV$
⟨*proof*⟩

61.5.5 The Constant *atLeastLessThan*

The orientation of the following 2 rules is tricky. The lhs is defined in terms of the rhs. Hence the chosen orientation makes sense in this theory — the reverse orientation complicates proofs (eg nontermination). But outside, when the definition of the lhs is rarely used, the opposite orientation seems preferable because it reduces a specific concept to a more general one.

lemma *atLeast0LessThan* [*code-abbrev*]: $\{0{::}nat..<n\} = \{..<n\}$
⟨*proof*⟩

lemma *atLeast0AtMost* [*code-abbrev*]: $\{0..n{::}nat\} = \{..n\}$
⟨*proof*⟩

lemma *lessThan-atLeast0*: $\{..<n\} = \{0{::}nat..<n\}$
⟨*proof*⟩

lemma *atMost-atLeast0*: $\{..n\} = \{0{::}nat..n\}$
⟨*proof*⟩

lemma *atLeastLessThan0*: $\{m..<0{::}nat\} = \{\}$
⟨*proof*⟩

lemma *atLeast0-lessThan-Suc*: $\{0..<Suc\ n\} = insert\ n\ \{0..<n\}$
⟨*proof*⟩

lemma *atLeast0-lessThan-Suc-eq-insert-0*: $\{0..<Suc\ n\} = insert\ 0\ (Suc\ `\{0..<n\})$
⟨*proof*⟩

61.5.6 The Constant *atLeastAtMost*

lemma *Icc-eq-insert-lb-nat*: $m \le n \Longrightarrow \{m..n\} = insert\ m\ \{Suc\ m..n\}$
⟨*proof*⟩

lemma *atLeast0-atMost-Suc*:
$\{0..Suc\ n\} = insert\ (Suc\ n)\ \{0..n\}$

⟨*proof*⟩

lemma *atLeast0-atMost-Suc-eq-insert-0*:
{*0*..*Suc n*} = *insert 0* (*Suc* ' {*0*..*n*})
⟨*proof*⟩

61.5.7 Intervals of nats with *Suc*

Not a simprule because the RHS is too messy.

lemma *atLeastLessThanSuc*:
{*m*..<*Suc n*} = (*if m* ≤ *n then insert n* {*m*..<*n*} *else* {})
⟨*proof*⟩

lemma *atLeastLessThan-singleton* [*simp*]: {*m*..<*Suc m*} = {*m*}
⟨*proof*⟩

lemma *atLeastLessThanSuc-atLeastAtMost*: {*l*..<*Suc u*} = {*l*..*u*}
⟨*proof*⟩

lemma *atLeastSucAtMost-greaterThanAtMost*: {*Suc l*..*u*} = {*l*<..*u*}
⟨*proof*⟩

lemma *atLeastSucLessThan-greaterThanLessThan*: {*Suc l*..<*u*} = {*l*<..<*u*}
⟨*proof*⟩

lemma *atLeastAtMostSuc-conv*: *m* ≤ *Suc n* ⟹ {*m*..*Suc n*} = *insert* (*Suc n*) {*m*..*n*}
⟨*proof*⟩

lemma *atLeastAtMost-insertL*: *m* ≤ *n* ⟹ *insert m* {*Suc m*..*n*} = {*m* ..*n*}
⟨*proof*⟩

The analogous result is useful on *int*:

lemma *atLeastAtMostPlus1-int-conv*:
m ≤ *1*+*n* ⟹ {*m*..*1*+*n*} = *insert* (*1*+*n*) {*m*..*n*::*int*}
⟨*proof*⟩

lemma *atLeastLessThan-add-Un*: *i* ≤ *j* ⟹ {*i*..<*j*+*k*} = {*i*..<*j*} ∪ {*j*..<*j*+*k*::*nat*}
⟨*proof*⟩

61.5.8 Intervals and numerals

lemma *lessThan-nat-numeral*: — Evaluation for specific numerals
lessThan (*numeral k* :: *nat*) = *insert* (*pred-numeral k*) (*lessThan* (*pred-numeral k*))
⟨*proof*⟩

lemma *atMost-nat-numeral*: — Evaluation for specific numerals
atMost (*numeral k* :: *nat*) = *insert* (*numeral k*) (*atMost* (*pred-numeral k*))

⟨*proof*⟩

lemma *atLeastLessThan-nat-numeral*: — Evaluation for specific numerals
atLeastLessThan m (*numeral k* :: *nat*) =
(*if m* ≤ (*pred-numeral k*) *then insert* (*pred-numeral k*) (*atLeastLessThan m* (*pred-numeral k*))
else {})
⟨*proof*⟩

61.5.9 Image

context *linordered-semidom*
begin

lemma *image-add-atLeast*[*simp*]: *plus k* ' {*i*..} = {*k* + *i*..}
⟨*proof*⟩

lemma *image-add-atLeastAtMost* [*simp*]:
plus k ' {*i*..*j*} = {*i* + *k*..*j* + *k*} (**is** *?A* = *?B*)
⟨*proof*⟩

lemma *image-add-atLeastAtMost'* [*simp*]:
(λ*n*. *n* + *k*) ' {*i*..*j*} = {*i* + *k*..*j* + *k*}
⟨*proof*⟩

lemma *image-add-atLeastLessThan* [*simp*]:
plus k ' {*i*..<*j*} = {*i* + *k*..<*j* + *k*}
⟨*proof*⟩

lemma *image-add-atLeastLessThan'* [*simp*]:
(λ*n*. *n* + *k*) ' {*i*..<*j*} = {*i* + *k*..<*j* + *k*}
⟨*proof*⟩

lemma *image-add-greaterThanAtMost*[*simp*]: (+) *c* ' {*a*<..*b*} = {*c* + *a*<..*c* + *b*}
⟨*proof*⟩

end

context *ordered-ab-group-add*
begin

lemma
fixes *x* :: *'a*
shows *image-uminus-greaterThan*[*simp*]: *uminus* ' {*x*<..} = {..<−*x*}
and *image-uminus-atLeast*[*simp*]: *uminus* ' {*x*..} = {..−*x*}
⟨*proof*⟩

lemma
fixes *x* :: *'a*

shows *image-uminus-lessThan*[*simp*]: *uminus* ‘ $\{..<x\} = \{-x<..\}$
and *image-uminus-atMost*[*simp*]: *uminus* ‘ $\{..x\} = \{-x..\}$
⟨*proof*⟩

lemma
fixes $x :: 'a$
shows *image-uminus-atLeastAtMost*[*simp*]: *uminus* ‘ $\{x..y\} = \{-y..-x\}$
and *image-uminus-greaterThanAtMost*[*simp*]: *uminus* ‘ $\{x<..y\} = \{-y..<-x\}$
and *image-uminus-atLeastLessThan*[*simp*]: *uminus* ‘ $\{x..<y\} = \{-y<..-x\}$
and *image-uminus-greaterThanLessThan*[*simp*]: *uminus* ‘ $\{x<..<y\} = \{-y<..<-x\}$
⟨*proof*⟩

lemma *image-add-atMost*[*simp*]: $(+)\ c$ ‘ $\{..a\} = \{..c + a\}$
⟨*proof*⟩

end

lemma *image-Suc-atLeastAtMost* [*simp*]:
Suc ‘ $\{i..j\} = \{Suc\ i..Suc\ j\}$
⟨*proof*⟩

lemma *image-Suc-atLeastLessThan* [*simp*]:
Suc ‘ $\{i..<j\} = \{Suc\ i..<Suc\ j\}$
⟨*proof*⟩

corollary *image-Suc-atMost*:
Suc ‘ $\{..n\} = \{1..Suc\ n\}$
⟨*proof*⟩

corollary *image-Suc-lessThan*:
Suc ‘ $\{..<n\} = \{1..n\}$
⟨*proof*⟩

lemma *image-diff-atLeastAtMost* [*simp*]:
fixes $d::'a::linordered\text{-}idom$ **shows** $((-)\ d$ ‘ $\{a..b\}) = \{d-b..d-a\}$
⟨*proof*⟩

lemma *image-diff-atLeastLessThan* [*simp*]:
fixes $a\ b\ c::'a::linordered\text{-}idom$
shows $(-)\ c$ ‘ $\{a..<b\} = \{c - b<..c - a\}$
⟨*proof*⟩

lemma *image-minus-const-greaterThanAtMost*[*simp*]:
fixes $a\ b\ c::'a::linordered\text{-}idom$
shows $(-)\ c$ ‘ $\{a<..b\} = \{c - b..<c - a\}$
⟨*proof*⟩

lemma *image-minus-const-atLeast*[*simp*]:
fixes $a\ c::'a::linordered\text{-}idom$

shows $(-)\ c\ `\ \{a..\} = \{..c - a\}$
⟨*proof*⟩

lemma *image-minus-const-AtMost*[*simp*]:
fixes $b\ c::'a::linordered\text{-}idom$
shows $(-)\ c\ `\ \{..b\} = \{c - b..\}$
⟨*proof*⟩

lemma *image-minus-const-atLeastAtMost'* [*simp*]:
$(\lambda t.\ t-d)`\{a..b\} = \{a-d..b-d\}$ **for** $d::'a::linordered\text{-}idom$
⟨*proof*⟩

context *linordered-field*
begin

lemma *image-mult-atLeastAtMost* [*simp*]:
$((*)\ d\ `\ \{a..b\}) = \{d*a..d*b\}$ **if** $d>0$
⟨*proof*⟩

lemma *image-divide-atLeastAtMost* [*simp*]:
$((\lambda c.\ c\ /\ d)\ `\ \{a..b\}) = \{a/d..b/d\}$ **if** $d>0$
⟨*proof*⟩

lemma *image-mult-atLeastAtMost-if*:
$(*)\ c\ `\ \{x\ ..\ y\} =$
$(if\ c > 0\ then\ \{c * x\ ..\ c * y\}\ else\ if\ x \le y\ then\ \{c * y\ ..\ c * x\}\ else\ \{\})$
⟨*proof*⟩

lemma *image-mult-atLeastAtMost-if'*:
$(\lambda x.\ x * c)\ `\ \{x..y\} =$
$(if\ x \le y\ then\ if\ c > 0\ then\ \{x * c\ ..\ y * c\}\ else\ \{y * c\ ..\ x * c\}\ else\ \{\})$
⟨*proof*⟩

lemma *image-affinity-atLeastAtMost*:
$((\lambda x.\ m * x + c)\ `\ \{a..b\}) = (if\ \{a..b\} = \{\}\ then\ \{\}$
$else\ if\ 0 \le m\ then\ \{m * a + c\ ..\ m * b + c\}$
$else\ \{m * b + c\ ..\ m * a + c\})$
⟨*proof*⟩

lemma *image-affinity-atLeastAtMost-diff*:
$((\lambda x.\ m*x - c)\ `\ \{a..b\}) = (if\ \{a..b\}=\{\}\ then\ \{\}$
$else\ if\ 0 \le m\ then\ \{m*a - c\ ..\ m*b - c\}$
$else\ \{m*b - c\ ..\ m*a - c\})$
⟨*proof*⟩

lemma *image-affinity-atLeastAtMost-div*:
$((\lambda x.\ x/m + c)\ `\ \{a..b\}) = (if\ \{a..b\}=\{\}\ then\ \{\}$
$else\ if\ 0 \le m\ then\ \{a/m + c\ ..\ b/m + c\}$
$else\ \{b/m + c\ ..\ a/m + c\})$

⟨*proof*⟩

lemma *image-affinity-atLeastAtMost-div-diff*:
((λ*x*. *x*/*m* − *c*) ' {*a*..*b*}) = (*if* {*a*..*b*}={} *then* {}
else if 0 ≤ *m then* {*a*/*m* − *c* .. *b*/*m* − *c*}
else {*b*/*m* − *c* .. *a*/*m* − *c*})
⟨*proof*⟩

end

lemma *atLeast1-lessThan-eq-remove0*:
{*Suc* 0..<*n*} = {..<*n*} − {0}
⟨*proof*⟩

lemma *atLeast1-atMost-eq-remove0*:
{*Suc* 0..*n*} = {..*n*} − {0}
⟨*proof*⟩

lemma *image-add-int-atLeastLessThan*:
(λ*x*. *x* + (*l*::*int*)) ' {0..<*u*−*l*} = {*l*..<*u*}
⟨*proof*⟩

lemma *image-minus-const-atLeastLessThan-nat*:
fixes *c* :: *nat*
shows (λ*i*. *i* − *c*) ' {*x* ..< *y*} =
(*if c* < *y then* {*x* − *c* ..< *y* − *c*} *else if x* < *y then* {0} *else* {})
(**is** - = *?right*)
⟨*proof*⟩

lemma *image-int-atLeastLessThan*:
int ' {*a*..<*b*} = {*int a*..<*int b*}
⟨*proof*⟩

lemma *image-int-atLeastAtMost*:
int ' {*a*..*b*} = {*int a*..*int b*}
⟨*proof*⟩

61.5.10 Finiteness

lemma *finite-lessThan* [*iff*]: **fixes** *k* :: *nat* **shows** *finite* {..<*k*}
⟨*proof*⟩

lemma *finite-atMost* [*iff*]: **fixes** *k* :: *nat* **shows** *finite* {..*k*}
⟨*proof*⟩

lemma *finite-greaterThanLessThan* [*iff*]:
fixes *l* :: *nat* **shows** *finite* {*l*<..<*u*}
⟨*proof*⟩

lemma *finite-atLeastLessThan* [*iff*]:
fixes l :: *nat* **shows** *finite* $\{l..<u\}$
⟨*proof*⟩

lemma *finite-greaterThanAtMost* [*iff*]:
fixes l :: *nat* **shows** *finite* $\{l<..u\}$
⟨*proof*⟩

lemma *finite-atLeastAtMost* [*iff*]:
fixes l :: *nat* **shows** *finite* $\{l..u\}$
⟨*proof*⟩

A bounded set of natural numbers is finite.

lemma *bounded-nat-set-is-finite*: $(\forall\, i{\in}N.\ i < (n{::}nat)) \Longrightarrow$ *finite* N
⟨*proof*⟩

A set of natural numbers is finite iff it is bounded.

lemma *finite-nat-set-iff-bounded*:
finite$(N{::}nat\ set) = (\exists\, m.\ \forall\, n{\in}N.\ n{<}m)$ (**is** *?F* = *?B*)
⟨*proof*⟩

lemma *finite-nat-set-iff-bounded-le*: *finite*$(N{::}nat\ set) = (\exists\, m.\ \forall\, n{\in}N.\ n{\leq}m)$
⟨*proof*⟩

lemma *finite-less-ub*:
$\bigwedge f{::}nat{\Rightarrow}nat.\ (!!n.\ n \leq f\ n) \Longrightarrow$ *finite* $\{n.\ f\ n \leq u\}$
⟨*proof*⟩

lemma *bounded-Max-nat*:
fixes P :: $nat \Rightarrow bool$
assumes x: $P\ x$ **and** M: $\bigwedge x.\ P\ x \Longrightarrow x \leq M$
obtains m **where** $P\ m\ \bigwedge x.\ P\ x \Longrightarrow x \leq m$
⟨*proof*⟩

Any subset of an interval of natural numbers the size of the subset is exactly that interval.

lemma *subset-card-intvl-is-intvl*:
assumes $A \subseteq \{k..<k + card\ A\}$
shows $A = \{k..<k + card\ A\}$
⟨*proof*⟩

61.5.11 Proving Inclusions and Equalities between Unions

lemma *UN-le-eq-Un0*:
$(\bigcup i{\leq}n{::}nat.\ M\ i) = (\bigcup i{\in}\{1..n\}.\ M\ i) \cup M\ 0$ (**is** *?A* = *?B*)
⟨*proof*⟩

lemma *UN-le-add-shift*:
$(\bigcup i{\leq}n{::}nat.\ M(i{+}k)) = (\bigcup i{\in}\{k..n{+}k\}.\ M\ i)$ (**is** *?A* = *?B*)

⟨*proof*⟩

lemma *UN-le-add-shift-strict*:
(⋃ *i*<*n*::*nat*. *M*(*i*+*k*)) = (⋃ *i*∈{*k*..<*n*+*k*}. *M i*) (**is** *?A* = *?B*)
⟨*proof*⟩

lemma *UN-UN-finite-eq*: (⋃ *n*::*nat*. ⋃ *i*∈{*0*..<*n*}. *A i*) = (⋃ *n*. *A n*)
⟨*proof*⟩

lemma *UN-finite-subset*:
(⋀*n*::*nat*. (⋃ *i*∈{*0*..<*n*}. *A i*) ⊆ *C*) ⟹ (⋃ *n*. *A n*) ⊆ *C*
⟨*proof*⟩

lemma *UN-finite2-subset*:
assumes ⋀*n*::*nat*. (⋃ *i*∈{*0*..<*n*}. *A i*) ⊆ (⋃ *i*∈{*0*..<*n* + *k*}. *B i*)
shows (⋃ *n*. *A n*) ⊆ (⋃ *n*. *B n*)
⟨*proof*⟩

lemma *UN-finite2-eq*:
assumes (⋀*n*::*nat*. (⋃ *i*∈{*0*..<*n*}. *A i*) = (⋃ *i*∈{*0*..<*n* + *k*}. *B i*))
shows (⋃ *n*. *A n*) = (⋃ *n*. *B n*)
⟨*proof*⟩

61.5.12 Cardinality

lemma *card-lessThan* [*simp*]: *card* {..<*u*} = *u*
⟨*proof*⟩

lemma *card-atMost* [*simp*]: *card* {..*u*} = *Suc u*
⟨*proof*⟩

lemma *card-atLeastLessThan* [*simp*]: *card* {*l*..<*u*} = *u* − *l*
⟨*proof*⟩

lemma *card-atLeastAtMost* [*simp*]: *card* {*l*..*u*} = *Suc u* − *l*
⟨*proof*⟩

lemma *card-greaterThanAtMost* [*simp*]: *card* {*l*<..*u*} = *u* − *l*
⟨*proof*⟩

lemma *card-greaterThanLessThan* [*simp*]: *card* {*l*<..<*u*} = *u* − *Suc l*
⟨*proof*⟩

lemma *subset-eq-atLeast0-lessThan-finite*:
fixes *n* :: *nat*
assumes *N* ⊆ {*0*..<*n*}
shows *finite N*
⟨*proof*⟩

lemma *subset-eq-atLeast0-atMost-finite*:
fixes $n :: nat$
assumes $N \subseteq \{0..n\}$
shows *finite* N
⟨*proof*⟩

lemma *ex-bij-betw-nat-finite*:
finite $M \Longrightarrow \exists h.$ *bij-betw* h $\{0..<card\ M\}$ M
⟨*proof*⟩

lemma *ex-bij-betw-finite-nat*:
finite $M \Longrightarrow \exists h.$ *bij-betw* h M $\{0..<card\ M\}$
⟨*proof*⟩

lemma *finite-same-card-bij*:
finite $A \Longrightarrow$ *finite* $B \Longrightarrow$ *card* $A =$ *card* $B \Longrightarrow \exists h.$ *bij-betw* h A B
⟨*proof*⟩

lemma *ex-bij-betw-nat-finite-1*:
finite $M \Longrightarrow \exists h.$ *bij-betw* h $\{1\ ..\ card\ M\}$ M
⟨*proof*⟩

lemma *bij-betw-iff-card*:
assumes *finite* A *finite* B
shows $(\exists f.$ *bij-betw* f A $B) \longleftrightarrow$ (*card* $A =$ *card* B)
⟨*proof*⟩

lemma *subset-eq-atLeast0-lessThan-card*:
fixes $n :: nat$
assumes $N \subseteq \{0..<n\}$
shows *card* $N \leq n$
⟨*proof*⟩

Relational version of *card-inj-on-le*:

lemma *card-le-if-inj-on-rel*:
assumes *finite* B
$\bigwedge a.\ a \in A \Longrightarrow \exists b.\ b \in B \wedge r\ a\ b$
$\bigwedge a1\ a2\ b.\ [\![\ a1 \in A;\ a2 \in A;\ b \in B;\ r\ a1\ b;\ r\ a2\ b\ ]\!] \Longrightarrow a1 = a2$
shows *card* $A \leq$ *card* B
⟨*proof*⟩

lemma *inj-on-funpow-least*:
‹*inj-on* $(\lambda k.\ (f \frown k)\ s)$ $\{0..<n\}$›
if ‹$(f \frown n)\ s = s$› ‹$\bigwedge m.\ 0 < m \Longrightarrow m < n \Longrightarrow (f \frown m)\ s \neq s$›
⟨*proof*⟩

61.6 Intervals of integers

lemma *atLeastLessThanPlusOne-atLeastAtMost-int*: $\{l..<u+1\} = \{l..(u::int)\}$

⟨*proof*⟩

lemma *atLeastPlusOneAtMost-greaterThanAtMost-int*: $\{l+1..u\} = \{l<..(u::int)\}$
⟨*proof*⟩

lemma *atLeastPlusOneLessThan-greaterThanLessThan-int*:
$\{l+1..<u\} = \{l<..<u::int\}$
⟨*proof*⟩

61.6.1 Finiteness

lemma *image-atLeastZeroLessThan-int*:
assumes $0 \leq u$
shows $\{(0::int)..<u\} = int\ `\ \{..<nat\ u\}$
⟨*proof*⟩

lemma *finite-atLeastZeroLessThan-int*: *finite* $\{(0::int)..<u\}$
⟨*proof*⟩

lemma *finite-atLeastLessThan-int* [*iff*]: *finite* $\{l..<u::int\}$
⟨*proof*⟩

lemma *finite-atLeastAtMost-int* [*iff*]: *finite* $\{l..(u::int)\}$
⟨*proof*⟩

lemma *finite-greaterThanAtMost-int* [*iff*]: *finite* $\{l<..(u::int)\}$
⟨*proof*⟩

lemma *finite-greaterThanLessThan-int* [*iff*]: *finite* $\{l<..<u::int\}$
⟨*proof*⟩

61.6.2 Cardinality

lemma *card-atLeastZeroLessThan-int*: *card* $\{(0::int)..<u\} = nat\ u$
⟨*proof*⟩

lemma *card-atLeastLessThan-int* [*simp*]: *card* $\{l..<u\} = nat\ (u - l)$
⟨*proof*⟩

lemma *card-atLeastAtMost-int* [*simp*]: *card* $\{l..u\} = nat\ (u - l + 1)$
⟨*proof*⟩

lemma *card-greaterThanAtMost-int* [*simp*]: *card* $\{l<..u\} = nat\ (u - l)$
⟨*proof*⟩

lemma *card-greaterThanLessThan-int* [*simp*]: *card* $\{l<..<u\} = nat\ (u - (l + 1))$
⟨*proof*⟩

lemma *finite-M-bounded-by-nat*: *finite* $\{k.\ P\ k \wedge k < (i::nat)\}$

⟨*proof*⟩

lemma *card-less*:
assumes *zero-in-M*: $0 \in M$
shows *card* $\{k \in M.\ k < Suc\ i\} \neq 0$
⟨*proof*⟩

lemma *card-less-Suc2*:
assumes $0 \notin M$ **shows** *card* $\{k.\ Suc\ k \in M \wedge k < i\} = card\ \{k \in M.\ k < Suc\ i\}$
⟨*proof*⟩

lemma *card-less-Suc*:
assumes $0 \in M$
shows $Suc\ (card\ \{k.\ Suc\ k \in M \wedge k < i\}) = card\ \{k \in M.\ k < Suc\ i\}$
⟨*proof*⟩

lemma *card-le-Suc-Max*: *finite* $S \Longrightarrow card\ S \leq Suc\ (Max\ S)$
⟨*proof*⟩

61.7 Lemmas useful with the summation operator sum

For examples, see Algebra/poly/UnivPoly2.thy

61.7.1 Disjoint Unions

Singletons and open intervals

lemma *ivl-disj-un-singleton*:
$\{l::'a::linorder\}\ Un\ \{l<..\} = \{l..\}$
$\{..<u\}\ Un\ \{u::'a::linorder\} = \{..u\}$
$(l::'a::linorder) < u \Longrightarrow \{l\}\ Un\ \{l<..<u\} = \{l..<u\}$
$(l::'a::linorder) < u \Longrightarrow \{l<..<u\}\ Un\ \{u\} = \{l<..u\}$
$(l::'a::linorder) \leq u \Longrightarrow \{l\}\ Un\ \{l<..u\} = \{l..u\}$
$(l::'a::linorder) \leq u \Longrightarrow \{l..<u\}\ Un\ \{u\} = \{l..u\}$
⟨*proof*⟩

One- and two-sided intervals

lemma *ivl-disj-un-one*:
$(l::'a::linorder) < u \Longrightarrow \{..l\}\ Un\ \{l<..<u\} = \{..<u\}$
$(l::'a::linorder) \leq u \Longrightarrow \{..<l\}\ Un\ \{l..<u\} = \{..<u\}$
$(l::'a::linorder) \leq u \Longrightarrow \{..l\}\ Un\ \{l<..u\} = \{..u\}$
$(l::'a::linorder) \leq u \Longrightarrow \{..<l\}\ Un\ \{l..u\} = \{..u\}$
$(l::'a::linorder) \leq u \Longrightarrow \{l<..u\}\ Un\ \{u<..\} = \{l<..\}$
$(l::'a::linorder) < u \Longrightarrow \{l<..<u\}\ Un\ \{u..\} = \{l<..\}$
$(l::'a::linorder) \leq u \Longrightarrow \{l..u\}\ Un\ \{u<..\} = \{l..\}$
$(l::'a::linorder) \leq u \Longrightarrow \{l..<u\}\ Un\ \{u..\} = \{l..\}$
⟨*proof*⟩

Two- and two-sided intervals

lemma *ivl-disj-un-two*:
$[\!|\ (l::'a::linorder) < m;\ m \le u\ |\!] ==> \{l<..<m\}\ Un\ \{m..<u\} = \{l<..<u\}$
$[\!|\ (l::'a::linorder) \le m;\ m < u\ |\!] ==> \{l<..m\}\ Un\ \{m<..<u\} = \{l<..<u\}$
$[\!|\ (l::'a::linorder) \le m;\ m \le u\ |\!] ==> \{l..<m\}\ Un\ \{m..<u\} = \{l..<u\}$
$[\!|\ (l::'a::linorder) \le m;\ m < u\ |\!] ==> \{l..m\}\ Un\ \{m<..<u\} = \{l..<u\}$
$[\!|\ (l::'a::linorder) < m;\ m \le u\ |\!] ==> \{l<..<m\}\ Un\ \{m..u\} = \{l<..u\}$
$[\!|\ (l::'a::linorder) \le m;\ m \le u\ |\!] ==> \{l<..m\}\ Un\ \{m<..u\} = \{l<..u\}$
$[\!|\ (l::'a::linorder) \le m;\ m \le u\ |\!] ==> \{l..<m\}\ Un\ \{m..u\} = \{l..u\}$
$[\!|\ (l::'a::linorder) \le m;\ m \le u\ |\!] ==> \{l..m\}\ Un\ \{m<..u\} = \{l..u\}$
⟨*proof*⟩

lemma *ivl-disj-un-two-touch*:
$[\!|\ (l::'a::linorder) < m;\ m < u\ |\!] ==> \{l<..m\}\ Un\ \{m..<u\} = \{l<..<u\}$
$[\!|\ (l::'a::linorder) \le m;\ m < u\ |\!] ==> \{l..m\}\ Un\ \{m..<u\} = \{l..<u\}$
$[\!|\ (l::'a::linorder) < m;\ m \le u\ |\!] ==> \{l<..m\}\ Un\ \{m..u\} = \{l<..u\}$
$[\!|\ (l::'a::linorder) \le m;\ m \le u\ |\!] ==> \{l..m\}\ Un\ \{m..u\} = \{l..u\}$
⟨*proof*⟩

lemmas *ivl-disj-un = ivl-disj-un-singleton ivl-disj-un-one ivl-disj-un-two ivl-disj-un-two-touch*

61.7.2 Disjoint Intersections

One- and two-sided intervals

lemma *ivl-disj-int-one*:
$\{..l::'a::order\}\ Int\ \{l<..<u\} = \{\}$
$\{..<l\}\ Int\ \{l..<u\} = \{\}$
$\{..l\}\ Int\ \{l<..u\} = \{\}$
$\{..<l\}\ Int\ \{l..u\} = \{\}$
$\{l<..u\}\ Int\ \{u<..\} = \{\}$
$\{l<..<u\}\ Int\ \{u..\} = \{\}$
$\{l..u\}\ Int\ \{u<..\} = \{\}$
$\{l..<u\}\ Int\ \{u..\} = \{\}$
⟨*proof*⟩

Two- and two-sided intervals

lemma *ivl-disj-int-two*:
$\{l::'a::order<..<m\}\ Int\ \{m..<u\} = \{\}$
$\{l<..m\}\ Int\ \{m<..<u\} = \{\}$
$\{l..<m\}\ Int\ \{m..<u\} = \{\}$
$\{l..m\}\ Int\ \{m<..<u\} = \{\}$
$\{l<..<m\}\ Int\ \{m..u\} = \{\}$
$\{l<..m\}\ Int\ \{m<..u\} = \{\}$
$\{l..<m\}\ Int\ \{m..u\} = \{\}$
$\{l..m\}\ Int\ \{m<..u\} = \{\}$
⟨*proof*⟩

lemmas *ivl-disj-int = ivl-disj-int-one ivl-disj-int-two*

61.7.3 Some Differences

lemma *ivl-diff*[*simp*]:
$i \leq n \Longrightarrow \{i..<m\} - \{i..<n\} = \{n..<(m::'a::linorder)\}$
⟨*proof*⟩

lemma (**in** *linorder*) *lessThan-minus-lessThan* [*simp*]:
$\{..< n\} - \{..< m\} = \{m\ ..< n\}$
⟨*proof*⟩

lemma (**in** *linorder*) *atLeastAtMost-diff-ends*:
$\{a..b\} - \{a, b\} = \{a<..<b\}$
⟨*proof*⟩

61.7.4 Some Subset Conditions

lemma *ivl-subset* [*simp*]: $(\{i..<j\} \subseteq \{m..<n\}) = (j \leq i \vee m \leq i \wedge j \leq (n::'a::linorder))$
⟨*proof*⟩

61.8 Generic big monoid operation over intervals

context *semiring-char-0*
begin

lemma *inj-on-of-nat* [*simp*]:
inj-on of-nat N
⟨*proof*⟩

lemma *bij-betw-of-nat* [*simp*]:
bij-betw of-nat N A $\longleftrightarrow$ *of-nat* ' *N* = *A*
⟨*proof*⟩

lemma *Nats-infinite*: *infinite* (**ℕ** :: *'a set*)
⟨*proof*⟩

end

context *comm-monoid-set*
begin

lemma *atLeastLessThan-reindex*:
F g {*h m..<h n*} = *F* (*g* ∘ *h*) {*m..<n*}
if *bij-betw h* {*m..<n*} {*h m..<h n*} **for** *m n* ::*nat*
⟨*proof*⟩

lemma *atLeastAtMost-reindex*:
F g {*h m..h n*} = *F* (*g* ∘ *h*) {*m..n*}
if *bij-betw h* {*m..n*} {*h m..h n*} **for** *m n* ::*nat*
⟨*proof*⟩

lemma *atLeastLessThan-shift-bounds*:
$F\ g\ \{m + k..<n + k\} = F\ (g \circ plus\ k)\ \{m..<n\}$
for $m\ n\ k :: nat$
⟨*proof*⟩

lemma *atLeastAtMost-shift-bounds*:
$F\ g\ \{m + k..n + k\} = F\ (g \circ plus\ k)\ \{m..n\}$
for $m\ n\ k :: nat$
⟨*proof*⟩

lemma *atLeast-Suc-lessThan-Suc-shift*:
$F\ g\ \{Suc\ m..<Suc\ n\} = F\ (g \circ Suc)\ \{m..<n\}$
⟨*proof*⟩

lemma *atLeast-Suc-atMost-Suc-shift*:
$F\ g\ \{Suc\ m..Suc\ n\} = F\ (g \circ Suc)\ \{m..n\}$
⟨*proof*⟩

lemma *atLeast-atMost-pred-shift*:
$F\ (g \circ (\lambda n.\ n - Suc\ 0))\ \{Suc\ m..Suc\ n\} = F\ g\ \{m..n\}$
⟨*proof*⟩

lemma *atLeast-lessThan-pred-shift*:
$F\ (g \circ (\lambda n.\ n - Suc\ 0))\ \{Suc\ m..<Suc\ n\} = F\ g\ \{m..<n\}$
⟨*proof*⟩

lemma *atLeast-int-lessThan-int-shift*:
$F\ g\ \{int\ m..<int\ n\} = F\ (g \circ int)\ \{m..<n\}$
⟨*proof*⟩

lemma *atLeast-int-atMost-int-shift*:
$F\ g\ \{int\ m..int\ n\} = F\ (g \circ int)\ \{m..n\}$
⟨*proof*⟩

lemma *atLeast0-lessThan-Suc*:
$F\ g\ \{0..<Suc\ n\} = F\ g\ \{0..<n\} * g\ n$
⟨*proof*⟩

lemma *atLeast0-atMost-Suc*:
$F\ g\ \{0..Suc\ n\} = F\ g\ \{0..n\} * g\ (Suc\ n)$
⟨*proof*⟩

lemma *atLeast0-lessThan-Suc-shift*:
$F\ g\ \{0..<Suc\ n\} = g\ 0 * F\ (g \circ Suc)\ \{0..<n\}$
⟨*proof*⟩

lemma *atLeast0-atMost-Suc-shift*:
$F\ g\ \{0..Suc\ n\} = g\ 0 * F\ (g \circ Suc)\ \{0..n\}$
⟨*proof*⟩

lemma *atLeast-Suc-lessThan*:
$F\ g\ \{m..<n\} = g\ m * F\ g\ \{Suc\ m..<n\}$ **if** $m < n$
⟨*proof*⟩

lemma *atLeast-Suc-atMost*:
$F\ g\ \{m..n\} = g\ m * F\ g\ \{Suc\ m..n\}$ **if** $m \leq n$
⟨*proof*⟩

lemma *ivl-cong*:
$a = c \Longrightarrow b = d \Longrightarrow (\bigwedge x.\ c \leq x \Longrightarrow x < d \Longrightarrow g\ x = h\ x)$
$\Longrightarrow F\ g\ \{a..<b\} = F\ h\ \{c..<d\}$
⟨*proof*⟩

lemma *atLeastLessThan-shift-0*:
fixes $m\ n\ p :: nat$
shows $F\ g\ \{m..<n\} = F\ (g \circ plus\ m)\ \{0..<n - m\}$
⟨*proof*⟩

lemma *atLeastAtMost-shift-0*:
fixes $m\ n\ p :: nat$
assumes $m \leq n$
shows $F\ g\ \{m..n\} = F\ (g \circ plus\ m)\ \{0..n - m\}$
⟨*proof*⟩

lemma *atLeastLessThan-concat*:
fixes $m\ n\ p :: nat$
shows $m \leq n \Longrightarrow n \leq p \Longrightarrow F\ g\ \{m..<n\} * F\ g\ \{n..<p\} = F\ g\ \{m..<p\}$
⟨*proof*⟩

lemma *atLeastLessThan-rev*:
$F\ g\ \{n..<m\} = F\ (\lambda i.\ g\ (m + n - Suc\ i))\ \{n..<m\}$
⟨*proof*⟩

lemma *atLeastAtMost-rev*:
fixes $n\ m :: nat$
shows $F\ g\ \{n..m\} = F\ (\lambda i.\ g\ (m + n - i))\ \{n..m\}$
⟨*proof*⟩

lemma *atLeastLessThan-rev-at-least-Suc-atMost*:
$F\ g\ \{n..<m\} = F\ (\lambda i.\ g\ (m + n - i))\ \{Suc\ n..m\}$
⟨*proof*⟩

end

61.9 Summation indexed over intervals

syntax (*ASCII*)
-from-to-sum :: $idt \Rightarrow 'a \Rightarrow 'a \Rightarrow 'b \Rightarrow 'b$ $((SUM\ -\ =\ -..-./\ -)\ [0,0,0,10]\ 10)$

-from-upto-sum :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((SUM - = -..<-./ -) [0,0,0,10] 10)
-upt-sum :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((SUM -<-./ -) [0,0,10] 10)
-upto-sum :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((SUM -<=-./ -) [0,0,10] 10)

syntax (*latex-sum* **output**)
-from-to-sum :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b*
$((3\sum_{-\,=\,-}^{-}$ -) [0,0,0,10] 10)
-from-upto-sum :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b*
$((3\sum_{-\,=\,-}^{<-}$ -) [0,0,0,10] 10)
-upt-sum :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b*
$((3\sum_{-\,<\,-}$ -) [0,0,10] 10)
-upto-sum :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b*
$((3\sum_{-\,\leq\,-}$ -) [0,0,10] 10)

syntax
-from-to-sum :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* (($3\sum$ - = -..-./ -) [0,0,0,10] 10)
-from-upto-sum :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* (($3\sum$ - = -..<-./ -) [0,0,0,10] 10)
-upt-sum :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* (($3\sum$ -<-./ -) [0,0,10] 10)
-upto-sum :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* (($3\sum$ -≤-./ -) [0,0,10] 10)

translations
$\sum x{=}a..b.\ t$ == *CONST sum* $(\lambda x.\ t)$ $\{a..b\}$
$\sum x{=}a..{<}b.\ t$ == *CONST sum* $(\lambda x.\ t)$ $\{a..{<}b\}$
$\sum i{\leq}n.\ t$ == *CONST sum* $(\lambda i.\ t)$ $\{..n\}$
$\sum i{<}n.\ t$ == *CONST sum* $(\lambda i.\ t)$ $\{..{<}n\}$

The above introduces some pretty alternative syntaxes for summation over intervals:

Old	New	LaTeX
$\sum x{\in}\{a..b\}.\ e$	$\sum x = a..b.\ e$	$\sum_{x\,=\,a}^{b}\ e$
$\sum x{\in}\{a..{<}b\}.\ e$	$\sum x = a..{<}b.\ e$	$\sum_{x\,=\,a}^{<b}\ e$
$\sum x{\in}\{..b\}.\ e$	$\sum x{\leq}b.\ e$	$\sum_{x\,\leq\,b}\ e$
$\sum x{\in}\{..{<}b\}.\ e$	$\sum x{<}b.\ e$	$\sum_{x\,<\,b}\ e$

The left column shows the term before introduction of the new syntax, the middle column shows the new (default) syntax, and the right column shows a special syntax. The latter is only meaningful for latex output and has to be activated explicitly by setting the print mode to *latex-sum* (e.g. via *mode* = *latex-sum* in antiquotations). It is not the default LaTeX output because it only works well with italic-style formulae, not tt-style.

Note that for uniformity on *nat* it is better to use $\sum x = 0..{<}n.\ e$ rather than $\sum x{<}n.\ e$: *sum* may not provide all lemmas available for $\{m..{<}n\}$ also in the special form for $\{..{<}n\}$.

This congruence rule should be used for sums over intervals as the standard theorem *sum.cong* does not work well with the simplifier who adds the unsimplified premise $x \in B$ to the context.

context *comm-monoid-set*
begin

lemma *zero-middle*:
assumes $1 \leq p$ $k \leq p$
shows $F\ (\lambda j.\ \textit{if } j < k \textit{ then } g\ j \textit{ else if } j = k \textit{ then } \mathbf{1} \textit{ else } h\ (j - \textit{Suc } 0))\ \{..p\}$
$= F\ (\lambda j.\ \textit{if } j < k \textit{ then } g\ j \textit{ else } h\ j)\ \{..p - \textit{Suc } 0\}$ (**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *atMost-Suc* [*simp*]:
$F\ g\ \{..\textit{Suc } n\} = F\ g\ \{..n\} * g\ (\textit{Suc } n)$
⟨*proof*⟩

lemma *lessThan-Suc* [*simp*]:
$F\ g\ \{..<\textit{Suc } n\} = F\ g\ \{..<n\} * g\ n$
⟨*proof*⟩

lemma *cl-ivl-Suc* [*simp*]:
$F\ g\ \{m..\textit{Suc } n\} = (\textit{if Suc } n < m \textit{ then } \mathbf{1} \textit{ else } F\ g\ \{m..n\} * g(\textit{Suc } n))$
⟨*proof*⟩

lemma *op-ivl-Suc* [*simp*]:
$F\ g\ \{m..<\textit{Suc } n\} = (\textit{if } n < m \textit{ then } \mathbf{1} \textit{ else } F\ g\ \{m..<n\} * g(n))$
⟨*proof*⟩

lemma *head*:
fixes $n :: \textit{nat}$
assumes *mn*: $m \leq n$
shows $F\ g\ \{m..n\} = g\ m * F\ g\ \{m<..n\}$ (**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *last-plus*:
fixes *n*::*nat* **shows** $m \leq n \Longrightarrow F\ g\ \{m..n\} = g\ n * F\ g\ \{m..<n\}$
⟨*proof*⟩

lemma *head-if*:
fixes $n :: \textit{nat}$
shows $F\ g\ \{m..n\} = (\textit{if } n < m \textit{ then } \mathbf{1} \textit{ else } F\ g\ \{m..<n\} * g(n))$
⟨*proof*⟩

lemma *ub-add-nat*:
assumes $(m::\textit{nat}) \leq n + 1$
shows $F\ g\ \{m..n + p\} = F\ g\ \{m..n\} * F\ g\ \{n + 1..n + p\}$
⟨*proof*⟩

lemma *nat-group*:
fixes *k*::*nat* **shows** $F\ (\lambda m.\ F\ g\ \{m * k\ ..<\ m*k + k\})\ \{..<n\} = F\ g\ \{..<\ n * k\}$
⟨*proof*⟩

lemma *triangle-reindex*:
fixes $n :: nat$
shows $F\ (\lambda(i,j).\ g\ i\ j)\ \{(i,j).\ i+j < n\} = F\ (\lambda k.\ F\ (\lambda i.\ g\ i\ (k - i))\ \{..k\})\ \{..<n\}$
⟨*proof*⟩

lemma *triangle-reindex-eq*:
fixes $n :: nat$
shows $F\ (\lambda(i,j).\ g\ i\ j)\ \{(i,j).\ i+j \leq n\} = F\ (\lambda k.\ F\ (\lambda i.\ g\ i\ (k - i))\ \{..k\})\ \{..n\}$
⟨*proof*⟩

lemma *nat-diff-reindex*: $F\ (\lambda i.\ g\ (n - Suc\ i))\ \{..<n\} = F\ g\ \{..<n\}$
⟨*proof*⟩

lemma *shift-bounds-nat-ivl*:
$F\ g\ \{m+k..<n+k\} = F\ (\lambda i.\ g(i + k))\{m..<n::nat\}$
⟨*proof*⟩

lemma *shift-bounds-cl-nat-ivl*:
$F\ g\ \{m+k..n+k\} = F\ (\lambda i.\ g(i + k))\{m..n::nat\}$
⟨*proof*⟩

corollary *shift-bounds-cl-Suc-ivl*:
$F\ g\ \{Suc\ m..Suc\ n\} = F\ (\lambda i.\ g(Suc\ i))\{m..n\}$
⟨*proof*⟩

corollary *Suc-reindex-ivl*: $m \leq n \Longrightarrow F\ g\ \{m..n\} * g\ (Suc\ n) = g\ m * F\ (\lambda i.\ g\ (Suc\ i))\ \{m..n\}$
⟨*proof*⟩

corollary *shift-bounds-Suc-ivl*:
$F\ g\ \{Suc\ m..<Suc\ n\} = F\ (\lambda i.\ g(Suc\ i))\{m..<n\}$
⟨*proof*⟩

lemma *atMost-Suc-shift*:
shows $F\ g\ \{..Suc\ n\} = g\ 0 * F\ (\lambda i.\ g\ (Suc\ i))\ \{..n\}$
⟨*proof*⟩

lemma *lessThan-Suc-shift*:
$F\ g\ \{..<Suc\ n\} = g\ 0 * F\ (\lambda i.\ g\ (Suc\ i))\ \{..<n\}$
⟨*proof*⟩

lemma *atMost-shift*:
$F\ g\ \{..n\} = g\ 0 * F\ (\lambda i.\ g\ (Suc\ i))\ \{..<n\}$
⟨*proof*⟩

lemma *nested-swap*:
$F\ (\lambda i.\ F\ (\lambda j.\ a\ i\ j)\ \{0..<i\})\ \{0..n\} = F\ (\lambda j.\ F\ (\lambda i.\ a\ i\ j)\ \{Suc\ j..n\})\ \{0..<n\}$
⟨*proof*⟩

lemma *nested-swap'*:
$F\ (\lambda i.\ F\ (\lambda j.\ a\ i\ j)\ \{..<i\})\ \{..n\} = F\ (\lambda j.\ F\ (\lambda i.\ a\ i\ j)\ \{Suc\ j..n\})\ \{..<n\}$
⟨*proof*⟩

lemma *atLeast1-atMost-eq*:
$F\ g\ \{Suc\ 0..n\} = F\ (\lambda k.\ g\ (Suc\ k))\ \{..<n\}$
⟨*proof*⟩

lemma *atLeastLessThan-Suc*: $a \leq b \Longrightarrow F\ g\ \{a..<Suc\ b\} = F\ g\ \{a..<b\} * g\ b$
⟨*proof*⟩

lemma *nat-ivl-Suc'*:
assumes $m \leq Suc\ n$
shows $F\ g\ \{m..Suc\ n\} = g\ (Suc\ n) * F\ g\ \{m..n\}$
⟨*proof*⟩

lemma *in-pairs*: $F\ g\ \{2*m..Suc(2*n)\} = F\ (\lambda i.\ g(2*i) * g(Suc(2*i)))\ \{m..n\}$
⟨*proof*⟩

lemma *in-pairs-0*: $F\ g\ \{..Suc(2*n)\} = F\ (\lambda i.\ g(2*i) * g(Suc(2*i)))\ \{..n\}$
⟨*proof*⟩

end

lemma *card-sum-le-nat-sum*: $\sum\ \{0..<card\ S\} \leq \sum\ S$
⟨*proof*⟩

lemma *sum-natinterval-diff*:
fixes f:: $nat \Rightarrow ('a::ab\text{-}group\text{-}add)$
shows $sum\ (\lambda k.\ f\ k - f(k + 1))\ \{(m::nat)\ ..\ n\} =$
$(if\ m \leq n\ then\ f\ m - f(n + 1)\ else\ 0)$
⟨*proof*⟩

lemma *sum-diff-nat-ivl*:
fixes $f :: nat \Rightarrow 'a::ab\text{-}group\text{-}add$
shows $[\![\ m \leq n;\ n \leq p\ ]\!] \Longrightarrow sum\ f\ \{m..<p\} - sum\ f\ \{m..<n\} = sum\ f\ \{n..<p\}$
⟨*proof*⟩

lemma *sum-diff-distrib*: $\forall x.\ Q\ x \leq P\ x\ \Longrightarrow (\sum x{<}n.\ P\ x) - (\sum x{<}n.\ Q\ x) = (\sum x{<}n.\ P\ x - Q\ x :: nat)$
⟨*proof*⟩

61.9.1 Shifting bounds

context *comm-monoid-add*
begin

context

fixes $f :: nat \Rightarrow 'a$
assumes $f\ 0 = 0$
begin

lemma *sum-shift-lb-Suc0-0-upt*:
$sum\ f\ \{Suc\ 0..<k\} = sum\ f\ \{0..<k\}$
⟨*proof*⟩

lemma *sum-shift-lb-Suc0-0*: $sum\ f\ \{Suc\ 0..k\} = sum\ f\ \{0..k\}$
⟨*proof*⟩

end

end

lemma *sum-Suc-diff*:
fixes $f :: nat \Rightarrow 'a::ab\text{-}group\text{-}add$
assumes $m \leq Suc\ n$
shows $(\sum i = m..n.\ f(Suc\ i) - f\ i) = f\ (Suc\ n) - f\ m$
⟨*proof*⟩

lemma *sum-Suc-diff'*:
fixes $f :: nat \Rightarrow 'a::ab\text{-}group\text{-}add$
assumes $m \leq n$
shows $(\sum i = m..<n.\ f\ (Suc\ i) - f\ i) = f\ n - f\ m$
⟨*proof*⟩

61.9.2 Telescoping

lemma *sum-telescope*:
fixes $f::nat \Rightarrow 'a::ab\text{-}group\text{-}add$
shows $sum\ (\lambda i.\ f\ i - f\ (Suc\ i))\ \{..\ i\} = f\ 0 - f\ (Suc\ i)$
⟨*proof*⟩

lemma *sum-telescope''*:
assumes $m \leq n$
shows $(\sum k \in \{Suc\ m..n\}.\ f\ k - f\ (k - 1)) = f\ n - (f\ m :: 'a :: ab\text{-}group\text{-}add)$
⟨*proof*⟩

lemma *sum-lessThan-telescope*:
$(\sum n<m.\ f\ (Suc\ n) - f\ n :: 'a :: ab\text{-}group\text{-}add) = f\ m - f\ 0$
⟨*proof*⟩

lemma *sum-lessThan-telescope'*:
$(\sum n<m.\ f\ n - f\ (Suc\ n) :: 'a :: ab\text{-}group\text{-}add) = f\ 0 - f\ m$
⟨*proof*⟩

61.9.3 The formula for geometric sums

lemma *sum-power2*: $(\sum i=0..<k.\ (2::nat)\hat{}i) = 2\hat{}k-1$

⟨*proof*⟩

lemma *geometric-sum*:
assumes $x \neq 1$
shows $(\sum i{<}n.\ x\ \hat{}\ i) = (x\ \hat{}\ n - 1)\ /\ (x - 1{::}'a{::}\mathit{field})$
⟨*proof*⟩

lemma *geometric-sum-less*:
assumes $0 < x\ x < 1$ *finite* S
shows $(\sum i{\in}S.\ x\ \hat{}\ i) < 1\ /\ (1 - x{::}'a{::}\mathit{linordered\text{-}field})$
⟨*proof*⟩

lemma *diff-power-eq-sum*:
fixes $y :: 'a{::}\{\mathit{comm\text{-}ring},\mathit{monoid\text{-}mult}\}$
shows
$x\ \hat{}\ (\mathit{Suc}\ n) - y\ \hat{}\ (\mathit{Suc}\ n) =$
$(x - y) * (\sum p{<}\mathit{Suc}\ n.\ (x\ \hat{}\ p) * y\ \hat{}\ (n - p))$
⟨*proof*⟩

corollary *power-diff-sumr2*: — *COMPLEX-POLYFUN* in HOL Light
fixes $x :: 'a{::}\{\mathit{comm\text{-}ring},\mathit{monoid\text{-}mult}\}$
shows $x\hat{}n - y\hat{}n = (x - y) * (\sum i{<}n.\ y\hat{}(n - \mathit{Suc}\ i) * x\hat{}i)$
⟨*proof*⟩

lemma *power-diff-1-eq*:
fixes $x :: 'a{::}\{\mathit{comm\text{-}ring},\mathit{monoid\text{-}mult}\}$
shows $x\hat{}n - 1 = (x - 1) * (\sum i{<}n.\ (x\hat{}i))$
⟨*proof*⟩

lemma *one-diff-power-eq'*:
fixes $x :: 'a{::}\{\mathit{comm\text{-}ring},\mathit{monoid\text{-}mult}\}$
shows $1 - x\hat{}n = (1 - x) * (\sum i{<}n.\ x\hat{}(n - \mathit{Suc}\ i))$
⟨*proof*⟩

lemma *one-diff-power-eq*:
fixes $x :: 'a{::}\{\mathit{comm\text{-}ring},\mathit{monoid\text{-}mult}\}$
shows $1 - x\hat{}n = (1 - x) * (\sum i{<}n.\ x\hat{}i)$
⟨*proof*⟩

lemma *sum-gp-basic*:
fixes $x :: 'a{::}\{\mathit{comm\text{-}ring},\mathit{monoid\text{-}mult}\}$
shows $(1 - x) * (\sum i{\leq}n.\ x\hat{}i) = 1 - x\hat{}\mathit{Suc}\ n$
⟨*proof*⟩

lemma *sum-power-shift*:
fixes $x :: 'a{::}\{\mathit{comm\text{-}ring},\mathit{monoid\text{-}mult}\}$
assumes $m \leq n$
shows $(\sum i{=}m..n.\ x\hat{}i) = x\hat{}m * (\sum i{\leq}n{-}m.\ x\hat{}i)$
⟨*proof*⟩

lemma *sum-gp-multiplied*:
fixes $x :: 'a::\{comm\text{-}ring,monoid\text{-}mult\}$
assumes $m \leq n$
shows $(1 - x) * (\sum i{=}m..n.\ x\hat{}i) = x\hat{}m - x\hat{}Suc\ n$
⟨*proof*⟩

lemma *sum-gp*:
fixes $x :: 'a::\{comm\text{-}ring,division\text{-}ring\}$
shows $(\sum i{=}m..n.\ x\hat{}i) =$
$(if\ n < m\ then\ 0$
$else\ if\ x = 1\ then\ of\text{-}nat((n + 1) - m)$
$else\ (x\hat{}m - x\hat{}Suc\ n) / (1 - x))$
⟨*proof*⟩

61.9.4 Geometric progressions

lemma *sum-gp0*:
fixes $x :: 'a::\{comm\text{-}ring,division\text{-}ring\}$
shows $(\sum i{\leq}n.\ x\hat{}i) = (if\ x = 1\ then\ of\text{-}nat(n + 1)\ else\ (1 - x\hat{}Suc\ n) / (1 - x))$
⟨*proof*⟩

lemma *sum-power-add*:
fixes $x :: 'a::\{comm\text{-}ring,monoid\text{-}mult\}$
shows $(\sum i{\in}I.\ x\hat{}(m+i)) = x\hat{}m * (\sum i{\in}I.\ x\hat{}i)$
⟨*proof*⟩

lemma *sum-gp-offset*:
fixes $x :: 'a::\{comm\text{-}ring,division\text{-}ring\}$
shows $(\sum i{=}m..m+n.\ x\hat{}i) =$
$(if\ x = 1\ then\ of\text{-}nat\ n + 1\ else\ x\hat{}m * (1 - x\hat{}Suc\ n) / (1 - x))$
⟨*proof*⟩

lemma *sum-gp-strict*:
fixes $x :: 'a::\{comm\text{-}ring,division\text{-}ring\}$
shows $(\sum i{<}n.\ x\hat{}i) = (if\ x = 1\ then\ of\text{-}nat\ n\ else\ (1 - x\hat{}n) / (1 - x))$
⟨*proof*⟩

61.9.5 The formulae for arithmetic sums

context *comm-semiring-1*
begin

lemma *double-gauss-sum*:
$2 * (\sum i = 0..n.\ of\text{-}nat\ i) = of\text{-}nat\ n * (of\text{-}nat\ n + 1)$
⟨*proof*⟩

lemma *double-gauss-sum-from-Suc-0*:
$2 * (\sum i = Suc\ 0..n.\ of\text{-}nat\ i) = of\text{-}nat\ n * (of\text{-}nat\ n + 1)$

⟨*proof*⟩

lemma *double-arith-series*:
$2 * (\sum i = 0..n.\ a + \textit{of-nat}\ i * d) = (\textit{of-nat}\ n + 1) * (2 * a + \textit{of-nat}\ n * d)$
⟨*proof*⟩

end

context *unique-euclidean-semiring-with-nat*
begin

lemma *gauss-sum*:
$(\sum i = 0..n.\ \textit{of-nat}\ i) = \textit{of-nat}\ n * (\textit{of-nat}\ n + 1)\ \textit{div}\ 2$
⟨*proof*⟩

lemma *gauss-sum-from-Suc-0*:
$(\sum i = \textit{Suc}\ 0..n.\ \textit{of-nat}\ i) = \textit{of-nat}\ n * (\textit{of-nat}\ n + 1)\ \textit{div}\ 2$
⟨*proof*⟩

lemma *arith-series*:
$(\sum i = 0..n.\ a + \textit{of-nat}\ i * d) = (\textit{of-nat}\ n + 1) * (2 * a + \textit{of-nat}\ n * d)\ \textit{div}\ 2$
⟨*proof*⟩

end

lemma *gauss-sum-nat*:
$\sum \{0..n\} = (n * \textit{Suc}\ n)\ \textit{div}\ 2$
⟨*proof*⟩

lemma *arith-series-nat*:
$(\sum i = 0..n.\ a + i * d) = \textit{Suc}\ n * (2 * a + n * d)\ \textit{div}\ 2$
⟨*proof*⟩

lemma *Sum-Icc-int*:
$\sum \{m..n\} = (n * (n + 1) - m * (m - 1))\ \textit{div}\ 2$
if $m \leq n$ **for** $m\ n :: \textit{int}$
⟨*proof*⟩

lemma *Sum-Icc-nat*:
$\sum \{m..n\} = (n * (n + 1) - m * (m - 1))\ \textit{div}\ 2$ **for** $m\ n :: \textit{nat}$
⟨*proof*⟩

lemma *Sum-Ico-nat*:
$\sum \{m..<n\} = (n * (n - 1) - m * (m - 1))\ \textit{div}\ 2$ **for** $m\ n :: \textit{nat}$
⟨*proof*⟩

61.9.6 Division remainder

lemma *range-mod*:

fixes $n :: nat$
assumes $n > 0$
shows *range* $(\lambda m.\ m\ mod\ n) = \{0..<n\}$ (**is** $?A = ?B$)
⟨*proof*⟩

61.10 Products indexed over intervals

syntax (*ASCII*)
-from-to-prod :: $idt \Rightarrow 'a \Rightarrow 'a \Rightarrow 'b \Rightarrow 'b$ ((*PROD - = -..-./ -*) [0,0,0,10] 10)
-from-upto-prod :: $idt \Rightarrow 'a \Rightarrow 'a \Rightarrow 'b \Rightarrow 'b$ ((*PROD - = -..<-./ -*) [0,0,0,10] 10)
-upt-prod :: $idt \Rightarrow 'a \Rightarrow 'b \Rightarrow 'b$ ((*PROD -<-./ -*) [0,0,10] 10)
-upto-prod :: $idt \Rightarrow 'a \Rightarrow 'b \Rightarrow 'b$ ((*PROD -<=-./ -*) [0,0,10] 10)

syntax (*latex-prod* **output**)
-from-to-prod :: $idt \Rightarrow 'a \Rightarrow 'a \Rightarrow 'b \Rightarrow 'b$
((3$\prod_{- = -}$ -) [0,0,0,10] 10)
-from-upto-prod :: $idt \Rightarrow 'a \Rightarrow 'a \Rightarrow 'b \Rightarrow 'b$
((3$\prod^{<-}_{- = -}$ -) [0,0,0,10] 10)
-upt-prod :: $idt \Rightarrow 'a \Rightarrow 'b \Rightarrow 'b$
((3$\prod_{- < -}$ -) [0,0,10] 10)
-upto-prod :: $idt \Rightarrow 'a \Rightarrow 'b \Rightarrow 'b$
((3$\prod_{- \le -}$ -) [0,0,10] 10)

syntax
-from-to-prod :: $idt \Rightarrow 'a \Rightarrow 'a \Rightarrow 'b \Rightarrow 'b$ ((3$\prod$ - = -..-./ -) [0,0,0,10] 10)
-from-upto-prod :: $idt \Rightarrow 'a \Rightarrow 'a \Rightarrow 'b \Rightarrow 'b$ ((3$\prod$ - = -..<-./ -) [0,0,0,10] 10)
-upt-prod :: $idt \Rightarrow 'a \Rightarrow 'b \Rightarrow 'b$ ((3$\prod$ -<-./ -) [0,0,10] 10)
-upto-prod :: $idt \Rightarrow 'a \Rightarrow 'b \Rightarrow 'b$ ((3$\prod$ -≤-./ -) [0,0,10] 10)

translations
$\prod x{=}a..b.\ t \rightleftharpoons CONST\ prod\ (\lambda x.\ t)\ \{a..b\}$
$\prod x{=}a..{<}b.\ t \rightleftharpoons CONST\ prod\ (\lambda x.\ t)\ \{a..<b\}$
$\prod i{\le}n.\ t \rightleftharpoons CONST\ prod\ (\lambda i.\ t)\ \{..n\}$
$\prod i{<}n.\ t \rightleftharpoons CONST\ prod\ (\lambda i.\ t)\ \{..<n\}$

lemma *prod-int-plus-eq*: $prod\ int\ \{i..i+j\} = \prod \{int\ i..int\ (i+j)\}$
⟨*proof*⟩

lemma *prod-int-eq*: $prod\ int\ \{i..j\} = \prod \{int\ i..int\ j\}$
⟨*proof*⟩

61.11 Efficient folding over intervals

function *fold-atLeastAtMost-nat* **where**
[*simp del*]: *fold-atLeastAtMost-nat f a* (*b*::*nat*) *acc* =
(*if a > b then acc else fold-atLeastAtMost-nat f* (*a*+1) *b* (*f a acc*))
⟨*proof*⟩
termination ⟨*proof*⟩

lemma *fold-atLeastAtMost-nat*:
assumes *comp-fun-commute f*
shows *fold-atLeastAtMost-nat f a b acc = Finite-Set.fold f acc {a..b}*
⟨*proof*⟩

lemma *sum-atLeastAtMost-code*:
sum f {a..b} = fold-atLeastAtMost-nat (λa acc. f a + acc) a b 0
⟨*proof*⟩

lemma *prod-atLeastAtMost-code*:
*prod f {a..b} = fold-atLeastAtMost-nat (λa acc. f a * acc) a b 1*
⟨*proof*⟩

end

62 Decision Procedure for Presburger Arithmetic

theory *Presburger*
imports *Groebner-Basis Set-Interval*
keywords *try0* :: *diag*
begin

⟨*ML*⟩

62.1 The $-\infty$ and $+\infty$ Properties

lemma *minf*:
⟦∃(*z* ::'*a*::*linorder*).∀ *x*<*z*. *P x* = *P*′ *x*; ∃ *z*.∀ *x*<*z*. *Q x* = *Q*′ *x*⟧
⟹ ∃ *z*.∀ *x*<*z*. (*P x* ∧ *Q x*) = (*P*′ *x* ∧ *Q*′ *x*)
⟦∃(*z* ::'*a*::*linorder*).∀ *x*<*z*. *P x* = *P*′ *x*; ∃ *z*.∀ *x*<*z*. *Q x* = *Q*′ *x*⟧
⟹ ∃ *z*.∀ *x*<*z*. (*P x* ∨ *Q x*) = (*P*′ *x* ∨ *Q*′ *x*)
∃(*z* ::'*a*::{*linorder*}).∀ *x*<*z*.(*x* = *t*) = *False*
∃(*z* ::'*a*::{*linorder*}).∀ *x*<*z*.(*x* ≠ *t*) = *True*
∃(*z* ::'*a*::{*linorder*}).∀ *x*<*z*.(*x* < *t*) = *True*
∃(*z* ::'*a*::{*linorder*}).∀ *x*<*z*.(*x* ≤ *t*) = *True*
∃(*z* ::'*a*::{*linorder*}).∀ *x*<*z*.(*x* > *t*) = *False*
∃(*z* ::'*a*::{*linorder*}).∀ *x*<*z*.(*x* ≥ *t*) = *False*
∃ *z*.∀(*x*::'*b*::{*linorder*,*plus*,*Rings.dvd*})<*z*. (*d dvd x* + *s*) = (*d dvd x* + *s*)
∃ *z*.∀(*x*::'*b*::{*linorder*,*plus*,*Rings.dvd*})<*z*. (¬ *d dvd x* + *s*) = (¬ *d dvd x* + *s*)
∃ *z*.∀ *x*<*z*. *F* = *F*
⟨*proof*⟩

lemma *pinf*:
⟦∃(*z* ::'*a*::*linorder*).∀ *x*>*z*. *P x* = *P*′ *x*; ∃ *z*.∀ *x*>*z*. *Q x* = *Q*′ *x*⟧
⟹ ∃ *z*.∀ *x*>*z*. (*P x* ∧ *Q x*) = (*P*′ *x* ∧ *Q*′ *x*)
⟦∃(*z* ::'*a*::*linorder*).∀ *x*>*z*. *P x* = *P*′ *x*; ∃ *z*.∀ *x*>*z*. *Q x* = *Q*′ *x*⟧
⟹ ∃ *z*.∀ *x*>*z*. (*P x* ∨ *Q x*) = (*P*′ *x* ∨ *Q*′ *x*)

$\exists (z ::'a::\{linorder\}).\forall x{>}z.(x = t) = False$
$\exists (z ::'a::\{linorder\}).\forall x{>}z.(x \neq t) = True$
$\exists (z ::'a::\{linorder\}).\forall x{>}z.(x < t) = False$
$\exists (z ::'a::\{linorder\}).\forall x{>}z.(x \leq t) = False$
$\exists (z ::'a::\{linorder\}).\forall x{>}z.(x > t) = True$
$\exists (z ::'a::\{linorder\}).\forall x{>}z.(x \geq t) = True$
$\exists z.\forall (x::'b::\{linorder,plus,Rings.dvd\}){>}z.\ (d\ dvd\ x + s) = (d\ dvd\ x + s)$
$\exists z.\forall (x::'b::\{linorder,plus,Rings.dvd\}){>}z.\ (\neg\ d\ dvd\ x + s) = (\neg\ d\ dvd\ x + s)$
$\exists z.\forall x{>}z.\ F = F$
⟨*proof*⟩

lemma *inf-period*:
$[\![\forall x\ k.\ P\ x = P\ (x - k{*}D); \forall x\ k.\ Q\ x = Q\ (x - k{*}D)]\!]$
$\Longrightarrow \forall x\ k.\ (P\ x \wedge Q\ x) = (P\ (x - k{*}D) \wedge Q\ (x - k{*}D))$
$[\![\forall x\ k.\ P\ x = P\ (x - k{*}D); \forall x\ k.\ Q\ x = Q\ (x - k{*}D)]\!]$
$\Longrightarrow \forall x\ k.\ (P\ x \vee Q\ x) = (P\ (x - k{*}D) \vee Q\ (x - k{*}D))$
$(d::'a::\{comm\text{-}ring,Rings.dvd\})\ dvd\ D \Longrightarrow \forall x\ k.\ (d\ dvd\ x + t) = (d\ dvd\ (x - k{*}D) + t)$
$(d::'a::\{comm\text{-}ring,Rings.dvd\})\ dvd\ D \Longrightarrow \forall x\ k.\ (\neg d\ dvd\ x + t) = (\neg d\ dvd\ (x - k{*}D) + t)$
$\forall x\ k.\ F = F$
⟨*proof*⟩

62.2 The A and B sets

lemma *bset*:
$[\![\forall x.(\forall j \in \{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j){\longrightarrow}\ P\ x \longrightarrow P(x - D)\ ;$
$\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j){\longrightarrow}\ Q\ x \longrightarrow Q(x - D)]\!] \Longrightarrow$
$\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (P\ x \wedge Q\ x) \longrightarrow (P(x - D) \wedge Q\ (x - D))$
$[\![\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j){\longrightarrow}\ P\ x \longrightarrow P(x - D)\ ;$
$\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j){\longrightarrow}\ Q\ x \longrightarrow Q(x - D)]\!] \Longrightarrow$
$\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j){\longrightarrow} (P\ x \vee Q\ x) \longrightarrow (P(x - D) \vee Q\ (x - D))$
$[\![D{>}0;\ t - 1{\in}\ B]\!] \Longrightarrow (\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j){\longrightarrow} (x = t) \longrightarrow (x - D = t))$
$[\![D{>}0\ ;\ t \in B]\!] \Longrightarrow (\forall (x::int).(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j){\longrightarrow} (x \neq t) \longrightarrow (x - D \neq t))$
$D{>}0 \Longrightarrow (\forall (x::int).(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j){\longrightarrow} (x < t) \longrightarrow (x - D < t))$
$D{>}0 \Longrightarrow (\forall (x::int).(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j){\longrightarrow} (x \leq t) \longrightarrow (x - D \leq t))$
$[\![D{>}0\ ;\ t \in B]\!] \Longrightarrow (\forall (x::int).(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j){\longrightarrow} (x > t) \longrightarrow (x - D > t))$
$[\![D{>}0\ ;\ t - 1 \in B]\!] \Longrightarrow (\forall (x::int).(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j){\longrightarrow} (x \geq t) \longrightarrow (x - D \geq t))$
$d\ dvd\ D \Longrightarrow (\forall (x::int).(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j){\longrightarrow} (d\ dvd\ x{+}t) \longrightarrow (d\ dvd\ (x - D) + t))$
$d\ dvd\ D \Longrightarrow (\forall (x::int).(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j){\longrightarrow} (\neg d\ dvd\ x{+}t) \longrightarrow (\neg$

d *dvd* $(x - D) + t))$
$\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow F \longrightarrow F$
⟨*proof*⟩

lemma *aset*:
$[\![\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow P\ x \longrightarrow P(x + D)\ ;$
$\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow Q\ x \longrightarrow Q(x + D)]\!] \Longrightarrow$
$\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (P\ x \wedge Q\ x) \longrightarrow (P(x + D) \wedge Q\ (x + D))$
$[\![\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow P\ x \longrightarrow P(x + D)\ ;$
$\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow Q\ x \longrightarrow Q(x + D)]\!] \Longrightarrow$
$\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (P\ x \vee Q\ x) \longrightarrow (P(x + D) \vee Q\ (x + D))$
$[\![D{>}0;\ t + 1{\in}\ A]\!] \Longrightarrow (\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x = t) \longrightarrow (x + D = t))$
$[\![D{>}0\ ;\ t \in A]\!] \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x \neq t) \longrightarrow (x + D \neq t))$
$[\![D{>}0;\ t{\in}\ A]\!] \Longrightarrow (\forall (x{::}int).\ (\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x < t) \longrightarrow (x + D < t))$
$[\![D{>}0;\ t + 1 \in A]\!] \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x \leq t) \longrightarrow (x + D \leq t))$
$D{>}0 \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x > t) \longrightarrow (x + D > t))$
$D{>}0 \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x \geq t) \longrightarrow (x + D \geq t))$
d *dvd* $D \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (d$ *dvd* $x{+}t) \longrightarrow (d$ *dvd* $(x + D) + t))$
d *dvd* $D \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (\neg d$ *dvd* $x{+}t) \longrightarrow (\neg d$ *dvd* $(x + D) + t))$
$\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow F \longrightarrow F$
⟨*proof*⟩

62.3 Cooper's Theorem $-\infty$ and $+\infty$ Version

62.3.1 First some trivial facts about periodic sets or predicates

lemma *periodic-finite-ex*:
assumes *dpos*: $(0{::}int) < d$ **and** *modd*: $\forall x\ k.\ P\ x = P(x - k{*}d)$
shows $(\exists x.\ P\ x) = (\exists j \in \{1..d\}.\ P\ j)$
(**is** *?LHS* = *?RHS*)
⟨*proof*⟩

62.3.2 The $-\infty$ Version

lemma *decr-lemma*: $0 < (d{::}int) \Longrightarrow x - (|x - z| + 1) * d < z$
⟨*proof*⟩

lemma *incr-lemma*: $0 < (d{::}int) \Longrightarrow z < x + (|x - z| + 1) * d$
⟨*proof*⟩

lemma *decr-mult-lemma*:
assumes *dpos*: $(0::int) < d$ **and** *minus*: $\forall x.\ P\ x \longrightarrow P(x - d)$ **and** *knneg*: $0 <= k$
shows $\forall x.\ P\ x \longrightarrow P(x - k*d)$
⟨*proof*⟩

lemma *minusinfinity*:
assumes *dpos*: $0 < d$ **and**
P1eqP1: $\forall x\ k.\ P1\ x = P1(x - k*d)$ **and** *ePeqP1*: $\exists z::int.\ \forall x.\ x < z \longrightarrow (P\ x = P1\ x)$
shows $(\exists x.\ P1\ x) \longrightarrow (\exists x.\ P\ x)$
⟨*proof*⟩

lemma *cpmi*:
assumes *dp*: $0 < D$ **and** *p1*:$\exists z.\ \forall\ x< z.\ P\ x = P'\ x$
and *nb*:$\forall x.(\forall\ j\in \{1..D\}.\ \forall (b::int) \in B.\ x \neq b+j) \longrightarrow P\ (x) \longrightarrow P\ (x - D)$
and *pd*: $\forall\ x\ k.\ P'\ x = P'\ (x-k*D)$
shows $(\exists x.\ P\ x) = ((\exists j \in \{1..D\}\ .\ P'\ j) \vee (\exists j \in \{1..D\}.\ \exists\ b \in B.\ P\ (b+j)))$
(**is** *?L* = (*?R1* ∨ *?R2*))
⟨*proof*⟩

62.3.3 The $+\infty$ Version

lemma *plusinfinity*:
assumes *dpos*: $(0::int) < d$ **and**
P1eqP1: $\forall x\ k.\ P'\ x = P'(x - k*d)$ **and** *ePeqP1*: $\exists\ z.\ \forall\ x>z.\ P\ x = P'\ x$
shows $(\exists\ x.\ P'\ x) \longrightarrow (\exists\ x.\ P\ x)$
⟨*proof*⟩

lemma *incr-mult-lemma*:
assumes *dpos*: $(0::int) < d$ **and** *plus*: $\forall x::int.\ P\ x \longrightarrow P(x + d)$ **and** *knneg*: $0 <= k$
shows $\forall x.\ P\ x \longrightarrow P(x + k*d)$
⟨*proof*⟩

lemma *cppi*:
assumes *dp*: $0 < D$ **and** *p1*:$\exists z.\ \forall\ x> z.\ P\ x = P'\ x$
and *nb*:$\forall x.(\forall\ j\in \{1..D\}.\ \forall (b::int) \in A.\ x \neq b - j) \longrightarrow P\ (x) \longrightarrow P\ (x + D)$
and *pd*: $\forall\ x\ k.\ P'\ x= P'\ (x-k*D)$
shows $(\exists x.\ P\ x) = ((\exists j \in \{1..D\}\ .\ P'\ j) \vee (\exists\ j \in \{1..D\}.\ \exists\ b\in A.\ P\ (b - j)))$
(**is** *?L* = (*?R1* ∨ *?R2*))
⟨*proof*⟩

lemma *simp-from-to*: $\{i..j::int\} = (if\ j < i\ then\ \{\}\ else\ insert\ i\ \{i+1..j\})$
⟨*proof*⟩

theorem *unity-coeff-ex*: $(\exists (x::'a::\{semiring\text{-}0,Rings.dvd\}).\ P\ (l * x)) \equiv (\exists x.\ l\ dvd\ (x + 0) \wedge P\ x)$
⟨*proof*⟩

lemma *zdvd-mono*:
fixes $k\ m\ t :: int$
assumes $k \neq 0$
shows $m\ dvd\ t \equiv k * m\ dvd\ k * t$
⟨*proof*⟩

lemma *uminus-dvd-conv*:
fixes $d\ t :: int$
shows $d\ dvd\ t \equiv -\ d\ dvd\ t$ **and** $d\ dvd\ t \equiv d\ dvd\ -\ t$
⟨*proof*⟩

Theorems for transforming predicates on nat to predicates on *int*

lemma *zdiff-int-split*: $P\ (int\ (x - y)) =$
$((y \leq x \longrightarrow P\ (int\ x - int\ y)) \wedge (x < y \longrightarrow P\ 0))$
⟨*proof*⟩

Specific instances of congruence rules, to prevent simplifier from looping.

theorem *imp-le-cong*:
$[\![x = x';\ 0 \leq x' \Longrightarrow P = P']\!] \Longrightarrow (0 \leq (x::int) \longrightarrow P) = (0 \leq x' \longrightarrow P')$
⟨*proof*⟩

theorem *conj-le-cong*:
$[\![x = x';\ 0 \leq x' \Longrightarrow P = P']\!] \Longrightarrow (0 \leq (x::int) \wedge P) = (0 \leq x' \wedge P')$
⟨*proof*⟩

⟨*ML*⟩

declare *mod-eq-0-iff-dvd* [*presburger*]
declare *mod-by-Suc-0* [*presburger*]
declare *mod-0* [*presburger*]
declare *mod-by-1* [*presburger*]
declare *mod-self* [*presburger*]
declare *div-by-0* [*presburger*]
declare *mod-by-0* [*presburger*]
declare *mod-div-trivial* [*presburger*]
declare *mult-div-mod-eq* [*presburger*]
declare *div-mult-mod-eq* [*presburger*]
declare *mod-mult-self1* [*presburger*]
declare *mod-mult-self2* [*presburger*]
declare *mod2-Suc-Suc* [*presburger*]
declare *not-mod-2-eq-0-eq-1* [*presburger*]
declare *nat-zero-less-power-iff* [*presburger*]

lemma [*presburger*, *algebra*]: $m\ mod\ 2 = (1::nat) \longleftrightarrow \neg\ 2\ dvd\ m$ ⟨*proof*⟩
lemma [*presburger*, *algebra*]: $m\ mod\ 2 = Suc\ 0 \longleftrightarrow \neg\ 2\ dvd\ m$ ⟨*proof*⟩
lemma [*presburger*, *algebra*]: $m\ mod\ (Suc\ (Suc\ 0)) = (1::nat) \longleftrightarrow \neg\ 2\ dvd\ m$
⟨*proof*⟩

lemma [*presburger*, *algebra*]: *m mod (Suc (Suc 0)) = Suc 0 ⟷ ¬ 2 dvd m* ⟨*proof*⟩
lemma [*presburger*, *algebra*]: *m mod 2 = (1::int) ⟷ ¬ 2 dvd m* ⟨*proof*⟩

context *semiring-parity*
begin

declare *even-mult-iff* [*presburger*]

declare *even-power* [*presburger*]

lemma [*presburger*]:
even (a + b) ⟷ even a ∧ even b ∨ odd a ∧ odd b
⟨*proof*⟩

end

context *ring-parity*
begin

declare *even-minus* [*presburger*]

end

context *linordered-idom*
begin

declare *zero-le-power-eq* [*presburger*]

declare *zero-less-power-eq* [*presburger*]

declare *power-less-zero-eq* [*presburger*]

declare *power-le-zero-eq* [*presburger*]

end

declare *even-Suc* [*presburger*]

lemma [*presburger*]:
Suc n div Suc (Suc 0) = n div Suc (Suc 0) ⟷ even n
⟨*proof*⟩

declare *even-diff-nat* [*presburger*]

lemma [*presburger*]:
fixes *k* :: *int*
shows *(k + 1) div 2 = k div 2 ⟷ even k*
⟨*proof*⟩

lemma [*presburger*]:
fixes $k :: int$
shows $(k + 1)\ div\ 2 = k\ div\ 2 + 1 \longleftrightarrow odd\ k$
⟨*proof*⟩

lemma [*presburger*]:
$even\ n \longleftrightarrow even\ (int\ n)$
⟨*proof*⟩

62.4 Nice facts about division by $4::'a$

lemma *even-even-mod-4-iff*:
$even\ (n::nat) \longleftrightarrow even\ (n\ mod\ 4)$
⟨*proof*⟩

lemma *odd-mod-4-div-2*:
$n\ mod\ 4 = (3::nat) \Longrightarrow odd\ ((n - Suc\ 0)\ div\ 2)$
⟨*proof*⟩

lemma *even-mod-4-div-2*:
$n\ mod\ 4 = Suc\ 0 \Longrightarrow even\ ((n - Suc\ 0)\ div\ 2)$
⟨*proof*⟩

62.5 Try0

⟨*ML*⟩

end

63 Misc lemmas on division, to be sorted out finally

theory *Divides*
imports *Parity*
begin

class *unique-euclidean-semiring-numeral* = *unique-euclidean-semiring-with-nat* + *linordered-semidom* +
assumes *div-less* [*no-atp*]: $0 \le a \Longrightarrow a < b \Longrightarrow a\ div\ b = 0$
and *mod-less* [*no-atp*]: $0 \le a \Longrightarrow a < b \Longrightarrow a\ mod\ b = a$
and *div-positive* [*no-atp*]: $0 < b \Longrightarrow b \le a \Longrightarrow a\ div\ b > 0$
and *mod-less-eq-dividend* [*no-atp*]: $0 \le a \Longrightarrow a\ mod\ b \le a$
and *pos-mod-bound* [*no-atp*]: $0 < b \Longrightarrow a\ mod\ b < b$
and *pos-mod-sign* [*no-atp*]: $0 < b \Longrightarrow 0 \le a\ mod\ b$
and *mod-mult2-eq* [*no-atp*]: $0 \le c \Longrightarrow a\ mod\ (b * c) = b * (a\ div\ b\ mod\ c) + a\ mod\ b$
and *div-mult2-eq* [*no-atp*]: $0 \le c \Longrightarrow a\ div\ (b * c) = a\ div\ b\ div\ c$
assumes *discrete* [*no-atp*]: $a < b \longleftrightarrow a + 1 \le b$

hide-fact (**open**) *div-less mod-less mod-less-eq-dividend mod-mult2-eq div-mult2-eq*

context *unique-euclidean-semiring-numeral*
begin

context
begin

lemma *divmod-digit-1* [*no-atp*]:
  assumes $0 \leq a$ $0 < b$ **and** $b \leq a \mathbin{mod} (2 * b)$
  shows $2 * (a \mathbin{div} (2 * b)) + 1 = a \mathbin{div} b$ (**is** *?P*)
    and $a \mathbin{mod} (2 * b) - b = a \mathbin{mod} b$ (**is** *?Q*)
⟨*proof*⟩

lemma *divmod-digit-0* [*no-atp*]:
  assumes $0 < b$ **and** $a \mathbin{mod} (2 * b) < b$
  shows $2 * (a \mathbin{div} (2 * b)) = a \mathbin{div} b$ (**is** *?P*)
    and $a \mathbin{mod} (2 * b) = a \mathbin{mod} b$ (**is** *?Q*)
⟨*proof*⟩

lemma *mod-double-modulus*: — This is actually useful and should be transferred to a suitable type class
  assumes $m > 0$ $x \geq 0$
  shows $x \mathbin{mod} (2 * m) = x \mathbin{mod} m \vee x \mathbin{mod} (2 * m) = x \mathbin{mod} m + m$
⟨*proof*⟩

end

end

instance *nat* :: *unique-euclidean-semiring-numeral*
  ⟨*proof*⟩

instance *int* :: *unique-euclidean-semiring-numeral*
  ⟨*proof*⟩

lemma *zmod-eq-0D* [*dest*!]: $\exists q.\ m = d * q$ **if** $m \mathbin{mod} d = 0$ **for** m d :: *int*
  ⟨*proof*⟩

lemma *div-geq* [*no-atp*]: $m \mathbin{div} n = Suc\ ((m - n) \mathbin{div} n)$ **if** $0 < n$ **and** $\neg\ m < n$ **for** m n :: *nat*
  ⟨*proof*⟩

lemma *mod-geq* [*no-atp*]: $m \mathbin{mod} n = (m - n) \mathbin{mod} n$ **if** $\neg\ m < n$ **for** m n :: *nat*
  ⟨*proof*⟩

lemma *mod-eq-0D* [*no-atp*]: $\exists q.\ m = d * q$ **if** $m \mathbin{mod} d = 0$ **for** m d :: *nat*
  ⟨*proof*⟩

lemma *pos-mod-conj* [*no-atp*]: $0 < b \Longrightarrow 0 \leq a \bmod b \wedge a \bmod b < b$ **for** $a\ b :: int$
⟨*proof*⟩

lemma *neg-mod-conj* [*no-atp*]: $b < 0 \Longrightarrow a \bmod b \leq 0 \wedge b < a \bmod b$ **for** $a\ b :: int$
⟨*proof*⟩

lemma *zmod-eq-0-iff* [*no-atp*]: $m \bmod d = 0 \longleftrightarrow (\exists q.\ m = d * q)$ **for** $m\ d :: int$
⟨*proof*⟩

lemma *div-positive-int* [*no-atp*]:
$k \operatorname{div} l > 0$ **if** $k \geq l$ **and** $l > 0$ **for** $k\ l :: int$
⟨*proof*⟩

code-identifier
code-module *Divides* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

end

64 Bindings to Satisfiability Modulo Theories (SMT) solvers based on SMT-LIB 2

theory *SMT*
imports *Divides Numeral-Simprocs*
keywords *smt-status* :: *diag*
begin

64.1 A skolemization tactic and proof method

lemma *choices*:
$\bigwedge Q.\ \forall x.\ \exists y\ ya.\ Q\ x\ y\ ya \Longrightarrow \exists f\ fa.\ \forall x.\ Q\ x\ (f\ x)\ (fa\ x)$
$\bigwedge Q.\ \forall x.\ \exists y\ ya\ yb.\ Q\ x\ y\ ya\ yb \Longrightarrow \exists f\ fa\ fb.\ \forall x.\ Q\ x\ (f\ x)\ (fa\ x)\ (fb\ x)$
$\bigwedge Q.\ \forall x.\ \exists y\ ya\ yb\ yc.\ Q\ x\ y\ ya\ yb\ yc \Longrightarrow \exists f\ fa\ fb\ fc.\ \forall x.\ Q\ x\ (f\ x)\ (fa\ x)\ (fb\ x)$
$(fc\ x)$
$\bigwedge Q.\ \forall x.\ \exists y\ ya\ yb\ yc\ yd.\ Q\ x\ y\ ya\ yb\ yc\ yd \Longrightarrow$
$\exists f\ fa\ fb\ fc\ fd.\ \forall x.\ Q\ x\ (f\ x)\ (fa\ x)\ (fb\ x)\ (fc\ x)\ (fd\ x)$
$\bigwedge Q.\ \forall x.\ \exists y\ ya\ yb\ yc\ yd\ ye.\ Q\ x\ y\ ya\ yb\ yc\ yd\ ye \Longrightarrow$
$\exists f\ fa\ fb\ fc\ fd\ fe.\ \forall x.\ Q\ x\ (f\ x)\ (fa\ x)\ (fb\ x)\ (fc\ x)\ (fd\ x)\ (fe\ x)$
$\bigwedge Q.\ \forall x.\ \exists y\ ya\ yb\ yc\ yd\ ye\ yf.\ Q\ x\ y\ ya\ yb\ yc\ yd\ ye\ yf \Longrightarrow$
$\exists f\ fa\ fb\ fc\ fd\ fe\ ff.\ \forall x.\ Q\ x\ (f\ x)\ (fa\ x)\ (fb\ x)\ (fc\ x)\ (fd\ x)\ (fe\ x)\ (ff\ x)$
$\bigwedge Q.\ \forall x.\ \exists y\ ya\ yb\ yc\ yd\ ye\ yf\ yg.\ Q\ x\ y\ ya\ yb\ yc\ yd\ ye\ yf\ yg \Longrightarrow$
$\exists f\ fa\ fb\ fc\ fd\ fe\ ff\ fg.\ \forall x.\ Q\ x\ (f\ x)\ (fa\ x)\ (fb\ x)\ (fc\ x)\ (fd\ x)\ (fe\ x)\ (ff\ x)\ (fg\ x)$
⟨*proof*⟩

lemma *bchoices*:
$\bigwedge Q.\ \forall x \in S.\ \exists y\ ya.\ Q\ x\ y\ ya \Longrightarrow \exists f\ fa.\ \forall x \in S.\ Q\ x\ (f\ x)\ (fa\ x)$

$\bigwedge Q.\ \forall x \in S.\ \exists y\ ya\ yb.\ Q\ x\ y\ ya\ yb \Longrightarrow \exists f\ fa\ fb.\ \forall x \in S.\ Q\ x\ (f\ x)\ (fa\ x)\ (fb\ x)$
$\bigwedge Q.\ \forall x \in S.\ \exists y\ ya\ yb\ yc.\ Q\ x\ y\ ya\ yb\ yc \Longrightarrow \exists f\ fa\ fb\ fc.\ \forall x \in S.\ Q\ x\ (f\ x)\ (fa\ x)\ (fb\ x)\ (fc\ x)$
$\bigwedge Q.\ \forall x \in S.\ \exists y\ ya\ yb\ yc\ yd.\ Q\ x\ y\ ya\ yb\ yc\ yd \Longrightarrow$
$\exists f\ fa\ fb\ fc\ fd.\ \forall x \in S.\ Q\ x\ (f\ x)\ (fa\ x)\ (fb\ x)\ (fc\ x)\ (fd\ x)$
$\bigwedge Q.\ \forall x \in S.\ \exists y\ ya\ yb\ yc\ yd\ ye.\ Q\ x\ y\ ya\ yb\ yc\ yd\ ye \Longrightarrow$
$\exists f\ fa\ fb\ fc\ fd\ fe.\ \forall x \in S.\ Q\ x\ (f\ x)\ (fa\ x)\ (fb\ x)\ (fc\ x)\ (fd\ x)\ (fe\ x)$
$\bigwedge Q.\ \forall x \in S.\ \exists y\ ya\ yb\ yc\ yd\ ye\ yf.\ Q\ x\ y\ ya\ yb\ yc\ yd\ ye\ yf \Longrightarrow$
$\exists f\ fa\ fb\ fc\ fd\ fe\ ff.\ \forall x \in S.\ Q\ x\ (f\ x)\ (fa\ x)\ (fb\ x)\ (fc\ x)\ (fd\ x)\ (fe\ x)\ (ff\ x)$
$\bigwedge Q.\ \forall x \in S.\ \exists y\ ya\ yb\ yc\ yd\ ye\ yf\ yg.\ Q\ x\ y\ ya\ yb\ yc\ yd\ ye\ yf\ yg \Longrightarrow$
$\exists f\ fa\ fb\ fc\ fd\ fe\ ff\ fg.\ \forall x \in S.\ Q\ x\ (f\ x)\ (fa\ x)\ (fb\ x)\ (fc\ x)\ (fd\ x)\ (fe\ x)\ (ff\ x)\ (fg\ x)$
$\langle proof \rangle$

$\langle ML \rangle$

hide-fact (**open**) *choices bchoices*

64.2 Triggers for quantifier instantiation

Some SMT solvers support patterns as a quantifier instantiation heuristics. Patterns may either be positive terms (tagged by "pat") triggering quantifier instantiations – when the solver finds a term matching a positive pattern, it instantiates the corresponding quantifier accordingly – or negative terms (tagged by "nopat") inhibiting quantifier instantiations. A list of patterns of the same kind is called a multipattern, and all patterns in a multipattern are considered conjunctively for quantifier instantiation. A list of multipatterns is called a trigger, and their multipatterns act disjunctively during quantifier instantiation. Each multipattern should mention at least all quantified variables of the preceding quantifier block.

typedecl *'a symb-list*

consts
Symb-Nil :: *'a symb-list*
Symb-Cons :: *'a* ⇒ *'a symb-list* ⇒ *'a symb-list*

typedecl *pattern*

consts
pat :: *'a* ⇒ *pattern*
nopat :: *'a* ⇒ *pattern*

definition *trigger* :: *pattern symb-list symb-list* ⇒ *bool* ⇒ *bool* **where**
trigger - P = P

64.3 Higher-order encoding

Application is made explicit for constants occurring with varying numbers of arguments. This is achieved by the introduction of the following constant.

definition *fun-app* :: *'a* ⇒ *'a* **where** *fun-app f* = *f*

Some solvers support a theory of arrays which can be used to encode higher-order functions. The following set of lemmas specifies the properties of such (extensional) arrays.

lemmas *array-rules* = *ext fun-upd-apply fun-upd-same fun-upd-other fun-upd-upd fun-app-def*

64.4 Normalization

lemma *case-bool-if*[*abs-def*]: *case-bool x y P* = (*if P then x else y*)
⟨*proof*⟩

lemmas *Ex1-def-raw* = *Ex1-def*[*abs-def*]
lemmas *Ball-def-raw* = *Ball-def*[*abs-def*]
lemmas *Bex-def-raw* = *Bex-def*[*abs-def*]
lemmas *abs-if-raw* = *abs-if*[*abs-def*]
lemmas *min-def-raw* = *min-def*[*abs-def*]
lemmas *max-def-raw* = *max-def*[*abs-def*]

lemma *nat-zero-as-int*:
0 = *nat 0*
⟨*proof*⟩

lemma *nat-one-as-int*:
1 = *nat 1*
⟨*proof*⟩

lemma *nat-numeral-as-int*: *numeral* = (λ*i*. *nat* (*numeral i*)) ⟨*proof*⟩
lemma *nat-less-as-int*: (<) = (λ*a b*. *int a* < *int b*) ⟨*proof*⟩
lemma *nat-leq-as-int*: (≤) = (λ*a b*. *int a* ≤ *int b*) ⟨*proof*⟩
lemma *Suc-as-int*: *Suc* = (λ*a*. *nat* (*int a* + *1*)) ⟨*proof*⟩
lemma *nat-plus-as-int*: (+) = (λ*a b*. *nat* (*int a* + *int b*)) ⟨*proof*⟩
lemma *nat-minus-as-int*: (−) = (λ*a b*. *nat* (*int a* − *int b*)) ⟨*proof*⟩
lemma *nat-times-as-int*: (∗) = (λ*a b*. *nat* (*int a* ∗ *int b*)) ⟨*proof*⟩
lemma *nat-div-as-int*: (*div*) = (λ*a b*. *nat* (*int a div int b*)) ⟨*proof*⟩
lemma *nat-mod-as-int*: (*mod*) = (λ*a b*. *nat* (*int a mod int b*)) ⟨*proof*⟩

lemma *int-Suc*: *int* (*Suc n*) = *int n* + *1* ⟨*proof*⟩
lemma *int-plus*: *int* (*n* + *m*) = *int n* + *int m* ⟨*proof*⟩
lemma *int-minus*: *int* (*n* − *m*) = *int* (*nat* (*int n* − *int m*)) ⟨*proof*⟩

lemma *nat-int-comparison*:
fixes *a b* :: *nat*
shows (*a* = *b*) = (*int a* = *int b*)

and $(a < b) = (int\ a < int\ b)$
and $(a \leq b) = (int\ a \leq int\ b)$
⟨*proof*⟩

lemma *int-ops*:
fixes $a\ b :: nat$
shows $int\ 0 = 0$
and $int\ 1 = 1$
and $int\ (numeral\ n) = numeral\ n$
and $int\ (Suc\ a) = int\ a + 1$
and $int\ (a + b) = int\ a + int\ b$
and $int\ (a - b) = (if\ int\ a < int\ b\ then\ 0\ else\ int\ a - int\ b)$
and $int\ (a * b) = int\ a * int\ b$
and $int\ (a\ div\ b) = int\ a\ div\ int\ b$
and $int\ (a\ mod\ b) = int\ a\ mod\ int\ b$
⟨*proof*⟩

lemma *int-if*:
fixes $a\ b :: nat$
shows $int\ (if\ P\ then\ a\ else\ b) = (if\ P\ then\ int\ a\ else\ int\ b)$
⟨*proof*⟩

64.5 Integer division and modulo for Z3

The following Z3-inspired definitions are overspecified for the case where $l = 0$. This Schönheitsfehler is corrected in the *div-as-z3div* and *mod-as-z3mod* theorems.

definition *z3div* :: $int \Rightarrow int \Rightarrow int$ **where**
$z3div\ k\ l = (if\ l \geq 0\ then\ k\ div\ l\ else\ -\ (k\ div\ -\ l))$

definition *z3mod* :: $int \Rightarrow int \Rightarrow int$ **where**
$z3mod\ k\ l = k\ mod\ (if\ l \geq 0\ then\ l\ else\ -\ l)$

lemma *div-as-z3div*:
$\forall k\ l.\ k\ div\ l = (if\ l = 0\ then\ 0\ else\ if\ l > 0\ then\ z3div\ k\ l\ else\ z3div\ (-\ k)\ (-\ l))$
⟨*proof*⟩

lemma *mod-as-z3mod*:
$\forall k\ l.\ k\ mod\ l = (if\ l = 0\ then\ k\ else\ if\ l > 0\ then\ z3mod\ k\ l\ else\ -\ z3mod\ (-\ k)\ (-\ l))$
⟨*proof*⟩

64.6 Extra theorems for veriT reconstruction

lemma *verit-sko-forall*: ‹$(\forall x.\ P\ x) \longleftrightarrow P\ (SOME\ x.\ \neg P\ x)$›
⟨*proof*⟩

lemma *verit-sko-forall'*: ‹$P\ (SOME\ x.\ \neg P\ x) = A \Longrightarrow (\forall x.\ P\ x) = A$›
⟨*proof*⟩

lemma *verit-sko-forall''*: ‹$B = A \Longrightarrow (SOME\ x.\ P\ x) = A \equiv (SOME\ x.\ P\ x) = B$›
⟨*proof*⟩

lemma *verit-sko-forall-indirect*: ‹$x = (SOME\ x.\ \neg P\ x) \Longrightarrow (\forall x.\ P\ x) \longleftrightarrow P\ x$›
⟨*proof*⟩

lemma *verit-sko-forall-indirect2*:
‹$x = (SOME\ x.\ \neg P\ x) \Longrightarrow (\bigwedge x :: 'a.\ (P\ x = P'\ x)) \Longrightarrow (\forall x.\ P'\ x) \longleftrightarrow P\ x$›
⟨*proof*⟩

lemma *verit-sko-ex*: ‹$(\exists x.\ P\ x) \longleftrightarrow P\ (SOME\ x.\ P\ x)$›
⟨*proof*⟩

lemma *verit-sko-ex'*: ‹$P\ (SOME\ x.\ P\ x) = A \Longrightarrow (\exists x.\ P\ x) = A$›
⟨*proof*⟩

lemma *verit-sko-ex-indirect*: ‹$x = (SOME\ x.\ P\ x) \Longrightarrow (\exists x.\ P\ x) \longleftrightarrow P\ x$›
⟨*proof*⟩

lemma *verit-sko-ex-indirect2*: ‹$x = (SOME\ x.\ P\ x) \Longrightarrow (\bigwedge x.\ P\ x = P'\ x) \Longrightarrow (\exists x.\ P'\ x) \longleftrightarrow P\ x$›
⟨*proof*⟩

lemma *verit-Pure-trans*:
‹$P \equiv Q \Longrightarrow Q \Longrightarrow P$›
⟨*proof*⟩

lemma *verit-if-cong*:
assumes ‹$b \equiv c$›
and ‹$c \Longrightarrow x \equiv u$›
and ‹$\neg\ c \Longrightarrow y \equiv v$›
shows ‹$(if\ b\ then\ x\ else\ y) \equiv (if\ c\ then\ u\ else\ v)$›
⟨*proof*⟩

lemma *verit-if-weak-cong'*:
‹$b \equiv c \Longrightarrow (if\ b\ then\ x\ else\ y) \equiv (if\ c\ then\ x\ else\ y)$›
⟨*proof*⟩

lemma *verit-or-neg*:
‹$(A \Longrightarrow B) \Longrightarrow B \vee \neg A$›
‹$(\neg A \Longrightarrow B) \Longrightarrow B \vee A$›
⟨*proof*⟩

lemma *verit-subst-bool*: ‹$P \Longrightarrow f\ True \Longrightarrow f\ P$›
⟨*proof*⟩

lemma *verit-and-pos*:

‹(a ⟹ ¬(b ∧ c) ∨ A) ⟹ ¬(a ∧ b ∧ c) ∨ A›
‹(a ⟹ b ⟹ A) ⟹ ¬(a ∧ b) ∨ A›
‹(a ⟹ A) ⟹ ¬a ∨ A›
‹(¬a ⟹ A) ⟹ a ∨ A›
⟨proof⟩

lemma *verit-or-pos*:
‹A ∧ A′ ⟹ (c ∧ A) ∨ (¬c ∧ A′)›
‹A ∧ A′ ⟹ (¬c ∧ A) ∨ (c ∧ A′)›
⟨proof⟩

lemma *verit-la-generic*:
‹(a::int) ≤ x ∨ a = x ∨ a ≥ x›
⟨proof⟩

lemma *verit-bfun-elim*:
‹(if b then P True else P False) = P b›
‹(∀ b. P′ b) = (P′ False ∧ P′ True)›
‹(∃ b. P′ b) = (P′ False ∨ P′ True)›
⟨proof⟩

lemma *verit-eq-true-simplify*:
‹(P = True) ≡ P›
⟨proof⟩

lemma *verit-and-neg*:
‹(a ⟹ ¬b ∨ A) ⟹ ¬(a ∧ b) ∨ A›
‹(a ⟹ A) ⟹ ¬a ∨ A›
‹(¬a ⟹ A) ⟹ a ∨ A›
⟨proof⟩

lemma *verit-forall-inst*:
‹A ⟷ B ⟹ ¬A ∨ B›
‹¬A ⟷ B ⟹ A ∨ B›
‹A ⟷ B ⟹ ¬B ∨ A›
‹A ⟷ ¬B ⟹ B ∨ A›
‹A ⟶ B ⟹ ¬A ∨ B›
‹¬A ⟶ B ⟹ A ∨ B›
⟨proof⟩

lemma *verit-eq-transitive*:
‹A = B ⟹ B = C ⟹ A = C›
‹A = B ⟹ C = B ⟹ A = C›
‹B = A ⟹ B = C ⟹ A = C›
‹B = A ⟹ C = B ⟹ A = C›
⟨proof⟩

lemma *verit-bool-simplify*:

‹¬(*P* ⟶ *Q*) ⟷ *P* ∧ ¬*Q*›
‹¬(*P* ∨ *Q*) ⟷ ¬*P* ∧ ¬*Q*›
‹¬(*P* ∧ *Q*) ⟷ ¬*P* ∨ ¬*Q*›
‹(*P* ⟶ (*Q* ⟶ *R*)) ⟷ ((*P* ∧ *Q*) ⟶ *R*)›
‹((*P* ⟶ *Q*) ⟶ *Q*) ⟷ *P* ∨ *Q*›
‹(*Q* ⟷ (*P* ∨ *Q*)) ⟷ (*P* ⟶ *Q*)› — This rule was inverted
‹*P* ∧ (*P* ⟶ *Q*) ⟷ *P* ∧ *Q*›
‹(*P* ⟶ *Q*) ∧ *P* ⟷ *P* ∧ *Q*›

⟨*proof*⟩

We need the last equation for ¬ (∀ *a b*. ¬ *P a b*)

lemma *verit-connective-def*: — the definition of XOR is missing as the operator is not generated by Isabelle
‹(*A* = *B*) ⟷ ((*A* ⟶ *B*) ∧ (*B* ⟶ *A*))›
‹(*If A B C*) = ((*A* ⟶ *B*) ∧ (¬*A* ⟶ *C*))›
‹(∃ *x*. *P x*) ⟷ ¬(∀ *x*. ¬*P x*)›
‹¬(∃ *x*. *P x*) ⟷ (∀ *x*. ¬*P x*)›
⟨*proof*⟩

lemma *verit-ite-simplify*:
‹(*If True B C*) = *B*›
‹(*If False B C*) = *C*›
‹(*If A' B B*) = *B*›
‹(*If* (¬*A'*) *B C*) = (*If A' C B*)›
‹(*If c* (*If c A B*) *C*) = (*If c A C*)›
‹(*If c C* (*If c A B*)) = (*If c C B*)›
‹(*If A' True False*) = *A'*›
‹(*If A' False True*) ⟷ ¬*A'*›
‹(*If A' True B'*) ⟷ *A'*∨*B'*›
‹(*If A' B' False*) ⟷ *A'*∧*B'*›
‹(*If A' False B'*) ⟷ ¬*A'*∧*B'*›
‹(*If A' B' True*) ⟷ ¬*A'*∨*B'*›
‹*x* ∧ *True* ⟷ *x*›
‹*x* ∨ *False* ⟷ *x*›
for *B C* :: *'a* **and** *A' B' C'* :: *bool*
⟨*proof*⟩

lemma *verit-and-simplify1*:
‹*True* ∧ *b* ⟷ *b*› ‹*b* ∧ *True* ⟷ *b*›
‹*False* ∧ *b* ⟷ *False*› ‹*b* ∧ *False* ⟷ *False*›
‹(*c* ∧ ¬*c*) ⟷ *False*› ‹(¬*c* ∧ *c*) ⟷ *False*›
‹¬¬*a* = *a*›
⟨*proof*⟩

lemmas *verit-and-simplify* = *conj-ac de-Morgan-conj disj-not1*

lemma *verit-or-simplify-1*:

‹*False* ∨ *b* ⟷ *b*› ‹*b* ∨ *False* ⟷ *b*›
‹*b* ∨ ¬*b*›
‹¬*b* ∨ *b*›
⟨*proof*⟩

lemmas *verit-or-simplify* = *disj-ac*

lemma *verit-not-simplify*:
‹¬ ¬*b* ⟷ *b*› ‹¬*True* ⟷ *False*› ‹¬*False* ⟷ *True*›
⟨*proof*⟩

lemma *verit-implies-simplify*:
‹(¬*a* ⟶ ¬*b*) ⟷ (*b* ⟶ *a*)›
‹(*False* ⟶ *a*) ⟷ *True*›
‹(*a* ⟶ *True*) ⟷ *True*›
‹(*True* ⟶ *a*) ⟷ *a*›
‹(*a* ⟶ *False*) ⟷ ¬*a*›
‹(*a* ⟶ *a*) ⟷ *True*›
‹(¬*a* ⟶ *a*) ⟷ *a*›
‹(*a* ⟶ ¬*a*) ⟷ ¬*a*›
‹((*a* ⟶ *b*) ⟶ *b*) ⟷ *a* ∨ *b*›
⟨*proof*⟩

lemma *verit-equiv-simplify*:
‹((¬*a*) = (¬*b*)) ⟷ (*a* = *b*)›
‹(*a* = *a*) ⟷ *True*›
‹(*a* = (¬*a*)) ⟷ *False*›
‹((¬*a*) = *a*) ⟷ *False*›
‹(*True* = *a*) ⟷ *a*›
‹(*a* = *True*) ⟷ *a*›
‹(*False* = *a*) ⟷ ¬*a*›
‹(*a* = *False*) ⟷ ¬*a*›
‹¬¬*a* ⟷ *a*›
‹(¬ *False*) = *True*›
for *a b* :: *bool*
⟨*proof*⟩

lemmas *verit-eq-simplify* =
semiring-char-0-class.eq-numeral-simps eq-refl zero-neq-one num.simps
neg-equal-zero equal-neg-zero one-neq-zero neg-equal-iff-equal

lemma *verit-minus-simplify*:
‹(*a* :: ′*a* :: *cancel-comm-monoid-add*) − *a* = *0*›
‹(*a* :: ′*a* :: *cancel-comm-monoid-add*) − *0* = *a*›
‹*0* − (*b* :: ′*b* :: {*group-add*}) = −*b*›
‹− (− (*b* :: ′*b* :: *group-add*)) = *b*›
⟨*proof*⟩

lemma *verit-sum-simplify*:
‹$(a :: 'a :: \textit{cancel-comm-monoid-add}) + 0 = a$›
⟨*proof*⟩

lemmas *verit-prod-simplify* =

mult-1
mult-1-right

lemma *verit-comp-simplify1*:
‹$(a :: 'a :: \textit{order}) < a \longleftrightarrow \textit{False}$›
‹$a \leq a$›
‹$\neg(b' \leq a') \longleftrightarrow (a' :: 'b :: \textit{linorder}) < b'$›
⟨*proof*⟩

lemmas *verit-comp-simplify* =
verit-comp-simplify1
le-numeral-simps
le-num-simps
less-numeral-simps
less-num-simps
zero-less-one
zero-le-one
less-neg-numeral-simps

lemma *verit-la-disequality*:
‹$(a :: 'a :: \textit{linorder}) = b \vee \neg a \leq b \vee \neg b \leq a$›
⟨*proof*⟩

context
begin

For the reconstruction, we need to keep the order of the arguments.

named-theorems *smt-arith-multiplication* ‹*Theorems to reconstruct arithmetic theorems.*›

named-theorems *smt-arith-combine* ‹*Theorems to reconstruct arithmetic theorems.*›

named-theorems *smt-arith-simplify* ‹*Theorems to combine theorems in the LA procedure*›

lemmas [*smt-arith-simplify*] =
div-add dvd-numeral-simp divmod-steps less-num-simps le-num-simps if-True if-False divmod-cancel
dvd-mult dvd-mult2 less-irrefl prod.case numeral-plus-one divmod-step-def order.refl le-zero-eq
le-numeral-simps less-numeral-simps mult.right-neutral simp-thms divides-aux-eq
mult-nonneg-nonneg dvd-imp-mod-0 dvd-add zero-less-one mod-mult-self4 nu-

meral-mod-numeral
divmod-trivial prod.sel mult.left-neutral div-pos-pos-trivial arith-simps div-add
div-mult-self1
add-le-cancel-left add-le-same-cancel2 not-one-le-zero le-numeral-simps add-le-same-cancel1
zero-neq-one zero-le-one le-num-simps add-Suc mod-div-trivial nat.distinct mult-minus-right
add.inverse-inverse distrib-left-numeral mult-num-simps numeral-times-numeral
add-num-simps
divmod-steps rel-simps if-True if-False numeral-div-numeral divmod-cancel prod.case
add-num-simps one-plus-numeral fst-conv arith-simps sub-num-simps dbl-inc-simps
dbl-simps mult-1 add-le-cancel-right left-diff-distrib-numeral add-uminus-conv-diff
zero-neq-one
zero-le-one One-nat-def add-Suc mod-div-trivial nat.distinct of-int-1 numerals
numeral-One
of-int-numeral add-uminus-conv-diff zle-diff1-eq add-less-same-cancel2 minus-add-distrib
add-uminus-conv-diff mult.left-neutral semiring-class.distrib-right
add-diff-cancel-left' add-diff-eq ring-distribs mult-minus-left minus-diff-eq

lemma [*smt-arith-simplify*]:
‹¬ (*a'* :: *'a* :: *linorder*) < *b'* ⟷ *b'* ≤ *a'*›
‹¬ (*a'* :: *'a* :: *linorder*) ≤ *b'* ⟷ *b'* < *a'*›
‹(*c*::*int*) *mod Numeral1* = *0*›
‹(*a*::*nat*) *mod Numeral1* = *0*›
‹(*c*::*int*) *div Numeral1* = *c*›
‹*a div Numeral1* = *a*›
‹(*c*::*int*) *mod 1* = *0*›
‹*a mod 1* = *0*›
‹(*c*::*int*) *div 1* = *c*›
‹*a div 1* = *a*›
‹¬(*a'* ≠ *b'*) ⟷ *a'* = *b'*›
⟨*proof*⟩

lemma *div-mod-decomp*: *A* = (*A div n*) ∗ *n* + (*A mod n*) **for** *A* :: *nat*
⟨*proof*⟩

lemma *div-less-mono*:
fixes *A B* :: *nat*
assumes *A* < *B 0* < *n* **and**
mod: *A mod n* = *0B mod n* = *0*
shows (*A div n*) < (*B div n*)
⟨*proof*⟩

lemma *verit-le-mono-div*:
fixes *A B* :: *nat*
assumes *A* < *B 0* < *n*
shows (*A div n*) + (*if B mod n* = *0 then 1 else 0*) ≤ (*B div n*)
⟨*proof*⟩

lemmas [*smt-arith-multiplication*] =

verit-le-mono-div[*THEN mult-le-mono1*, *unfolded add-mult-distrib*]
div-le-mono[*THEN mult-le-mono2*, *unfolded add-mult-distrib*]

lemma *div-mod-decomp-int*: $A = (A\ div\ n) * n + (A\ mod\ n)$ **for** $A :: int$
⟨*proof*⟩

lemma *zdiv-mono-strict*:
fixes $A\ B :: int$
assumes $A < B\ 0 < n$ **and**
mod: $A\ mod\ n = 0 B\ mod\ n = 0$
shows $(A\ div\ n) < (B\ div\ n)$
⟨*proof*⟩

lemma *verit-le-mono-div-int*:
‹$A\ div\ n + (if\ B\ mod\ n = 0\ then\ 1\ else\ 0) \leq B\ div\ n$›
if ‹$A < B$› ‹$0 < n$›
for $A\ B\ n :: int$
⟨*proof*⟩

lemma *verit-less-mono-div-int2*:
fixes $A\ B :: int$
assumes $A \leq B\ 0 < -n$
shows $(A\ div\ n) \geq (B\ div\ n)$
⟨*proof*⟩

lemmas [*smt-arith-multiplication*] =
verit-le-mono-div-int[*THEN mult-left-mono*, *unfolded int-distrib*]
zdiv-mono1[*THEN mult-left-mono*, *unfolded int-distrib*]

lemmas [*smt-arith-multiplication*] =
arg-cong[*of* - - ‹$\lambda a :: nat.\ a\ div\ n * p$›] **for** $n\ p :: nat$, *THEN sym*]
arg-cong[*of* - - ‹$\lambda a :: int.\ a\ div\ n * p$›] **for** $n\ p :: int$, *THEN sym*]

lemma [*smt-arith-combine*]:
$a < b \Longrightarrow c < d \Longrightarrow a + c + 2 \leq b + d$
$a < b \Longrightarrow c \leq d \Longrightarrow a + c + 1 \leq b + d$
$a \leq b \Longrightarrow c < d \Longrightarrow a + c + 1 \leq b + d$ **for** $a\ b\ c :: int$
⟨*proof*⟩

lemma [*smt-arith-combine*]:
$a < b \Longrightarrow c < d \Longrightarrow a + c + 2 \leq b + d$
$a < b \Longrightarrow c \leq d \Longrightarrow a + c + 1 \leq b + d$
$a \leq b \Longrightarrow c < d \Longrightarrow a + c + 1 \leq b + d$ **for** $a\ b\ c :: nat$
⟨*proof*⟩

lemmas [*smt-arith-combine*] =
add-strict-mono
add-less-le-mono
add-mono

add-le-less-mono

lemma [*smt-arith-combine*]:
‹$m < n \Longrightarrow c = d \Longrightarrow m + c < n + d$›
‹$m \leq n \Longrightarrow c = d \Longrightarrow m + c \leq n + d$›
‹$c = d \Longrightarrow m < n \Longrightarrow m + c < n + d$›
‹$c = d \Longrightarrow m \leq n \Longrightarrow m + c \leq n + d$›
for m :: ‹$'a$ :: *ordered-cancel-ab-semigroup-add*›
⟨*proof*⟩

lemma *verit-negate-coefficient*:
‹$a \leq (b :: 'a :: \{\textit{ordered-ab-group-add}\}) \Longrightarrow -a \geq -b$›
‹$a < b \Longrightarrow -a > -b$›
‹$a = b \Longrightarrow -a = -b$›
⟨*proof*⟩

end

lemma *verit-ite-intro*:
‹(*if a then P* (*if a then a′ else b′*) *else Q*) ⟷ (*if a then P a′ else Q*)›
‹(*if a then P′ else Q′* (*if a then a′ else b′*)) ⟷ (*if a then P′ else Q′ b′*)›
‹$A = f$ (*if a then R else S*) ⟷ (*if a then A = f R else A = f S*)›
⟨*proof*⟩

lemma *verit-ite-if-cong*:
fixes *x y* :: *bool*
assumes *b*=*c*
and $c \equiv \textit{True} \Longrightarrow x = u$
and $c \equiv \textit{False} \Longrightarrow y = v$
shows (*if b then x else y*) ≡ (*if c then u else v*)
⟨*proof*⟩

64.7 Setup

⟨*ML*⟩

64.8 Configuration

The current configuration can be printed by the command *smt-status*, which shows the values of most options.

64.9 General configuration options

The option *smt-solver* can be used to change the target SMT solver. The possible values can be obtained from the *smt-status* command.

declare [[*smt-solver* = *z3*]]

Since SMT solvers are potentially nonterminating, there is a timeout (given in seconds) to restrict their runtime.

declare [[*smt-timeout = 0*]]

SMT solvers apply randomized heuristics. In case a problem is not solvable by an SMT solver, changing the following option might help.

declare [[*smt-random-seed = 1*]]

In general, the binding to SMT solvers runs as an oracle, i.e, the SMT solvers are fully trusted without additional checks. The following option can cause the SMT solver to run in proof-producing mode, giving a checkable certificate. This is currently implemented only for veriT and Z3.

declare [[*smt-oracle = false*]]

Each SMT solver provides several command-line options to tweak its behaviour. They can be passed to the solver by setting the following options.

declare [[*cvc4-options =]]*
declare [[*cvc5-options = −−proof−format−mode=alethe −−proof−granularity=dsl−rewrite*]]
declare [[*verit-options =]]*
declare [[*z3-options =]]*

The SMT method provides an inference mechanism to detect simple triggers in quantified formulas, which might increase the number of problems solvable by SMT solvers (note: triggers guide quantifier instantiations in the SMT solver). To turn it on, set the following option.

declare [[*smt-infer-triggers = false*]]

Enable the following option to use built-in support for datatypes, codatatypes, and records in CVC4 and cvc5. Currently, this is implemented only in oracle mode.

declare [[*cvc-extensions = false*]]

Enable the following option to use built-in support for div/mod, datatypes, and records in Z3. Currently, this is implemented only in oracle mode.

declare [[*z3-extensions = false*]]

64.10 Certificates

By setting the option *smt-certificates* to the name of a file, all following applications of an SMT solver a cached in that file. Any further application of the same SMT solver (using the very same configuration) re-uses the cached certificate instead of invoking the solver. An empty string disables caching certificates.

The filename should be given as an explicit path. It is good practice to use the name of the current theory (with ending *.certs* instead of *.thy*) as the

certificates file. Certificate files should be used at most once in a certain theory context, to avoid race conditions with other concurrent accesses.

declare [[*smt-certificates* =]]

The option *smt-read-only-certificates* controls whether only stored certificates should be used or invocation of an SMT solver is allowed. When set to *true*, no SMT solver will ever be invoked and only the existing certificates found in the configured cache are used; when set to *false* and there is no cached certificate for some proposition, then the configured SMT solver is invoked.

declare [[*smt-read-only-certificates* = *false*]]

64.11 Tracing

The SMT method, when applied, traces important information. To make it entirely silent, set the following option to *false*.

declare [[*smt-verbose* = *true*]]

For tracing the generated problem file given to the SMT solver as well as the returned result of the solver, the option *smt-trace* should be set to *true*.

declare [[*smt-trace* = *false*]]

64.12 Schematic rules for Z3 proof reconstruction

Several prof rules of Z3 are not very well documented. There are two lemma groups which can turn failing Z3 proof reconstruction attempts into succeeding ones: the facts in *z3-rule* are tried prior to any implemented reconstruction procedure for all uncertain Z3 proof rules; the facts in *z3-simp* are only fed to invocations of the simplifier when reconstructing theory-specific proof steps.

lemmas [*z3-rule*] =
refl eq-commute conj-commute disj-commute simp-thms nnf-simps
ring-distribs field-simps times-divide-eq-right times-divide-eq-left
if-True if-False not-not
NO-MATCH-def

lemma [*z3-rule*]:
$(P \wedge Q) = (\neg\ (\neg\ P \vee \neg\ Q))$
$(P \wedge Q) = (\neg\ (\neg\ Q \vee \neg\ P))$
$(\neg\ P \wedge Q) = (\neg\ (P \vee \neg\ Q))$
$(\neg\ P \wedge Q) = (\neg\ (\neg\ Q \vee P))$
$(P \wedge \neg\ Q) = (\neg\ (\neg\ P \vee Q))$
$(P \wedge \neg\ Q) = (\neg\ (Q \vee \neg\ P))$
$(\neg\ P \wedge \neg\ Q) = (\neg\ (P \vee Q))$
$(\neg\ P \wedge \neg\ Q) = (\neg\ (Q \vee P))$

⟨*proof*⟩

lemma [*z3-rule*]:
$(P \longrightarrow Q) = (Q \vee \neg P)$
$(\neg P \longrightarrow Q) = (P \vee Q)$
$(\neg P \longrightarrow Q) = (Q \vee P)$
$(True \longrightarrow P) = P$
$(P \longrightarrow True) = True$
$(False \longrightarrow P) = True$
$(P \longrightarrow P) = True$
$(\neg (A \longleftrightarrow \neg B)) \longleftrightarrow (A \longleftrightarrow B)$
⟨*proof*⟩

lemma [*z3-rule*]:
$((P = Q) \longrightarrow R) = (R \vee (Q = (\neg P)))$
⟨*proof*⟩

lemma [*z3-rule*]:
$(\neg True) = False$
$(\neg False) = True$
$(x = x) = True$
$(P = True) = P$
$(True = P) = P$
$(P = False) = (\neg P)$
$(False = P) = (\neg P)$
$((\neg P) = P) = False$
$(P = (\neg P)) = False$
$((\neg P) = (\neg Q)) = (P = Q)$
$\neg (P = (\neg Q)) = (P = Q)$
$\neg ((\neg P) = Q) = (P = Q)$
$(P \neq Q) = (Q = (\neg P))$
$(P = Q) = ((\neg P \vee Q) \wedge (P \vee \neg Q))$
$(P \neq Q) = ((\neg P \vee \neg Q) \wedge (P \vee Q))$
⟨*proof*⟩

lemma [*z3-rule*]:
$(if\ P\ then\ P\ else\ \neg P) = True$
$(if\ \neg P\ then\ \neg P\ else\ P) = True$
$(if\ P\ then\ True\ else\ False) = P$
$(if\ P\ then\ False\ else\ True) = (\neg P)$
$(if\ P\ then\ Q\ else\ True) = ((\neg P) \vee Q)$
$(if\ P\ then\ Q\ else\ True) = (Q \vee (\neg P))$
$(if\ P\ then\ Q\ else\ \neg Q) = (P = Q)$
$(if\ P\ then\ Q\ else\ \neg Q) = (Q = P)$
$(if\ P\ then\ \neg Q\ else\ Q) = (P = (\neg Q))$
$(if\ P\ then\ \neg Q\ else\ Q) = ((\neg Q) = P)$
$(if\ \neg P\ then\ x\ else\ y) = (if\ P\ then\ y\ else\ x)$
$(if\ P\ then\ (if\ Q\ then\ x\ else\ y)\ else\ x) = (if\ P \wedge (\neg Q)\ then\ y\ else\ x)$
$(if\ P\ then\ (if\ Q\ then\ x\ else\ y)\ else\ x) = (if\ (\neg Q) \wedge P\ then\ y\ else\ x)$

$(if\ P\ then\ (if\ Q\ then\ x\ else\ y)\ else\ y) = (if\ P \wedge Q\ then\ x\ else\ y)$
$(if\ P\ then\ (if\ Q\ then\ x\ else\ y)\ else\ y) = (if\ Q \wedge P\ then\ x\ else\ y)$
$(if\ P\ then\ x\ else\ if\ P\ then\ y\ else\ z) = (if\ P\ then\ x\ else\ z)$
$(if\ P\ then\ x\ else\ if\ Q\ then\ x\ else\ y) = (if\ P \vee Q\ then\ x\ else\ y)$
$(if\ P\ then\ x\ else\ if\ Q\ then\ x\ else\ y) = (if\ Q \vee P\ then\ x\ else\ y)$
$(if\ P\ then\ x = y\ else\ x = z) = (x = (if\ P\ then\ y\ else\ z))$
$(if\ P\ then\ x = y\ else\ y = z) = (y = (if\ P\ then\ x\ else\ z))$
$(if\ P\ then\ x = y\ else\ z = y) = (y = (if\ P\ then\ x\ else\ z))$
⟨*proof*⟩

lemma [*z3-rule*]:
$0 + (x::int) = x$
$x + 0 = x$
$x + x = 2 * x$
$0 * x = 0$
$1 * x = x$
$x + y = y + x$
⟨*proof*⟩

lemma [*z3-rule*]:
$P = Q \vee P \vee Q$
$P = Q \vee \neg P \vee \neg Q$
$(\neg P) = Q \vee \neg P \vee Q$
$(\neg P) = Q \vee P \vee \neg Q$
$P = (\neg Q) \vee \neg P \vee Q$
$P = (\neg Q) \vee P \vee \neg Q$
$P \neq Q \vee P \vee \neg Q$
$P \neq Q \vee \neg P \vee Q$
$P \neq (\neg Q) \vee P \vee Q$
$(\neg P) \neq Q \vee P \vee Q$
$P \vee Q \vee P \neq (\neg Q)$
$P \vee Q \vee (\neg P) \neq Q$
$P \vee \neg Q \vee P \neq Q$
$\neg P \vee Q \vee P \neq Q$
$P \vee y = (if\ P\ then\ x\ else\ y)$
$P \vee (if\ P\ then\ x\ else\ y) = y$
$\neg P \vee x = (if\ P\ then\ x\ else\ y)$
$\neg P \vee (if\ P\ then\ x\ else\ y) = x$
$P \vee R \vee \neg (if\ P\ then\ Q\ else\ R)$
$\neg P \vee Q \vee \neg (if\ P\ then\ Q\ else\ R)$
$\neg (if\ P\ then\ Q\ else\ R) \vee \neg P \vee Q$
$\neg (if\ P\ then\ Q\ else\ R) \vee P \vee R$
$(if\ P\ then\ Q\ else\ R) \vee \neg P \vee \neg Q$
$(if\ P\ then\ Q\ else\ R) \vee P \vee \neg R$
$(if\ P\ then\ \neg Q\ else\ R) \vee \neg P \vee Q$
$(if\ P\ then\ Q\ else\ \neg R) \vee P \vee R$
⟨*proof*⟩

hide-type (**open**) *symb-list pattern*

hide-const (**open**) *Symb-Nil Symb-Cons trigger pat nopat fun-app z3div z3mod*

end

65 Sledgehammer: Isabelle–ATP Linkup

theory *Sledgehammer*
imports *Presburger SMT*
keywords
sledgehammer :: *diag* **and**
sledgehammer-params :: *thy-decl*
begin

⟨*ML*⟩

end

66 Setup for Lifting/Transfer for the set type

theory *Lifting-Set*
imports *Lifting Groups-Big*
begin

66.1 Relator and predicator properties

lemma *rel-setD1*: ⟦ *rel-set R A B*; $x \in A$ ⟧ ⟹ $\exists y \in B.$ *R x y*
and *rel-setD2*: ⟦ *rel-set R A B*; $y \in B$ ⟧ ⟹ $\exists x \in A.$ *R x y*
⟨*proof*⟩

lemma *rel-set-conversep* [*simp*]: *rel-set* A^{-1-1} = (*rel-set A*)$^{-1-1}$
⟨*proof*⟩

lemma *rel-set-eq* [*relator-eq*]: *rel-set* (=) = (=)
⟨*proof*⟩

lemma *rel-set-mono*[*relator-mono*]:
assumes $A \leq B$
shows *rel-set A* ≤ *rel-set B*
⟨*proof*⟩

lemma *rel-set-OO*[*relator-distr*]: *rel-set R OO rel-set S* = *rel-set* (*R OO S*)
⟨*proof*⟩

lemma *Domainp-set*[*relator-domain*]:
Domainp (*rel-set T*) = (λ*A*. *Ball A* (*Domainp T*))
⟨*proof*⟩

lemma *left-total-rel-set*[*transfer-rule*]:

left-total A $\Longrightarrow$ *left-total* (*rel-set A*)
⟨*proof*⟩

lemma *left-unique-rel-set*[*transfer-rule*]:
left-unique A $\Longrightarrow$ *left-unique* (*rel-set A*)
⟨*proof*⟩

lemma *right-total-rel-set* [*transfer-rule*]:
right-total A $\Longrightarrow$ *right-total* (*rel-set A*)
⟨*proof*⟩

lemma *right-unique-rel-set* [*transfer-rule*]:
right-unique A $\Longrightarrow$ *right-unique* (*rel-set A*)
⟨*proof*⟩

lemma *bi-total-rel-set* [*transfer-rule*]:
bi-total A $\Longrightarrow$ *bi-total* (*rel-set A*)
⟨*proof*⟩

lemma *bi-unique-rel-set* [*transfer-rule*]:
bi-unique A $\Longrightarrow$ *bi-unique* (*rel-set A*)
⟨*proof*⟩

lemma *set-relator-eq-onp* [*relator-eq-onp*]:
rel-set (*eq-onp P*) = *eq-onp* (λA. *Ball A P*)
⟨*proof*⟩

lemma *bi-unique-rel-set-lemma*:
assumes *bi-unique R* **and** *rel-set R X Y*
obtains *f* **where** *Y* = *image f X* **and** *inj-on f X* **and** $\forall x{\in}X.$ *R x* (*f x*)
⟨*proof*⟩

66.2 Quotient theorem for the Lifting package

lemma *Quotient-set*[*quot-map*]:
assumes *Quotient R Abs Rep T*
shows *Quotient* (*rel-set R*) (*image Abs*) (*image Rep*) (*rel-set T*)
⟨*proof*⟩

66.3 Transfer rules for the Transfer package

66.3.1 Unconditional transfer rules

context includes *lifting-syntax*
begin

lemma *empty-transfer* [*transfer-rule*]: (*rel-set A*) {} {}
⟨*proof*⟩

lemma *insert-transfer* [*transfer-rule*]:

(*A* ===> *rel-set A* ===> *rel-set A*) *insert insert*
⟨*proof*⟩

lemma *union-transfer* [*transfer-rule*]:
(*rel-set A* ===> *rel-set A* ===> *rel-set A*) *union union*
⟨*proof*⟩

lemma *Union-transfer* [*transfer-rule*]:
(*rel-set* (*rel-set A*) ===> *rel-set A*) *Union Union*
⟨*proof*⟩

lemma *image-transfer* [*transfer-rule*]:
((*A* ===> *B*) ===> *rel-set A* ===> *rel-set B*) *image image*
⟨*proof*⟩

lemma *UNION-transfer* [*transfer-rule*]: — TODO deletion candidate
(*rel-set A* ===> (*A* ===> *rel-set B*) ===> *rel-set B*) (λ*A f*. ⋃(*f* ‘ *A*)) (λ*A f*. ⋃(*f* ‘ *A*))
⟨*proof*⟩

lemma *Ball-transfer* [*transfer-rule*]:
(*rel-set A* ===> (*A* ===> (=)) ===> (=)) *Ball Ball*
⟨*proof*⟩

lemma *Bex-transfer* [*transfer-rule*]:
(*rel-set A* ===> (*A* ===> (=)) ===> (=)) *Bex Bex*
⟨*proof*⟩

lemma *Pow-transfer* [*transfer-rule*]:
(*rel-set A* ===> *rel-set* (*rel-set A*)) *Pow Pow*
⟨*proof*⟩

lemma *rel-set-transfer* [*transfer-rule*]:
((*A* ===> *B* ===> (=)) ===> *rel-set A* ===> *rel-set B* ===> (=)) *rel-set rel-set*
⟨*proof*⟩

lemma *bind-transfer* [*transfer-rule*]:
(*rel-set A* ===> (*A* ===> *rel-set B*) ===> *rel-set B*) *Set.bind Set.bind*
⟨*proof*⟩

lemma *INF-parametric* [*transfer-rule*]: — TODO deletion candidate
(*rel-set A* ===> (*A* ===> *HOL.eq*) ===> *HOL.eq*) (λ*A f*. *Inf* (*f* ‘ *A*)) (λ*A f*. *Inf* (*f* ‘ *A*))
⟨*proof*⟩

lemma *SUP-parametric* [*transfer-rule*]: — TODO deletion candidate
(*rel-set R* ===> (*R* ===> *HOL.eq*) ===> *HOL.eq*) (λ*A f*. *Sup* (*f* ‘ *A*)) (λ*A f*. *Sup* (*f* ‘ *A*))

⟨*proof*⟩

66.3.2 Rules requiring bi-unique, bi-total or right-total relations

lemma *member-transfer* [*transfer-rule*]:
assumes *bi-unique A*
shows (*A* ===> *rel-set A* ===> (=)) (∈) (∈)
⟨*proof*⟩

lemma *right-total-Collect-transfer*[*transfer-rule*]:
assumes *right-total A*
shows ((*A* ===> (=)) ===> *rel-set A*) (λ*P*. *Collect* (λ*x*. *P x* ∧ *Domainp A x*)) *Collect*
⟨*proof*⟩

lemma *Collect-transfer* [*transfer-rule*]:
assumes *bi-total A*
shows ((*A* ===> (=)) ===> *rel-set A*) *Collect Collect*
⟨*proof*⟩

lemma *inter-transfer* [*transfer-rule*]:
assumes *bi-unique A*
shows (*rel-set A* ===> *rel-set A* ===> *rel-set A*) *inter inter*
⟨*proof*⟩

lemma *Diff-transfer* [*transfer-rule*]:
assumes *bi-unique A*
shows (*rel-set A* ===> *rel-set A* ===> *rel-set A*) (−) (−)
⟨*proof*⟩

lemma *subset-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*rel-set A* ===> *rel-set A* ===> (=)) (⊆) (⊆)
⟨*proof*⟩

context
includes *lifting-syntax*
begin

lemma *strict-subset-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*rel-set A* ===> *rel-set A* ===> (=)) (⊂) (⊂)
⟨*proof*⟩

end

declare *right-total-UNIV-transfer*[*transfer-rule*]

lemma *UNIV-transfer* [*transfer-rule*]:

assumes *bi-total A*
shows (*rel-set A*) *UNIV UNIV*
⟨*proof*⟩

lemma *right-total-Compl-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A* **and** [*transfer-rule*]: *right-total A*
shows (*rel-set A* ===> *rel-set A*) (λ*S*. *uminus S* ∩ *Collect* (*Domainp A*)) *uminus*
⟨*proof*⟩

lemma *Compl-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A* **and** [*transfer-rule*]: *bi-total A*
shows (*rel-set A* ===> *rel-set A*) *uminus uminus*
⟨*proof*⟩

lemma *right-total-Inter-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A* **and** [*transfer-rule*]: *right-total A*
shows (*rel-set* (*rel-set A*) ===> *rel-set A*) (λ*S*. ⋂*S* ∩ *Collect* (*Domainp A*)) *Inter*
⟨*proof*⟩

lemma *Inter-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A* **and** [*transfer-rule*]: *bi-total A*
shows (*rel-set* (*rel-set A*) ===> *rel-set A*) *Inter Inter*
⟨*proof*⟩

lemma *filter-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows ((*A* ===> (=)) ===> *rel-set A* ===> *rel-set A*) *Set.filter Set.filter*
⟨*proof*⟩

lemma *finite-transfer* [*transfer-rule*]:
bi-unique A ⟹ (*rel-set A* ===> (=)) *finite finite*
⟨*proof*⟩

lemma *card-transfer* [*transfer-rule*]:
bi-unique A ⟹ (*rel-set A* ===> (=)) *card card*
⟨*proof*⟩

context
includes *lifting-syntax*
begin

lemma *vimage-right-total-transfer*[*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique B right-total A*
shows ((*A* ===> *B*) ===> *rel-set B* ===> *rel-set A*) (λ*f X*. *f* −‘ *X* ∩ *Collect* (*Domainp A*)) *vimage*
⟨*proof*⟩

end

lemma *vimage-parametric* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-total A bi-unique B*
shows ((*A* ===> *B*) ===> *rel-set B* ===> *rel-set A*) *vimage vimage*
⟨*proof*⟩

lemma *Image-parametric* [*transfer-rule*]:
assumes *bi-unique A*
shows (*rel-set* (*rel-prod A B*) ===> *rel-set A* ===> *rel-set B*) (‘) (‘)
⟨*proof*⟩

lemma *inj-on-transfer*[*transfer-rule*]:
((*A* ===> *B*) ===> *rel-set A* ===> (=)) *inj-on inj-on*
if [*transfer-rule*]: *bi-unique A bi-unique B*
⟨*proof*⟩

end

lemma (**in** *comm-monoid-set*) *F-parametric* [*transfer-rule*]:
fixes *A* :: *'b* ⇒ *'c* ⇒ *bool*
assumes *bi-unique A*
shows *rel-fun* (*rel-fun A* (=)) (*rel-fun* (*rel-set A*) (=)) *F F*
⟨*proof*⟩

lemmas *sum-parametric* = *sum.F-parametric*
lemmas *prod-parametric* = *prod.F-parametric*

lemma *rel-set-UNION*:
assumes [*transfer-rule*]: *rel-set Q A B rel-fun Q* (*rel-set R*) *f g*
shows *rel-set R* (⋃(*f* ‘ *A*)) (⋃(*g* ‘ *B*))
⟨*proof*⟩

context
includes *lifting-syntax*
begin

lemma *fold-graph-transfer*[*transfer-rule*]:
assumes *bi-unique R right-total R*
shows ((*R* ===> (=) ===> (=)) ===> (=) ===> *rel-set R* ===> (=) ===> (=)) *fold-graph fold-graph*
⟨*proof*⟩

lemma *fold-transfer*[*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique R right-total R*
shows ((*R* ===> (=) ===> (=)) ===> (=) ===> *rel-set R* ===> (=)) *Finite-Set.fold Finite-Set.fold*
⟨*proof*⟩

end

end

67 The datatype of finite lists

theory *List*
imports *Sledgehammer Lifting-Set*
begin

datatype (*set*: *'a*) *list* =
  Nil ([])
| *Cons* (*hd*: *'a*) (*tl*: *'a list*) (**infixr** # *65*)
for
  map: *map*
  rel: *list-all2*
  pred: *list-all*
where
  tl [] = []

datatype-compat *list*

lemma [*case-names Nil Cons*, *cases type*: *list*]:
  — for backward compatibility – names of variables differ
  (*y* = [] ⟹ *P*) ⟹ (⋀*a list*. *y* = *a* # *list* ⟹ *P*) ⟹ *P*
⟨*proof*⟩

lemma [*case-names Nil Cons*, *induct type*: *list*]:
  — for backward compatibility – names of variables differ
  P [] ⟹ (⋀*a list*. *P list* ⟹ *P* (*a* # *list*)) ⟹ *P list*
⟨*proof*⟩

Compatibility:

⟨*ML*⟩

lemmas *inducts* = *list.induct*
lemmas *recs* = *list.rec*
lemmas *cases* = *list.case*

⟨*ML*⟩

lemmas *set-simps* = *list.set*

syntax
  — list Enumeration
  -*list* :: *args* => *'a list* ([(-)])

translations

$[x, xs] == x\#[xs]$
$[x] == x\#[]$

67.1 Basic list processing functions

primrec (*nonexhaustive*) *last* :: *'a list* ⇒ *'a* **where**
last (*x* # *xs*) = (*if xs* = [] *then x else last xs*)

primrec *butlast* :: *'a list* ⇒ *'a list* **where**
butlast [] = [] |
butlast (*x* # *xs*) = (*if xs* = [] *then* [] *else x* # *butlast xs*)

lemma *set-rec*: *set xs* = *rec-list* {} (λ*x* -. *insert x*) *xs*
⟨*proof*⟩

definition *coset* :: *'a list* ⇒ *'a set* **where**
[*simp*]: *coset xs* = − *set xs*

primrec *append* :: *'a list* ⇒ *'a list* ⇒ *'a list* (**infixr** @ *65*) **where**
append-Nil: [] @ *ys* = *ys* |
append-Cons: (*x*#*xs*) @ *ys* = *x* # *xs* @ *ys*

primrec *rev* :: *'a list* ⇒ *'a list* **where**
rev [] = [] |
rev (*x* # *xs*) = *rev xs* @ [*x*]

primrec *filter*:: (*'a* ⇒ *bool*) ⇒ *'a list* ⇒ *'a list* **where**
filter P [] = [] |
filter P (*x* # *xs*) = (*if P x then x* # *filter P xs else filter P xs*)

Special input syntax for filter:

syntax (*ASCII*)
-filter :: [*pttrn*, *'a list*, *bool*] => *'a list* ((*1*[-<−-./ -]))
syntax
-filter :: [*pttrn*, *'a list*, *bool*] => *'a list* ((*1*[-←- ./ -]))
translations
[*x*<−*xs* . *P*] ⇀ *CONST filter* (λ*x*. *P*) *xs*

primrec *fold* :: (*'a* ⇒ *'b* ⇒ *'b*) ⇒ *'a list* ⇒ *'b* ⇒ *'b* **where**
fold-Nil: *fold f* [] = *id* |
fold-Cons: *fold f* (*x* # *xs*) = *fold f xs* ∘ *f x*

primrec *foldr* :: (*'a* ⇒ *'b* ⇒ *'b*) ⇒ *'a list* ⇒ *'b* ⇒ *'b* **where**
foldr-Nil: *foldr f* [] = *id* |
foldr-Cons: *foldr f* (*x* # *xs*) = *f x* ∘ *foldr f xs*

primrec *foldl* :: (*'b* ⇒ *'a* ⇒ *'b*) ⇒ *'b* ⇒ *'a list* ⇒ *'b* **where**
foldl-Nil: *foldl f a* [] = *a* |
foldl-Cons: *foldl f a* (*x* # *xs*) = *foldl f* (*f a x*) *xs*

primrec *concat*:: *'a list list ⇒ 'a list* **where**
concat [] = [] |
concat (*x # xs*) = *x @ concat xs*

primrec *drop*:: *nat ⇒ 'a list ⇒ 'a list* **where**
drop-Nil: *drop n* [] = [] |
drop-Cons: *drop n* (*x # xs*) = (*case n of 0 ⇒ x # xs | Suc m ⇒ drop m xs*)
— Warning: simpset does not contain this definition, but separate theorems for *n = 0* and *n = Suc k*

primrec *take*:: *nat ⇒ 'a list ⇒ 'a list* **where**
take-Nil:*take n* [] = [] |
take-Cons: *take n* (*x # xs*) = (*case n of 0 ⇒* [] *| Suc m ⇒ x # take m xs*)
— Warning: simpset does not contain this definition, but separate theorems for *n = 0* and *n = Suc k*

primrec (*nonexhaustive*) *nth* :: *'a list => nat => 'a* (**infixl** ! *100*) **where**
nth-Cons: (*x # xs*) ! *n* = (*case n of 0 ⇒ x | Suc k ⇒ xs ! k*)
— Warning: simpset does not contain this definition, but separate theorems for *n = 0* and *n = Suc k*

primrec *list-update* :: *'a list ⇒ nat ⇒ 'a ⇒ 'a list* **where**
list-update [] *i v* = [] |
list-update (*x # xs*) *i v* =
(*case i of 0 ⇒ v # xs | Suc j ⇒ x # list-update xs j v*)

nonterminal *lupdbinds* **and** *lupdbind*

syntax
-lupdbind:: [*'a*, *'a*] => *lupdbind* ((*2-* :=/ -))
:: *lupdbind* => *lupdbinds* (-)
-lupdbinds :: [*lupdbind*, *lupdbinds*] => *lupdbinds* (-,/ -)
-LUpdate :: [*'a*, *lupdbinds*] => *'a* (-/[(-)] [*1000*,*0*] *900*)

translations
-LUpdate xs (*-lupdbinds b bs*) == *-LUpdate* (*-LUpdate xs b*) *bs*
xs[*i*:=*x*] == *CONST list-update xs i x*

primrec *takeWhile* :: (*'a ⇒ bool*) ⇒ *'a list ⇒ 'a list* **where**
takeWhile P [] = [] |
takeWhile P (*x # xs*) = (*if P x then x # takeWhile P xs else* [])

primrec *dropWhile* :: (*'a ⇒ bool*) ⇒ *'a list ⇒ 'a list* **where**
dropWhile P [] = [] |
dropWhile P (*x # xs*) = (*if P x then dropWhile P xs else x # xs*)

primrec *zip* :: *'a list ⇒ 'b list ⇒* (*'a × 'b*) *list* **where**
zip xs [] = [] |

zip-Cons: *zip xs* (*y # ys*) =
(*case xs of* [] ⇒ [] | *z # zs* ⇒ (*z*, *y*) # *zip zs ys*)
— Warning: simpset does not contain this definition, but separate theorems for *xs* = [] and *xs* = *z # zs*

abbreviation *map2* :: ('*a* ⇒ '*b* ⇒ '*c*) ⇒ '*a list* ⇒ '*b list* ⇒ '*c list* **where**
map2 f xs ys ≡ *map* (λ(*x*,*y*). *f x y*) (*zip xs ys*)

primrec *product* :: '*a list* ⇒ '*b list* ⇒ ('*a* × '*b*) *list* **where**
product [] - = [] |
product (*x*#*xs*) *ys* = *map* (*Pair x*) *ys* @ *product xs ys*

hide-const (**open**) *product*

primrec *product-lists* :: '*a list list* ⇒ '*a list list* **where**
product-lists [] = [[]] |
product-lists (*xs # xss*) = *concat* (*map* (λ*x*. *map* (*Cons x*) (*product-lists xss*)) *xs*)

primrec *upt* :: *nat* ⇒ *nat* ⇒ *nat list* ((*1*[-..</-'])) **where**
upt-0: [*i*..<*0*] = [] |
upt-Suc: [*i*..<(*Suc j*)] = (*if i* ≤ *j then* [*i*..<*j*] @ [*j*] *else* [])

definition *insert* :: '*a* ⇒ '*a list* ⇒ '*a list* **where**
insert x xs = (*if x* ∈ *set xs then xs else x # xs*)

definition *union* :: '*a list* ⇒ '*a list* ⇒ '*a list* **where**
union = *fold insert*

hide-const (**open**) *insert union*
hide-fact (**open**) *insert-def union-def*

primrec *find* :: ('*a* ⇒ *bool*) ⇒ '*a list* ⇒ '*a option* **where**
find - [] = *None* |
find P (*x*#*xs*) = (*if P x then Some x else find P xs*)

In the context of multisets, *count-list* is equivalent to *count* ∘ *mset* and it it advisable to use the latter.

primrec *count-list* :: '*a list* ⇒ '*a* ⇒ *nat* **where**
count-list [] *y* = *0* |
count-list (*x*#*xs*) *y* = (*if x*=*y then count-list xs y* + *1 else count-list xs y*)

definition
extract :: ('*a* ⇒ *bool*) ⇒ '*a list* ⇒ ('*a list* ∗ '*a* ∗ '*a list*) *option*
where *extract P xs* =
(*case dropWhile* (*Not* ∘ *P*) *xs of*
[] ⇒ *None* |
y#*ys* ⇒ *Some*(*takeWhile* (*Not* ∘ *P*) *xs*, *y*, *ys*))

hide-const (**open**) *extract*

primrec *those* :: *'a option list* ⇒ *'a list option*
where
those [] = *Some* [] |
those (*x* # *xs*) = (*case x of*
  None ⇒ *None*
| *Some y* ⇒ *map-option* (*Cons y*) (*those xs*))

primrec *remove1* :: *'a* ⇒ *'a list* ⇒ *'a list* **where**
remove1 x [] = [] |
remove1 x (*y* # *xs*) = (*if x* = *y then xs else y* # *remove1 x xs*)

primrec *removeAll* :: *'a* ⇒ *'a list* ⇒ *'a list* **where**
removeAll x [] = [] |
removeAll x (*y* # *xs*) = (*if x* = *y then removeAll x xs else y* # *removeAll x xs*)

primrec *distinct* :: *'a list* ⇒ *bool* **where**
distinct [] ⟷ *True* |
distinct (*x* # *xs*) ⟷ *x* ∉ *set xs* ∧ *distinct xs*

fun *successively* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *'a list* ⇒ *bool* **where**
successively P [] = *True* |
successively P [*x*] = *True* |
successively P (*x* # *y* # *xs*) = (*P x y* ∧ *successively P* (*y*#*xs*))

definition *distinct-adj* **where**
distinct-adj = *successively* (≠)

primrec *remdups* :: *'a list* ⇒ *'a list* **where**
remdups [] = [] |
remdups (*x* # *xs*) = (*if x* ∈ *set xs then remdups xs else x* # *remdups xs*)

fun *remdups-adj* :: *'a list* ⇒ *'a list* **where**
remdups-adj [] = [] |
remdups-adj [*x*] = [*x*] |
remdups-adj (*x* # *y* # *xs*) = (*if x* = *y then remdups-adj* (*x* # *xs*) *else x* # *remdups-adj* (*y* # *xs*))

primrec *replicate* :: *nat* ⇒ *'a* ⇒ *'a list* **where**
replicate-0: *replicate 0 x* = [] |
replicate-Suc: *replicate* (*Suc n*) *x* = *x* # *replicate n x*

Function *size* is overloaded for all datatypes. Users may refer to the list version as *length*.

abbreviation *length* :: *'a list* ⇒ *nat* **where**
length ≡ *size*

definition *enumerate* :: *nat* ⇒ *'a list* ⇒ (*nat* × *'a*) *list* **where**
enumerate-eq-zip: *enumerate n xs* = *zip* [*n*..<*n* + *length xs*] *xs*

primrec *rotate1* :: *'a list* ⇒ *'a list* **where**
rotate1 [] = [] |
rotate1 (*x* # *xs*) = *xs* @ [*x*]

definition *rotate* :: *nat* ⇒ *'a list* ⇒ *'a list* **where**
rotate n = *rotate1* ^^ *n*

definition *nths* :: *'a list* => *nat set* => *'a list* **where**
nths xs A = *map fst* (*filter* (λ*p*. *snd p* ∈ *A*) (*zip xs* [*0*..<*size xs*]))

primrec *subseqs* :: *'a list* ⇒ *'a list list* **where**
subseqs [] = [[]] |
subseqs (*x*#*xs*) = (*let xss* = *subseqs xs in map* (*Cons x*) *xss* @ *xss*)

primrec *n-lists* :: *nat* ⇒ *'a list* ⇒ *'a list list* **where**
n-lists 0 xs = [[]] |
n-lists (*Suc n*) *xs* = *concat* (*map* (λ*ys*. *map* (λ*y*. *y* # *ys*) *xs*) (*n-lists n xs*))

hide-const (**open**) *n-lists*

function *splice* :: *'a list* ⇒ *'a list* ⇒ *'a list* **where**
splice [] *ys* = *ys* |
splice (*x*#*xs*) *ys* = *x* # *splice ys xs*
⟨*proof*⟩

termination
⟨*proof*⟩

function *shuffles* **where**
  shuffles [] *ys* = {*ys*}
| *shuffles xs* [] = {*xs*}
| *shuffles* (*x* # *xs*) (*y* # *ys*) = (#) *x* ‘ *shuffles xs* (*y* # *ys*) ∪ (#) *y* ‘ *shuffles* (*x* # *xs*) *ys*
  ⟨*proof*⟩
termination ⟨*proof*⟩

Use only if you cannot use *Min* instead:

fun *min-list* :: *'a*::*ord list* ⇒ *'a* **where**
min-list (*x* # *xs*) = (*case xs of* [] ⇒ *x* | - ⇒ *min x* (*min-list xs*))

Returns first minimum:

fun *arg-min-list* :: (*'a* ⇒ (*'b*::*linorder*)) ⇒ *'a list* ⇒ *'a* **where**
arg-min-list f [*x*] = *x* |
arg-min-list f (*x*#*y*#*zs*) = (*let m* = *arg-min-list f* (*y*#*zs*) *in if f x* ≤ *f m then x else m*)

Figure 1 shows characteristic examples that should give an intuitive understanding of the above functions.

$[a, b] @ [c, d] = [a, b, c, d]$
length $[a, b, c] = 3$
set $[a, b, c] = \{a, b, c\}$
map f $[a, b, c] = [f\ a, f\ b, f\ c]$
rev $[a, b, c] = [c, b, a]$
hd $[a, b, c, d] = a$
tl $[a, b, c, d] = [b, c, d]$
last $[a, b, c, d] = d$
butlast $[a, b, c, d] = [a, b, c]$
filter $(\lambda n{::}nat.\ n{<}2)\ [0,2,1] = [0,1]$
concat $[[a, b], [c, d, e], [], [f]] = [a, b, c, d, e, f]$
fold f $[a, b, c]\ x = f\ c\ (f\ b\ (f\ a\ x))$
foldr f $[a, b, c]\ x = f\ a\ (f\ b\ (f\ c\ x))$
foldl f x $[a, b, c] = f\ (f\ (f\ x\ a)\ b)\ c$
successively $(\neq)$ $[True, False, True, False]$
zip $[a, b, c]\ [x, y, z] = [(a, x), (b, y), (c, z)]$
zip $[a, b]\ [x, y, z] = [(a, x), (b, y)]$
enumerate 3 $[a, b, c] = [(3, a), (4, b), (5, c)]$
List.product $[a, b]\ [c, d] = [(a, c), (a, d), (b, c), (b, d)]$
product-lists $[[a, b], [c], [d, e]] = [[a, c, d], [a, c, e], [b, c, d], [b, c, e]]$
splice $[a, b, c]\ [x, y, z] = [a, x, b, y, c, z]$
splice $[a, b, c, d]\ [x, y] = [a, x, b, y, c, d]$
shuffles $[a, b]\ [c, d] = \{[a, b, c, d], [a, c, b, d], [a, c, d, b], [c, a, b, d], [c, a, d, b], [c, d, a, b]\}$
take 2 $[a, b, c, d] = [a, b]$
take 6 $[a, b, c, d] = [a, b, c, d]$
drop 2 $[a, b, c, d] = [c, d]$
drop 6 $[a, b, c, d] = []$
takeWhile $(\lambda n.\ n < 3)\ [1, 2, 3, 0] = [1, 2]$
dropWhile $(\lambda n.\ n < 3)\ [1, 2, 3, 0] = [3, 0]$
distinct $[2, 0, 1]$
remdups $[2, 0, 2, 1, 2] = [0, 1, 2]$
remdups-adj $[2, 2, 3, 1, 1, 2, 1] = [2, 3, 1, 2, 1]$
List.insert 2 $[0, 1, 2] = [0, 1, 2]$
List.insert 3 $[0, 1, 2] = [3, 0, 1, 2]$
List.union $[2, 3, 4]\ [0, 1, 2] = [4, 3, 0, 1, 2]$
find $((<)\ 0)\ [0, 0] = None$
find $((<)\ 0)\ [0, 1, 0, 2] = Some\ 1$
count-list $[0, 1, 0, 2]\ 0 = 2$
List.extract $((<)\ 0)\ [0, 0] = None$
List.extract $((<)\ 0)\ [0, 1, 0, 2] = Some\ ([0], 1, [0, 2])$
remove1 2 $[2, 0, 2, 1, 2] = [0, 2, 1, 2]$
removeAll 2 $[2, 0, 2, 1, 2] = [0, 1]$
$[a, b, c, d]\ !\ 2 = c$
$[a, b, c, d][2 := x] = [a, b, x, d]$
nths $[a, b, c, d, e]\ \{0, 2, 3\} = [a, c, d]$
subseqs $[a, b] = [[a, b], [a], [b], []]$
List.n-lists 2 $[a, b, c] = [[a, a], [b, a], [c, a], [a, b], [b, b], [c, b], [a, c], [b, c], [c, c]]$
rotate1 $[a, b, c, d] = [b, c, d, a]$
rotate 3 $[a, b, c, d] = [d, a, b, c]$
replicate 4 $a = [a, a, a, a]$
$[2..{<}5] = [2, 3, 4]$
min-list $[3, 1, -2] = -2$
arg-min-list $(\lambda i.\ i * i)\ [3, -1, 1, -2] = -1$

The following simple sort(ed) functions are intended for proofs, not for efficient implementations.

A sorted predicate w.r.t. a relation:

fun *sorted-wrt* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a\ list \Rightarrow bool$ **where**
sorted-wrt P [] = *True* |
sorted-wrt P (*x* # *ys*) = (($\forall$ *y* $\in$ *set ys*. *P x y*) $\wedge$ *sorted-wrt P ys*)

A class-based sorted predicate:

context *linorder*
begin

abbreviation *sorted* :: $'a\ list \Rightarrow bool$ **where**
sorted $\equiv$ *sorted-wrt* ($\leq$)

lemma *sorted-simps*: *sorted* [] = *True sorted* (*x* # *ys*) = (($\forall$ *y* $\in$ *set ys*. *x*$\leq$*y*) $\wedge$ *sorted ys*)
⟨*proof*⟩

lemma *strict-sorted-simps*: *sorted-wrt* ($<$) [] = *True sorted-wrt* ($<$) (*x* # *ys*) = (($\forall$ *y* $\in$ *set ys*. *x*$<$*y*) $\wedge$ *sorted-wrt* ($<$) *ys*)
⟨*proof*⟩

primrec *insort-key* :: $('b \Rightarrow 'a) \Rightarrow 'b \Rightarrow 'b\ list \Rightarrow 'b\ list$ **where**
insort-key f x [] = [*x*] |
insort-key f x (*y*#*ys*) =
(*if f x* $\leq$ *f y then* (*x*#*y*#*ys*) *else y*#(*insort-key f x ys*))

definition *sort-key* :: $('b \Rightarrow 'a) \Rightarrow 'b\ list \Rightarrow 'b\ list$ **where**
sort-key f xs = *foldr* (*insort-key f*) *xs* []

definition *insort-insert-key* :: $('b \Rightarrow 'a) \Rightarrow 'b \Rightarrow 'b\ list \Rightarrow 'b\ list$ **where**
insort-insert-key f x xs =
(*if f x* $\in$ *f* ‘ *set xs then xs else insort-key f x xs*)

abbreviation *sort* $\equiv$ *sort-key* ($\lambda x.\ x$)
abbreviation *insort* $\equiv$ *insort-key* ($\lambda x.\ x$)
abbreviation *insort-insert* $\equiv$ *insort-insert-key* ($\lambda x.\ x$)

definition *stable-sort-key* :: $(('b \Rightarrow 'a) \Rightarrow 'b\ list \Rightarrow 'b\ list) \Rightarrow bool$ **where**
stable-sort-key sk =
($\forall$ *f xs k*. *filter* ($\lambda y.\ f\ y = k$) (*sk f xs*) = *filter* ($\lambda y.\ f\ y = k$) *xs*)

lemma *strict-sorted-iff*: *sorted-wrt* ($<$) *l* $\longleftrightarrow$ *sorted l* $\wedge$ *distinct l*
⟨*proof*⟩

lemma *strict-sorted-imp-sorted*: *sorted-wrt* ($<$) *xs* $\Longrightarrow$ *sorted xs*
⟨*proof*⟩

end

67.1.1 List comprehension

Input syntax for Haskell-like list comprehension notation. Typical example: $[(x,y).\ x \leftarrow xs,\ y \leftarrow ys,\ x \neq y]$, the list of all pairs of distinct elements from *xs* and *ys*. The syntax is as in Haskell, except that | becomes a dot (like in Isabelle's set comprehension): $[e.\ x \leftarrow xs,\ \ldots]$ rather than `[e| x <- xs, ...]`.
The qualifiers after the dot are

generators $p \leftarrow xs$, where p is a pattern and *xs* an expression of list type, or

guards b, where b is a boolean expression.

Just like in Haskell, list comprehension is just a shorthand. To avoid misunderstandings, the translation into desugared form is not reversed upon output. Note that the translation of $[e.\ x \leftarrow xs]$ is optmized to $map\ (\lambda x.\ e)\ xs$.
It is easy to write short list comprehensions which stand for complex expressions. During proofs, they may become unreadable (and mangled). In such cases it can be advisable to introduce separate definitions for the list comprehensions in question.

nonterminal *lc-qual* **and** *lc-quals*

syntax
-listcompr :: $'a \Rightarrow$ *lc-qual* $\Rightarrow$ *lc-quals* $\Rightarrow$ $'a$ *list* ([- . --)
-lc-gen :: $'a \Rightarrow$ $'a$ *list* $\Rightarrow$ *lc-qual* (- $\leftarrow$ -)
-lc-test :: *bool* $\Rightarrow$ *lc-qual* (-)

-lc-end :: *lc-quals* (])
-lc-quals :: *lc-qual* $\Rightarrow$ *lc-quals* $\Rightarrow$ *lc-quals* (, --)

syntax (*ASCII*)
-lc-gen :: $'a \Rightarrow$ $'a$ *list* $\Rightarrow$ *lc-qual* (- <− -)

⟨*ML*⟩

code-datatype *set coset*
hide-const (**open**) *coset*

67.1.2 [] and (#)

lemma *not-Cons-self* [*simp*]:
$xs \neq x \# xs$
⟨*proof*⟩

lemma *not-Cons-self2* [*simp*]: $x \# xs \neq xs$
⟨*proof*⟩

lemma *neq-Nil-conv*: $(xs \neq []) = (\exists y\ ys.\ xs = y \# ys)$
⟨*proof*⟩

lemma *tl-Nil*: $tl\ xs = [] \longleftrightarrow xs = [] \vee (\exists x.\ xs = [x])$
⟨*proof*⟩

lemmas *Nil-tl* = *tl-Nil*[*THEN eq-iff-swap*]

lemma *length-induct*:
$(\bigwedge xs.\ \forall ys.\ length\ ys < length\ xs \longrightarrow P\ ys \Longrightarrow P\ xs) \Longrightarrow P\ xs$
⟨*proof*⟩

lemma *induct-list012*:
$[\![P\ []; \bigwedge x.\ P\ [x]; \bigwedge x\ y\ zs.\ [\![\ P\ zs; P\ (y \# zs)\ ]\!] \Longrightarrow P\ (x \# y \# zs)]\!] \Longrightarrow P\ xs$
⟨*proof*⟩

lemma *list-nonempty-induct* [*consumes 1*, *case-names single cons*]:
$[\![\ xs \neq []; \bigwedge x.\ P\ [x]; \bigwedge x\ xs.\ xs \neq [] \Longrightarrow P\ xs \Longrightarrow P\ (x \# xs)]\!] \Longrightarrow P\ xs$
⟨*proof*⟩

lemma *inj-split-Cons*: *inj-on* $(\lambda(xs,\ n).\ n\#xs)\ X$
⟨*proof*⟩

lemma *inj-on-Cons1* [*simp*]: *inj-on* $((\#)\ x)\ A$
⟨*proof*⟩

67.1.3 *length*

Needs to come before @ because of theorem *append-eq-append-conv*.

lemma *length-append* [*simp*]: $length\ (xs\ @\ ys) = length\ xs + length\ ys$
⟨*proof*⟩

lemma *length-map* [*simp*]: $length\ (map\ f\ xs) = length\ xs$
⟨*proof*⟩

lemma *length-rev* [*simp*]: $length\ (rev\ xs) = length\ xs$
⟨*proof*⟩

lemma *length-tl* [*simp*]: $length\ (tl\ xs) = length\ xs - 1$
⟨*proof*⟩

lemma *length-0-conv* [*iff*]: $(length\ xs = 0) = (xs = [])$
⟨*proof*⟩

lemma *length-greater-0-conv* [*iff*]: $(0 < length\ xs) = (xs \neq [])$

⟨*proof*⟩

lemma *length-pos-if-in-set*: $x \in set\ xs \Longrightarrow length\ xs > 0$
⟨*proof*⟩

lemma *length-Suc-conv*: $(length\ xs = Suc\ n) = (\exists\, y\ ys.\ xs = y \#\ ys \wedge length\ ys = n)$
⟨*proof*⟩

lemmas *Suc-length-conv* = *length-Suc-conv*[*THEN eq-iff-swap*]

lemma *Suc-le-length-iff*:
$(Suc\ n \leq length\ xs) = (\exists\, x\ ys.\ xs = x \#\ ys \wedge n \leq length\ ys)$
⟨*proof*⟩

lemma *impossible-Cons*: $length\ xs \leq length\ ys \Longrightarrow xs = x \#\ ys = False$
⟨*proof*⟩

lemma *list-induct2* [*consumes 1*, *case-names Nil Cons*]:
$length\ xs = length\ ys \Longrightarrow P\ []\ [] \Longrightarrow$
$(\bigwedge x\ xs\ y\ ys.\ length\ xs = length\ ys \Longrightarrow P\ xs\ ys \Longrightarrow P\ (x\#xs)\ (y\#ys))$
$\Longrightarrow P\ xs\ ys$
⟨*proof*⟩

lemma *list-induct3* [*consumes 2*, *case-names Nil Cons*]:
$length\ xs = length\ ys \Longrightarrow length\ ys = length\ zs \Longrightarrow P\ []\ []\ [] \Longrightarrow$
$(\bigwedge x\ xs\ y\ ys\ z\ zs.\ length\ xs = length\ ys \Longrightarrow length\ ys = length\ zs \Longrightarrow P\ xs\ ys\ zs \Longrightarrow P\ (x\#xs)\ (y\#ys)\ (z\#zs))$
$\Longrightarrow P\ xs\ ys\ zs$
⟨*proof*⟩

lemma *list-induct4* [*consumes 3*, *case-names Nil Cons*]:
$length\ xs = length\ ys \Longrightarrow length\ ys = length\ zs \Longrightarrow length\ zs = length\ ws \Longrightarrow$
$P\ []\ []\ []\ [] \Longrightarrow (\bigwedge x\ xs\ y\ ys\ z\ zs\ w\ ws.\ length\ xs = length\ ys \Longrightarrow$
$length\ ys = length\ zs \Longrightarrow length\ zs = length\ ws \Longrightarrow P\ xs\ ys\ zs\ ws \Longrightarrow$
$P\ (x\#xs)\ (y\#ys)\ (z\#zs)\ (w\#ws)) \Longrightarrow P\ xs\ ys\ zs\ ws$
⟨*proof*⟩

lemma *list-induct2'*:
$[\![\ P\ []\ [];$
$\bigwedge x\ xs.\ P\ (x\#xs)\ [];$
$\bigwedge y\ ys.\ P\ []\ (y\#ys);$
$\bigwedge x\ xs\ y\ ys.\ P\ xs\ ys \Longrightarrow P\ (x\#xs)\ (y\#ys)\ ]\!]$
$\Longrightarrow P\ xs\ ys$
⟨*proof*⟩

lemma *list-all2-iff*:
$list\text{-}all2\ P\ xs\ ys \longleftrightarrow length\ xs = length\ ys \wedge (\forall\, (x,\ y) \in set\ (zip\ xs\ ys).\ P\ x\ y)$
⟨*proof*⟩

lemma *neq-if-length-neq*: *length xs* ≠ *length ys* ⟹ (*xs* = *ys*) == *False*
⟨*proof*⟩

67.1.4 @ − append

global-interpretation *append*: *monoid append Nil*
⟨*proof*⟩

lemma *append-assoc* [*simp*]: (*xs* @ *ys*) @ *zs* = *xs* @ (*ys* @ *zs*)
⟨*proof*⟩

lemma *append-Nil2*: *xs* @ [] = *xs*
⟨*proof*⟩

lemma *append-is-Nil-conv* [*iff*]: (*xs* @ *ys* = []) = (*xs* = [] ∧ *ys* = [])
⟨*proof*⟩

lemmas *Nil-is-append-conv* [*iff*] = *append-is-Nil-conv*[*THEN eq-iff-swap*]

lemma *append-self-conv* [*iff*]: (*xs* @ *ys* = *xs*) = (*ys* = [])
⟨*proof*⟩

lemmas *self-append-conv* [*iff*] = *append-self-conv*[*THEN eq-iff-swap*]

lemma *append-eq-append-conv* [*simp*]:
length xs = *length ys* ∨ *length us* = *length vs*
⟹ (*xs*@*us* = *ys*@*vs*) = (*xs*=*ys* ∧ *us*=*vs*)
⟨*proof*⟩

lemma *append-eq-append-conv2*: (*xs* @ *ys* = *zs* @ *ts*) =
(∃ *us*. *xs* = *zs* @ *us* ∧ *us* @ *ys* = *ts* ∨ *xs* @ *us* = *zs* ∧ *ys* = *us* @ *ts*)
⟨*proof*⟩

lemma *same-append-eq* [*iff*, *induct-simp*]: (*xs* @ *ys* = *xs* @ *zs*) = (*ys* = *zs*)
⟨*proof*⟩

lemma *append1-eq-conv* [*iff*]: (*xs* @ [*x*] = *ys* @ [*y*]) = (*xs* = *ys* ∧ *x* = *y*)
⟨*proof*⟩

lemma *append-same-eq* [*iff*, *induct-simp*]: (*ys* @ *xs* = *zs* @ *xs*) = (*ys* = *zs*)
⟨*proof*⟩

lemma *append-self-conv2* [*iff*]: (*xs* @ *ys* = *ys*) = (*xs* = [])
⟨*proof*⟩

lemmas *self-append-conv2* [*iff*] = *append-self-conv2*[*THEN eq-iff-swap*]

lemma *hd-Cons-tl*: *xs* ≠ [] ⟹ *hd xs* # *tl xs* = *xs*

⟨*proof*⟩

lemma *hd-append*: *hd* (*xs* @ *ys*) = (*if xs* = [] *then hd ys else hd xs*)
⟨*proof*⟩

lemma *hd-append2* [*simp*]: *xs* ≠ [] ⟹ *hd* (*xs* @ *ys*) = *hd xs*
⟨*proof*⟩

lemma *tl-append*: *tl* (*xs* @ *ys*) = (*case xs of* [] ⇒ *tl ys* | *z*#*zs* ⇒ *zs* @ *ys*)
⟨*proof*⟩

lemma *tl-append2* [*simp*]: *xs* ≠ [] ⟹ *tl* (*xs* @ *ys*) = *tl xs* @ *ys*
⟨*proof*⟩

lemma *tl-append-if*: *tl* (*xs* @ *ys*) = (*if xs* = [] *then tl ys else tl xs* @ *ys*)
⟨*proof*⟩

lemma *Cons-eq-append-conv*: *x*#*xs* = *ys*@*zs* =
(*ys* = [] ∧ *x*#*xs* = *zs* ∨ (∃ *ys'*. *x*#*ys'* = *ys* ∧ *xs* = *ys'*@*zs*))
⟨*proof*⟩

lemma *append-eq-Cons-conv*: (*ys*@*zs* = *x*#*xs*) =
(*ys* = [] ∧ *zs* = *x*#*xs* ∨ (∃ *ys'*. *ys* = *x*#*ys'* ∧ *ys'*@*zs* = *xs*))
⟨*proof*⟩

lemma *longest-common-prefix*:
∃ *ps xs' ys'*. *xs* = *ps* @ *xs'* ∧ *ys* = *ps* @ *ys'*
∧ (*xs'* = [] ∨ *ys'* = [] ∨ *hd xs'* ≠ *hd ys'*)
⟨*proof*⟩

Trivial rules for solving @-equations automatically.

lemma *eq-Nil-appendI*: *xs* = *ys* ⟹ *xs* = [] @ *ys*
⟨*proof*⟩

lemma *Cons-eq-appendI*: ⟦*x* # *xs1* = *ys*; *xs* = *xs1* @ *zs*⟧ ⟹ *x* # *xs* = *ys* @ *zs*
⟨*proof*⟩

lemma *append-eq-appendI*: ⟦*xs* @ *xs1* = *zs*; *ys* = *xs1* @ *us*⟧ ⟹ *xs* @ *ys* = *zs* @ *us*
⟨*proof*⟩

Simplification procedure for all list equalities. Currently only tries to rearrange @ to see if - both lists end in a singleton list, - or both lists end in the same list.

⟨*ML*⟩

67.1.5 *map*

lemma *hd-map*: *xs* ≠ [] ⟹ *hd* (*map f xs*) = *f* (*hd xs*)

⟨*proof*⟩

lemma *map-tl*: *map f (tl xs) = tl (map f xs)*
⟨*proof*⟩

lemma *map-ext*: *(⋀x. x ∈ set xs ⟶ f x = g x) ⟹ map f xs = map g xs*
⟨*proof*⟩

lemma *map-ident* [*simp*]: *map (λx. x) = (λxs. xs)*
⟨*proof*⟩

lemma *map-append* [*simp*]: *map f (xs @ ys) = map f xs @ map f ys*
⟨*proof*⟩

lemma *map-map* [*simp*]: *map f (map g xs) = map (f ∘ g) xs*
⟨*proof*⟩

lemma *map-comp-map*[*simp*]: *((map f) ∘ (map g)) = map(f ∘ g)*
⟨*proof*⟩

lemma *rev-map*: *rev (map f xs) = map f (rev xs)*
⟨*proof*⟩

lemma *map-eq-conv*[*simp*]: *(map f xs = map g xs) = (∀x ∈ set xs. f x = g x)*
⟨*proof*⟩

lemma *map-cong* [*fundef-cong*]:
xs = ys ⟹ (⋀x. x ∈ set ys ⟹ f x = g x) ⟹ map f xs = map g ys
⟨*proof*⟩

lemma *map-is-Nil-conv* [*iff*]: *(map f xs = []) = (xs = [])*
⟨*proof*⟩

lemmas *Nil-is-map-conv* [*iff*] = *map-is-Nil-conv*[*THEN eq-iff-swap*]

lemma *map-eq-Cons-conv*:
(map f xs = y#ys) = (∃z zs. xs = z#zs ∧ f z = y ∧ map f zs = ys)
⟨*proof*⟩

lemma *Cons-eq-map-conv*:
(x#xs = map f ys) = (∃z zs. ys = z#zs ∧ x = f z ∧ xs = map f zs)
⟨*proof*⟩

lemmas *map-eq-Cons-D* = *map-eq-Cons-conv* [*THEN iffD1*]
lemmas *Cons-eq-map-D* = *Cons-eq-map-conv* [*THEN iffD1*]
declare *map-eq-Cons-D* [*dest*!] *Cons-eq-map-D* [*dest*!]

lemma *ex-map-conv*:
(∃xs. ys = map f xs) = (∀y ∈ set ys. ∃x. y = f x)

⟨*proof*⟩

lemma *map-eq-imp-length-eq*:
assumes *map f xs* = *map g ys*
shows *length xs* = *length ys*
⟨*proof*⟩

lemma *map-inj-on*:
assumes *map*: *map f xs* = *map f ys* **and** *inj*: *inj-on f* (*set xs Un set ys*)
shows *xs* = *ys*
⟨*proof*⟩

lemma *inj-on-map-eq-map*:
inj-on f (*set xs Un set ys*) ⟹ (*map f xs* = *map f ys*) = (*xs* = *ys*)
⟨*proof*⟩

lemma *map-injective*:
map f xs = *map f ys* ⟹ *inj f* ⟹ *xs* = *ys*
⟨*proof*⟩

lemma *inj-map-eq-map*[*simp*]: *inj f* ⟹ (*map f xs* = *map f ys*) = (*xs* = *ys*)
⟨*proof*⟩

lemma *inj-mapI*: *inj f* ⟹ *inj* (*map f*)
⟨*proof*⟩

lemma *inj-mapD*: *inj* (*map f*) ⟹ *inj f*
⟨*proof*⟩

lemma *inj-map*[*iff*]: *inj* (*map f*) = *inj f*
⟨*proof*⟩

lemma *inj-on-mapI*: *inj-on f* (⋃(*set* ' *A*)) ⟹ *inj-on* (*map f*) *A*
⟨*proof*⟩

lemma *map-idI*: (⋀*x*. *x* ∈ *set xs* ⟹ *f x* = *x*) ⟹ *map f xs* = *xs*
⟨*proof*⟩

lemma *map-fun-upd* [*simp*]: *y* ∉ *set xs* ⟹ *map* (*f*(*y*:=*v*)) *xs* = *map f xs*
⟨*proof*⟩

lemma *map-fst-zip*[*simp*]:
length xs = *length ys* ⟹ *map fst* (*zip xs ys*) = *xs*
⟨*proof*⟩

lemma *map-snd-zip*[*simp*]:
length xs = *length ys* ⟹ *map snd* (*zip xs ys*) = *ys*
⟨*proof*⟩

lemma *map-fst-zip-take*:
map fst (*zip xs ys*) = *take* (*min* (*length xs*) (*length ys*)) *xs*
⟨*proof*⟩

lemma *map-snd-zip-take*:
map snd (*zip xs ys*) = *take* (*min* (*length xs*) (*length ys*)) *ys*
⟨*proof*⟩

lemma *map2-map-map*: *map2 h* (*map f xs*) (*map g xs*) = *map* (λ*x*. *h* (*f x*) (*g x*)) *xs*
⟨*proof*⟩

functor *map*: *map*
⟨*proof*⟩

declare *map.id* [*simp*]

67.1.6 *rev*

lemma *rev-append* [*simp*]: *rev* (*xs* @ *ys*) = *rev ys* @ *rev xs*
⟨*proof*⟩

lemma *rev-rev-ident* [*simp*]: *rev* (*rev xs*) = *xs*
⟨*proof*⟩

lemma *rev-swap*: (*rev xs* = *ys*) = (*xs* = *rev ys*)
⟨*proof*⟩

lemma *rev-is-Nil-conv* [*iff*]: (*rev xs* = []) = (*xs* = [])
⟨*proof*⟩

lemmas *Nil-is-rev-conv* [*iff*] = *rev-is-Nil-conv*[*THEN eq-iff-swap*]

lemma *rev-singleton-conv* [*simp*]: (*rev xs* = [*x*]) = (*xs* = [*x*])
⟨*proof*⟩

lemma *singleton-rev-conv* [*simp*]: ([*x*] = *rev xs*) = ([*x*] = *xs*)
⟨*proof*⟩

lemma *rev-is-rev-conv* [*iff*]: (*rev xs* = *rev ys*) = (*xs* = *ys*)
⟨*proof*⟩

lemma *inj-on-rev*[*iff*]: *inj-on rev A*
⟨*proof*⟩

lemma *rev-induct* [*case-names Nil snoc*]:
assumes *P* [] **and** ⋀*x xs*. *P xs* ⟹ *P* (*xs* @ [*x*])
shows *P xs*
⟨*proof*⟩

lemma *rev-exhaust* [*case-names Nil snoc*]:
(*xs* = [] ⟹ *P*) ⟹(⋀*ys y*. *xs* = *ys* @ [*y*] ⟹ *P*) ⟹ *P*
⟨*proof*⟩

lemmas *rev-cases* = *rev-exhaust*

lemma *rev-nonempty-induct* [*consumes 1*, *case-names single snoc*]:
assumes *xs* ≠ []
and *single*: ⋀*x*. *P* [*x*]
and *snoc'*: ⋀*x xs*. *xs* ≠ [] ⟹ *P xs* ⟹ *P* (*xs*@[*x*])
shows *P xs*
⟨*proof*⟩

lemma *rev-eq-Cons-iff*[*iff*]: (*rev xs* = *y*#*ys*) = (*xs* = *rev ys* @ [*y*])
⟨*proof*⟩

lemma *length-Suc-conv-rev*: (*length xs* = *Suc n*) = (∃ *y ys*. *xs* = *ys* @ [*y*] ∧ *length ys* = *n*)
⟨*proof*⟩

67.1.7 *set*

declare *list.set*[*code-post*] — pretty output

lemma *finite-set* [*iff*]: *finite* (*set xs*)
⟨*proof*⟩

lemma *set-append* [*simp*]: *set* (*xs* @ *ys*) = (*set xs* ∪ *set ys*)
⟨*proof*⟩

lemma *hd-in-set*[*simp*]: *xs* ≠ [] ⟹ *hd xs* ∈ *set xs*
⟨*proof*⟩

lemma *set-subset-Cons*: *set xs* ⊆ *set* (*x* # *xs*)
⟨*proof*⟩

lemma *set-ConsD*: *y* ∈ *set* (*x* # *xs*) ⟹ *y*=*x* ∨ *y* ∈ *set xs*
⟨*proof*⟩

lemma *set-empty* [*iff*]: (*set xs* = {}) = (*xs* = [])
⟨*proof*⟩

lemmas *set-empty2*[*iff*] = *set-empty*[*THEN eq-iff-swap*]

lemma *set-rev* [*simp*]: *set* (*rev xs*) = *set xs*
⟨*proof*⟩

lemma *set-map* [*simp*]: *set* (*map f xs*) = *f*‘(*set xs*)

⟨*proof*⟩

lemma *set-filter* [*simp*]: *set* (*filter P xs*) = {*x*. *x* ∈ *set xs* ∧ *P x*}
⟨*proof*⟩

lemma *set-upt* [*simp*]: *set*[*i*..<*j*] = {*i*..<*j*}
⟨*proof*⟩

lemma *split-list*: *x* ∈ *set xs* ⟹ ∃ *ys zs*. *xs* = *ys* @ *x* # *zs*
⟨*proof*⟩

lemma *in-set-conv-decomp*: *x* ∈ *set xs* ⟷ (∃ *ys zs*. *xs* = *ys* @ *x* # *zs*)
⟨*proof*⟩

lemma *split-list-first*: *x* ∈ *set xs* ⟹ ∃ *ys zs*. *xs* = *ys* @ *x* # *zs* ∧ *x* ∉ *set ys*
⟨*proof*⟩

lemma *in-set-conv-decomp-first*:
(*x* ∈ *set xs*) = (∃ *ys zs*. *xs* = *ys* @ *x* # *zs* ∧ *x* ∉ *set ys*)
⟨*proof*⟩

lemma *split-list-last*: *x* ∈ *set xs* ⟹ ∃ *ys zs*. *xs* = *ys* @ *x* # *zs* ∧ *x* ∉ *set zs*
⟨*proof*⟩

lemma *in-set-conv-decomp-last*:
(*x* ∈ *set xs*) = (∃ *ys zs*. *xs* = *ys* @ *x* # *zs* ∧ *x* ∉ *set zs*)
⟨*proof*⟩

lemma *split-list-prop*: ∃ *x* ∈ *set xs*. *P x* ⟹ ∃ *ys x zs*. *xs* = *ys* @ *x* # *zs* ∧ *P x*
⟨*proof*⟩

lemma *split-list-propE*:
assumes ∃ *x* ∈ *set xs*. *P x*
obtains *ys x zs* **where** *xs* = *ys* @ *x* # *zs* **and** *P x*
⟨*proof*⟩

lemma *split-list-first-prop*:
∃ *x* ∈ *set xs*. *P x* ⟹
∃ *ys x zs*. *xs* = *ys*@*x*#*zs* ∧ *P x* ∧ (∀ *y* ∈ *set ys*. ¬ *P y*)
⟨*proof*⟩

lemma *split-list-first-propE*:
assumes ∃ *x* ∈ *set xs*. *P x*
obtains *ys x zs* **where** *xs* = *ys* @ *x* # *zs* **and** *P x* **and** ∀ *y* ∈ *set ys*. ¬ *P y*
⟨*proof*⟩

lemma *split-list-first-prop-iff*:
(∃ *x* ∈ *set xs*. *P x*) ⟷

$(\exists\, ys\ x\ zs.\ xs = ys@x\#zs \wedge P\ x \wedge (\forall\, y \in set\ ys.\ \neg\ P\ y))$
⟨*proof*⟩

lemma *split-list-last-prop*:
$\exists\, x \in set\ xs.\ P\ x \Longrightarrow$
$\exists\, ys\ x\ zs.\ xs = ys@x\#zs \wedge P\ x \wedge (\forall\, z \in set\ zs.\ \neg\ P\ z)$
⟨*proof*⟩

lemma *split-list-last-propE*:
assumes $\exists\, x \in set\ xs.\ P\ x$
obtains *ys x zs* **where** $xs = ys\ @\ x\ \#\ zs$ **and** $P\ x$ **and** $\forall\, z \in set\ zs.\ \neg\ P\ z$
⟨*proof*⟩

lemma *split-list-last-prop-iff*:
$(\exists\, x \in set\ xs.\ P\ x) \longleftrightarrow$
$(\exists\, ys\ x\ zs.\ xs = ys@x\#zs \wedge P\ x \wedge (\forall\, z \in set\ zs.\ \neg\ P\ z))$
⟨*proof*⟩

lemma *finite-list*: *finite* $A \Longrightarrow \exists\, xs.\ set\ xs = A$
⟨*proof*⟩

lemma *card-length*: *card* $(set\ xs) \leq length\ xs$
⟨*proof*⟩

lemma *set-minus-filter-out*:
$set\ xs - \{y\} = set\ (filter\ (\lambda x.\ \neg\ (x = y))\ xs)$
⟨*proof*⟩

lemma *append-Cons-eq-iff*:
$[\![\ x \notin set\ xs;\ x \notin set\ ys\ ]\!] \Longrightarrow$
$xs\ @\ x\ \#\ ys = xs'\ @\ x\ \#\ ys' \longleftrightarrow (xs = xs' \wedge ys = ys')$
⟨*proof*⟩

67.1.8 *concat*

lemma *concat-append* [*simp*]: $concat\ (xs\ @\ ys) = concat\ xs\ @\ concat\ ys$
⟨*proof*⟩

lemma *concat-eq-Nil-conv* [*simp*]: $(concat\ xss = []) = (\forall\, xs \in set\ xss.\ xs = [])$
⟨*proof*⟩

lemmas *Nil-eq-concat-conv* [*simp*] = *concat-eq-Nil-conv*[*THEN eq-iff-swap*]

lemma *set-concat* [*simp*]: $set\ (concat\ xs) = (\bigcup x{\in}set\ xs.\ set\ x)$
⟨*proof*⟩

lemma *concat-map-singleton*[*simp*]: $concat(map\ (\%x.\ [f\ x])\ xs) = map\ f\ xs$
⟨*proof*⟩

lemma *map-concat*: *map f* (*concat xs*) = *concat* (*map* (*map f*) *xs*)
⟨*proof*⟩

lemma *rev-concat*: *rev* (*concat xs*) = *concat* (*map rev* (*rev xs*))
⟨*proof*⟩

lemma *length-concat-rev*[*simp*]: *length* (*concat* (*rev xs*)) = *length* (*concat xs*)
⟨*proof*⟩

lemma *concat-eq-concat-iff*: ∀(*x*, *y*) ∈ *set* (*zip xs ys*). *length x* = *length y* ⟹ *length xs* = *length ys* ⟹ (*concat xs* = *concat ys*) = (*xs* = *ys*)
⟨*proof*⟩

lemma *concat-injective*: *concat xs* = *concat ys* ⟹ *length xs* = *length ys* ⟹ ∀(*x*, *y*) ∈ *set* (*zip xs ys*). *length x* = *length y* ⟹ *xs* = *ys*
⟨*proof*⟩

lemma *concat-eq-appendD*:
assumes *concat xss* = *ys* @ *zs xss* ≠ []
shows ∃ *xss1 xs xs′ xss2*. *xss* = *xss1* @ (*xs* @ *xs′*) # *xss2* ∧ *ys* = *concat xss1* @ *xs* ∧ *zs* = *xs′* @ *concat xss2*
⟨*proof*⟩

lemma *concat-eq-append-conv*:
concat xss = *ys* @ *zs* ⟷
(*if xss* = [] *then ys* = [] ∧ *zs* = []
else ∃ *xss1 xs xs′ xss2*. *xss* = *xss1* @ (*xs* @ *xs′*) # *xss2* ∧ *ys* = *concat xss1* @ *xs* ∧ *zs* = *xs′* @ *concat xss2*)
⟨*proof*⟩

lemma *hd-concat*: ⟦*xs* ≠ []; *hd xs* ≠ []⟧ ⟹ *hd* (*concat xs*) = *hd* (*hd xs*)
⟨*proof*⟩

⟨*ML*⟩

67.1.9 *filter*

lemma *filter-append* [*simp*]: *filter P* (*xs* @ *ys*) = *filter P xs* @ *filter P ys*
⟨*proof*⟩

lemma *rev-filter*: *rev* (*filter P xs*) = *filter P* (*rev xs*)
⟨*proof*⟩

lemma *filter-filter* [*simp*]: *filter P* (*filter Q xs*) = *filter* (λ*x*. *Q x* ∧ *P x*) *xs*
⟨*proof*⟩

lemma *filter-concat*: *filter p* (*concat xs*) = *concat* (*map* (*filter p*) *xs*)

⟨*proof*⟩

lemma *length-filter-le* [*simp*]: *length* (*filter P xs*) ≤ *length xs*
⟨*proof*⟩

lemma *sum-length-filter-compl*:
length(*filter P xs*) + *length*(*filter* (λ*x*. ¬*P x*) *xs*) = *length xs*
⟨*proof*⟩

lemma *filter-True* [*simp*]: ∀ *x* ∈ *set xs*. *P x* ⟹ *filter P xs* = *xs*
⟨*proof*⟩

lemma *filter-False* [*simp*]: ∀ *x* ∈ *set xs*. ¬ *P x* ⟹ *filter P xs* = []
⟨*proof*⟩

lemma *filter-empty-conv*: (*filter P xs* = []) = (∀ *x*∈*set xs*. ¬ *P x*)
⟨*proof*⟩

lemmas *empty-filter-conv* = *filter-empty-conv*[*THEN eq-iff-swap*]

lemma *filter-id-conv*: (*filter P xs* = *xs*) = (∀ *x*∈*set xs*. *P x*)
⟨*proof*⟩

lemma *filter-map*: *filter P* (*map f xs*) = *map f* (*filter* (*P* ∘ *f*) *xs*)
⟨*proof*⟩

lemma *length-filter-map*[*simp*]:
length (*filter P* (*map f xs*)) = *length*(*filter* (*P* ∘ *f*) *xs*)
⟨*proof*⟩

lemma *filter-is-subset* [*simp*]: *set* (*filter P xs*) ≤ *set xs*
⟨*proof*⟩

lemma *length-filter-less*:
⟦ *x* ∈ *set xs*; ¬ *P x* ⟧ ⟹ *length*(*filter P xs*) < *length xs*
⟨*proof*⟩

lemma *length-filter-conv-card*:
length(*filter p xs*) = *card*{*i*. *i* < *length xs* ∧ *p*(*xs*!*i*)}
⟨*proof*⟩

lemma *Cons-eq-filterD*:
x#*xs* = *filter P ys* ⟹
∃ *us vs*. *ys* = *us* @ *x* # *vs* ∧ (∀ *u*∈*set us*. ¬ *P u*) ∧ *P x* ∧ *xs* = *filter P vs*
(**is** - ⟹ ∃ *us vs*. *?P ys us vs*)
⟨*proof*⟩

lemma *filter-eq-ConsD*:
filter P ys = *x*#*xs* ⟹

$\exists us\ vs.\ ys = us\ @\ x\ \#\ vs \wedge (\forall u{\in}set\ us.\ \neg\ P\ u) \wedge P\ x \wedge xs = filter\ P\ vs$
⟨*proof*⟩

lemma *filter-eq-Cons-iff*:
$(filter\ P\ ys = x\#xs) =$
$(\exists us\ vs.\ ys = us\ @\ x\ \#\ vs \wedge (\forall u{\in}set\ us.\ \neg\ P\ u) \wedge P\ x \wedge xs = filter\ P\ vs)$
⟨*proof*⟩

lemmas *Cons-eq-filter-iff* = *filter-eq-Cons-iff*[*THEN eq-iff-swap*]

lemma *inj-on-filter-key-eq*:
assumes *inj-on f* (*insert y* (*set xs*))
shows *filter* (λ*x*. *f y* = *f x*) *xs* = *filter* (*HOL.eq y*) *xs*
⟨*proof*⟩

lemma *filter-cong*[*fundef-cong*]:
$xs = ys \Longrightarrow (\bigwedge x.\ x \in set\ ys \Longrightarrow P\ x = Q\ x) \Longrightarrow filter\ P\ xs = filter\ Q\ ys$
⟨*proof*⟩

67.1.10 List partitioning

primrec *partition* :: ('*a* ⇒ *bool*) ⇒'*a list* ⇒ '*a list* × '*a list* **where**
partition P [] = ([], [])
partition P (*x* # *xs*) =
(*let* (*yes*, *no*) = *partition P xs*
in if P x then (*x* # *yes*, *no*) *else* (*yes*, *x* # *no*))

lemma *partition-filter1*: *fst* (*partition P xs*) = *filter P xs*
⟨*proof*⟩

lemma *partition-filter2*: *snd* (*partition P xs*) = *filter* (*Not* ∘ *P*) *xs*
⟨*proof*⟩

lemma *partition-P*:
assumes *partition P xs* = (*yes*, *no*)
shows (∀ *p* ∈ *set yes*. *P p*) ∧ (∀ *p* ∈ *set no*. ¬ *P p*)
⟨*proof*⟩

lemma *partition-set*:
assumes *partition P xs* = (*yes*, *no*)
shows *set yes* ∪ *set no* = *set xs*
⟨*proof*⟩

lemma *partition-filter-conv*[*simp*]:
partition f xs = (*filter f xs*,*filter* (*Not* ∘ *f*) *xs*)
⟨*proof*⟩

declare *partition.simps*[*simp del*]

67.1.11 (!)

lemma *nth-Cons-0* [*simp*, *code*]: $(x \# xs)!0 = x$
⟨*proof*⟩

lemma *nth-Cons-Suc* [*simp*, *code*]: $(x \# xs)!(Suc\ n) = xs!n$
⟨*proof*⟩

declare *nth.simps* [*simp del*]

lemma *nth-Cons-pos*[*simp*]: $0 < n \Longrightarrow (x\#xs)\ !\ n = xs\ !\ (n - 1)$
⟨*proof*⟩

lemma *nth-append*:
$(xs\ @\ ys)!n = (if\ n < length\ xs\ then\ xs!n\ else\ ys!(n - length\ xs))$
⟨*proof*⟩

lemma *nth-append-length* [*simp*]: $(xs\ @\ x \# ys)\ !\ length\ xs = x$
⟨*proof*⟩

lemma *nth-append-length-plus*[*simp*]: $(xs\ @\ ys)\ !\ (length\ xs + n) = ys\ !\ n$
⟨*proof*⟩

lemma *nth-map* [*simp*]: $n < length\ xs \Longrightarrow (map\ f\ xs)!n = f(xs!n)$
⟨*proof*⟩

lemma *nth-tl*: $n < length\ (tl\ xs) \Longrightarrow tl\ xs\ !\ n = xs\ !\ Suc\ n$
⟨*proof*⟩

lemma *hd-conv-nth*: $xs \neq [] \Longrightarrow hd\ xs = xs!0$
⟨*proof*⟩

lemma *list-eq-iff-nth-eq*:
$(xs = ys) = (length\ xs = length\ ys \wedge (\forall i{<}length\ xs.\ xs!i = ys!i))$
⟨*proof*⟩

lemma *map-equality-iff*:
$map\ f\ xs = map\ g\ ys \longleftrightarrow length\ xs = length\ ys \wedge (\forall i{<}length\ ys.\ f\ (xs!i) = g\ (ys!i))$
⟨*proof*⟩

lemma *set-conv-nth*: $set\ xs = \{xs!i \mid i.\ i < length\ xs\}$
⟨*proof*⟩

lemma *in-set-conv-nth*: $(x \in set\ xs) = (\exists i < length\ xs.\ xs!i = x)$
⟨*proof*⟩

lemma *nth-equal-first-eq*:
assumes $x \notin set\ xs$
assumes $n \leq length\ xs$

shows $(x \# xs) ! n = x \longleftrightarrow n = 0$ (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *nth-non-equal-first-eq*:
assumes $x \neq y$
shows $(x \# xs) ! n = y \longleftrightarrow xs ! (n - 1) = y \wedge n > 0$ (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *list-ball-nth*: $[\![n < length\ xs; \forall x \in set\ xs.\ P\ x]\!] \Longrightarrow P(xs!n)$
⟨*proof*⟩

lemma *nth-mem* [*simp*]: $n < length\ xs \Longrightarrow xs!n \in set\ xs$
⟨*proof*⟩

lemma *all-nth-imp-all-set*:
$[\![\forall i < length\ xs.\ P(xs!i); x \in set\ xs]\!] \Longrightarrow P\ x$
⟨*proof*⟩

lemma *all-set-conv-all-nth*:
$(\forall x \in set\ xs.\ P\ x) = (\forall i.\ i < length\ xs \longrightarrow P\ (xs\ !\ i))$
⟨*proof*⟩

lemma *rev-nth*:
$n < size\ xs \Longrightarrow rev\ xs\ !\ n = xs\ !\ (length\ xs - Suc\ n)$
⟨*proof*⟩

lemma *Skolem-list-nth*:
$(\forall i{<}k.\ \exists x.\ P\ i\ x) = (\exists xs.\ size\ xs = k \wedge (\forall i{<}k.\ P\ i\ (xs!i)))$
(**is** - = $(\exists xs.\ ?P\ k\ xs)$)
⟨*proof*⟩

67.1.12 *list-update*

lemma *length-list-update* [*simp*]: $length(xs[i{:=}x]) = length\ xs$
⟨*proof*⟩

lemma *nth-list-update*:
$i < length\ xs \Longrightarrow (xs[i{:=}x])!j = (if\ i = j\ then\ x\ else\ xs!j)$
⟨*proof*⟩

lemma *nth-list-update-eq* [*simp*]: $i < length\ xs \Longrightarrow (xs[i{:=}x])!i = x$
⟨*proof*⟩

lemma *nth-list-update-neq* [*simp*]: $i \neq j \Longrightarrow xs[i{:=}x]!j = xs!j$
⟨*proof*⟩

lemma *list-update-id*[*simp*]: $xs[i := xs!i] = xs$
⟨*proof*⟩

lemma *list-update-beyond*[*simp*]: *length xs* $\leq i \Longrightarrow xs[i{:=}x] = xs$
⟨*proof*⟩

lemma *list-update-nonempty*[*simp*]: $xs[k{:=}x] = [] \longleftrightarrow xs{=}[]$
⟨*proof*⟩

lemma *list-update-same-conv*:
$i <$ *length xs* $\Longrightarrow (xs[i := x] = xs) = (xs!i = x)$
⟨*proof*⟩

lemma *list-update-append1*:
$i <$ *size xs* $\Longrightarrow (xs \mathbin{@} ys)[i{:=}x] = xs[i{:=}x] \mathbin{@} ys$
⟨*proof*⟩

lemma *list-update-append*:
$(xs \mathbin{@} ys)\ [n{:=}\ x] =$
(*if* $n <$ *length xs then* $xs[n{:=}\ x] \mathbin{@} ys$ *else* $xs \mathbin{@} (ys\ [n{-}$*length xs*$:=\ x])$)
⟨*proof*⟩

lemma *list-update-length* [*simp*]:
$(xs \mathbin{@} x \# ys)[$*length xs* $:= y] = (xs \mathbin{@} y \# ys)$
⟨*proof*⟩

lemma *map-update*: *map f* $(xs[k{:=}\ y]) =$ *(map f xs)*$[k := f\ y]$
⟨*proof*⟩

lemma *rev-update*:
$k <$ *length xs* $\Longrightarrow$ *rev* $(xs[k{:=}\ y]) =$ *(rev xs)*$[$*length xs* $-\ k - 1 := y]$
⟨*proof*⟩

lemma *update-zip*:
$($*zip xs ys*$)[i{:=}xy] =$ *zip* $(xs[i{:=}$*fst xy*$])\ (ys[i{:=}$*snd xy*$])$
⟨*proof*⟩

lemma *set-update-subset-insert*: $set(xs[i{:=}x]) \leq$ *insert x* $($*set xs*$)$
⟨*proof*⟩

lemma *set-update-subsetI*: $[\![$*set xs* $\subseteq A$; $x \in A]\!] \Longrightarrow set(xs[i := x]) \subseteq A$
⟨*proof*⟩

lemma *set-update-memI*: $n <$ *length xs* $\Longrightarrow x \in$ *set* $(xs[n := x])$
⟨*proof*⟩

lemma *list-update-overwrite*[*simp*]:
$xs\ [i := x,\ i := y] = xs\ [i := y]$
⟨*proof*⟩

lemma *list-update-swap*:
$i \neq i' \Longrightarrow xs\ [i := x,\ i' := x'] = xs\ [i' := x',\ i := x]$

⟨*proof*⟩

lemma *list-update-code* [*code*]:
[][*i* := *y*] = []
(*x* # *xs*)[0 := *y*] = *y* # *xs*
(*x* # *xs*)[*Suc i* := *y*] = *x* # *xs*[*i* := *y*]
⟨*proof*⟩

67.1.13 *last* and *butlast*

lemma *hd-Nil-eq-last*: *hd Nil* = *last Nil*
⟨*proof*⟩

lemma *last-snoc* [*simp*]: *last* (*xs* @ [*x*]) = *x*
⟨*proof*⟩

lemma *butlast-snoc* [*simp*]: *butlast* (*xs* @ [*x*]) = *xs*
⟨*proof*⟩

lemma *last-ConsL*: *xs* = [] ⟹ *last*(*x*#*xs*) = *x*
⟨*proof*⟩

lemma *last-ConsR*: *xs* ≠ [] ⟹ *last*(*x*#*xs*) = *last xs*
⟨*proof*⟩

lemma *last-append*: *last*(*xs* @ *ys*) = (*if ys* = [] *then last xs else last ys*)
⟨*proof*⟩

lemma *last-appendL*[*simp*]: *ys* = [] ⟹ *last*(*xs* @ *ys*) = *last xs*
⟨*proof*⟩

lemma *last-appendR*[*simp*]: *ys* ≠ [] ⟹ *last*(*xs* @ *ys*) = *last ys*
⟨*proof*⟩

lemma *last-tl*: *xs* = [] ∨ *tl xs* ≠ [] ⟹*last* (*tl xs*) = *last xs*
⟨*proof*⟩

lemma *butlast-tl*: *butlast* (*tl xs*) = *tl* (*butlast xs*)
⟨*proof*⟩

lemma *hd-rev*: *hd*(*rev xs*) = *last xs*
⟨*proof*⟩

lemma *last-rev*: *last*(*rev xs*) = *hd xs*
⟨*proof*⟩

lemma *last-in-set*[*simp*]: *as* ≠ [] ⟹ *last as* ∈ *set as*
⟨*proof*⟩

lemma *length-butlast* [*simp*]: *length* (*butlast xs*) = *length xs* − *1*
⟨*proof*⟩

lemma *butlast-append*:
butlast (*xs* @ *ys*) = (*if ys* = [] *then butlast xs else xs* @ *butlast ys*)
⟨*proof*⟩

lemma *append-butlast-last-id* [*simp*]:
xs ≠ [] ⟹ *butlast xs* @ [*last xs*] = *xs*
⟨*proof*⟩

lemma *in-set-butlastD*: *x* ∈ *set* (*butlast xs*) ⟹ *x* ∈ *set xs*
⟨*proof*⟩

lemma *in-set-butlast-appendI*:
x ∈ *set* (*butlast xs*) ∨ *x* ∈ *set* (*butlast ys*) ⟹ *x* ∈ *set* (*butlast* (*xs* @ *ys*))
⟨*proof*⟩

lemma *last-drop*[*simp*]: *n* < *length xs* ⟹ *last* (*drop n xs*) = *last xs*
⟨*proof*⟩

lemma *nth-butlast*:
assumes *n* < *length* (*butlast xs*) **shows** *butlast xs* ! *n* = *xs* ! *n*
⟨*proof*⟩

lemma *last-conv-nth*: *xs*≠[] ⟹ *last xs* = *xs*!(*length xs* − *1*)
⟨*proof*⟩

lemma *butlast-conv-take*: *butlast xs* = *take* (*length xs* − *1*) *xs*
⟨*proof*⟩

lemma *last-list-update*:
xs ≠ [] ⟹ *last*(*xs*[*k*:=*x*]) = (*if k* = *size xs* − *1 then x else last xs*)
⟨*proof*⟩

lemma *butlast-list-update*:
butlast(*xs*[*k*:=*x*]) =
(*if k* = *size xs* − *1 then butlast xs else* (*butlast xs*)[*k*:=*x*])
⟨*proof*⟩

lemma *last-map*: *xs* ≠ [] ⟹ *last* (*map f xs*) = *f* (*last xs*)
⟨*proof*⟩

lemma *map-butlast*: *map f* (*butlast xs*) = *butlast* (*map f xs*)
⟨*proof*⟩

lemma *snoc-eq-iff-butlast*:
xs @ [*x*] = *ys* ⟷ (*ys* ≠ [] ∧ *butlast ys* = *xs* ∧ *last ys* = *x*)
⟨*proof*⟩

corollary *longest-common-suffix*:
$\exists ss\ xs'\ ys'.\ xs = xs' @ ss \wedge ys = ys' @ ss$
$\quad \wedge (xs' = [] \vee ys' = [] \vee last\ xs' \neq last\ ys')$
⟨*proof*⟩

lemma *butlast-rev* [*simp*]: *butlast* (*rev xs*) = *rev* (*tl xs*)
⟨*proof*⟩

67.1.14 *take* and *drop*

lemma *take-0*: *take 0 xs* = []
⟨*proof*⟩

lemma *drop-0*: *drop 0 xs* = *xs*
⟨*proof*⟩

lemma *take0*[*simp*]: *take 0* = (λ*xs*. [])
⟨*proof*⟩

lemma *drop0*[*simp*]: *drop 0* = (λ*x*. *x*)
⟨*proof*⟩

lemma *take-Suc-Cons* [*simp*]: *take* (*Suc n*) (*x* # *xs*) = *x* # *take n xs*
⟨*proof*⟩

lemma *drop-Suc-Cons* [*simp*]: *drop* (*Suc n*) (*x* # *xs*) = *drop n xs*
⟨*proof*⟩

declare *take-Cons* [*simp del*] **and** *drop-Cons* [*simp del*]

lemma *take-Suc*: *xs* ≠ [] ⟹ *take* (*Suc n*) *xs* = *hd xs* # *take n* (*tl xs*)
⟨*proof*⟩

lemma *drop-Suc*: *drop* (*Suc n*) *xs* = *drop n* (*tl xs*)
⟨*proof*⟩

lemma *hd-take*[*simp*]: *j* > *0* ⟹ *hd* (*take j xs*) = *hd xs*
⟨*proof*⟩

lemma *take-tl*: *take n* (*tl xs*) = *tl* (*take* (*Suc n*) *xs*)
⟨*proof*⟩

lemma *drop-tl*: *drop n* (*tl xs*) = *tl*(*drop n xs*)
⟨*proof*⟩

lemma *tl-take*: *tl* (*take n xs*) = *take* (*n* − *1*) (*tl xs*)
⟨*proof*⟩

lemma *tl-drop*: *tl* (*drop n xs*) = *drop n* (*tl xs*)
⟨*proof*⟩

lemma *nth-via-drop*: *drop n xs* = *y*#*ys* ⟹ *xs*!*n* = *y*
⟨*proof*⟩

lemma *take-Suc-conv-app-nth*:
i < *length xs* ⟹ *take* (*Suc i*) *xs* = *take i xs* @ [*xs*!*i*]
⟨*proof*⟩

lemma *Cons-nth-drop-Suc*:
i < *length xs* ⟹ (*xs*!*i*) # (*drop* (*Suc i*) *xs*) = *drop i xs*
⟨*proof*⟩

lemma *length-take* [*simp*]: *length* (*take n xs*) = *min* (*length xs*) *n*
⟨*proof*⟩

lemma *length-drop* [*simp*]: *length* (*drop n xs*) = (*length xs* − *n*)
⟨*proof*⟩

lemma *take-all* [*simp*]: *length xs* ≤ *n* ⟹ *take n xs* = *xs*
⟨*proof*⟩

lemma *drop-all* [*simp*]: *length xs* ≤ *n* ⟹ *drop n xs* = []
⟨*proof*⟩

lemma *take-all-iff* [*simp*]: *take n xs* = *xs* ⟷ *length xs* ≤ *n*
⟨*proof*⟩

lemma *take-eq-Nil*[*simp*]: (*take n xs* = []) = (*n* = *0* ∨ *xs* = [])
⟨*proof*⟩

lemmas *take-eq-Nil2*[*simp*] = *take-eq-Nil*[*THEN eq-iff-swap*]

lemma *drop-eq-Nil* [*simp*]: *drop n xs* = [] ⟷ *length xs* ≤ *n*
⟨*proof*⟩

lemmas *drop-eq-Nil2* [*simp*] = *drop-eq-Nil*[*THEN eq-iff-swap*]

lemma *take-append* [*simp*]:
take n (*xs* @ *ys*) = (*take n xs* @ *take* (*n* − *length xs*) *ys*)
⟨*proof*⟩

lemma *drop-append* [*simp*]:
drop n (*xs* @ *ys*) = *drop n xs* @ *drop* (*n* − *length xs*) *ys*
⟨*proof*⟩

lemma *take-take* [*simp*]: *take n* (*take m xs*) = *take* (*min n m*) *xs*
⟨*proof*⟩

lemma *drop-drop* [*simp*]: *drop n* (*drop m xs*) = *drop* (*n* + *m*) *xs*
⟨*proof*⟩

lemma *take-drop*: *take n* (*drop m xs*) = *drop m* (*take* (*n* + *m*) *xs*)
⟨*proof*⟩

lemma *drop-take*: *drop n* (*take m xs*) = *take* (*m*−*n*) (*drop n xs*)
⟨*proof*⟩

lemma *append-take-drop-id* [*simp*]: *take n xs* @ *drop n xs* = *xs*
⟨*proof*⟩

lemma *take-map*: *take n* (*map f xs*) = *map f* (*take n xs*)
⟨*proof*⟩

lemma *drop-map*: *drop n* (*map f xs*) = *map f* (*drop n xs*)
⟨*proof*⟩

lemma *rev-take*: *rev* (*take i xs*) = *drop* (*length xs* − *i*) (*rev xs*)
⟨*proof*⟩

lemma *rev-drop*: *rev* (*drop i xs*) = *take* (*length xs* − *i*) (*rev xs*)
⟨*proof*⟩

lemma *drop-rev*: *drop n* (*rev xs*) = *rev* (*take* (*length xs* − *n*) *xs*)
⟨*proof*⟩

lemma *take-rev*: *take n* (*rev xs*) = *rev* (*drop* (*length xs* − *n*) *xs*)
⟨*proof*⟩

lemma *nth-take* [*simp*]: *i* < *n* ⟹ (*take n xs*)!*i* = *xs*!*i*
⟨*proof*⟩

lemma *nth-drop* [*simp*]:
n ≤ *length xs* ⟹ (*drop n xs*)!*i* = *xs*!(*n* + *i*)
⟨*proof*⟩

lemma *butlast-take*:
n ≤ *length xs* ⟹ *butlast* (*take n xs*) = *take* (*n* − *1*) *xs*
⟨*proof*⟩

lemma *butlast-drop*: *butlast* (*drop n xs*) = *drop n* (*butlast xs*)
⟨*proof*⟩

lemma *take-butlast*: *n* < *length xs* ⟹ *take n* (*butlast xs*) = *take n xs*
⟨*proof*⟩

lemma *drop-butlast*: *drop n (butlast xs) = butlast (drop n xs)*
⟨*proof*⟩

lemma *butlast-power*: *(butlast ^^ n) xs = take (length xs − n) xs*
⟨*proof*⟩

lemma *hd-drop-conv-nth*: *n < length xs ⟹ hd(drop n xs) = xs!n*
⟨*proof*⟩

lemma *set-take-subset-set-take*:
m ≤ n ⟹ set(take m xs) ≤ set(take n xs)
⟨*proof*⟩

lemma *set-take-subset*: *set(take n xs) ⊆ set xs*
⟨*proof*⟩

lemma *set-drop-subset*: *set(drop n xs) ⊆ set xs*
⟨*proof*⟩

lemma *set-drop-subset-set-drop*:
m ≥ n ⟹ set(drop m xs) ≤ set(drop n xs)
⟨*proof*⟩

lemma *in-set-takeD*: *x ∈ set(take n xs) ⟹ x ∈ set xs*
⟨*proof*⟩

lemma *in-set-dropD*: *x ∈ set(drop n xs) ⟹ x ∈ set xs*
⟨*proof*⟩

lemma *append-eq-conv-conj*:
(xs @ ys = zs) = (xs = take (length xs) zs ∧ ys = drop (length xs) zs)
⟨*proof*⟩

lemma *map-eq-append-conv*:
map f xs = ys @ zs ⟷ (∃ us vs. xs = us @ vs ∧ ys = map f us ∧ zs = map f vs)
⟨*proof*⟩

lemmas *append-eq-map-conv = map-eq-append-conv[THEN eq-iff-swap]*

lemma *take-add*: *take (i+j) xs = take i xs @ take j (drop i xs)*
⟨*proof*⟩

lemma *append-eq-append-conv-if*:
(xs₁ @ xs₂ = ys₁ @ ys₂) =
(if size xs₁ ≤ size ys₁
then xs₁ = take (size xs₁) ys₁ ∧ xs₂ = drop (size xs₁) ys₁ @ ys₂
else take (size ys₁) xs₁ = ys₁ ∧ drop (size ys₁) xs₁ @ xs₂ = ys₂)

⟨*proof*⟩

lemma *take-hd-drop*:
n < *length xs* ⟹ *take n xs* @ [*hd* (*drop n xs*)] = *take* (*Suc n*) *xs*
⟨*proof*⟩

lemma *id-take-nth-drop*:
i < *length xs* ⟹ *xs* = *take i xs* @ *xs*!*i* # *drop* (*Suc i*) *xs*
⟨*proof*⟩

lemma *take-update-cancel*[*simp*]: *n* ≤ *m* ⟹ *take n* (*xs*[*m* := *y*]) = *take n xs*
⟨*proof*⟩

lemma *drop-update-cancel*[*simp*]: *n* < *m* ⟹ *drop m* (*xs*[*n* := *x*]) = *drop m xs*
⟨*proof*⟩

lemma *upd-conv-take-nth-drop*:
i < *length xs* ⟹ *xs*[*i*:=*a*] = *take i xs* @ *a* # *drop* (*Suc i*) *xs*
⟨*proof*⟩

lemma *take-update-swap*: *take m* (*xs*[*n* := *x*]) = (*take m xs*)[*n* := *x*]
⟨*proof*⟩

lemma *drop-update-swap*:
assumes *m* ≤ *n* **shows** *drop m* (*xs*[*n* := *x*]) = (*drop m xs*)[*n*−*m* := *x*]
⟨*proof*⟩

lemma *nth-image*: *l* ≤ *size xs* ⟹ *nth xs* ‘ {*0*..<*l*} = *set*(*take l xs*)
⟨*proof*⟩

67.1.15 *takeWhile* and *dropWhile*

lemma *length-takeWhile-le*: *length* (*takeWhile P xs*) ≤ *length xs*
⟨*proof*⟩

lemma *takeWhile-dropWhile-id* [*simp*]: *takeWhile P xs* @ *dropWhile P xs* = *xs*
⟨*proof*⟩

lemma *takeWhile-append1* [*simp*]:
⟦*x* ∈ *set xs*; ¬*P*(*x*)⟧ ⟹ *takeWhile P* (*xs* @ *ys*) = *takeWhile P xs*
⟨*proof*⟩

lemma *takeWhile-append2* [*simp*]:
(⋀*x*. *x* ∈ *set xs* ⟹ *P x*) ⟹ *takeWhile P* (*xs* @ *ys*) = *xs* @ *takeWhile P ys*
⟨*proof*⟩

lemma *takeWhile-append*:
takeWhile P (*xs* @ *ys*) = (*if* ∀*x*∈*set xs*. *P x then xs* @ *takeWhile P ys else takeWhile P xs*)

⟨*proof*⟩

lemma *takeWhile-tail*: ¬ *P x* ⟹ *takeWhile P* (*xs* @ (*x*#*l*)) = *takeWhile P xs*
⟨*proof*⟩

lemma *takeWhile-eq-Nil-iff*: *takeWhile P xs* = [] ⟷ *xs* = [] ∨ ¬*P* (*hd xs*)
⟨*proof*⟩

lemma *takeWhile-nth*: *j* < *length* (*takeWhile P xs*) ⟹ *takeWhile P xs* ! *j* = *xs* ! *j*
⟨*proof*⟩

lemma *takeWhile-takeWhile*: *takeWhile Q* (*takeWhile P xs*) = *takeWhile* (λ*x*. *P x* ∧ *Q x*) *xs*
⟨*proof*⟩

lemma *dropWhile-nth*: *j* < *length* (*dropWhile P xs*) ⟹
dropWhile P xs ! *j* = *xs* ! (*j* + *length* (*takeWhile P xs*))
⟨*proof*⟩

lemma *length-dropWhile-le*: *length* (*dropWhile P xs*) ≤ *length xs*
⟨*proof*⟩

lemma *dropWhile-append1* [*simp*]:
⟦*x* ∈ *set xs*; ¬*P*(*x*)⟧ ⟹ *dropWhile P* (*xs* @ *ys*) = (*dropWhile P xs*)@*ys*
⟨*proof*⟩

lemma *dropWhile-append2* [*simp*]:
(⋀*x*. *x* ∈ *set xs* ⟹ *P*(*x*)) ⟹ *dropWhile P* (*xs* @ *ys*) = *dropWhile P ys*
⟨*proof*⟩

lemma *dropWhile-append3*:
¬ *P y* ⟹*dropWhile P* (*xs* @ *y* # *ys*) = *dropWhile P xs* @ *y* # *ys*
⟨*proof*⟩

lemma *dropWhile-append*:
dropWhile P (*xs* @ *ys*) = (*if* ∀ *x*∈*set xs*. *P x then dropWhile P ys else dropWhile P xs* @ *ys*)
⟨*proof*⟩

lemma *dropWhile-last*:
x ∈ *set xs* ⟹ ¬ *P x* ⟹ *last* (*dropWhile P xs*) = *last xs*
⟨*proof*⟩

lemma *set-dropWhileD*: *x* ∈ *set* (*dropWhile P xs*) ⟹ *x* ∈ *set xs*
⟨*proof*⟩

lemma *set-takeWhileD*: *x* ∈ *set* (*takeWhile P xs*) ⟹ *x* ∈ *set xs* ∧ *P x*
⟨*proof*⟩

lemma *takeWhile-eq-all-conv*[*simp*]:
(*takeWhile P xs* = *xs*) = (∀ *x* ∈ *set xs*. *P x*)
⟨*proof*⟩

lemma *dropWhile-eq-Nil-conv*[*simp*]:
(*dropWhile P xs* = []) = (∀ *x* ∈ *set xs*. *P x*)
⟨*proof*⟩

lemma *dropWhile-eq-Cons-conv*:
(*dropWhile P xs* = *y*#*ys*) = (*xs* = *takeWhile P xs* @ *y* # *ys* ∧ ¬ *P y*)
⟨*proof*⟩

lemma *dropWhile-eq-self-iff*: *dropWhile P xs* = *xs* ⟷ *xs* = [] ∨ ¬*P* (*hd xs*)
⟨*proof*⟩

lemma *dropWhile-dropWhile1*: (⋀*x*. *Q x* ⟹ *P x*) ⟹ *dropWhile Q* (*dropWhile P xs*) = *dropWhile P xs*
⟨*proof*⟩

lemma *dropWhile-dropWhile2*: (⋀*x*. *P x* ⟹ *Q x*) ⟹ *takeWhile P* (*takeWhile Q xs*) = *takeWhile P xs*
⟨*proof*⟩

lemma *dropWhile-takeWhile*:
(⋀*x*. *P x* ⟹ *Q x*) ⟹ *dropWhile P* (*takeWhile Q xs*) = *takeWhile Q* (*dropWhile P xs*)
⟨*proof*⟩

lemma *distinct-takeWhile*[*simp*]: *distinct xs* ⟹ *distinct* (*takeWhile P xs*)
⟨*proof*⟩

lemma *distinct-dropWhile*[*simp*]: *distinct xs* ⟹ *distinct* (*dropWhile P xs*)
⟨*proof*⟩

lemma *takeWhile-map*: *takeWhile P* (*map f xs*) = *map f* (*takeWhile* (*P* ∘ *f*) *xs*)
⟨*proof*⟩

lemma *dropWhile-map*: *dropWhile P* (*map f xs*) = *map f* (*dropWhile* (*P* ∘ *f*) *xs*)
⟨*proof*⟩

lemma *takeWhile-eq-take*: *takeWhile P xs* = *take* (*length* (*takeWhile P xs*)) *xs*
⟨*proof*⟩

lemma *dropWhile-eq-drop*: *dropWhile P xs* = *drop* (*length* (*takeWhile P xs*)) *xs*
⟨*proof*⟩

lemma *hd-dropWhile*: *dropWhile P xs* ≠ [] ⟹ ¬ *P* (*hd* (*dropWhile P xs*))
⟨*proof*⟩

lemma *takeWhile-eq-filter*:
assumes $\bigwedge x.\ x \in set\ (dropWhile\ P\ xs) \Longrightarrow \neg\ P\ x$
shows $takeWhile\ P\ xs = filter\ P\ xs$
⟨*proof*⟩

lemma *takeWhile-eq-take-P-nth*:
$[\![\bigwedge i.\ [\![i < n\ ;\ i < length\ xs]\!] \Longrightarrow P\ (xs\ !\ i)\ ;\ n < length\ xs \Longrightarrow \neg\ P\ (xs\ !\ n)]\!]$
$\Longrightarrow$
$takeWhile\ P\ xs = take\ n\ xs$
⟨*proof*⟩

lemma *nth-length-takeWhile*:
$length\ (takeWhile\ P\ xs) < length\ xs \Longrightarrow \neg\ P\ (xs\ !\ length\ (takeWhile\ P\ xs))$
⟨*proof*⟩

lemma *length-takeWhile-less-P-nth*:
assumes *all*: $\bigwedge i.\ i < j \Longrightarrow P\ (xs\ !\ i)$ **and** $j \leq length\ xs$
shows $j \leq length\ (takeWhile\ P\ xs)$
⟨*proof*⟩

lemma *takeWhile-neq-rev*: $[\![distinct\ xs;\ x \in set\ xs]\!] \Longrightarrow$
$takeWhile\ (\lambda y.\ y \neq x)\ (rev\ xs) = rev\ (tl\ (dropWhile\ (\lambda y.\ y \neq x)\ xs))$
⟨*proof*⟩

lemma *dropWhile-neq-rev*: $[\![distinct\ xs;\ x \in set\ xs]\!] \Longrightarrow$
$dropWhile\ (\lambda y.\ y \neq x)\ (rev\ xs) = x\ \#\ rev\ (takeWhile\ (\lambda y.\ y \neq x)\ xs)$
⟨*proof*⟩

lemma *takeWhile-not-last*:
$distinct\ xs \Longrightarrow takeWhile\ (\lambda y.\ y \neq last\ xs)\ xs = butlast\ xs$
⟨*proof*⟩

lemma *takeWhile-cong* [*fundef-cong*]:
$[\![l = k;\ \bigwedge x.\ x \in set\ l \Longrightarrow P\ x = Q\ x]\!]$
$\Longrightarrow takeWhile\ P\ l = takeWhile\ Q\ k$
⟨*proof*⟩

lemma *dropWhile-cong* [*fundef-cong*]:
$[\![l = k;\ \bigwedge x.\ x \in set\ l \Longrightarrow P\ x = Q\ x]\!]$
$\Longrightarrow dropWhile\ P\ l = dropWhile\ Q\ k$
⟨*proof*⟩

lemma *takeWhile-idem* [*simp*]:
$takeWhile\ P\ (takeWhile\ P\ xs) = takeWhile\ P\ xs$
⟨*proof*⟩

lemma *dropWhile-idem* [*simp*]:
$dropWhile\ P\ (dropWhile\ P\ xs) = dropWhile\ P\ xs$

⟨*proof*⟩

67.1.16 *zip*

lemma *zip-Nil* [*simp*]: *zip* [] *ys* = []
⟨*proof*⟩

lemma *zip-Cons-Cons* [*simp*]: *zip* (*x* # *xs*) (*y* # *ys*) = (*x*, *y*) # *zip xs ys*
⟨*proof*⟩

declare *zip-Cons* [*simp del*]

lemma [*code*]:
zip [] *ys* = []
zip xs [] = []
zip (*x* # *xs*) (*y* # *ys*) = (*x*, *y*) # *zip xs ys*
⟨*proof*⟩

lemma *zip-Cons1*:
zip (*x*#*xs*) *ys* = (*case ys of* [] ⇒ [] | *y*#*ys* ⇒ (*x*,*y*)#*zip xs ys*)
⟨*proof*⟩

lemma *length-zip* [*simp*]:
length (*zip xs ys*) = *min* (*length xs*) (*length ys*)
⟨*proof*⟩

lemma *zip-obtain-same-length*:
assumes ⋀*zs ws n*. *length zs* = *length ws* ⟹ *n* = *min* (*length xs*) (*length ys*)
⟹ *zs* = *take n xs* ⟹ *ws* = *take n ys* ⟹ *P* (*zip zs ws*)
shows *P* (*zip xs ys*)
⟨*proof*⟩

lemma *zip-append1*:
zip (*xs* @ *ys*) *zs* =
zip xs (*take* (*length xs*) *zs*) @ *zip ys* (*drop* (*length xs*) *zs*)
⟨*proof*⟩

lemma *zip-append2*:
zip xs (*ys* @ *zs*) =
zip (*take* (*length ys*) *xs*) *ys* @ *zip* (*drop* (*length ys*) *xs*) *zs*
⟨*proof*⟩

lemma *zip-append* [*simp*]:
⟦*length xs* = *length us*⟧ ⟹
zip (*xs*@*ys*) (*us*@*vs*) = *zip xs us* @ *zip ys vs*
⟨*proof*⟩

lemma *zip-rev*:
length xs = *length ys* ⟹ *zip* (*rev xs*) (*rev ys*) = *rev* (*zip xs ys*)

⟨*proof*⟩

lemma *zip-map-map*:
zip (*map f xs*) (*map g ys*) = *map* (λ (*x*, *y*). (*f x*, *g y*)) (*zip xs ys*)
⟨*proof*⟩

lemma *zip-map1*:
zip (*map f xs*) *ys* = *map* (λ(*x*, *y*). (*f x*, *y*)) (*zip xs ys*)
⟨*proof*⟩

lemma *zip-map2*:
zip xs (*map f ys*) = *map* (λ(*x*, *y*). (*x*, *f y*)) (*zip xs ys*)
⟨*proof*⟩

lemma *map-zip-map*:
map f (*zip* (*map g xs*) *ys*) = *map* (%(*x*,*y*). *f*(*g x*, *y*)) (*zip xs ys*)
⟨*proof*⟩

lemma *map-zip-map2*:
map f (*zip xs* (*map g ys*)) = *map* (%(*x*,*y*). *f*(*x*, *g y*)) (*zip xs ys*)
⟨*proof*⟩

Courtesy of Andreas Lochbihler:

lemma *zip-same-conv-map*: *zip xs xs* = *map* (λ*x*. (*x*, *x*)) *xs*
⟨*proof*⟩

lemma *nth-zip* [*simp*]:
⟦*i* < *length xs*; *i* < *length ys*⟧ ⟹ (*zip xs ys*)!*i* = (*xs*!*i*, *ys*!*i*)
⟨*proof*⟩

lemma *set-zip*:
set (*zip xs ys*) = {(*xs*!*i*, *ys*!*i*) | *i*. *i* < *min* (*length xs*) (*length ys*)}
⟨*proof*⟩

lemma *zip-same*: ((*a*,*b*) ∈ *set* (*zip xs xs*)) = (*a* ∈ *set xs* ∧ *a* = *b*)
⟨*proof*⟩

lemma *zip-update*: *zip* (*xs*[*i*:=*x*]) (*ys*[*i*:=*y*]) = (*zip xs ys*)[*i*:=(*x*,*y*)]
⟨*proof*⟩

lemma *zip-replicate* [*simp*]:
zip (*replicate i x*) (*replicate j y*) = *replicate* (*min i j*) (*x*,*y*)
⟨*proof*⟩

lemma *zip-replicate1*: *zip* (*replicate n x*) *ys* = *map* (*Pair x*) (*take n ys*)
⟨*proof*⟩

lemma *take-zip*: *take n* (*zip xs ys*) = *zip* (*take n xs*) (*take n ys*)
⟨*proof*⟩

lemma *drop-zip*: *drop n (zip xs ys) = zip (drop n xs) (drop n ys)*
⟨*proof*⟩

lemma *zip-takeWhile-fst*: *zip (takeWhile P xs) ys = takeWhile (P ∘ fst) (zip xs ys)*
⟨*proof*⟩

lemma *zip-takeWhile-snd*: *zip xs (takeWhile P ys) = takeWhile (P ∘ snd) (zip xs ys)*
⟨*proof*⟩

lemma *set-zip-leftD*: *(x,y)∈ set (zip xs ys) ⟹ x ∈ set xs*
⟨*proof*⟩

lemma *set-zip-rightD*: *(x,y)∈ set (zip xs ys) ⟹ y ∈ set ys*
⟨*proof*⟩

lemma *in-set-zipE*:
(x,y) ∈ set(zip xs ys) ⟹ (⟦ x ∈ set xs; y ∈ set ys ⟧ ⟹ R) ⟹ R
⟨*proof*⟩

lemma *zip-map-fst-snd*: *zip (map fst zs) (map snd zs) = zs*
⟨*proof*⟩

lemma *zip-eq-conv*:
length xs = length ys ⟹ zip xs ys = zs ⟷ map fst zs = xs ∧ map snd zs = ys
⟨*proof*⟩

lemma *in-set-zip*:
p ∈ set (zip xs ys) ⟷ (∃ n. xs ! n = fst p ∧ ys ! n = snd p
∧ n < length xs ∧ n < length ys)
⟨*proof*⟩

lemma *in-set-impl-in-set-zip1*:
assumes *length xs = length ys*
assumes *x ∈ set xs*
obtains *y* **where** *(x, y) ∈ set (zip xs ys)*
⟨*proof*⟩

lemma *in-set-impl-in-set-zip2*:
assumes *length xs = length ys*
assumes *y ∈ set ys*
obtains *x* **where** *(x, y) ∈ set (zip xs ys)*
⟨*proof*⟩

lemma *zip-eq-Nil-iff*[*simp*]:
zip xs ys = [] ⟷ xs = [] ∨ ys = []
⟨*proof*⟩

lemmas *Nil-eq-zip-iff*[*simp*] = *zip-eq-Nil-iff*[*THEN eq-iff-swap*]

lemma *zip-eq-ConsE*:
assumes *zip xs ys* = *xy* # *xys*
obtains *x xs′ y ys′* **where** *xs* = *x* # *xs′*
and *ys* = *y* # *ys′* **and** *xy* = (*x*, *y*)
and *xys* = *zip xs′ ys′*
⟨*proof*⟩

lemma *semilattice-map2*:
semilattice (*map2* (∗)) **if** *semilattice* (∗)
for *f* (**infixl** ∗ *70*)
⟨*proof*⟩

lemma *pair-list-eqI*:
assumes *map fst xs* = *map fst ys* **and** *map snd xs* = *map snd ys*
shows *xs* = *ys*
⟨*proof*⟩

lemma *hd-zip*:
‹*hd* (*zip xs ys*) = (*hd xs*, *hd ys*)› **if** ‹*xs* ≠ []› **and** ‹*ys* ≠ []›
⟨*proof*⟩

lemma *last-zip*:
‹*last* (*zip xs ys*) = (*last xs*, *last ys*)› **if** ‹*xs* ≠ []› **and** ‹*ys* ≠ []›
and ‹*length xs* = *length ys*›
⟨*proof*⟩

67.1.17 *list-all2*

lemma *list-all2-lengthD* [*intro?*]:
list-all2 P xs ys ⟹ *length xs* = *length ys*
⟨*proof*⟩

lemma *list-all2-Nil* [*iff*, *code*]: *list-all2 P* [] *ys* = (*ys* = [])
⟨*proof*⟩

lemma *list-all2-Nil2* [*iff*, *code*]: *list-all2 P xs* [] = (*xs* = [])
⟨*proof*⟩

lemma *list-all2-Cons* [*iff*, *code*]:
list-all2 P (*x* # *xs*) (*y* # *ys*) = (*P x y* ∧ *list-all2 P xs ys*)
⟨*proof*⟩

lemma *list-all2-Cons1*:
list-all2 P (*x* # *xs*) *ys* = (∃ *z zs*. *ys* = *z* # *zs* ∧ *P x z* ∧ *list-all2 P xs zs*)
⟨*proof*⟩

lemma *list-all2-Cons2*:
list-all2 P xs (*y # ys*) = (∃ *z zs*. *xs* = *z # zs* ∧ *P z y* ∧ *list-all2 P zs ys*)
⟨*proof*⟩

lemma *list-all2-induct*
[*consumes 1*, *case-names Nil Cons*, *induct set*: *list-all2*]:
assumes *P*: *list-all2 P xs ys*
assumes *Nil*: *R* [] []
assumes *Cons*: ⋀*x xs y ys*.
⟦*P x y*; *list-all2 P xs ys*; *R xs ys*⟧ ⟹ *R* (*x # xs*) (*y # ys*)
shows *R xs ys*
⟨*proof*⟩

lemma *list-all2-rev* [*iff*]:
list-all2 P (*rev xs*) (*rev ys*) = *list-all2 P xs ys*
⟨*proof*⟩

lemma *list-all2-rev1*:
list-all2 P (*rev xs*) *ys* = *list-all2 P xs* (*rev ys*)
⟨*proof*⟩

lemma *list-all2-append1*:
list-all2 P (*xs* @ *ys*) *zs* =
(∃ *us vs*. *zs* = *us* @ *vs* ∧ *length us* = *length xs* ∧ *length vs* = *length ys* ∧
list-all2 P xs us ∧ *list-all2 P ys vs*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *list-all2-append2*:
list-all2 P xs (*ys* @ *zs*) =
(∃ *us vs*. *xs* = *us* @ *vs* ∧ *length us* = *length ys* ∧ *length vs* = *length zs* ∧
list-all2 P us ys ∧ *list-all2 P vs zs*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *list-all2-append*:
length xs = *length ys* ⟹
list-all2 P (*xs*@*us*) (*ys*@*vs*) = (*list-all2 P xs ys* ∧ *list-all2 P us vs*)
⟨*proof*⟩

lemma *list-all2-appendI* [*intro?*, *trans*]:
⟦ *list-all2 P a b*; *list-all2 P c d* ⟧ ⟹ *list-all2 P* (*a*@*c*) (*b*@*d*)
⟨*proof*⟩

lemma *list-all2-conv-all-nth*:
list-all2 P xs ys =
(*length xs* = *length ys* ∧ (∀ *i* < *length xs*. *P* (*xs*!*i*) (*ys*!*i*)))
⟨*proof*⟩

lemma *list-all2-trans*:
assumes *tr*: !!*a b c*. *P1 a b* ⟹ *P2 b c* ⟹ *P3 a c*

shows !!*bs cs. list-all2 P1 as bs* ⟹ *list-all2 P2 bs cs* ⟹ *list-all2 P3 as cs*
(**is** !!*bs cs. PROP ?Q as bs cs*)
⟨*proof*⟩

lemma *list-all2-all-nthI* [*intro?*]:
length a = *length b* ⟹ (⋀*n. n* < *length a* ⟹ *P* (*a*!*n*) (*b*!*n*)) ⟹ *list-all2 P a b*
⟨*proof*⟩

lemma *list-all2I*:
∀ *x* ∈ *set* (*zip a b*). *case-prod P x* ⟹ *length a* = *length b* ⟹ *list-all2 P a b*
⟨*proof*⟩

lemma *list-all2-nthD*:
⟦ *list-all2 P xs ys*; *p* < *size xs* ⟧ ⟹ *P* (*xs*!*p*) (*ys*!*p*)
⟨*proof*⟩

lemma *list-all2-nthD2*:
⟦*list-all2 P xs ys*; *p* < *size ys*⟧ ⟹ *P* (*xs*!*p*) (*ys*!*p*)
⟨*proof*⟩

lemma *list-all2-map1*:
list-all2 P (*map f as*) *bs* = *list-all2* (λ*x y. P* (*f x*) *y*) *as bs*
⟨*proof*⟩

lemma *list-all2-map2*:
list-all2 P as (*map f bs*) = *list-all2* (λ*x y. P x* (*f y*)) *as bs*
⟨*proof*⟩

lemma *list-all2-refl* [*intro?*]:
(⋀*x. P x x*) ⟹ *list-all2 P xs xs*
⟨*proof*⟩

lemma *list-all2-update-cong*:
⟦ *list-all2 P xs ys*; *P x y* ⟧ ⟹ *list-all2 P* (*xs*[*i*:=*x*]) (*ys*[*i*:=*y*])
⟨*proof*⟩

lemma *list-all2-takeI* [*simp*,*intro?*]:
list-all2 P xs ys ⟹ *list-all2 P* (*take n xs*) (*take n ys*)
⟨*proof*⟩

lemma *list-all2-dropI* [*simp*,*intro?*]:
list-all2 P xs ys ⟹ *list-all2 P* (*drop n xs*) (*drop n ys*)
⟨*proof*⟩

lemma *list-all2-mono* [*intro?*]:
list-all2 P xs ys ⟹ (⋀*xs ys. P xs ys* ⟹ *Q xs ys*) ⟹ *list-all2 Q xs ys*
⟨*proof*⟩

lemma *list-all2-eq*:

xs = ys ⟷ list-all2 (=) xs ys
⟨*proof*⟩

lemma *list-eq-iff-zip-eq*:
xs = ys ⟷ length xs = length ys ∧ (∀(x,y) ∈ set (zip xs ys). x = y)
⟨*proof*⟩

lemma *list-all2-same*: *list-all2 P xs xs ⟷ (∀x∈set xs. P x x)*
⟨*proof*⟩

lemma *zip-assoc*:
zip xs (zip ys zs) = map (λ((x, y), z). (x, y, z)) (zip (zip xs ys) zs)
⟨*proof*⟩

lemma *zip-commute*: *zip xs ys = map (λ(x, y). (y, x)) (zip ys xs)*
⟨*proof*⟩

lemma *zip-left-commute*:
zip xs (zip ys zs) = map (λ(y, (x, z)). (x, y, z)) (zip ys (zip xs zs))
⟨*proof*⟩

lemma *zip-replicate2*: *zip xs (replicate n y) = map (λx. (x, y)) (take n xs)*
⟨*proof*⟩

67.1.18 *List.product* and *product-lists*

lemma *product-concat-map*:
List.product xs ys = concat (map (λx. map (λy. (x,y)) ys) xs)
⟨*proof*⟩

lemma *set-product*[*simp*]: *set (List.product xs ys) = set xs × set ys*
⟨*proof*⟩

lemma *length-product* [*simp*]:
*length (List.product xs ys) = length xs * length ys*
⟨*proof*⟩

lemma *product-nth*:
assumes *n < length xs * length ys*
shows *List.product xs ys ! n = (xs ! (n div length ys), ys ! (n mod length ys))*
⟨*proof*⟩

lemma *in-set-product-lists-length*:
xs ∈ set (product-lists xss) ⟹ length xs = length xss
⟨*proof*⟩

lemma *product-lists-set*:
set (product-lists xss) = {xs. list-all2 (λx ys. x ∈ set ys) xs xss} (**is** *?L = Collect ?R*)

⟨*proof*⟩

67.1.19 *fold* with natural argument order

lemma *fold-simps* [*code*]: — eta-expanded variant for generated code – enables tail-recursion optimisation in Scala
fold f [] *s* = *s*
fold f (*x* # *xs*) *s* = *fold f xs* (*f x s*)
⟨*proof*⟩

lemma *fold-remove1-split*:
⟦ ⋀*x y*. *x* ∈ *set xs* ⟹ *y* ∈ *set xs* ⟹ *f x* ∘ *f y* = *f y* ∘ *f x*;
x ∈ *set xs* ⟧
⟹ *fold f xs* = *fold f* (*remove1 x xs*) ∘ *f x*
⟨*proof*⟩

lemma *fold-cong* [*fundef-cong*]:
a = *b* ⟹ *xs* = *ys* ⟹ (⋀*x*. *x* ∈ *set xs* ⟹ *f x* = *g x*)
⟹ *fold f xs a* = *fold g ys b*
⟨*proof*⟩

lemma *fold-id*: (⋀*x*. *x* ∈ *set xs* ⟹ *f x* = *id*) ⟹ *fold f xs* = *id*
⟨*proof*⟩

lemma *fold-commute*:
(⋀*x*. *x* ∈ *set xs* ⟹ *h* ∘ *g x* = *f x* ∘ *h*) ⟹ *h* ∘ *fold g xs* = *fold f xs* ∘ *h*
⟨*proof*⟩

lemma *fold-commute-apply*:
assumes ⋀*x*. *x* ∈ *set xs* ⟹ *h* ∘ *g x* = *f x* ∘ *h*
shows *h* (*fold g xs s*) = *fold f xs* (*h s*)
⟨*proof*⟩

lemma *fold-invariant*:
⟦ ⋀*x*. *x* ∈ *set xs* ⟹ *Q x*; *P s*; ⋀*x s*. *Q x* ⟹ *P s* ⟹ *P* (*f x s*) ⟧
⟹ *P* (*fold f xs s*)
⟨*proof*⟩

lemma *fold-append* [*simp*]: *fold f* (*xs* @ *ys*) = *fold f ys* ∘ *fold f xs*
⟨*proof*⟩

lemma *fold-map* [*code-unfold*]: *fold g* (*map f xs*) = *fold* (*g* ∘ *f*) *xs*
⟨*proof*⟩

lemma *fold-filter*:
fold f (*filter P xs*) = *fold* (λ*x*. *if P x then f x else id*) *xs*
⟨*proof*⟩

lemma *fold-rev*:

$(\bigwedge x\ y.\ x \in set\ xs \Longrightarrow y \in set\ xs \Longrightarrow f\ y \circ f\ x = f\ x \circ f\ y)$
$\Longrightarrow fold\ f\ (rev\ xs) = fold\ f\ xs$
⟨*proof*⟩

lemma *fold-Cons-rev*: $fold\ Cons\ xs = append\ (rev\ xs)$
⟨*proof*⟩

lemma *rev-conv-fold* [*code*]: $rev\ xs = fold\ Cons\ xs\ []$
⟨*proof*⟩

lemma *fold-append-concat-rev*: $fold\ append\ xss = append\ (concat\ (rev\ xss))$
⟨*proof*⟩

Finite-Set.fold and *fold*

lemma (**in** *comp-fun-commute-on*) *fold-set-fold-remdups*:
assumes $set\ xs \subseteq S$
shows $Finite\text{-}Set.fold\ f\ y\ (set\ xs) = fold\ f\ (remdups\ xs)\ y$
⟨*proof*⟩

lemma (**in** *comp-fun-idem-on*) *fold-set-fold*:
assumes $set\ xs \subseteq S$
shows $Finite\text{-}Set.fold\ f\ y\ (set\ xs) = fold\ f\ xs\ y$
⟨*proof*⟩

lemma *union-set-fold* [*code*]: $set\ xs \cup A = fold\ Set.insert\ xs\ A$
⟨*proof*⟩

lemma *union-coset-filter* [*code*]:
$List.coset\ xs \cup A = List.coset\ (List.filter\ (\lambda x.\ x \notin A)\ xs)$
⟨*proof*⟩

lemma *minus-set-fold* [*code*]: $A - set\ xs = fold\ Set.remove\ xs\ A$
⟨*proof*⟩

lemma *minus-coset-filter* [*code*]:
$A - List.coset\ xs = set\ (List.filter\ (\lambda x.\ x \in A)\ xs)$
⟨*proof*⟩

lemma *inter-set-filter* [*code*]:
$A \cap set\ xs = set\ (List.filter\ (\lambda x.\ x \in A)\ xs)$
⟨*proof*⟩

lemma *inter-coset-fold* [*code*]:
$A \cap List.coset\ xs = fold\ Set.remove\ xs\ A$
⟨*proof*⟩

lemma (**in** *semilattice-set*) *set-eq-fold* [*code*]:
$F\ (set\ (x\ \#\ xs)) = fold\ f\ xs\ x$
⟨*proof*⟩

lemma (**in** *complete-lattice*) *Inf-set-fold*:
Inf (*set xs*) = *fold inf xs top*
⟨*proof*⟩

declare *Inf-set-fold* [**where** *'a* = *'a set*, *code*]

lemma (**in** *complete-lattice*) *Sup-set-fold*:
Sup (*set xs*) = *fold sup xs bot*
⟨*proof*⟩

declare *Sup-set-fold* [**where** *'a* = *'a set*, *code*]

lemma (**in** *complete-lattice*) *INF-set-fold*:
⨅(*f* ‘ *set xs*) = *fold* (*inf* ∘ *f*) *xs top*
⟨*proof*⟩

lemma (**in** *complete-lattice*) *SUP-set-fold*:
⨆(*f* ‘ *set xs*) = *fold* (*sup* ∘ *f*) *xs bot*
⟨*proof*⟩

67.1.20 Fold variants: *foldr* and *foldl*

Correspondence

lemma *foldr-conv-fold* [*code-abbrev*]: *foldr f xs* = *fold f* (*rev xs*)
⟨*proof*⟩

lemma *foldl-conv-fold*: *foldl f s xs* = *fold* (λ*x s*. *f s x*) *xs s*
⟨*proof*⟩

lemma *foldr-conv-foldl*: — The “Third Duality Theorem” in Bird & Wadler:
foldr f xs a = *foldl* (λ*x y*. *f y x*) *a* (*rev xs*)
⟨*proof*⟩

lemma *foldl-conv-foldr*:
foldl f a xs = *foldr* (λ*x y*. *f y x*) (*rev xs*) *a*
⟨*proof*⟩

lemma *foldr-fold*:
(⋀*x y*. *x* ∈ *set xs* ⟹ *y* ∈ *set xs* ⟹ *f y* ∘ *f x* = *f x* ∘ *f y*)
⟹ *foldr f xs* = *fold f xs*
⟨*proof*⟩

lemma *foldr-cong* [*fundef-cong*]:
a = *b* ⟹ *l* = *k* ⟹ (⋀*a x*. *x* ∈ *set l* ⟹ *f x a* = *g x a*) ⟹ *foldr f l a* = *foldr g k b*
⟨*proof*⟩

lemma *foldl-cong* [*fundef-cong*]:

$a = b \Longrightarrow l = k \Longrightarrow (\bigwedge a\ x.\ x \in set\ l \Longrightarrow f\ a\ x = g\ a\ x) \Longrightarrow foldl\ f\ a\ l = foldl\ g\ b\ k$
⟨*proof*⟩

lemma *foldr-append* [*simp*]: $foldr\ f\ (xs\ @\ ys)\ a = foldr\ f\ xs\ (foldr\ f\ ys\ a)$
⟨*proof*⟩

lemma *foldl-append* [*simp*]: $foldl\ f\ a\ (xs\ @\ ys) = foldl\ f\ (foldl\ f\ a\ xs)\ ys$
⟨*proof*⟩

lemma *foldr-map* [*code-unfold*]: $foldr\ g\ (map\ f\ xs)\ a = foldr\ (g \circ f)\ xs\ a$
⟨*proof*⟩

lemma *foldr-filter*:
$foldr\ f\ (filter\ P\ xs) = foldr\ (\lambda x.\ if\ P\ x\ then\ f\ x\ else\ id)\ xs$
⟨*proof*⟩

lemma *foldl-map* [*code-unfold*]:
$foldl\ g\ a\ (map\ f\ xs) = foldl\ (\lambda a\ x.\ g\ a\ (f\ x))\ a\ xs$
⟨*proof*⟩

lemma *concat-conv-foldr* [*code*]:
$concat\ xss = foldr\ append\ xss\ []$
⟨*proof*⟩

67.1.21 *upt*

lemma *upt-rec*[*code*]: $[i..<j] = (if\ i<j\ then\ i\#[Suc\ i..<j]\ else\ [])$
— simp does not terminate!
⟨*proof*⟩

lemmas *upt-rec-numeral*[*simp*] = *upt-rec*[*of numeral m numeral n*] **for** *m n*

lemma *upt-conv-Nil* [*simp*]: $j \le i \Longrightarrow [i..<j] = []$
⟨*proof*⟩

lemma *upt-eq-Nil-conv*[*simp*]: $([i..<j] = []) = (j = 0 \vee j \le i)$
⟨*proof*⟩

lemma *upt-eq-Cons-conv*:
$([i..<j] = x\#xs) = (i < j \wedge i = x \wedge [i+1..<j] = xs)$
⟨*proof*⟩

lemma *upt-Suc-append*: $i \le j \Longrightarrow [i..<(Suc\ j)] = [i..<j]@[j]$
— Only needed if *upt-Suc* is deleted from the simpset.
⟨*proof*⟩

lemma *upt-conv-Cons*: $i < j \Longrightarrow [i..<j] = i\ \#\ [Suc\ i..<j]$
⟨*proof*⟩

lemma *upt-conv-Cons-Cons*: — no precondition
$m \# n \# ns = [m..<q] \longleftrightarrow n \# ns = [Suc\ m..<q]$
⟨*proof*⟩

lemma *upt-add-eq-append*: $i<=j \Longrightarrow [i..<j+k] = [i..<j]@[j..<j+k]$
— LOOPS as a simprule, since $j \leq j$.
⟨*proof*⟩

lemma *length-upt* [*simp*]: $length\ [i..<j] = j - i$
⟨*proof*⟩

lemma *nth-upt* [*simp*]: $i + k < j \Longrightarrow [i..<j] ! k = i + k$
⟨*proof*⟩

lemma *hd-upt*[*simp*]: $i < j \Longrightarrow hd[i..<j] = i$
⟨*proof*⟩

lemma *tl-upt* [*simp*]: $tl\ [m..<n] = [Suc\ m..<n]$
⟨*proof*⟩

lemma *last-upt*[*simp*]: $i < j \Longrightarrow last[i..<j] = j - 1$
⟨*proof*⟩

lemma *take-upt* [*simp*]: $i+m \leq n \Longrightarrow take\ m\ [i..<n] = [i..<i+m]$
⟨*proof*⟩

lemma *drop-upt*[*simp*]: $drop\ m\ [i..<j] = [i+m..<j]$
⟨*proof*⟩

lemma *map-Suc-upt*: $map\ Suc\ [m..<n] = [Suc\ m..<Suc\ n]$
⟨*proof*⟩

lemma *map-add-upt*: $map\ (\lambda i.\ i + n)\ [0..<m] = [n..<m + n]$
⟨*proof*⟩

lemma *nth-map-upt*: $i < n-m \Longrightarrow (map\ f\ [m..<n]) ! i = f(m+i)$
⟨*proof*⟩

lemma *map-decr-upt*: $map\ (\lambda n.\ n - Suc\ 0)\ [Suc\ m..<Suc\ n] = [m..<n]$
⟨*proof*⟩

lemma *map-upt-Suc*: $map\ f\ [0\ ..<\ Suc\ n] = f\ 0\ \#\ map\ (\lambda i.\ f\ (Suc\ i))\ [0\ ..<\ n]$
⟨*proof*⟩

lemma *nth-take-lemma*:
$k \leq length\ xs \Longrightarrow k \leq length\ ys \Longrightarrow$
$(\bigwedge i.\ i < k \longrightarrow xs!i = ys!i) \Longrightarrow take\ k\ xs = take\ k\ ys$
⟨*proof*⟩

lemma *nth-equalityI*:
⟦*length xs* = *length ys*; ⋀*i*. *i* < *length xs* ⟹ *xs*!*i* = *ys*!*i*⟧ ⟹ *xs* = *ys*
⟨*proof*⟩

lemma *map-nth*:
map (λ*i*. *xs* ! *i*) [*0*..<*length xs*] = *xs*
⟨*proof*⟩

lemma *list-all2-antisym*:
⟦ (⋀*x y*. ⟦*P x y*; *Q y x*⟧ ⟹ *x* = *y*); *list-all2 P xs ys*; *list-all2 Q ys xs* ⟧
⟹ *xs* = *ys*
⟨*proof*⟩

lemma *take-equalityI*: (∀ *i*. *take i xs* = *take i ys*) ⟹ *xs* = *ys*
— The famous take-lemma.
⟨*proof*⟩

lemma *take-Cons'*:
take n (*x* # *xs*) = (*if n* = *0 then* [] *else x* # *take* (*n* − *1*) *xs*)
⟨*proof*⟩

lemma *drop-Cons'*:
drop n (*x* # *xs*) = (*if n* = *0 then x* # *xs else drop* (*n* − *1*) *xs*)
⟨*proof*⟩

lemma *nth-Cons'*: (*x* # *xs*)!*n* = (*if n* = *0 then x else xs*!(*n* − *1*))
⟨*proof*⟩

lemma *take-Cons-numeral* [*simp*]:
take (*numeral v*) (*x* # *xs*) = *x* # *take* (*numeral v* − *1*) *xs*
⟨*proof*⟩

lemma *drop-Cons-numeral* [*simp*]:
drop (*numeral v*) (*x* # *xs*) = *drop* (*numeral v* − *1*) *xs*
⟨*proof*⟩

lemma *nth-Cons-numeral* [*simp*]:
(*x* # *xs*) ! *numeral v* = *xs* ! (*numeral v* − *1*)
⟨*proof*⟩

lemma *map-upt-eqI*:
‹*map f* [*m*..<*n*] = *xs*› **if** ‹*length xs* = *n* − *m*›
‹⋀*i*. *i* < *length xs* ⟹ *xs* ! *i* = *f* (*m* + *i*)›
⟨*proof*⟩

67.1.22 *upto*: interval-list on *int*

function *upto* :: *int* ⇒ *int* ⇒ *int list* ((*1*[-../-])) **where**

upto i j = (if i ≤ j then i # [i+1..j] else [])
⟨*proof*⟩
termination
⟨*proof*⟩

declare *upto.simps*[*simp del*]

lemmas *upto-rec-numeral* [*simp*] =
upto.simps[*of numeral m numeral n*]
upto.simps[*of numeral m − numeral n*]
upto.simps[*of − numeral m numeral n*]
upto.simps[*of − numeral m − numeral n*] **for** *m n*

lemma *upto-empty*[*simp*]: *j < i ⟹ [i..j] = []*
⟨*proof*⟩

lemma *upto-single*[*simp*]: *[i..i] = [i]*
⟨*proof*⟩

lemma *upto-Nil*[*simp*]: *[i..j] = [] ⟷ j < i*
⟨*proof*⟩

lemmas *upto-Nil2*[*simp*] = *upto-Nil*[*THEN eq-iff-swap*]

lemma *upto-rec1*: *i ≤ j ⟹ [i..j] = i#[i+1..j]*
⟨*proof*⟩

lemma *upto-rec2*: *i ≤ j ⟹ [i..j] = [i..j − 1]@[j]*
⟨*proof*⟩

lemma *length-upto*[*simp*]: *length [i..j] = nat(j − i + 1)*
⟨*proof*⟩

lemma *set-upto*[*simp*]: *set[i..j] = {i..j}*
⟨*proof*⟩

lemma *nth-upto*[*simp*]: *i + int k ≤ j ⟹ [i..j] ! k = i + int k*
⟨*proof*⟩

lemma *upto-split1*:
i ≤ j ⟹ j ≤ k ⟹ [i..k] = [i..j−1] @ [j..k]
⟨*proof*⟩

lemma *upto-split2*:
i ≤ j ⟹ j ≤ k ⟹ [i..k] = [i..j] @ [j+1..k]
⟨*proof*⟩

lemma *upto-split3*: *⟦ i ≤ j; j ≤ k ⟧ ⟹ [i..k] = [i..j−1] @ j # [j+1..k]*
⟨*proof*⟩

Tail recursive version for code generation:

definition *upto-aux* :: *int* ⇒ *int* ⇒ *int list* ⇒ *int list* **where**
upto-aux i j js = [*i..j*] @ *js*

lemma *upto-aux-rec* [*code*]:
upto-aux i j js = (*if j*<*i then js else upto-aux i* (*j* − *1*) (*j*#*js*))
⟨*proof*⟩

lemma *upto-code*[*code*]: [*i..j*] = *upto-aux i j* []
⟨*proof*⟩

67.1.23 *successively*

lemma *successively-Cons*:
successively P (*x* # *xs*) ⟷ *xs* = [] ∨ *P x* (*hd xs*) ∧ *successively P xs*
⟨*proof*⟩

lemma *successively-cong* [*cong*]:
assumes ⋀*x y*. *x* ∈ *set xs* ⟹ *y* ∈ *set xs* ⟹ *P x y* ⟷ *Q x y xs* = *ys*
shows *successively P xs* ⟷ *successively Q ys*
⟨*proof*⟩

lemma *successively-append-iff*:
successively P (*xs* @ *ys*) ⟷
successively P xs ∧ *successively P ys* ∧
(*xs* = [] ∨ *ys* = [] ∨ *P* (*last xs*) (*hd ys*))
⟨*proof*⟩

lemma *successively-if-sorted-wrt*: *sorted-wrt P xs* ⟹ *successively P xs*
⟨*proof*⟩

lemma *successively-iff-sorted-wrt-strong*:
assumes ⋀*x y z*. *x* ∈ *set xs* ⟹ *y* ∈ *set xs* ⟹ *z* ∈ *set xs* ⟹
P x y ⟹ *P y z* ⟹ *P x z*
shows *successively P xs* ⟷ *sorted-wrt P xs*
⟨*proof*⟩

lemma *successively-conv-sorted-wrt*:
assumes *transp P*
shows *successively P xs* ⟷ *sorted-wrt P xs*
⟨*proof*⟩

lemma *successively-rev* [*simp*]: *successively P* (*rev xs*) ⟷ *successively* (λ*x y*. *P y x*) *xs*
⟨*proof*⟩

lemma *successively-map*: *successively P* (*map f xs*) ⟷ *successively* (λ*x y*. *P* (*f*

$x)$ $(f\ y))$ xs
⟨*proof*⟩

lemma *successively-mono*:
assumes *successively P xs*
assumes $\bigwedge x\ y.\ x \in set\ xs \Longrightarrow y \in set\ xs \Longrightarrow P\ x\ y \Longrightarrow Q\ x\ y$
shows *successively Q xs*
⟨*proof*⟩

lemma *successively-altdef*:
successively = $(\lambda P.$ *rec-list True* $(\lambda x\ xs\ b.$ *case xs of* [] ⇒ *True* | *y* # - ⇒ *P x y* ∧ *b*))
⟨*proof*⟩

67.1.24 *distinct* and *remdups* and *remdups-adj*

lemma *distinct-tl*: *distinct xs* ⟹ *distinct* (*tl xs*)
⟨*proof*⟩

lemma *distinct-append* [*simp*]:
distinct (*xs* @ *ys*) = (*distinct xs* ∧ *distinct ys* ∧ *set xs* ∩ *set ys* = {})
⟨*proof*⟩

lemma *distinct-rev*[*simp*]: *distinct*(*rev xs*) = *distinct xs*
⟨*proof*⟩

lemma *set-remdups* [*simp*]: *set* (*remdups xs*) = *set xs*
⟨*proof*⟩

lemma *distinct-remdups* [*iff*]: *distinct* (*remdups xs*)
⟨*proof*⟩

lemma *distinct-remdups-id*: *distinct xs* ⟹ *remdups xs* = *xs*
⟨*proof*⟩

lemma *remdups-id-iff-distinct* [*simp*]: *remdups xs* = *xs* ⟷ *distinct xs*
⟨*proof*⟩

lemma *finite-distinct-list*: *finite A* ⟹ ∃ *xs*. *set xs* = *A* ∧ *distinct xs*
⟨*proof*⟩

lemma *remdups-eq-nil-iff* [*simp*]: (*remdups x* = []) = (*x* = [])
⟨*proof*⟩

lemmas *remdups-eq-nil-right-iff* [*simp*] = *remdups-eq-nil-iff*[*THEN eq-iff-swap*]

lemma *length-remdups-leq*[*iff*]: *length*(*remdups xs*) ≤ *length xs*
⟨*proof*⟩

lemma *length-remdups-eq*[*iff*]:
(*length* (*remdups xs*) = *length xs*) = (*remdups xs* = *xs*)
⟨*proof*⟩

lemma *remdups-filter*: *remdups*(*filter P xs*) = *filter P* (*remdups xs*)
⟨*proof*⟩

lemma *distinct-map*:
distinct(*map f xs*) = (*distinct xs* ∧ *inj-on f* (*set xs*))
⟨*proof*⟩

lemma *distinct-map-filter*:
distinct (*map f xs*) ⟹ *distinct* (*map f* (*filter P xs*))
⟨*proof*⟩

lemma *distinct-filter* [*simp*]: *distinct xs* ⟹ *distinct* (*filter P xs*)
⟨*proof*⟩

lemma *distinct-upt*[*simp*]: *distinct*[*i*..<*j*]
⟨*proof*⟩

lemma *distinct-upto*[*simp*]: *distinct*[*i*..*j*]
⟨*proof*⟩

lemma *distinct-take*[*simp*]: *distinct xs* ⟹ *distinct* (*take i xs*)
⟨*proof*⟩

lemma *distinct-drop*[*simp*]: *distinct xs* ⟹ *distinct* (*drop i xs*)
⟨*proof*⟩

lemma *distinct-list-update*:
assumes *d*: *distinct xs* **and** *a*: *a* ∉ *set xs* − {*xs*!*i*}
shows *distinct* (*xs*[*i*:=*a*])
⟨*proof*⟩

lemma *distinct-concat*:
⟦ *distinct xs*;
⋀ *ys*. *ys* ∈ *set xs* ⟹ *distinct ys*;
⋀ *ys zs*. ⟦ *ys* ∈ *set xs* ; *zs* ∈ *set xs* ; *ys* ≠ *zs* ⟧ ⟹ *set ys* ∩ *set zs* = {}
⟧ ⟹ *distinct* (*concat xs*)
⟨*proof*⟩

An iff-version of ⟦*distinct ?xs*; ⋀*ys*. *ys* ∈ *set ?xs* ⟹ *distinct ys*; ⋀*ys zs*. ⟦*ys* ∈ *set ?xs*; *zs* ∈ *set ?xs*; *ys* ≠ *zs*⟧ ⟹ *set ys* ∩ *set zs* = {}⟧ ⟹ *distinct* (*concat ?xs*) is available further down as *distinct-concat-iff*.

It is best to avoid the following indexed version of distinct, but sometimes it is useful.

lemma *distinct-conv-nth*: *distinct xs* = (∀ *i* < *size xs*. ∀ *j* < *size xs*. *i* ≠ *j* ⟶ *xs*!*i*

$\neq xs!j$)
⟨*proof*⟩

lemma *nth-eq-iff-index-eq*:
⟦ *distinct xs*; $i < length\ xs$; $j < length\ xs$ ⟧ ⟹ $(xs!i = xs!j) = (i = j)$
⟨*proof*⟩

lemma *distinct-Ex1*:
distinct xs ⟹ $x \in set\ xs$ ⟹ $(\exists!i.\ i < length\ xs \wedge xs\ !\ i = x)$
⟨*proof*⟩

lemma *inj-on-nth*: *distinct xs* ⟹ $\forall i \in I.\ i < length\ xs$ ⟹ *inj-on* (*nth xs*) *I*
⟨*proof*⟩

lemma *bij-betw-nth*:
assumes *distinct xs* $A = \{..<length\ xs\}$ $B = set\ xs$
shows *bij-betw* ((!) *xs*) *A B*
⟨*proof*⟩

lemma *set-update-distinct*: ⟦ *distinct xs*; $n < length\ xs$ ⟧ ⟹
$set(xs[n := x]) = insert\ x\ (set\ xs - \{xs!n\})$
⟨*proof*⟩

lemma *distinct-swap*[*simp*]: ⟦ $i < size\ xs$; $j < size\ xs$⟧ ⟹
$distinct(xs[i := xs!j, j := xs!i]) = distinct\ xs$
⟨*proof*⟩

lemma *set-swap*[*simp*]:
⟦ $i < size\ xs$; $j < size\ xs$ ⟧ ⟹ $set(xs[i := xs!j, j := xs!i]) = set\ xs$
⟨*proof*⟩

lemma *distinct-card*: *distinct xs* ⟹ *card* (*set xs*) = *size xs*
⟨*proof*⟩

lemma *card-distinct*: *card* (*set xs*) = *size xs* ⟹ *distinct xs*
⟨*proof*⟩

lemma *distinct-length-filter*: *distinct xs* ⟹ *length* (*filter P xs*) = *card* ({*x*. *P x*} *Int set xs*)
⟨*proof*⟩

lemma *not-distinct-decomp*: ¬ *distinct ws* ⟹ $\exists\ xs\ ys\ zs\ y.\ ws = xs@[y]@ys@[y]@zs$
⟨*proof*⟩

lemma *not-distinct-conv-prefix*:
defines *dec as xs y ys* ≡ $y \in set\ xs \wedge distinct\ xs \wedge as = xs\ @\ y\ \#\ ys$
shows ¬*distinct as* ⟷ $(\exists\ xs\ y\ ys.\ dec\ as\ xs\ y\ ys)$ (**is** *?L* = *?R*)
⟨*proof*⟩

lemma *distinct-product*:
distinct xs $\Longrightarrow$ *distinct ys* $\Longrightarrow$ *distinct* (*List.product xs ys*)
⟨*proof*⟩

lemma *distinct-product-lists*:
assumes $\forall$ *xs* $\in$ *set xss*. *distinct xs*
shows *distinct* (*product-lists xss*)
⟨*proof*⟩

lemma *length-remdups-concat*:
length (*remdups* (*concat xss*)) = *card* ($\bigcup$ *xs*$\in$*set xss*. *set xs*)
⟨*proof*⟩

lemma *remdups-append2*:
remdups (*xs* @ *remdups ys*) = *remdups* (*xs* @ *ys*)
⟨*proof*⟩

lemma *length-remdups-card-conv*: *length*(*remdups xs*) = *card*(*set xs*)
⟨*proof*⟩

lemma *remdups-remdups*: *remdups* (*remdups xs*) = *remdups xs*
⟨*proof*⟩

lemma *distinct-butlast*:
assumes *distinct xs*
shows *distinct* (*butlast xs*)
⟨*proof*⟩

lemma *remdups-map-remdups*:
remdups (*map f* (*remdups xs*)) = *remdups* (*map f xs*)
⟨*proof*⟩

lemma *distinct-zipI1*:
assumes *distinct xs*
shows *distinct* (*zip xs ys*)
⟨*proof*⟩

lemma *distinct-zipI2*:
assumes *distinct ys*
shows *distinct* (*zip xs ys*)
⟨*proof*⟩

lemma *set-take-disj-set-drop-if-distinct*:
distinct vs $\Longrightarrow$ *i* $\leq$ *j* $\Longrightarrow$ *set* (*take i vs*) $\cap$ *set* (*drop j vs*) = {}
⟨*proof*⟩

lemma *distinct-singleton*: *distinct* [*x*] ⟨*proof*⟩

lemma *distinct-length-2-or-more*:
distinct (*a* # *b* # *xs*) ⟷ (*a* ≠ *b* ∧ *distinct* (*a* # *xs*) ∧ *distinct* (*b* # *xs*))
⟨*proof*⟩

lemma *remdups-adj-altdef*: (*remdups-adj xs* = *ys*) ⟷
(∃*f*::*nat* => *nat*. *mono f* ∧ *f* ' {*0* ..< *size xs*} = {*0* ..< *size ys*}
∧ (∀ *i* < *size xs*. *xs*!*i* = *ys*!(*f i*))
∧ (∀ *i*. *i* + *1* < *size xs* ⟶ (*xs*!*i* = *xs*!(*i*+*1*) ⟷ *f i* = *f*(*i*+*1*)))) (**is** *?L* ⟷
(∃*f*. *?p f xs ys*))
⟨*proof*⟩

lemma *hd-remdups-adj*[*simp*]: *hd* (*remdups-adj xs*) = *hd xs*
⟨*proof*⟩

lemma *remdups-adj-Cons*: *remdups-adj* (*x* # *xs*) =
(*case remdups-adj xs of* [] ⇒ [*x*] | *y* # *xs* ⇒ *if x* = *y then y* # *xs else x* # *y* #
xs)
⟨*proof*⟩

lemma *remdups-adj-append-two*:
remdups-adj (*xs* @ [*x*,*y*]) = *remdups-adj* (*xs* @ [*x*]) @ (*if x* = *y then* [] *else* [*y*])
⟨*proof*⟩

lemma *remdups-adj-adjacent*:
Suc i < *length* (*remdups-adj xs*) ⟹ *remdups-adj xs* ! *i* ≠ *remdups-adj xs* ! *Suc i*
⟨*proof*⟩

lemma *remdups-adj-rev*[*simp*]: *remdups-adj* (*rev xs*) = *rev* (*remdups-adj xs*)
⟨*proof*⟩

lemma *remdups-adj-length*[*simp*]: *length* (*remdups-adj xs*) ≤ *length xs*
⟨*proof*⟩

lemma *remdups-adj-length-ge1*[*simp*]: *xs* ≠ [] ⟹ *length* (*remdups-adj xs*) ≥ *Suc*
0
⟨*proof*⟩

lemma *remdups-adj-Nil-iff*[*simp*]: *remdups-adj xs* = [] ⟷ *xs* = []
⟨*proof*⟩

lemma *remdups-adj-set*[*simp*]: *set* (*remdups-adj xs*) = *set xs*
⟨*proof*⟩

lemma *last-remdups-adj* [*simp*]: *last* (*remdups-adj xs*) = *last xs*
⟨*proof*⟩

lemma *remdups-adj-Cons-alt*[*simp*]: *x* # *tl* (*remdups-adj* (*x* # *xs*)) = *remdups-adj*
(*x* # *xs*)

⟨*proof*⟩

lemma *remdups-adj-distinct*: *distinct xs* ⟹ *remdups-adj xs* = *xs*
⟨*proof*⟩

lemma *remdups-adj-append*:
remdups-adj (*xs*$_1$ @ *x* # *xs*$_2$) = *remdups-adj* (*xs*$_1$ @ [*x*]) @ *tl* (*remdups-adj* (*x* # *xs*$_2$))
⟨*proof*⟩

lemma *remdups-adj-singleton*:
remdups-adj xs = [*x*] ⟹ *xs* = *replicate* (*length xs*) *x*
⟨*proof*⟩

lemma *remdups-adj-map-injective*:
assumes *inj f*
shows *remdups-adj* (*map f xs*) = *map f* (*remdups-adj xs*)
⟨*proof*⟩

lemma *remdups-adj-replicate*:
remdups-adj (*replicate n x*) = (*if n* = *0 then* [] *else* [*x*])
⟨*proof*⟩

lemma *remdups-upt* [*simp*]: *remdups* [*m*..<*n*] = [*m*..<*n*]
⟨*proof*⟩

lemma *successively-remdups-adjI*:
successively P xs ⟹ *successively P* (*remdups-adj xs*)
⟨*proof*⟩

lemma *successively-remdups-adj-iff*:
(⋀*x*. *x* ∈ *set xs* ⟹ *P x x*) ⟹
successively P (*remdups-adj xs*) ⟷ *successively P xs*
⟨*proof*⟩

lemma *remdups-adj-Cons'*:
remdups-adj (*x* # *xs*) = *x* # *remdups-adj* (*dropWhile* (λ*y*. *y* = *x*) *xs*)
⟨*proof*⟩

lemma *remdups-adj-singleton-iff*:
length (*remdups-adj xs*) = *Suc 0* ⟷ *xs* ≠ [] ∧ *xs* = *replicate* (*length xs*) (*hd xs*)
⟨*proof*⟩

lemma *tl-remdups-adj*:
ys ≠ [] ⟹ *tl* (*remdups-adj ys*) = *remdups-adj* (*dropWhile* (λ*x*. *x* = *hd ys*) (*tl ys*))
⟨*proof*⟩

lemma *remdups-adj-append-dropWhile*:

remdups-adj (*xs* @ *y* # *ys*) = *remdups-adj* (*xs* @ [*y*]) @ *remdups-adj* (*dropWhile* (λ*x*. *x* = *y*) *ys*)
⟨*proof*⟩

lemma *remdups-adj-append'*:
assumes *xs* = [] ∨ *ys* = [] ∨ *last xs* ≠ *hd ys*
shows *remdups-adj* (*xs* @ *ys*) = *remdups-adj xs* @ *remdups-adj ys*
⟨*proof*⟩

lemma *remdups-adj-append''*: *xs* ≠ []
⟹ *remdups-adj* (*xs* @ *ys*) = *remdups-adj xs* @ *remdups-adj* (*dropWhile* (λ*y*. *y* = *last xs*) *ys*)
⟨*proof*⟩

67.2 *distinct-adj*

lemma *distinct-adj-Nil* [*simp*]: *distinct-adj* []
and *distinct-adj-singleton* [*simp*]: *distinct-adj* [*x*]
and *distinct-adj-Cons-Cons* [*simp*]: *distinct-adj* (*x* # *y* # *xs*) ⟷ *x* ≠ *y* ∧ *distinct-adj* (*y* # *xs*)
⟨*proof*⟩

lemma *distinct-adj-Cons*: *distinct-adj* (*x* # *xs*) ⟷ *xs* = [] ∨ *x* ≠ *hd xs* ∧ *distinct-adj xs*
⟨*proof*⟩

lemma *distinct-adj-ConsD*: *distinct-adj* (*x* # *xs*) ⟹ *distinct-adj xs*
⟨*proof*⟩

lemma *distinct-adj-remdups-adj*[*simp*]: *distinct-adj* (*remdups-adj xs*)
⟨*proof*⟩

lemma *distinct-adj-altdef*: *distinct-adj xs* ⟷ *remdups-adj xs* = *xs*
⟨*proof*⟩

lemma *distinct-adj-rev* [*simp*]: *distinct-adj* (*rev xs*) ⟷ *distinct-adj xs*
⟨*proof*⟩

lemma *distinct-adj-append-iff*:
distinct-adj (*xs* @ *ys*) ⟷
distinct-adj xs ∧ *distinct-adj ys* ∧ (*xs* = [] ∨ *ys* = [] ∨ *last xs* ≠ *hd ys*)
⟨*proof*⟩

lemma *distinct-adj-appendD1* [*dest*]: *distinct-adj* (*xs* @ *ys*) ⟹ *distinct-adj xs*
and *distinct-adj-appendD2* [*dest*]: *distinct-adj* (*xs* @ *ys*) ⟹ *distinct-adj ys*
⟨*proof*⟩

lemma *distinct-adj-mapI*: *distinct-adj xs* ⟹ *inj-on f* (*set xs*) ⟹ *distinct-adj* (*map f xs*)

⟨*proof*⟩

lemma *distinct-adj-mapD*: *distinct-adj* (*map f xs*) ⟹ *distinct-adj xs*
⟨*proof*⟩

lemma *distinct-adj-map-iff*: *inj-on f* (*set xs*) ⟹ *distinct-adj* (*map f xs*) ⟷ *distinct-adj xs*
⟨*proof*⟩

67.2.1 *insert*

lemma *in-set-insert* [*simp*]:
x ∈ *set xs* ⟹ *List.insert x xs* = *xs*
⟨*proof*⟩

lemma *not-in-set-insert* [*simp*]:
x ∉ *set xs* ⟹ *List.insert x xs* = *x* # *xs*
⟨*proof*⟩

lemma *insert-Nil* [*simp*]: *List.insert x* [] = [*x*]
⟨*proof*⟩

lemma *set-insert* [*simp*]: *set* (*List.insert x xs*) = *insert x* (*set xs*)
⟨*proof*⟩

lemma *distinct-insert* [*simp*]: *distinct* (*List.insert x xs*) = *distinct xs*
⟨*proof*⟩

lemma *insert-remdups*:
List.insert x (*remdups xs*) = *remdups* (*List.insert x xs*)
⟨*proof*⟩

67.2.2 *List.union*

This is all one should need to know about union:

lemma *set-union*[*simp*]: *set* (*List.union xs ys*) = *set xs* ∪ *set ys*
⟨*proof*⟩

lemma *distinct-union*[*simp*]: *distinct*(*List.union xs ys*) = *distinct ys*
⟨*proof*⟩

67.2.3 *find*

lemma *find-None-iff*: *List.find P xs* = *None* ⟷ ¬ (∃ *x*. *x* ∈ *set xs* ∧ *P x*)
⟨*proof*⟩

lemmas *find-None-iff2* = *find-None-iff*[*THEN eq-iff-swap*]

lemma *find-Some-iff*:

List.find P xs = Some x ⟷
(∃ i<length xs. P (xs!i) ∧ x = xs!i ∧ (∀ j<i. ¬ P (xs!j)))
⟨*proof*⟩

lemmas *find-Some-iff2 = find-Some-iff*[*THEN eq-iff-swap*]

lemma *find-cong*[*fundef-cong*]:
assumes *xs = ys* **and** ⋀*x. x ∈ set ys ⟹ P x = Q x*
shows *List.find P xs = List.find Q ys*
⟨*proof*⟩

lemma *find-dropWhile*:
List.find P xs = (case dropWhile (Not ∘ P) xs
of [] ⇒ None
| x # - ⇒ Some x)
⟨*proof*⟩

67.2.4 *count-list*

This library is intentionally minimal. See the remark about multisets at the point above where *count-list* is defined.

lemma *count-list-append*[*simp*]: *count-list (xs @ ys) x = count-list xs x + count-list ys x*
⟨*proof*⟩

lemma *count-list-0-iff*: *count-list xs x = 0 ⟷ x ∉ set xs*
⟨*proof*⟩

lemma *count-notin*[*simp*]: *x ∉ set xs ⟹ count-list xs x = 0*
⟨*proof*⟩

lemma *count-le-length*: *count-list xs x ≤ length xs*
⟨*proof*⟩

lemma *count-list-map-ge*: *count-list xs x ≤ count-list (map f xs) (f x)*
⟨*proof*⟩

lemma *count-list-inj-map*:
⟦ inj-on f (set xs); x ∈ set xs ⟧ ⟹ count-list (map f xs) (f x) = count-list xs x
⟨*proof*⟩

lemma *count-list-rev*[*simp*]: *count-list (rev xs) x = count-list xs x*
⟨*proof*⟩

lemma *sum-count-set*:
set xs ⊆ X ⟹ finite X ⟹ sum (count-list xs) X = length xs
⟨*proof*⟩

67.2.5 List.extract

lemma *extract-None-iff*: *List.extract P xs = None* $\longleftrightarrow$ $\neg$ ($\exists$ *x*$\in$*set xs. P x*)
⟨*proof*⟩

lemma *extract-SomeE*:
List.extract P xs = Some (ys, y, zs) $\Longrightarrow$
xs = ys @ y # zs $\wedge$ *P y* $\wedge$ $\neg$ ($\exists$ *y* $\in$ *set ys. P y*)
⟨*proof*⟩

lemma *extract-Some-iff*:
List.extract P xs = Some (ys, y, zs) $\longleftrightarrow$
xs = ys @ y # zs $\wedge$ *P y* $\wedge$ $\neg$ ($\exists$ *y* $\in$ *set ys. P y*)
⟨*proof*⟩

lemma *extract-Nil-code*[*code*]: *List.extract P [] = None*
⟨*proof*⟩

lemma *extract-Cons-code*[*code*]:
List.extract P (x # xs) = (if P x then Some ([], x, xs) else
(case List.extract P xs of
None $\Rightarrow$ *None* |
Some (ys, y, zs) $\Rightarrow$ *Some (x#ys, y, zs)))*
⟨*proof*⟩

67.2.6 remove1

lemma *remove1-append*:
remove1 x (xs @ ys) =
(if x $\in$ *set xs then remove1 x xs @ ys else xs @ remove1 x ys)*
⟨*proof*⟩

lemma *remove1-commute*: *remove1 x (remove1 y zs) = remove1 y (remove1 x zs)*
⟨*proof*⟩

lemma *in-set-remove1*[*simp*]:
a $\neq$ *b* $\Longrightarrow$ *a* $\in$ *set*(*remove1 b xs*) = (*a* $\in$ *set xs*)
⟨*proof*⟩

lemma *set-remove1-subset*: *set*(*remove1 x xs*) $\subseteq$ *set xs*
⟨*proof*⟩

lemma *set-remove1-eq* [*simp*]: *distinct xs* $\Longrightarrow$ *set*(*remove1 x xs*) = *set xs* $-$ {*x*}
⟨*proof*⟩

lemma *length-remove1*:
length(*remove1 x xs*) = (*if x* $\in$ *set xs then length xs* $-$ *1 else length xs*)
⟨*proof*⟩

lemma *remove1-filter-not*[*simp*]:

$\neg\ P\ x \Longrightarrow remove1\ x\ (filter\ P\ xs) = filter\ P\ xs$
⟨*proof*⟩

lemma *filter-remove1*:
$filter\ Q\ (remove1\ x\ xs) = remove1\ x\ (filter\ Q\ xs)$
⟨*proof*⟩

lemma *notin-set-remove1*[*simp*]: $x \notin set\ xs \Longrightarrow x \notin set(remove1\ y\ xs)$
⟨*proof*⟩

lemma *distinct-remove1*[*simp*]: $distinct\ xs \Longrightarrow distinct(remove1\ x\ xs)$
⟨*proof*⟩

lemma *remove1-remdups*:
$distinct\ xs \Longrightarrow remove1\ x\ (remdups\ xs) = remdups\ (remove1\ x\ xs)$
⟨*proof*⟩

lemma *remove1-idem*: $x \notin set\ xs \Longrightarrow remove1\ x\ xs = xs$
⟨*proof*⟩

lemma *remove1-split*:
$a \in set\ xs \Longrightarrow remove1\ a\ xs = ys \longleftrightarrow (\exists\ ls\ rs.\ xs = ls\ @\ a\ \#\ rs \wedge a \notin set\ ls \wedge ys = ls\ @\ rs)$
⟨*proof*⟩

67.2.7 *removeAll*

lemma *removeAll-filter-not-eq*:
$removeAll\ x = filter\ (\lambda y.\ x \neq y)$
⟨*proof*⟩

lemma *removeAll-append*[*simp*]:
$removeAll\ x\ (xs\ @\ ys) = removeAll\ x\ xs\ @\ removeAll\ x\ ys$
⟨*proof*⟩

lemma *set-removeAll*[*simp*]: $set(removeAll\ x\ xs) = set\ xs - \{x\}$
⟨*proof*⟩

lemma *removeAll-id*[*simp*]: $x \notin set\ xs \Longrightarrow removeAll\ x\ xs = xs$
⟨*proof*⟩

lemma *removeAll-filter-not*[*simp*]:
$\neg\ P\ x \Longrightarrow removeAll\ x\ (filter\ P\ xs) = filter\ P\ xs$
⟨*proof*⟩

lemma *distinct-removeAll*:
$distinct\ xs \Longrightarrow distinct\ (removeAll\ x\ xs)$

⟨*proof*⟩

lemma *distinct-remove1-removeAll*:
distinct xs ⟹ *remove1 x xs* = *removeAll x xs*
⟨*proof*⟩

lemma *map-removeAll-inj-on*: *inj-on f* (*insert x* (*set xs*)) ⟹
map f (*removeAll x xs*) = *removeAll* (*f x*) (*map f xs*)
⟨*proof*⟩

lemma *map-removeAll-inj*: *inj f* ⟹
map f (*removeAll x xs*) = *removeAll* (*f x*) (*map f xs*)
⟨*proof*⟩

lemma *length-removeAll-less-eq* [*simp*]:
length (*removeAll x xs*) ≤ *length xs*
⟨*proof*⟩

lemma *length-removeAll-less* [*termination-simp*]:
x ∈ *set xs* ⟹ *length* (*removeAll x xs*) < *length xs*
⟨*proof*⟩

lemma *distinct-concat-iff*: *distinct* (*concat xs*) ⟷
distinct (*removeAll* [] *xs*) ∧
(∀ *ys*. *ys* ∈ *set xs* ⟶ *distinct ys*) ∧
(∀ *ys zs*. *ys* ∈ *set xs* ∧ *zs* ∈ *set xs* ∧ *ys* ≠ *zs* ⟶ *set ys* ∩ *set zs* = {})
⟨*proof*⟩

67.2.8 *replicate*

lemma *length-replicate* [*simp*]: *length* (*replicate n x*) = *n*
⟨*proof*⟩

lemma *replicate-eqI*:
assumes *length xs* = *n* **and** ⋀*y*. *y* ∈ *set xs* ⟹ *y* = *x*
shows *xs* = *replicate n x*
⟨*proof*⟩

lemma *Ex-list-of-length*: ∃ *xs*. *length xs* = *n*
⟨*proof*⟩

lemma *map-replicate* [*simp*]: *map f* (*replicate n x*) = *replicate n* (*f x*)
⟨*proof*⟩

lemma *map-replicate-const*:
map (λ *x*. *k*) *lst* = *replicate* (*length lst*) *k*
⟨*proof*⟩

lemma *replicate-app-Cons-same*:

$(replicate\ n\ x)\ @\ (x\ \#\ xs) = x\ \#\ replicate\ n\ x\ @\ xs$
⟨*proof*⟩

lemma *rev-replicate* [*simp*]: $rev\ (replicate\ n\ x) = replicate\ n\ x$
⟨*proof*⟩

lemma *replicate-add*: $replicate\ (n + m)\ x = replicate\ n\ x\ @\ replicate\ m\ x$
⟨*proof*⟩

Courtesy of Matthias Daum:

lemma *append-replicate-commute*:
$replicate\ n\ x\ @\ replicate\ k\ x = replicate\ k\ x\ @\ replicate\ n\ x$
⟨*proof*⟩

Courtesy of Andreas Lochbihler:

lemma *filter-replicate*:
$filter\ P\ (replicate\ n\ x) = (if\ P\ x\ then\ replicate\ n\ x\ else\ [])$
⟨*proof*⟩

lemma *hd-replicate* [*simp*]: $n \neq 0 \Longrightarrow hd\ (replicate\ n\ x) = x$
⟨*proof*⟩

lemma *tl-replicate* [*simp*]: $tl\ (replicate\ n\ x) = replicate\ (n - 1)\ x$
⟨*proof*⟩

lemma *last-replicate* [*simp*]: $n \neq 0 \Longrightarrow last\ (replicate\ n\ x) = x$
⟨*proof*⟩

lemma *nth-replicate*[*simp*]: $i < n \Longrightarrow (replicate\ n\ x)!i = x$
⟨*proof*⟩

Courtesy of Matthias Daum (2 lemmas):

lemma *take-replicate*[*simp*]: $take\ i\ (replicate\ k\ x) = replicate\ (min\ i\ k)\ x$
⟨*proof*⟩

lemma *drop-replicate*[*simp*]: $drop\ i\ (replicate\ k\ x) = replicate\ (k-i)\ x$
⟨*proof*⟩

lemma *set-replicate-Suc*: $set\ (replicate\ (Suc\ n)\ x) = \{x\}$
⟨*proof*⟩

lemma *set-replicate* [*simp*]: $n \neq 0 \Longrightarrow set\ (replicate\ n\ x) = \{x\}$
⟨*proof*⟩

lemma *set-replicate-conv-if*: $set\ (replicate\ n\ x) = (if\ n = 0\ then\ \{\}\ else\ \{x\})$
⟨*proof*⟩

lemma *in-set-replicate*[*simp*]: $(x \in set\ (replicate\ n\ y)) = (x = y \wedge n \neq 0)$
⟨*proof*⟩

lemma *card-set-1-iff-replicate*:
card(*set xs*) = *Suc 0* ⟷ *xs* ≠ [] ∧ (∃ *x*. *xs* = *replicate* (*length xs*) *x*)
⟨*proof*⟩

lemma *Ball-set-replicate*[*simp*]:
(∀ *x* ∈ *set*(*replicate n a*). *P x*) = (*P a* ∨ *n*=*0*)
⟨*proof*⟩

lemma *Bex-set-replicate*[*simp*]:
(∃ *x* ∈ *set*(*replicate n a*). *P x*) = (*P a* ∧ *n*≠*0*)
⟨*proof*⟩

lemma *replicate-append-same*:
replicate i x @ [*x*] = *x* # *replicate i x*
⟨*proof*⟩

lemma *map-replicate-trivial*:
map (λ*i*. *x*) [*0*..<*i*] = *replicate i x*
⟨*proof*⟩

lemma *concat-replicate-trivial*[*simp*]:
concat (*replicate i* []) = []
⟨*proof*⟩

lemma *replicate-empty*[*simp*]: (*replicate n x* = []) ⟷ *n*=*0*
⟨*proof*⟩

lemmas *empty-replicate*[*simp*] = *replicate-empty*[*THEN eq-iff-swap*]

lemma *replicate-eq-replicate*[*simp*]:
(*replicate m x* = *replicate n y*) ⟷ (*m*=*n* ∧ (*m*≠*0* ⟶ *x*=*y*))
⟨*proof*⟩

lemma *takeWhile-replicate*[*simp*]:
takeWhile P (*replicate n x*) = (*if P x then replicate n x else* [])
⟨*proof*⟩

lemma *dropWhile-replicate*[*simp*]:
dropWhile P (*replicate n x*) = (*if P x then* [] *else replicate n x*)
⟨*proof*⟩

lemma *replicate-length-filter*:
replicate (*length* (*filter* (λ*y*. *x* = *y*) *xs*)) *x* = *filter* (λ*y*. *x* = *y*) *xs*
⟨*proof*⟩

lemma *comm-append-are-replicate*:
xs @ *ys* = *ys* @ *xs* ⟹ ∃ *m n zs*. *concat* (*replicate m zs*) = *xs* ∧ *concat* (*replicate n zs*) = *ys*

⟨*proof*⟩

lemma *comm-append-is-replicate*:
fixes *xs ys* :: *'a list*
assumes *xs* ≠ [] *ys* ≠ []
assumes *xs* @ *ys* = *ys* @ *xs*
shows ∃ *n zs*. *n* > *1* ∧ *concat* (*replicate n zs*) = *xs* @ *ys*
⟨*proof*⟩

lemma *Cons-replicate-eq*:
x # *xs* = *replicate n y* ⟷ *x* = *y* ∧ *n* > *0* ∧ *xs* = *replicate* (*n* − *1*) *x*
⟨*proof*⟩

lemma *replicate-length-same*:
(∀ *y*∈*set xs*. *y* = *x*) ⟹ *replicate* (*length xs*) *x* = *xs*
⟨*proof*⟩

lemma *foldr-replicate* [*simp*]:
foldr f (*replicate n x*) = *f x* ^^ *n*
⟨*proof*⟩

lemma *fold-replicate* [*simp*]:
fold f (*replicate n x*) = *f x* ^^ *n*
⟨*proof*⟩

67.2.9 *enumerate*

lemma *enumerate-simps* [*simp*, *code*]:
enumerate n [] = []
enumerate n (*x* # *xs*) = (*n*, *x*) # *enumerate* (*Suc n*) *xs*
⟨*proof*⟩

lemma *length-enumerate* [*simp*]:
length (*enumerate n xs*) = *length xs*
⟨*proof*⟩

lemma *map-fst-enumerate* [*simp*]:
map fst (*enumerate n xs*) = [*n*..<*n* + *length xs*]
⟨*proof*⟩

lemma *map-snd-enumerate* [*simp*]:
map snd (*enumerate n xs*) = *xs*
⟨*proof*⟩

lemma *in-set-enumerate-eq*:
p ∈ *set* (*enumerate n xs*) ⟷ *n* ≤ *fst p* ∧ *fst p* < *length xs* + *n* ∧ *nth xs* (*fst p* − *n*) = *snd p*
⟨*proof*⟩

lemma *nth-enumerate-eq*: $m < \mathit{length}\ \mathit{xs} \Longrightarrow \mathit{enumerate}\ n\ \mathit{xs}\ !\ m = (n + m,\ \mathit{xs}\ !\ m)$
⟨*proof*⟩

lemma *enumerate-replicate-eq*:
$\mathit{enumerate}\ n\ (\mathit{replicate}\ m\ a) = \mathit{map}\ (\lambda q.\ (q,\ a))\ [n..<n + m]$
⟨*proof*⟩

lemma *enumerate-Suc-eq*:
$\mathit{enumerate}\ (\mathit{Suc}\ n)\ \mathit{xs} = \mathit{map}\ (\mathit{apfst}\ \mathit{Suc})\ (\mathit{enumerate}\ n\ \mathit{xs})$
⟨*proof*⟩

lemma *distinct-enumerate* [*simp*]:
$\mathit{distinct}\ (\mathit{enumerate}\ n\ \mathit{xs})$
⟨*proof*⟩

lemma *enumerate-append-eq*:
$\mathit{enumerate}\ n\ (\mathit{xs}\ @\ \mathit{ys}) = \mathit{enumerate}\ n\ \mathit{xs}\ @\ \mathit{enumerate}\ (n + \mathit{length}\ \mathit{xs})\ \mathit{ys}$
⟨*proof*⟩

lemma *enumerate-map-upt*:
$\mathit{enumerate}\ n\ (\mathit{map}\ f\ [n..<m]) = \mathit{map}\ (\lambda k.\ (k,\ f\ k))\ [n..<m]$
⟨*proof*⟩

67.2.10 *rotate1* and *rotate*

lemma *rotate0*[*simp*]: $\mathit{rotate}\ 0 = \mathit{id}$
⟨*proof*⟩

lemma *rotate-Suc*[*simp*]: $\mathit{rotate}\ (\mathit{Suc}\ n)\ \mathit{xs} = \mathit{rotate1}(\mathit{rotate}\ n\ \mathit{xs})$
⟨*proof*⟩

lemma *rotate-add*:
$\mathit{rotate}\ (m+n) = \mathit{rotate}\ m \circ \mathit{rotate}\ n$
⟨*proof*⟩

lemma *rotate-rotate*: $\mathit{rotate}\ m\ (\mathit{rotate}\ n\ \mathit{xs}) = \mathit{rotate}\ (m+n)\ \mathit{xs}$
⟨*proof*⟩

lemma *rotate1-map*: $\mathit{rotate1}\ (\mathit{map}\ f\ \mathit{xs}) = \mathit{map}\ f\ (\mathit{rotate1}\ \mathit{xs})$
⟨*proof*⟩

lemma *rotate1-rotate-swap*: $\mathit{rotate1}\ (\mathit{rotate}\ n\ \mathit{xs}) = \mathit{rotate}\ n\ (\mathit{rotate1}\ \mathit{xs})$
⟨*proof*⟩

lemma *rotate1-length01*[*simp*]: $\mathit{length}\ \mathit{xs} \leq 1 \Longrightarrow \mathit{rotate1}\ \mathit{xs} = \mathit{xs}$
⟨*proof*⟩

lemma *rotate-length01*[*simp*]: $\mathit{length}\ \mathit{xs} \leq 1 \Longrightarrow \mathit{rotate}\ n\ \mathit{xs} = \mathit{xs}$

⟨*proof*⟩

lemma *rotate1-hd-tl*: *xs* ≠ [] ⟹ *rotate1 xs* = *tl xs* @ [*hd xs*]
⟨*proof*⟩

lemma *rotate-drop-take*:
rotate n xs = *drop* (*n mod length xs*) *xs* @ *take* (*n mod length xs*) *xs*
⟨*proof*⟩

lemma *rotate-conv-mod*: *rotate n xs* = *rotate* (*n mod length xs*) *xs*
⟨*proof*⟩

lemma *rotate-id*[*simp*]: *n mod length xs* = *0* ⟹ *rotate n xs* = *xs*
⟨*proof*⟩

lemma *length-rotate1*[*simp*]: *length*(*rotate1 xs*) = *length xs*
⟨*proof*⟩

lemma *length-rotate*[*simp*]: *length*(*rotate n xs*) = *length xs*
⟨*proof*⟩

lemma *distinct1-rotate*[*simp*]: *distinct*(*rotate1 xs*) = *distinct xs*
⟨*proof*⟩

lemma *distinct-rotate*[*simp*]: *distinct*(*rotate n xs*) = *distinct xs*
⟨*proof*⟩

lemma *rotate-map*: *rotate n* (*map f xs*) = *map f* (*rotate n xs*)
⟨*proof*⟩

lemma *set-rotate1*[*simp*]: *set*(*rotate1 xs*) = *set xs*
⟨*proof*⟩

lemma *set-rotate*[*simp*]: *set*(*rotate n xs*) = *set xs*
⟨*proof*⟩

lemma *rotate1-replicate*[*simp*]: *rotate1* (*replicate n a*) = *replicate n a*
⟨*proof*⟩

lemma *rotate1-is-Nil-conv*[*simp*]: (*rotate1 xs* = []) = (*xs* = [])
⟨*proof*⟩

lemma *rotate-is-Nil-conv*[*simp*]: (*rotate n xs* = []) = (*xs* = [])
⟨*proof*⟩

lemma *rotate-rev*:
rotate n (*rev xs*) = *rev*(*rotate* (*length xs* − (*n mod length xs*)) *xs*)
⟨*proof*⟩

lemma *hd-rotate-conv-nth*:
assumes $xs \neq []$ **shows** $hd(rotate\ n\ xs) = xs!(n\ mod\ length\ xs)$
⟨*proof*⟩

lemma *rotate-append*: $rotate\ (length\ l)\ (l\ @\ q) = q\ @\ l$
⟨*proof*⟩

lemma *nth-rotate*:
$‹rotate\ m\ xs\ !\ n = xs\ !\ ((m + n)\ mod\ length\ xs)›$ **if** $‹n < length\ xs›$
⟨*proof*⟩

lemma *nth-rotate1*:
$‹rotate1\ xs\ !\ n = xs\ !\ (Suc\ n\ mod\ length\ xs)›$ **if** $‹n < length\ xs›$
⟨*proof*⟩

lemma *inj-rotate1*: *inj rotate1*
⟨*proof*⟩

lemma *surj-rotate1*: *surj rotate1*
⟨*proof*⟩

lemma *bij-rotate1*: $bij\ (rotate1 :: 'a\ list \Rightarrow 'a\ list)$
⟨*proof*⟩

lemma *rotate1-fixpoint-card*: $rotate1\ xs = xs \Longrightarrow xs = [] \vee card(set\ xs) = 1$
⟨*proof*⟩

67.2.11 *nths* — a generalization of (!) to sets

lemma *nths-empty* [*simp*]: $nths\ xs\ \{\} = []$
⟨*proof*⟩

lemma *nths-nil* [*simp*]: $nths\ []\ A = []$
⟨*proof*⟩

lemma *nths-all*: $\forall i < length\ xs.\ i \in I \Longrightarrow nths\ xs\ I = xs$
⟨*proof*⟩

lemma *length-nths*:
$length\ (nths\ xs\ I) = card\{i.\ i < length\ xs \wedge i \in I\}$
⟨*proof*⟩

lemma *nths-shift-lemma-Suc*:
$map\ fst\ (filter\ (\lambda p.\ P(Suc(snd\ p)))\ (zip\ xs\ is)) =$
$map\ fst\ (filter\ (\lambda p.\ P(snd\ p))\ (zip\ xs\ (map\ Suc\ is)))$
⟨*proof*⟩

lemma *nths-shift-lemma*:
$map\ fst\ (filter\ (\lambda p.\ snd\ p \in A)\ (zip\ xs\ [i..<i + length\ xs])) =$

map fst (*filter* (λ*p*. *snd p* + *i* ∈ *A*) (*zip xs* [*0*..<*length xs*]))
⟨*proof*⟩

lemma *nths-append*:
nths (*l* @ *l'*) *A* = *nths l A* @ *nths l'* {*j*. *j* + *length l* ∈ *A*}
⟨*proof*⟩

lemma *nths-Cons*:
nths (*x* # *l*) *A* = (*if 0* ∈ *A then* [*x*] *else* []) @ *nths l* {*j*. *Suc j* ∈ *A*}
⟨*proof*⟩

lemma *nths-map*: *nths* (*map f xs*) *I* = *map f* (*nths xs I*)
⟨*proof*⟩

lemma *set-nths*: *set*(*nths xs I*) = {*xs*!*i*|*i*. *i*<*size xs* ∧ *i* ∈ *I*}
⟨*proof*⟩

lemma *set-nths-subset*: *set*(*nths xs I*) ⊆ *set xs*
⟨*proof*⟩

lemma *notin-set-nthsI*[*simp*]: *x* ∉ *set xs* ⟹ *x* ∉ *set*(*nths xs I*)
⟨*proof*⟩

lemma *in-set-nthsD*: *x* ∈ *set*(*nths xs I*) ⟹ *x* ∈ *set xs*
⟨*proof*⟩

lemma *nths-singleton* [*simp*]: *nths* [*x*] *A* = (*if 0* ∈ *A then* [*x*] *else* [])
⟨*proof*⟩

lemma *distinct-nthsI*[*simp*]: *distinct xs* ⟹ *distinct* (*nths xs I*)
⟨*proof*⟩

lemma *nths-upt-eq-take* [*simp*]: *nths l* {..<*n*} = *take n l*
⟨*proof*⟩

lemma *nths-nths*: *nths* (*nths xs A*) *B* = *nths xs* {*i* ∈ *A*. ∃*j* ∈ *B*. *card* {*i'* ∈ *A*. *i'* < *i*} = *j*}
⟨*proof*⟩

lemma *drop-eq-nths*: *drop n xs* = *nths xs* {*i*. *i* ≥ *n*}
⟨*proof*⟩

lemma *nths-drop*: *nths* (*drop n xs*) *I* = *nths xs* ((+) *n* ‘ *I*)
⟨*proof*⟩

lemma *filter-eq-nths*: *filter P xs* = *nths xs* {*i*. *i*<*length xs* ∧ *P*(*xs*!*i*)}
⟨*proof*⟩

lemma *filter-in-nths*:

distinct xs $\Longrightarrow$ *filter* (%*x*. *x* $\in$ *set* (*nths xs s*)) *xs* = *nths xs s*
⟨*proof*⟩

67.2.12 *subseqs* and *List.n-lists*

lemma *length-subseqs*: *length* (*subseqs xs*) = 2 ^ *length xs*
⟨*proof*⟩

lemma *subseqs-powset*: *set* ‘ *set* (*subseqs xs*) = *Pow* (*set xs*)
⟨*proof*⟩

lemma *distinct-set-subseqs*:
assumes *distinct xs*
shows *distinct* (*map set* (*subseqs xs*))
⟨*proof*⟩

lemma *n-lists-Nil* [*simp*]: *List.n-lists n* [] = (*if n* = *0 then* [[]] *else* [])
⟨*proof*⟩

lemma *length-n-lists-elem*: *ys* $\in$ *set* (*List.n-lists n xs*) $\Longrightarrow$ *length ys* = *n*
⟨*proof*⟩

lemma *set-n-lists*: *set* (*List.n-lists n xs*) = {*ys*. *length ys* = *n* $\wedge$ *set ys* $\subseteq$ *set xs*}
⟨*proof*⟩

lemma *subseqs-refl*: *xs* $\in$ *set* (*subseqs xs*)
⟨*proof*⟩

lemma *subset-subseqs*: *X* $\subseteq$ *set xs* $\Longrightarrow$ *X* $\in$ *set* ‘ *set* (*subseqs xs*)
⟨*proof*⟩

lemma *Cons-in-subseqsD*: *y* # *ys* $\in$ *set* (*subseqs xs*) $\Longrightarrow$ *ys* $\in$ *set* (*subseqs xs*)
⟨*proof*⟩

lemma *subseqs-distinctD*: ⟦ *ys* $\in$ *set* (*subseqs xs*); *distinct xs* ⟧ $\Longrightarrow$ *distinct ys*
⟨*proof*⟩

67.2.13 *splice*

lemma *splice-Nil2* [*simp*]: *splice xs* [] = *xs*
⟨*proof*⟩

lemma *length-splice*[*simp*]: *length*(*splice xs ys*) = *length xs* + *length ys*
⟨*proof*⟩

lemma *split-Nil-iff*[*simp*]: *splice xs ys* = [] $\longleftrightarrow$ *xs* = [] $\wedge$ *ys* = []
⟨*proof*⟩

lemma *splice-replicate*[*simp*]: *splice* (*replicate m x*) (*replicate n x*) = *replicate* (*m*+*n*) *x*

⟨*proof*⟩

67.2.14 *shuffles*

lemma *shuffles-commutes*: *shuffles xs ys* = *shuffles ys xs*
⟨*proof*⟩

lemma *Nil-in-shuffles*[*simp*]: [] ∈ *shuffles xs ys* ⟷ *xs* = [] ∧ *ys* = []
⟨*proof*⟩

lemma *shufflesE*:
zs ∈ *shuffles xs ys* ⟹
(*zs* = *xs* ⟹ *ys* = [] ⟹ *P*) ⟹
(*zs* = *ys* ⟹ *xs* = [] ⟹ *P*) ⟹
(⋀*x xs*′ *z zs*′. *xs* = *x* # *xs*′ ⟹ *zs* = *z* # *zs*′ ⟹ *x* = *z* ⟹ *zs*′ ∈ *shuffles xs*′ *ys* ⟹ *P*) ⟹
(⋀*y ys*′ *z zs*′. *ys* = *y* # *ys*′ ⟹ *zs* = *z* # *zs*′ ⟹ *y* = *z* ⟹ *zs*′ ∈ *shuffles xs ys*′ ⟹ *P*) ⟹ *P*
⟨*proof*⟩

lemma *Cons-in-shuffles-iff*:
z # *zs* ∈ *shuffles xs ys* ⟷
(*xs* ≠ [] ∧ *hd xs* = *z* ∧ *zs* ∈ *shuffles* (*tl xs*) *ys* ∨
ys ≠ [] ∧ *hd ys* = *z* ∧ *zs* ∈ *shuffles xs* (*tl ys*))
⟨*proof*⟩

lemma *splice-in-shuffles* [*simp*, *intro*]: *splice xs ys* ∈ *shuffles xs ys*
⟨*proof*⟩

lemma *Nil-in-shufflesI*: *xs* = [] ⟹ *ys* = [] ⟹ [] ∈ *shuffles xs ys*
⟨*proof*⟩

lemma *Cons-in-shuffles-leftI*: *zs* ∈ *shuffles xs ys* ⟹ *z* # *zs* ∈ *shuffles* (*z* # *xs*) *ys*
⟨*proof*⟩

lemma *Cons-in-shuffles-rightI*: *zs* ∈ *shuffles xs ys* ⟹ *z* # *zs* ∈ *shuffles xs* (*z* # *ys*)
⟨*proof*⟩

lemma *finite-shuffles* [*simp*, *intro*]: *finite* (*shuffles xs ys*)
⟨*proof*⟩

lemma *length-shuffles*: *zs* ∈ *shuffles xs ys* ⟹ *length zs* = *length xs* + *length ys*
⟨*proof*⟩

lemma *set-shuffles*: *zs* ∈ *shuffles xs ys* ⟹ *set zs* = *set xs* ∪ *set ys*
⟨*proof*⟩

lemma *distinct-disjoint-shuffles*:
assumes *distinct xs distinct ys set xs ∩ set ys = {} zs ∈ shuffles xs ys*
shows *distinct zs*
⟨*proof*⟩

lemma *Cons-shuffles-subset1*: *(#) x ‘ shuffles xs ys ⊆ shuffles (x # xs) ys*
⟨*proof*⟩

lemma *Cons-shuffles-subset2*: *(#) y ‘ shuffles xs ys ⊆ shuffles xs (y # ys)*
⟨*proof*⟩

lemma *filter-shuffles*:
filter P ‘ shuffles xs ys = shuffles (filter P xs) (filter P ys)
⟨*proof*⟩

lemma *filter-shuffles-disjoint1*:
assumes *set xs ∩ set ys = {} zs ∈ shuffles xs ys*
shows *filter (λx. x ∈ set xs) zs = xs* (**is** *filter ?P - = -*)
and *filter (λx. x ∉ set xs) zs = ys* (**is** *filter ?Q - = -*)
⟨*proof*⟩

lemma *filter-shuffles-disjoint2*:
assumes *set xs ∩ set ys = {} zs ∈ shuffles xs ys*
shows *filter (λx. x ∈ set ys) zs = ys filter (λx. x ∉ set ys) zs = xs*
⟨*proof*⟩

lemma *partition-in-shuffles*:
xs ∈ shuffles (filter P xs) (filter (λx. ¬P x) xs)
⟨*proof*⟩

lemma *inv-image-partition*:
assumes *⋀x. x ∈ set xs ⟹ P x ⋀y. y ∈ set ys ⟹ ¬P y*
shows *partition P −‘ {(xs, ys)} = shuffles xs ys*
⟨*proof*⟩

67.2.15 Transpose

function *transpose* **where**
transpose [] = [] |
transpose ([] # xss) = *transpose xss* |
transpose ((x#xs) # xss) =
(x # [h. (h#t) ← xss]) # transpose (xs # [t. (h#t) ← xss])
⟨*proof*⟩

lemma *transpose-aux-filter-head*:
concat (map (case-list [] (λh t. [h])) xss) =
map (λxs. hd xs) (filter (λys. ys ≠ []) xss)
⟨*proof*⟩

lemma *transpose-aux-filter-tail*:
concat (*map* (*case-list* [] (λ*h t*. [*t*])) *xss*) =
map (λ*xs*. *tl xs*) (*filter* (λ*ys*. *ys* ≠ []) *xss*)
⟨*proof*⟩

lemma *transpose-aux-max*:
max (*Suc* (*length xs*)) (*foldr* (λ*xs*. *max* (*length xs*)) *xss 0*) =
Suc (*max* (*length xs*) (*foldr* (λ*x*. *max* (*length x* − *Suc 0*)) (*filter* (λ*ys*. *ys* ≠ []) *xss*) *0*))
(**is** *max* - *?foldB* = *Suc* (*max* - *?foldA*))
⟨*proof*⟩

termination *transpose*
⟨*proof*⟩

lemma *transpose-empty*: (*transpose xs* = []) ⟷ (∀ *x* ∈ *set xs*. *x* = [])
⟨*proof*⟩

lemma *length-transpose*:
fixes *xs* :: *'a list list*
shows *length* (*transpose xs*) = *foldr* (λ*xs*. *max* (*length xs*)) *xs 0*
⟨*proof*⟩

lemma *nth-transpose*:
fixes *xs* :: *'a list list*
assumes *i* < *length* (*transpose xs*)
shows *transpose xs* ! *i* = *map* (λ*xs*. *xs* ! *i*) (*filter* (λ*ys*. *i* < *length ys*) *xs*)
⟨*proof*⟩

lemma *transpose-map-map*:
transpose (*map* (*map f*) *xs*) = *map* (*map f*) (*transpose xs*)
⟨*proof*⟩

67.2.16 *min* and *arg-min*

lemma *min-list-Min*: *xs* ≠ [] ⟹ *min-list xs* = *Min* (*set xs*)
⟨*proof*⟩

lemma *f-arg-min-list-f*: *xs* ≠ [] ⟹ *f* (*arg-min-list f xs*) = *Min* (*f* ‘ (*set xs*))
⟨*proof*⟩

lemma *arg-min-list-in*: *xs* ≠ [] ⟹ *arg-min-list f xs* ∈ *set xs*
⟨*proof*⟩

67.2.17 (In)finiteness

lemma *finite-maxlen*:
finite (*M*::*'a list set*) ⟹ ∃ *n*. ∀ *s*∈*M*. *size s* < *n*
⟨*proof*⟩

lemma *lists-length-Suc-eq*:
{*xs*. *set xs* ⊆ *A* ∧ *length xs* = *Suc n*} =
(λ(*xs*, *n*). *n*#*xs*) ‘ ({*xs*. *set xs* ⊆ *A* ∧ *length xs* = *n*} × *A*)
⟨*proof*⟩

lemma
assumes *finite A*
shows *finite-lists-length-eq*: *finite* {*xs*. *set xs* ⊆ *A* ∧ *length xs* = *n*}
and *card-lists-length-eq*: *card* {*xs*. *set xs* ⊆ *A* ∧ *length xs* = *n*} = (*card A*)^*n*
⟨*proof*⟩

lemma *finite-lists-length-le*:
assumes *finite A* **shows** *finite* {*xs*. *set xs* ⊆ *A* ∧ *length xs* ≤ *n*}
(**is** *finite ?S*)
⟨*proof*⟩

lemma *card-lists-length-le*:
assumes *finite A* **shows** *card* {*xs*. *set xs* ⊆ *A* ∧ *length xs* ≤ *n*} = (∑ *i*≤*n*. *card A*^*i*)
⟨*proof*⟩

lemma *finite-lists-distinct-length-eq* [*intro*]:
assumes *finite A*
shows *finite* {*xs*. *length xs* = *n* ∧ *distinct xs* ∧ *set xs* ⊆ *A*} (**is** *finite ?S*)
⟨*proof*⟩

lemma *card-lists-distinct-length-eq*:
assumes *finite A k* ≤ *card A*
shows *card* {*xs*. *length xs* = *k* ∧ *distinct xs* ∧ *set xs* ⊆ *A*} = ∏ {*card A* − *k* + *1* .. *card A*}
⟨*proof*⟩

lemma *card-lists-distinct-length-eq′*:
assumes *k* < *card A*
shows *card* {*xs*. *length xs* = *k* ∧ *distinct xs* ∧ *set xs* ⊆ *A*} = ∏ {*card A* − *k* + *1* .. *card A*}
⟨*proof*⟩

lemma *infinite-UNIV-listI*: ¬ *finite*(*UNIV*::*′a list set*)
⟨*proof*⟩

lemma *same-length-different*:
assumes *xs* ≠ *ys* **and** *length xs* = *length ys*
shows ∃ *pre x xs′ y ys′*. *x*≠*y* ∧ *xs* = *pre* @ [*x*] @ *xs′* ∧ *ys* = *pre* @ [*y*] @ *ys′*
⟨*proof*⟩

67.3 Sorting

67.3.1 *sorted-wrt*

Sometimes the second equation in the definition of *sorted-wrt* is too aggressive because it relates each list element to *all* its successors. Then this equation should be removed and *sorted-wrt2-simps* should be added instead.

lemma *sorted-wrt1*: *sorted-wrt P* [*x*] = *True*
⟨*proof*⟩

lemma *sorted-wrt2*: *transp P* ⟹ *sorted-wrt P* (*x* # *y* # *zs*) = (*P x y* ∧ *sorted-wrt P* (*y* # *zs*))
⟨*proof*⟩

lemmas *sorted-wrt2-simps* = *sorted-wrt1 sorted-wrt2*

lemma *sorted-wrt-true* [*simp*]:
sorted-wrt (λ- -. *True*) *xs*
⟨*proof*⟩

lemma *sorted-wrt-append*:
sorted-wrt P (*xs* @ *ys*) ⟷
sorted-wrt P xs ∧ *sorted-wrt P ys* ∧ (∀ *x*∈*set xs*. ∀ *y*∈*set ys*. *P x y*)
⟨*proof*⟩

lemma *sorted-wrt-map*:
sorted-wrt R (*map f xs*) = *sorted-wrt* (λ*x y*. *R* (*f x*) (*f y*)) *xs*
⟨*proof*⟩

lemma
assumes *sorted-wrt f xs*
shows *sorted-wrt-take*: *sorted-wrt f* (*take n xs*)
and *sorted-wrt-drop*: *sorted-wrt f* (*drop n xs*)
⟨*proof*⟩

lemma *sorted-wrt-filter*:
sorted-wrt f xs ⟹ *sorted-wrt f* (*filter P xs*)
⟨*proof*⟩

lemma *sorted-wrt-rev*:
sorted-wrt P (*rev xs*) = *sorted-wrt* (λ*x y*. *P y x*) *xs*
⟨*proof*⟩

lemma *sorted-wrt-mono-rel*:
(⋀*x y*. ⟦ *x* ∈ *set xs*; *y* ∈ *set xs*; *P x y* ⟧ ⟹ *Q x y*) ⟹ *sorted-wrt P xs* ⟹ *sorted-wrt Q xs*
⟨*proof*⟩

lemma *sorted-wrt01*: *length xs* ≤ *1* ⟹ *sorted-wrt P xs*

⟨*proof*⟩

lemma *sorted-wrt-iff-nth-less*:
sorted-wrt P xs = (∀ *i j*. *i* < *j* ⟶ *j* < *length xs* ⟶ *P* (*xs* ! *i*) (*xs* ! *j*))
⟨*proof*⟩

lemma *sorted-wrt-nth-less*:
⟦ *sorted-wrt P xs*; *i* < *j*; *j* < *length xs* ⟧ ⟹ *P* (*xs* ! *i*) (*xs* ! *j*)
⟨*proof*⟩

lemma *sorted-wrt-iff-nth-Suc-transp*: **assumes** *transp P*
shows *sorted-wrt P xs* ⟷ (∀ *i*. *Suc i* < *length xs* ⟶ *P* (*xs*!*i*) (*xs*!(*Suc i*))) (**is** *?L* = *?R*)
⟨*proof*⟩

lemma *sorted-wrt-upt*[*simp*]: *sorted-wrt* (<) [*m*..<*n*]
⟨*proof*⟩

lemma *sorted-wrt-upto*[*simp*]: *sorted-wrt* (<) [*i*..*j*]
⟨*proof*⟩

Each element is greater or equal to its index:

lemma *sorted-wrt-less-idx*:
sorted-wrt (<) *ns* ⟹ *i* < *length ns* ⟹ *i* ≤ *ns*!*i*
⟨*proof*⟩

67.3.2 *sorted*

context *linorder*
begin

Sometimes the second equation in the definition of *sorted* is too aggressive because it relates each list element to *all* its successors. Then this equation should be removed and *sorted2-simps* should be added instead. Executable code is one such use case.

lemma *sorted0*: *sorted* [] = *True*
⟨*proof*⟩

lemma *sorted1*: *sorted* [*x*] = *True*
⟨*proof*⟩

lemma *sorted2*: *sorted* (*x* # *y* # *zs*) = (*x* ≤ *y* ∧ *sorted* (*y* # *zs*))
⟨*proof*⟩

lemmas *sorted2-simps* = *sorted1 sorted2*

lemma *sorted-append*:
sorted (*xs*@*ys*) = (*sorted xs* ∧ *sorted ys* ∧ (∀ *x* ∈ *set xs*. ∀ *y* ∈ *set ys*. *x*≤*y*))
⟨*proof*⟩

lemma *sorted-map*:
sorted (*map f xs*) = *sorted-wrt* (λ*x y*. *f x* ≤ *f y*) *xs*
⟨*proof*⟩

lemma *sorted01*: *length xs* ≤ *1* ⟹ *sorted xs*
⟨*proof*⟩

lemma *sorted-tl*:
sorted xs ⟹ *sorted* (*tl xs*)
⟨*proof*⟩

lemma *sorted-iff-nth-mono-less*:
sorted xs = (∀ *i j*. *i* < *j* ⟶ *j* < *length xs* ⟶ *xs* ! *i* ≤ *xs* ! *j*)
⟨*proof*⟩

lemma *sorted-iff-nth-mono*:
sorted xs = (∀ *i j*. *i* ≤ *j* ⟶ *j* < *length xs* ⟶ *xs* ! *i* ≤ *xs* ! *j*)
⟨*proof*⟩

lemma *sorted-nth-mono*:
sorted xs ⟹ *i* ≤ *j* ⟹ *j* < *length xs* ⟹ *xs*!*i* ≤ *xs*!*j*
⟨*proof*⟩

lemma *sorted-iff-nth-Suc*:
sorted xs ⟷ (∀ *i*. *Suc i* < *length xs* ⟶ *xs*!*i* ≤ *xs*!(*Suc i*))
⟨*proof*⟩

lemma *sorted-rev-nth-mono*:
sorted (*rev xs*) ⟹ *i* ≤ *j* ⟹ *j* < *length xs* ⟹ *xs*!*j* ≤ *xs*!*i*
⟨*proof*⟩

lemma *sorted-rev-iff-nth-mono*:
sorted (*rev xs*) ⟷ (∀ *i j*. *i* ≤ *j* ⟶ *j* < *length xs* ⟶ *xs*!*j* ≤ *xs*!*i*) (**is** *?L* = *?R*)
⟨*proof*⟩

lemma *sorted-rev-iff-nth-Suc*:
sorted (*rev xs*) ⟷ (∀ *i*. *Suc i* < *length xs* ⟶ *xs*!(*Suc i*) ≤ *xs*!*i*)
⟨*proof*⟩

lemma *sorted-map-remove1*:
sorted (*map f xs*) ⟹ *sorted* (*map f* (*remove1 x xs*))
⟨*proof*⟩

lemma *sorted-remove1*: *sorted xs* ⟹ *sorted* (*remove1 a xs*)
⟨*proof*⟩

lemma *sorted-butlast*:
assumes *sorted xs*

shows *sorted* (*butlast xs*)
⟨*proof*⟩

lemma *sorted-replicate* [*simp*]: *sorted*(*replicate n x*)
⟨*proof*⟩

lemma *sorted-remdups*[*simp*]:
sorted xs ⟹ *sorted* (*remdups xs*)
⟨*proof*⟩

lemma *sorted-remdups-adj*[*simp*]:
sorted xs ⟹ *sorted* (*remdups-adj xs*)
⟨*proof*⟩

lemma *sorted-nths*: *sorted xs* ⟹ *sorted* (*nths xs I*)
⟨*proof*⟩

lemma *sorted-distinct-set-unique*:
assumes *sorted xs distinct xs sorted ys distinct ys set xs* = *set ys*
shows *xs* = *ys*
⟨*proof*⟩

lemma *map-sorted-distinct-set-unique*:
assumes *inj-on f* (*set xs* ∪ *set ys*)
assumes *sorted* (*map f xs*) *distinct* (*map f xs*)
sorted (*map f ys*) *distinct* (*map f ys*)
assumes *set xs* = *set ys*
shows *xs* = *ys*
⟨*proof*⟩

lemma *sorted-dropWhile*: *sorted xs* ⟹ *sorted* (*dropWhile P xs*)
⟨*proof*⟩

lemma *sorted-takeWhile*: *sorted xs* ⟹ *sorted* (*takeWhile P xs*)
⟨*proof*⟩

lemma *sorted-filter*:
sorted (*map f xs*) ⟹ *sorted* (*map f* (*filter P xs*))
⟨*proof*⟩

lemma *foldr-max-sorted*:
assumes *sorted* (*rev xs*)
shows *foldr max xs y* = (*if xs* = [] *then y else max* (*xs* ! *0*) *y*)
⟨*proof*⟩

lemma *filter-equals-takeWhile-sorted-rev*:
assumes *sorted*: *sorted* (*rev* (*map f xs*))
shows *filter* (λ*x*. *t* < *f x*) *xs* = *takeWhile* (λ *x*. *t* < *f x*) *xs*
(**is** *filter ?P xs* = *?tW*)

⟨*proof*⟩

lemma *sorted-map-same*:
sorted (*map f* (*filter* (λ*x*. *f x* = *g xs*) *xs*))
⟨*proof*⟩

lemma *sorted-same*:
sorted (*filter* (λ*x*. *x* = *g xs*) *xs*)
⟨*proof*⟩

end

lemma *sorted-upt*[*simp*]: *sorted* [*m*..<*n*]
⟨*proof*⟩

lemma *sorted-upto*[*simp*]: *sorted* [*m*..*n*]
⟨*proof*⟩

67.3.3 Sorting functions

Currently it is not shown that *sort* returns a permutation of its input because the nicest proof is via multisets, which are not part of Main. Alternatively one could define a function that counts the number of occurrences of an element in a list and use that instead of multisets to state the correctness property.

context *linorder*
begin

lemma *set-insort-key*:
set (*insort-key f x xs*) = *insert x* (*set xs*)
⟨*proof*⟩

lemma *length-insort* [*simp*]:
length (*insort-key f x xs*) = *Suc* (*length xs*)
⟨*proof*⟩

lemma *insort-key-left-comm*:
assumes *f x* ≠ *f y*
shows *insort-key f y* (*insort-key f x xs*) = *insort-key f x* (*insort-key f y xs*)
⟨*proof*⟩

lemma *insort-left-comm*:
insort x (*insort y xs*) = *insort y* (*insort x xs*)
⟨*proof*⟩

lemma *comp-fun-commute-insort*: *comp-fun-commute insort*
⟨*proof*⟩

lemma *sort-key-simps* [*simp*]:
sort-key f [] = []
sort-key f (*x*#*xs*) = *insort-key f x* (*sort-key f xs*)
⟨*proof*⟩

lemma *sort-key-conv-fold*:
assumes *inj-on f* (*set xs*)
shows *sort-key f xs* = *fold* (*insort-key f*) *xs* []
⟨*proof*⟩

lemma *sort-conv-fold*:
sort xs = *fold insort xs* []
⟨*proof*⟩

lemma *length-sort*[*simp*]: *length* (*sort-key f xs*) = *length xs*
⟨*proof*⟩

lemma *set-sort*[*simp*]: *set*(*sort-key f xs*) = *set xs*
⟨*proof*⟩

lemma *distinct-insort*: *distinct* (*insort-key f x xs*) = (*x* ∉ *set xs* ∧ *distinct xs*)
⟨*proof*⟩

lemma *distinct-insort-key*:
distinct (*map f* (*insort-key f x xs*)) = (*f x* ∉ *f* ' *set xs* ∧ (*distinct* (*map f xs*)))
⟨*proof*⟩

lemma *distinct-sort*[*simp*]: *distinct* (*sort-key f xs*) = *distinct xs*
⟨*proof*⟩

lemma *sorted-insort-key*: *sorted* (*map f* (*insort-key f x xs*)) = *sorted* (*map f xs*)
⟨*proof*⟩

lemma *sorted-insort*: *sorted* (*insort x xs*) = *sorted xs*
⟨*proof*⟩

theorem *sorted-sort-key* [*simp*]: *sorted* (*map f* (*sort-key f xs*))
⟨*proof*⟩

theorem *sorted-sort* [*simp*]: *sorted* (*sort xs*)
⟨*proof*⟩

lemma *insort-not-Nil* [*simp*]:
insort-key f a xs ≠ []
⟨*proof*⟩

lemma *insort-is-Cons*: ∀ *x*∈*set xs*. *f a* ≤ *f x* ⟹ *insort-key f a xs* = *a* # *xs*
⟨*proof*⟩

lemma *sort-key-id-if-sorted*: *sorted* (*map f xs*) $\Longrightarrow$ *sort-key f xs* = *xs*
⟨*proof*⟩

Subsumed by *sorted* (*map ?f ?xs*) $\Longrightarrow$ *sort-key ?f ?xs* = *?xs* but easier to find:

lemma *sorted-sort-id*: *sorted xs* $\Longrightarrow$ *sort xs* = *xs*
⟨*proof*⟩

lemma *insort-key-remove1*:
assumes *a* ∈ *set xs* **and** *sorted* (*map f xs*) **and** *hd* (*filter* (λ*x*. *f a* = *f x*) *xs*) = *a*
shows *insort-key f a* (*remove1 a xs*) = *xs*
⟨*proof*⟩

lemma *insort-remove1*:
assumes *a* ∈ *set xs* **and** *sorted xs*
shows *insort a* (*remove1 a xs*) = *xs*
⟨*proof*⟩

lemma *finite-sorted-distinct-unique*:
assumes *finite A* **shows** ∃!*xs*. *set xs* = *A* ∧ *sorted xs* ∧ *distinct xs*
⟨*proof*⟩

lemma *insort-insert-key-triv*:
f x ∈ *f* ‘ *set xs* $\Longrightarrow$ *insort-insert-key f x xs* = *xs*
⟨*proof*⟩

lemma *insort-insert-triv*:
x ∈ *set xs* $\Longrightarrow$ *insort-insert x xs* = *xs*
⟨*proof*⟩

lemma *insort-insert-insort-key*:
f x ∉ *f* ‘ *set xs* $\Longrightarrow$ *insort-insert-key f x xs* = *insort-key f x xs*
⟨*proof*⟩

lemma *insort-insert-insort*:
x ∉ *set xs* $\Longrightarrow$ *insort-insert x xs* = *insort x xs*
⟨*proof*⟩

lemma *set-insort-insert*:
set (*insort-insert x xs*) = *insert x* (*set xs*)
⟨*proof*⟩

lemma *distinct-insort-insert*:
assumes *distinct xs*
shows *distinct* (*insort-insert-key f x xs*)
⟨*proof*⟩

lemma *sorted-insort-insert-key*:
assumes *sorted* (*map f xs*)

shows *sorted* (*map f* (*insort-insert-key f x xs*))
⟨*proof*⟩

lemma *sorted-insort-insert*:
assumes *sorted xs*
shows *sorted* (*insort-insert x xs*)
⟨*proof*⟩

lemma *filter-insort-triv*:
¬ *P x* ⟹ *filter P* (*insort-key f x xs*) = *filter P xs*
⟨*proof*⟩

lemma *filter-insort*:
sorted (*map f xs*) ⟹ *P x* ⟹ *filter P* (*insort-key f x xs*) = *insort-key f x* (*filter P xs*)
⟨*proof*⟩

lemma *filter-sort*:
filter P (*sort-key f xs*) = *sort-key f* (*filter P xs*)
⟨*proof*⟩

lemma *remove1-insort-key* [*simp*]:
remove1 x (*insort-key f x xs*) = *xs*
⟨*proof*⟩

end

lemma *sort-upt* [*simp*]: *sort* [*m*..<*n*] = [*m*..<*n*]
⟨*proof*⟩

lemma *sort-upto* [*simp*]: *sort* [*i*..*j*] = [*i*..*j*]
⟨*proof*⟩

lemma *sorted-find-Min*:
sorted xs ⟹ ∃ *x* ∈ *set xs*. *P x* ⟹ *List.find P xs* = *Some* (*Min* {*x*∈*set xs*. *P x*})
⟨*proof*⟩

lemma *sorted-enumerate* [*simp*]: *sorted* (*map fst* (*enumerate n xs*))
⟨*proof*⟩

lemma *sorted-insort-is-snoc*: *sorted xs* ⟹ ∀ *x* ∈ *set xs*. *a* ≥ *x* ⟹ *insort a xs* = *xs* @ [*a*]
⟨*proof*⟩

Stability of *sort-key*:

lemma *sort-key-stable*: *filter* (λ*y*. *f y* = *k*) (*sort-key f xs*) = *filter* (λ*y*. *f y* = *k*) *xs*
⟨*proof*⟩

corollary *stable-sort-key-sort-key*: *stable-sort-key sort-key*

⟨*proof*⟩

lemma *sort-key-const*: *sort-key* (λ*x*. *c*) *xs* = *xs*
⟨*proof*⟩

67.3.4 *transpose* on sorted lists

lemma *sorted-transpose*[*simp*]: *sorted* (*rev* (*map length* (*transpose xs*)))
⟨*proof*⟩

lemma *transpose-max-length*:
foldr (λ*xs*. *max* (*length xs*)) (*transpose xs*) *0* = *length* (*filter* (λ*x*. *x* ≠ []) *xs*)
(**is** *?L* = *?R*)
⟨*proof*⟩

lemma *length-transpose-sorted*:
fixes *xs* :: *'a list list*
assumes *sorted*: *sorted* (*rev* (*map length xs*))
shows *length* (*transpose xs*) = (*if xs* = [] *then 0 else length* (*xs* ! *0*))
⟨*proof*⟩

lemma *nth-nth-transpose-sorted*[*simp*]:
fixes *xs* :: *'a list list*
assumes *sorted*: *sorted* (*rev* (*map length xs*))
and *i*: *i* < *length* (*transpose xs*)
and *j*: *j* < *length* (*filter* (λ*ys*. *i* < *length ys*) *xs*)
shows *transpose xs* ! *i* ! *j* = *xs* ! *j* ! *i*
⟨*proof*⟩

lemma *transpose-column-length*:
fixes *xs* :: *'a list list*
assumes *sorted*: *sorted* (*rev* (*map length xs*)) **and** *i* < *length xs*
shows *length* (*filter* (λ*ys*. *i* < *length ys*) (*transpose xs*)) = *length* (*xs* ! *i*)
⟨*proof*⟩

lemma *transpose-column*:
fixes *xs* :: *'a list list*
assumes *sorted*: *sorted* (*rev* (*map length xs*)) **and** *i* < *length xs*
shows *map* (λ*ys*. *ys* ! *i*) (*filter* (λ*ys*. *i* < *length ys*) (*transpose xs*))
= *xs* ! *i* (**is** *?R* = -)
⟨*proof*⟩

lemma *transpose-transpose*:
fixes *xs* :: *'a list list*
assumes *sorted*: *sorted* (*rev* (*map length xs*))
shows *transpose* (*transpose xs*) = *takeWhile* (λ*x*. *x* ≠ []) *xs* (**is** *?L* = *?R*)
⟨*proof*⟩

theorem *transpose-rectangle*:

assumes $xs = [] \Longrightarrow n = 0$
assumes *rect*: $\bigwedge i.\ i < length\ xs \Longrightarrow length\ (xs\ !\ i) = n$
shows *transpose xs* = *map* ($\lambda\ i.\ map\ (\lambda\ j.\ xs\ !\ j\ !\ i)\ [0..<length\ xs]$) $[0..<n]$
(**is** *?trans* = *?map*)
⟨*proof*⟩

67.3.5 *sorted-key-list-of-set*

This function maps (finite) linearly ordered sets to sorted lists. The linear order is obtained by a key function that maps the elements of the set to a type that is linearly ordered. Warning: in most cases it is not a good idea to convert from sets to lists but one should convert in the other direction (via *set*).

Note: this is a generalisation of the older *sorted-list-of-set* that is obtained by setting the key function to the identity. Consequently, new theorems should be added to the locale below. They should also be aliased to more convenient names for use with *sorted-list-of-set* as seen further below.

definition (**in** *linorder*) *sorted-key-list-of-set* :: $('b \Rightarrow 'a) \Rightarrow 'b\ set \Rightarrow 'b\ list$
where *sorted-key-list-of-set f* ≡ *folding-on.F* (*insort-key f*) []

locale *folding-insort-key* = *lo?*: *linorder less-eq* :: $'a \Rightarrow 'a \Rightarrow bool\ less$
for *less-eq* (**infix** $\preceq$ *50*) **and** *less* (**infix** $\prec$ *50*) +
fixes *S*
fixes *f* :: $'b \Rightarrow 'a$
assumes *inj-on*: *inj-on f S*
begin

lemma *insort-key-commute*:
$x \in S \Longrightarrow y \in S \Longrightarrow$ *insort-key f y o insort-key f x* = *insort-key f x o insort-key f y*
⟨*proof*⟩

sublocale *fold-insort-key*: *folding-on S insort-key f* []
rewrites *folding-on.F* (*insort-key f*) [] = *sorted-key-list-of-set f*
⟨*proof*⟩

lemma *idem-if-sorted-distinct*:
assumes *set xs* ⊆ *S* **and** *sorted* (*map f xs*) *distinct xs*
shows *sorted-key-list-of-set f* (*set xs*) = *xs*
⟨*proof*⟩

lemma *sorted-key-list-of-set-empty*:
sorted-key-list-of-set f {} = []
⟨*proof*⟩

lemma *sorted-key-list-of-set-insert*:
assumes *insert x A* ⊆ *S* **and** *finite A x* ∉ *A*
shows *sorted-key-list-of-set f* (*insert x A*)

$= \mathit{insort\text{-}key}\ f\ x\ (\mathit{sorted\text{-}key\text{-}list\text{-}of\text{-}set}\ f\ A)$
⟨*proof*⟩

lemma *sorted-key-list-of-set-insert-remove* [*simp*]:
assumes $\mathit{insert}\ x\ A \subseteq S$ **and** *finite A*
shows $\mathit{sorted\text{-}key\text{-}list\text{-}of\text{-}set}\ f\ (\mathit{insert}\ x\ A)$
$= \mathit{insort\text{-}key}\ f\ x\ (\mathit{sorted\text{-}key\text{-}list\text{-}of\text{-}set}\ f\ (A - \{x\}))$
⟨*proof*⟩

lemma *sorted-key-list-of-set-eq-Nil-iff* [*simp*]:
assumes $A \subseteq S$ **and** *finite A*
shows $\mathit{sorted\text{-}key\text{-}list\text{-}of\text{-}set}\ f\ A = [] \longleftrightarrow A = \{\}$
⟨*proof*⟩

lemma *set-sorted-key-list-of-set* [*simp*]:
assumes $A \subseteq S$ **and** *finite A*
shows $\mathit{set}\ (\mathit{sorted\text{-}key\text{-}list\text{-}of\text{-}set}\ f\ A) = A$
⟨*proof*⟩

lemma *sorted-sorted-key-list-of-set* [*simp*]:
assumes $A \subseteq S$
shows $\mathit{sorted}\ (\mathit{map}\ f\ (\mathit{sorted\text{-}key\text{-}list\text{-}of\text{-}set}\ f\ A))$
⟨*proof*⟩

lemma *distinct-if-distinct-map*: $\mathit{distinct}\ (\mathit{map}\ f\ xs) \Longrightarrow \mathit{distinct}\ xs$
⟨*proof*⟩

lemma *distinct-sorted-key-list-of-set* [*simp*]:
assumes $A \subseteq S$
shows $\mathit{distinct}\ (\mathit{map}\ f\ (\mathit{sorted\text{-}key\text{-}list\text{-}of\text{-}set}\ f\ A))$
⟨*proof*⟩

lemma *length-sorted-key-list-of-set* [*simp*]:
assumes $A \subseteq S$
shows $\mathit{length}\ (\mathit{sorted\text{-}key\text{-}list\text{-}of\text{-}set}\ f\ A) = \mathit{card}\ A$
⟨*proof*⟩

lemmas *sorted-key-list-of-set* =
set-sorted-key-list-of-set sorted-sorted-key-list-of-set distinct-sorted-key-list-of-set

lemma *sorted-key-list-of-set-remove*:
assumes $\mathit{insert}\ x\ A \subseteq S$ **and** *finite A*
shows $\mathit{sorted\text{-}key\text{-}list\text{-}of\text{-}set}\ f\ (A - \{x\}) = \mathit{remove1}\ x\ (\mathit{sorted\text{-}key\text{-}list\text{-}of\text{-}set}\ f\ A)$
⟨*proof*⟩

lemma *strict-sorted-key-list-of-set* [*simp*]:
$A \subseteq S \Longrightarrow \mathit{sorted\text{-}wrt}\ (\prec)\ (\mathit{map}\ f\ (\mathit{sorted\text{-}key\text{-}list\text{-}of\text{-}set}\ f\ A))$
⟨*proof*⟩

lemma *finite-set-strict-sorted*:
assumes $A \subseteq S$ **and** *finite A*
obtains *l* **where** *sorted-wrt* $(\prec)$ *(map f l) set l = A length l = card A*
⟨*proof*⟩

lemma (**in** *linorder*) *strict-sorted-equal*:
assumes *sorted-wrt* $(<)$ *xs*
and *sorted-wrt* $(<)$ *ys*
and *set ys = set xs*
shows *ys = xs*
⟨*proof*⟩

lemma (**in** *linorder*) *strict-sorted-equal-Uniq*: $\exists_{\leq 1} xs.$ *sorted-wrt* $(<)$ *xs* $\wedge$ *set xs = A*
⟨*proof*⟩

lemma *sorted-key-list-of-set-inject*:
assumes $A \subseteq S$ $B \subseteq S$
assumes *sorted-key-list-of-set f A = sorted-key-list-of-set f B finite A finite B*
shows *A = B*
⟨*proof*⟩

lemma *sorted-key-list-of-set-unique*:
assumes $A \subseteq S$ **and** *finite A*
shows *sorted-wrt* $(\prec)$ *(map f l)* $\wedge$ *set l = A* $\wedge$ *length l = card A*
$\longleftrightarrow$ *sorted-key-list-of-set f A = l*
⟨*proof*⟩

end

context *linorder*
begin

definition *sorted-list-of-set* $\equiv$ *sorted-key-list-of-set* $(\lambda x::'a.\ x)$

We abuse the *rewrites* functionality of locales to remove trivial assumptions that result from instantiating the key function to the identity.

sublocale *sorted-list-of-set*: *folding-insort-key* $(\leq)$ $(<)$ *UNIV* $(\lambda x.\ x)$
rewrites *sorted-key-list-of-set* $(\lambda x.\ x)$ = *sorted-list-of-set*
and $\bigwedge xs.$ *map* $(\lambda x.\ x)$ *xs* $\equiv$ *xs*
and $\bigwedge X.$ $(X \subseteq UNIV) \equiv True$
and $\bigwedge x.$ $x \in UNIV \equiv True$
and $\bigwedge P.$ $(True \Longrightarrow P) \equiv Trueprop\ P$
and $\bigwedge P\ Q.$ $(True \Longrightarrow PROP\ P \Longrightarrow PROP\ Q) \equiv (PROP\ P \Longrightarrow True \Longrightarrow PROP\ Q)$
⟨*proof*⟩

Alias theorems for backwards compatibility and ease of use.

lemmas *sorted-list-of-set = sorted-list-of-set.sorted-key-list-of-set* **and**

sorted-list-of-set-empty = *sorted-list-of-set.sorted-key-list-of-set-empty* **and**
sorted-list-of-set-insert = *sorted-list-of-set.sorted-key-list-of-set-insert* **and**
sorted-list-of-set-insert-remove = *sorted-list-of-set.sorted-key-list-of-set-insert-remove*
and
sorted-list-of-set-eq-Nil-iff = *sorted-list-of-set.sorted-key-list-of-set-eq-Nil-iff*
and
set-sorted-list-of-set = *sorted-list-of-set.set-sorted-key-list-of-set* **and**
sorted-sorted-list-of-set = *sorted-list-of-set.sorted-sorted-key-list-of-set* **and**
distinct-sorted-list-of-set = *sorted-list-of-set.distinct-sorted-key-list-of-set* **and**
length-sorted-list-of-set = *sorted-list-of-set.length-sorted-key-list-of-set* **and**
sorted-list-of-set-remove = *sorted-list-of-set.sorted-key-list-of-set-remove* **and**
strict-sorted-list-of-set = *sorted-list-of-set.strict-sorted-key-list-of-set* **and**
sorted-list-of-set-inject = *sorted-list-of-set.sorted-key-list-of-set-inject* **and**
sorted-list-of-set-unique = *sorted-list-of-set.sorted-key-list-of-set-unique* **and**
finite-set-strict-sorted = *sorted-list-of-set.finite-set-strict-sorted*

lemma *sorted-list-of-set-sort-remdups* [*code*]:
sorted-list-of-set (*set xs*) = *sort* (*remdups xs*)
⟨*proof*⟩

end

lemma *sorted-list-of-set-range* [*simp*]:
sorted-list-of-set {*m*..<*n*} = [*m*..<*n*]
⟨*proof*⟩

lemma *sorted-list-of-set-lessThan-Suc* [*simp*]:
sorted-list-of-set {..<*Suc k*} = *sorted-list-of-set* {..<*k*} @ [*k*]
⟨*proof*⟩

lemma *sorted-list-of-set-atMost-Suc* [*simp*]:
sorted-list-of-set {..*Suc k*} = *sorted-list-of-set* {..*k*} @ [*Suc k*]
⟨*proof*⟩

lemma *sorted-list-of-set-nonempty*:
assumes *finite A A* ≠ {}
shows *sorted-list-of-set A* = *Min A* # *sorted-list-of-set* (*A* − {*Min A*})
⟨*proof*⟩

lemma *sorted-list-of-set-greaterThanLessThan*:
assumes *Suc i* < *j*
shows *sorted-list-of-set* {*i*<..<*j*} = *Suc i* # *sorted-list-of-set* {*Suc i*<..<*j*}
⟨*proof*⟩

lemma *sorted-list-of-set-greaterThanAtMost*:
assumes *Suc i* ≤ *j*
shows *sorted-list-of-set* {*i*<..*j*} = *Suc i* # *sorted-list-of-set* {*Suc i*<..*j*}
⟨*proof*⟩

lemma *nth-sorted-list-of-set-greaterThanLessThan*:
$n < j - Suc\ i \Longrightarrow$ *sorted-list-of-set* $\{i<..<j\}\ !\ n = Suc\ (i+n)$
⟨*proof*⟩

lemma *nth-sorted-list-of-set-greaterThanAtMost*:
$n < j - i \Longrightarrow$ *sorted-list-of-set* $\{i<..j\}\ !\ n = Suc\ (i+n)$
⟨*proof*⟩

67.3.6 *lists*: the list-forming operator over sets

inductive-set
lists :: *'a set => 'a list set*
for *A* :: *'a set*
where
Nil [*intro*!, *simp*]: [] ∈ *lists A*
| *Cons* [*intro*!, *simp*]: ⟦*a* ∈ *A*; *l* ∈ *lists A*⟧ ⟹ *a#l* ∈ *lists A*

inductive-cases *listsE* [*elim*!]: *x#l* ∈ *lists A*
inductive-cases *listspE* [*elim*!]: *listsp A* (*x # l*)

inductive-simps *listsp-simps*[*code*]:
listsp A []
listsp A (*x # xs*)

lemma *listsp-mono* [*mono*]: *A* ≤ *B* ⟹ *listsp A* ≤ *listsp B*
⟨*proof*⟩

lemmas *lists-mono* = *listsp-mono* [*to-set*]

lemma *listsp-infI*:
assumes *l*: *listsp A l* **shows** *listsp B l* ⟹ *listsp* (*inf A B*) *l* ⟨*proof*⟩

lemmas *lists-IntI* = *listsp-infI* [*to-set*]

lemma *listsp-inf-eq* [*simp*]: *listsp* (*inf A B*) = *inf* (*listsp A*) (*listsp B*)
⟨*proof*⟩

lemmas *listsp-conj-eq* [*simp*] = *listsp-inf-eq* [*simplified inf-fun-def inf-bool-def*]

lemmas *lists-Int-eq* [*simp*] = *listsp-inf-eq* [*to-set*]

lemma *Cons-in-lists-iff*[*simp*]: *x#xs* ∈ *lists A* ⟷ *x* ∈ *A* ∧ *xs* ∈ *lists A*
⟨*proof*⟩

lemma *append-in-listsp-conv* [*iff*]:
(*listsp A* (*xs* @ *ys*)) = (*listsp A xs* ∧ *listsp A ys*)
⟨*proof*⟩

lemmas *append-in-lists-conv* [*iff*] = *append-in-listsp-conv* [*to-set*]

lemma *in-listsp-conv-set*: (*listsp A xs*) = (∀ *x* ∈ *set xs*. *A x*)
— eliminate *listsp* in favour of *set*
⟨*proof*⟩

lemmas *in-lists-conv-set* [*code-unfold*] = *in-listsp-conv-set* [*to-set*]

lemma *in-listspD* [*dest*!]: *listsp A xs* ⟹ ∀ *x*∈*set xs*. *A x*
⟨*proof*⟩

lemmas *in-listsD* [*dest*!] = *in-listspD* [*to-set*]

lemma *in-listspI* [*intro*!]: ∀ *x*∈*set xs*. *A x* ⟹ *listsp A xs*
⟨*proof*⟩

lemmas *in-listsI* [*intro*!] = *in-listspI* [*to-set*]

lemma *lists-eq-set*: *lists A* = {*xs*. *set xs* ≤ *A*}
⟨*proof*⟩

lemma *lists-empty* [*simp*]: *lists* {} = {[]}
⟨*proof*⟩

lemma *lists-UNIV* [*simp*]: *lists UNIV* = *UNIV*
⟨*proof*⟩

lemma *lists-image*: *lists* (*f*‘*A*) = *map f* ‘ *lists A*
⟨*proof*⟩

67.3.7 Inductive definition for membership

inductive *ListMem* :: ′*a* ⇒ ′*a list* ⇒ *bool*
where
elem: *ListMem x* (*x* # *xs*)
| *insert*: *ListMem x xs* ⟹ *ListMem x* (*y* # *xs*)

lemma *ListMem-iff*: (*ListMem x xs*) = (*x* ∈ *set xs*)
⟨*proof*⟩

67.3.8 Lists as Cartesian products

set-Cons A Xs: the set of lists with head drawn from *A* and tail drawn from *Xs*.

definition *set-Cons* :: ′*a set* ⇒ ′*a list set* ⇒ ′*a list set* **where**
set-Cons A XS = {*z*. ∃ *x xs*. *z* = *x* # *xs* ∧ *x* ∈ *A* ∧ *xs* ∈ *XS*}

lemma *set-Cons-sing-Nil* [*simp*]: *set-Cons A* {[]} = (%*x*. [*x*])‘*A*
⟨*proof*⟩

Yields the set of lists, all of the same length as the argument and with elements drawn from the corresponding element of the argument.

primrec *listset* :: *'a set list* ⇒ *'a list set* **where**
listset [] = {[]} |
listset (*A* # *As*) = *set-Cons A* (*listset As*)

67.4 Relations on Lists

67.4.1 Length Lexicographic Ordering

These orderings preserve well-foundedness: shorter lists precede longer lists. These ordering are not used in dictionaries.

primrec — The lexicographic ordering for lists of the specified length
lexn :: (*'a* × *'a*) *set* ⇒ *nat* ⇒ (*'a list* × *'a list*) *set* **where**
lexn r 0 = {} |
lexn r (*Suc n*) =
(*map-prod* (%(*x*, *xs*). *x*#*xs*) (%(*x*, *xs*). *x*#*xs*) ‘ (*r* <∗*lex*∗> *lexn r n*)) *Int*
{(*xs*, *ys*). *length xs* = *Suc n* ∧ *length ys* = *Suc n*}

definition *lex* :: (*'a* × *'a*) *set* ⇒ (*'a list* × *'a list*) *set* **where**
lex r = (⋃ *n*. *lexn r n*) — Holds only between lists of the same length

definition *lenlex* :: (*'a* × *'a*) *set* => (*'a list* × *'a list*) *set* **where**
lenlex r = *inv-image* (*less-than* <∗*lex*∗> *lex r*) (λ*xs*. (*length xs*, *xs*))
— Compares lists by their length and then lexicographically

lemma *wf-lexn*: **assumes** *wf r* **shows** *wf* (*lexn r n*)
⟨*proof*⟩

lemma *lexn-length*:
(*xs*, *ys*) ∈ *lexn r n* ⟹ *length xs* = *n* ∧ *length ys* = *n*
⟨*proof*⟩

lemma *wf-lex* [*intro*!]:
assumes *wf r* **shows** *wf* (*lex r*)
⟨*proof*⟩

lemma *lexn-conv*:
lexn r n =
{(*xs*,*ys*). *length xs* = *n* ∧ *length ys* = *n* ∧
(∃ *xys x y xs′ ys′*. *xs*= *xys* @ *x*#*xs′* ∧ *ys*= *xys* @ *y* # *ys′* ∧ (*x*, *y*) ∈ *r*)}
⟨*proof*⟩

By Mathias Fleury:

proposition *lexn-transI*:
assumes *trans r* **shows** *trans* (*lexn r n*)
⟨*proof*⟩

corollary *lex-transI*:
  assumes *trans r* **shows** *trans* (*lex r*)
⟨*proof*⟩

lemma *lex-conv*:
  lex r =
    {(*xs*,*ys*). *length xs* = *length ys* ∧
    (∃ *xys x y xs′ ys′*. *xs* = *xys* @ *x* # *xs′* ∧ *ys* = *xys* @ *y* # *ys′* ∧ (*x*, *y*) ∈ *r*)}
⟨*proof*⟩

lemma *wf-lenlex* [*intro*!]: *wf r* ⟹ *wf* (*lenlex r*)
⟨*proof*⟩

lemma *lenlex-conv*:
  lenlex r = {(*xs*,*ys*). *length xs* < *length ys* ∨
      length xs = *length ys* ∧ (*xs*, *ys*) ∈ *lex r*}
⟨*proof*⟩

lemma *total-lenlex*:
  assumes *total r*
  shows *total* (*lenlex r*)
⟨*proof*⟩

lemma *lenlex-transI* [*intro*]: *trans r* ⟹ *trans* (*lenlex r*)
⟨*proof*⟩

lemma *Nil-notin-lex* [*iff*]: ([], *ys*) ∉ *lex r*
⟨*proof*⟩

lemma *Nil2-notin-lex* [*iff*]: (*xs*, []) ∉ *lex r*
⟨*proof*⟩

lemma *Cons-in-lex* [*simp*]:
  (*x* # *xs*, *y* # *ys*) ∈ *lex r* ⟷ (*x*, *y*) ∈ *r* ∧ *length xs* = *length ys* ∨ *x* = *y* ∧ (*xs*, *ys*) ∈ *lex r*
 (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *Nil-lenlex-iff1* [*simp*]: ([], *ns*) ∈ *lenlex r* ⟷ *ns* ≠ []
  and *Nil-lenlex-iff2* [*simp*]: (*ns*,[]) ∉ *lenlex r*
  ⟨*proof*⟩

lemma *Cons-lenlex-iff*:
  ((*m* # *ms*, *n* # *ns*) ∈ *lenlex r*) ⟷
    length ms < *length ns*
  ∨ *length ms* = *length ns* ∧ (*m*,*n*) ∈ *r*
  ∨ (*m* = *n* ∧ (*ms*,*ns*) ∈ *lenlex r*)
  ⟨*proof*⟩

lemma *lenlex-irreflexive*: $(\bigwedge x.\ (x,x) \notin r) \Longrightarrow (xs,xs) \notin lenlex\ r$
⟨*proof*⟩

lemma *lenlex-trans*:
$[\![(x,y) \in lenlex\ r;\ (y,z) \in lenlex\ r;\ trans\ r]\!] \Longrightarrow (x,z) \in lenlex\ r$
⟨*proof*⟩

lemma *lenlex-length*: $(ms,\ ns) \in lenlex\ r \Longrightarrow length\ ms \leq length\ ns$
⟨*proof*⟩

lemma *lex-append-rightI*:
$(xs,\ ys) \in lex\ r \Longrightarrow length\ vs = length\ us \Longrightarrow (xs\ @\ us,\ ys\ @\ vs) \in lex\ r$
⟨*proof*⟩

lemma *lex-append-leftI*:
$(ys,\ zs) \in lex\ r \Longrightarrow (xs\ @\ ys,\ xs\ @\ zs) \in lex\ r$
⟨*proof*⟩

lemma *lex-append-leftD*:
$\forall x.\ (x,x) \notin r \Longrightarrow (xs\ @\ ys,\ xs\ @\ zs) \in lex\ r \Longrightarrow (ys,\ zs) \in lex\ r$
⟨*proof*⟩

lemma *lex-append-left-iff*:
$\forall x.\ (x,x) \notin r \Longrightarrow (xs\ @\ ys,\ xs\ @\ zs) \in lex\ r \longleftrightarrow (ys,\ zs) \in lex\ r$
⟨*proof*⟩

lemma *lex-take-index*:
assumes $(xs,\ ys) \in lex\ r$
obtains i **where** $i < length\ xs$ **and** $i < length\ ys$ **and** $take\ i\ xs = take\ i\ ys$
and $(xs\ !\ i,\ ys\ !\ i) \in r$
⟨*proof*⟩

lemma *irrefl-lex*: $irrefl\ r \Longrightarrow irrefl\ (lex\ r)$
⟨*proof*⟩

lemma *lexl-not-refl* [*simp*]: $irrefl\ r \Longrightarrow (x,x) \notin lex\ r$
⟨*proof*⟩

67.4.2 Lexicographic Ordering

Classical lexicographic ordering on lists, ie. "a" < "ab" < "b". This ordering does *not* preserve well-foundedness. Author: N. Voelker, March 2005.

definition *lexord* :: $('a \times 'a)\ set \Rightarrow ('a\ list \times 'a\ list)\ set$ **where**
$lexord\ r = \{(x,y).\ \exists\ a\ v.\ y = x\ @\ a\ \#\ v \vee$
$(\exists\ u\ a\ b\ v\ w.\ (a,b) \in r \wedge x = u\ @\ (a\ \#\ v) \wedge y = u\ @\ (b\ \#\ w))\}$

lemma *lexord-Nil-left*[*simp*]: $([],y) \in lexord\ r = (\exists\ a\ x.\ y = a\ \#\ x)$
⟨*proof*⟩

lemma *lexord-Nil-right*[*simp*]: $(x,[]) \notin lexord\ r$
⟨*proof*⟩

lemma *lexord-cons-cons*[*simp*]:
$(a \# x,\ b \# y) \in lexord\ r \longleftrightarrow (a,b) \in r \vee (a = b \wedge (x,y) \in lexord\ r)$ (**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemmas *lexord-simps = lexord-Nil-left lexord-Nil-right lexord-cons-cons*

lemma *lexord-same-pref-iff*:
$(xs\ @\ ys,\ xs\ @\ zs) \in lexord\ r \longleftrightarrow (\exists x \in set\ xs.\ (x,x) \in r) \vee (ys,\ zs) \in lexord\ r$
⟨*proof*⟩

lemma *lexord-same-pref-if-irrefl*[*simp*]:
$irrefl\ r \Longrightarrow (xs\ @\ ys,\ xs\ @\ zs) \in lexord\ r \longleftrightarrow (ys,\ zs) \in lexord\ r$
⟨*proof*⟩

lemma *lexord-append-rightI*: $\exists\ b\ z.\ y = b \# z \Longrightarrow (x,\ x\ @\ y) \in lexord\ r$
⟨*proof*⟩

lemma *lexord-append-left-rightI*:
$(a,b) \in r \Longrightarrow (u\ @\ a \# x,\ u\ @\ b \# y) \in lexord\ r$
⟨*proof*⟩

lemma *lexord-append-leftI*: $(u,v) \in lexord\ r \Longrightarrow (x\ @\ u,\ x\ @\ v) \in lexord\ r$
⟨*proof*⟩

lemma *lexord-append-leftD*:
$[\![(x\ @\ u,\ x\ @\ v) \in lexord\ r;\ (\forall a.\ (a,a) \notin r)\ ]\!] \Longrightarrow (u,v) \in lexord\ r$
⟨*proof*⟩

lemma *lexord-take-index-conv*:
$((x,y) \in lexord\ r) =$
$((length\ x < length\ y \wedge take\ (length\ x)\ y = x) \vee$
$(\exists i.\ i < min(length\ x)(length\ y) \wedge take\ i\ x = take\ i\ y \wedge (x!i,y!i) \in r))$
⟨*proof*⟩
lemma *lexord-lex*: $(x,y) \in lex\ r = ((x,y) \in lexord\ r \wedge length\ x = length\ y)$
⟨*proof*⟩

lemma *lexord-sufI*:
assumes $(u,w) \in lexord\ r\ length\ w \leq length\ u$
shows $(u@v,w@z) \in lexord\ r$
⟨*proof*⟩

lemma *lexord-sufE*:
assumes $(xs@zs,ys@qs) \in lexord\ r\ xs \neq ys\ length\ xs = length\ ys\ length\ zs = length\ qs$
shows $(xs,ys) \in lexord\ r$

⟨*proof*⟩

lemma *lexord-irreflexive*: $\forall x.\ (x,x) \notin r \Longrightarrow (xs,xs) \notin lexord\ r$
⟨*proof*⟩

By René Thiemann:

lemma *lexord-partial-trans*:
$(\bigwedge x\ y\ z.\ x \in set\ xs \Longrightarrow (x,y) \in r \Longrightarrow (y,z) \in r \Longrightarrow (x,z) \in r)$
$\Longrightarrow (xs,ys) \in lexord\ r \Longrightarrow (ys,zs) \in lexord\ r \Longrightarrow (xs,zs) \in lexord\ r$
⟨*proof*⟩

lemma *lexord-trans*:
$[\![(x,\ y) \in lexord\ r;\ (y,\ z) \in lexord\ r;\ trans\ r\ ]\!] \Longrightarrow (x,\ z) \in lexord\ r$
⟨*proof*⟩

lemma *lexord-transI*: $trans\ r \Longrightarrow trans\ (lexord\ r)$
⟨*proof*⟩

lemma *total-lexord*: $total\ r \Longrightarrow total\ (lexord\ r)$
⟨*proof*⟩

corollary *lexord-linear*: $(\forall a\ b.\ (a,b) \in r \vee a = b \vee (b,a) \in r) \Longrightarrow (x,y) \in lexord\ r \vee x = y \vee (y,x) \in lexord\ r$
⟨*proof*⟩

lemma *lexord-irrefl*:
$irrefl\ R \Longrightarrow irrefl\ (lexord\ R)$
⟨*proof*⟩

lemma *lexord-asym*:
assumes *asym R*
shows *asym* (*lexord R*)
⟨*proof*⟩

lemma *lexord-asymmetric*:
assumes *asym R*
assumes *hyp*: $(a,\ b) \in lexord\ R$
shows $(b,\ a) \notin lexord\ R$
⟨*proof*⟩

lemma *asym-lex*: $asym\ R \Longrightarrow asym\ (lex\ R)$
⟨*proof*⟩

lemma *asym-lenlex*: $asym\ R \Longrightarrow asym\ (lenlex\ R)$
⟨*proof*⟩

lemma *lenlex-append1*:
assumes *len*: $(us,xs) \in lenlex\ R$ **and** *eq*: $length\ vs = length\ ys$
shows $(us\ @\ vs,\ xs\ @\ ys) \in lenlex\ R$

⟨*proof*⟩

lemma *lenlex-append2* [*simp*]:
assumes *irrefl R*
shows (*us* @ *xs*, *us* @ *ys*) ∈ *lenlex R* ⟷ (*xs*, *ys*) ∈ *lenlex R*
⟨*proof*⟩

Predicate version of lexicographic order integrated with Isabelle's order type classes. Author: Andreas Lochbihler

context *ord*
begin

context
notes [[*inductive-internals*]]
begin

inductive *lexordp* :: 'a *list* ⇒ 'a *list* ⇒ *bool*
where
Nil: *lexordp* [] (*y* # *ys*)
| *Cons*: *x* < *y* ⟹ *lexordp* (*x* # *xs*) (*y* # *ys*)
| *Cons-eq*:
⟦ ¬ *x* < *y*; ¬ *y* < *x*; *lexordp xs ys* ⟧ ⟹ *lexordp* (*x* # *xs*) (*y* # *ys*)

end

lemma *lexordp-simps* [*simp*, *code*]:
lexordp [] *ys* = (*ys* ≠ [])
lexordp xs [] = *False*
lexordp (*x* # *xs*) (*y* # *ys*) ⟷ *x* < *y* ∨ ¬ *y* < *x* ∧ *lexordp xs ys*
⟨*proof*⟩

inductive *lexordp-eq* :: 'a *list* ⇒ 'a *list* ⇒ *bool* **where**
Nil: *lexordp-eq* [] *ys*
| *Cons*: *x* < *y* ⟹ *lexordp-eq* (*x* # *xs*) (*y* # *ys*)
| *Cons-eq*: ⟦ ¬ *x* < *y*; ¬ *y* < *x*; *lexordp-eq xs ys* ⟧ ⟹ *lexordp-eq* (*x* # *xs*) (*y* # *ys*)

lemma *lexordp-eq-simps* [*simp*, *code*]:
lexordp-eq [] *ys* = *True*
lexordp-eq xs [] ⟷ *xs* = []
lexordp-eq (*x* # *xs*) [] = *False*
lexordp-eq (*x* # *xs*) (*y* # *ys*) ⟷ *x* < *y* ∨ ¬ *y* < *x* ∧ *lexordp-eq xs ys*
⟨*proof*⟩

lemma *lexordp-append-rightI*: *ys* ≠ *Nil* ⟹ *lexordp xs* (*xs* @ *ys*)
⟨*proof*⟩

lemma *lexordp-append-left-rightI*: *x* < *y* ⟹ *lexordp* (*us* @ *x* # *xs*) (*us* @ *y* # *ys*)
⟨*proof*⟩

lemma *lexordp-eq-refl*: *lexordp-eq xs xs*
⟨*proof*⟩

lemma *lexordp-append-leftI*: *lexordp us vs* ⟹ *lexordp* (*xs* @ *us*) (*xs* @ *vs*)
⟨*proof*⟩

lemma *lexordp-append-leftD*: ⟦ *lexordp* (*xs* @ *us*) (*xs* @ *vs*); ∀ *a*. ¬ *a* < *a* ⟧ ⟹ *lexordp us vs*
⟨*proof*⟩

lemma *lexordp-irreflexive*:
assumes *irrefl*: ∀ *x*. ¬ *x* < *x*
shows ¬ *lexordp xs xs*
⟨*proof*⟩

lemma *lexordp-into-lexordp-eq*:
lexordp xs ys ⟹ *lexordp-eq xs ys*
⟨*proof*⟩

lemma *lexordp-eq-pref*: *lexordp-eq u* (*u* @ *v*)
⟨*proof*⟩

end

declare *ord.lexordp-simps* [*simp*, *code*]
declare *ord.lexordp-eq-simps* [*simp*, *code*]

context *order*
begin

lemma *lexordp-antisym*:
assumes *lexordp xs ys lexordp ys xs*
shows *False*
⟨*proof*⟩

lemma *lexordp-irreflexive'*: ¬ *lexordp xs xs*
⟨*proof*⟩

end

context *linorder* **begin**

lemma *lexordp-cases* [*consumes 1*, *case-names Nil Cons Cons-eq*, *cases pred*: *lexordp*]:
assumes *lexordp xs ys*
obtains (*Nil*) *y ys'* **where** *xs* = [] *ys* = *y* # *ys'*
| (*Cons*) *x xs' y ys'* **where** *xs* = *x* # *xs' ys* = *y* # *ys' x* < *y*
| (*Cons-eq*) *x xs' ys'* **where** *xs* = *x* # *xs' ys* = *x* # *ys' lexordp xs' ys'*
⟨*proof*⟩

lemma *lexordp-induct* [*consumes 1*, *case-names Nil Cons Cons-eq*, *induct pred*: *lexordp*]:
assumes *major*: *lexordp xs ys*
and *Nil*: $\bigwedge y\ ys.\ P\ []\ (y\ \#\ ys)$
and *Cons*: $\bigwedge x\ xs\ y\ ys.\ x < y \Longrightarrow P\ (x\ \#\ xs)\ (y\ \#\ ys)$
and *Cons-eq*: $\bigwedge x\ xs\ ys.\ [\![\ lexordp\ xs\ ys;\ P\ xs\ ys\ ]\!] \Longrightarrow P\ (x\ \#\ xs)\ (x\ \#\ ys)$
shows *P xs ys*
⟨*proof*⟩

lemma *lexordp-iff*:
$lexordp\ xs\ ys \longleftrightarrow (\exists x\ vs.\ ys = xs\ @\ x\ \#\ vs) \vee (\exists us\ a\ b\ vs\ ws.\ a < b \wedge xs = us\ @\ a\ \#\ vs \wedge ys = us\ @\ b\ \#\ ws)$
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *lexordp-conv-lexord*:
$lexordp\ xs\ ys \longleftrightarrow (xs,\ ys) \in lexord\ \{(x,\ y).\ x < y\}$
⟨*proof*⟩

lemma *lexordp-eq-antisym*:
assumes *lexordp-eq xs ys lexordp-eq ys xs*
shows *xs = ys*
⟨*proof*⟩

lemma *lexordp-eq-trans*:
assumes *lexordp-eq xs ys* **and** *lexordp-eq ys zs*
shows *lexordp-eq xs zs*
⟨*proof*⟩

lemma *lexordp-trans*:
assumes *lexordp xs ys lexordp ys zs*
shows *lexordp xs zs*
⟨*proof*⟩

lemma *lexordp-linear*: $lexordp\ xs\ ys \vee xs = ys \vee lexordp\ ys\ xs$
⟨*proof*⟩

lemma *lexordp-conv-lexordp-eq*: $lexordp\ xs\ ys \longleftrightarrow lexordp\text{-}eq\ xs\ ys \wedge \neg\ lexordp\text{-}eq\ ys\ xs$
(**is** *?lhs ⟷ ?rhs*)
⟨*proof*⟩

lemma *lexordp-eq-conv-lexord*: $lexordp\text{-}eq\ xs\ ys \longleftrightarrow xs = ys \vee lexordp\ xs\ ys$
⟨*proof*⟩

lemma *lexordp-eq-linear*: $lexordp\text{-}eq\ xs\ ys \vee lexordp\text{-}eq\ ys\ xs$
⟨*proof*⟩

lemma *lexordp-linorder*: *class.linorder lexordp-eq lexordp*
⟨*proof*⟩

end

67.4.3 Lexicographic combination of measure functions

These are useful for termination proofs

definition *measures fs* = *inv-image* (*lex less-than*) (%*a*. *map* (%*f*. *f a*) *fs*)

lemma *wf-measures*[*simp*]: *wf* (*measures fs*)
⟨*proof*⟩

lemma *in-measures*[*simp*]:
(*x*, *y*) ∈ *measures* [] = *False*
(*x*, *y*) ∈ *measures* (*f* # *fs*)
= (*f x* < *f y* ∨ (*f x* = *f y* ∧ (*x*, *y*) ∈ *measures fs*))
⟨*proof*⟩

lemma *measures-less*: *f x* < *f y* ⟹ (*x*, *y*) ∈ *measures* (*f*#*fs*)
⟨*proof*⟩

lemma *measures-lesseq*: *f x* ≤ *f y* ⟹ (*x*, *y*) ∈ *measures fs* ⟹ (*x*, *y*) ∈ *measures* (*f*#*fs*)
⟨*proof*⟩

67.4.4 Lifting Relations to Lists: one element

definition *listrel1* :: ('*a* × '*a*) *set* ⇒ ('*a list* × '*a list*) *set* **where**
listrel1 r = {(*xs*,*ys*).
∃ *us z z' vs*. *xs* = *us* @ *z* # *vs* ∧ (*z*,*z'*) ∈ *r* ∧ *ys* = *us* @ *z'* # *vs*}

lemma *listrel1I*:
⟦ (*x*, *y*) ∈ *r*; *xs* = *us* @ *x* # *vs*; *ys* = *us* @ *y* # *vs* ⟧ ⟹
(*xs*, *ys*) ∈ *listrel1 r*
⟨*proof*⟩

lemma *listrel1E*:
⟦ (*xs*, *ys*) ∈ *listrel1 r*;
!!*x y us vs*. ⟦ (*x*, *y*) ∈ *r*; *xs* = *us* @ *x* # *vs*; *ys* = *us* @ *y* # *vs* ⟧ ⟹ *P*
⟧ ⟹ *P*
⟨*proof*⟩

lemma *not-Nil-listrel1* [*iff*]: ([], *xs*) ∉ *listrel1 r*
⟨*proof*⟩

lemma *not-listrel1-Nil* [*iff*]: (*xs*, []) ∉ *listrel1 r*
⟨*proof*⟩

lemma *Cons-listrel1-Cons* [*iff*]:
$(x \# xs,\ y \# ys) \in listrel1\ r \longleftrightarrow$
$(x,y) \in r \wedge xs = ys \vee x = y \wedge (xs,\ ys) \in listrel1\ r$
⟨*proof*⟩

lemma *listrel1I1*: $(x,y) \in r \Longrightarrow (x \# xs,\ y \# xs) \in listrel1\ r$
⟨*proof*⟩

lemma *listrel1I2*: $(xs,\ ys) \in listrel1\ r \Longrightarrow (x \# xs,\ x \# ys) \in listrel1\ r$
⟨*proof*⟩

lemma *append-listrel1I*:
$(xs,\ ys) \in listrel1\ r \wedge us = vs \vee xs = ys \wedge (us,\ vs) \in listrel1\ r$
$\Longrightarrow (xs\ @\ us,\ ys\ @\ vs) \in listrel1\ r$
⟨*proof*⟩

lemma *Cons-listrel1E1*[*elim*!]:
assumes $(x \# xs,\ ys) \in listrel1\ r$
and $\bigwedge y.\ ys = y \# xs \Longrightarrow (x,\ y) \in r \Longrightarrow R$
and $\bigwedge zs.\ ys = x \# zs \Longrightarrow (xs,\ zs) \in listrel1\ r \Longrightarrow R$
shows R
⟨*proof*⟩

lemma *Cons-listrel1E2*[*elim*!]:
assumes $(xs,\ y \# ys) \in listrel1\ r$
and $\bigwedge x.\ xs = x \# ys \Longrightarrow (x,\ y) \in r \Longrightarrow R$
and $\bigwedge zs.\ xs = y \# zs \Longrightarrow (zs,\ ys) \in listrel1\ r \Longrightarrow R$
shows R
⟨*proof*⟩

lemma *snoc-listrel1-snoc-iff*:
$(xs\ @\ [x],\ ys\ @\ [y]) \in listrel1\ r$
$\longleftrightarrow (xs,\ ys) \in listrel1\ r \wedge x = y \vee xs = ys \wedge (x,y) \in r$ (**is** *?L* $\longleftrightarrow$ *?R*)
⟨*proof*⟩

lemma *listrel1-eq-len*: $(xs,ys) \in listrel1\ r \Longrightarrow length\ xs = length\ ys$
⟨*proof*⟩

lemma *listrel1-mono*:
$r \subseteq s \Longrightarrow listrel1\ r \subseteq listrel1\ s$
⟨*proof*⟩

lemma *listrel1-converse*: $listrel1\ (r^{-1}) = (listrel1\ r)^{-1}$
⟨*proof*⟩

lemma *in-listrel1-converse*:
$(x,y) \in listrel1\ (r^{-1}) \longleftrightarrow (x,y) \in (listrel1\ r)^{-1}$
⟨*proof*⟩

lemma *listrel1-iff-update*:
(*xs*,*ys*) ∈ (*listrel1 r*)
⟷ (∃ *y n*. (*xs* ! *n*, *y*) ∈ *r* ∧ *n* < *length xs* ∧ *ys* = *xs*[*n*:=*y*]) (**is** *?L* ⟷ *?R*)
⟨*proof*⟩

Accessible part and wellfoundedness:

lemma *Cons-acc-listrel1I* [*intro*!]:
x ∈ *Wellfounded.acc r* ⟹ *xs* ∈ *Wellfounded.acc* (*listrel1 r*) ⟹ (*x* # *xs*) ∈ *Wellfounded.acc* (*listrel1 r*)
⟨*proof*⟩

lemma *lists-accD*: *xs* ∈ *lists* (*Wellfounded.acc r*) ⟹ *xs* ∈ *Wellfounded.acc* (*listrel1 r*)
⟨*proof*⟩

lemma *lists-accI*: *xs* ∈ *Wellfounded.acc* (*listrel1 r*) ⟹ *xs* ∈ *lists* (*Wellfounded.acc r*)
⟨*proof*⟩

lemma *wf-listrel1-iff*[*simp*]: *wf*(*listrel1 r*) = *wf r*
⟨*proof*⟩

67.4.5 Lifting Relations to Lists: all elements

inductive-set
listrel :: ('*a* × '*b*) *set* ⇒ ('*a list* × '*b list*) *set*
for *r* :: ('*a* × '*b*) *set*
where
Nil: ([],[]) ∈ *listrel r*
| *Cons*: ⟦(*x*,*y*) ∈ *r*; (*xs*,*ys*) ∈ *listrel r*⟧ ⟹ (*x*#*xs*, *y*#*ys*) ∈ *listrel r*

inductive-cases *listrel-Nil1* [*elim*!]: ([],*xs*) ∈ *listrel r*
inductive-cases *listrel-Nil2* [*elim*!]: (*xs*,[]) ∈ *listrel r*
inductive-cases *listrel-Cons1* [*elim*!]: (*y*#*ys*,*xs*) ∈ *listrel r*
inductive-cases *listrel-Cons2* [*elim*!]: (*xs*,*y*#*ys*) ∈ *listrel r*

lemma *listrel-eq-len*: (*xs*, *ys*) ∈ *listrel r* ⟹ *length xs* = *length ys*
⟨*proof*⟩

lemma *listrel-iff-zip* [*code-unfold*]: (*xs*,*ys*) ∈ *listrel r* ⟷
length xs = *length ys* ∧ (∀ (*x*,*y*) ∈ *set*(*zip xs ys*). (*x*,*y*) ∈ *r*) (**is** *?L* ⟷ *?R*)
⟨*proof*⟩

lemma *listrel-iff-nth*: (*xs*,*ys*) ∈ *listrel r* ⟷
length xs = *length ys* ∧ (∀ *n* < *length xs*. (*xs*!*n*, *ys*!*n*) ∈ *r*) (**is** *?L* ⟷ *?R*)
⟨*proof*⟩

lemma *listrel-mono*: $r \subseteq s \Longrightarrow$ *listrel* $r \subseteq$ *listrel* s
⟨*proof*⟩

lemma *listrel-subset*:
assumes $r \subseteq A \times A$ **shows** *listrel* $r \subseteq$ *lists* $A \times$ *lists* A
⟨*proof*⟩

lemma *listrel-refl-on*:
assumes *refl-on* A r **shows** *refl-on* (*lists* A) (*listrel* r)
⟨*proof*⟩

lemma *listrel-sym*: *sym* $r \Longrightarrow$ *sym* (*listrel* r)
⟨*proof*⟩

lemma *listrel-trans*:
assumes *trans* r **shows** *trans* (*listrel* r)
⟨*proof*⟩

theorem *equiv-listrel*: *equiv* A $r \Longrightarrow$ *equiv* (*lists* A) (*listrel* r)
⟨*proof*⟩

lemma *listrel-rtrancl-refl*[*iff*]: $(xs,xs) \in$ *listrel*(r^*)
⟨*proof*⟩

lemma *listrel-rtrancl-trans*:
$[\![(xs,ys) \in$ *listrel*(r^*); $(ys,zs) \in$ *listrel*$(r^*)]\!] \Longrightarrow (xs,zs) \in$ *listrel*(r^*)
⟨*proof*⟩

lemma *listrel-Nil* [*simp*]: *listrel* r `` $\{[]\} = \{[]\}$
⟨*proof*⟩

lemma *listrel-Cons*:
listrel r `` $\{x\#xs\} =$ *set-Cons* ($r$``$\{x\}$) (*listrel* r `` $\{xs\}$)
⟨*proof*⟩

Relating *listrel1*, *listrel* and closures:

lemma *listrel1-rtrancl-subset-rtrancl-listrel1*: *listrel1* $(r^*) \subseteq$ (*listrel1* r)*
⟨*proof*⟩

lemma *rtrancl-listrel1-eq-len*: $(x,y) \in$ (*listrel1* r)$^* \Longrightarrow$ *length* $x =$ *length* y
⟨*proof*⟩

lemma *rtrancl-listrel1-ConsI1*:
$(xs,ys) \in$ (*listrel1* r)$^* \Longrightarrow (x\#xs,x\#ys) \in$ (*listrel1* r)*
⟨*proof*⟩

lemma *rtrancl-listrel1-ConsI2*:
$(x,y) \in r^* \Longrightarrow (xs,\ ys) \in$ (*listrel1* r)$^* \Longrightarrow (x\ \#\ xs,\ y\ \#\ ys) \in$ (*listrel1* r)*
⟨*proof*⟩

lemma *listrel1-subset-listrel*:
$r \subseteq r' \Longrightarrow \mathit{refl}\ r' \Longrightarrow \mathit{listrel1}\ r \subseteq \mathit{listrel}(r')$
⟨*proof*⟩

lemma *listrel-reflcl-if-listrel1*:
$(xs,ys) \in \mathit{listrel1}\ r \Longrightarrow (xs,ys) \in \mathit{listrel}(r^*)$
⟨*proof*⟩

lemma *listrel-rtrancl-eq-rtrancl-listrel1*: $\mathit{listrel}\ (r^*) = (\mathit{listrel1}\ r)^*$
⟨*proof*⟩

lemma *rtrancl-listrel1-if-listrel*:
$(xs,ys) \in \mathit{listrel}\ r \Longrightarrow (xs,ys) \in (\mathit{listrel1}\ r)^*$
⟨*proof*⟩

lemma *listrel-subset-rtrancl-listrel1*: $\mathit{listrel}\ r \subseteq (\mathit{listrel1}\ r)^*$
⟨*proof*⟩

67.5 Size function

lemma [*measure-function*]: $\mathit{is\text{-}measure}\ f \Longrightarrow \mathit{is\text{-}measure}\ (\mathit{size\text{-}list}\ f)$
⟨*proof*⟩

lemma [*measure-function*]: $\mathit{is\text{-}measure}\ f \Longrightarrow \mathit{is\text{-}measure}\ (\mathit{size\text{-}option}\ f)$
⟨*proof*⟩

lemma *size-list-estimation*[*termination-simp*]:
$x \in \mathit{set}\ xs \Longrightarrow y < f\ x \Longrightarrow y < \mathit{size\text{-}list}\ f\ xs$
⟨*proof*⟩

lemma *size-list-estimation'*[*termination-simp*]:
$x \in \mathit{set}\ xs \Longrightarrow y \leq f\ x \Longrightarrow y \leq \mathit{size\text{-}list}\ f\ xs$
⟨*proof*⟩

lemma *size-list-map*[*simp*]: $\mathit{size\text{-}list}\ f\ (\mathit{map}\ g\ xs) = \mathit{size\text{-}list}\ (f \circ g)\ xs$
⟨*proof*⟩

lemma *size-list-append*[*simp*]: $\mathit{size\text{-}list}\ f\ (xs\ @\ ys) = \mathit{size\text{-}list}\ f\ xs + \mathit{size\text{-}list}\ f\ ys$
⟨*proof*⟩

lemma *size-list-pointwise*[*termination-simp*]:
$(\bigwedge x.\ x \in \mathit{set}\ xs \Longrightarrow f\ x \leq g\ x) \Longrightarrow \mathit{size\text{-}list}\ f\ xs \leq \mathit{size\text{-}list}\ g\ xs$
⟨*proof*⟩

67.6 Monad operation

definition $\mathit{bind} :: 'a\ \mathit{list} \Rightarrow ('a \Rightarrow 'b\ \mathit{list}) \Rightarrow 'b\ \mathit{list}$ **where**
$\mathit{bind}\ xs\ f = \mathit{concat}\ (\mathit{map}\ f\ xs)$

hide-const (**open**) *bind*

lemma *bind-simps* [*simp*]:
List.bind [] *f* = []
List.bind (*x* # *xs*) *f* = *f x* @ *List.bind xs f*
⟨*proof*⟩

lemma *list-bind-cong* [*fundef-cong*]:
assumes *xs* = *ys* (⋀*x*. *x* ∈ *set xs* ⟹ *f x* = *g x*)
shows *List.bind xs f* = *List.bind ys g*
⟨*proof*⟩

lemma *set-list-bind*: *set* (*List.bind xs f*) = (⋃*x*∈*set xs*. *set* (*f x*))
⟨*proof*⟩

67.7 Code generation

Optional tail recursive version of *map*. Can avoid stack overflow in some target languages.

fun *map-tailrec-rev* :: ('*a* ⇒ '*b*) ⇒ '*a list* ⇒ '*b list* ⇒ '*b list* **where**
map-tailrec-rev f [] *bs* = *bs* |
map-tailrec-rev f (*a*#*as*) *bs* = *map-tailrec-rev f as* (*f a* # *bs*)

lemma *map-tailrec-rev*:
map-tailrec-rev f as bs = *rev*(*map f as*) @ *bs*
⟨*proof*⟩

definition *map-tailrec* :: ('*a* ⇒ '*b*) ⇒ '*a list* ⇒ '*b list* **where**
map-tailrec f as = *rev* (*map-tailrec-rev f as* [])

Code equation:

lemma *map-eq-map-tailrec*: *map* = *map-tailrec*
⟨*proof*⟩

67.7.1 Counterparts for set-related operations

definition *member* :: '*a list* ⇒ '*a* ⇒ *bool* **where**
[*code-abbrev*]: *member xs x* ⟷ *x* ∈ *set xs*

Use *member* only for generating executable code. Otherwise use *x* ∈ *set xs* instead — it is much easier to reason about.

lemma *member-rec* [*code*]:
member (*x* # *xs*) *y* ⟷ *x* = *y* ∨ *member xs y*
member [] *y* ⟷ *False*
⟨*proof*⟩

lemma *in-set-member* :
x ∈ *set xs* ⟷ *member xs x*

⟨*proof*⟩

lemmas *list-all-iff* [*code-abbrev*] = *fun-cong*[*OF list.pred-set*]

definition *list-ex* :: ('*a* ⇒ *bool*) ⇒ '*a list* ⇒ *bool* **where**
list-ex-iff [*code-abbrev*]: *list-ex P xs* ⟷ *Bex* (*set xs*) *P*

definition *list-ex1* :: ('*a* ⇒ *bool*) ⇒ '*a list* ⇒ *bool* **where**
list-ex1-iff [*code-abbrev*]: *list-ex1 P xs* ⟷ (∃! *x*. *x* ∈ *set xs* ∧ *P x*)

Usually you should prefer ∀ *x*∈*set xs*, ∃ *x*∈*set xs* and ∃!*x*. *x*∈*set xs* ∧ - over *list-all*, *list-ex* and *list-ex1* in specifications.

lemma *list-all-simps* [*code*]:
list-all P (*x* # *xs*) ⟷ *P x* ∧ *list-all P xs*
list-all P [] ⟷ *True*
⟨*proof*⟩

lemma *list-ex-simps* [*simp*, *code*]:
list-ex P (*x* # *xs*) ⟷ *P x* ∨ *list-ex P xs*
list-ex P [] ⟷ *False*
⟨*proof*⟩

lemma *list-ex1-simps* [*simp*, *code*]:
list-ex1 P [] = *False*
list-ex1 P (*x* # *xs*) = (*if P x then list-all* (λ*y*. ¬ *P y* ∨ *x* = *y*) *xs else list-ex1 P xs*)
⟨*proof*⟩

lemma *Ball-set-list-all*:
Ball (*set xs*) *P* ⟷ *list-all P xs*
⟨*proof*⟩

lemma *Bex-set-list-ex*:
Bex (*set xs*) *P* ⟷ *list-ex P xs*
⟨*proof*⟩

lemma *list-all-append* [*simp*]:
list-all P (*xs* @ *ys*) ⟷ *list-all P xs* ∧ *list-all P ys*
⟨*proof*⟩

lemma *list-ex-append* [*simp*]:
list-ex P (*xs* @ *ys*) ⟷ *list-ex P xs* ∨ *list-ex P ys*
⟨*proof*⟩

lemma *list-all-rev* [*simp*]:
list-all P (*rev xs*) ⟷ *list-all P xs*
⟨*proof*⟩

lemma *list-ex-rev* [*simp*]:

list-ex P (rev xs) ⟷ *list-ex P xs*
⟨*proof*⟩

lemma *list-all-length*:
list-all P xs ⟷ (∀ *n* < *length xs*. *P* (*xs* ! *n*))
⟨*proof*⟩

lemma *list-ex-length*:
list-ex P xs ⟷ (∃ *n* < *length xs*. *P* (*xs* ! *n*))
⟨*proof*⟩

lemmas *list-all-cong* [*fundef-cong*] = *list.pred-cong*

lemma *list-ex-cong* [*fundef-cong*]:
xs = *ys* ⟹ (⋀*x*. *x* ∈ *set ys* ⟹ *f x* = *g x*) ⟹ *list-ex f xs* = *list-ex g ys*
⟨*proof*⟩

definition *can-select* :: ('*a* ⇒ *bool*) ⇒ '*a set* ⇒ *bool* **where**
[*code-abbrev*]: *can-select P A* = (∃ !*x*∈*A*. *P x*)

lemma *can-select-set-list-ex1* [*code*]:
can-select P (*set A*) = *list-ex1 P A*
⟨*proof*⟩

Executable checks for relations on sets

definition *listrel1p* :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ '*a list* ⇒ '*a list* ⇒ *bool* **where**
listrel1p r xs ys = ((*xs*, *ys*) ∈ *listrel1* {(*x*, *y*). *r x y*})

lemma [*code-unfold*]:
(*xs*, *ys*) ∈ *listrel1 r* = *listrel1p* (λ*x y*. (*x*, *y*) ∈ *r*) *xs ys*
⟨*proof*⟩

lemma [*code*]:
listrel1p r [] *xs* = *False*
listrel1p r xs [] = *False*
listrel1p r (*x* # *xs*) (*y* # *ys*) ⟷
r x y ∧ *xs* = *ys* ∨ *x* = *y* ∧ *listrel1p r xs ys*
⟨*proof*⟩

definition
lexordp :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ '*a list* ⇒ '*a list* ⇒ *bool* **where**
lexordp r xs ys = ((*xs*, *ys*) ∈ *lexord* {(*x*, *y*). *r x y*})

lemma [*code-unfold*]:
(*xs*, *ys*) ∈ *lexord r* = *lexordp* (λ*x y*. (*x*, *y*) ∈ *r*) *xs ys*
⟨*proof*⟩

lemma [*code*]:
lexordp r xs [] = *False*

lexordp r [] (y#ys) = True
lexordp r (x # xs) (y # ys) = (r x y ∨ (x = y ∧ lexordp r xs ys))
⟨*proof*⟩

Bounded quantification and summation over nats.

lemma *atMost-upto* [*code-unfold*]:
{..n} = set [0..<Suc n]
⟨*proof*⟩

lemma *atLeast-upt* [*code-unfold*]:
{..<n} = set [0..<n]
⟨*proof*⟩

lemma *greaterThanLessThan-upt* [*code-unfold*]:
{n<..<m} = set [Suc n..<m]
⟨*proof*⟩

lemmas *atLeastLessThan-upt* [*code-unfold*] = *set-upt* [*symmetric*]

lemma *greaterThanAtMost-upt* [*code-unfold*]:
{n<..m} = set [Suc n..<Suc m]
⟨*proof*⟩

lemma *atLeastAtMost-upt* [*code-unfold*]:
{n..m} = set [n..<Suc m]
⟨*proof*⟩

lemma *all-nat-less-eq* [*code-unfold*]:
(∀ m<n::nat. P m) ⟷ (∀ m ∈ {0..<n}. P m)
⟨*proof*⟩

lemma *ex-nat-less-eq* [*code-unfold*]:
(∃ m<n::nat. P m) ⟷ (∃ m ∈ {0..<n}. P m)
⟨*proof*⟩

lemma *all-nat-less* [*code-unfold*]:
(∀ m≤n::nat. P m) ⟷ (∀ m ∈ {0..n}. P m)
⟨*proof*⟩

lemma *ex-nat-less* [*code-unfold*]:
(∃ m≤n::nat. P m) ⟷ (∃ m ∈ {0..n}. P m)
⟨*proof*⟩

Bounded *LEAST* operator:

definition *Bleast S P* = *(LEAST x. x ∈ S ∧ P x)*

definition *abort-Bleast S P* = *(LEAST x. x ∈ S ∧ P x)*

declare [[*code abort*: *abort-Bleast*]]

lemma *Bleast-code* [*code*]:
Bleast (*set xs*) *P* = (*case filter P* (*sort xs*) *of*
x#*xs* ⇒ *x* |
[] ⇒ *abort-Bleast* (*set xs*) *P*)
⟨*proof*⟩

declare *Bleast-def*[*symmetric*, *code-unfold*]

Summation over ints.

lemma *greaterThanLessThan-upto* [*code-unfold*]:
{*i*<..<*j*::*int*} = *set* [*i*+*1*..*j* − *1*]
⟨*proof*⟩

lemma *atLeastLessThan-upto* [*code-unfold*]:
{*i*..<*j*::*int*} = *set* [*i*..*j* − *1*]
⟨*proof*⟩

lemma *greaterThanAtMost-upto* [*code-unfold*]:
{*i*<..*j*::*int*} = *set* [*i*+*1*..*j*]
⟨*proof*⟩

lemmas *atLeastAtMost-upto* [*code-unfold*] = *set-upto* [*symmetric*]

67.7.2 Optimizing by rewriting

definition *null* :: *'a list* ⇒ *bool* **where**
[*code-abbrev*]: *null xs* ⟷ *xs* = []

Efficient emptyness check is implemented by *null*.

lemma *null-rec* [*code*]:
null (*x* # *xs*) ⟷ *False*
null [] ⟷ *True*
⟨*proof*⟩

lemma *eq-Nil-null*:
xs = [] ⟷ *null xs*
⟨*proof*⟩

lemma *equal-Nil-null* [*code-unfold*]:
HOL.equal xs [] ⟷ *null xs*
HOL.equal [] = *null*
⟨*proof*⟩

definition *maps* :: (*'a* ⇒ *'b list*) ⇒ *'a list* ⇒ *'b list* **where**
[*code-abbrev*]: *maps f xs* = *concat* (*map f xs*)

definition *map-filter* :: (*'a* ⇒ *'b option*) ⇒ *'a list* ⇒ *'b list* **where**
[*code-post*]: *map-filter f xs* = *map* (*the* ∘ *f*) (*filter* (λ*x*. *f x* ≠ *None*) *xs*)

Operations *maps* and *map-filter* avoid intermediate lists on execution – do not use for proving.

lemma *maps-simps* [*code*]:
maps f (x # xs) = f x @ maps f xs
maps f [] = []
⟨*proof*⟩

lemma *map-filter-simps* [*code*]:
map-filter f (x # xs) = (case f x of None ⇒ map-filter f xs | Some y ⇒ y # map-filter f xs)
map-filter f [] = []
⟨*proof*⟩

lemma *concat-map-maps*:
concat (map f xs) = maps f xs
⟨*proof*⟩

lemma *map-filter-map-filter* [*code-unfold*]:
map f (filter P xs) = map-filter (λx. if P x then Some (f x) else None) xs
⟨*proof*⟩

Optimized code for $\forall\, i{\in}\{a..b::int\}$ and $\forall\, n{:}\{a..{<}b::nat\}$ and similiarly for $\exists$.

definition *all-interval-nat* :: *(nat ⇒ bool) ⇒ nat ⇒ nat ⇒ bool* **where**
all-interval-nat P i j ⟷ (∀ n ∈ {i..<j}. P n)

lemma [*code*]:
all-interval-nat P i j ⟷ i ≥ j ∨ P i ∧ all-interval-nat P (Suc i) j
⟨*proof*⟩

lemma *list-all-iff-all-interval-nat* [*code-unfold*]:
list-all P [i..<j] ⟷ all-interval-nat P i j
⟨*proof*⟩

lemma *list-ex-iff-not-all-inverval-nat* [*code-unfold*]:
list-ex P [i..<j] ⟷ ¬ (all-interval-nat (Not ∘ P) i j)
⟨*proof*⟩

definition *all-interval-int* :: *(int ⇒ bool) ⇒ int ⇒ int ⇒ bool* **where**
all-interval-int P i j ⟷ (∀ k ∈ {i..j}. P k)

lemma [*code*]:
all-interval-int P i j ⟷ i > j ∨ P i ∧ all-interval-int P (i + 1) j
⟨*proof*⟩

lemma *list-all-iff-all-interval-int* [*code-unfold*]:
list-all P [i..j] ⟷ all-interval-int P i j
⟨*proof*⟩

lemma *list-ex-iff-not-all-inverval-int* [*code-unfold*]:

list-ex P [i..j] ⟷ ¬ (*all-interval-int* (*Not* ∘ *P*) *i j*)
⟨*proof*⟩

optimized code (tail-recursive) for *length*

definition *gen-length* :: *nat* ⇒ *'a list* ⇒ *nat*
where *gen-length n xs* = *n* + *length xs*

lemma *gen-length-code* [*code*]:
gen-length n [] = *n*
gen-length n (*x* # *xs*) = *gen-length* (*Suc n*) *xs*
⟨*proof*⟩

declare *list.size*(*3*−*4*)[*code del*]

lemma *length-code* [*code*]: *length* = *gen-length 0*
⟨*proof*⟩

hide-const (**open**) *member null maps map-filter all-interval-nat all-interval-int gen-length*

67.7.3 Pretty lists

⟨*ML*⟩

code-printing
type-constructor *list* ⇀
(*SML*) - *list*
and (*OCaml*) - *list*
and (*Haskell*) ![(-)]
and (*Scala*) *List*[(-)]
| **constant** *Nil* ⇀
(*SML*) []
and (*OCaml*) []
and (*Haskell*) []
and (*Scala*) !*Nil*
| **class-instance** *list* :: *equal* ⇀
(*Haskell*) −
| **constant** *HOL.equal* :: *'a list* ⇒ *'a list* ⇒ *bool* ⇀
(*Haskell*) **infix** *4* ==

⟨*ML*⟩

code-reserved *SML*
list

code-reserved *OCaml*
list

67.7.4 Use convenient predefined operations

code-printing
  constant (@) ⇀
    (*SML*) **infixr** *7* @
    and (*OCaml*) **infixr** *6* @
    and (*Haskell*) **infixr** *5* ++
    and (*Scala*) **infixl** *7* ++
| **constant** *map* ⇀
    (*Haskell*) *map*
| **constant** *filter* ⇀
    (*Haskell*) *filter*
| **constant** *concat* ⇀
    (*Haskell*) *concat*
| **constant** *List.maps* ⇀
    (*Haskell*) *concatMap*
| **constant** *rev* ⇀
    (*Haskell*) *reverse*
| **constant** *zip* ⇀
    (*Haskell*) *zip*
| **constant** *List.null* ⇀
    (*Haskell*) *null*
| **constant** *takeWhile* ⇀
    (*Haskell*) *takeWhile*
| **constant** *dropWhile* ⇀
    (*Haskell*) *dropWhile*
| **constant** *list-all* ⇀
    (*Haskell*) *all*
| **constant** *list-ex* ⇀
    (*Haskell*) *any*

67.7.5 Implementation of sets by lists

lemma *is-empty-set* [*code*]:
  Set.is-empty (*set xs*) ⟷ *List.null xs*
  ⟨*proof*⟩

lemma *empty-set* [*code*]:
  {} = *set* []
  ⟨*proof*⟩

lemma *UNIV-coset* [*code*]:
  UNIV = *List.coset* []
  ⟨*proof*⟩

lemma *compl-set* [*code*]:
  − *set xs* = *List.coset xs*
  ⟨*proof*⟩

lemma *compl-coset* [*code*]:

– *List.coset xs = set xs*
⟨*proof*⟩

lemma [*code*]:
x ∈ set xs ⟷ List.member xs x
x ∈ List.coset xs ⟷ ¬ List.member xs x
⟨*proof*⟩

lemma *insert-code* [*code*]:
insert x (set xs) = set (List.insert x xs)
insert x (List.coset xs) = List.coset (removeAll x xs)
⟨*proof*⟩

lemma *remove-code* [*code*]:
Set.remove x (set xs) = set (removeAll x xs)
Set.remove x (List.coset xs) = List.coset (List.insert x xs)
⟨*proof*⟩

lemma *filter-set* [*code*]:
Set.filter P (set xs) = set (filter P xs)
⟨*proof*⟩

lemma *image-set* [*code*]:
image f (set xs) = set (map f xs)
⟨*proof*⟩

lemma *subset-code* [*code*]:
set xs ≤ B ⟷ (∀ x∈set xs. x ∈ B)
A ≤ List.coset ys ⟷ (∀ y∈set ys. y ∉ A)
List.coset [] ⊆ set [] ⟷ False
⟨*proof*⟩

A frequent case – avoid intermediate sets

lemma [*code-unfold*]:
set xs ⊆ set ys ⟷ list-all (λx. x ∈ set ys) xs
⟨*proof*⟩

lemma *Ball-set* [*code*]:
Ball (set xs) P ⟷ list-all P xs
⟨*proof*⟩

lemma *Bex-set* [*code*]:
Bex (set xs) P ⟷ list-ex P xs
⟨*proof*⟩

lemma *card-set* [*code*]:
card (set xs) = length (remdups xs)
⟨*proof*⟩

lemma *the-elem-set* [*code*]:
the-elem (*set* [*x*]) = *x*
⟨*proof*⟩

lemma *Pow-set* [*code*]:
Pow (*set* []) = {{}}
Pow (*set* (*x* # *xs*)) = (*let A* = *Pow* (*set xs*) *in A* ∪ *insert x* ' *A*)
⟨*proof*⟩

definition *map-project* :: ('*a* ⇒ '*b option*) ⇒ '*a set* ⇒ '*b set* **where**
map-project f A = {*b*. ∃ *a* ∈ *A*. *f a* = *Some b*}

lemma [*code*]:
map-project f (*set xs*) = *set* (*List.map-filter f xs*)
⟨*proof*⟩

hide-const (**open**) *map-project*

Operations on relations

lemma *product-code* [*code*]:
Product-Type.product (*set xs*) (*set ys*) = *set* [(*x*, *y*). *x* ← *xs*, *y* ← *ys*]
⟨*proof*⟩

lemma *Id-on-set* [*code*]:
Id-on (*set xs*) = *set* [(*x*, *x*). *x* ← *xs*]
⟨*proof*⟩

lemma [*code*]:
R `` *S* = *List.map-project* (λ(*x*, *y*). *if x* ∈ *S then Some y else None*) *R*
⟨*proof*⟩

lemma *trancl-set-ntrancl* [*code*]:
trancl (*set xs*) = *ntrancl* (*card* (*set xs*) − *1*) (*set xs*)
⟨*proof*⟩

lemma *set-relcomp* [*code*]:
set xys O set yzs = *set* ([(*fst xy*, *snd yz*). *xy* ← *xys*, *yz* ← *yzs*, *snd xy* = *fst yz*])
⟨*proof*⟩

lemma *wf-set* [*code*]:
wf (*set xs*) = *acyclic* (*set xs*)
⟨*proof*⟩

67.8 Setup for Lifting/Transfer

67.8.1 Transfer rules for the Transfer package

context includes *lifting-syntax*
begin

lemma *tl-transfer* [*transfer-rule*]:
(*list-all2 A* ===> *list-all2 A*) *tl tl*
⟨*proof*⟩

lemma *butlast-transfer* [*transfer-rule*]:
(*list-all2 A* ===> *list-all2 A*) *butlast butlast*
⟨*proof*⟩

lemma *map-rec*: *map f xs* = *rec-list Nil* (%*x - y. Cons* (*f x*) *y*) *xs*
⟨*proof*⟩

lemma *append-transfer* [*transfer-rule*]:
(*list-all2 A* ===> *list-all2 A* ===> *list-all2 A*) *append append*
⟨*proof*⟩

lemma *rev-transfer* [*transfer-rule*]:
(*list-all2 A* ===> *list-all2 A*) *rev rev*
⟨*proof*⟩

lemma *filter-transfer* [*transfer-rule*]:
((*A* ===> (=)) ===> *list-all2 A* ===> *list-all2 A*) *filter filter*
⟨*proof*⟩

lemma *fold-transfer* [*transfer-rule*]:
((*A* ===> *B* ===> *B*) ===> *list-all2 A* ===> *B* ===> *B*) *fold fold*
⟨*proof*⟩

lemma *foldr-transfer* [*transfer-rule*]:
((*A* ===> *B* ===> *B*) ===> *list-all2 A* ===> *B* ===> *B*) *foldr foldr*
⟨*proof*⟩

lemma *foldl-transfer* [*transfer-rule*]:
((*B* ===> *A* ===> *B*) ===> *B* ===> *list-all2 A* ===> *B*) *foldl foldl*
⟨*proof*⟩

lemma *concat-transfer* [*transfer-rule*]:
(*list-all2* (*list-all2 A*) ===> *list-all2 A*) *concat concat*
⟨*proof*⟩

lemma *drop-transfer* [*transfer-rule*]:
((=) ===> *list-all2 A* ===> *list-all2 A*) *drop drop*
⟨*proof*⟩

lemma *take-transfer* [*transfer-rule*]:
((=) ===> *list-all2 A* ===> *list-all2 A*) *take take*
⟨*proof*⟩

lemma *list-update-transfer* [*transfer-rule*]:
(*list-all2 A* ===> (=) ===> *A* ===> *list-all2 A*) *list-update list-update*

⟨*proof*⟩

lemma *takeWhile-transfer* [*transfer-rule*]:
((*A* ===> (=)) ===> *list-all2 A* ===> *list-all2 A*) *takeWhile takeWhile*
⟨*proof*⟩

lemma *dropWhile-transfer* [*transfer-rule*]:
((*A* ===> (=)) ===> *list-all2 A* ===> *list-all2 A*) *dropWhile dropWhile*
⟨*proof*⟩

lemma *zip-transfer* [*transfer-rule*]:
(*list-all2 A* ===> *list-all2 B* ===> *list-all2* (*rel-prod A B*)) *zip zip*
⟨*proof*⟩

lemma *product-transfer* [*transfer-rule*]:
(*list-all2 A* ===> *list-all2 B* ===> *list-all2* (*rel-prod A B*)) *List.product List.product*
⟨*proof*⟩

lemma *product-lists-transfer* [*transfer-rule*]:
(*list-all2* (*list-all2 A*) ===> *list-all2* (*list-all2 A*)) *product-lists product-lists*
⟨*proof*⟩

lemma *insert-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*A* ===> *list-all2 A* ===> *list-all2 A*) *List.insert List.insert*
⟨*proof*⟩

lemma *find-transfer* [*transfer-rule*]:
((*A* ===> (=)) ===> *list-all2 A* ===> *rel-option A*) *List.find List.find*
⟨*proof*⟩

lemma *those-transfer* [*transfer-rule*]:
(*list-all2* (*rel-option P*) ===> *rel-option* (*list-all2 P*)) *those those*
⟨*proof*⟩

lemma *remove1-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*A* ===> *list-all2 A* ===> *list-all2 A*) *remove1 remove1*
⟨*proof*⟩

lemma *removeAll-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*A* ===> *list-all2 A* ===> *list-all2 A*) *removeAll removeAll*
⟨*proof*⟩

lemma *successively-transfer* [*transfer-rule*]:
((*A* ===> *A* ===> (=)) ===> *list-all2 A* ===> (=)) *successively successively*
⟨*proof*⟩

lemma *distinct-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*list-all2 A* ===> (=)) *distinct distinct*
⟨*proof*⟩

lemma *distinct-adj-transfer* [*transfer-rule*]:
assumes *bi-unique A*
shows (*list-all2 A* ===> (=)) *distinct-adj distinct-adj*
⟨*proof*⟩

lemma *remdups-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*list-all2 A* ===> *list-all2 A*) *remdups remdups*
⟨*proof*⟩

lemma *remdups-adj-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*list-all2 A* ===> *list-all2 A*) *remdups-adj remdups-adj*
⟨*proof*⟩

lemma *replicate-transfer* [*transfer-rule*]:
((=) ===> *A* ===> *list-all2 A*) *replicate replicate*
⟨*proof*⟩

lemma *length-transfer* [*transfer-rule*]:
(*list-all2 A* ===> (=)) *length length*
⟨*proof*⟩

lemma *rotate1-transfer* [*transfer-rule*]:
(*list-all2 A* ===> *list-all2 A*) *rotate1 rotate1*
⟨*proof*⟩

lemma *rotate-transfer* [*transfer-rule*]:
((=) ===> *list-all2 A* ===> *list-all2 A*) *rotate rotate*
⟨*proof*⟩

lemma *nths-transfer* [*transfer-rule*]:
(*list-all2 A* ===> *rel-set* (=) ===> *list-all2 A*) *nths nths*
⟨*proof*⟩

lemma *subseqs-transfer* [*transfer-rule*]:
(*list-all2 A* ===> *list-all2* (*list-all2 A*)) *subseqs subseqs*
⟨*proof*⟩

lemma *partition-transfer* [*transfer-rule*]:
((*A* ===> (=)) ===> *list-all2 A* ===> *rel-prod* (*list-all2 A*) (*list-all2 A*))
partition partition
⟨*proof*⟩

lemma *lists-transfer* [*transfer-rule*]:
(*rel-set A* ===> *rel-set* (*list-all2 A*)) *lists lists*
⟨*proof*⟩

lemma *set-Cons-transfer* [*transfer-rule*]:
(*rel-set A* ===> *rel-set* (*list-all2 A*) ===> *rel-set* (*list-all2 A*))
set-Cons set-Cons
⟨*proof*⟩

lemma *listset-transfer* [*transfer-rule*]:
(*list-all2* (*rel-set A*) ===> *rel-set* (*list-all2 A*)) *listset listset*
⟨*proof*⟩

lemma *null-transfer* [*transfer-rule*]:
(*list-all2 A* ===> (=)) *List.null List.null*
⟨*proof*⟩

lemma *list-all-transfer* [*transfer-rule*]:
((*A* ===> (=)) ===> *list-all2 A* ===> (=)) *list-all list-all*
⟨*proof*⟩

lemma *list-ex-transfer* [*transfer-rule*]:
((*A* ===> (=)) ===> *list-all2 A* ===> (=)) *list-ex list-ex*
⟨*proof*⟩

lemma *splice-transfer* [*transfer-rule*]:
(*list-all2 A* ===> *list-all2 A* ===> *list-all2 A*) *splice splice*
⟨*proof*⟩

lemma *shuffles-transfer* [*transfer-rule*]:
(*list-all2 A* ===> *list-all2 A* ===> *rel-set* (*list-all2 A*)) *shuffles shuffles*
⟨*proof*⟩

lemma *rtrancl-parametric* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A bi-total A*
shows (*rel-set* (*rel-prod A A*) ===> *rel-set* (*rel-prod A A*)) *rtrancl rtrancl*
⟨*proof*⟩

lemma *monotone-parametric* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-total A*
shows ((*A* ===> *A* ===> (=)) ===> (*B* ===> *B* ===> (=)) ===> (*A* ===> *B*) ===> (=)) *monotone monotone*
⟨*proof*⟩

lemma *fun-ord-parametric* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-total C*
shows ((*A* ===> *B* ===> (=)) ===> (*C* ===> *A*) ===> (*C* ===> *B*) ===> (=)) *fun-ord fun-ord*
⟨*proof*⟩

lemma *fun-lub-parametric* [*transfer-rule*]:
  assumes [*transfer-rule*]: *bi-total A* *bi-unique A*
  shows ((*rel-set A* ===> *B*) ===> *rel-set* (*C* ===> *A*) ===> *C* ===> *B*)
fun-lub fun-lub
⟨*proof*⟩

end

end

68 Sum and product over lists

theory *Groups-List*
imports *List*
begin

locale *monoid-list* = *monoid*
begin

definition *F* :: *'a list* ⇒ *'a*
where
  eq-foldr [*code*]: *F xs* = *foldr f xs* **1**

lemma *Nil* [*simp*]:
  F [] = **1**
  ⟨*proof*⟩

lemma *Cons* [*simp*]:
  F (*x* # *xs*) = *x* ∗ *F xs*
  ⟨*proof*⟩

lemma *append* [*simp*]:
  F (*xs* @ *ys*) = *F xs* ∗ *F ys*
  ⟨*proof*⟩

end

locale *comm-monoid-list* = *comm-monoid* + *monoid-list*
begin

lemma *rev* [*simp*]:
  F (*rev xs*) = *F xs*
  ⟨*proof*⟩

end

locale *comm-monoid-list-set* = *list*: *comm-monoid-list* + *set*: *comm-monoid-set*
begin

lemma *distinct-set-conv-list*:
distinct xs $\Longrightarrow$ *set.F g* (*set xs*) = *list.F* (*map g xs*)
⟨*proof*⟩

lemma *set-conv-list* [*code*]:
set.F g (*set xs*) = *list.F* (*map g* (*remdups xs*))
⟨*proof*⟩

end

68.1 List summation

context *monoid-add*
begin

sublocale *sum-list*: *monoid-list plus 0*
defines
sum-list = *sum-list.F* ⟨*proof*⟩

end

context *comm-monoid-add*
begin

sublocale *sum-list*: *comm-monoid-list plus 0*
rewrites
monoid-list.F plus 0 = *sum-list*
⟨*proof*⟩

sublocale *sum*: *comm-monoid-list-set plus 0*
rewrites
monoid-list.F plus 0 = *sum-list*
and *comm-monoid-set.F plus 0* = *sum*
⟨*proof*⟩

end

Some syntactic sugar for summing a function over a list:

syntax (*ASCII*)
-sum-list :: *pttrn* => *'a list* => *'b* => *'b* (("3SUM -<-. -) [0, 51, 10] 10)
syntax
-sum-list :: *pttrn* => *'a list* => *'b* => *'b* (("3$\sum$ -←-. -) [0, 51, 10] 10)
translations — Beware of argument permutation!
$\sum x \leftarrow xs.\ b$ == *CONST sum-list* (*CONST map* ($\lambda x.\ b$) *xs*)

context
includes *lifting-syntax*
begin

lemma *sum-list-transfer* [*transfer-rule*]:
(*list-all2 A* ===> *A*) *sum-list sum-list*
if [*transfer-rule*]: *A 0 0* (*A* ===> *A* ===> *A*) (+) (+)
⟨*proof*⟩

end

TODO duplicates

lemmas *sum-list-simps* = *sum-list.Nil sum-list.Cons*
lemmas *sum-list-append* = *sum-list.append*
lemmas *sum-list-rev* = *sum-list.rev*

lemma (**in** *monoid-add*) *fold-plus-sum-list-rev*:
fold plus xs = *plus* (*sum-list* (*rev xs*))
⟨*proof*⟩

lemma (**in** *comm-monoid-add*) *sum-list-map-remove1*:
x ∈ *set xs* ⟹ *sum-list* (*map f xs*) = *f x* + *sum-list* (*map f* (*remove1 x xs*))
⟨*proof*⟩

lemma (**in** *monoid-add*) *size-list-conv-sum-list*:
size-list f xs = *sum-list* (*map f xs*) + *size xs*
⟨*proof*⟩

lemma (**in** *monoid-add*) *length-concat*:
length (*concat xss*) = *sum-list* (*map length xss*)
⟨*proof*⟩

lemma (**in** *monoid-add*) *length-product-lists*:
length (*product-lists xss*) = *foldr* (∗) (*map length xss*) *1*
⟨*proof*⟩

lemma (**in** *monoid-add*) *sum-list-map-filter*:
assumes ⋀*x*. *x* ∈ *set xs* ⟹ ¬ *P x* ⟹ *f x* = *0*
shows *sum-list* (*map f* (*filter P xs*)) = *sum-list* (*map f xs*)
⟨*proof*⟩

lemma *sum-list-filter-le-nat*:
fixes *f* :: *'a* ⇒ *nat*
shows *sum-list* (*map f* (*filter P xs*)) ≤ *sum-list* (*map f xs*)
⟨*proof*⟩

lemma (**in** *comm-monoid-add*) *distinct-sum-list-conv-Sum*:
distinct xs ⟹ *sum-list xs* = *Sum* (*set xs*)
⟨*proof*⟩

lemma *sum-list-upt*[*simp*]:
m ≤ *n* ⟹ *sum-list* [*m*..<*n*] = ∑ {*m*..<*n*}

⟨*proof*⟩

context *ordered-comm-monoid-add*
begin

lemma *sum-list-nonneg*: $(\bigwedge x.\ x \in set\ xs \Longrightarrow 0 \leq x) \Longrightarrow 0 \leq sum\text{-}list\ xs$
⟨*proof*⟩

lemma *sum-list-nonpos*: $(\bigwedge x.\ x \in set\ xs \Longrightarrow x \leq 0) \Longrightarrow sum\text{-}list\ xs \leq 0$
⟨*proof*⟩

lemma *sum-list-nonneg-eq-0-iff*:
$(\bigwedge x.\ x \in set\ xs \Longrightarrow 0 \leq x) \Longrightarrow sum\text{-}list\ xs = 0 \longleftrightarrow (\forall x{\in}\ set\ xs.\ x = 0)$
⟨*proof*⟩

end

context *canonically-ordered-monoid-add*
begin

lemma *sum-list-eq-0-iff* [*simp*]:
$sum\text{-}list\ ns = 0 \longleftrightarrow (\forall n \in set\ ns.\ n = 0)$
⟨*proof*⟩

lemma *member-le-sum-list*:
$x \in set\ xs \Longrightarrow x \leq sum\text{-}list\ xs$
⟨*proof*⟩

lemma *elem-le-sum-list*:
$k < size\ ns \Longrightarrow ns\ !\ k \leq sum\text{-}list\ (ns)$
⟨*proof*⟩

end

lemma (**in** *ordered-cancel-comm-monoid-diff*) *sum-list-update*:
$k < size\ xs \Longrightarrow sum\text{-}list\ (xs[k := x]) = sum\text{-}list\ xs + x - xs\ !\ k$
⟨*proof*⟩

lemma (**in** *monoid-add*) *sum-list-triv*:
$(\sum x{\leftarrow}xs.\ r) = of\text{-}nat\ (length\ xs) * r$
⟨*proof*⟩

lemma (**in** *monoid-add*) *sum-list-0* [*simp*]:
$(\sum x{\leftarrow}xs.\ 0) = 0$
⟨*proof*⟩

For non-Abelian groups *xs* needs to be reversed on one side:

lemma (**in** *ab-group-add*) *uminus-sum-list-map*:
$-\ sum\text{-}list\ (map\ f\ xs) = sum\text{-}list\ (map\ (uminus \circ f)\ xs)$

⟨*proof*⟩

lemma (**in** *comm-monoid-add*) *sum-list-addf*:
(∑ *x*←*xs*. *f x* + *g x*) = *sum-list* (*map f xs*) + *sum-list* (*map g xs*)
⟨*proof*⟩

lemma (**in** *ab-group-add*) *sum-list-subtractf*:
(∑ *x*←*xs*. *f x* − *g x*) = *sum-list* (*map f xs*) − *sum-list* (*map g xs*)
⟨*proof*⟩

lemma (**in** *semiring-0*) *sum-list-const-mult*:
(∑ *x*←*xs*. *c* ∗ *f x*) = *c* ∗ (∑ *x*←*xs*. *f x*)
⟨*proof*⟩

lemma (**in** *semiring-0*) *sum-list-mult-const*:
(∑ *x*←*xs*. *f x* ∗ *c*) = (∑ *x*←*xs*. *f x*) ∗ *c*
⟨*proof*⟩

lemma (**in** *ordered-ab-group-add-abs*) *sum-list-abs*:
|*sum-list xs*| ≤ *sum-list* (*map abs xs*)
⟨*proof*⟩

lemma *sum-list-mono*:
fixes *f g* :: *'a* ⇒ *'b*::{*monoid-add*, *ordered-ab-semigroup-add*}
shows (⋀*x*. *x* ∈ *set xs* ⟹ *f x* ≤ *g x*) ⟹ (∑ *x*←*xs*. *f x*) ≤ (∑ *x*←*xs*. *g x*)
⟨*proof*⟩

lemma *sum-list-strict-mono*:
fixes *f g* :: *'a* ⇒ *'b*::{*monoid-add*, *strict-ordered-ab-semigroup-add*}
shows ⟦ *xs* ≠ []; ⋀*x*. *x* ∈ *set xs* ⟹ *f x* < *g x* ⟧
⟹ *sum-list* (*map f xs*) < *sum-list* (*map g xs*)
⟨*proof*⟩

A much more general version of this monotonicity lemma can be formulated with multisets and the multiset order

lemma *sum-list-mono2*: **fixes** *xs* :: *'a* ::*ordered-comm-monoid-add list*
shows ⟦ *length xs* = *length ys*; ⋀*i*. *i* < *length xs* ⟶ *xs*!*i* ≤ *ys*!*i* ⟧
⟹ *sum-list xs* ≤ *sum-list ys*
⟨*proof*⟩

lemma (**in** *monoid-add*) *sum-list-distinct-conv-sum-set*:
distinct xs ⟹ *sum-list* (*map f xs*) = *sum f* (*set xs*)
⟨*proof*⟩

lemma (**in** *monoid-add*) *interv-sum-list-conv-sum-set-nat*:
sum-list (*map f* [*m*..<*n*]) = *sum f* (*set* [*m*..<*n*])
⟨*proof*⟩

lemma (**in** *monoid-add*) *interv-sum-list-conv-sum-set-int*:

sum-list (*map f* [*k..l*]) = *sum f* (*set* [*k..l*])
⟨*proof*⟩

General equivalence between *sum-list* and *sum*

lemma (**in** *monoid-add*) *sum-list-sum-nth*:
sum-list xs = (∑ *i* = *0* ..< *length xs*. *xs* ! *i*)
⟨*proof*⟩

lemma *sum-list-map-eq-sum-count*:
sum-list (*map f xs*) = *sum* (λ*x*. *count-list xs x* ∗ *f x*) (*set xs*)
⟨*proof*⟩

lemma *sum-list-map-eq-sum-count2*:
assumes *set xs* ⊆ *X finite X*
shows *sum-list* (*map f xs*) = *sum* (λ*x*. *count-list xs x* ∗ *f x*) *X*
⟨*proof*⟩

lemma *sum-list-replicate*: *sum-list* (*replicate n c*) = *of-nat n* ∗ *c*
⟨*proof*⟩

lemma *sum-list-nonneg*:
(⋀*x*. *x* ∈ *set xs* ⟹ (*x* :: *'a* :: *ordered-comm-monoid-add*) ≥ *0*) ⟹ *sum-list xs* ≥ *0*
⟨*proof*⟩

lemma *sum-list-Suc*:
sum-list (*map* (λ*x*. *Suc*(*f x*)) *xs*) = *sum-list* (*map f xs*) + *length xs*
⟨*proof*⟩

lemma (**in** *monoid-add*) *sum-list-map-filter'*:
sum-list (*map f* (*filter P xs*)) = *sum-list* (*map* (λ*x*. *if P x then f x else 0*) *xs*)
⟨*proof*⟩

Summation of a strictly ascending sequence with length n can be upper-bounded by summation over $\{0..<n\}$.

lemma *sorted-wrt-less-sum-mono-lowerbound*:
fixes *f* :: *nat* ⇒ (*'b*::*ordered-comm-monoid-add*)
assumes *mono*: ⋀*x y*. *x*≤*y* ⟹ *f x* ≤ *f y*
shows *sorted-wrt* (<) *ns* ⟹
(∑ *i*∈{*0..<length ns*}. *f i*) ≤ (∑ *i*←*ns*. *f i*)
⟨*proof*⟩

68.2 Horner sums

context *comm-semiring-0*
begin

definition *horner-sum* :: ‹('*b* ⇒ '*a*) ⇒ '*a* ⇒ '*b list* ⇒ '*a*›

where *horner-sum-foldr*: ‹*horner-sum f a xs = foldr* (λ*x b*. *f x* + *a* ∗ *b*) *xs 0*›

lemma *horner-sum-simps* [*simp*]:
‹*horner-sum f a* [] = *0*›
‹*horner-sum f a* (*x* # *xs*) = *f x* + *a* ∗ *horner-sum f a xs*›
⟨*proof*⟩

lemma *horner-sum-eq-sum-funpow*:
‹*horner-sum f a xs* = (∑ *n* = *0*..<*length xs*. ((∗) *a* ^^ *n*) (*f* (*xs* ! *n*)))›
⟨*proof*⟩

end

context
includes *lifting-syntax*
begin

lemma *horner-sum-transfer* [*transfer-rule*]:
‹((*B* ===> *A*) ===> *A* ===> *list-all2 B* ===> *A*) *horner-sum horner-sum*›
if [*transfer-rule*]: ‹*A 0 0*›
and [*transfer-rule*]: ‹(*A* ===> *A* ===> *A*) (+) (+)›
and [*transfer-rule*]: ‹(*A* ===> *A* ===> *A*) (∗) (∗)›
⟨*proof*⟩

end

context *comm-semiring-1*
begin

lemma *horner-sum-eq-sum*:
‹*horner-sum f a xs* = (∑ *n* = *0*..<*length xs*. *f* (*xs* ! *n*) ∗ *a* ^ *n*)›
⟨*proof*⟩

lemma *horner-sum-append*:
‹*horner-sum f a* (*xs* @ *ys*) = *horner-sum f a xs* + *a* ^ *length xs* ∗ *horner-sum f a ys*›
⟨*proof*⟩

end

context *linordered-semidom*
begin

lemma *horner-sum-nonnegative*:
‹*0* ≤ *horner-sum of-bool 2 bs*›
⟨*proof*⟩

end

context *unique-euclidean-semiring-numeral*
begin

lemma *horner-sum-bound*:
‹*horner-sum of-bool 2 bs* < *2* ^ *length bs*›
⟨*proof*⟩

end

lemma *nat-horner-sum* [*simp*]:
‹*nat* (*horner-sum of-bool 2 bs*) = *horner-sum of-bool 2 bs*›
⟨*proof*⟩

context *unique-euclidean-semiring-numeral*
begin

lemma *horner-sum-less-eq-iff-lexordp-eq*:
‹*horner-sum of-bool 2 bs* ≤ *horner-sum of-bool 2 cs* ⟷ *lexordp-eq* (*rev bs*) (*rev cs*)›
if ‹*length bs* = *length cs*›
⟨*proof*⟩

lemma *horner-sum-less-iff-lexordp*:
‹*horner-sum of-bool 2 bs* < *horner-sum of-bool 2 cs* ⟷ *ord-class.lexordp* (*rev bs*) (*rev cs*)›
if ‹*length bs* = *length cs*›
⟨*proof*⟩

end

68.3 Further facts about *List.n-lists*

lemma *length-n-lists*: *length* (*List.n-lists n xs*) = *length xs* ^ *n*
⟨*proof*⟩

lemma *distinct-n-lists*:
assumes *distinct xs*
shows *distinct* (*List.n-lists n xs*)
⟨*proof*⟩

68.4 Tools setup

lemmas *sum-code* = *sum.set-conv-list*

lemma *sum-set-upto-conv-sum-list-int* [*code-unfold*]:
sum f (*set* [*i*..*j*::*int*]) = *sum-list* (*map f* [*i*..*j*])
⟨*proof*⟩

lemma *sum-set-upt-conv-sum-list-nat* [*code-unfold*]:
sum f (*set* [*m*..<*n*]) = *sum-list* (*map f* [*m*..<*n*])

⟨*proof*⟩

68.5 List product

context *monoid-mult*
begin

sublocale *prod-list*: *monoid-list times 1*
defines
prod-list = *prod-list.F* ⟨*proof*⟩

end

context *comm-monoid-mult*
begin

sublocale *prod-list*: *comm-monoid-list times 1*
rewrites
monoid-list.F times 1 = *prod-list*
⟨*proof*⟩

sublocale *prod*: *comm-monoid-list-set times 1*
rewrites
monoid-list.F times 1 = *prod-list*
and *comm-monoid-set.F times 1* = *prod*
⟨*proof*⟩

end

Some syntactic sugar:

syntax (*ASCII*)
-prod-list :: *pttrn* => *'a list* => *'b* => *'b* ((*3PROD -<−-. -*) [*0*, *51*, *10*] *10*)
syntax
-prod-list :: *pttrn* => *'a list* => *'b* => *'b* ((*3*∏ *-←-. -*) [*0*, *51*, *10*] *10*)
translations — Beware of argument permutation!
∏ *x*←*xs*. *b* ⇌ *CONST prod-list* (*CONST map* (λ*x*. *b*) *xs*)

context
includes *lifting-syntax*
begin

lemma *prod-list-transfer* [*transfer-rule*]:
(*list-all2 A* ===> *A*) *prod-list prod-list*
if [*transfer-rule*]: *A 1 1* (*A* ===> *A* ===> *A*) (∗) (∗)
⟨*proof*⟩

end

lemma *prod-list-zero-iff*:

prod-list xs = 0 ⟷ (0 :: 'a :: {semiring-no-zero-divisors, semiring-1}) ∈ set xs
⟨*proof*⟩

end

69 Bit operations in suitable algebraic structures

theory *Bit-Operations*
imports *Presburger Groups-List*
begin

69.1 Abstract bit structures

class *semiring-bits = semiring-parity +*
assumes *bits-induct* [*case-names stable rec*]:
‹(⋀*a*. *a div 2 = a ⟹ P a*)
⟹ (⋀*a b*. *P a ⟹ (of-bool b + 2 * a) div 2 = a ⟹ P (of-bool b + 2 * a)*)
⟹ *P a*›
assumes *bits-div-0* [*simp*]: ‹*0 div a = 0*›
and *bits-div-by-1* [*simp*]: ‹*a div 1 = a*›
and *bits-mod-div-trivial* [*simp*]: ‹*a mod b div b = 0*›
and *even-succ-div-2* [*simp*]: ‹*even a ⟹ (1 + a) div 2 = a div 2*›
and *even-mask-div-iff*: ‹*even ((2 ^ m − 1) div 2 ^ n) ⟷ 2 ^ n = 0 ∨ m ≤ n*›
and *exp-div-exp-eq*: ‹*2 ^ m div 2 ^ n = of-bool (2 ^ m ≠ 0 ∧ m ≥ n) * 2 ^ (m − n)*›
and *div-exp-eq*: ‹*a div 2 ^ m div 2 ^ n = a div 2 ^ (m + n)*›
and *mod-exp-eq*: ‹*a mod 2 ^ m mod 2 ^ n = a mod 2 ^ min m n*›
and *mult-exp-mod-exp-eq*: ‹*m ≤ n ⟹ (a * 2 ^ m) mod (2 ^ n) = (a mod 2 ^ (n − m)) * 2 ^ m*›
and *div-exp-mod-exp-eq*: ‹*a div 2 ^ n mod 2 ^ m = a mod (2 ^ (n + m)) div 2 ^ n*›
and *even-mult-exp-div-exp-iff*: ‹*even (a * 2 ^ m div 2 ^ n) ⟷ m > n ∨ 2 ^ n = 0 ∨ (m ≤ n ∧ even (a div 2 ^ (n − m)))*›
fixes *bit* :: ‹*'a ⇒ nat ⇒ bool*›
assumes *bit-iff-odd*: ‹*bit a n ⟷ odd (a div 2 ^ n)*›
begin

Having *bit* as definitional class operation takes into account that specific instances can be implemented differently wrt. code generation.

lemma *bits-div-by-0* [*simp*]:
‹*a div 0 = 0*›
⟨*proof*⟩

lemma *bits-1-div-2* [*simp*]:
‹*1 div 2 = 0*›
⟨*proof*⟩

lemma *bits-1-div-exp* [*simp*]:

‹*1 div 2 ^ n = of-bool (n = 0)*›
⟨*proof*⟩

lemma *even-succ-div-exp* [*simp*]:
‹*(1 + a) div 2 ^ n = a div 2 ^ n*› **if** ‹*even a*› **and** ‹*n > 0*›
⟨*proof*⟩

lemma *even-succ-mod-exp* [*simp*]:
‹*(1 + a) mod 2 ^ n = 1 + (a mod 2 ^ n)*› **if** ‹*even a*› **and** ‹*n > 0*›
⟨*proof*⟩

lemma *bits-mod-by-1* [*simp*]:
‹*a mod 1 = 0*›
⟨*proof*⟩

lemma *bits-mod-0* [*simp*]:
‹*0 mod a = 0*›
⟨*proof*⟩

lemma *bits-one-mod-two-eq-one* [*simp*]:
‹*1 mod 2 = 1*›
⟨*proof*⟩

lemma *bit-0*:
‹*bit a 0 ⟷ odd a*›
⟨*proof*⟩

lemma *bit-Suc*:
‹*bit a (Suc n) ⟷ bit (a div 2) n*›
⟨*proof*⟩

lemma *bit-rec*:
‹*bit a n ⟷ (if n = 0 then odd a else bit (a div 2) (n − 1))*›
⟨*proof*⟩

lemma *bit-0-eq* [*simp*]:
‹*bit 0 = bot*›
⟨*proof*⟩

context
fixes *a*
assumes *stable*: ‹*a div 2 = a*›
begin

lemma *bits-stable-imp-add-self*:
‹*a + a mod 2 = 0*›
⟨*proof*⟩

lemma *stable-imp-bit-iff-odd*:

‹*bit a n* ⟷ *odd a*›
⟨*proof*⟩

end

lemma *bit-iff-idd-imp-stable*:
‹*a div 2 = a*› **if** ‹⋀*n*. *bit a n* ⟷ *odd a*›
⟨*proof*⟩

lemma *exp-eq-0-imp-not-bit*:
‹¬ *bit a n*› **if** ‹*2 ^ n = 0*›
⟨*proof*⟩

definition
possible-bit :: *'a itself* ⇒ *nat* ⇒ *bool*
where
possible-bit tyrep n = (*2 ^ n* ≠ (*0* :: *'a*))

lemma *possible-bit-0*[*simp*]:
possible-bit ty 0
⟨*proof*⟩

lemma *fold-possible-bit*:
2 ^ n = (*0* :: *'a*) ⟷ ¬ *possible-bit TYPE*(*'a*) *n*
⟨*proof*⟩

lemmas *impossible-bit* = *exp-eq-0-imp-not-bit*[*simplified fold-possible-bit*]

lemma *bit-imp-possible-bit*:
bit a n ⟹ *possible-bit TYPE*(*'a*) *n*
⟨*proof*⟩

lemma *possible-bit-less-imp*:
possible-bit tyrep i ⟹ *j* ≤ *i* ⟹ *possible-bit tyrep j*
⟨*proof*⟩

lemma *possible-bit-min*[*simp*]:
possible-bit tyrep (*min i j*) ⟷ *possible-bit tyrep i* ∨ *possible-bit tyrep j*
⟨*proof*⟩

lemma *bit-eqI*:
‹*a = b*› **if** ‹⋀*n*. *possible-bit TYPE*(*'a*) *n* ⟹ *bit a n* ⟷ *bit b n*›
⟨*proof*⟩

lemma *bit-eq-iff*:
‹*a = b* ⟷ (∀ *n*. *possible-bit TYPE*(*'a*) *n* ⟶ *bit a n* ⟷ *bit b n*)›
⟨*proof*⟩

named-theorems *bit-simps* ‹*Simplification rules for* ***const*** ‹*bit*››

lemma *bit-exp-iff* [*bit-simps*]:
‹*bit* (*2* ^ *m*) *n* ⟷ *possible-bit TYPE*('*a*) *n* ∧ *m* = *n*›
⟨*proof*⟩

lemma *bit-1-iff* [*bit-simps*]:
‹*bit 1 n* ⟷ *n* = *0*›
⟨*proof*⟩

lemma *bit-2-iff* [*bit-simps*]:
‹*bit 2 n* ⟷ *possible-bit TYPE*('*a*) *1* ∧ *n* = *1*›
⟨*proof*⟩

lemma *even-bit-succ-iff*:
‹*bit* (*1* + *a*) *n* ⟷ *bit a n* ∨ *n* = *0*› **if** ‹*even a*›
⟨*proof*⟩

lemma *bit-double-iff* [*bit-simps*]:
‹*bit* (*2* * *a*) *n* ⟷ *bit a* (*n* − *1*) ∧ *n* ≠ *0* ∧ *possible-bit TYPE*('*a*) *n*›
⟨*proof*⟩

lemma *odd-bit-iff-bit-pred*:
‹*bit a n* ⟷ *bit* (*a* − *1*) *n* ∨ *n* = *0*› **if** ‹*odd a*›
⟨*proof*⟩

lemma *bit-eq-rec*:
‹*a* = *b* ⟷ (*even a* ⟷ *even b*) ∧ *a div 2* = *b div 2*› (**is** ‹*?P* = *?Q*›)
⟨*proof*⟩

lemma *bit-mod-2-iff* [*simp*]:
‹*bit* (*a mod 2*) *n* ⟷ *n* = *0* ∧ *odd a*›
⟨*proof*⟩

lemma *bit-mask-sub-iff*:
‹*bit* (*2* ^ *m* − *1*) *n* ⟷ *possible-bit TYPE*('*a*) *n* ∧ *n* < *m*›
⟨*proof*⟩

lemma *exp-add-not-zero-imp*:
‹*2* ^ *m* ≠ *0*› **and** ‹*2* ^ *n* ≠ *0*› **if** ‹*2* ^ (*m* + *n*) ≠ *0*›
⟨*proof*⟩

lemma *bit-disjunctive-add-iff*:
‹*bit* (*a* + *b*) *n* ⟷ *bit a n* ∨ *bit b n*›
if ‹⋀*n*. ¬ *bit a n* ∨ ¬ *bit b n*›
⟨*proof*⟩

lemma
exp-add-not-zero-imp-left: ‹*2* ^ *m* ≠ *0*›
and *exp-add-not-zero-imp-right*: ‹*2* ^ *n* ≠ *0*›

if ‹*2* ^ (*m* + *n*) ≠ *0*›
⟨*proof*⟩

lemma *exp-not-zero-imp-exp-diff-not-zero*:
‹*2* ^ (*n* − *m*) ≠ *0*› **if** ‹*2* ^ *n* ≠ *0*›
⟨*proof*⟩

end

lemma *nat-bit-induct* [*case-names zero even odd*]:
P n **if** *zero*: *P 0*
and *even*: ⋀*n*. *P n* ⟹ *n* > *0* ⟹ *P* (*2* ∗ *n*)
and *odd*: ⋀*n*. *P n* ⟹ *P* (*Suc* (*2* ∗ *n*))
⟨*proof*⟩

instantiation *nat* :: *semiring-bits*
begin

definition *bit-nat* :: ‹*nat* ⇒ *nat* ⇒ *bool*›
where ‹*bit-nat m n* ⟷ *odd* (*m div 2* ^ *n*)›

instance
⟨*proof*⟩

end

lemma *possible-bit-nat*[*simp*]:
possible-bit TYPE(*nat*) *n*
⟨*proof*⟩

lemma *not-bit-Suc-0-Suc* [*simp*]:
‹¬ *bit* (*Suc 0*) (*Suc n*)›
⟨*proof*⟩

lemma *not-bit-Suc-0-numeral* [*simp*]:
‹¬ *bit* (*Suc 0*) (*numeral n*)›
⟨*proof*⟩

lemma *int-bit-induct* [*case-names zero minus even odd*]:
P k **if** *zero-int*: *P 0*
and *minus-int*: *P* (− *1*)
and *even-int*: ⋀*k*. *P k* ⟹ *k* ≠ *0* ⟹ *P* (*k* ∗ *2*)
and *odd-int*: ⋀*k*. *P k* ⟹ *k* ≠ − *1* ⟹ *P* (*1* + (*k* ∗ *2*)) **for** *k* :: *int*
⟨*proof*⟩

context *semiring-bits*
begin

lemma *bit-of-bool-iff* [*bit-simps*]:

‹*bit* (*of-bool b*) *n* ⟷ *b* ∧ *n* = *0*›
⟨*proof*⟩

lemma *bit-of-nat-iff* [*bit-simps*]:
‹*bit* (*of-nat m*) *n* ⟷ *possible-bit TYPE*('*a*) *n* ∧ *bit m n*›
⟨*proof*⟩

end

instantiation *int* :: *semiring-bits*
begin

definition *bit-int* :: ‹*int* ⇒ *nat* ⇒ *bool*›
where ‹*bit-int k n* ⟷ *odd* (*k div* 2 ^ *n*)›

instance
⟨*proof*⟩

end

lemma *possible-bit-int*[*simp*]:
possible-bit TYPE(*int*) *n*
⟨*proof*⟩

lemma *bit-not-int-iff'*:
‹*bit* (− *k* − *1*) *n* ⟷ ¬ *bit k n*›
for *k* :: *int*
⟨*proof*⟩

lemma *bit-nat-iff* [*bit-simps*]:
‹*bit* (*nat k*) *n* ⟷ *k* ≥ *0* ∧ *bit k n*›
⟨*proof*⟩

69.2 Bit operations

class *semiring-bit-operations* = *semiring-bits* +
fixes *and* :: ‹'*a* ⇒ '*a* ⇒ '*a*› (**infixr** ‹*AND*› *64*)
and *or* :: ‹'*a* ⇒ '*a* ⇒ '*a*› (**infixr** ‹*OR*› *59*)
and *xor* :: ‹'*a* ⇒ '*a* ⇒ '*a*› (**infixr** ‹*XOR*› *59*)
and *mask* :: ‹*nat* ⇒ '*a*›
and *set-bit* :: ‹*nat* ⇒ '*a* ⇒ '*a*›
and *unset-bit* :: ‹*nat* ⇒ '*a* ⇒ '*a*›
and *flip-bit* :: ‹*nat* ⇒ '*a* ⇒ '*a*›
and *push-bit* :: ‹*nat* ⇒ '*a* ⇒ '*a*›
and *drop-bit* :: ‹*nat* ⇒ '*a* ⇒ '*a*›
and *take-bit* :: ‹*nat* ⇒ '*a* ⇒ '*a*›
assumes *bit-and-iff* [*bit-simps*]: ‹*bit* (*a AND b*) *n* ⟷ *bit a n* ∧ *bit b n*›
and *bit-or-iff* [*bit-simps*]: ‹*bit* (*a OR b*) *n* ⟷ *bit a n* ∨ *bit b n*›
and *bit-xor-iff* [*bit-simps*]: ‹*bit* (*a XOR b*) *n* ⟷ *bit a n* ≠ *bit b n*›

and *mask-eq-exp-minus-1*: ‹*mask n = 2 ^ n − 1*›
and *set-bit-eq-or*: ‹*set-bit n a = a OR push-bit n 1*›
and *bit-unset-bit-iff* [*bit-simps*]: ‹*bit* (*unset-bit m a*) *n* ⟷ *bit a n* ∧ *m* ≠ *n*›
and *flip-bit-eq-xor*: ‹*flip-bit n a = a XOR push-bit n 1*›
and *push-bit-eq-mult*: ‹*push-bit n a = a * 2 ^ n*›
and *drop-bit-eq-div*: ‹*drop-bit n a = a div 2 ^ n*›
and *take-bit-eq-mod*: ‹*take-bit n a = a mod 2 ^ n*›
begin

We want the bitwise operations to bind slightly weaker than + and −.

Logically, *push-bit*, *drop-bit* and *take-bit* are just aliases; having them as separate operations makes proofs easier, otherwise proof automation would fiddle with concrete expressions $(2::'a)^n$ in a way obfuscating the basic algebraic relationships between those operations.

For the sake of code generation operations are specified as definitional class operations, taking into account that specific instances of these can be implemented differently wrt. code generation.

sublocale *and*: *semilattice* ‹(*AND*)›
⟨*proof*⟩

sublocale *or*: *semilattice-neutr* ‹(*OR*)› *0*
⟨*proof*⟩

sublocale *xor*: *comm-monoid* ‹(*XOR*)› *0*
⟨*proof*⟩

lemma *even-and-iff*:
‹*even* (*a AND b*) ⟷ *even a* ∨ *even b*›
⟨*proof*⟩

lemma *even-or-iff*:
‹*even* (*a OR b*) ⟷ *even a* ∧ *even b*›
⟨*proof*⟩

lemma *even-xor-iff*:
‹*even* (*a XOR b*) ⟷ (*even a* ⟷ *even b*)›
⟨*proof*⟩

lemma *zero-and-eq* [*simp*]:
‹*0 AND a = 0*›
⟨*proof*⟩

lemma *and-zero-eq* [*simp*]:
‹*a AND 0 = 0*›
⟨*proof*⟩

lemma *one-and-eq*:
‹*1 AND a = a mod 2*›

⟨*proof*⟩

lemma *and-one-eq*:
‹*a AND 1 = a mod 2*›
⟨*proof*⟩

lemma *one-or-eq*:
‹*1 OR a = a + of-bool (even a)*›
⟨*proof*⟩

lemma *or-one-eq*:
‹*a OR 1 = a + of-bool (even a)*›
⟨*proof*⟩

lemma *one-xor-eq*:
‹*1 XOR a = a + of-bool (even a) − of-bool (odd a)*›
⟨*proof*⟩

lemma *xor-one-eq*:
‹*a XOR 1 = a + of-bool (even a) − of-bool (odd a)*›
⟨*proof*⟩

lemma *xor-self-eq* [*simp*]:
‹*a XOR a = 0*›
⟨*proof*⟩

lemma *bit-iff-odd-drop-bit*:
‹*bit a n ⟷ odd (drop-bit n a)*›
⟨*proof*⟩

lemma *even-drop-bit-iff-not-bit*:
‹*even (drop-bit n a) ⟷ ¬ bit a n*›
⟨*proof*⟩

lemma *div-push-bit-of-1-eq-drop-bit*:
‹*a div push-bit n 1 = drop-bit n a*›
⟨*proof*⟩

lemma *bits-ident*:
push-bit n (drop-bit n a) + take-bit n a = a
⟨*proof*⟩

lemma *push-bit-push-bit* [*simp*]:
push-bit m (push-bit n a) = push-bit (m + n) a
⟨*proof*⟩

lemma *push-bit-0-id* [*simp*]:
push-bit 0 = id
⟨*proof*⟩

lemma *push-bit-of-0* [*simp*]:
push-bit n 0 = 0
⟨*proof*⟩

lemma *push-bit-of-1* [*simp*]:
push-bit n 1 = 2 ^ n
⟨*proof*⟩

lemma *push-bit-Suc* [*simp*]:
*push-bit (Suc n) a = push-bit n (a * 2)*
⟨*proof*⟩

lemma *push-bit-double*:
*push-bit n (a * 2) = push-bit n a * 2*
⟨*proof*⟩

lemma *push-bit-add*:
push-bit n (a + b) = push-bit n a + push-bit n b
⟨*proof*⟩

lemma *push-bit-numeral* [*simp*]:
‹*push-bit (numeral l) (numeral k) = push-bit (pred-numeral l) (numeral (Num.Bit0 k))*›
⟨*proof*⟩

lemma *take-bit-0* [*simp*]:
take-bit 0 a = 0
⟨*proof*⟩

lemma *take-bit-Suc*:
‹*take-bit (Suc n) a = take-bit n (a div 2) * 2 + a mod 2*›
⟨*proof*⟩

lemma *take-bit-rec*:
‹*take-bit n a = (if n = 0 then 0 else take-bit (n − 1) (a div 2) * 2 + a mod 2)*›
⟨*proof*⟩

lemma *take-bit-Suc-0* [*simp*]:
‹*take-bit (Suc 0) a = a mod 2*›
⟨*proof*⟩

lemma *take-bit-of-0* [*simp*]:
take-bit n 0 = 0
⟨*proof*⟩

lemma *take-bit-of-1* [*simp*]:
take-bit n 1 = of-bool (n > 0)
⟨*proof*⟩

lemma *drop-bit-of-0* [*simp*]:
drop-bit n 0 = 0
⟨*proof*⟩

lemma *drop-bit-of-1* [*simp*]:
drop-bit n 1 = of-bool (n = 0)
⟨*proof*⟩

lemma *drop-bit-0* [*simp*]:
drop-bit 0 = id
⟨*proof*⟩

lemma *drop-bit-Suc*:
drop-bit (Suc n) a = drop-bit n (a div 2)
⟨*proof*⟩

lemma *drop-bit-rec*:
drop-bit n a = (if n = 0 then a else drop-bit (n − 1) (a div 2))
⟨*proof*⟩

lemma *drop-bit-half*:
drop-bit n (a div 2) = drop-bit n a div 2
⟨*proof*⟩

lemma *drop-bit-of-bool* [*simp*]:
drop-bit n (of-bool b) = of-bool (n = 0 ∧ b)
⟨*proof*⟩

lemma *even-take-bit-eq* [*simp*]:
‹*even (take-bit n a) ⟷ n = 0 ∨ even a*›
⟨*proof*⟩

lemma *take-bit-take-bit* [*simp*]:
take-bit m (take-bit n a) = take-bit (min m n) a
⟨*proof*⟩

lemma *drop-bit-drop-bit* [*simp*]:
drop-bit m (drop-bit n a) = drop-bit (m + n) a
⟨*proof*⟩

lemma *push-bit-take-bit*:
push-bit m (take-bit n a) = take-bit (m + n) (push-bit m a)
⟨*proof*⟩

lemma *take-bit-push-bit*:
take-bit m (push-bit n a) = push-bit n (take-bit (m − n) a)
⟨*proof*⟩

lemma *take-bit-drop-bit*:
take-bit m (drop-bit n a) = drop-bit n (take-bit (m + n) a)
⟨*proof*⟩

lemma *drop-bit-take-bit*:
drop-bit m (take-bit n a) = take-bit (n − m) (drop-bit m a)
⟨*proof*⟩

lemma *even-push-bit-iff* [*simp*]:
‹*even (push-bit n a) ⟷ n ≠ 0 ∨ even a*›
⟨*proof*⟩

lemma *bit-push-bit-iff* [*bit-simps*]:
‹*bit (push-bit m a) n ⟷ m ≤ n ∧ possible-bit TYPE('a) n ∧ bit a (n − m)*›
⟨*proof*⟩

lemma *bit-drop-bit-eq* [*bit-simps*]:
‹*bit (drop-bit n a) = bit a ∘ (+) n*›
⟨*proof*⟩

lemma *bit-take-bit-iff* [*bit-simps*]:
‹*bit (take-bit m a) n ⟷ n < m ∧ bit a n*›
⟨*proof*⟩

lemma *stable-imp-drop-bit-eq*:
‹*drop-bit n a = a*›
if ‹*a div 2 = a*›
⟨*proof*⟩

lemma *stable-imp-take-bit-eq*:
‹*take-bit n a = (if even a then 0 else 2 ^ n − 1)*›
if ‹*a div 2 = a*›
⟨*proof*⟩

lemma *exp-dvdE*:
assumes ‹*2 ^ n dvd a*›
obtains *b* **where** ‹*a = push-bit n b*›
⟨*proof*⟩

lemma *take-bit-eq-0-iff*:
‹*take-bit n a = 0 ⟷ 2 ^ n dvd a*› (**is** ‹*?P ⟷ ?Q*›)
⟨*proof*⟩

lemma *take-bit-tightened*:
‹*take-bit m a = take-bit m b*› **if** ‹*take-bit n a = take-bit n b*› **and** ‹*m ≤ n*›
⟨*proof*⟩

lemma *take-bit-eq-self-iff-drop-bit-eq-0*:
‹*take-bit n a = a ⟷ drop-bit n a = 0*› (**is** ‹*?P ⟷ ?Q*›)

⟨*proof*⟩

lemma *drop-bit-exp-eq*:
‹*drop-bit m* (*2* ^ *n*) = *of-bool* (*m* ≤ *n* ∧ *possible-bit TYPE*('*a*) *n*) * *2* ^ (*n* − *m*)›
⟨*proof*⟩

lemma *take-bit-and* [*simp*]:
‹*take-bit n* (*a AND b*) = *take-bit n a AND take-bit n b*›
⟨*proof*⟩

lemma *take-bit-or* [*simp*]:
‹*take-bit n* (*a OR b*) = *take-bit n a OR take-bit n b*›
⟨*proof*⟩

lemma *take-bit-xor* [*simp*]:
‹*take-bit n* (*a XOR b*) = *take-bit n a XOR take-bit n b*›
⟨*proof*⟩

lemma *push-bit-and* [*simp*]:
‹*push-bit n* (*a AND b*) = *push-bit n a AND push-bit n b*›
⟨*proof*⟩

lemma *push-bit-or* [*simp*]:
‹*push-bit n* (*a OR b*) = *push-bit n a OR push-bit n b*›
⟨*proof*⟩

lemma *push-bit-xor* [*simp*]:
‹*push-bit n* (*a XOR b*) = *push-bit n a XOR push-bit n b*›
⟨*proof*⟩

lemma *drop-bit-and* [*simp*]:
‹*drop-bit n* (*a AND b*) = *drop-bit n a AND drop-bit n b*›
⟨*proof*⟩

lemma *drop-bit-or* [*simp*]:
‹*drop-bit n* (*a OR b*) = *drop-bit n a OR drop-bit n b*›
⟨*proof*⟩

lemma *drop-bit-xor* [*simp*]:
‹*drop-bit n* (*a XOR b*) = *drop-bit n a XOR drop-bit n b*›
⟨*proof*⟩

lemma *bit-mask-iff* [*bit-simps*]:
‹*bit* (*mask m*) *n* ⟷ *possible-bit TYPE*('*a*) *n* ∧ *n* < *m*›
⟨*proof*⟩

lemma *even-mask-iff*:
‹*even* (*mask n*) ⟷ *n* = *0*›
⟨*proof*⟩

lemma *mask-0* [*simp*]:
‹*mask 0 = 0*›
⟨*proof*⟩

lemma *mask-Suc-0* [*simp*]:
‹*mask (Suc 0) = 1*›
⟨*proof*⟩

lemma *mask-Suc-exp*:
‹*mask (Suc n) = 2 ^ n OR mask n*›
⟨*proof*⟩

lemma *mask-Suc-double*:
‹*mask (Suc n) = 1 OR 2 * mask n*›
⟨*proof*⟩

lemma *mask-numeral*:
‹*mask (numeral n) = 1 + 2 * mask (pred-numeral n)*›
⟨*proof*⟩

lemma *take-bit-of-mask* [*simp*]:
‹*take-bit m (mask n) = mask (min m n)*›
⟨*proof*⟩

lemma *take-bit-eq-mask*:
‹*take-bit n a = a AND mask n*›
⟨*proof*⟩

lemma *or-eq-0-iff*:
‹*a OR b = 0 ⟷ a = 0 ∧ b = 0*›
⟨*proof*⟩

lemma *disjunctive-add*:
‹*a + b = a OR b*› **if** ‹⋀*n*. ¬ *bit a n* ∨ ¬ *bit b n*›
⟨*proof*⟩

lemma *bit-iff-and-drop-bit-eq-1*:
‹*bit a n ⟷ drop-bit n a AND 1 = 1*›
⟨*proof*⟩

lemma *bit-iff-and-push-bit-not-eq-0*:
‹*bit a n ⟷ a AND push-bit n 1 ≠ 0*›
⟨*proof*⟩

lemmas *set-bit-def = set-bit-eq-or*

lemma *bit-set-bit-iff* [*bit-simps*]:
‹*bit (set-bit m a) n ⟷ bit a n ∨ (m = n ∧ possible-bit TYPE('a) n)*›

⟨*proof*⟩

lemma *even-set-bit-iff*:
‹*even* (*set-bit m a*) ⟷ *even a* ∧ *m* ≠ *0*›
⟨*proof*⟩

lemma *even-unset-bit-iff*:
‹*even* (*unset-bit m a*) ⟷ *even a* ∨ *m* = *0*›
⟨*proof*⟩

lemma *and-exp-eq-0-iff-not-bit*:
‹*a AND* 2 ^ *n* = *0* ⟷ ¬ *bit a n*› (**is** ‹*?P* ⟷ *?Q*›)
⟨*proof*⟩

lemmas *flip-bit-def* = *flip-bit-eq-xor*

lemma *bit-flip-bit-iff* [*bit-simps*]:
‹*bit* (*flip-bit m a*) *n* ⟷ (*m* = *n* ⟷ ¬ *bit a n*) ∧ *possible-bit TYPE*('*a*) *n*›
⟨*proof*⟩

lemma *even-flip-bit-iff*:
‹*even* (*flip-bit m a*) ⟷ ¬ (*even a* ⟷ *m* = *0*)›
⟨*proof*⟩

lemma *set-bit-0* [*simp*]:
‹*set-bit 0 a* = *1* + *2* ∗ (*a div 2*)›
⟨*proof*⟩

lemma *bit-sum-mult-2-cases*:
assumes *a*: ∀ *j*. ¬ *bit a* (*Suc j*)
shows *bit* (*a* + *2* ∗ *b*) *n* = (*if n* = *0 then odd a else bit* (*2* ∗ *b*) *n*)
⟨*proof*⟩

lemma *set-bit-Suc*:
‹*set-bit* (*Suc n*) *a* = *a mod 2* + *2* ∗ *set-bit n* (*a div 2*)›
⟨*proof*⟩

lemma *unset-bit-0* [*simp*]:
‹*unset-bit 0 a* = *2* ∗ (*a div 2*)›
⟨*proof*⟩

lemma *unset-bit-Suc*:
‹*unset-bit* (*Suc n*) *a* = *a mod 2* + *2* ∗ *unset-bit n* (*a div 2*)›
⟨*proof*⟩

lemma *flip-bit-0* [*simp*]:
‹*flip-bit 0 a* = *of-bool* (*even a*) + *2* ∗ (*a div 2*)›
⟨*proof*⟩

lemma *flip-bit-Suc*:
‹*flip-bit* (*Suc n*) *a* = *a mod 2* + *2* ∗ *flip-bit n* (*a div 2*)›
⟨*proof*⟩

lemma *flip-bit-eq-if*:
‹*flip-bit n a* = (*if bit a n then unset-bit else set-bit*) *n a*›
⟨*proof*⟩

lemma *take-bit-set-bit-eq*:
‹*take-bit n* (*set-bit m a*) = (*if n* ≤ *m then take-bit n a else set-bit m* (*take-bit n a*))›
⟨*proof*⟩

lemma *take-bit-unset-bit-eq*:
‹*take-bit n* (*unset-bit m a*) = (*if n* ≤ *m then take-bit n a else unset-bit m* (*take-bit n a*))›
⟨*proof*⟩

lemma *take-bit-flip-bit-eq*:
‹*take-bit n* (*flip-bit m a*) = (*if n* ≤ *m then take-bit n a else flip-bit m* (*take-bit n a*))›
⟨*proof*⟩

lemma *bit-1-0* [*simp*]:
‹*bit 1 0*›
⟨*proof*⟩

lemma *not-bit-1-Suc* [*simp*]:
‹¬ *bit 1* (*Suc n*)›
⟨*proof*⟩

lemma *push-bit-Suc-numeral* [*simp*]:
‹*push-bit* (*Suc n*) (*numeral k*) = *push-bit n* (*numeral* (*Num.Bit0 k*))›
⟨*proof*⟩

lemma *mask-eq-0-iff* [*simp*]:
‹*mask n* = *0* ⟷ *n* = *0*›
⟨*proof*⟩

end

class *ring-bit-operations* = *semiring-bit-operations* + *ring-parity* +
fixes *not* :: ‹'*a* ⇒ '*a*› (‹*NOT*›)
assumes *bit-not-iff-eq*: ‹⋀*n*. *bit* (*NOT a*) *n* ⟷ *2* ^ *n* ≠ *0* ∧ ¬ *bit a n*›
assumes *minus-eq-not-minus-1*: ‹− *a* = *NOT* (*a* − *1*)›
begin

lemmas *bit-not-iff*[*bit-simps*] = *bit-not-iff-eq*[*unfolded fold-possible-bit*]

For the sake of code generation *NOT* is specified as definitional class op-

eration. Note that *NOT* has no sensible definition for unlimited but only positive bit strings (type *nat*).

lemma *bits-minus-1-mod-2-eq* [*simp*]:
‹(− 1) mod 2 = 1›
⟨*proof*⟩

lemma *not-eq-complement*:
‹NOT a = − a − 1›
⟨*proof*⟩

lemma *minus-eq-not-plus-1*:
‹− a = NOT a + 1›
⟨*proof*⟩

lemma *bit-minus-iff* [*bit-simps*]:
‹bit (− a) n ⟷ possible-bit TYPE('a) n ∧ ¬ bit (a − 1) n›
⟨*proof*⟩

lemma *even-not-iff* [*simp*]:
‹even (NOT a) ⟷ odd a›
⟨*proof*⟩

lemma *bit-not-exp-iff* [*bit-simps*]:
‹bit (NOT (2 ^ m)) n ⟷ possible-bit TYPE('a) n ∧ n ≠ m›
⟨*proof*⟩

lemma *bit-minus-1-iff* [*simp*]:
‹bit (− 1) n ⟷ possible-bit TYPE('a) n›
⟨*proof*⟩

lemma *bit-minus-exp-iff* [*bit-simps*]:
‹bit (− (2 ^ m)) n ⟷ possible-bit TYPE('a) n ∧ n ≥ m›
⟨*proof*⟩

lemma *bit-minus-2-iff* [*simp*]:
‹bit (− 2) n ⟷ possible-bit TYPE('a) n ∧ n > 0›
⟨*proof*⟩

lemma *not-one-eq* [*simp*]:
‹NOT 1 = − 2›
⟨*proof*⟩

sublocale *and*: *semilattice-neutr* ‹(AND)› ‹− 1›
⟨*proof*⟩

sublocale *bit*: *abstract-boolean-algebra* ‹(AND)› ‹(OR)› *NOT 0* ‹− 1›
⟨*proof*⟩

sublocale *bit*: *abstract-boolean-algebra-sym-diff* ‹(AND)› ‹(OR)› *NOT 0* ‹− 1›

‹(*XOR*)›
⟨*proof*⟩

lemma *and-eq-not-not-or*:
‹*a AND b = NOT (NOT a OR NOT b)*›
⟨*proof*⟩

lemma *or-eq-not-not-and*:
‹*a OR b = NOT (NOT a AND NOT b)*›
⟨*proof*⟩

lemma *not-add-distrib*:
‹*NOT (a + b) = NOT a − b*›
⟨*proof*⟩

lemma *not-diff-distrib*:
‹*NOT (a − b) = NOT a + b*›
⟨*proof*⟩

lemma *and-eq-minus-1-iff*:
‹*a AND b = − 1 ⟷ a = − 1 ∧ b = − 1*›
⟨*proof*⟩

lemma *disjunctive-diff*:
‹*a − b = a AND NOT b*› **if** ‹⋀*n*. *bit b n ⟹ bit a n*›
⟨*proof*⟩

lemma *push-bit-minus*:
‹*push-bit n (− a) = − push-bit n a*›
⟨*proof*⟩

lemma *take-bit-not-take-bit*:
‹*take-bit n (NOT (take-bit n a)) = take-bit n (NOT a)*›
⟨*proof*⟩

lemma *take-bit-not-iff*:
‹*take-bit n (NOT a) = take-bit n (NOT b) ⟷ take-bit n a = take-bit n b*›
⟨*proof*⟩

lemma *take-bit-not-eq-mask-diff*:
‹*take-bit n (NOT a) = mask n − take-bit n a*›
⟨*proof*⟩

lemma *mask-eq-take-bit-minus-one*:
‹*mask n = take-bit n (− 1)*›
⟨*proof*⟩

lemma *take-bit-minus-one-eq-mask* [*simp*]:
‹*take-bit n (− 1) = mask n*›

⟨*proof*⟩

lemma *minus-exp-eq-not-mask*:
‹− (2 ^ *n*) = *NOT* (*mask n*)›
⟨*proof*⟩

lemma *push-bit-minus-one-eq-not-mask* [*simp*]:
‹*push-bit n* (− *1*) = *NOT* (*mask n*)›
⟨*proof*⟩

lemma *take-bit-not-mask-eq-0*:
‹*take-bit m* (*NOT* (*mask n*)) = *0*› **if** ‹*n* ≥ *m*›
⟨*proof*⟩

lemma *unset-bit-eq-and-not*:
‹*unset-bit n a* = *a AND NOT* (*push-bit n 1*)›
⟨*proof*⟩

lemmas *unset-bit-def* = *unset-bit-eq-and-not*

lemma *push-bit-Suc-minus-numeral* [*simp*]:
‹*push-bit* (*Suc n*) (− *numeral k*) = *push-bit n* (− *numeral* (*Num.Bit0 k*))›
⟨*proof*⟩

lemma *push-bit-minus-numeral* [*simp*]:
‹*push-bit* (*numeral l*) (− *numeral k*) = *push-bit* (*pred-numeral l*) (− *numeral* (*Num.Bit0 k*))›
⟨*proof*⟩

lemma *take-bit-Suc-minus-1-eq*:
‹*take-bit* (*Suc n*) (− *1*) = *2* ^ *Suc n* − *1*›
⟨*proof*⟩

lemma *take-bit-numeral-minus-1-eq*:
‹*take-bit* (*numeral k*) (− *1*) = *2* ^ *numeral k* − *1*›
⟨*proof*⟩

lemma *push-bit-mask-eq*:
‹*push-bit m* (*mask n*) = *mask* (*n* + *m*) *AND NOT* (*mask m*)›
⟨*proof*⟩

lemma *slice-eq-mask*:
‹*push-bit n* (*take-bit m* (*drop-bit n a*)) = *a AND mask* (*m* + *n*) *AND NOT* (*mask n*)›
⟨*proof*⟩

lemma *push-bit-numeral-minus-1* [*simp*]:
‹*push-bit* (*numeral n*) (− *1*) = − (*2* ^ *numeral n*)›
⟨*proof*⟩

end

69.3 Instance *int*

instantiation *int* :: *ring-bit-operations*
begin

definition *not-int* :: ‹*int* ⇒ *int*›
  where ‹*not-int k* = − *k* − *1*›

lemma *not-int-rec*:
  ‹*NOT k* = *of-bool* (*even k*) + *2* ∗ *NOT* (*k div 2*)› **for** *k* :: *int*
  ⟨*proof*⟩

lemma *even-not-iff-int*:
  ‹*even* (*NOT k*) ⟷ *odd k*› **for** *k* :: *int*
  ⟨*proof*⟩

lemma *not-int-div-2*:
  ‹*NOT k div 2* = *NOT* (*k div 2*)› **for** *k* :: *int*
  ⟨*proof*⟩

lemma *bit-not-int-iff*:
  ‹*bit* (*NOT k*) *n* ⟷ ¬ *bit k n*›
  for *k* :: *int*
  ⟨*proof*⟩

function *and-int* :: ‹*int* ⇒ *int* ⇒ *int*›
  where ‹(*k*::*int*) *AND l* = (*if k* ∈ {*0*, − *1*} ∧ *l* ∈ {*0*, − *1*}
    then − *of-bool* (*odd k* ∧ *odd l*)
    else of-bool (*odd k* ∧ *odd l*) + *2* ∗ ((*k div 2*) *AND* (*l div 2*)))›
  ⟨*proof*⟩

termination ⟨*proof*⟩

declare *and-int.simps* [*simp del*]

lemma *and-int-rec*:
  ‹*k AND l* = *of-bool* (*odd k* ∧ *odd l*) + *2* ∗ ((*k div 2*) *AND* (*l div 2*))›
    for *k l* :: *int*
⟨*proof*⟩

lemma *bit-and-int-iff*:
  ‹*bit* (*k AND l*) *n* ⟷ *bit k n* ∧ *bit l n*› **for** *k l* :: *int*
⟨*proof*⟩

lemma *even-and-iff-int*:
  ‹*even* (*k AND l*) ⟷ *even k* ∨ *even l*› **for** *k l* :: *int*

⟨*proof*⟩

definition *or-int* :: ‹*int* ⇒ *int* ⇒ *int*›
where ‹*k OR l* = *NOT* (*NOT k AND NOT l*)› **for** *k l* :: *int*

lemma *or-int-rec*:
‹*k OR l* = *of-bool* (*odd k* ∨ *odd l*) + 2 ∗ ((*k div* 2) *OR* (*l div* 2))›
for *k l* :: *int*
⟨*proof*⟩

lemma *bit-or-int-iff*:
‹*bit* (*k OR l*) *n* ⟷ *bit k n* ∨ *bit l n*› **for** *k l* :: *int*
⟨*proof*⟩

definition *xor-int* :: ‹*int* ⇒ *int* ⇒ *int*›
where ‹*k XOR l* = *k AND NOT l OR NOT k AND l*› **for** *k l* :: *int*

lemma *xor-int-rec*:
‹*k XOR l* = *of-bool* (*odd k* ≠ *odd l*) + 2 ∗ ((*k div* 2) *XOR* (*l div* 2))›
for *k l* :: *int*
⟨*proof*⟩

lemma *bit-xor-int-iff*:
‹*bit* (*k XOR l*) *n* ⟷ *bit k n* ≠ *bit l n*› **for** *k l* :: *int*
⟨*proof*⟩

definition *mask-int* :: ‹*nat* ⇒ *int*›
where ‹*mask n* = (2 :: *int*) ^ *n* − 1›

definition *push-bit-int* :: ‹*nat* ⇒ *int* ⇒ *int*›
where ‹*push-bit-int n k* = *k* ∗ 2 ^ *n*›

definition *drop-bit-int* :: ‹*nat* ⇒ *int* ⇒ *int*›
where ‹*drop-bit-int n k* = *k div* 2 ^ *n*›

definition *take-bit-int* :: ‹*nat* ⇒ *int* ⇒ *int*›
where ‹*take-bit-int n k* = *k mod* 2 ^ *n*›

definition *set-bit-int* :: ‹*nat* ⇒ *int* ⇒ *int*›
where ‹*set-bit n k* = *k OR push-bit n* 1› **for** *k* :: *int*

definition *unset-bit-int* :: ‹*nat* ⇒ *int* ⇒ *int*›
where ‹*unset-bit n k* = *k AND NOT* (*push-bit n* 1)› **for** *k* :: *int*

definition *flip-bit-int* :: ‹*nat* ⇒ *int* ⇒ *int*›
where ‹*flip-bit n k* = *k XOR push-bit n* 1› **for** *k* :: *int*

instance ⟨*proof*⟩

end

lemma *bit-push-bit-iff-int*:
‹*bit* (*push-bit m k*) *n* ⟷ *m* ≤ *n* ∧ *bit k* (*n* − *m*)› **for** *k* :: *int*
⟨*proof*⟩

lemma *take-bit-nonnegative* [*simp*]:
‹*take-bit n k* ≥ *0*› **for** *k* :: *int*
⟨*proof*⟩

lemma *not-take-bit-negative* [*simp*]:
‹¬ *take-bit n k* < *0*› **for** *k* :: *int*
⟨*proof*⟩

lemma *take-bit-int-less-exp* [*simp*]:
‹*take-bit n k* < *2* ^ *n*› **for** *k* :: *int*
⟨*proof*⟩

lemma *take-bit-int-eq-self-iff*:
‹*take-bit n k* = *k* ⟷ *0* ≤ *k* ∧ *k* < *2* ^ *n*› (**is** ‹*?P* ⟷ *?Q*›)
for *k* :: *int*
⟨*proof*⟩

lemma *take-bit-int-eq-self*:
‹*take-bit n k* = *k*› **if** ‹*0* ≤ *k*› ‹*k* < *2* ^ *n*› **for** *k* :: *int*
⟨*proof*⟩

lemma *mask-half-int*:
‹*mask n div 2* = (*mask* (*n* − *1*) :: *int*)›
⟨*proof*⟩

lemma *mask-nonnegative-int* [*simp*]:
‹*mask n* ≥ (*0*::*int*)›
⟨*proof*⟩

lemma *not-mask-negative-int* [*simp*]:
‹¬ *mask n* < (*0*::*int*)›
⟨*proof*⟩

lemma *not-nonnegative-int-iff* [*simp*]:
‹*NOT k* ≥ *0* ⟷ *k* < *0*› **for** *k* :: *int*
⟨*proof*⟩

lemma *not-negative-int-iff* [*simp*]:
‹*NOT k* < *0* ⟷ *k* ≥ *0*› **for** *k* :: *int*
⟨*proof*⟩

lemma *and-nonnegative-int-iff* [*simp*]:
‹*k AND l* ≥ *0* ⟷ *k* ≥ *0* ∨ *l* ≥ *0*› **for** *k l* :: *int*

⟨*proof*⟩

lemma *and-negative-int-iff* [*simp*]:
‹*k AND l* < *0* ⟷ *k* < *0* ∧ *l* < *0*› **for** *k l* :: *int*
⟨*proof*⟩

lemma *and-less-eq*:
‹*k AND l* ≤ *k*› **if** ‹*l* < *0*› **for** *k l* :: *int*
⟨*proof*⟩

lemma *or-nonnegative-int-iff* [*simp*]:
‹*k OR l* ≥ *0* ⟷ *k* ≥ *0* ∧ *l* ≥ *0*› **for** *k l* :: *int*
⟨*proof*⟩

lemma *or-negative-int-iff* [*simp*]:
‹*k OR l* < *0* ⟷ *k* < *0* ∨ *l* < *0*› **for** *k l* :: *int*
⟨*proof*⟩

lemma *or-greater-eq*:
‹*k OR l* ≥ *k*› **if** ‹*l* ≥ *0*› **for** *k l* :: *int*
⟨*proof*⟩

lemma *xor-nonnegative-int-iff* [*simp*]:
‹*k XOR l* ≥ *0* ⟷ (*k* ≥ *0* ⟷ *l* ≥ *0*)› **for** *k l* :: *int*
⟨*proof*⟩

lemma *xor-negative-int-iff* [*simp*]:
‹*k XOR l* < *0* ⟷ (*k* < *0*) ≠ (*l* < *0*)› **for** *k l* :: *int*
⟨*proof*⟩

lemma *OR-upper*:
fixes *x y* :: *int*
assumes ‹*0* ≤ *x*› ‹*x* < *2* ^ *n*› ‹*y* < *2* ^ *n*›
shows ‹*x OR y* < *2* ^ *n*›
⟨*proof*⟩

lemma *XOR-upper*:
fixes *x y* :: *int*
assumes ‹*0* ≤ *x*› ‹*x* < *2* ^ *n*› ‹*y* < *2* ^ *n*›
shows ‹*x XOR y* < *2* ^ *n*›
⟨*proof*⟩

lemma *AND-lower* [*simp*]:
fixes *x y* :: *int*
assumes ‹*0* ≤ *x*›
shows ‹*0* ≤ *x AND y*›
⟨*proof*⟩

lemma *OR-lower* [*simp*]:

fixes *x y* :: *int*
assumes ‹*0* ≤ *x*› ‹*0* ≤ *y*›
shows ‹*0* ≤ *x OR y*›
⟨*proof*⟩

lemma *XOR-lower* [*simp*]:
fixes *x y* :: *int*
assumes ‹*0* ≤ *x*› ‹*0* ≤ *y*›
shows ‹*0* ≤ *x XOR y*›
⟨*proof*⟩

lemma *AND-upper1* [*simp*]:
fixes *x y* :: *int*
assumes ‹*0* ≤ *x*›
shows ‹*x AND y* ≤ *x*›
⟨*proof*⟩

lemmas *AND-upper1'* [*simp*] = *order-trans* [*OF AND-upper1*]
lemmas *AND-upper1''* [*simp*] = *order-le-less-trans* [*OF AND-upper1*]

lemma *AND-upper2* [*simp*]:
fixes *x y* :: *int*
assumes ‹*0* ≤ *y*›
shows ‹*x AND y* ≤ *y*›
⟨*proof*⟩

lemmas *AND-upper2'* [*simp*] = *order-trans* [*OF AND-upper2*]
lemmas *AND-upper2''* [*simp*] = *order-le-less-trans* [*OF AND-upper2*]

lemma *plus-and-or*: ‹(*x AND y*) + (*x OR y*) = *x* + *y*› **for** *x y* :: *int*
⟨*proof*⟩

lemma *push-bit-minus-one*:
push-bit n (− *1* :: *int*) = − (*2* ^ *n*)
⟨*proof*⟩

lemma *minus-1-div-exp-eq-int*:
‹− *1 div* (*2* :: *int*) ^ *n* = − *1*›
⟨*proof*⟩

lemma *drop-bit-minus-one* [*simp*]:
‹*drop-bit n* (− *1* :: *int*) = − *1*›
⟨*proof*⟩

lemma *take-bit-Suc-from-most*:
‹*take-bit* (*Suc n*) *k* = *2* ^ *n* ∗ *of-bool* (*bit k n*) + *take-bit n k*› **for** *k* :: *int*
⟨*proof*⟩

lemma *take-bit-minus*:

‹*take-bit n* (− *take-bit n k*) = *take-bit n* (− *k*)›
for *k* :: *int*
⟨*proof*⟩

lemma *take-bit-diff*:
‹*take-bit n* (*take-bit n k* − *take-bit n l*) = *take-bit n* (*k* − *l*)›
for *k l* :: *int*
⟨*proof*⟩

lemma *bit-imp-take-bit-positive*:
‹*0* < *take-bit m k*› **if** ‹*n* < *m*› **and** ‹*bit k n*› **for** *k* :: *int*
⟨*proof*⟩

lemma *take-bit-mult*:
‹*take-bit n* (*take-bit n k* ∗ *take-bit n l*) = *take-bit n* (*k* ∗ *l*)›
for *k l* :: *int*
⟨*proof*⟩

lemma (**in** *ring-1*) *of-nat-nat-take-bit-eq* [*simp*]:
‹*of-nat* (*nat* (*take-bit n k*)) = *of-int* (*take-bit n k*)›
⟨*proof*⟩

lemma *take-bit-minus-small-eq*:
‹*take-bit n* (− *k*) = *2* ^ *n* − *k*› **if** ‹*0* < *k*› ‹*k* ≤ *2* ^ *n*› **for** *k* :: *int*
⟨*proof*⟩

lemma *drop-bit-push-bit-int*:
‹*drop-bit m* (*push-bit n k*) = *drop-bit* (*m* − *n*) (*push-bit* (*n* − *m*) *k*)› **for** *k* :: *int*
⟨*proof*⟩

lemma *push-bit-nonnegative-int-iff* [*simp*]:
‹*push-bit n k* ≥ *0* ⟷ *k* ≥ *0*› **for** *k* :: *int*
⟨*proof*⟩

lemma *push-bit-negative-int-iff* [*simp*]:
‹*push-bit n k* < *0* ⟷ *k* < *0*› **for** *k* :: *int*
⟨*proof*⟩

lemma *drop-bit-nonnegative-int-iff* [*simp*]:
‹*drop-bit n k* ≥ *0* ⟷ *k* ≥ *0*› **for** *k* :: *int*
⟨*proof*⟩

lemma *drop-bit-negative-int-iff* [*simp*]:
‹*drop-bit n k* < *0* ⟷ *k* < *0*› **for** *k* :: *int*
⟨*proof*⟩

lemma *set-bit-nonnegative-int-iff* [*simp*]:
‹*set-bit n k* ≥ *0* ⟷ *k* ≥ *0*› **for** *k* :: *int*
⟨*proof*⟩

lemma *set-bit-negative-int-iff* [*simp*]:
‹*set-bit n k* < *0* ⟷ *k* < *0*› **for** *k* :: *int*
⟨*proof*⟩

lemma *unset-bit-nonnegative-int-iff* [*simp*]:
‹*unset-bit n k* ≥ *0* ⟷ *k* ≥ *0*› **for** *k* :: *int*
⟨*proof*⟩

lemma *unset-bit-negative-int-iff* [*simp*]:
‹*unset-bit n k* < *0* ⟷ *k* < *0*› **for** *k* :: *int*
⟨*proof*⟩

lemma *flip-bit-nonnegative-int-iff* [*simp*]:
‹*flip-bit n k* ≥ *0* ⟷ *k* ≥ *0*› **for** *k* :: *int*
⟨*proof*⟩

lemma *flip-bit-negative-int-iff* [*simp*]:
‹*flip-bit n k* < *0* ⟷ *k* < *0*› **for** *k* :: *int*
⟨*proof*⟩

lemma *set-bit-greater-eq*:
‹*set-bit n k* ≥ *k*› **for** *k* :: *int*
⟨*proof*⟩

lemma *unset-bit-less-eq*:
‹*unset-bit n k* ≤ *k*› **for** *k* :: *int*
⟨*proof*⟩

lemma *set-bit-eq*:
‹*set-bit n k* = *k* + *of-bool* (¬ *bit k n*) ∗ *2* ^ *n*› **for** *k* :: *int*
⟨*proof*⟩

lemma *unset-bit-eq*:
‹*unset-bit n k* = *k* − *of-bool* (*bit k n*) ∗ *2* ^ *n*› **for** *k* :: *int*
⟨*proof*⟩

lemma *and-int-unfold*:
‹*k AND l* = (*if k* = *0* ∨ *l* = *0 then 0 else if k* = − *1 then l else if l* = − *1 then k*
else (*k mod 2*) ∗ (*l mod 2*) + *2* ∗ ((*k div 2*) *AND* (*l div 2*)))› **for** *k l* :: *int*
⟨*proof*⟩

lemma *or-int-unfold*:
‹*k OR l* = (*if k* = − *1* ∨ *l* = − *1 then* − *1 else if k* = *0 then l else if l* = *0 then*
k
else max (*k mod 2*) (*l mod 2*) + *2* ∗ ((*k div 2*) *OR* (*l div 2*)))› **for** *k l* :: *int*
⟨*proof*⟩

lemma *xor-int-unfold*:

‹*k XOR l = (if k = − 1 then NOT l else if l = − 1 then NOT k else if k = 0 then l else if l = 0 then k*
*else |k mod 2 − l mod 2| + 2 * ((k div 2) XOR (l div 2)))*› **for** *k l* :: *int*
⟨*proof*⟩

lemma *bit-minus-int-iff*:
‹*bit (− k) n ⟷ bit (NOT (k − 1)) n*›
for *k* :: *int*
⟨*proof*⟩

lemma *take-bit-incr-eq*:
‹*take-bit n (k + 1) = 1 + take-bit n k*› **if** ‹*take-bit n k ≠ 2 ^ n − 1*›
for *k* :: *int*
⟨*proof*⟩

lemma *take-bit-decr-eq*:
‹*take-bit n (k − 1) = take-bit n k − 1*› **if** ‹*take-bit n k ≠ 0*›
for *k* :: *int*
⟨*proof*⟩

lemma *take-bit-int-greater-eq*:
‹*k + 2 ^ n ≤ take-bit n k*› **if** ‹*k < 0*› **for** *k* :: *int*
⟨*proof*⟩

lemma *take-bit-int-less-eq*:
‹*take-bit n k ≤ k − 2 ^ n*› **if** ‹*2 ^ n ≤ k*› **and** ‹*n > 0*› **for** *k* :: *int*
⟨*proof*⟩

lemma *take-bit-int-less-eq-self-iff*:
‹*take-bit n k ≤ k ⟷ 0 ≤ k*› (**is** ‹*?P ⟷ ?Q*›)
for *k* :: *int*
⟨*proof*⟩

lemma *take-bit-int-less-self-iff*:
‹*take-bit n k < k ⟷ 2 ^ n ≤ k*›
for *k* :: *int*
⟨*proof*⟩

lemma *take-bit-int-greater-self-iff*:
‹*k < take-bit n k ⟷ k < 0*›
for *k* :: *int*
⟨*proof*⟩

lemma *take-bit-int-greater-eq-self-iff*:
‹*k ≤ take-bit n k ⟷ k < 2 ^ n*›
for *k* :: *int*
⟨*proof*⟩

lemma *not-exp-less-eq-0-int* [*simp*]:

‹¬ 2 ^ $n \leq (0::int)$›
⟨*proof*⟩

lemma *int-bit-bound*:
fixes $k :: int$
obtains n **where** ‹$\bigwedge m.\ n \leq m \Longrightarrow bit\ k\ m \longleftrightarrow bit\ k\ n$›
and ‹$n > 0 \Longrightarrow bit\ k\ (n - 1) \neq bit\ k\ n$›
⟨*proof*⟩

lemma *take-bit-tightened-less-eq-int*:
‹*take-bit* $m\ k \leq$ *take-bit* $n\ k$› **if** ‹$m \leq n$› **for** $k :: int$
⟨*proof*⟩

context *ring-bit-operations*
begin

lemma *even-of-int-iff*:
‹*even* (*of-int* k) $\longleftrightarrow$ *even* k›
⟨*proof*⟩

lemma *bit-of-int-iff* [*bit-simps*]:
‹*bit* (*of-int* k) $n \longleftrightarrow$ *possible-bit* $TYPE('a)\ n \wedge$ *bit* $k\ n$›
⟨*proof*⟩

lemma *push-bit-of-int*:
‹*push-bit* n (*of-int* k) = *of-int* (*push-bit* $n\ k$)›
⟨*proof*⟩

lemma *of-int-push-bit*:
‹*of-int* (*push-bit* $n\ k$) = *push-bit* n (*of-int* k)›
⟨*proof*⟩

lemma *take-bit-of-int*:
‹*take-bit* n (*of-int* k) = *of-int* (*take-bit* $n\ k$)›
⟨*proof*⟩

lemma *of-int-take-bit*:
‹*of-int* (*take-bit* $n\ k$) = *take-bit* n (*of-int* k)›
⟨*proof*⟩

lemma *of-int-not-eq*:
‹*of-int* ($NOT\ k$) = NOT (*of-int* k)›
⟨*proof*⟩

lemma *of-int-not-numeral*:
‹*of-int* (NOT (*numeral* k)) = NOT (*numeral* k)›
⟨*proof*⟩

lemma *of-int-and-eq*:

‹*of-int* (*k AND l*) = *of-int k AND of-int l*›
⟨*proof*⟩

lemma *of-int-or-eq*:
‹*of-int* (*k OR l*) = *of-int k OR of-int l*›
⟨*proof*⟩

lemma *of-int-xor-eq*:
‹*of-int* (*k XOR l*) = *of-int k XOR of-int l*›
⟨*proof*⟩

lemma *of-int-mask-eq*:
‹*of-int* (*mask n*) = *mask n*›
⟨*proof*⟩

end

69.4 Instance *nat*

instantiation *nat* :: *semiring-bit-operations*
begin

definition *and-nat* :: ‹*nat* ⇒ *nat* ⇒ *nat*›
where ‹*m AND n* = *nat* (*int m AND int n*)› **for** *m n* :: *nat*

definition *or-nat* :: ‹*nat* ⇒ *nat* ⇒ *nat*›
where ‹*m OR n* = *nat* (*int m OR int n*)› **for** *m n* :: *nat*

definition *xor-nat* :: ‹*nat* ⇒ *nat* ⇒ *nat*›
where ‹*m XOR n* = *nat* (*int m XOR int n*)› **for** *m n* :: *nat*

definition *mask-nat* :: ‹*nat* ⇒ *nat*›
where ‹*mask n* = (*2* :: *nat*) ^ *n* − *1*›

definition *push-bit-nat* :: ‹*nat* ⇒ *nat* ⇒ *nat*›
where ‹*push-bit-nat n m* = *m* ∗ *2* ^ *n*›

definition *drop-bit-nat* :: ‹*nat* ⇒ *nat* ⇒ *nat*›
where ‹*drop-bit-nat n m* = *m div 2* ^ *n*›

definition *take-bit-nat* :: ‹*nat* ⇒ *nat* ⇒ *nat*›
where ‹*take-bit-nat n m* = *m mod 2* ^ *n*›

definition *set-bit-nat* :: ‹*nat* ⇒ *nat* ⇒ *nat*›
where ‹*set-bit m n* = *n OR push-bit m 1*› **for** *m n* :: *nat*

definition *unset-bit-nat* :: ‹*nat* ⇒ *nat* ⇒ *nat*›
where ‹*unset-bit m n* = *nat* (*unset-bit m* (*int n*))› **for** *m n* :: *nat*

definition *flip-bit-nat* :: ‹*nat* ⇒ *nat* ⇒ *nat*›
  where ‹*flip-bit m n* = *n XOR push-bit m 1*› **for** *m n* :: *nat*

instance ⟨*proof*⟩

end

lemma *take-bit-nat-less-exp* [*simp*]:
  ‹*take-bit n m* < *2* ^ *n*› **for** *n m* ::*nat*
  ⟨*proof*⟩

lemma *take-bit-nat-eq-self-iff*:
  ‹*take-bit n m* = *m* ⟷ *m* < *2* ^ *n*› (**is** ‹*?P* ⟷ *?Q*›)
  for *n m* :: *nat*
⟨*proof*⟩

lemma *take-bit-nat-eq-self*:
  ‹*take-bit n m* = *m*› **if** ‹*m* < *2* ^ *n*› **for** *m n* :: *nat*
  ⟨*proof*⟩

lemma *take-bit-nat-less-eq-self* [*simp*]:
  ‹*take-bit n m* ≤ *m*› **for** *n m* :: *nat*
  ⟨*proof*⟩

lemma *take-bit-nat-less-self-iff*:
  ‹*take-bit n m* < *m* ⟷ *2* ^ *n* ≤ *m*› (**is** ‹*?P* ⟷ *?Q*›)
  for *m n* :: *nat*
⟨*proof*⟩

lemma *bit-push-bit-iff-nat*:
  ‹*bit* (*push-bit m q*) *n* ⟷ *m* ≤ *n* ∧ *bit q* (*n* − *m*)› **for** *q* :: *nat*
  ⟨*proof*⟩

lemma *and-nat-rec*:
  ‹*m AND n* = *of-bool* (*odd m* ∧ *odd n*) + *2* ∗ ((*m div 2*) *AND* (*n div 2*))› **for** *m n* :: *nat*
  ⟨*proof*⟩

lemma *or-nat-rec*:
  ‹*m OR n* = *of-bool* (*odd m* ∨ *odd n*) + *2* ∗ ((*m div 2*) *OR* (*n div 2*))› **for** *m n* :: *nat*
  ⟨*proof*⟩

lemma *xor-nat-rec*:
  ‹*m XOR n* = *of-bool* (*odd m* ≠ *odd n*) + *2* ∗ ((*m div 2*) *XOR* (*n div 2*))› **for** *m n* :: *nat*
  ⟨*proof*⟩

lemma *Suc-0-and-eq* [*simp*]:

‹*Suc 0 AND n = n mod 2*›
⟨*proof*⟩

lemma *and-Suc-0-eq* [*simp*]:
‹*n AND Suc 0 = n mod 2*›
⟨*proof*⟩

lemma *Suc-0-or-eq*:
‹*Suc 0 OR n = n + of-bool (even n)*›
⟨*proof*⟩

lemma *or-Suc-0-eq*:
‹*n OR Suc 0 = n + of-bool (even n)*›
⟨*proof*⟩

lemma *Suc-0-xor-eq*:
‹*Suc 0 XOR n = n + of-bool (even n) − of-bool (odd n)*›
⟨*proof*⟩

lemma *xor-Suc-0-eq*:
‹*n XOR Suc 0 = n + of-bool (even n) − of-bool (odd n)*›
⟨*proof*⟩

lemma *and-nat-unfold* [*code*]:
‹*m AND n = (if m = 0 ∨ n = 0 then 0 else (m mod 2) ∗ (n mod 2) + 2 ∗ ((m div 2) AND (n div 2)))*›
for *m n* :: *nat*
⟨*proof*⟩

lemma *or-nat-unfold* [*code*]:
‹*m OR n = (if m = 0 then n else if n = 0 then m*
else max (m mod 2) (n mod 2) + 2 ∗ ((m div 2) OR (n div 2)))› **for** *m n* :: *nat*
⟨*proof*⟩

lemma *xor-nat-unfold* [*code*]:
‹*m XOR n = (if m = 0 then n else if n = 0 then m*
else (m mod 2 + n mod 2) mod 2 + 2 ∗ ((m div 2) XOR (n div 2)))› **for** *m n* :: *nat*
⟨*proof*⟩

lemma [*code*]:
‹*unset-bit 0 m = 2 ∗ (m div 2)*›
‹*unset-bit (Suc n) m = m mod 2 + 2 ∗ unset-bit n (m div 2)*› **for** *m n* :: *nat*
⟨*proof*⟩

lemma *push-bit-of-Suc-0* [*simp*]:
‹*push-bit n (Suc 0) = 2 ^ n*›
⟨*proof*⟩

lemma *take-bit-of-Suc-0* [*simp*]:
‹*take-bit n* (*Suc 0*) = *of-bool* (*0* < *n*)›
⟨*proof*⟩

lemma *drop-bit-of-Suc-0* [*simp*]:
‹*drop-bit n* (*Suc 0*) = *of-bool* (*n* = *0*)›
⟨*proof*⟩

lemma *Suc-mask-eq-exp*:
‹*Suc* (*mask n*) = *2* ^ *n*›
⟨*proof*⟩

lemma *less-eq-mask*:
‹*n* ≤ *mask n*›
⟨*proof*⟩

lemma *less-mask*:
‹*n* < *mask n*› **if** ‹*Suc 0* < *n*›
⟨*proof*⟩

lemma *mask-nat-less-exp* [*simp*]:
‹(*mask n* :: *nat*) < *2* ^ *n*›
⟨*proof*⟩

lemma *mask-nat-positive-iff* [*simp*]:
‹(*0*::*nat*) < *mask n* ⟷ *0* < *n*›
⟨*proof*⟩

lemma *take-bit-tightened-less-eq-nat*:
‹*take-bit m q* ≤ *take-bit n q*› **if** ‹*m* ≤ *n*› **for** *q* :: *nat*
⟨*proof*⟩

lemma *push-bit-nat-eq*:
‹*push-bit n* (*nat k*) = *nat* (*push-bit n k*)›
⟨*proof*⟩

lemma *drop-bit-nat-eq*:
‹*drop-bit n* (*nat k*) = *nat* (*drop-bit n k*)›
⟨*proof*⟩

lemma *take-bit-nat-eq*:
‹*take-bit n* (*nat k*) = *nat* (*take-bit n k*)› **if** ‹*k* ≥ *0*›
⟨*proof*⟩

lemma *nat-take-bit-eq*:
‹*nat* (*take-bit n k*) = *take-bit n* (*nat k*)›
if ‹*k* ≥ *0*›
⟨*proof*⟩

context *semiring-bit-operations*
begin

lemma *push-bit-of-nat*:
‹*push-bit n* (*of-nat m*) = *of-nat* (*push-bit n m*)›
⟨*proof*⟩

lemma *of-nat-push-bit*:
‹*of-nat* (*push-bit m n*) = *push-bit m* (*of-nat n*)›
⟨*proof*⟩

lemma *take-bit-of-nat*:
‹*take-bit n* (*of-nat m*) = *of-nat* (*take-bit n m*)›
⟨*proof*⟩

lemma *of-nat-take-bit*:
‹*of-nat* (*take-bit n m*) = *take-bit n* (*of-nat m*)›
⟨*proof*⟩

end

context *semiring-bit-operations*
begin

lemma *of-nat-and-eq*:
‹*of-nat* (*m AND n*) = *of-nat m AND of-nat n*›
⟨*proof*⟩

lemma *of-nat-or-eq*:
‹*of-nat* (*m OR n*) = *of-nat m OR of-nat n*›
⟨*proof*⟩

lemma *of-nat-xor-eq*:
‹*of-nat* (*m XOR n*) = *of-nat m XOR of-nat n*›
⟨*proof*⟩

lemma *of-nat-mask-eq*:
‹*of-nat* (*mask n*) = *mask n*›
⟨*proof*⟩

end

lemma *nat-mask-eq*:
‹*nat* (*mask n*) = *mask n*›
⟨*proof*⟩

69.5 Common algebraic structure

class *unique-euclidean-semiring-with-bit-operations* =
unique-euclidean-semiring-with-nat + *semiring-bit-operations*
begin

lemma *possible-bit* [*simp*]:
‹*possible-bit TYPE('a) n*›
⟨*proof*⟩

lemma *take-bit-of-exp* [*simp*]:
‹*take-bit m (2 ^ n) = of-bool (n < m) * 2 ^ n*›
⟨*proof*⟩

lemma *take-bit-of-2* [*simp*]:
‹*take-bit n 2 = of-bool (2 ≤ n) * 2*›
⟨*proof*⟩

lemma *push-bit-eq-0-iff* [*simp*]:
push-bit n a = 0 ⟷ a = 0
⟨*proof*⟩

lemma *take-bit-add*:
take-bit n (take-bit n a + take-bit n b) = take-bit n (a + b)
⟨*proof*⟩

lemma *take-bit-of-1-eq-0-iff* [*simp*]:
take-bit n 1 = 0 ⟷ n = 0
⟨*proof*⟩

lemma *drop-bit-Suc-bit0* [*simp*]:
‹*drop-bit (Suc n) (numeral (Num.Bit0 k)) = drop-bit n (numeral k)*›
⟨*proof*⟩

lemma *drop-bit-Suc-bit1* [*simp*]:
‹*drop-bit (Suc n) (numeral (Num.Bit1 k)) = drop-bit n (numeral k)*›
⟨*proof*⟩

lemma *drop-bit-numeral-bit0* [*simp*]:
‹*drop-bit (numeral l) (numeral (Num.Bit0 k)) = drop-bit (pred-numeral l) (numeral k)*›
⟨*proof*⟩

lemma *drop-bit-numeral-bit1* [*simp*]:
‹*drop-bit (numeral l) (numeral (Num.Bit1 k)) = drop-bit (pred-numeral l) (numeral k)*›
⟨*proof*⟩

lemma *take-bit-Suc-1* [*simp*]:
‹*take-bit (Suc n) 1 = 1*›

⟨*proof*⟩

lemma *take-bit-Suc-bit0*:
‹*take-bit* (*Suc n*) (*numeral* (*Num.Bit0 k*)) = *take-bit n* (*numeral k*) ∗ *2*›
⟨*proof*⟩

lemma *take-bit-Suc-bit1*:
‹*take-bit* (*Suc n*) (*numeral* (*Num.Bit1 k*)) = *take-bit n* (*numeral k*) ∗ *2* + *1*›
⟨*proof*⟩

lemma *take-bit-numeral-1* [*simp*]:
‹*take-bit* (*numeral l*) *1* = *1*›
⟨*proof*⟩

lemma *take-bit-numeral-bit0*:
‹*take-bit* (*numeral l*) (*numeral* (*Num.Bit0 k*)) = *take-bit* (*pred-numeral l*) (*numeral k*) ∗ *2*›
⟨*proof*⟩

lemma *take-bit-numeral-bit1*:
‹*take-bit* (*numeral l*) (*numeral* (*Num.Bit1 k*)) = *take-bit* (*pred-numeral l*) (*numeral k*) ∗ *2* + *1*›
⟨*proof*⟩

lemma *bit-of-nat-iff-bit* [*bit-simps*]:
‹*bit* (*of-nat m*) *n* ⟷ *bit m n*›
⟨*proof*⟩

lemma *drop-bit-mask-eq*:
‹*drop-bit m* (*mask n*) = *mask* (*n* − *m*)›
⟨*proof*⟩

lemma *drop-bit-of-nat*:
drop-bit n (*of-nat m*) = *of-nat* (*drop-bit n m*)
⟨*proof*⟩

lemma *of-nat-drop-bit*:
‹*of-nat* (*drop-bit m n*) = *drop-bit m* (*of-nat n*)›
⟨*proof*⟩

lemma *take-bit-sum*:
take-bit n a = (∑ *k* = *0*..<*n*. *push-bit k* (*of-bool* (*bit a k*)))
for *n* :: *nat*
⟨*proof*⟩

end

instance *nat* :: *unique-euclidean-semiring-with-bit-operations* ⟨*proof*⟩

instance *int* :: *unique-euclidean-semiring-with-bit-operations* ⟨*proof*⟩

lemma *drop-bit-Suc-minus-bit0* [*simp*]:
‹*drop-bit* (*Suc n*) (− *numeral* (*Num.Bit0 k*)) = *drop-bit n* (− *numeral k* :: *int*)›
⟨*proof*⟩

lemma *drop-bit-Suc-minus-bit1* [*simp*]:
‹*drop-bit* (*Suc n*) (− *numeral* (*Num.Bit1 k*)) = *drop-bit n* (− *numeral* (*Num.inc k*) :: *int*)›
⟨*proof*⟩

lemma *drop-bit-numeral-minus-bit0* [*simp*]:
‹*drop-bit* (*numeral l*) (− *numeral* (*Num.Bit0 k*)) = *drop-bit* (*pred-numeral l*) (− *numeral k* :: *int*)›
⟨*proof*⟩

lemma *drop-bit-numeral-minus-bit1* [*simp*]:
‹*drop-bit* (*numeral l*) (− *numeral* (*Num.Bit1 k*)) = *drop-bit* (*pred-numeral l*) (− *numeral* (*Num.inc k*) :: *int*)›
⟨*proof*⟩

lemma *take-bit-Suc-minus-bit0*:
‹*take-bit* (*Suc n*) (− *numeral* (*Num.Bit0 k*)) = *take-bit n* (− *numeral k*) ∗ (*2* :: *int*)›
⟨*proof*⟩

lemma *take-bit-Suc-minus-bit1*:
‹*take-bit* (*Suc n*) (− *numeral* (*Num.Bit1 k*)) = *take-bit n* (− *numeral* (*Num.inc k*)) ∗ *2* + (*1* :: *int*)›
⟨*proof*⟩

lemma *take-bit-numeral-minus-bit0*:
‹*take-bit* (*numeral l*) (− *numeral* (*Num.Bit0 k*)) = *take-bit* (*pred-numeral l*) (− *numeral k*) ∗ (*2* :: *int*)›
⟨*proof*⟩

lemma *take-bit-numeral-minus-bit1*:
‹*take-bit* (*numeral l*) (− *numeral* (*Num.Bit1 k*)) = *take-bit* (*pred-numeral l*) (− *numeral* (*Num.inc k*)) ∗ *2* + (*1* :: *int*)›
⟨*proof*⟩

69.6 Symbolic computations on numeral expressions

context *semiring-bits*
begin

lemma *not-bit-numeral-Bit0-0* [*simp*]:
‹¬ *bit* (*numeral* (*Num.Bit0 m*)) *0*›
⟨*proof*⟩

lemma *bit-numeral-Bit1-0* [*simp*]:
‹*bit* (*numeral* (*Num.Bit1 m*)) *0*›
⟨*proof*⟩

end

context *ring-bit-operations*
begin

lemma *not-bit-minus-numeral-Bit0-0* [*simp*]:
‹¬ *bit* (− *numeral* (*Num.Bit0 m*)) *0*›
⟨*proof*⟩

lemma *bit-minus-numeral-Bit1-0* [*simp*]:
‹*bit* (− *numeral* (*Num.Bit1 m*)) *0*›
⟨*proof*⟩

end

context *unique-euclidean-semiring-with-bit-operations*
begin

lemma *bit-numeral-iff*:
‹*bit* (*numeral m*) *n* ⟷ *bit* (*numeral m* :: *nat*) *n*›
⟨*proof*⟩

lemma *bit-numeral-Bit0-Suc-iff* [*simp*]:
‹*bit* (*numeral* (*Num.Bit0 m*)) (*Suc n*) ⟷ *bit* (*numeral m*) *n*›
⟨*proof*⟩

lemma *bit-numeral-Bit1-Suc-iff* [*simp*]:
‹*bit* (*numeral* (*Num.Bit1 m*)) (*Suc n*) ⟷ *bit* (*numeral m*) *n*›
⟨*proof*⟩

lemma *bit-numeral-rec*:
‹*bit* (*numeral* (*Num.Bit0 w*)) *n* ⟷ (*case n of 0* ⇒ *False* | *Suc m* ⇒ *bit* (*numeral w*) *m*)›
‹*bit* (*numeral* (*Num.Bit1 w*)) *n* ⟷ (*case n of 0* ⇒ *True* | *Suc m* ⇒ *bit* (*numeral w*) *m*)›
⟨*proof*⟩

lemma *bit-numeral-simps* [*simp*]:
‹¬ *bit 1* (*numeral n*)›
‹*bit* (*numeral* (*Num.Bit0 w*)) (*numeral n*) ⟷ *bit* (*numeral w*) (*pred-numeral n*)›
‹*bit* (*numeral* (*Num.Bit1 w*)) (*numeral n*) ⟷ *bit* (*numeral w*) (*pred-numeral n*)›
⟨*proof*⟩

lemma *and-numerals* [*simp*]:

‹*1 AND numeral (Num.Bit0 y) = 0*›
‹*1 AND numeral (Num.Bit1 y) = 1*›
‹*numeral (Num.Bit0 x) AND numeral (Num.Bit0 y) = 2 * (numeral x AND numeral y)*›
‹*numeral (Num.Bit0 x) AND numeral (Num.Bit1 y) = 2 * (numeral x AND numeral y)*›
‹*numeral (Num.Bit0 x) AND 1 = 0*›
‹*numeral (Num.Bit1 x) AND numeral (Num.Bit0 y) = 2 * (numeral x AND numeral y)*›
‹*numeral (Num.Bit1 x) AND numeral (Num.Bit1 y) = 1 + 2 * (numeral x AND numeral y)*›
‹*numeral (Num.Bit1 x) AND 1 = 1*›
⟨*proof*⟩

fun *and-num* :: ‹*num ⇒ num ⇒ num option*›
where
‹*and-num num.One num.One = Some num.One*›
| ‹*and-num num.One (num.Bit0 n) = None*›
| ‹*and-num num.One (num.Bit1 n) = Some num.One*›
| ‹*and-num (num.Bit0 m) num.One = None*›
| ‹*and-num (num.Bit0 m) (num.Bit0 n) = map-option num.Bit0 (and-num m n)*›
| ‹*and-num (num.Bit0 m) (num.Bit1 n) = map-option num.Bit0 (and-num m n)*›
| ‹*and-num (num.Bit1 m) num.One = Some num.One*›
| ‹*and-num (num.Bit1 m) (num.Bit0 n) = map-option num.Bit0 (and-num m n)*›
| ‹*and-num (num.Bit1 m) (num.Bit1 n) = (case and-num m n of None ⇒ Some num.One | Some n′ ⇒ Some (num.Bit1 n′))*›

lemma *numeral-and-num*:
‹*numeral m AND numeral n = (case and-num m n of None ⇒ 0 | Some n′ ⇒ numeral n′)*›
⟨*proof*⟩

lemma *and-num-eq-None-iff*:
‹*and-num m n = None ⟷ numeral m AND numeral n = 0*›
⟨*proof*⟩

lemma *and-num-eq-Some-iff*:
‹*and-num m n = Some q ⟷ numeral m AND numeral n = numeral q*›
⟨*proof*⟩

lemma *or-numerals* [*simp*]:
‹*1 OR numeral (Num.Bit0 y) = numeral (Num.Bit1 y)*›
‹*1 OR numeral (Num.Bit1 y) = numeral (Num.Bit1 y)*›
‹*numeral (Num.Bit0 x) OR numeral (Num.Bit0 y) = 2 * (numeral x OR numeral y)*›
‹*numeral (Num.Bit0 x) OR numeral (Num.Bit1 y) = 1 + 2 * (numeral x OR numeral y)*›
‹*numeral (Num.Bit0 x) OR 1 = numeral (Num.Bit1 x)*›
‹*numeral (Num.Bit1 x) OR numeral (Num.Bit0 y) = 1 + 2 * (numeral x OR*

numeral y)›
‹*numeral (Num.Bit1 x) OR numeral (Num.Bit1 y) = 1 + 2 * (numeral x OR numeral y)*›
‹*numeral (Num.Bit1 x) OR 1 = numeral (Num.Bit1 x)*›
⟨*proof*⟩

fun *or-num* :: ‹*num ⇒ num ⇒ num*›
where
‹*or-num num.One num.One = num.One*›
| ‹*or-num num.One (num.Bit0 n) = num.Bit1 n*›
| ‹*or-num num.One (num.Bit1 n) = num.Bit1 n*›
| ‹*or-num (num.Bit0 m) num.One = num.Bit1 m*›
| ‹*or-num (num.Bit0 m) (num.Bit0 n) = num.Bit0 (or-num m n)*›
| ‹*or-num (num.Bit0 m) (num.Bit1 n) = num.Bit1 (or-num m n)*›
| ‹*or-num (num.Bit1 m) num.One = num.Bit1 m*›
| ‹*or-num (num.Bit1 m) (num.Bit0 n) = num.Bit1 (or-num m n)*›
| ‹*or-num (num.Bit1 m) (num.Bit1 n) = num.Bit1 (or-num m n)*›

lemma *numeral-or-num*:
‹*numeral m OR numeral n = numeral (or-num m n)*›
⟨*proof*⟩

lemma *numeral-or-num-eq*:
‹*numeral (or-num m n) = numeral m OR numeral n*›
⟨*proof*⟩

lemma *xor-numerals* [*simp*]:
‹*1 XOR numeral (Num.Bit0 y) = numeral (Num.Bit1 y)*›
‹*1 XOR numeral (Num.Bit1 y) = numeral (Num.Bit0 y)*›
‹*numeral (Num.Bit0 x) XOR numeral (Num.Bit0 y) = 2 * (numeral x XOR numeral y)*›
‹*numeral (Num.Bit0 x) XOR numeral (Num.Bit1 y) = 1 + 2 * (numeral x XOR numeral y)*›
‹*numeral (Num.Bit0 x) XOR 1 = numeral (Num.Bit1 x)*›
‹*numeral (Num.Bit1 x) XOR numeral (Num.Bit0 y) = 1 + 2 * (numeral x XOR numeral y)*›
‹*numeral (Num.Bit1 x) XOR numeral (Num.Bit1 y) = 2 * (numeral x XOR numeral y)*›
‹*numeral (Num.Bit1 x) XOR 1 = numeral (Num.Bit0 x)*›
⟨*proof*⟩

fun *xor-num* :: ‹*num ⇒ num ⇒ num option*›
where
‹*xor-num num.One num.One = None*›
| ‹*xor-num num.One (num.Bit0 n) = Some (num.Bit1 n)*›
| ‹*xor-num num.One (num.Bit1 n) = Some (num.Bit0 n)*›
| ‹*xor-num (num.Bit0 m) num.One = Some (num.Bit1 m)*›
| ‹*xor-num (num.Bit0 m) (num.Bit0 n) = map-option num.Bit0 (xor-num m n)*›
| ‹*xor-num (num.Bit0 m) (num.Bit1 n) = Some (case xor-num m n of None ⇒*

num.One | Some n′ ⇒ num.Bit1 n′)›
| ‹*xor-num (num.Bit1 m) num.One = Some (num.Bit0 m)*›
| ‹*xor-num (num.Bit1 m) (num.Bit0 n) = Some (case xor-num m n of None ⇒ num.One | Some n′ ⇒ num.Bit1 n′)*›
| ‹*xor-num (num.Bit1 m) (num.Bit1 n) = map-option num.Bit0 (xor-num m n)*›

lemma *numeral-xor-num*:
‹*numeral m XOR numeral n = (case xor-num m n of None ⇒ 0 | Some n′ ⇒ numeral n′)*›
⟨*proof*⟩

lemma *xor-num-eq-None-iff*:
‹*xor-num m n = None ⟷ numeral m XOR numeral n = 0*›
⟨*proof*⟩

lemma *xor-num-eq-Some-iff*:
‹*xor-num m n = Some q ⟷ numeral m XOR numeral n = numeral q*›
⟨*proof*⟩

end

lemma *bit-Suc-0-iff* [*bit-simps*]:
‹*bit (Suc 0) n ⟷ n = 0*›
⟨*proof*⟩

lemma *and-nat-numerals* [*simp*]:
‹*Suc 0 AND numeral (Num.Bit0 y) = 0*›
‹*Suc 0 AND numeral (Num.Bit1 y) = 1*›
‹*numeral (Num.Bit0 x) AND Suc 0 = 0*›
‹*numeral (Num.Bit1 x) AND Suc 0 = 1*›
⟨*proof*⟩

lemma *or-nat-numerals* [*simp*]:
‹*Suc 0 OR numeral (Num.Bit0 y) = numeral (Num.Bit1 y)*›
‹*Suc 0 OR numeral (Num.Bit1 y) = numeral (Num.Bit1 y)*›
‹*numeral (Num.Bit0 x) OR Suc 0 = numeral (Num.Bit1 x)*›
‹*numeral (Num.Bit1 x) OR Suc 0 = numeral (Num.Bit1 x)*›
⟨*proof*⟩

lemma *xor-nat-numerals* [*simp*]:
‹*Suc 0 XOR numeral (Num.Bit0 y) = numeral (Num.Bit1 y)*›
‹*Suc 0 XOR numeral (Num.Bit1 y) = numeral (Num.Bit0 y)*›
‹*numeral (Num.Bit0 x) XOR Suc 0 = numeral (Num.Bit1 x)*›
‹*numeral (Num.Bit1 x) XOR Suc 0 = numeral (Num.Bit0 x)*›
⟨*proof*⟩

context *ring-bit-operations*
begin

lemma *minus-numeral-inc-eq*:
‹− *numeral* (*Num.inc n*) = *NOT* (*numeral n*)›
⟨*proof*⟩

lemma *sub-one-eq-not-neg*:
‹*Num.sub n num.One* = *NOT* (− *numeral n*)›
⟨*proof*⟩

lemma *minus-numeral-eq-not-sub-one*:
‹− *numeral n* = *NOT* (*Num.sub n num.One*)›
⟨*proof*⟩

lemma *not-numeral-eq* [*simp*]:
‹*NOT* (*numeral n*) = − *numeral* (*Num.inc n*)›
⟨*proof*⟩

lemma *not-minus-numeral-eq* [*simp*]:
‹*NOT* (− *numeral n*) = *Num.sub n num.One*›
⟨*proof*⟩

lemma *minus-not-numeral-eq* [*simp*]:
‹− (*NOT* (*numeral n*)) = *numeral* (*Num.inc n*)›
⟨*proof*⟩

lemma *not-numeral-BitM-eq*:
‹*NOT* (*numeral* (*Num.BitM n*)) = − *numeral* (*num.Bit0 n*)›
⟨*proof*⟩

lemma *not-numeral-Bit0-eq*:
‹*NOT* (*numeral* (*Num.Bit0 n*)) = − *numeral* (*num.Bit1 n*)›
⟨*proof*⟩

end

lemma *bit-minus-numeral-int* [*simp*]:
‹*bit* (− *numeral* (*num.Bit0 w*) :: *int*) (*numeral n*) ⟷ *bit* (− *numeral w* :: *int*) (*pred-numeral n*)›
‹*bit* (− *numeral* (*num.Bit1 w*) :: *int*) (*numeral n*) ⟷ ¬ *bit* (*numeral w* :: *int*) (*pred-numeral n*)›
⟨*proof*⟩

lemma *bit-minus-numeral-Bit0-Suc-iff* [*simp*]:
‹*bit* (− *numeral* (*num.Bit0 w*) :: *int*) (*Suc n*) ⟷ *bit* (− *numeral w* :: *int*) *n*›
⟨*proof*⟩

lemma *bit-minus-numeral-Bit1-Suc-iff* [*simp*]:
‹*bit* (− *numeral* (*num.Bit1 w*) :: *int*) (*Suc n*) ⟷ ¬ *bit* (*numeral w* :: *int*) *n*›
⟨*proof*⟩

lemma *and-not-numerals*:
‹*1 AND NOT 1 = (0 :: int)*›
‹*1 AND NOT (numeral (Num.Bit0 n)) = (1 :: int)*›
‹*1 AND NOT (numeral (Num.Bit1 n)) = (0 :: int)*›
‹*numeral (Num.Bit0 m) AND NOT (1 :: int) = numeral (Num.Bit0 m)*›
‹*numeral (Num.Bit0 m) AND NOT (numeral (Num.Bit0 n)) = (2 :: int) * (numeral m AND NOT (numeral n))*›
‹*numeral (Num.Bit0 m) AND NOT (numeral (Num.Bit1 n)) = (2 :: int) * (numeral m AND NOT (numeral n))*›
‹*numeral (Num.Bit1 m) AND NOT (1 :: int) = numeral (Num.Bit0 m)*›
‹*numeral (Num.Bit1 m) AND NOT (numeral (Num.Bit0 n)) = 1 + (2 :: int) * (numeral m AND NOT (numeral n))*›
‹*numeral (Num.Bit1 m) AND NOT (numeral (Num.Bit1 n)) = (2 :: int) * (numeral m AND NOT (numeral n))*›
⟨*proof*⟩

fun *and-not-num* :: ‹*num ⇒ num ⇒ num option*›
where
‹*and-not-num num.One num.One = None*›
| ‹*and-not-num num.One (num.Bit0 n) = Some num.One*›
| ‹*and-not-num num.One (num.Bit1 n) = None*›
| ‹*and-not-num (num.Bit0 m) num.One = Some (num.Bit0 m)*›
| ‹*and-not-num (num.Bit0 m) (num.Bit0 n) = map-option num.Bit0 (and-not-num m n)*›
| ‹*and-not-num (num.Bit0 m) (num.Bit1 n) = map-option num.Bit0 (and-not-num m n)*›
| ‹*and-not-num (num.Bit1 m) num.One = Some (num.Bit0 m)*›
| ‹*and-not-num (num.Bit1 m) (num.Bit0 n) = (case and-not-num m n of None ⇒ Some num.One | Some n′ ⇒ Some (num.Bit1 n′))*›
| ‹*and-not-num (num.Bit1 m) (num.Bit1 n) = map-option num.Bit0 (and-not-num m n)*›

lemma *int-numeral-and-not-num*:
‹*numeral m AND NOT (numeral n) = (case and-not-num m n of None ⇒ 0 :: int | Some n′ ⇒ numeral n′)*›
⟨*proof*⟩

lemma *int-numeral-not-and-num*:
‹*NOT (numeral m) AND numeral n = (case and-not-num n m of None ⇒ 0 :: int | Some n′ ⇒ numeral n′)*›
⟨*proof*⟩

lemma *and-not-num-eq-None-iff*:
‹*and-not-num m n = None ⟷ numeral m AND NOT (numeral n) = (0 :: int)*›
⟨*proof*⟩

lemma *and-not-num-eq-Some-iff*:
‹*and-not-num m n = Some q ⟷ numeral m AND NOT (numeral n) = (numeral q :: int)*›

⟨*proof*⟩

lemma *and-minus-numerals* [*simp*]:
‹*1 AND − (numeral (num.Bit0 n)) = (0::int)*›
‹*1 AND − (numeral (num.Bit1 n)) = (1::int)*›
‹*numeral m AND − (numeral (num.Bit0 n)) = (case and-not-num m (Num.BitM n) of None ⇒ 0 :: int | Some n′ ⇒ numeral n′)*›
‹*numeral m AND − (numeral (num.Bit1 n)) = (case and-not-num m (Num.Bit0 n) of None ⇒ 0 :: int | Some n′ ⇒ numeral n′)*›
‹*− (numeral (num.Bit0 n)) AND 1 = (0::int)*›
‹*− (numeral (num.Bit1 n)) AND 1 = (1::int)*›
‹*− (numeral (num.Bit0 n)) AND numeral m = (case and-not-num m (Num.BitM n) of None ⇒ 0 :: int | Some n′ ⇒ numeral n′)*›
‹*− (numeral (num.Bit1 n)) AND numeral m = (case and-not-num m (Num.Bit0 n) of None ⇒ 0 :: int | Some n′ ⇒ numeral n′)*›
⟨*proof*⟩

lemma *and-minus-minus-numerals* [*simp*]:
‹*− (numeral m :: int) AND − (numeral n :: int) = NOT ((numeral m − 1) OR (numeral n − 1))*›
⟨*proof*⟩

lemma *or-not-numerals*:
‹*1 OR NOT 1 = NOT (0 :: int)*›
‹*1 OR NOT (numeral (Num.Bit0 n)) = NOT (numeral (Num.Bit0 n) :: int)*›
‹*1 OR NOT (numeral (Num.Bit1 n)) = NOT (numeral (Num.Bit0 n) :: int)*›
‹*numeral (Num.Bit0 m) OR NOT (1 :: int) = NOT (1 :: int)*›
‹*numeral (Num.Bit0 m) OR NOT (numeral (Num.Bit0 n)) = 1 + (2 :: int) * (numeral m OR NOT (numeral n))*›
‹*numeral (Num.Bit0 m) OR NOT (numeral (Num.Bit1 n)) = (2 :: int) * (numeral m OR NOT (numeral n))*›
‹*numeral (Num.Bit1 m) OR NOT (1 :: int) = NOT (0 :: int)*›
‹*numeral (Num.Bit1 m) OR NOT (numeral (Num.Bit0 n)) = 1 + (2 :: int) * (numeral m OR NOT (numeral n))*›
‹*numeral (Num.Bit1 m) OR NOT (numeral (Num.Bit1 n)) = 1 + (2 :: int) * (numeral m OR NOT (numeral n))*›
⟨*proof*⟩

fun *or-not-num-neg* :: ‹*num ⇒ num ⇒ num*›
where
‹*or-not-num-neg num.One num.One = num.One*›
| ‹*or-not-num-neg num.One (num.Bit0 m) = num.Bit1 m*›
| ‹*or-not-num-neg num.One (num.Bit1 m) = num.Bit1 m*›
| ‹*or-not-num-neg (num.Bit0 n) num.One = num.Bit0 num.One*›
| ‹*or-not-num-neg (num.Bit0 n) (num.Bit0 m) = Num.BitM (or-not-num-neg n m)*›
| ‹*or-not-num-neg (num.Bit0 n) (num.Bit1 m) = num.Bit0 (or-not-num-neg n m)*›
| ‹*or-not-num-neg (num.Bit1 n) num.One = num.One*›
| ‹*or-not-num-neg (num.Bit1 n) (num.Bit0 m) = Num.BitM (or-not-num-neg n*

m)›
| ‹*or-not-num-neg* (*num.Bit1 n*) (*num.Bit1 m*) = *Num.BitM* (*or-not-num-neg n m*)›

lemma *int-numeral-or-not-num-neg*:
‹*numeral m OR NOT* (*numeral n* :: *int*) = − *numeral* (*or-not-num-neg m n*)›
⟨*proof*⟩

lemma *int-numeral-not-or-num-neg*:
‹*NOT* (*numeral m*) *OR* (*numeral n* :: *int*) = − *numeral* (*or-not-num-neg n m*)›
⟨*proof*⟩

lemma *numeral-or-not-num-eq*:
‹*numeral* (*or-not-num-neg m n*) = − (*numeral m OR NOT* (*numeral n* :: *int*))›
⟨*proof*⟩

lemma *or-minus-numerals* [*simp*]:
‹*1 OR* − (*numeral* (*num.Bit0 n*)) = − (*numeral* (*or-not-num-neg num.One* (*Num.BitM n*)) :: *int*)›
‹*1 OR* − (*numeral* (*num.Bit1 n*)) = − (*numeral* (*num.Bit1 n*) :: *int*)›
‹*numeral m OR* − (*numeral* (*num.Bit0 n*)) = − (*numeral* (*or-not-num-neg m* (*Num.BitM n*)) :: *int*)›
‹*numeral m OR* − (*numeral* (*num.Bit1 n*)) = − (*numeral* (*or-not-num-neg m* (*Num.Bit0 n*)) :: *int*)›
‹− (*numeral* (*num.Bit0 n*)) *OR 1* = − (*numeral* (*or-not-num-neg num.One* (*Num.BitM n*)) :: *int*)›
‹− (*numeral* (*num.Bit1 n*)) *OR 1* = − (*numeral* (*num.Bit1 n*) :: *int*)›
‹− (*numeral* (*num.Bit0 n*)) *OR numeral m* = − (*numeral* (*or-not-num-neg m* (*Num.BitM n*)) :: *int*)›
‹− (*numeral* (*num.Bit1 n*)) *OR numeral m* = − (*numeral* (*or-not-num-neg m* (*Num.Bit0 n*)) :: *int*)›
⟨*proof*⟩

lemma *or-minus-minus-numerals* [*simp*]:
‹− (*numeral m* :: *int*) *OR* − (*numeral n* :: *int*) = *NOT* ((*numeral m* − *1*) *AND* (*numeral n* − *1*))›
⟨*proof*⟩

lemma *xor-minus-numerals* [*simp*]:
‹− *numeral n XOR k* = *NOT* (*neg-numeral-class.sub n num.One XOR k*)›
‹*k XOR* − *numeral n* = *NOT* (*k XOR* (*neg-numeral-class.sub n num.One*))› **for** *k* :: *int*
⟨*proof*⟩

definition *take-bit-num* :: ‹*nat* ⇒ *num* ⇒ *num option*›
where ‹*take-bit-num n m* =
(*if take-bit n* (*numeral m* :: *nat*) = *0 then None else Some* (*num-of-nat* (*take-bit n* (*numeral m* :: *nat*))))›

lemma *take-bit-num-simps*:
‹*take-bit-num 0 m = None*›
‹*take-bit-num (Suc n) Num.One =*
Some Num.One›
‹*take-bit-num (Suc n) (Num.Bit0 m) =*
(*case take-bit-num n m of None ⇒ None | Some q ⇒ Some (Num.Bit0 q)*)›
‹*take-bit-num (Suc n) (Num.Bit1 m) =*
Some (case take-bit-num n m of None ⇒ Num.One | Some q ⇒ Num.Bit1 q)›
‹*take-bit-num (numeral r) Num.One =*
Some Num.One›
‹*take-bit-num (numeral r) (Num.Bit0 m) =*
(*case take-bit-num (pred-numeral r) m of None ⇒ None | Some q ⇒ Some (Num.Bit0 q)*)›
‹*take-bit-num (numeral r) (Num.Bit1 m) =*
Some (case take-bit-num (pred-numeral r) m of None ⇒ Num.One | Some q ⇒ Num.Bit1 q)›
⟨*proof*⟩

lemma *take-bit-num-code* [*code*]:
— Ocaml-style pattern matching is more robust wrt. different representations of *nat*
‹*take-bit-num n m = (case (n, m)*
of (0, -) ⇒ None
| (Suc n, Num.One) ⇒ Some Num.One
| (Suc n, Num.Bit0 m) ⇒ (case take-bit-num n m of None ⇒ None | Some q ⇒ Some (Num.Bit0 q))
| (Suc n, Num.Bit1 m) ⇒ Some (case take-bit-num n m of None ⇒ Num.One | Some q ⇒ Num.Bit1 q))›
⟨*proof*⟩

context *semiring-bit-operations*
begin

lemma *take-bit-num-eq-None-imp*:
‹*take-bit m (numeral n) = 0*› **if** ‹*take-bit-num m n = None*›
⟨*proof*⟩

lemma *take-bit-num-eq-Some-imp*:
‹*take-bit m (numeral n) = numeral q*› **if** ‹*take-bit-num m n = Some q*›
⟨*proof*⟩

lemma *take-bit-numeral-numeral*:
‹*take-bit (numeral m) (numeral n) =*
(*case take-bit-num (numeral m) n of None ⇒ 0 | Some q ⇒ numeral q*)›
⟨*proof*⟩

end

lemma *take-bit-numeral-minus-numeral-int*:

‹*take-bit* (*numeral m*) (− *numeral n* :: *int*) =
(*case take-bit-num* (*numeral m*) *n of None* ⇒ *0* | *Some q* ⇒ *take-bit* (*numeral m*) (*2* ^ *numeral m* − *numeral q*))› (**is** ‹*?lhs* = *?rhs*›)
⟨*proof*⟩

declare *take-bit-num-simps* [*simp*]
take-bit-numeral-numeral [*simp*]
take-bit-numeral-minus-numeral-int [*simp*]

69.7 More properties

lemma *take-bit-eq-mask-iff*:
‹*take-bit n k* = *mask n* ⟷ *take-bit n* (*k* + *1*) = *0*› (**is** ‹*?P* ⟷ *?Q*›)
for *k* :: *int*
⟨*proof*⟩

lemma *take-bit-eq-mask-iff-exp-dvd*:
‹*take-bit n k* = *mask n* ⟷ *2* ^ *n dvd k* + *1*›
for *k* :: *int*
⟨*proof*⟩

69.8 Bit concatenation

definition *concat-bit* :: ‹*nat* ⇒ *int* ⇒ *int* ⇒ *int*›
where ‹*concat-bit n k l* = *take-bit n k OR push-bit n l*›

lemma *bit-concat-bit-iff* [*bit-simps*]:
‹*bit* (*concat-bit m k l*) *n* ⟷ *n* < *m* ∧ *bit k n* ∨ *m* ≤ *n* ∧ *bit l* (*n* − *m*)›
⟨*proof*⟩

lemma *concat-bit-eq*:
‹*concat-bit n k l* = *take-bit n k* + *push-bit n l*›
⟨*proof*⟩

lemma *concat-bit-0* [*simp*]:
‹*concat-bit 0 k l* = *l*›
⟨*proof*⟩

lemma *concat-bit-Suc*:
‹*concat-bit* (*Suc n*) *k l* = *k mod 2* + *2* ∗ *concat-bit n* (*k div 2*) *l*›
⟨*proof*⟩

lemma *concat-bit-of-zero-1* [*simp*]:
‹*concat-bit n 0 l* = *push-bit n l*›
⟨*proof*⟩

lemma *concat-bit-of-zero-2* [*simp*]:
‹*concat-bit n k 0* = *take-bit n k*›
⟨*proof*⟩

lemma *concat-bit-nonnegative-iff* [*simp*]:
‹*concat-bit n k l* ≥ *0* ⟷ *l* ≥ *0*›
⟨*proof*⟩

lemma *concat-bit-negative-iff* [*simp*]:
‹*concat-bit n k l* < *0* ⟷ *l* < *0*›
⟨*proof*⟩

lemma *concat-bit-assoc*:
‹*concat-bit n k* (*concat-bit m l r*) = *concat-bit* (*m* + *n*) (*concat-bit n k l*) *r*›
⟨*proof*⟩

lemma *concat-bit-assoc-sym*:
‹*concat-bit m* (*concat-bit n k l*) *r* = *concat-bit* (*min m n*) *k* (*concat-bit* (*m* − *n*) *l r*)›
⟨*proof*⟩

lemma *concat-bit-eq-iff*:
‹*concat-bit n k l* = *concat-bit n r s*
⟷ *take-bit n k* = *take-bit n r* ∧ *l* = *s*› (**is** ‹*?P* ⟷ *?Q*›)
⟨*proof*⟩

lemma *take-bit-concat-bit-eq*:
‹*take-bit m* (*concat-bit n k l*) = *concat-bit* (*min m n*) *k* (*take-bit* (*m* − *n*) *l*)›
⟨*proof*⟩

lemma *concat-bit-take-bit-eq*:
‹*concat-bit n* (*take-bit n b*) = *concat-bit n b*›
⟨*proof*⟩

69.9 Taking bits with sign propagation

context *ring-bit-operations*
begin

definition *signed-take-bit* :: ‹*nat* ⇒ *'a* ⇒ *'a*›
where ‹*signed-take-bit n a* = *take-bit n a OR* (*of-bool* (*bit a n*) ∗ *NOT* (*mask n*))›

lemma *signed-take-bit-eq-if-positive*:
‹*signed-take-bit n a* = *take-bit n a*› **if** ‹¬ *bit a n*›
⟨*proof*⟩

lemma *signed-take-bit-eq-if-negative*:
‹*signed-take-bit n a* = *take-bit n a OR NOT* (*mask n*)› **if** ‹*bit a n*›
⟨*proof*⟩

lemma *even-signed-take-bit-iff*:
‹*even* (*signed-take-bit m a*) ⟷ *even a*›

⟨*proof*⟩

lemma *bit-signed-take-bit-iff* [*bit-simps*]:
‹*bit* (*signed-take-bit m a*) *n* ⟷ *possible-bit TYPE*('*a*) *n* ∧ *bit a* (*min m n*)›
⟨*proof*⟩

lemma *signed-take-bit-0* [*simp*]:
‹*signed-take-bit 0 a* = − (*a mod 2*)›
⟨*proof*⟩

lemma *signed-take-bit-Suc*:
‹*signed-take-bit* (*Suc n*) *a* = *a mod 2* + *2* ∗ *signed-take-bit n* (*a div 2*)›
⟨*proof*⟩

lemma *signed-take-bit-of-0* [*simp*]:
‹*signed-take-bit n 0* = *0*›
⟨*proof*⟩

lemma *signed-take-bit-of-minus-1* [*simp*]:
‹*signed-take-bit n* (− *1*) = − *1*›
⟨*proof*⟩

lemma *signed-take-bit-Suc-1* [*simp*]:
‹*signed-take-bit* (*Suc n*) *1* = *1*›
⟨*proof*⟩

lemma *signed-take-bit-numeral-of-1* [*simp*]:
‹*signed-take-bit* (*numeral k*) *1* = *1*›
⟨*proof*⟩

lemma *signed-take-bit-rec*:
‹*signed-take-bit n a* = (*if n* = *0 then* − (*a mod 2*) *else a mod 2* + *2* ∗ *signed-take-bit* (*n* − *1*) (*a div 2*))›
⟨*proof*⟩

lemma *signed-take-bit-eq-iff-take-bit-eq*:
‹*signed-take-bit n a* = *signed-take-bit n b* ⟷ *take-bit* (*Suc n*) *a* = *take-bit* (*Suc n*) *b*›
⟨*proof*⟩

lemma *signed-take-bit-signed-take-bit* [*simp*]:
‹*signed-take-bit m* (*signed-take-bit n a*) = *signed-take-bit* (*min m n*) *a*›
⟨*proof*⟩

lemma *signed-take-bit-take-bit*:
‹*signed-take-bit m* (*take-bit n a*) = (*if n* ≤ *m then take-bit n else signed-take-bit m*) *a*›
⟨*proof*⟩

lemma *take-bit-signed-take-bit*:
‹*take-bit m* (*signed-take-bit n a*) = *take-bit m a*› **if** ‹*m* ≤ *Suc n*›
⟨*proof*⟩

end

Modulus centered around 0

lemma *signed-take-bit-eq-concat-bit*:
‹*signed-take-bit n k* = *concat-bit n k* (− *of-bool* (*bit k n*))›
⟨*proof*⟩

lemma *signed-take-bit-add*:
‹*signed-take-bit n* (*signed-take-bit n k* + *signed-take-bit n l*) = *signed-take-bit n* (*k* + *l*)›
for *k l* :: *int*
⟨*proof*⟩

lemma *signed-take-bit-diff*:
‹*signed-take-bit n* (*signed-take-bit n k* − *signed-take-bit n l*) = *signed-take-bit n* (*k* − *l*)›
for *k l* :: *int*
⟨*proof*⟩

lemma *signed-take-bit-minus*:
‹*signed-take-bit n* (− *signed-take-bit n k*) = *signed-take-bit n* (− *k*)›
for *k* :: *int*
⟨*proof*⟩

lemma *signed-take-bit-mult*:
‹*signed-take-bit n* (*signed-take-bit n k* ∗ *signed-take-bit n l*) = *signed-take-bit n* (*k* ∗ *l*)›
for *k l* :: *int*
⟨*proof*⟩

lemma *signed-take-bit-eq-take-bit-minus*:
‹*signed-take-bit n k* = *take-bit* (*Suc n*) *k* − *2* ^ *Suc n* ∗ *of-bool* (*bit k n*)›
for *k* :: *int*
⟨*proof*⟩

lemma *signed-take-bit-eq-take-bit-shift*:
‹*signed-take-bit n k* = *take-bit* (*Suc n*) (*k* + *2* ^ *n*) − *2* ^ *n*›
for *k* :: *int*
⟨*proof*⟩

lemma *signed-take-bit-nonnegative-iff* [*simp*]:
‹*0* ≤ *signed-take-bit n k* ⟷ ¬ *bit k n*›
for *k* :: *int*
⟨*proof*⟩

lemma *signed-take-bit-negative-iff* [*simp*]:
‹*signed-take-bit n k* < *0* ⟷ *bit k n*›
for *k* :: *int*
⟨*proof*⟩

lemma *signed-take-bit-int-greater-eq-minus-exp* [*simp*]:
‹− (*2* ^ *n*) ≤ *signed-take-bit n k*›
for *k* :: *int*
⟨*proof*⟩

lemma *signed-take-bit-int-less-exp* [*simp*]:
‹*signed-take-bit n k* < *2* ^ *n*›
for *k* :: *int*
⟨*proof*⟩

lemma *signed-take-bit-int-eq-self-iff*:
‹*signed-take-bit n k* = *k* ⟷ − (*2* ^ *n*) ≤ *k* ∧ *k* < *2* ^ *n*›
for *k* :: *int*
⟨*proof*⟩

lemma *signed-take-bit-int-eq-self*:
‹*signed-take-bit n k* = *k*› **if** ‹− (*2* ^ *n*) ≤ *k*› ‹*k* < *2* ^ *n*›
for *k* :: *int*
⟨*proof*⟩

lemma *signed-take-bit-int-less-eq-self-iff*:
‹*signed-take-bit n k* ≤ *k* ⟷ − (*2* ^ *n*) ≤ *k*›
for *k* :: *int*
⟨*proof*⟩

lemma *signed-take-bit-int-less-self-iff*:
‹*signed-take-bit n k* < *k* ⟷ *2* ^ *n* ≤ *k*›
for *k* :: *int*
⟨*proof*⟩

lemma *signed-take-bit-int-greater-self-iff*:
‹*k* < *signed-take-bit n k* ⟷ *k* < − (*2* ^ *n*)›
for *k* :: *int*
⟨*proof*⟩

lemma *signed-take-bit-int-greater-eq-self-iff*:
‹*k* ≤ *signed-take-bit n k* ⟷ *k* < *2* ^ *n*›
for *k* :: *int*
⟨*proof*⟩

lemma *signed-take-bit-int-greater-eq*:
‹*k* + *2* ^ *Suc n* ≤ *signed-take-bit n k*› **if** ‹*k* < − (*2* ^ *n*)›
for *k* :: *int*
⟨*proof*⟩

lemma *signed-take-bit-int-less-eq*:
‹*signed-take-bit n k* ≤ *k* − *2* ^ *Suc n*› **if** ‹*k* ≥ *2* ^ *n*›
for *k* :: *int*
⟨*proof*⟩

lemma *signed-take-bit-Suc-bit0* [*simp*]:
‹*signed-take-bit* (*Suc n*) (*numeral* (*Num.Bit0 k*)) = *signed-take-bit n* (*numeral k*) ∗ (*2* :: *int*)›
⟨*proof*⟩

lemma *signed-take-bit-Suc-bit1* [*simp*]:
‹*signed-take-bit* (*Suc n*) (*numeral* (*Num.Bit1 k*)) = *signed-take-bit n* (*numeral k*) ∗ *2* + (*1* :: *int*)›
⟨*proof*⟩

lemma *signed-take-bit-Suc-minus-bit0* [*simp*]:
‹*signed-take-bit* (*Suc n*) (− *numeral* (*Num.Bit0 k*)) = *signed-take-bit n* (− *numeral k*) ∗ (*2* :: *int*)›
⟨*proof*⟩

lemma *signed-take-bit-Suc-minus-bit1* [*simp*]:
‹*signed-take-bit* (*Suc n*) (− *numeral* (*Num.Bit1 k*)) = *signed-take-bit n* (− *numeral k* − *1*) ∗ *2* + (*1* :: *int*)›
⟨*proof*⟩

lemma *signed-take-bit-numeral-bit0* [*simp*]:
‹*signed-take-bit* (*numeral l*) (*numeral* (*Num.Bit0 k*)) = *signed-take-bit* (*pred-numeral l*) (*numeral k*) ∗ (*2* :: *int*)›
⟨*proof*⟩

lemma *signed-take-bit-numeral-bit1* [*simp*]:
‹*signed-take-bit* (*numeral l*) (*numeral* (*Num.Bit1 k*)) = *signed-take-bit* (*pred-numeral l*) (*numeral k*) ∗ *2* + (*1* :: *int*)›
⟨*proof*⟩

lemma *signed-take-bit-numeral-minus-bit0* [*simp*]:
‹*signed-take-bit* (*numeral l*) (− *numeral* (*Num.Bit0 k*)) = *signed-take-bit* (*pred-numeral l*) (− *numeral k*) ∗ (*2* :: *int*)›
⟨*proof*⟩

lemma *signed-take-bit-numeral-minus-bit1* [*simp*]:
‹*signed-take-bit* (*numeral l*) (− *numeral* (*Num.Bit1 k*)) = *signed-take-bit* (*pred-numeral l*) (− *numeral k* − *1*) ∗ *2* + (*1* :: *int*)›
⟨*proof*⟩

lemma *signed-take-bit-code* [*code*]:
‹*signed-take-bit n a* =
(*let l* = *take-bit* (*Suc n*) *a*

in if bit l n then l + push-bit (Suc n) (− 1) else l)›
⟨*proof*⟩

69.10 Horner sums

context *semiring-bit-operations*
begin

lemma *horner-sum-bit-eq-take-bit*:
‹*horner-sum of-bool 2 (map (bit a) [0..<n]) = take-bit n a*›
⟨*proof*⟩

end

context *unique-euclidean-semiring-with-bit-operations*
begin

lemma *bit-horner-sum-bit-iff* [*bit-simps*]:
‹*bit (horner-sum of-bool 2 bs) n ⟷ n < length bs ∧ bs ! n*›
⟨*proof*⟩

lemma *take-bit-horner-sum-bit-eq*:
‹*take-bit n (horner-sum of-bool 2 bs) = horner-sum of-bool 2 (take n bs)*›
⟨*proof*⟩

end

lemma *horner-sum-of-bool-2-less*:
‹(*horner-sum of-bool 2 bs* :: *int*) < *2* ^ *length bs*›
⟨*proof*⟩

69.11 Symbolic computations for code generation

lemma *bit-int-code* [*code*]:
‹*bit* (*0*::*int*) *n* ⟷ *False*›
‹*bit* (*Int.Neg num.One*) *n* ⟷ *True*›
‹*bit* (*Int.Pos num.One*) *0* ⟷ *True*›
‹*bit* (*Int.Pos* (*num.Bit0 m*)) *0* ⟷ *False*›
‹*bit* (*Int.Pos* (*num.Bit1 m*)) *0* ⟷ *True*›
‹*bit* (*Int.Neg* (*num.Bit0 m*)) *0* ⟷ *False*›
‹*bit* (*Int.Neg* (*num.Bit1 m*)) *0* ⟷ *True*›
‹*bit* (*Int.Pos num.One*) (*Suc n*) ⟷ *False*›
‹*bit* (*Int.Pos* (*num.Bit0 m*)) (*Suc n*) ⟷ *bit* (*Int.Pos m*) *n*›
‹*bit* (*Int.Pos* (*num.Bit1 m*)) (*Suc n*) ⟷ *bit* (*Int.Pos m*) *n*›
‹*bit* (*Int.Neg* (*num.Bit0 m*)) (*Suc n*) ⟷ *bit* (*Int.Neg m*) *n*›
‹*bit* (*Int.Neg* (*num.Bit1 m*)) (*Suc n*) ⟷ *bit* (*Int.Neg* (*Num.inc m*)) *n*›
⟨*proof*⟩

lemma *not-int-code* [*code*]:
‹*NOT* (*0* :: *int*) = − *1*›

‹*NOT* (*Int.Pos n*) = *Int.Neg* (*Num.inc n*)›
‹*NOT* (*Int.Neg n*) = *Num.sub n num.One*›
⟨*proof*⟩

lemma *and-int-code* [*code*]:
fixes *i j* :: *int* **shows**
‹*0 AND j* = *0*›
‹*i AND 0* = *0*›
‹*Int.Pos n AND Int.Pos m* = (*case and-num n m of None* ⇒ *0* | *Some n′* ⇒ *Int.Pos n′*)›
‹*Int.Neg n AND Int.Neg m* = *NOT* (*Num.sub n num.One OR Num.sub m num.One*)›
‹*Int.Pos n AND Int.Neg num.One* = *Int.Pos n*›
‹*Int.Pos n AND Int.Neg* (*num.Bit0 m*) = *Num.sub* (*or-not-num-neg* (*Num.BitM m*) *n*) *num.One*›
‹*Int.Pos n AND Int.Neg* (*num.Bit1 m*) = *Num.sub* (*or-not-num-neg* (*num.Bit0 m*) *n*) *num.One*›
‹*Int.Neg num.One AND Int.Pos m* = *Int.Pos m*›
‹*Int.Neg* (*num.Bit0 n*) *AND Int.Pos m* = *Num.sub* (*or-not-num-neg* (*Num.BitM n*) *m*) *num.One*›
‹*Int.Neg* (*num.Bit1 n*) *AND Int.Pos m* = *Num.sub* (*or-not-num-neg* (*num.Bit0 n*) *m*) *num.One*›
⟨*proof*⟩

lemma *or-int-code* [*code*]:
fixes *i j* :: *int* **shows**
‹*0 OR j* = *j*›
‹*i OR 0* = *i*›
‹*Int.Pos n OR Int.Pos m* = *Int.Pos* (*or-num n m*)›
‹*Int.Neg n OR Int.Neg m* = *NOT* (*Num.sub n num.One AND Num.sub m num.One*)›
‹*Int.Pos n OR Int.Neg num.One* = *Int.Neg num.One*›
‹*Int.Pos n OR Int.Neg* (*num.Bit0 m*) = (*case and-not-num* (*Num.BitM m*) *n of None* ⇒ −*1* | *Some n′* ⇒ *Int.Neg* (*Num.inc n′*))›
‹*Int.Pos n OR Int.Neg* (*num.Bit1 m*) = (*case and-not-num* (*num.Bit0 m*) *n of None* ⇒ −*1* | *Some n′* ⇒ *Int.Neg* (*Num.inc n′*))›
‹*Int.Neg num.One OR Int.Pos m* = *Int.Neg num.One*›
‹*Int.Neg* (*num.Bit0 n*) *OR Int.Pos m* = (*case and-not-num* (*Num.BitM n*) *m of None* ⇒ −*1* | *Some n′* ⇒ *Int.Neg* (*Num.inc n′*))›
‹*Int.Neg* (*num.Bit1 n*) *OR Int.Pos m* = (*case and-not-num* (*num.Bit0 n*) *m of None* ⇒ −*1* | *Some n′* ⇒ *Int.Neg* (*Num.inc n′*))›
⟨*proof*⟩

lemma *xor-int-code* [*code*]:
fixes *i j* :: *int* **shows**
‹*0 XOR j* = *j*›
‹*i XOR 0* = *i*›
‹*Int.Pos n XOR Int.Pos m* = (*case xor-num n m of None* ⇒ *0* | *Some n′* ⇒ *Int.Pos n′*)›

‹*Int.Neg n XOR Int.Neg m = Num.sub n num.One XOR Num.sub m num.One*›
‹*Int.Neg n XOR Int.Pos m = NOT (Num.sub n num.One XOR Int.Pos m)*›
‹*Int.Pos n XOR Int.Neg m = NOT (Int.Pos n XOR Num.sub m num.One)*›
⟨*proof*⟩

lemma *push-bit-int-code* [*code*]:
‹*push-bit 0 i = i*›
‹*push-bit (Suc n) i = push-bit n (Int.dup i)*›
⟨*proof*⟩

lemma *drop-bit-int-code* [*code*]:
fixes *i* :: *int* **shows**
‹*drop-bit 0 i = i*›
‹*drop-bit (Suc n) 0 = (0 :: int)*›
‹*drop-bit (Suc n) (Int.Pos num.One) = 0*›
‹*drop-bit (Suc n) (Int.Pos (num.Bit0 m)) = drop-bit n (Int.Pos m)*›
‹*drop-bit (Suc n) (Int.Pos (num.Bit1 m)) = drop-bit n (Int.Pos m)*›
‹*drop-bit (Suc n) (Int.Neg num.One) = − 1*›
‹*drop-bit (Suc n) (Int.Neg (num.Bit0 m)) = drop-bit n (Int.Neg m)*›
‹*drop-bit (Suc n) (Int.Neg (num.Bit1 m)) = drop-bit n (Int.Neg (Num.inc m))*›
⟨*proof*⟩

69.12 Key ideas of bit operations

When formalizing bit operations, it is tempting to represent bit values as explicit lists over a binary type. This however is a bad idea, mainly due to the inherent ambiguities in representation concerning repeating leading bits.

Hence this approach avoids such explicit lists altogether following an algebraic path:

- Bit values are represented by numeric types: idealized unbounded bit values can be represented by type *int*, bounded bit values by quotient types over *int*.
- (A special case are idealized unbounded bit values ending in *0* which can be represented by type *nat* but only support a restricted set of operations).
- From this idea follows that
  - multiplication by *2* is a bit shift to the left and
  - division by *2* is a bit shift to the right.
- Concerning bounded bit values, iterated shifts to the left may result in eliminating all bits by shifting them all beyond the boundary. The property $2^n \neq 0$ represents that n is *not* beyond that boundary.

- The projection on a single bit is then *bit a n = odd (a div 2^n)*.
- This leads to the most fundamental properties of bit values:
  - Equality rule: *(⋀n. possible-bit TYPE(int) n ⟹ bit a n = bit b n) ⟹ a = b*
  - Induction rule: *⟦⋀a. a div 2 = a ⟹ P a; ⋀a b. ⟦P a; (of-bool b + 2 * a) div 2 = a⟧ ⟹ P (of-bool b + 2 * a)⟧ ⟹ P a*
- Typical operations are characterized as follows:
  - Singleton *n*th bit: *2^n*
  - Bit mask upto bit *n*: *mask n = 2^n - 1*
  - Left shift: *push-bit n a = a * 2^n*
  - Right shift: *drop-bit n a = a div 2^n*
  - Truncation: *take-bit n a = a mod 2^n*
  - Negation: *bit (NOT a) n = (¬ ¬ possible-bit TYPE(int) n ∧ ¬ bit a n)*
  - And: *bit (a AND b) n = (bit a n ∧ bit b n)*
  - Or: *bit (a OR b) n = (bit a n ∨ bit b n)*
  - Xor: *bit (a XOR b) n = (bit a n ≠ bit b n)*
  - Set a single bit: *set-bit n a = a OR push-bit n 1*
  - Unset a single bit: *unset-bit n a = a AND NOT (push-bit n 1)*
  - Flip a single bit: *flip-bit n a = a XOR push-bit n 1*
  - Signed truncation, or modulus centered around *0*: *signed-take-bit n a = take-bit n a OR of-bool (bit a n) * NOT (mask n)*
  - Bit concatenation: *concat-bit n k l = take-bit n k OR push-bit n l*
  - (Bounded) conversion from and to a list of bits: *horner-sum of-bool 2 (map (bit a) [0..<n]) = take-bit n a*

no-notation
not (‹*NOT*›)
and *and* (**infixr** ‹*AND*› *64*)
and *or* (**infixr** ‹*OR*› *59*)
and *xor* (**infixr** ‹*XOR*› *59*)

bundle *bit-operations-syntax*
begin

notation

not (‹*NOT*›)
and *and* (**infixr** ‹*AND*› *64*)
and *or* (**infixr** ‹*OR*› *59*)
and *xor* (**infixr** ‹*XOR*› *59*)

end

end

70 Numeric types for code generation onto target language numerals only

theory *Code-Numeral*
imports *Divides Lifting Bit-Operations*
begin

70.1 Type of target language integers

typedef *integer* = *UNIV* :: *int set*
morphisms *int-of-integer integer-of-int* ⟨*proof*⟩

setup-lifting *type-definition-integer*

lemma *integer-eq-iff*:
$k = l \longleftrightarrow$ *int-of-integer* $k =$ *int-of-integer* l
⟨*proof*⟩

lemma *integer-eqI*:
int-of-integer $k =$ *int-of-integer* $l \Longrightarrow k = l$
⟨*proof*⟩

lemma *int-of-integer-integer-of-int* [*simp*]:
int-of-integer (*integer-of-int* k) $= k$
⟨*proof*⟩

lemma *integer-of-int-int-of-integer* [*simp*]:
integer-of-int (*int-of-integer* k) $= k$
⟨*proof*⟩

instantiation *integer* :: *ring-1*
begin

lift-definition *zero-integer* :: *integer*
is *0* :: *int*
⟨*proof*⟩

declare *zero-integer.rep-eq* [*simp*]

lift-definition *one-integer* :: *integer*
  is *1* :: *int*
  ⟨*proof*⟩

declare *one-integer.rep-eq* [*simp*]

lift-definition *plus-integer* :: *integer* ⇒ *integer* ⇒ *integer*
  is *plus* :: *int* ⇒ *int* ⇒ *int*
  ⟨*proof*⟩

declare *plus-integer.rep-eq* [*simp*]

lift-definition *uminus-integer* :: *integer* ⇒ *integer*
  is *uminus* :: *int* ⇒ *int*
  ⟨*proof*⟩

declare *uminus-integer.rep-eq* [*simp*]

lift-definition *minus-integer* :: *integer* ⇒ *integer* ⇒ *integer*
  is *minus* :: *int* ⇒ *int* ⇒ *int*
  ⟨*proof*⟩

declare *minus-integer.rep-eq* [*simp*]

lift-definition *times-integer* :: *integer* ⇒ *integer* ⇒ *integer*
  is *times* :: *int* ⇒ *int* ⇒ *int*
  ⟨*proof*⟩

declare *times-integer.rep-eq* [*simp*]

instance ⟨*proof*⟩

end

instance *integer* :: *Rings.dvd* ⟨*proof*⟩

context
  includes *lifting-syntax*
  notes *transfer-rule-numeral* [*transfer-rule*]
begin

lemma [*transfer-rule*]:
  (*pcr-integer* ===> *pcr-integer* ===> (⟷)) (*dvd*) (*dvd*)
  ⟨*proof*⟩

lemma [*transfer-rule*]:
  ((⟷) ===> *pcr-integer*) *of-bool of-bool*
  ⟨*proof*⟩

lemma [*transfer-rule*]:
(=) ===> *pcr-integer*) *int of-nat*
⟨*proof*⟩

lemma [*transfer-rule*]:
(=) ===> *pcr-integer*) (λ*k*. *k*) *of-int*
⟨*proof*⟩

lemma [*transfer-rule*]:
(=) ===> *pcr-integer*) *numeral numeral*
⟨*proof*⟩

lemma [*transfer-rule*]:
(=) ===> (=) ===> *pcr-integer*) *Num.sub Num.sub*
⟨*proof*⟩

lemma [*transfer-rule*]:
(*pcr-integer* ===> (=) ===> *pcr-integer*) (^) (^)
⟨*proof*⟩

end

lemma *int-of-integer-of-nat* [*simp*]:
int-of-integer (*of-nat n*) = *of-nat n*
⟨*proof*⟩

lift-definition *integer-of-nat* :: *nat* ⇒ *integer*
is *of-nat* :: *nat* ⇒ *int*
⟨*proof*⟩

lemma *integer-of-nat-eq-of-nat* [*code*]:
integer-of-nat = *of-nat*
⟨*proof*⟩

lemma *int-of-integer-integer-of-nat* [*simp*]:
int-of-integer (*integer-of-nat n*) = *of-nat n*
⟨*proof*⟩

lift-definition *nat-of-integer* :: *integer* ⇒ *nat*
is *Int.nat*
⟨*proof*⟩

lemma *nat-of-integer-of-nat* [*simp*]:
nat-of-integer (*of-nat n*) = *n*
⟨*proof*⟩

lemma *int-of-integer-of-int* [*simp*]:
int-of-integer (*of-int k*) = *k*
⟨*proof*⟩

lemma *nat-of-integer-integer-of-nat* [*simp*]:
nat-of-integer (*integer-of-nat n*) = *n*
⟨*proof*⟩

lemma *integer-of-int-eq-of-int* [*simp*, *code-abbrev*]:
integer-of-int = *of-int*
⟨*proof*⟩

lemma *of-int-integer-of* [*simp*]:
of-int (*int-of-integer k*) = (*k* :: *integer*)
⟨*proof*⟩

lemma *int-of-integer-numeral* [*simp*]:
int-of-integer (*numeral k*) = *numeral k*
⟨*proof*⟩

lemma *int-of-integer-sub* [*simp*]:
int-of-integer (*Num.sub k l*) = *Num.sub k l*
⟨*proof*⟩

definition *integer-of-num* :: *num* ⇒ *integer*
where [*simp*]: *integer-of-num* = *numeral*

lemma *integer-of-num* [*code*]:
integer-of-num Num.One = *1*
integer-of-num (*Num.Bit0 n*) = (*let k* = *integer-of-num n in k* + *k*)
integer-of-num (*Num.Bit1 n*) = (*let k* = *integer-of-num n in k* + *k* + *1*)
⟨*proof*⟩

lemma *integer-of-num-triv*:
integer-of-num Num.One = *1*
integer-of-num (*Num.Bit0 Num.One*) = *2*
⟨*proof*⟩

instantiation *integer* :: {*linordered-idom*, *equal*}
begin

lift-definition *abs-integer* :: *integer* ⇒ *integer*
is *abs* :: *int* ⇒ *int*
⟨*proof*⟩

declare *abs-integer.rep-eq* [*simp*]

lift-definition *sgn-integer* :: *integer* ⇒ *integer*
is *sgn* :: *int* ⇒ *int*
⟨*proof*⟩

declare *sgn-integer.rep-eq* [*simp*]

lift-definition *less-eq-integer* :: *integer* ⇒ *integer* ⇒ *bool*
is *less-eq* :: *int* ⇒ *int* ⇒ *bool*
⟨*proof*⟩

lemma *integer-less-eq-iff*:
$k \leq l \longleftrightarrow$ *int-of-integer* $k \leq$ *int-of-integer* l
⟨*proof*⟩

lift-definition *less-integer* :: *integer* ⇒ *integer* ⇒ *bool*
is *less* :: *int* ⇒ *int* ⇒ *bool*
⟨*proof*⟩

lemma *integer-less-iff*:
$k < l \longleftrightarrow$ *int-of-integer* $k <$ *int-of-integer* l
⟨*proof*⟩

lift-definition *equal-integer* :: *integer* ⇒ *integer* ⇒ *bool*
is *HOL.equal* :: *int* ⇒ *int* ⇒ *bool*
⟨*proof*⟩

instance
⟨*proof*⟩

end

context
includes *lifting-syntax*
begin

lemma [*transfer-rule*]:
‹(*pcr-integer* ===> *pcr-integer* ===> *pcr-integer*) *min min*›
⟨*proof*⟩

lemma [*transfer-rule*]:
‹(*pcr-integer* ===> *pcr-integer* ===> *pcr-integer*) *max max*›
⟨*proof*⟩

end

lemma *int-of-integer-min* [*simp*]:
int-of-integer (*min k l*) = *min* (*int-of-integer k*) (*int-of-integer l*)
⟨*proof*⟩

lemma *int-of-integer-max* [*simp*]:
int-of-integer (*max k l*) = *max* (*int-of-integer k*) (*int-of-integer l*)
⟨*proof*⟩

lemma *nat-of-integer-non-positive* [*simp*]:

$k \leq 0 \Longrightarrow$ *nat-of-integer* $k = 0$
⟨*proof*⟩

lemma *of-nat-of-integer* [*simp*]:
of-nat (*nat-of-integer* k) = *max* 0 k
⟨*proof*⟩

instantiation *integer* :: *unique-euclidean-ring*
begin

lift-definition *divide-integer* :: *integer* ⇒ *integer* ⇒ *integer*
is *divide* :: *int* ⇒ *int* ⇒ *int*
⟨*proof*⟩

declare *divide-integer.rep-eq* [*simp*]

lift-definition *modulo-integer* :: *integer* ⇒ *integer* ⇒ *integer*
is *modulo* :: *int* ⇒ *int* ⇒ *int*
⟨*proof*⟩

declare *modulo-integer.rep-eq* [*simp*]

lift-definition *euclidean-size-integer* :: *integer* ⇒ *nat*
is *euclidean-size* :: *int* ⇒ *nat*
⟨*proof*⟩

declare *euclidean-size-integer.rep-eq* [*simp*]

lift-definition *division-segment-integer* :: *integer* ⇒ *integer*
is *division-segment* :: *int* ⇒ *int*
⟨*proof*⟩

declare *division-segment-integer.rep-eq* [*simp*]

instance
⟨*proof*⟩

end

lemma [*code*]:
euclidean-size = *nat-of-integer* ∘ *abs*
⟨*proof*⟩

lemma [*code*]:
division-segment (*k* :: *integer*) = (*if* $k \geq 0$ *then* 1 *else* − 1)
⟨*proof*⟩

instance *integer* :: *unique-euclidean-ring-with-nat*
⟨*proof*⟩

instantiation *integer* :: *ring-bit-operations*
begin

lift-definition *bit-integer* :: ‹*integer* ⇒ *nat* ⇒ *bool*›
  is *bit* ⟨*proof*⟩

lift-definition *not-integer* :: ‹*integer* ⇒ *integer*›
  is *not* ⟨*proof*⟩

lift-definition *and-integer* :: ‹*integer* ⇒ *integer* ⇒ *integer*›
  is ‹*and*› ⟨*proof*⟩

lift-definition *or-integer* :: ‹*integer* ⇒ *integer* ⇒ *integer*›
  is *or* ⟨*proof*⟩

lift-definition *xor-integer* :: ‹*integer* ⇒ *integer* ⇒ *integer*›
  is *xor* ⟨*proof*⟩

lift-definition *mask-integer* :: ‹*nat* ⇒ *integer*›
  is *mask* ⟨*proof*⟩

lift-definition *set-bit-integer* :: ‹*nat* ⇒ *integer* ⇒ *integer*›
  is *set-bit* ⟨*proof*⟩

lift-definition *unset-bit-integer* :: ‹*nat* ⇒ *integer* ⇒ *integer*›
  is *unset-bit* ⟨*proof*⟩

lift-definition *flip-bit-integer* :: ‹*nat* ⇒ *integer* ⇒ *integer*›
  is *flip-bit* ⟨*proof*⟩

lift-definition *push-bit-integer* :: ‹*nat* ⇒ *integer* ⇒ *integer*›
  is *push-bit* ⟨*proof*⟩

lift-definition *drop-bit-integer* :: ‹*nat* ⇒ *integer* ⇒ *integer*›
  is *drop-bit* ⟨*proof*⟩

lift-definition *take-bit-integer* :: ‹*nat* ⇒ *integer* ⇒ *integer*›
  is *take-bit* ⟨*proof*⟩

instance ⟨*proof*⟩

end

instance *integer* :: *unique-euclidean-semiring-with-bit-operations* ⟨*proof*⟩

context
  includes *bit-operations-syntax*
begin

lemma [*code*]:
‹*bit k n* ⟷ *odd* (*drop-bit n k*)›
‹*NOT k* = − *k* − *1*›
‹*mask n* = *2* ^ *n* − (*1* :: *integer*)›
‹*set-bit n k* = *k OR push-bit n 1*›
‹*unset-bit n k* = *k AND NOT* (*push-bit n 1*)›
‹*flip-bit n k* = *k XOR push-bit n 1*›
‹*push-bit n k* = *k* ∗ *2* ^ *n*›
‹*drop-bit n k* = *k div 2* ^ *n*›
‹*take-bit n k* = *k mod 2* ^ *n*› **for** *k* :: *integer*
⟨*proof*⟩

lemma [*code*]:
‹*k AND l* = (*if k* = *0* ∨ *l* = *0 then 0 else if k* = − *1 then l else if l* = − *1 then k*
else (*k mod 2*) ∗ (*l mod 2*) + *2* ∗ ((*k div 2*) *AND* (*l div 2*)))› **for** *k l* :: *integer*
⟨*proof*⟩

lemma [*code*]:
‹*k OR l* = (*if k* = − *1* ∨ *l* = − *1 then* − *1 else if k* = *0 then l else if l* = *0 then*
k
else max (*k mod 2*) (*l mod 2*) + *2* ∗ ((*k div 2*) *OR* (*l div 2*)))› **for** *k l* :: *integer*
⟨*proof*⟩

lemma [*code*]:
‹*k XOR l* = (*if k* = − *1 then NOT l else if l* = − *1 then NOT k else if k* = *0*
then l else if l = *0 then k*
else |*k mod 2* − *l mod 2*| + *2* ∗ ((*k div 2*) *XOR* (*l div 2*)))› **for** *k l* :: *integer*
⟨*proof*⟩

end

instantiation *integer* :: *unique-euclidean-semiring-with-nat-division*
begin

definition *divmod-integer* :: *num* ⇒ *num* ⇒ *integer* × *integer*
where
divmod-integer'-def: *divmod-integer m n* = (*numeral m div numeral n*, *numeral*
m mod numeral n)

definition *divmod-step-integer* :: *integer* ⇒ *integer* × *integer* ⇒ *integer* × *integer*
where
divmod-step-integer l qr = (*let* (*q*, *r*) = *qr*
in if |*l*| ≤ |*r*| *then* (*2* ∗ *q* + *1*, *r* − *l*)
else (*2* ∗ *q*, *r*))

instance ⟨*proof*⟩

end

declare *divmod-algorithm-code* [**where** *?'a = integer*,
folded integer-of-num-def, *unfolded integer-of-num-triv*,
code]

lemma *integer-of-nat-0*: *integer-of-nat 0 = 0*
⟨*proof*⟩

lemma *integer-of-nat-1*: *integer-of-nat 1 = 1*
⟨*proof*⟩

lemma *integer-of-nat-numeral*:
integer-of-nat (*numeral n*) = *numeral n*
⟨*proof*⟩

70.2 Code theorems for target language integers

Constructors

definition *Pos* :: *num* ⇒ *integer*
where
[*simp*, *code-post*]: *Pos* = *numeral*

context
includes *lifting-syntax*
begin

lemma [*transfer-rule*]:
‹((=) ===> *pcr-integer*) *numeral Pos›*
⟨*proof*⟩

end

lemma *Pos-fold* [*code-unfold*]:
numeral Num.One = *Pos Num.One*
numeral (*Num.Bit0 k*) = *Pos* (*Num.Bit0 k*)
numeral (*Num.Bit1 k*) = *Pos* (*Num.Bit1 k*)
⟨*proof*⟩

definition *Neg* :: *num* ⇒ *integer*
where
[*simp*, *code-abbrev*]: *Neg n* = − *Pos n*

context
includes *lifting-syntax*
begin

lemma [*transfer-rule*]:
‹((=) ===> *pcr-integer*) (λ*n*. − *numeral n*) *Neg›*
⟨*proof*⟩

end

code-datatype *0*::*integer Pos Neg*

A further pair of constructors for generated computations

context
begin

qualified definition *positive* :: *num* ⇒ *integer*
  where [*simp*]: *positive* = *numeral*

qualified definition *negative* :: *num* ⇒ *integer*
  where [*simp*]: *negative* = *uminus* ∘ *numeral*

lemma [*code-computation-unfold*]:
  numeral = *positive*
  Pos = *positive*
  Neg = *negative*
  ⟨*proof*⟩

end

Auxiliary operations

lift-definition *dup* :: *integer* ⇒ *integer*
  is λ*k*::*int*. *k* + *k*
  ⟨*proof*⟩

lemma *dup-code* [*code*]:
  dup 0 = *0*
  dup (*Pos n*) = *Pos* (*Num.Bit0 n*)
  dup (*Neg n*) = *Neg* (*Num.Bit0 n*)
  ⟨*proof*⟩

lift-definition *sub* :: *num* ⇒ *num* ⇒ *integer*
  is λ*m n*. *numeral m* − *numeral n* :: *int*
  ⟨*proof*⟩

lemma *sub-code* [*code*]:
  sub Num.One Num.One = *0*
  sub (*Num.Bit0 m*) *Num.One* = *Pos* (*Num.BitM m*)
  sub (*Num.Bit1 m*) *Num.One* = *Pos* (*Num.Bit0 m*)
  sub Num.One (*Num.Bit0 n*) = *Neg* (*Num.BitM n*)
  sub Num.One (*Num.Bit1 n*) = *Neg* (*Num.Bit0 n*)
  sub (*Num.Bit0 m*) (*Num.Bit0 n*) = *dup* (*sub m n*)
  sub (*Num.Bit1 m*) (*Num.Bit1 n*) = *dup* (*sub m n*)
  sub (*Num.Bit1 m*) (*Num.Bit0 n*) = *dup* (*sub m n*) + *1*
  sub (*Num.Bit0 m*) (*Num.Bit1 n*) = *dup* (*sub m n*) − *1*
  ⟨*proof*⟩

Implementations

lemma *one-integer-code* [*code*, *code-unfold*]:
1 = Pos Num.One
⟨*proof*⟩

lemma *plus-integer-code* [*code*]:
k + 0 = (k::integer)
0 + l = (l::integer)
Pos m + Pos n = Pos (m + n)
Pos m + Neg n = sub m n
Neg m + Pos n = sub n m
Neg m + Neg n = Neg (m + n)
⟨*proof*⟩

lemma *uminus-integer-code* [*code*]:
uminus 0 = (0::integer)
uminus (Pos m) = Neg m
uminus (Neg m) = Pos m
⟨*proof*⟩

lemma *minus-integer-code* [*code*]:
k − 0 = (k::integer)
0 − l = uminus (l::integer)
Pos m − Pos n = sub m n
Pos m − Neg n = Pos (m + n)
Neg m − Pos n = Neg (m + n)
Neg m − Neg n = sub n m
⟨*proof*⟩

lemma *abs-integer-code* [*code*]:
|k| = (if (k::integer) < 0 then − k else k)
⟨*proof*⟩

lemma *sgn-integer-code* [*code*]:
sgn k = (if k = 0 then 0 else if (k::integer) < 0 then − 1 else 1)
⟨*proof*⟩

lemma *times-integer-code* [*code*]:
*k * 0 = (0::integer)*
*0 * l = (0::integer)*
*Pos m * Pos n = Pos (m * n)*
*Pos m * Neg n = Neg (m * n)*
*Neg m * Pos n = Neg (m * n)*
*Neg m * Neg n = Pos (m * n)*
⟨*proof*⟩

definition *divmod-integer* :: *integer* ⇒ *integer* ⇒ *integer* × *integer*
where
divmod-integer k l = (k div l, k mod l)

lemma *fst-divmod-integer* [*simp*]:
fst (*divmod-integer k l*) = *k div l*
⟨*proof*⟩

lemma *snd-divmod-integer* [*simp*]:
snd (*divmod-integer k l*) = *k mod l*
⟨*proof*⟩

definition *divmod-abs* :: *integer* ⇒ *integer* ⇒ *integer* × *integer*
where
divmod-abs k l = (|*k*| *div* |*l*|, |*k*| *mod* |*l*|)

lemma *fst-divmod-abs* [*simp*]:
fst (*divmod-abs k l*) = |*k*| *div* |*l*|
⟨*proof*⟩

lemma *snd-divmod-abs* [*simp*]:
snd (*divmod-abs k l*) = |*k*| *mod* |*l*|
⟨*proof*⟩

lemma *divmod-abs-code* [*code*]:
divmod-abs (*Pos k*) (*Pos l*) = *divmod k l*
divmod-abs (*Neg k*) (*Neg l*) = *divmod k l*
divmod-abs (*Neg k*) (*Pos l*) = *divmod k l*
divmod-abs (*Pos k*) (*Neg l*) = *divmod k l*
divmod-abs j 0 = (0, |*j*|)
divmod-abs 0 *j* = (0, 0)
⟨*proof*⟩

lemma *divmod-integer-eq-cases*:
divmod-integer k l =
(*if k* = 0 *then* (0, 0) *else if l* = 0 *then* (0, *k*) *else*
(*apsnd* ∘ *times* ∘ *sgn*) *l* (*if sgn k* = *sgn l*
then divmod-abs k l
else (*let* (*r*, *s*) = *divmod-abs k l in*
if s = 0 *then* (− *r*, 0) *else* (− *r* − 1, |*l*| − *s*))))
⟨*proof*⟩

lemma *divmod-integer-code* [*code*]:
divmod-integer k l =
(*if k* = 0 *then* (0, 0)
else if l > 0 *then*
(*if k* > 0 *then Code-Numeral.divmod-abs k l*
else case Code-Numeral.divmod-abs k l of (*r*, *s*) ⇒
if s = 0 *then* (− *r*, 0) *else* (− *r* − 1, *l* − *s*))
else if l = 0 *then* (0, *k*)
else apsnd uminus
(*if k* < 0 *then Code-Numeral.divmod-abs k l*

else case Code-Numeral.divmod-abs k l of $(r, s) \Rightarrow$
if $s = 0$ *then* $(-r, 0)$ *else* $(-r - 1, -l - s)))$
⟨*proof*⟩

lemma *div-integer-code* [*code*]:
k *div* l = *fst* (*divmod-integer k l*)
⟨*proof*⟩

lemma *mod-integer-code* [*code*]:
k *mod* l = *snd* (*divmod-integer k l*)
⟨*proof*⟩

definition *bit-cut-integer* :: *integer* ⇒ *integer* × *bool*
where *bit-cut-integer k* = (k *div* 2, *odd k*)

lemma *bit-cut-integer-code* [*code*]:
bit-cut-integer k = (*if k* = 0 *then* (0, *False*)
else let (r, s) = *Code-Numeral.divmod-abs k* 2
in (*if k* > 0 *then r else* $-r - s$, $s = 1$))
⟨*proof*⟩

lemma *equal-integer-code* [*code*]:
HOL.equal 0 (0::*integer*) ⟷ *True*
HOL.equal 0 (*Pos l*) ⟷ *False*
HOL.equal 0 (*Neg l*) ⟷ *False*
HOL.equal (*Pos k*) 0 ⟷ *False*
HOL.equal (*Pos k*) (*Pos l*) ⟷ *HOL.equal k l*
HOL.equal (*Pos k*) (*Neg l*) ⟷ *False*
HOL.equal (*Neg k*) 0 ⟷ *False*
HOL.equal (*Neg k*) (*Pos l*) ⟷ *False*
HOL.equal (*Neg k*) (*Neg l*) ⟷ *HOL.equal k l*
⟨*proof*⟩

lemma *equal-integer-refl* [*code nbe*]:
HOL.equal (*k*::*integer*) *k* ⟷ *True*
⟨*proof*⟩

lemma *less-eq-integer-code* [*code*]:
$0 \leq$ (0::*integer*) ⟷ *True*
$0 \leq$ *Pos l* ⟷ *True*
$0 \leq$ *Neg l* ⟷ *False*
Pos k ≤ 0 ⟷ *False*
Pos k $\leq$ *Pos l* ⟷ $k \leq l$
Pos k $\leq$ *Neg l* ⟷ *False*
Neg k ≤ 0 ⟷ *True*
Neg k $\leq$ *Pos l* ⟷ *True*
Neg k $\leq$ *Neg l* ⟷ $l \leq k$
⟨*proof*⟩

lemma *less-integer-code* [*code*]:
0 < (0::integer) ⟷ False
0 < Pos l ⟷ True
0 < Neg l ⟷ False
Pos k < 0 ⟷ False
Pos k < Pos l ⟷ k < l
Pos k < Neg l ⟷ False
Neg k < 0 ⟷ True
Neg k < Pos l ⟷ True
Neg k < Neg l ⟷ l < k
⟨*proof*⟩

lift-definition *num-of-integer* :: *integer ⇒ num*
is *num-of-nat ∘ nat*
⟨*proof*⟩

lemma *num-of-integer-code* [*code*]:
num-of-integer k = (if k ≤ 1 then Num.One
else let
(l, j) = divmod-integer k 2;
l′ = num-of-integer l;
l″ = l′ + l′
in if j = 0 then l″ else l″ + Num.One)
⟨*proof*⟩

lemma *nat-of-integer-code* [*code*]:
nat-of-integer k = (if k ≤ 0 then 0
else let
(l, j) = divmod-integer k 2;
l′ = nat-of-integer l;
l″ = l′ + l′
in if j = 0 then l″ else l″ + 1)
⟨*proof*⟩

lemma *int-of-integer-code* [*code*]:
int-of-integer k = (if k < 0 then − (int-of-integer (− k))
else if k = 0 then 0
else let
(l, j) = divmod-integer k 2;
l′ = 2 ∗ int-of-integer l
in if j = 0 then l′ else l′ + 1)
⟨*proof*⟩

lemma *integer-of-int-code* [*code*]:
integer-of-int k = (if k < 0 then − (integer-of-int (− k))
else if k = 0 then 0
else let
l = 2 ∗ integer-of-int (k div 2);
j = k mod 2

in if j = 0 then l else l + 1)
⟨*proof*⟩

hide-const (**open**) *Pos Neg sub dup divmod-abs*

70.3 Serializer setup for target language integers

code-reserved *Eval int Integer abs*

code-printing
type-constructor *integer* ⇀
(*SML*) *IntInf.int*
and (*OCaml*) *Z.t*
and (*Haskell*) *Integer*
and (*Scala*) *BigInt*
and (*Eval*) *int*
| **class-instance** *integer* :: *equal* ⇀
(*Haskell*) –

code-printing
constant *0*::*integer* ⇀
(*SML*) !(*0*/ :/ *IntInf.int*)
and (*OCaml*) *Z.zero*
and (*Haskell*) !(*0*/ ::/ *Integer*)
and (*Scala*) *BigInt*(*0*)

⟨*ML*⟩

code-printing
constant *plus* :: *integer* ⇒ - ⇒ - ⇀
(*SML*) *IntInf.+* ((-), (-))
and (*OCaml*) *Z.add*
and (*Haskell*) **infixl** *6* +
and (*Scala*) **infixl** *7* +
and (*Eval*) **infixl** *8* +
| **constant** *uminus* :: *integer* ⇒ - ⇀
(*SML*) *IntInf.~*
and (*OCaml*) *Z.neg*
and (*Haskell*) *negate*
and (*Scala*) !(– -)
and (*Eval*) ~/ -
| **constant** *minus* :: *integer* ⇒ - ⇀
(*SML*) *IntInf.–* ((-), (-))
and (*OCaml*) *Z.sub*
and (*Haskell*) **infixl** *6* –
and (*Scala*) **infixl** *7* –
and (*Eval*) **infixl** *8* –
| **constant** *Code-Numeral.dup* ⇀
(*SML*) *IntInf.*∗/ (*2*,/ (-))

and (*OCaml*) *Z.shift'-left/ -/ 1*
and (*Haskell*) !(*2 ∗ -*)
and (*Scala*) !(*2 ∗ -*)
and (*Eval*) !(*2 ∗ -*)
| **constant** *Code-Numeral.sub* ⇀
(*SML*) !(*raise/ Fail/ sub*)
and (*OCaml*) *failwith/ sub*
and (*Haskell*) *error/ sub*
and (*Scala*) !*sys.error*(*sub*)
| **constant** *times* :: *integer* ⇒ *-* ⇒ *-* ⇀
(*SML*) *IntInf.∗* ((*-*), (*-*))
and (*OCaml*) *Z.mul*
and (*Haskell*) **infixl** *7 ∗*
and (*Scala*) **infixl** *8 ∗*
and (*Eval*) **infixl** *9 ∗*
| **constant** *Code-Numeral.divmod-abs* ⇀
(*SML*) *IntInf.divMod/* (*IntInf.abs -,/ IntInf.abs -*)
and (*OCaml*) !(*fun k l −>/ if Z.equal Z.zero l then/* (*Z.zero*, *l*) *else/ Z.div'-rem/* (*Z.abs k*)*/* (*Z.abs l*))
and (*Haskell*) *divMod/* (*abs -*)*/* (*abs -*)
and (*Scala*) !((*k*: *BigInt*) *=>* (*l*: *BigInt*) *=>/ if* (*l == 0*)*/* (*BigInt*(*0*), *k*) *else/* (*k.abs '/% l.abs*))
and (*Eval*) *Integer.div'-mod/* (*abs -*)*/* (*abs -*)
| **constant** *HOL.equal* :: *integer* ⇒ *-* ⇒ *bool* ⇀
(*SML*) !((*-* : *IntInf.int*) = *-*)
and (*OCaml*) *Z.equal*
and (*Haskell*) **infix** *4 ==*
and (*Scala*) **infixl** *5 ==*
and (*Eval*) **infixl** *6 =*
| **constant** *less-eq* :: *integer* ⇒ *-* ⇒ *bool* ⇀
(*SML*) *IntInf.<=* ((*-*), (*-*))
and (*OCaml*) *Z.leq*
and (*Haskell*) **infix** *4 <=*
and (*Scala*) **infixl** *4 <=*
and (*Eval*) **infixl** *6 <=*
| **constant** *less* :: *integer* ⇒ *-* ⇒ *bool* ⇀
(*SML*) *IntInf.<* ((*-*), (*-*))
and (*OCaml*) *Z.lt*
and (*Haskell*) **infix** *4 <*
and (*Scala*) **infixl** *4 <*
and (*Eval*) **infixl** *6 <*
| **constant** *abs* :: *integer* ⇒ *-* ⇀
(*SML*) *IntInf.abs*
and (*OCaml*) *Z.abs*
and (*Haskell*) *Prelude.abs*
and (*Scala*) *-.abs*
and (*Eval*) *abs*

code-identifier

code-module *Code-Numeral* ⇀ (*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

70.4 Type of target language naturals

typedef *natural* = *UNIV* :: *nat set*
  morphisms *nat-of-natural natural-of-nat* ⟨*proof*⟩

setup-lifting *type-definition-natural*

lemma *natural-eq-iff* [*termination-simp*]:
  m = *n* ⟷ *nat-of-natural m* = *nat-of-natural n*
  ⟨*proof*⟩

lemma *natural-eqI*:
  nat-of-natural m = *nat-of-natural n* ⟹ *m* = *n*
  ⟨*proof*⟩

lemma *nat-of-natural-of-nat-inverse* [*simp*]:
  nat-of-natural (*natural-of-nat n*) = *n*
  ⟨*proof*⟩

lemma *natural-of-nat-of-natural-inverse* [*simp*]:
  natural-of-nat (*nat-of-natural n*) = *n*
  ⟨*proof*⟩

instantiation *natural* :: {*comm-monoid-diff*, *semiring-1*}
begin

lift-definition *zero-natural* :: *natural*
  is *0* :: *nat*
  ⟨*proof*⟩

declare *zero-natural.rep-eq* [*simp*]

lift-definition *one-natural* :: *natural*
  is *1* :: *nat*
  ⟨*proof*⟩

declare *one-natural.rep-eq* [*simp*]

lift-definition *plus-natural* :: *natural* ⇒ *natural* ⇒ *natural*
  is *plus* :: *nat* ⇒ *nat* ⇒ *nat*
  ⟨*proof*⟩

declare *plus-natural.rep-eq* [*simp*]

lift-definition *minus-natural* :: *natural* ⇒ *natural* ⇒ *natural*
  is *minus* :: *nat* ⇒ *nat* ⇒ *nat*

⟨*proof*⟩

declare *minus-natural.rep-eq* [*simp*]

lift-definition *times-natural* :: *natural* ⇒ *natural* ⇒ *natural*
is *times* :: *nat* ⇒ *nat* ⇒ *nat*
⟨*proof*⟩

declare *times-natural.rep-eq* [*simp*]

instance ⟨*proof*⟩

end

instance *natural* :: *Rings.dvd* ⟨*proof*⟩

context
includes *lifting-syntax*
begin

lemma [*transfer-rule*]:
‹(*pcr-natural* ===> *pcr-natural* ===> (⟷)) (*dvd*) (*dvd*)›
⟨*proof*⟩

lemma [*transfer-rule*]:
‹((⟷) ===> *pcr-natural*) *of-bool of-bool*›
⟨*proof*⟩

lemma [*transfer-rule*]:
‹((=) ===> *pcr-natural*) (λ*n*. *n*) *of-nat*›
⟨*proof*⟩

lemma [*transfer-rule*]:
‹((=) ===> *pcr-natural*) *numeral numeral*›
⟨*proof*⟩

lemma [*transfer-rule*]:
‹(*pcr-natural* ===> (=) ===> *pcr-natural*) (^) (^)›
⟨*proof*⟩

end

lemma *nat-of-natural-of-nat* [*simp*]:
nat-of-natural (*of-nat n*) = *n*
⟨*proof*⟩

lemma *natural-of-nat-of-nat* [*simp*, *code-abbrev*]:
natural-of-nat = *of-nat*
⟨*proof*⟩

lemma *of-nat-of-natural* [*simp*]:
of-nat (*nat-of-natural n*) = *n*
⟨*proof*⟩

lemma *nat-of-natural-numeral* [*simp*]:
nat-of-natural (*numeral k*) = *numeral k*
⟨*proof*⟩

instantiation *natural* :: {*linordered-semiring*, *equal*}
begin

lift-definition *less-eq-natural* :: *natural* ⇒ *natural* ⇒ *bool*
is *less-eq* :: *nat* ⇒ *nat* ⇒ *bool*
⟨*proof*⟩

declare *less-eq-natural.rep-eq* [*termination-simp*]

lift-definition *less-natural* :: *natural* ⇒ *natural* ⇒ *bool*
is *less* :: *nat* ⇒ *nat* ⇒ *bool*
⟨*proof*⟩

declare *less-natural.rep-eq* [*termination-simp*]

lift-definition *equal-natural* :: *natural* ⇒ *natural* ⇒ *bool*
is *HOL.equal* :: *nat* ⇒ *nat* ⇒ *bool*
⟨*proof*⟩

instance ⟨*proof*⟩

end

context
includes *lifting-syntax*
begin

lemma [*transfer-rule*]:
‹(*pcr-natural* ===> *pcr-natural* ===> *pcr-natural*) *min min*›
⟨*proof*⟩

lemma [*transfer-rule*]:
‹(*pcr-natural* ===> *pcr-natural* ===> *pcr-natural*) *max max*›
⟨*proof*⟩

end

lemma *nat-of-natural-min* [*simp*]:
nat-of-natural (*min k l*) = *min* (*nat-of-natural k*) (*nat-of-natural l*)
⟨*proof*⟩

lemma *nat-of-natural-max* [*simp*]:
nat-of-natural (*max k l*) = *max* (*nat-of-natural k*) (*nat-of-natural l*)
⟨*proof*⟩

instantiation *natural* :: *unique-euclidean-semiring*
begin

lift-definition *divide-natural* :: *natural* ⇒ *natural* ⇒ *natural*
is *divide* :: *nat* ⇒ *nat* ⇒ *nat*
⟨*proof*⟩

declare *divide-natural.rep-eq* [*simp*]

lift-definition *modulo-natural* :: *natural* ⇒ *natural* ⇒ *natural*
is *modulo* :: *nat* ⇒ *nat* ⇒ *nat*
⟨*proof*⟩

declare *modulo-natural.rep-eq* [*simp*]

lift-definition *euclidean-size-natural* :: *natural* ⇒ *nat*
is *euclidean-size* :: *nat* ⇒ *nat*
⟨*proof*⟩

declare *euclidean-size-natural.rep-eq* [*simp*]

lift-definition *division-segment-natural* :: *natural* ⇒ *natural*
is *division-segment* :: *nat* ⇒ *nat*
⟨*proof*⟩

declare *division-segment-natural.rep-eq* [*simp*]

instance
⟨*proof*⟩

end

lemma [*code*]:
euclidean-size = *nat-of-natural*
⟨*proof*⟩

lemma [*code*]:
division-segment (*n*::*natural*) = *1*
⟨*proof*⟩

instance *natural* :: *linordered-semidom*
⟨*proof*⟩

instance *natural* :: *unique-euclidean-semiring-with-nat*

⟨*proof*⟩

instantiation *natural* :: *semiring-bit-operations*
begin

lift-definition *bit-natural* :: ‹*natural* ⇒ *nat* ⇒ *bool*›
is *bit* ⟨*proof*⟩

lift-definition *and-natural* :: ‹*natural* ⇒ *natural* ⇒ *natural*›
is ‹*and*› ⟨*proof*⟩

lift-definition *or-natural* :: ‹*natural* ⇒ *natural* ⇒ *natural*›
is *or* ⟨*proof*⟩

lift-definition *xor-natural* :: ‹*natural* ⇒ *natural* ⇒ *natural*›
is *xor* ⟨*proof*⟩

lift-definition *mask-natural* :: ‹*nat* ⇒ *natural*›
is *mask* ⟨*proof*⟩

lift-definition *set-bit-natural* :: ‹*nat* ⇒ *natural* ⇒ *natural*›
is *set-bit* ⟨*proof*⟩

lift-definition *unset-bit-natural* :: ‹*nat* ⇒ *natural* ⇒ *natural*›
is *unset-bit* ⟨*proof*⟩

lift-definition *flip-bit-natural* :: ‹*nat* ⇒ *natural* ⇒ *natural*›
is *flip-bit* ⟨*proof*⟩

lift-definition *push-bit-natural* :: ‹*nat* ⇒ *natural* ⇒ *natural*›
is *push-bit* ⟨*proof*⟩

lift-definition *drop-bit-natural* :: ‹*nat* ⇒ *natural* ⇒ *natural*›
is *drop-bit* ⟨*proof*⟩

lift-definition *take-bit-natural* :: ‹*nat* ⇒ *natural* ⇒ *natural*›
is *take-bit* ⟨*proof*⟩

instance ⟨*proof*⟩

end

instance *natural* :: *unique-euclidean-semiring-with-bit-operations* ⟨*proof*⟩

context
includes *bit-operations-syntax*
begin

lemma [*code*]:

‹*bit m n* ⟷ *odd* (*drop-bit n m*)›
‹*mask n* = *2* ^ *n* − (*1* :: *integer*)›
‹*set-bit n m* = *m OR push-bit n 1*›
‹*flip-bit n m* = *m XOR push-bit n 1*›
‹*push-bit n m* = *m* ∗ *2* ^ *n*›
‹*drop-bit n m* = *m div 2* ^ *n*›
‹*take-bit n m* = *m mod 2* ^ *n*› **for** *m* :: *natural*
⟨*proof*⟩

lemma [*code*]:
‹*m AND n* = (*if m* = *0* ∨ *n* = *0 then 0*
else (*m mod 2*) ∗ (*n mod 2*) + *2* ∗ ((*m div 2*) *AND* (*n div 2*)))› **for** *m n* :: *natural*
⟨*proof*⟩

lemma [*code*]:
‹*m OR n* = (*if m* = *0 then n else if n* = *0 then m*
else max (*m mod 2*) (*n mod 2*) + *2* ∗ ((*m div 2*) *OR* (*n div 2*)))› **for** *m n* :: *natural*
⟨*proof*⟩

lemma [*code*]:
‹*m XOR n* = (*if m* = *0 then n else if n* = *0 then m*
else (*m mod 2* + *n mod 2*) *mod 2* + *2* ∗ ((*m div 2*) *XOR* (*n div 2*)))› **for** *m n* :: *natural*
⟨*proof*⟩

lemma [*code*]:
‹*unset-bit 0 m* = *2* ∗ (*m div 2*)›
‹*unset-bit* (*Suc n*) *m* = *m mod 2* + *2* ∗ *unset-bit n* (*m div 2*)› **for** *m* :: *natural*
⟨*proof*⟩

end

lift-definition *natural-of-integer* :: *integer* ⇒ *natural*
is *nat* :: *int* ⇒ *nat*
⟨*proof*⟩

lift-definition *integer-of-natural* :: *natural* ⇒ *integer*
is *of-nat* :: *nat* ⇒ *int*
⟨*proof*⟩

lemma *natural-of-integer-of-natural* [*simp*]:
natural-of-integer (*integer-of-natural n*) = *n*
⟨*proof*⟩

lemma *integer-of-natural-of-integer* [*simp*]:
integer-of-natural (*natural-of-integer k*) = *max 0 k*
⟨*proof*⟩

lemma *int-of-integer-of-natural* [*simp*]:
int-of-integer (*integer-of-natural n*) = *of-nat* (*nat-of-natural n*)
⟨*proof*⟩

lemma *integer-of-natural-of-nat* [*simp*]:
integer-of-natural (*of-nat n*) = *of-nat n*
⟨*proof*⟩

lemma [*measure-function*]:
is-measure nat-of-natural
⟨*proof*⟩

70.5 Inductive representation of target language naturals

lift-definition *Suc* :: *natural* ⇒ *natural*
is *Nat.Suc*
⟨*proof*⟩

declare *Suc.rep-eq* [*simp*]

old-rep-datatype *0*::*natural Suc*
⟨*proof*⟩

lemma *natural-cases* [*case-names nat*, *cases type*: *natural*]:
fixes *m* :: *natural*
assumes ⋀*n*. *m* = *of-nat n* ⟹ *P*
shows *P*
⟨*proof*⟩

instantiation *natural* :: *size*
begin

definition *size-nat* **where** [*simp*, *code*]: *size-nat* = *nat-of-natural*

instance ⟨*proof*⟩

end

lemma *natural-decr* [*termination-simp*]:
n ≠ *0* ⟹ *nat-of-natural n* − *Nat.Suc 0* < *nat-of-natural n*
⟨*proof*⟩

lemma *natural-zero-minus-one*: (*0*::*natural*) − *1* = *0*
⟨*proof*⟩

lemma *Suc-natural-minus-one*: *Suc n* − *1* = *n*
⟨*proof*⟩

hide-const (**open**) *Suc*

70.6 Code refinement for target language naturals

lift-definition *Nat* :: *integer* ⇒ *natural*
is *nat*
⟨*proof*⟩

lemma [*code-post*]:
Nat 0 = 0
Nat 1 = 1
Nat (*numeral k*) = *numeral k*
⟨*proof*⟩

lemma [*code abstype*]:
Nat (*integer-of-natural n*) = *n*
⟨*proof*⟩

lemma [*code*]:
natural-of-nat n = *natural-of-integer* (*integer-of-nat n*)
⟨*proof*⟩

lemma [*code abstract*]:
integer-of-natural (*natural-of-integer k*) = *max 0 k*
⟨*proof*⟩

lemma [*code*]:
‹*integer-of-natural* (*mask n*) = *mask n*›
⟨*proof*⟩

lemma [*code-abbrev*]:
natural-of-integer (*Code-Numeral.Pos k*) = *numeral k*
⟨*proof*⟩

lemma [*code abstract*]:
integer-of-natural 0 = 0
⟨*proof*⟩

lemma [*code abstract*]:
integer-of-natural 1 = 1
⟨*proof*⟩

lemma [*code abstract*]:
integer-of-natural (*Code-Numeral.Suc n*) = *integer-of-natural n + 1*
⟨*proof*⟩

lemma [*code*]:
nat-of-natural = *nat-of-integer* ∘ *integer-of-natural*
⟨*proof*⟩

lemma [*code*, *code-unfold*]:
case-natural f g n = (*if n* = *0 then f else g* (*n* − *1*))
⟨*proof*⟩

declare *natural.rec* [*code del*]

lemma [*code abstract*]:
integer-of-natural (*m* + *n*) = *integer-of-natural m* + *integer-of-natural n*
⟨*proof*⟩

lemma [*code abstract*]:
integer-of-natural (*m* − *n*) = *max 0* (*integer-of-natural m* − *integer-of-natural n*)
⟨*proof*⟩

lemma [*code abstract*]:
integer-of-natural (*m* ∗ *n*) = *integer-of-natural m* ∗ *integer-of-natural n*
⟨*proof*⟩

lemma [*code abstract*]:
integer-of-natural (*m div n*) = *integer-of-natural m div integer-of-natural n*
⟨*proof*⟩

lemma [*code abstract*]:
integer-of-natural (*m mod n*) = *integer-of-natural m mod integer-of-natural n*
⟨*proof*⟩

lemma [*code*]:
HOL.equal m n ⟷ *HOL.equal* (*integer-of-natural m*) (*integer-of-natural n*)
⟨*proof*⟩

lemma [*code nbe*]: *HOL.equal n* (*n*::*natural*) ⟷ *True*
⟨*proof*⟩

lemma [*code*]: *m* ≤ *n* ⟷ *integer-of-natural m* ≤ *integer-of-natural n*
⟨*proof*⟩

lemma [*code*]: *m* < *n* ⟷ *integer-of-natural m* < *integer-of-natural n*
⟨*proof*⟩

hide-const (**open**) *Nat*

code-reflect *Code-Numeral*
datatypes *natural*
functions *Code-Numeral.Suc 0* :: *natural 1* :: *natural*
plus :: *natural* ⇒ - *minus* :: *natural* ⇒ -
times :: *natural* ⇒ - *divide* :: *natural* ⇒ -
modulo :: *natural* ⇒ -

integer-of-natural natural-of-integer

lifting-update *integer.lifting*
lifting-forget *integer.lifting*

lifting-update *natural.lifting*
lifting-forget *natural.lifting*

end

71 A HOL random engine

theory *Random*
imports *List Groups-List Code-Numeral*
begin

71.1 Auxiliary functions

fun *log* :: *natural* ⇒ *natural* ⇒ *natural* **where**
log b i = (*if b* ≤ *1* ∨ *i* < *b then 1 else 1* + *log b* (*i div b*))

definition *inc-shift* :: *natural* ⇒ *natural* ⇒ *natural* **where**
inc-shift v k = (*if v* = *k then 1 else k* + *1*)

definition *minus-shift* :: *natural* ⇒ *natural* ⇒ *natural* ⇒ *natural* **where**
minus-shift r k l = (*if k* < *l then r* + *k* − *l else k* − *l*)

71.2 Random seeds

type-synonym *seed* = *natural* × *natural*

primrec *next* :: *seed* ⇒ *natural* × *seed* **where**
next (*v*, *w*) = (*let*
k = *v div 53668*;
v' = *minus-shift 2147483563* ((*v mod 53668*) ∗ *40014*) (*k* ∗ *12211*);
l = *w div 52774*;
w' = *minus-shift 2147483399* ((*w mod 52774*) ∗ *40692*) (*l* ∗ *3791*);
z = *minus-shift 2147483562 v'* (*w'* + *1*) + *1*
in (*z*, (*v'*, *w'*)))

definition *split-seed* :: *seed* ⇒ *seed* × *seed* **where**
split-seed s = (*let*
(*v*, *w*) = *s*;
(*v'*, *w'*) = *snd* (*next s*);
v'' = *inc-shift 2147483562 v*;
w'' = *inc-shift 2147483398 w*
in ((*v''*, *w'*), (*v'*, *w''*)))

71.3 Base selectors

context
includes *state-combinator-syntax*
begin

fun *iterate* :: *natural* ⇒ ('*b* ⇒ '*a* ⇒ '*b* × '*a*) ⇒ '*b* ⇒ '*a* ⇒ '*b* × '*a* **where**
iterate k f x = (*if k* = *0 then Pair x else f x* ∘→ *iterate* (*k* − *1*) *f*)

definition *range* :: *natural* ⇒ *seed* ⇒ *natural* × *seed* **where**
range k = *iterate* (*log 2147483561 k*)
(λ*l*. *next* ∘→ (λ*v*. *Pair* (*v* + *l* ∗ *2147483561*))) *1*
∘→ (λ*v*. *Pair* (*v mod k*))

lemma *range*:
k > *0* ⟹ *fst* (*range k s*) < *k*
⟨*proof*⟩

definition *select* :: '*a list* ⇒ *seed* ⇒ '*a* × *seed* **where**
select xs = *range* (*natural-of-nat* (*length xs*))
∘→ (λ*k*. *Pair* (*nth xs* (*nat-of-natural k*)))

lemma *select*:
assumes *xs* ≠ []
shows *fst* (*select xs s*) ∈ *set xs*
⟨*proof*⟩

primrec *pick* :: (*natural* × '*a*) *list* ⇒ *natural* ⇒ '*a* **where**
pick (*x* # *xs*) *i* = (*if i* < *fst x then snd x else pick xs* (*i* − *fst x*))

lemma *pick-member*:
i < *sum-list* (*map fst xs*) ⟹ *pick xs i* ∈ *set* (*map snd xs*)
⟨*proof*⟩

lemma *pick-drop-zero*:
pick (*filter* (λ(*k*, -). *k* > *0*) *xs*) = *pick xs*
⟨*proof*⟩

lemma *pick-same*:
l < *length xs* ⟹ *Random.pick* (*map* (*Pair 1*) *xs*) (*natural-of-nat l*) = *nth xs l*
⟨*proof*⟩

definition *select-weight* :: (*natural* × '*a*) *list* ⇒ *seed* ⇒ '*a* × *seed* **where**
select-weight xs = *range* (*sum-list* (*map fst xs*))
∘→ (λ*k*. *Pair* (*pick xs k*))

lemma *select-weight-member*:
assumes *0* < *sum-list* (*map fst xs*)
shows *fst* (*select-weight xs s*) ∈ *set* (*map snd xs*)
⟨*proof*⟩

lemma *select-weight-cons-zero*:
select-weight ((*0*, *x*) # *xs*) = *select-weight xs*
⟨*proof*⟩

lemma *select-weight-drop-zero*:
select-weight (*filter* (λ(*k*, -). *k* > *0*) *xs*) = *select-weight xs*
⟨*proof*⟩

lemma *select-weight-select*:
assumes *xs* ≠ []
shows *select-weight* (*map* (*Pair 1*) *xs*) = *select xs*
⟨*proof*⟩

end

71.4 *ML* interface

code-reflect *Random-Engine*
functions *range select select-weight*

⟨*ML*⟩

hide-type (**open**) *seed*
hide-const (**open**) *inc-shift minus-shift log next split-seed*
iterate range select pick select-weight
hide-fact (**open**) *range-def*

end

72 Maps

theory *Map*
imports *List*
abbrevs (= = ⊆$_m$
begin

type-synonym ('*a*, '*b*) *map* = '*a* ⇒ '*b option* (**infixr** ⇀ *0*)

abbreviation (*input*)
empty :: '*a* ⇀ '*b* **where**
empty ≡ λ*x*. *None*

definition
map-comp :: ('*b* ⇀ '*c*) ⇒ ('*a* ⇀ '*b*) ⇒ ('*a* ⇀ '*c*) (**infixl** ∘$_m$ *55*) **where**
f ∘$_m$ *g* = (λ*k*. *case g k of None* ⇒ *None* | *Some v* ⇒ *f v*)

definition
map-add :: ('*a* ⇀ '*b*) ⇒ ('*a* ⇀ '*b*) ⇒ ('*a* ⇀ '*b*) (**infixl** ++ *100*) **where**

m1 ++ m2 = (λx. case m2 x of None ⇒ m1 x | Some y ⇒ Some y)

definition
restrict-map :: *('a ⇀ 'b) ⇒ 'a set ⇒ ('a ⇀ 'b)* (**infixl** |` *110*) **where**
m|`A = (λx. if x ∈ A then m x else None)

notation (*latex* **output**)
restrict-map (-|`- [*111*,*110*] *110*)

definition
dom :: *('a ⇀ 'b) ⇒ 'a set* **where**
dom m = {a. m a ≠ None}

definition
ran :: *('a ⇀ 'b) ⇒ 'b set* **where**
ran m = {b. ∃ a. m a = Some b}

definition
graph :: *('a ⇀ 'b) ⇒ ('a × 'b) set* **where**
graph m = {(a, b) | a b. m a = Some b}

definition
map-le :: *('a ⇀ 'b) ⇒ ('a ⇀ 'b) ⇒ bool* (**infix** $\subseteq_m$ *50*) **where**
$(m_1 \subseteq_m m_2) \longleftrightarrow (\forall a \in dom\ m_1.\ m_1\ a = m_2\ a)$

Function update syntax $f(x := y, \ldots)$ is extended with $x \mapsto y$, which is short for $x := Some\ y$. := and $\mapsto$ can be mixed freely. The syntax $[x \mapsto y, \ldots]$ is short for $Map.empty(x \mapsto y, \ldots)$ but must only contain $\mapsto$, not :=, because $[x{:=}y]$ clashes with the list update syntax $xs[i{:=}x]$.

nonterminal *maplet* **and** *maplets*

syntax
-maplet :: *['a, 'a] ⇒ maplet* (- /↦/ -)
:: *maplet ⇒ updbind* (-)
:: *maplet ⇒ maplets* (-)
-Maplets :: *[maplet, maplets] ⇒ maplets* (-,/ -)
-Map :: *maplets ⇒ 'a ⇀ 'b* ((*1*[-]))

syntax (*ASCII*)
-maplet :: *['a, 'a] ⇒ maplet* (- /|->/ -)

translations
-Update f (-maplet x y) ⇌ f(x := CONST Some y)
-Maplets m ms ⇀ -updbinds m ms
-Map ms ⇀ -Update (CONST empty) ms

-Map (-maplet x y) ↽ -Update (λu. CONST None) (-maplet x y)

-Map (*-updbinds m* (*-maplet x y*)) ↽ *-Update* (*-Map m*) (*-maplet x y*)

Updating with lists:

primrec *map-of* :: (*'a* × *'b*) *list* ⇒ *'a* ⇀ *'b* **where**
map-of [] = *empty*
| *map-of* (*p* # *ps*) = (*map-of ps*)(*fst p* ↦ *snd p*)

lemma *map-of-Cons-code* [*code*]:
map-of [] *k* = *None*
map-of ((*l*, *v*) # *ps*) *k* = (*if l* = *k then Some v else map-of ps k*)
⟨*proof*⟩

definition *map-upds* :: (*'a* ⇀ *'b*) ⇒ *'a list* ⇒ *'b list* ⇒ *'a* ⇀ *'b* **where**
map-upds m xs ys = *m* ++ *map-of* (*rev* (*zip xs ys*))

There is also the more specialized update syntax *xs* [↦] *ys* for lists *xs* and *ys*.

syntax
-maplets :: [*'a*, *'a*] ⇒ *maplet* (- /[↦]/ -)

syntax (*ASCII*)
-maplets :: [*'a*, *'a*] ⇒ *maplet* (- /[|->]/ -)

translations
-Update m (*-maplets xs ys*) ⇌ *CONST map-upds m xs ys*

-Map (*-maplets xs ys*) ↽ *-Update* (λ*u*. *CONST None*) (*-maplets xs ys*)
-Map (*-updbinds m* (*-maplets xs ys*)) ↽ *-Update* (*-Map m*) (*-maplets xs ys*)

72.1 *empty*

lemma *empty-upd-none* [*simp*]: *empty*(*x* := *None*) = *empty*
⟨*proof*⟩

72.2 *map-upd*

lemma *map-upd-triv*: *t k* = *Some x* ⟹ *t*(*k*↦*x*) = *t*
⟨*proof*⟩

lemma *map-upd-nonempty* [*simp*]: *t*(*k*↦*x*) ≠ *empty*
⟨*proof*⟩

lemma *map-upd-eqD1*:
assumes *m*(*a*↦*x*) = *n*(*a*↦*y*)
shows *x* = *y*
⟨*proof*⟩

lemma *map-upd-Some-unfold*:
((*m*(*a*↦*b*)) *x* = *Some y*) = (*x* = *a* ∧ *b* = *y* ∨ *x* ≠ *a* ∧ *m x* = *Some y*)

⟨*proof*⟩

lemma *image-map-upd* [*simp*]: $x \notin A \Longrightarrow m(x \mapsto y)\ `\ A = m\ `\ A$
⟨*proof*⟩

lemma *finite-range-updI*:
assumes *finite* (*range f*) **shows** *finite* (*range* ($f(a \mapsto b)$))
⟨*proof*⟩

72.3 *map-of*

lemma *map-of-eq-empty-iff* [*simp*]:
map-of xys = *empty* ⟷ *xys* = []
⟨*proof*⟩

lemma *empty-eq-map-of-iff* [*simp*]:
empty = *map-of xys* ⟷ *xys* = []
⟨*proof*⟩

lemma *map-of-eq-None-iff*:
(*map-of xys x* = *None*) = (*x* ∉ *fst* ` (*set xys*))
⟨*proof*⟩

lemma *map-of-eq-Some-iff* [*simp*]:
distinct(*map fst xys*) ⟹ (*map-of xys x* = *Some y*) = ((*x*,*y*) ∈ *set xys*)
⟨*proof*⟩

lemma *Some-eq-map-of-iff* [*simp*]:
distinct(*map fst xys*) ⟹ (*Some y* = *map-of xys x*) = ((*x*,*y*) ∈ *set xys*)
⟨*proof*⟩

lemma *map-of-is-SomeI* [*simp*]:
⟦*distinct*(*map fst xys*); (*x*,*y*) ∈ *set xys*⟧ ⟹ *map-of xys x* = *Some y*
⟨*proof*⟩

lemma *map-of-zip-is-None* [*simp*]:
length xs = *length ys* ⟹ (*map-of* (*zip xs ys*) *x* = *None*) = (*x* ∉ *set xs*)
⟨*proof*⟩

lemma *map-of-zip-is-Some*:
assumes *length xs* = *length ys*
shows *x* ∈ *set xs* ⟷ (∃ *y*. *map-of* (*zip xs ys*) *x* = *Some y*)
⟨*proof*⟩

lemma *map-of-zip-upd*:
fixes *x* :: *'a* **and** *xs* :: *'a list* **and** *ys zs* :: *'b list*
assumes *length ys* = *length xs*
and *length zs* = *length xs*
and *x* ∉ *set xs*

and (*map-of* (*zip xs ys*))($x \mapsto y$) = (*map-of* (*zip xs zs*))($x \mapsto z$)
shows *map-of* (*zip xs ys*) = *map-of* (*zip xs zs*)
⟨*proof*⟩

lemma *map-of-zip-inject*:
assumes *length ys* = *length xs*
and *length zs* = *length xs*
and *dist*: *distinct xs*
and *map-of*: *map-of* (*zip xs ys*) = *map-of* (*zip xs zs*)
shows *ys* = *zs*
⟨*proof*⟩

lemma *map-of-zip-nth*:
assumes *length xs* = *length ys*
assumes *distinct xs*
assumes *i* < *length ys*
shows *map-of* (*zip xs ys*) (*xs* ! *i*) = *Some* (*ys* ! *i*)
⟨*proof*⟩

lemma *map-of-zip-map*:
map-of (*zip xs* (*map f xs*)) = (λ*x*. *if x* ∈ *set xs then Some* (*f x*) *else None*)
⟨*proof*⟩

lemma *finite-range-map-of*: *finite* (*range* (*map-of xys*))
⟨*proof*⟩

lemma *map-of-SomeD*: *map-of xs k* = *Some y* ⟹ (*k*, *y*) ∈ *set xs*
⟨*proof*⟩

lemma *map-of-mapk-SomeI*:
inj f ⟹ *map-of t k* = *Some x* ⟹
map-of (*map* (*case-prod* (λ*k*. *Pair* (*f k*))) *t*) (*f k*) = *Some x*
⟨*proof*⟩

lemma *weak-map-of-SomeI*: (*k*, *x*) ∈ *set l* ⟹ ∃ *x*. *map-of l k* = *Some x*
⟨*proof*⟩

lemma *map-of-filter-in*:
map-of xs k = *Some z* ⟹ *P k z* ⟹ *map-of* (*filter* (*case-prod P*) *xs*) *k* = *Some z*
⟨*proof*⟩

lemma *map-of-map*:
map-of (*map* (λ(*k*, *v*). (*k*, *f v*)) *xs*) = *map-option f* ∘ *map-of xs*
⟨*proof*⟩

lemma *dom-map-option*:
dom (λ*k*. *map-option* (*f k*) (*m k*)) = *dom m*
⟨*proof*⟩

lemma *dom-map-option-comp* [*simp*]:
dom (*map-option g* ∘ *m*) = *dom m*
⟨*proof*⟩

72.4 *map-option* related

lemma *map-option-o-empty* [*simp*]: *map-option f* ∘ *empty* = *empty*
⟨*proof*⟩

lemma *map-option-o-map-upd* [*simp*]:
map-option f ∘ *m*(*a*↦*b*) = (*map-option f* ∘ *m*)(*a*↦*f b*)
⟨*proof*⟩

72.5 *map-comp* related

lemma *map-comp-empty* [*simp*]:
m ∘$_m$ *empty* = *empty*
empty ∘$_m$ *m* = *empty*
⟨*proof*⟩

lemma *map-comp-simps* [*simp*]:
m2 k = *None* ⟹ (*m1* ∘$_m$ *m2*) *k* = *None*
m2 k = *Some k′* ⟹ (*m1* ∘$_m$ *m2*) *k* = *m1 k′*
⟨*proof*⟩

lemma *map-comp-Some-iff*:
((*m1* ∘$_m$ *m2*) *k* = *Some v*) = (∃ *k′*. *m2 k* = *Some k′* ∧ *m1 k′* = *Some v*)
⟨*proof*⟩

lemma *map-comp-None-iff*:
((*m1* ∘$_m$ *m2*) *k* = *None*) = (*m2 k* = *None* ∨ (∃ *k′*. *m2 k* = *Some k′* ∧ *m1 k′* = *None*))
⟨*proof*⟩

72.6 ++

lemma *map-add-empty*[*simp*]: *m* ++ *empty* = *m*
⟨*proof*⟩

lemma *empty-map-add*[*simp*]: *empty* ++ *m* = *m*
⟨*proof*⟩

lemma *map-add-assoc*[*simp*]: *m1* ++ (*m2* ++ *m3*) = (*m1* ++ *m2*) ++ *m3*
⟨*proof*⟩

lemma *map-add-Some-iff*:
((*m* ++ *n*) *k* = *Some x*) = (*n k* = *Some x* ∨ *n k* = *None* ∧ *m k* = *Some x*)
⟨*proof*⟩

lemma *map-add-SomeD* [*dest*!]:
(*m* ++ *n*) *k* = *Some x* ⟹ *n k* = *Some x* ∨ *n k* = *None* ∧ *m k* = *Some x*
⟨*proof*⟩

lemma *map-add-find-right* [*simp*]: *n k* = *Some xx* ⟹ (*m* ++ *n*) *k* = *Some xx*
⟨*proof*⟩

lemma *map-add-None* [*iff*]: ((*m* ++ *n*) *k* = *None*) = (*n k* = *None* ∧ *m k* = *None*)
⟨*proof*⟩

lemma *map-add-upd*[*simp*]: *f* ++ *g*(*x*↦*y*) = (*f* ++ *g*)(*x*↦*y*)
⟨*proof*⟩

lemma *map-add-upds*[*simp*]: *m1* ++ (*m2*(*xs*[↦]*ys*)) = (*m1*++*m2*)(*xs*[↦]*ys*)
⟨*proof*⟩

lemma *map-add-upd-left*: *m*∉*dom e2* ⟹ *e1*(*m* ↦ *u1*) ++ *e2* = (*e1* ++ *e2*)(*m* ↦ *u1*)
⟨*proof*⟩

lemma *map-of-append*[*simp*]: *map-of* (*xs* @ *ys*) = *map-of ys* ++ *map-of xs*
⟨*proof*⟩

lemma *finite-range-map-of-map-add*:
finite (*range f*) ⟹ *finite* (*range* (*f* ++ *map-of l*))
⟨*proof*⟩

lemma *inj-on-map-add-dom* [*iff*]:
inj-on (*m* ++ *m*′) (*dom m*′) = *inj-on m*′ (*dom m*′)
⟨*proof*⟩

lemma *map-upds-fold-map-upd*:
m(*ks*[↦]*vs*) = *foldl* (λ*m* (*k*, *v*). *m*(*k* ↦ *v*)) *m* (*zip ks vs*)
⟨*proof*⟩

lemma *map-add-map-of-foldr*:
m ++ *map-of ps* = *foldr* (λ(*k*, *v*) *m*. *m*(*k* ↦ *v*)) *ps m*
⟨*proof*⟩

72.7 *restrict-map*

lemma *restrict-map-to-empty* [*simp*]: *m*|‘{} = *empty*
⟨*proof*⟩

lemma *restrict-map-insert*: *f* |‘ (*insert a A*) = (*f* |‘ *A*)(*a* := *f a*)
⟨*proof*⟩

lemma *restrict-map-empty* [*simp*]: *empty*|‘*D* = *empty*
⟨*proof*⟩

lemma *restrict-in* [*simp*]: $x \in A \Longrightarrow (m|`A)\ x = m\ x$
⟨*proof*⟩

lemma *restrict-out* [*simp*]: $x \notin A \Longrightarrow (m|`A)\ x = None$
⟨*proof*⟩

lemma *ran-restrictD*: $y \in ran\ (m|`A) \Longrightarrow \exists\, x{\in}A.\ m\ x = Some\ y$
⟨*proof*⟩

lemma *dom-restrict* [*simp*]: $dom\ (m|`A) = dom\ m \cap A$
⟨*proof*⟩

lemma *restrict-upd-same* [*simp*]: $m(x{\mapsto}y)|`(-\{x\}) = m|`(-\{x\})$
⟨*proof*⟩

lemma *restrict-restrict* [*simp*]: $m|`A|`B = m|`(A{\cap}B)$
⟨*proof*⟩

lemma *restrict-fun-upd* [*simp*]:
$m(x := y)|`D = (if\ x \in D\ then\ (m|`(D{-}\{x\}))(x := y)\ else\ m|`D)$
⟨*proof*⟩

lemma *fun-upd-None-restrict* [*simp*]:
$(m|`D)(x := None) = (if\ x \in D\ then\ m|`(D - \{x\})\ else\ m|`D)$
⟨*proof*⟩

lemma *fun-upd-restrict*: $(m|`D)(x := y) = (m|`(D{-}\{x\}))(x := y)$
⟨*proof*⟩

lemma *fun-upd-restrict-conv* [*simp*]:
$x \in D \Longrightarrow (m|`D)(x := y) = (m|`(D{-}\{x\}))(x := y)$
⟨*proof*⟩

lemma *map-of-map-restrict*:
$map\text{-}of\ (map\ (\lambda k.\ (k, f\ k))\ ks) = (Some \circ f)\ |`\ set\ ks$
⟨*proof*⟩

lemma *restrict-complement-singleton-eq*:
$f\ |`\ (-\ \{x\}) = f(x := None)$
⟨*proof*⟩

72.8 *map-upds*

lemma *map-upds-Nil1* [*simp*]: $m([]\ [{\mapsto}]\ bs) = m$
⟨*proof*⟩

lemma *map-upds-Nil2* [*simp*]: $m(as\ [{\mapsto}]\ []) = m$
⟨*proof*⟩

lemma *map-upds-Cons* [*simp*]: $m(a\#as\ [\mapsto]\ b\#bs) = (m(a\mapsto b))(as[\mapsto]bs)$
⟨*proof*⟩

lemma *map-upds-append1* [*simp*]:
$size\ xs < size\ ys \Longrightarrow m(xs@[x]\ [\mapsto]\ ys) = m(xs\ [\mapsto]\ ys,\ x \mapsto ys!size\ xs)$
⟨*proof*⟩

lemma *map-upds-list-update2-drop* [*simp*]:
$size\ xs \leq i \Longrightarrow m(xs[\mapsto]ys[i:=y]) = m(xs[\mapsto]ys)$
⟨*proof*⟩

Something weirdly sensitive about this proof, which needs only four lines in apply style

lemma *map-upd-upds-conv-if*:
$(f(x\mapsto y))(xs\ [\mapsto]\ ys) =$
$(if\ x \in set(take\ (length\ ys)\ xs)\ then\ f(xs\ [\mapsto]\ ys)$
$else\ (f(xs\ [\mapsto]\ ys))(x\mapsto y))$
⟨*proof*⟩

lemma *map-upds-twist* [*simp*]:
$a \notin set\ as \Longrightarrow m(a\mapsto b,\ as[\mapsto]bs) = m(as[\mapsto]bs,\ a\mapsto b)$
⟨*proof*⟩

lemma *map-upds-apply-nontin* [*simp*]:
$x \notin set\ xs \Longrightarrow (f(xs[\mapsto]ys))\ x = f\ x$
⟨*proof*⟩

lemma *fun-upds-append-drop* [*simp*]:
$size\ xs = size\ ys \Longrightarrow m(xs@zs[\mapsto]ys) = m(xs[\mapsto]ys)$
⟨*proof*⟩

lemma *fun-upds-append2-drop* [*simp*]:
$size\ xs = size\ ys \Longrightarrow m(xs[\mapsto]ys@zs) = m(xs[\mapsto]ys)$
⟨*proof*⟩

lemma *restrict-map-upds*[*simp*]:
$[\![\ length\ xs = length\ ys;\ set\ xs \subseteq D\ ]\!]$
$\Longrightarrow m(xs\ [\mapsto]\ ys)|`D = (m|`(D - set\ xs))(xs\ [\mapsto]\ ys)$
⟨*proof*⟩

72.9 *dom*

lemma *dom-eq-empty-conv* [*simp*]: $dom\ f = \{\} \longleftrightarrow f = empty$
⟨*proof*⟩

lemma *domI*: $m\ a = Some\ b \Longrightarrow a \in dom\ m$
⟨*proof*⟩

lemma *domD*: $a \in dom\ m \Longrightarrow \exists\, b.\ m\ a = Some\ b$
⟨*proof*⟩

lemma *domIff* [*iff*, *simp del*, *code-unfold*]: $a \in dom\ m \longleftrightarrow m\ a \neq None$
⟨*proof*⟩

lemma *dom-empty* [*simp*]: $dom\ empty = \{\}$
⟨*proof*⟩

lemma *dom-fun-upd* [*simp*]:
$dom(f(x := y)) = (if\ y = None\ then\ dom\ f - \{x\}\ else\ insert\ x\ (dom\ f))$
⟨*proof*⟩

lemma *dom-if*:
$dom\ (\lambda x.\ if\ P\ x\ then\ f\ x\ else\ g\ x) = dom\ f \cap \{x.\ P\ x\} \cup dom\ g \cap \{x.\ \neg\ P\ x\}$
⟨*proof*⟩

lemma *dom-map-of-conv-image-fst*:
$dom\ (map\text{-}of\ xys) = fst\ `\ set\ xys$
⟨*proof*⟩

lemma *dom-map-of-zip* [*simp*]: $length\ xs = length\ ys \Longrightarrow dom\ (map\text{-}of\ (zip\ xs\ ys))$
$= set\ xs$
⟨*proof*⟩

lemma *finite-dom-map-of*: $finite\ (dom\ (map\text{-}of\ l))$
⟨*proof*⟩

lemma *dom-map-upds* [*simp*]:
$dom(m(xs[\mapsto]ys)) = set(take\ (length\ ys)\ xs) \cup dom\ m$
⟨*proof*⟩

lemma *dom-map-add* [*simp*]: $dom\ (m\ ++\ n) = dom\ n \cup dom\ m$
⟨*proof*⟩

lemma *dom-override-on* [*simp*]:
$dom\ (override\text{-}on\ f\ g\ A) =$
$(dom\ f\ - \{a.\ a \in A - dom\ g\}) \cup \{a.\ a \in A \cap dom\ g\}$
⟨*proof*⟩

lemma *map-add-comm*: $dom\ m1 \cap dom\ m2 = \{\} \Longrightarrow m1\ ++\ m2 = m2\ ++\ m1$
⟨*proof*⟩

lemma *map-add-dom-app-simps*:
$m \in dom\ l2 \Longrightarrow (l1\ ++\ l2)\ m = l2\ m$
$m \notin dom\ l1 \Longrightarrow (l1\ ++\ l2)\ m = l2\ m$

$m \notin dom\ l2 \Longrightarrow (l1\ ++\ l2)\ m = l1\ m$
⟨*proof*⟩

lemma *dom-const* [*simp*]:
$dom\ (\lambda x.\ Some\ (f\ x)) = UNIV$
⟨*proof*⟩

lemma *finite-map-freshness*:
$finite\ (dom\ (f :: 'a \rightharpoonup 'b)) \Longrightarrow \neg\ finite\ (UNIV :: 'a\ set) \Longrightarrow$
$\exists x.\ f\ x = None$
⟨*proof*⟩

lemma *dom-minus*:
$f\ x = None \Longrightarrow dom\ f - insert\ x\ A = dom\ f - A$
⟨*proof*⟩

lemma *insert-dom*:
$f\ x = Some\ y \Longrightarrow insert\ x\ (dom\ f) = dom\ f$
⟨*proof*⟩

lemma *map-of-map-keys*:
$set\ xs = dom\ m \Longrightarrow map\text{-}of\ (map\ (\lambda k.\ (k,\ the\ (m\ k)))\ xs) = m$
⟨*proof*⟩

lemma *map-of-eqI*:
assumes *set-eq*: $set\ (map\ fst\ xs) = set\ (map\ fst\ ys)$
assumes *map-eq*: $\forall k \in set\ (map\ fst\ xs).\ map\text{-}of\ xs\ k = map\text{-}of\ ys\ k$
shows $map\text{-}of\ xs = map\text{-}of\ ys$
⟨*proof*⟩

lemma *map-of-eq-dom*:
assumes $map\text{-}of\ xs = map\text{-}of\ ys$
shows $fst\ `\ set\ xs = fst\ `\ set\ ys$
⟨*proof*⟩

lemma *finite-set-of-finite-maps*:
assumes *finite A finite B*
shows $finite\ \{m.\ dom\ m = A \wedge ran\ m \subseteq B\}$ (**is** *finite ?S*)
⟨*proof*⟩

72.10 *ran*

lemma *ranI*: $m\ a = Some\ b \Longrightarrow b \in ran\ m$
⟨*proof*⟩

lemma *ran-empty* [*simp*]: $ran\ empty = \{\}$
⟨*proof*⟩

lemma *ran-map-upd* [*simp*]: $m\ a = None \Longrightarrow ran(m(a{\mapsto}b)) = insert\ b\ (ran\ m)$
⟨*proof*⟩

lemma *fun-upd-None-if-notin-dom*[*simp*]: $k \notin dom\ m \Longrightarrow m(k := None) = m$
⟨*proof*⟩

lemma *ran-map-upd-Some*:
$[\![\ m\ x = Some\ y;\ inj\text{-}on\ m\ (dom\ m);\ z \notin ran\ m\ ]\!] \Longrightarrow ran(m(x := Some\ z)) = ran\ m - \{y\} \cup \{z\}$
⟨*proof*⟩

lemma *ran-map-add*:
assumes $dom\ m1 \cap dom\ m2 = \{\}$
shows $ran\ (m1\ ++\ m2) = ran\ m1 \cup ran\ m2$
⟨*proof*⟩

lemma *finite-ran*:
assumes *finite* (*dom p*)
shows *finite* (*ran p*)
⟨*proof*⟩

lemma *ran-distinct*:
assumes *dist*: *distinct* (*map fst al*)
shows *ran* (*map-of al*) = *snd* ' *set al*
⟨*proof*⟩

lemma *ran-map-of-zip*:
assumes *length xs* = *length ys distinct xs*
shows *ran* (*map-of* (*zip xs ys*)) = *set ys*
⟨*proof*⟩

lemma *ran-map-option*: *ran* (λ*x*. *map-option f* (*m x*)) = *f* ' *ran m*
⟨*proof*⟩

72.11 *graph*

lemma *graph-empty*[*simp*]: *graph empty* = {}
⟨*proof*⟩

lemma *in-graphI*: *m k* = *Some v* ⟹ (*k*, *v*) ∈ *graph m*
⟨*proof*⟩

lemma *in-graphD*: (*k*, *v*) ∈ *graph m* ⟹ *m k* = *Some v*
⟨*proof*⟩

lemma *graph-map-upd*[*simp*]: *graph* (*m*(*k* ↦ *v*)) = *insert* (*k*, *v*) (*graph* (*m*(*k* := *None*)))
⟨*proof*⟩

lemma *graph-fun-upd-None*: *graph* $(m(k := None)) = \{e \in graph\ m.\ fst\ e \neq k\}$
⟨*proof*⟩

lemma *graph-restrictD*:
assumes $(k, v) \in graph\ (m \mid` A)$
shows $k \in A$ **and** $m\ k = Some\ v$
⟨*proof*⟩

lemma *graph-map-comp*[*simp*]: *graph* $(m1 \circ_m m2) = graph\ m2\ O\ graph\ m1$
⟨*proof*⟩

lemma *graph-map-add*: $dom\ m1 \cap dom\ m2 = \{\} \Longrightarrow graph\ (m1 +\!\!+ m2) = graph\ m1 \cup graph\ m2$
⟨*proof*⟩

lemma *graph-eq-to-snd-dom*: *graph* $m = (\lambda x.\ (x, the\ (m\ x)))\ `\ dom\ m$
⟨*proof*⟩

lemma *fst-graph-eq-dom*: $fst\ `\ graph\ m = dom\ m$
⟨*proof*⟩

lemma *graph-domD*: $x \in graph\ m \Longrightarrow fst\ x \in dom\ m$
⟨*proof*⟩

lemma *snd-graph-ran*: $snd\ `\ graph\ m = ran\ m$
⟨*proof*⟩

lemma *graph-ranD*: $x \in graph\ m \Longrightarrow snd\ x \in ran\ m$
⟨*proof*⟩

lemma *finite-graph-map-of*: *finite* (*graph* (*map-of al*))
⟨*proof*⟩

lemma *graph-map-of-if-distinct-dom*: $distinct\ (map\ fst\ al) \Longrightarrow graph\ (map\text{-}of\ al) = set\ al$
⟨*proof*⟩

lemma *finite-graph-iff-finite-dom*[*simp*]: *finite* (*graph m*) = *finite* (*dom m*)
⟨*proof*⟩

lemma *inj-on-fst-graph*: *inj-on fst* (*graph m*)
⟨*proof*⟩

72.12 *map-le*

lemma *map-le-empty* [*simp*]: $empty \subseteq_m g$
⟨*proof*⟩

lemma *upd-None-map-le* [*simp*]: $f(x := None) \subseteq_m f$
⟨*proof*⟩

lemma *map-le-upd*[*simp*]: $f \subseteq_m g \Longrightarrow f(a := b) \subseteq_m g(a := b)$
⟨*proof*⟩

lemma *map-le-imp-upd-le* [*simp*]: $m1 \subseteq_m m2 \Longrightarrow m1(x := None) \subseteq_m m2(x \mapsto y)$
⟨*proof*⟩

lemma *map-le-upds* [*simp*]:
$f \subseteq_m g \Longrightarrow f(as\ [\mapsto]\ bs) \subseteq_m g(as\ [\mapsto]\ bs)$
⟨*proof*⟩

lemma *map-le-implies-dom-le*: $(f \subseteq_m g) \Longrightarrow (dom\ f \subseteq dom\ g)$
⟨*proof*⟩

lemma *map-le-refl* [*simp*]: $f \subseteq_m f$
⟨*proof*⟩

lemma *map-le-trans*[*trans*]: $[\![m1 \subseteq_m m2; m2 \subseteq_m m3]\!] \Longrightarrow m1 \subseteq_m m3$
⟨*proof*⟩

lemma *map-le-antisym*: $[\![f \subseteq_m g; g \subseteq_m f]\!] \Longrightarrow f = g$
⟨*proof*⟩

lemma *map-le-map-add* [*simp*]: $f \subseteq_m g +\!\!+ f$
⟨*proof*⟩

lemma *map-le-iff-map-add-commute*: $f \subseteq_m f +\!\!+ g \longleftrightarrow f +\!\!+ g = g +\!\!+ f$
⟨*proof*⟩

lemma *map-add-le-mapE*: $f +\!\!+ g \subseteq_m h \Longrightarrow g \subseteq_m h$
⟨*proof*⟩

lemma *map-add-le-mapI*: $[\![f \subseteq_m h; g \subseteq_m h]\!] \Longrightarrow f +\!\!+ g \subseteq_m h$
⟨*proof*⟩

lemma *map-add-subsumed1*: $f \subseteq_m g \Longrightarrow f +\!\!+ g = g$
⟨*proof*⟩

lemma *map-add-subsumed2*: $f \subseteq_m g \Longrightarrow g +\!\!+ f = g$
⟨*proof*⟩

lemma *dom-eq-singleton-conv*: $dom\ f = \{x\} \longleftrightarrow (\exists\, v.\ f = [x \mapsto v])$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *map-add-eq-empty-iff*[*simp*]:

$(f{+}{+}g = empty) \longleftrightarrow f = empty \wedge g = empty$
⟨*proof*⟩

lemma *empty-eq-map-add-iff*[*simp*]:
$(empty = f{+}{+}g) \longleftrightarrow f = empty \wedge g = empty$
⟨*proof*⟩

72.13 Various

lemma *set-map-of-compr*:
assumes *distinct*: *distinct* (*map fst xs*)
shows *set xs* = {(*k*, *v*). *map-of xs k* = *Some v*}
⟨*proof*⟩

lemma *eq-key-imp-eq-value*:
v1 = *v2*
if *distinct* (*map fst xs*) (*k*, *v1*) ∈ *set xs* (*k*, *v2*) ∈ *set xs*
⟨*proof*⟩

lemma *map-of-inject-set*:
assumes *distinct*: *distinct* (*map fst xs*) *distinct* (*map fst ys*)
shows *map-of xs* = *map-of ys* ⟷ *set xs* = *set ys* (**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *finite-Map-induct*[*consumes 1*, *case-names empty update*]:
assumes *finite* (*dom m*)
assumes *P Map.empty*
assumes ⋀*k v m*. *finite* (*dom m*) ⟹ *k* ∉ *dom m* ⟹ *P m* ⟹ *P* (*m*(*k* ↦ *v*))
shows *P m*
⟨*proof*⟩

hide-const (**open**) *Map.empty Map.graph*

end

73 Finite types as explicit enumerations

theory *Enum*
imports *Map Groups-List*
begin

73.1 Class *enum*

class *enum* =
fixes *enum* :: *'a list*
fixes *enum-all* :: (*'a* ⇒ *bool*) ⇒ *bool*
fixes *enum-ex* :: (*'a* ⇒ *bool*) ⇒ *bool*
assumes *UNIV-enum*: *UNIV* = *set enum*
and *enum-distinct*: *distinct enum*

assumes *enum-all-UNIV*: *enum-all P* $\longleftrightarrow$ *Ball UNIV P*
assumes *enum-ex-UNIV*: *enum-ex P* $\longleftrightarrow$ *Bex UNIV P*
— tailored towards simple instantiation
begin

subclass *finite* ⟨*proof*⟩

lemma *enum-UNIV*:
set enum = *UNIV*
⟨*proof*⟩

lemma *in-enum*: *x* ∈ *set enum*
⟨*proof*⟩

lemma *enum-eq-I*:
assumes ⋀*x*. *x* ∈ *set xs*
shows *set enum* = *set xs*
⟨*proof*⟩

lemma *card-UNIV-length-enum*:
card (*UNIV* :: *'a set*) = *length enum*
⟨*proof*⟩

lemma *enum-all* [*simp*]:
enum-all = *HOL.All*
⟨*proof*⟩

lemma *enum-ex* [*simp*]:
enum-ex = *HOL.Ex*
⟨*proof*⟩

end

73.2 Implementations using *enum*

73.2.1 Unbounded operations and quantifiers

lemma *Collect-code* [*code*]:
Collect P = *set* (*filter P enum*)
⟨*proof*⟩

lemma *vimage-code* [*code*]:
f −‘ *B* = *set* (*filter* (λ*x*. *f x* ∈ *B*) *enum-class.enum*)
⟨*proof*⟩

definition *card-UNIV* :: *'a itself* ⇒ *nat*
where
[*code del*]: *card-UNIV TYPE*(*'a*) = *card* (*UNIV* :: *'a set*)

lemma [*code*]:

card-UNIV TYPE('a :: enum) = card (set (Enum.enum :: 'a list))
⟨*proof*⟩

lemma *all-code* [*code*]: $(\forall x.\ P\ x) \longleftrightarrow$ *enum-all P*
⟨*proof*⟩

lemma *exists-code* [*code*]: $(\exists x.\ P\ x) \longleftrightarrow$ *enum-ex P*
⟨*proof*⟩

lemma *exists1-code* [*code*]: $(\exists!x.\ P\ x) \longleftrightarrow$ *list-ex1 P enum*
⟨*proof*⟩

73.2.2 An executable choice operator

definition
[*code del*]: *enum-the = The*

lemma [*code*]:
The P = (case filter P enum of [x] ⇒ x | - ⇒ enum-the P)
⟨*proof*⟩

declare [[*code abort*: *enum-the*]]

code-printing
constant *enum-the* ⇀ (*Eval*) (*fn '- => raise Match*)

73.2.3 Equality and order on functions

instantiation *fun* :: (*enum*, *equal*) *equal*
begin

definition
HOL.equal f g $\longleftrightarrow$ $(\forall x \in$ *set enum. f x = g x*$)$

instance ⟨*proof*⟩

end

lemma [*code*]:
HOL.equal f g $\longleftrightarrow$ *enum-all* (%*x. f x = g x*)
⟨*proof*⟩

lemma [*code nbe*]:
HOL.equal (*f* :: - ⇒ -) *f* $\longleftrightarrow$ *True*
⟨*proof*⟩

lemma *order-fun* [*code*]:
fixes *f g* :: 'a::*enum* ⇒ 'b::*order*
shows $f \leq g \longleftrightarrow$ *enum-all* $(\lambda x.\ f\ x \leq g\ x)$
and $f < g \longleftrightarrow f \leq g \wedge$ *enum-ex* $(\lambda x.\ f\ x \neq g\ x)$

⟨*proof*⟩

73.2.4 Operations on relations

lemma [*code*]:
Id = *image* (λ*x*. (*x*, *x*)) (*set Enum.enum*)
⟨*proof*⟩

lemma *tranclp-unfold* [*code*]:
tranclp r a b ⟷ (*a*, *b*) ∈ *trancl* {(*x*, *y*). *r x y*}
⟨*proof*⟩

lemma *rtranclp-rtrancl-eq* [*code*]:
rtranclp r x y ⟷ (*x*, *y*) ∈ *rtrancl* {(*x*, *y*). *r x y*}
⟨*proof*⟩

lemma *max-ext-eq* [*code*]:
max-ext R = {(*X*, *Y*). *finite X* ∧ *finite Y* ∧ *Y* ≠ {} ∧ (∀ *x*. *x* ∈ *X* ⟶ (∃ *xa* ∈ *Y*. (*x*, *xa*) ∈ *R*))}
⟨*proof*⟩

lemma *max-extp-eq* [*code*]:
max-extp r x y ⟷ (*x*, *y*) ∈ *max-ext* {(*x*, *y*). *r x y*}
⟨*proof*⟩

lemma *mlex-eq* [*code*]:
f <∗*mlex*∗> *R* = {(*x*, *y*). *f x* < *f y* ∨ (*f x* ≤ *f y* ∧ (*x*, *y*) ∈ *R*)}
⟨*proof*⟩

73.2.5 Bounded accessible part

primrec *bacc* :: ('*a* × '*a*) *set* ⇒ *nat* ⇒ '*a set*
where
bacc r 0 = {*x*. ∀ *y*. (*y*, *x*) ∉ *r*}
| *bacc r* (*Suc n*) = (*bacc r n* ∪ {*x*. ∀ *y*. (*y*, *x*) ∈ *r* ⟶ *y* ∈ *bacc r n*})

lemma *bacc-subseteq-acc*:
bacc r n ⊆ *Wellfounded.acc r*
⟨*proof*⟩

lemma *bacc-mono*:
n ≤ *m* ⟹ *bacc r n* ⊆ *bacc r m*
⟨*proof*⟩

lemma *bacc-upper-bound*:
bacc (*r* :: ('*a* × '*a*) *set*) (*card* (*UNIV* :: '*a*::*finite set*)) = (⋃ *n*. *bacc r n*)
⟨*proof*⟩

lemma *acc-subseteq-bacc*:
assumes *finite r*

shows *Wellfounded.acc r* ⊆ (⋃ *n*. *bacc r n*)
⟨*proof*⟩

lemma *acc-bacc-eq*:
fixes *A* :: (*'a* :: *finite* × *'a*) *set*
assumes *finite A*
shows *Wellfounded.acc A* = *bacc A* (*card* (*UNIV* :: *'a set*))
⟨*proof*⟩

lemma [*code*]:
fixes *xs* :: (*'a*::*finite* × *'a*) *list*
shows *Wellfounded.acc* (*set xs*) = *bacc* (*set xs*) (*card-UNIV TYPE*(*'a*))
⟨*proof*⟩

73.3 Default instances for *enum*

lemma *map-of-zip-enum-is-Some*:
assumes *length ys* = *length* (*enum* :: *'a*::*enum list*)
shows ∃ *y*. *map-of* (*zip* (*enum* :: *'a*::*enum list*) *ys*) *x* = *Some y*
⟨*proof*⟩

lemma *map-of-zip-enum-inject*:
fixes *xs ys* :: *'b*::*enum list*
assumes *length*: *length xs* = *length* (*enum* :: *'a*::*enum list*)
length ys = *length* (*enum* :: *'a*::*enum list*)
and *map-of*: *the* ∘ *map-of* (*zip* (*enum* :: *'a*::*enum list*) *xs*) = *the* ∘ *map-of* (*zip* (*enum* :: *'a*::*enum list*) *ys*)
shows *xs* = *ys*
⟨*proof*⟩

definition *all-n-lists* :: ((*'a* :: *enum*) *list* ⇒ *bool*) ⇒ *nat* ⇒ *bool*
where
all-n-lists P n ⟷ (∀ *xs* ∈ *set* (*List.n-lists n enum*). *P xs*)

lemma [*code*]:
all-n-lists P n ⟷ (*if n* = *0 then P* [] *else enum-all* (%*x*. *all-n-lists* (%*xs*. *P* (*x* # *xs*)) (*n* − *1*)))
⟨*proof*⟩

definition *ex-n-lists* :: ((*'a* :: *enum*) *list* ⇒ *bool*) ⇒ *nat* ⇒ *bool*
where
ex-n-lists P n ⟷ (∃ *xs* ∈ *set* (*List.n-lists n enum*). *P xs*)

lemma [*code*]:
ex-n-lists P n ⟷ (*if n* = *0 then P* [] *else enum-ex* (%*x*. *ex-n-lists* (%*xs*. *P* (*x* # *xs*)) (*n* − *1*)))
⟨*proof*⟩

instantiation *fun* :: (*enum*, *enum*) *enum*

begin

definition
enum = *map* (λ*ys*. *the* ∘ *map-of* (*zip* (*enum*::′*a list*) *ys*)) (*List.n-lists* (*length* (*enum*::′*a*::*enum list*)) *enum*)

definition
enum-all P = *all-n-lists* (λ*bs*. *P* (*the* ∘ *map-of* (*zip enum bs*))) (*length* (*enum* :: ′*a list*))

definition
enum-ex P = *ex-n-lists* (λ*bs*. *P* (*the* ∘ *map-of* (*zip enum bs*))) (*length* (*enum* :: ′*a list*))

instance ⟨*proof*⟩

end

lemma *enum-fun-code* [*code*]: *enum* = (*let enum-a* = (*enum* :: ′*a*::{*enum*, *equal*} *list*)
in map (λ*ys*. *the* ∘ *map-of* (*zip enum-a ys*)) (*List.n-lists* (*length enum-a*) *enum*))
⟨*proof*⟩

lemma *enum-all-fun-code* [*code*]:
enum-all P = (*let enum-a* = (*enum* :: ′*a*::{*enum*, *equal*} *list*)
in all-n-lists (λ*bs*. *P* (*the* ∘ *map-of* (*zip enum-a bs*))) (*length enum-a*))
⟨*proof*⟩

lemma *enum-ex-fun-code* [*code*]:
enum-ex P = (*let enum-a* = (*enum* :: ′*a*::{*enum*, *equal*} *list*)
in ex-n-lists (λ*bs*. *P* (*the* ∘ *map-of* (*zip enum-a bs*))) (*length enum-a*))
⟨*proof*⟩

instantiation *set* :: (*enum*) *enum*
begin

definition
enum = *map set* (*subseqs enum*)

definition
enum-all P ⟷ (∀ *A*∈*set enum*. *P* (*A*::′*a set*))

definition
enum-ex P ⟷ (∃ *A*∈*set enum*. *P* (*A*::′*a set*))

instance ⟨*proof*⟩

end

instantiation *unit* :: *enum*
begin

definition
enum = [()]

definition
enum-all P = *P* ()

definition
enum-ex P = *P* ()

instance ⟨*proof*⟩

end

instantiation *bool* :: *enum*
begin

definition
enum = [*False*, *True*]

definition
enum-all P ⟷ *P False* ∧ *P True*

definition
enum-ex P ⟷ *P False* ∨ *P True*

instance ⟨*proof*⟩

end

instantiation *prod* :: (*enum*, *enum*) *enum*
begin

definition
enum = *List.product enum enum*

definition
enum-all P = *enum-all* (%*x*. *enum-all* (%*y*. *P* (*x*, *y*)))

definition
enum-ex P = *enum-ex* (%*x*. *enum-ex* (%*y*. *P* (*x*, *y*)))

instance
⟨*proof*⟩

end

instantiation *sum* :: (*enum*, *enum*) *enum*
begin

definition
enum = *map Inl enum* @ *map Inr enum*

definition
enum-all P ⟷ *enum-all* (λ*x*. *P* (*Inl x*)) ∧ *enum-all* (λ*x*. *P* (*Inr x*))

definition
enum-ex P ⟷ *enum-ex* (λ*x*. *P* (*Inl x*)) ∨ *enum-ex* (λ*x*. *P* (*Inr x*))

instance ⟨*proof*⟩

end

instantiation *option* :: (*enum*) *enum*
begin

definition
enum = *None* # *map Some enum*

definition
enum-all P ⟷ *P None* ∧ *enum-all* (λ*x*. *P* (*Some x*))

definition
enum-ex P ⟷ *P None* ∨ *enum-ex* (λ*x*. *P* (*Some x*))

instance ⟨*proof*⟩

end

73.4 Small finite types

We define small finite types for use in Quickcheck

datatype (*plugins only*: *code quickcheck extraction*) *finite-1* =
a_1

notation (**output**) a_1 (a_1)

lemma *UNIV-finite-1*:
UNIV = {a_1}
⟨*proof*⟩

instantiation *finite-1* :: *enum*
begin

definition

enum = [a_1]

definition
enum-all P = P a_1

definition
enum-ex P = P a_1

instance ⟨*proof*⟩

end

instantiation *finite-1* :: *linorder*
begin

definition *less-finite-1* :: *finite-1* ⇒ *finite-1* ⇒ *bool*
where
x < (y :: *finite-1*) ⟷ *False*

definition *less-eq-finite-1* :: *finite-1* ⇒ *finite-1* ⇒ *bool*
where
x ≤ (y :: *finite-1*) ⟷ *True*

instance
⟨*proof*⟩

end

instance *finite-1* :: {*dense-linorder*, *wellorder*}
⟨*proof*⟩

instantiation *finite-1* :: *complete-lattice*
begin

definition [*simp*]: *Inf* = (λ-. a_1)
definition [*simp*]: *Sup* = (λ-. a_1)
definition [*simp*]: *bot* = a_1
definition [*simp*]: *top* = a_1
definition [*simp*]: *inf* = (λ- -. a_1)
definition [*simp*]: *sup* = (λ- -. a_1)

instance ⟨*proof*⟩
end

instance *finite-1* :: *complete-distrib-lattice*
⟨*proof*⟩

instance *finite-1* :: *complete-linorder* ⟨*proof*⟩

lemma *finite-1-eq*: $x = a_1$
⟨*proof*⟩

⟨*ML*⟩

instantiation *finite-1* :: *complete-boolean-algebra*
begin
definition [*simp*]: $(-) = (\lambda\text{-}\ \text{-}.\ a_1)$
definition [*simp*]: *uminus* $= (\lambda\text{-}.\ a_1)$
instance ⟨*proof*⟩
end

instantiation *finite-1* ::
{*linordered-ring-strict*, *linordered-comm-semiring-strict*, *ordered-comm-ring*,
ordered-cancel-comm-monoid-diff, *comm-monoid-mult*, *ordered-ring-abs*,
one, *modulo*, *sgn*, *inverse*}
begin
definition [*simp*]: *Groups.zero* $= a_1$
definition [*simp*]: *Groups.one* $= a_1$
definition [*simp*]: $(+) = (\lambda\text{-}\ \text{-}.\ a_1)$
definition [*simp*]: $(*) = (\lambda\text{-}\ \text{-}.\ a_1)$
definition [*simp*]: $(mod) = (\lambda\text{-}\ \text{-}.\ a_1)$
definition [*simp*]: *abs* $= (\lambda\text{-}.\ a_1)$
definition [*simp*]: *sgn* $= (\lambda\text{-}.\ a_1)$
definition [*simp*]: *inverse* $= (\lambda\text{-}.\ a_1)$
definition [*simp*]: *divide* $= (\lambda\text{-}\ \text{-}.\ a_1)$

instance ⟨*proof*⟩
end

declare [[*simproc del*: *finite-1-eq*]]
hide-const (**open**) a_1

datatype (*plugins only*: *code quickcheck extraction*) *finite-2* =
a_1 | a_2

notation (**output**) a_1 (a_1)
notation (**output**) a_2 (a_2)

lemma *UNIV-finite-2*:
UNIV $= \{a_1, a_2\}$
⟨*proof*⟩

instantiation *finite-2* :: *enum*
begin

definition
enum $= [a_1, a_2]$

definition
enum-all $P \longleftrightarrow P\ a_1 \wedge P\ a_2$

definition
enum-ex $P \longleftrightarrow P\ a_1 \vee P\ a_2$

instance ⟨*proof*⟩

end

instantiation *finite-2* :: *linorder*
begin

definition *less-finite-2* :: *finite-2* ⇒ *finite-2* ⇒ *bool*
where
$x < y \longleftrightarrow x = a_1 \wedge y = a_2$

definition *less-eq-finite-2* :: *finite-2* ⇒ *finite-2* ⇒ *bool*
where
$x \leq y \longleftrightarrow x = y \vee x < (y :: \textit{finite-2})$

instance
⟨*proof*⟩

end

instance *finite-2* :: *wellorder*
⟨*proof*⟩

instantiation *finite-2* :: *complete-lattice*
begin

definition $\bigsqcap A = (\textit{if}\ a_1 \in A\ \textit{then}\ a_1\ \textit{else}\ a_2)$
definition $\bigsqcup A = (\textit{if}\ a_2 \in A\ \textit{then}\ a_2\ \textit{else}\ a_1)$
definition [*simp*]: $\textit{bot} = a_1$
definition [*simp*]: $\textit{top} = a_2$
definition $x \sqcap y = (\textit{if}\ x = a_1 \vee y = a_1\ \textit{then}\ a_1\ \textit{else}\ a_2)$
definition $x \sqcup y = (\textit{if}\ x = a_2 \vee y = a_2\ \textit{then}\ a_2\ \textit{else}\ a_1)$

lemma *neq-finite-2-a₁-iff* [*simp*]: $x \neq a_1 \longleftrightarrow x = a_2$
⟨*proof*⟩

lemma *neq-finite-2-a₁-iff'* [*simp*]: $a_1 \neq x \longleftrightarrow x = a_2$
⟨*proof*⟩

lemma *neq-finite-2-a₂-iff* [*simp*]: $x \neq a_2 \longleftrightarrow x = a_1$
⟨*proof*⟩

lemma *neq-finite-2-a₂-iff'* [*simp*]: $a_2 \neq x \longleftrightarrow x = a_1$

⟨*proof*⟩

instance
⟨*proof*⟩
end

instance *finite-2* :: *complete-linorder* ⟨*proof*⟩

instance *finite-2* :: *complete-distrib-lattice* ⟨*proof*⟩

instantiation *finite-2* :: {*field*, *idom-abs-sgn*, *idom-modulo*} **begin**
definition [*simp*]: $0 = a_1$
definition [*simp*]: $1 = a_2$
definition $x + y = (case\ (x,\ y)\ of\ (a_1,\ a_1) \Rightarrow a_1 \mid (a_2,\ a_2) \Rightarrow a_1 \mid \text{-} \Rightarrow a_2)$
definition $uminus = (\lambda x :: finite\text{-}2.\ x)$
definition $(-) = ((+) :: finite\text{-}2 \Rightarrow \text{-})$
definition $x * y = (case\ (x,\ y)\ of\ (a_2,\ a_2) \Rightarrow a_2 \mid \text{-} \Rightarrow a_1)$
definition $inverse = (\lambda x :: finite\text{-}2.\ x)$
definition $divide = ((*) :: finite\text{-}2 \Rightarrow \text{-})$
definition $x\ mod\ y = (case\ (x,\ y)\ of\ (a_2,\ a_1) \Rightarrow a_2 \mid \text{-} \Rightarrow a_1)$
definition $abs = (\lambda x :: finite\text{-}2.\ x)$
definition $sgn = (\lambda x :: finite\text{-}2.\ x)$
instance
⟨*proof*⟩
end

lemma *two-finite-2* [*simp*]:
$2 = a_1$
⟨*proof*⟩

lemma *dvd-finite-2-unfold*:
$x\ dvd\ y \longleftrightarrow x = a_2 \vee y = a_1$
⟨*proof*⟩

instantiation *finite-2* :: {*normalization-semidom*, *unique-euclidean-semiring*} **begin**
definition [*simp*]: $normalize = (id :: finite\text{-}2 \Rightarrow \text{-})$
definition [*simp*]: $unit\text{-}factor = (id :: finite\text{-}2 \Rightarrow \text{-})$
definition [*simp*]: $euclidean\text{-}size\ x = (case\ x\ of\ a_1 \Rightarrow 0 \mid a_2 \Rightarrow 1)$
definition [*simp*]: $division\text{-}segment\ (x :: finite\text{-}2) = 1$
instance
⟨*proof*⟩
end

hide-const (**open**) $a_1\ a_2$

datatype (*plugins only*: *code quickcheck extraction*) *finite-3* =
$a_1 \mid a_2 \mid a_3$

notation (**output**) a_1 (a_1)
notation (**output**) a_2 (a_2)
notation (**output**) a_3 (a_3)

lemma *UNIV-finite-3*:
$UNIV = \{a_1, a_2, a_3\}$
⟨*proof*⟩

instantiation *finite-3* :: *enum*
begin

definition
$enum = [a_1, a_2, a_3]$

definition
$enum\text{-}all\ P \longleftrightarrow P\ a_1 \wedge P\ a_2 \wedge P\ a_3$

definition
$enum\text{-}ex\ P \longleftrightarrow P\ a_1 \vee P\ a_2 \vee P\ a_3$

instance ⟨*proof*⟩

end

lemma *finite-3-not-eq-unfold*:
$x \neq a_1 \longleftrightarrow x \in \{a_2, a_3\}$
$x \neq a_2 \longleftrightarrow x \in \{a_1, a_3\}$
$x \neq a_3 \longleftrightarrow x \in \{a_1, a_2\}$
⟨*proof*⟩

instantiation *finite-3* :: *linorder*
begin

definition *less-finite-3* :: *finite-3* ⇒ *finite-3* ⇒ *bool*
where
$x < y = (case\ x\ of\ a_1 \Rightarrow y \neq a_1 \mid a_2 \Rightarrow y = a_3 \mid a_3 \Rightarrow False)$

definition *less-eq-finite-3* :: *finite-3* ⇒ *finite-3* ⇒ *bool*
where
$x \leq y \longleftrightarrow x = y \vee x < (y :: finite\text{-}3)$

instance ⟨*proof*⟩

end

instance *finite-3* :: *wellorder*
⟨*proof*⟩

class *finite-lattice* = *finite* + *lattice* + *Inf* + *Sup* + *bot* + *top* +
assumes *Inf-finite-empty*: $Inf\ \{\} = Sup\ UNIV$
assumes *Inf-finite-insert*: $Inf\ (insert\ a\ A) = a \sqcap Inf\ A$
assumes *Sup-finite-empty*: $Sup\ \{\} = Inf\ UNIV$
assumes *Sup-finite-insert*: $Sup\ (insert\ a\ A) = a \sqcup Sup\ A$
assumes *bot-finite-def*: $bot = Inf\ UNIV$
assumes *top-finite-def*: $top = Sup\ UNIV$
begin

subclass *complete-lattice*
⟨*proof*⟩
end

class *finite-distrib-lattice* = *finite-lattice* + *distrib-lattice*
begin
lemma *finite-inf-Sup*: $a \sqcap (Sup\ A) = Sup\ \{a \sqcap b \mid b\ .\ b \in A\}$
⟨*proof*⟩

lemma *finite-Inf-Sup*: $\bigsqcap (Sup\ `\ A) \leq \bigsqcup (Inf\ `\ \{f\ `\ A\ |f.\ \forall\ Y{\in}A.\ f\ Y\ \in\ Y\})$
⟨*proof*⟩

subclass *complete-distrib-lattice*
⟨*proof*⟩
end

instantiation *finite-3* :: *finite-lattice*
begin

definition $\bigsqcap A = (if\ a_1 \in A\ then\ a_1\ else\ if\ a_2 \in A\ then\ a_2\ else\ a_3)$
definition $\bigsqcup A = (if\ a_3 \in A\ then\ a_3\ else\ if\ a_2 \in A\ then\ a_2\ else\ a_1)$
definition [*simp*]: $bot = a_1$
definition [*simp*]: $top = a_3$
definition [*simp*]: $inf = (min :: finite\text{-}3 \Rightarrow \text{-})$
definition [*simp*]: $sup = (max :: finite\text{-}3 \Rightarrow \text{-})$

instance
⟨*proof*⟩
end

instance *finite-3* :: *complete-lattice* ⟨*proof*⟩

instance *finite-3* :: *finite-distrib-lattice*
⟨*proof*⟩

instance *finite-3* :: *complete-distrib-lattice* ⟨*proof*⟩

instance *finite-3* :: *complete-linorder* ⟨*proof*⟩

instantiation *finite-3* :: {*field*, *idom-abs-sgn*, *idom-modulo*} **begin**

definition [*simp*]: $0 = a_1$
definition [*simp*]: $1 = a_2$
definition
$x + y = (\mathit{case}\ (x,\ y)\ \mathit{of}$
$(a_1,\ a_1) \Rightarrow a_1 \mid (a_2,\ a_3) \Rightarrow a_1 \mid (a_3,\ a_2) \Rightarrow a_1$
$\mid (a_1,\ a_2) \Rightarrow a_2 \mid (a_2,\ a_1) \Rightarrow a_2 \mid (a_3,\ a_3) \Rightarrow a_2$
$\mid \text{-} \Rightarrow a_3)$
definition $- x = (\mathit{case}\ x\ \mathit{of}\ a_1 \Rightarrow a_1 \mid a_2 \Rightarrow a_3 \mid a_3 \Rightarrow a_2)$
definition $x - y = x + (- y :: \mathit{finite\text{-}3})$
definition $x * y = (\mathit{case}\ (x,\ y)\ \mathit{of}\ (a_2,\ a_2) \Rightarrow a_2 \mid (a_3,\ a_3) \Rightarrow a_2 \mid (a_2,\ a_3) \Rightarrow a_3$
$\mid (a_3,\ a_2) \Rightarrow a_3 \mid \text{-} \Rightarrow a_1)$
definition $\mathit{inverse} = (\lambda x :: \mathit{finite\text{-}3}.\ x)$
definition $x\ \mathit{div}\ y = x * \mathit{inverse}\ (y :: \mathit{finite\text{-}3})$
definition $x\ \mathit{mod}\ y = (\mathit{case}\ y\ \mathit{of}\ a_1 \Rightarrow x \mid \text{-} \Rightarrow a_1)$
definition $\mathit{abs} = (\lambda x.\ \mathit{case}\ x\ \mathit{of}\ a_3 \Rightarrow a_2 \mid \text{-} \Rightarrow x)$
definition $\mathit{sgn} = (\lambda x :: \mathit{finite\text{-}3}.\ x)$
instance
⟨*proof*⟩
end

lemma *two-finite-3* [*simp*]:
$2 = a_3$
⟨*proof*⟩

lemma *dvd-finite-3-unfold*:
$x\ \mathit{dvd}\ y \longleftrightarrow x = a_2 \vee x = a_3 \vee y = a_1$
⟨*proof*⟩

instantiation *finite-3* :: {*normalization-semidom*, *unique-euclidean-semiring*} **begin**
definition [*simp*]: $\mathit{normalize}\ x = (\mathit{case}\ x\ \mathit{of}\ a_3 \Rightarrow a_2 \mid \text{-} \Rightarrow x)$
definition [*simp*]: $\mathit{unit\text{-}factor} = (\mathit{id} :: \mathit{finite\text{-}3} \Rightarrow \text{-})$
definition [*simp*]: $\mathit{euclidean\text{-}size}\ x = (\mathit{case}\ x\ \mathit{of}\ a_1 \Rightarrow 0 \mid \text{-} \Rightarrow 1)$
definition [*simp*]: $\mathit{division\text{-}segment}\ (x :: \mathit{finite\text{-}3}) = 1$
instance
⟨*proof*⟩
end

hide-const (**open**) $a_1\ a_2\ a_3$

datatype (*plugins only*: *code quickcheck extraction*) *finite-4* =
$a_1 \mid a_2 \mid a_3 \mid a_4$

notation (**output**) a_1 (a_1)
notation (**output**) a_2 (a_2)
notation (**output**) a_3 (a_3)
notation (**output**) a_4 (a_4)

lemma *UNIV-finite-4*:

$UNIV = \{a_1,\ a_2,\ a_3,\ a_4\}$
⟨*proof*⟩

instantiation *finite-4* :: *enum*
begin

definition
$enum = [a_1,\ a_2,\ a_3,\ a_4]$

definition
$enum\text{-}all\ P \longleftrightarrow P\ a_1 \wedge P\ a_2 \wedge P\ a_3 \wedge P\ a_4$

definition
$enum\text{-}ex\ P \longleftrightarrow P\ a_1 \vee P\ a_2 \vee P\ a_3 \vee P\ a_4$

instance ⟨*proof*⟩

end

instantiation *finite-4* :: *finite-distrib-lattice* **begin**

$a_1 < a_2$,$a_3 < a_4$, but a_2 and a_3 are incomparable.

definition
$x < y \longleftrightarrow (case\ (x,\ y)\ of$
$(a_1,\ a_1) \Rightarrow False \mid (a_1,\ \text{-}) \Rightarrow True$
$\mid\ (a_2,\ a_4) \Rightarrow True$
$\mid\ (a_3,\ a_4) \Rightarrow True\ \mid \text{-} \Rightarrow False)$

definition
$x \leq y \longleftrightarrow (case\ (x,\ y)\ of$
$(a_1,\ \text{-}) \Rightarrow True$
$\mid (a_2,\ a_2) \Rightarrow True \mid (a_2,\ a_4) \Rightarrow True$
$\mid (a_3,\ a_3) \Rightarrow True \mid (a_3,\ a_4) \Rightarrow True$
$\mid (a_4,\ a_4) \Rightarrow True \mid \text{-} \Rightarrow False)$

definition
$\bigsqcap A = (if\ a_1 \in A \vee a_2 \in A \wedge a_3 \in A\ then\ a_1\ else\ if\ a_2 \in A\ then\ a_2\ else\ if\ a_3 \in A\ then\ a_3\ else\ a_4)$
definition
$\bigsqcup A = (if\ a_4 \in A \vee a_2 \in A \wedge a_3 \in A\ then\ a_4\ else\ if\ a_2 \in A\ then\ a_2\ else\ if\ a_3 \in A\ then\ a_3\ else\ a_1)$
definition [*simp*]: $bot = a_1$
definition [*simp*]: $top = a_4$
definition
$x \sqcap y = (case\ (x,\ y)\ of$
$(a_1,\ \text{-}) \Rightarrow a_1 \mid (\text{-},\ a_1) \Rightarrow a_1 \mid (a_2,\ a_3) \Rightarrow a_1 \mid (a_3,\ a_2) \Rightarrow a_1$
$\mid (a_2,\ \text{-}) \Rightarrow a_2 \mid (\text{-},\ a_2) \Rightarrow a_2$
$\mid (a_3,\ \text{-}) \Rightarrow a_3 \mid (\text{-},\ a_3) \Rightarrow a_3$
$\mid \text{-} \Rightarrow a_4)$

definition
$x \sqcup y = (\mathit{case}\ (x,\ y)\ \mathit{of}$
$(a_4,\ \text{-}) \Rightarrow a_4 \mid (\text{-},\ a_4) \Rightarrow a_4 \mid (a_2,\ a_3) \Rightarrow a_4 \mid (a_3,\ a_2) \Rightarrow a_4$
$\mid (a_2,\ \text{-}) \Rightarrow a_2 \mid (\text{-},\ a_2) \Rightarrow a_2$
$\mid (a_3,\ \text{-}) \Rightarrow a_3 \mid (\text{-},\ a_3) \Rightarrow a_3$
$\mid \text{-} \Rightarrow a_1)$

instance
⟨*proof*⟩
end

instance *finite-4* :: *complete-lattice* ⟨*proof*⟩

instance *finite-4* :: *complete-distrib-lattice* ⟨*proof*⟩

instantiation *finite-4* :: *complete-boolean-algebra* **begin**
definition $-\ x = (\mathit{case}\ x\ \mathit{of}\ a_1 \Rightarrow a_4 \mid a_2 \Rightarrow a_3 \mid a_3 \Rightarrow a_2 \mid a_4 \Rightarrow a_1)$
definition $x - y = x \sqcap -\ (y :: \mathit{finite\text{-}4})$
instance
⟨*proof*⟩
end

hide-const (**open**) $a_1\ a_2\ a_3\ a_4$

datatype (*plugins only*: *code quickcheck extraction*) *finite-5* =
$a_1 \mid a_2 \mid a_3 \mid a_4 \mid a_5$

notation (**output**) a_1 (a_1)
notation (**output**) a_2 (a_2)
notation (**output**) a_3 (a_3)
notation (**output**) a_4 (a_4)
notation (**output**) a_5 (a_5)

lemma *UNIV-finite-5*:
$\mathit{UNIV} = \{a_1,\ a_2,\ a_3,\ a_4,\ a_5\}$
⟨*proof*⟩

instantiation *finite-5* :: *enum*
begin

definition
$\mathit{enum} = [a_1,\ a_2,\ a_3,\ a_4,\ a_5]$

definition
$\mathit{enum\text{-}all}\ P \longleftrightarrow P\ a_1 \wedge P\ a_2 \wedge P\ a_3 \wedge P\ a_4 \wedge P\ a_5$

definition
$\mathit{enum\text{-}ex}\ P \longleftrightarrow P\ a_1 \vee P\ a_2 \vee P\ a_3 \vee P\ a_4 \vee P\ a_5$

instance ⟨*proof*⟩

end

instantiation *finite-5* :: *finite-lattice*
begin

The non-distributive pentagon lattice N_5

definition
$x < y \longleftrightarrow$ (*case* $(x,\ y)$ *of*
$(a_1,\ a_1) \Rightarrow$ *False* | $(a_1,$ -$) \Rightarrow$ *True*
| $(a_2,\ a_3) \Rightarrow$ *True* | $(a_2,\ a_5) \Rightarrow$ *True*
| $(a_3,\ a_5) \Rightarrow$ *True*
| $(a_4,\ a_5) \Rightarrow$ *True* | - $\Rightarrow$ *False*)

definition
$x \leq y \longleftrightarrow$ (*case* $(x,\ y)$ *of*
$(a_1,$ -$) \Rightarrow$ *True*
| $(a_2,\ a_2) \Rightarrow$ *True* | $(a_2,\ a_3) \Rightarrow$ *True* | $(a_2,\ a_5) \Rightarrow$ *True*
| $(a_3,\ a_3) \Rightarrow$ *True* | $(a_3,\ a_5) \Rightarrow$ *True*
| $(a_4,\ a_4) \Rightarrow$ *True* | $(a_4,\ a_5) \Rightarrow$ *True*
| $(a_5,\ a_5) \Rightarrow$ *True* | - $\Rightarrow$ *False*)

definition
$\bigsqcap A =$
(*if* $a_1 \in A \vee a_4 \in A \wedge (a_2 \in A \vee a_3 \in A)$ *then* a_1
else if $a_2 \in A$ *then* a_2
else if $a_3 \in A$ *then* a_3
else if $a_4 \in A$ *then* a_4
else a_5)
definition
$\bigsqcup A =$
(*if* $a_5 \in A \vee a_4 \in A \wedge (a_2 \in A \vee a_3 \in A)$ *then* a_5
else if $a_3 \in A$ *then* a_3
else if $a_2 \in A$ *then* a_2
else if $a_4 \in A$ *then* a_4
else a_1)
definition [*simp*]: $bot = a_1$
definition [*simp*]: $top = a_5$
definition
$x \sqcap y =$ (*case* $(x,\ y)$ *of*
$(a_1,$ -$) \Rightarrow a_1$ | $(-,\ a_1) \Rightarrow a_1$ | $(a_2,\ a_4) \Rightarrow a_1$ | $(a_4,\ a_2) \Rightarrow a_1$ | $(a_3,\ a_4) \Rightarrow a_1$ | $(a_4,\ a_3) \Rightarrow a_1$
| $(a_2,$ -$) \Rightarrow a_2$ | $(-,\ a_2) \Rightarrow a_2$
| $(a_3,$ -$) \Rightarrow a_3$ | $(-,\ a_3) \Rightarrow a_3$
| $(a_4,$ -$) \Rightarrow a_4$ | $(-,\ a_4) \Rightarrow a_4$
| - $\Rightarrow a_5$)
definition
$x \sqcup y =$ (*case* $(x,\ y)$ *of*

$(a_5,\ \text{-}) \Rightarrow a_5 \mid (\text{-},\ a_5) \Rightarrow a_5 \mid (a_2,\ a_4) \Rightarrow a_5 \mid (a_4,\ a_2) \Rightarrow a_5 \mid (a_3,\ a_4) \Rightarrow a_5 \mid (a_4,\ a_3) \Rightarrow a_5$
$\mid (a_3,\ \text{-}) \Rightarrow a_3 \mid (\text{-},\ a_3) \Rightarrow a_3$
$\mid (a_2,\ \text{-}) \Rightarrow a_2 \mid (\text{-},\ a_2) \Rightarrow a_2$
$\mid (a_4,\ \text{-}) \Rightarrow a_4 \mid (\text{-},\ a_4) \Rightarrow a_4$
$\mid \text{-} \Rightarrow a_1)$

instance
⟨*proof*⟩
end

instance *finite-5* :: *complete-lattice* ⟨*proof*⟩

hide-const (**open**) a_1 a_2 a_3 a_4 a_5

73.5 Closing up

hide-type (**open**) *finite-1 finite-2 finite-3 finite-4 finite-5*
hide-const (**open**) *enum enum-all enum-ex all-n-lists ex-n-lists ntrancl*

end

74 Character and string types

theory *String*
imports *Enum Bit-Operations Code-Numeral*
begin

74.1 Strings as list of bytes

When modelling strings, we follow the approach given in https://utf8everywhere.org/:

- Strings are a list of bytes (8 bit).
- Byte values from 0 to 127 are US-ASCII.
- Byte values from 128 to 255 are uninterpreted blobs.

74.1.1 Bytes as datatype

datatype *char* =
Char (*digit0*: *bool*) (*digit1*: *bool*) (*digit2*: *bool*) (*digit3*: *bool*)
(*digit4*: *bool*) (*digit5*: *bool*) (*digit6*: *bool*) (*digit7*: *bool*)

context *comm-semiring-1*
begin

definition *of-char* :: ‹*char* ⇒ *'a*›
  where ‹*of-char c* = *horner-sum of-bool 2* [*digit0 c*, *digit1 c*, *digit2 c*, *digit3 c*, *digit4 c*, *digit5 c*, *digit6 c*, *digit7 c*]›

lemma *of-char-Char* [*simp*]:
  ‹*of-char* (*Char b0 b1 b2 b3 b4 b5 b6 b7*) =
    horner-sum of-bool 2 [*b0*, *b1*, *b2*, *b3*, *b4*, *b5*, *b6*, *b7*]›
  ⟨*proof*⟩

end

lemma (**in** *comm-semiring-1*) *of-nat-of-char*:
  ‹*of-nat* (*of-char c*) = *of-char c*›
  ⟨*proof*⟩

lemma (**in** *comm-ring-1*) *of-int-of-char*:
  ‹*of-int* (*of-char c*) = *of-char c*›
  ⟨*proof*⟩

lemma *nat-of-char* [*simp*]:
  ‹*nat* (*of-char c*) = *of-char c*›
  ⟨*proof*⟩

context *unique-euclidean-semiring-with-bit-operations*
begin

definition *char-of* :: ‹*'a* ⇒ *char*›
  where ‹*char-of n* = *Char* (*bit n 0*) (*bit n 1*) (*bit n 2*) (*bit n 3*) (*bit n 4*) (*bit n 5*) (*bit n 6*) (*bit n 7*)›

lemma *char-of-take-bit-eq*:
  ‹*char-of* (*take-bit n m*) = *char-of m*› **if** ‹*n* ≥ *8*›
  ⟨*proof*⟩

lemma *char-of-char* [*simp*]:
  ‹*char-of* (*of-char c*) = *c*›
  ⟨*proof*⟩

lemma *char-of-comp-of-char* [*simp*]:
  char-of ∘ *of-char* = *id*
  ⟨*proof*⟩

lemma *inj-of-char*:
  ‹*inj of-char*›
⟨*proof*⟩

lemma *of-char-eqI*:
‹*c = d*› **if** ‹*of-char c = of-char d*›
⟨*proof*⟩

lemma *of-char-eq-iff* [*simp*]:
‹*of-char c = of-char d ⟷ c = d*›
⟨*proof*⟩

lemma *of-char-of* [*simp*]:
‹*of-char* (*char-of a*) = *a mod 256*›
⟨*proof*⟩

lemma *char-of-mod-256* [*simp*]:
‹*char-of* (*n mod 256*) = *char-of n*›
⟨*proof*⟩

lemma *of-char-mod-256* [*simp*]:
‹*of-char c mod 256 = of-char c*›
⟨*proof*⟩

lemma *char-of-quasi-inj* [*simp*]:
‹*char-of m = char-of n ⟷ m mod 256 = n mod 256*› (**is** ‹*?P ⟷ ?Q*›)
⟨*proof*⟩

lemma *char-of-eq-iff*:
‹*char-of n = c ⟷ take-bit 8 n = of-char c*›
⟨*proof*⟩

lemma *char-of-nat* [*simp*]:
‹*char-of* (*of-nat n*) = *char-of n*›
⟨*proof*⟩

end

lemma *inj-on-char-of-nat* [*simp*]:
inj-on char-of {*0*::*nat*..<*256*}
⟨*proof*⟩

lemma *nat-of-char-less-256* [*simp*]:
of-char c < (*256* :: *nat*)
⟨*proof*⟩

lemma *range-nat-of-char*:
range of-char = {*0*::*nat*..<*256*}
⟨*proof*⟩

lemma *UNIV-char-of-nat*:
UNIV = *char-of* ' {*0*::*nat*..<*256*}

⟨*proof*⟩

lemma *card-UNIV-char*:
card (*UNIV* :: *char set*) = *256*
⟨*proof*⟩

context
includes *lifting-syntax integer.lifting natural.lifting*
begin

lemma [*transfer-rule*]:
‹(*pcr-integer* ===> (=)) *char-of char-of*›
⟨*proof*⟩

lemma [*transfer-rule*]:
‹((=) ===> *pcr-integer*) *of-char of-char*›
⟨*proof*⟩

lemma [*transfer-rule*]:
‹(*pcr-natural* ===> (=)) *char-of char-of*›
⟨*proof*⟩

lemma [*transfer-rule*]:
‹((=) ===> *pcr-natural*) *of-char of-char*›
⟨*proof*⟩

end

lifting-update *integer.lifting*
lifting-forget *integer.lifting*

lifting-update *natural.lifting*
lifting-forget *natural.lifting*

lemma *size-char-eq-0* [*simp*, *code*]:
‹*size c* = *0*› **for** *c* :: *char*
⟨*proof*⟩

lemma *size'-char-eq-0* [*simp*, *code*]:
‹*size-char c* = *0*›
⟨*proof*⟩

syntax
-*Char* :: *str-position* ⇒ *char* (*CHR* -)
-*Char-ord* :: *num-const* ⇒ *char* (*CHR* -)

type-synonym *string* = *char list*

syntax

-String :: *str-position* ⇒ *string* (-)

⟨*ML*⟩

instantiation *char* :: *enum*
begin

definition
Enum.enum = [
CHR 0x00, *CHR 0x01*, *CHR 0x02*, *CHR 0x03*,
CHR 0x04, *CHR 0x05*, *CHR 0x06*, *CHR 0x07*,
CHR 0x08, *CHR 0x09*, *CHR ''⏎''*, *CHR 0x0B*,
CHR 0x0C, *CHR 0x0D*, *CHR 0x0E*, *CHR 0x0F*,
CHR 0x10, *CHR 0x11*, *CHR 0x12*, *CHR 0x13*,
CHR 0x14, *CHR 0x15*, *CHR 0x16*, *CHR 0x17*,
CHR 0x18, *CHR 0x19*, *CHR 0x1A*, *CHR 0x1B*,
CHR 0x1C, *CHR 0x1D*, *CHR 0x1E*, *CHR 0x1F*,
CHR '' '', *CHR ''!''*, *CHR 0x22*, *CHR ''#''*,
CHR ''$'', *CHR ''%''*, *CHR ''&''*, *CHR 0x27*,
CHR ''('', *CHR '')''*, *CHR ''*''*, *CHR ''+''*,
CHR '','', *CHR ''−''*, *CHR ''.''*, *CHR ''/''*,
CHR ''0'', *CHR ''1''*, *CHR ''2''*, *CHR ''3''*,
CHR ''4'', *CHR ''5''*, *CHR ''6''*, *CHR ''7''*,
CHR ''8'', *CHR ''9''*, *CHR '':''*, *CHR '';''*,
CHR ''<'', *CHR ''=''*, *CHR ''>''*, *CHR ''?''*,
CHR ''@'', *CHR ''A''*, *CHR ''B''*, *CHR ''C''*,
CHR ''D'', *CHR ''E''*, *CHR ''F''*, *CHR ''G''*,
CHR ''H'', *CHR ''I''*, *CHR ''J''*, *CHR ''K''*,
CHR ''L'', *CHR ''M''*, *CHR ''N''*, *CHR ''O''*,
CHR ''P'', *CHR ''Q''*, *CHR ''R''*, *CHR ''S''*,
CHR ''T'', *CHR ''U''*, *CHR ''V''*, *CHR ''W''*,
CHR ''X'', *CHR ''Y''*, *CHR ''Z''*, *CHR ''[''*,
CHR 0x5C, *CHR '']''*, *CHR ''^''*, *CHR ''-''*,
CHR 0x60, *CHR ''a''*, *CHR ''b''*, *CHR ''c''*,
CHR ''d'', *CHR ''e''*, *CHR ''f''*, *CHR ''g''*,
CHR ''h'', *CHR ''i''*, *CHR ''j''*, *CHR ''k''*,
CHR ''l'', *CHR ''m''*, *CHR ''n''*, *CHR ''o''*,
CHR ''p'', *CHR ''q''*, *CHR ''r''*, *CHR ''s''*,
CHR ''t'', *CHR ''u''*, *CHR ''v''*, *CHR ''w''*,
CHR ''x'', *CHR ''y''*, *CHR ''z''*, *CHR ''{''*,
CHR ''|'', *CHR ''}''*, *CHR ''~''*, *CHR 0x7F*,
CHR 0x80, *CHR 0x81*, *CHR 0x82*, *CHR 0x83*,
CHR 0x84, *CHR 0x85*, *CHR 0x86*, *CHR 0x87*,
CHR 0x88, *CHR 0x89*, *CHR 0x8A*, *CHR 0x8B*,
CHR 0x8C, *CHR 0x8D*, *CHR 0x8E*, *CHR 0x8F*,
CHR 0x90, *CHR 0x91*, *CHR 0x92*, *CHR 0x93*,
CHR 0x94, *CHR 0x95*, *CHR 0x96*, *CHR 0x97*,
CHR 0x98, *CHR 0x99*, *CHR 0x9A*, *CHR 0x9B*,
CHR 0x9C, *CHR 0x9D*, *CHR 0x9E*, *CHR 0x9F*,

CHR 0xA0, *CHR 0xA1*, *CHR 0xA2*, *CHR 0xA3*,
CHR 0xA4, *CHR 0xA5*, *CHR 0xA6*, *CHR 0xA7*,
CHR 0xA8, *CHR 0xA9*, *CHR 0xAA*, *CHR 0xAB*,
CHR 0xAC, *CHR 0xAD*, *CHR 0xAE*, *CHR 0xAF*,
CHR 0xB0, *CHR 0xB1*, *CHR 0xB2*, *CHR 0xB3*,
CHR 0xB4, *CHR 0xB5*, *CHR 0xB6*, *CHR 0xB7*,
CHR 0xB8, *CHR 0xB9*, *CHR 0xBA*, *CHR 0xBB*,
CHR 0xBC, *CHR 0xBD*, *CHR 0xBE*, *CHR 0xBF*,
CHR 0xC0, *CHR 0xC1*, *CHR 0xC2*, *CHR 0xC3*,
CHR 0xC4, *CHR 0xC5*, *CHR 0xC6*, *CHR 0xC7*,
CHR 0xC8, *CHR 0xC9*, *CHR 0xCA*, *CHR 0xCB*,
CHR 0xCC, *CHR 0xCD*, *CHR 0xCE*, *CHR 0xCF*,
CHR 0xD0, *CHR 0xD1*, *CHR 0xD2*, *CHR 0xD3*,
CHR 0xD4, *CHR 0xD5*, *CHR 0xD6*, *CHR 0xD7*,
CHR 0xD8, *CHR 0xD9*, *CHR 0xDA*, *CHR 0xDB*,
CHR 0xDC, *CHR 0xDD*, *CHR 0xDE*, *CHR 0xDF*,
CHR 0xE0, *CHR 0xE1*, *CHR 0xE2*, *CHR 0xE3*,
CHR 0xE4, *CHR 0xE5*, *CHR 0xE6*, *CHR 0xE7*,
CHR 0xE8, *CHR 0xE9*, *CHR 0xEA*, *CHR 0xEB*,
CHR 0xEC, *CHR 0xED*, *CHR 0xEE*, *CHR 0xEF*,
CHR 0xF0, *CHR 0xF1*, *CHR 0xF2*, *CHR 0xF3*,
CHR 0xF4, *CHR 0xF5*, *CHR 0xF6*, *CHR 0xF7*,
CHR 0xF8, *CHR 0xF9*, *CHR 0xFA*, *CHR 0xFB*,
CHR 0xFC, *CHR 0xFD*, *CHR 0xFE*, *CHR 0xFF*]

definition
Enum.enum-all P $\longleftrightarrow$ *list-all P* (*Enum.enum* :: *char list*)

definition
Enum.enum-ex P $\longleftrightarrow$ *list-ex P* (*Enum.enum* :: *char list*)

lemma *enum-char-unfold*:
Enum.enum = *map char-of* [*0*..<*256*]
⟨*proof*⟩

instance ⟨*proof*⟩

end

lemma *linorder-char*:
class.linorder (λ*c d*. *of-char c* ≤ (*of-char d* :: *nat*)) (λ*c d*. *of-char c* < (*of-char d* :: *nat*))
⟨*proof*⟩

Optimized version for execution

definition *char-of-integer* :: *integer* ⇒ *char*
where [*code-abbrev*]: *char-of-integer* = *char-of*

definition *integer-of-char* :: *char* ⇒ *integer*

where [*code-abbrev*]: *integer-of-char* = *of-char*

lemma *char-of-integer-code* [*code*]:
char-of-integer k = (*let*
(*q0*, *b0*) = *bit-cut-integer k*;
(*q1*, *b1*) = *bit-cut-integer q0*;
(*q2*, *b2*) = *bit-cut-integer q1*;
(*q3*, *b3*) = *bit-cut-integer q2*;
(*q4*, *b4*) = *bit-cut-integer q3*;
(*q5*, *b5*) = *bit-cut-integer q4*;
(*q6*, *b6*) = *bit-cut-integer q5*;
(-, *b7*) = *bit-cut-integer q6*
in Char b0 b1 b2 b3 b4 b5 b6 b7)
⟨*proof*⟩

lemma *integer-of-char-code* [*code*]:
integer-of-char (*Char b0 b1 b2 b3 b4 b5 b6 b7*) =
((((((*of-bool b7* ∗ *2* + *of-bool b6*) ∗ *2* +
of-bool b5) ∗ *2* + *of-bool b4*) ∗ *2* +
of-bool b3) ∗ *2* + *of-bool b2*) ∗ *2* +
of-bool b1) ∗ *2* + *of-bool b0*
⟨*proof*⟩

74.2 Strings as dedicated type for target language code generation

74.2.1 Logical specification

context
begin

qualified definition *ascii-of* :: *char* ⇒ *char*
where *ascii-of c* = *Char* (*digit0 c*) (*digit1 c*) (*digit2 c*) (*digit3 c*) (*digit4 c*) (*digit5 c*) (*digit6 c*) *False*

qualified lemma *ascii-of-Char* [*simp*]:
ascii-of (*Char b0 b1 b2 b3 b4 b5 b6 b7*) = *Char b0 b1 b2 b3 b4 b5 b6 False*
⟨*proof*⟩ **lemma** *digit0-ascii-of-iff* [*simp*]:
digit0 (*String.ascii-of c*) ⟷ *digit0 c*
⟨*proof*⟩ **lemma** *digit1-ascii-of-iff* [*simp*]:
digit1 (*String.ascii-of c*) ⟷ *digit1 c*
⟨*proof*⟩ **lemma** *digit2-ascii-of-iff* [*simp*]:
digit2 (*String.ascii-of c*) ⟷ *digit2 c*
⟨*proof*⟩ **lemma** *digit3-ascii-of-iff* [*simp*]:
digit3 (*String.ascii-of c*) ⟷ *digit3 c*
⟨*proof*⟩ **lemma** *digit4-ascii-of-iff* [*simp*]:
digit4 (*String.ascii-of c*) ⟷ *digit4 c*
⟨*proof*⟩ **lemma** *digit5-ascii-of-iff* [*simp*]:
digit5 (*String.ascii-of c*) ⟷ *digit5 c*
⟨*proof*⟩ **lemma** *digit6-ascii-of-iff* [*simp*]:

digit6 (*String.ascii-of c*) ⟷ *digit6 c*
⟨*proof*⟩ **lemma** *not-digit7-ascii-of* [*simp*]:
¬ *digit7* (*ascii-of c*)
⟨*proof*⟩ **lemma** *ascii-of-idem*:
ascii-of c = *c* **if** ¬ *digit7 c*
⟨*proof*⟩ **typedef** *literal* = {*cs*. ∀ *c*∈*set cs*. ¬ *digit7 c*}
morphisms *explode Abs-literal*
⟨*proof*⟩ **setup-lifting** *type-definition-literal*

qualified lift-definition *implode* :: *string* ⇒ *literal*
is *map ascii-of*
⟨*proof*⟩ **lemma** *implode-explode-eq* [*simp*]:
String.implode (*String.explode s*) = *s*
⟨*proof*⟩ **lemma** *explode-implode-eq* [*simp*]:
String.explode (*String.implode cs*) = *map ascii-of cs*
⟨*proof*⟩

end

context *unique-euclidean-semiring-with-bit-operations*
begin

context
begin

qualified lemma *char-of-ascii-of* [*simp*]:
‹*of-char* (*String.ascii-of c*) = *take-bit* 7 (*of-char c*)›
⟨*proof*⟩ **lemma** *ascii-of-char-of*:
‹*String.ascii-of* (*char-of a*) = *char-of* (*take-bit* 7 *a*)›
⟨*proof*⟩

end

end

74.2.2 Syntactic representation

Logical ground representations for literals are:

1. *0* for the empty literal;
2. *Literal b0 ... b6 s* for a literal starting with one character and continued by another literal.

   Syntactic representations for literals are:

3. Printable text as string prefixed with *STR*;
4. A single ascii value as numerical hexadecimal value prefixed with *STR*.

instantiation *String.literal* :: *zero*
begin

context
begin

qualified lift-definition *zero-literal* :: *String.literal*
is *Nil*
⟨*proof*⟩

instance ⟨*proof*⟩

end

end

context
begin

qualified abbreviation (**output**) *empty-literal* :: *String.literal*
where *empty-literal* ≡ *0*

qualified lift-definition *Literal* :: *bool* ⇒ *bool* ⇒ *bool* ⇒ *bool* ⇒ *bool* ⇒ *bool* ⇒ *bool* ⇒ *String.literal* ⇒ *String.literal*
is λ*b0 b1 b2 b3 b4 b5 b6 cs. Char b0 b1 b2 b3 b4 b5 b6 False # cs*
⟨*proof*⟩ **lemma** *Literal-eq-iff* [*simp*]:
Literal b0 b1 b2 b3 b4 b5 b6 s = Literal c0 c1 c2 c3 c4 c5 c6 t
⟷ (*b0* ⟷ *c0*) ∧ (*b1* ⟷ *c1*) ∧ (*b2* ⟷ *c2*) ∧ (*b3* ⟷ *c3*)
∧ (*b4* ⟷ *c4*) ∧ (*b5* ⟷ *c5*) ∧ (*b6* ⟷ *c6*) ∧ *s* = *t*
⟨*proof*⟩ **lemma** *empty-neq-Literal* [*simp*]:
empty-literal ≠ *Literal b0 b1 b2 b3 b4 b5 b6 s*
⟨*proof*⟩ **lemma** *Literal-neq-empty* [*simp*]:
Literal b0 b1 b2 b3 b4 b5 b6 s ≠ *empty-literal*
⟨*proof*⟩

end

code-datatype *0* :: *String.literal String.Literal*

syntax
-Literal :: *str-position* ⇒ *String.literal* (*STR* -)
-Ascii :: *num-const* ⇒ *String.literal* (*STR* -)

⟨*ML*⟩

74.2.3 Operations

instantiation *String.literal* :: *plus*
begin

context
begin

qualified lift-definition *plus-literal* :: *String.literal* ⇒ *String.literal* ⇒ *String.literal*
  is (@)
  ⟨*proof*⟩

instance ⟨*proof*⟩

end

end

instance *String.literal* :: *monoid-add*
  ⟨*proof*⟩

instantiation *String.literal* :: *size*
begin

context
  includes *literal.lifting*
begin

lift-definition *size-literal* :: *String.literal* ⇒ *nat*
  is *length* ⟨*proof*⟩

end

instance ⟨*proof*⟩

end

instantiation *String.literal* :: *equal*
begin

context
begin

qualified lift-definition *equal-literal* :: *String.literal* ⇒ *String.literal* ⇒ *bool*
  is *HOL.equal* ⟨*proof*⟩

instance
  ⟨*proof*⟩

end

end

instantiation *String.literal* :: *linorder*
begin

context
begin

qualified lift-definition *less-eq-literal* :: *String.literal* ⇒ *String.literal* ⇒ *bool*
is *ord.lexordp-eq* (λ*c d*. *of-char c* < (*of-char d* :: *nat*))
⟨*proof*⟩ **lift-definition** *less-literal* :: *String.literal* ⇒ *String.literal* ⇒ *bool*
is *ord.lexordp* (λ*c d*. *of-char c* < (*of-char d* :: *nat*))
⟨*proof*⟩

instance ⟨*proof*⟩

end

end

lemma *infinite-literal*:
infinite (*UNIV* :: *String.literal set*)
⟨*proof*⟩

74.2.4 Executable conversions

context
begin

qualified lift-definition *asciis-of-literal* :: *String.literal* ⇒ *integer list*
is *map of-char*
⟨*proof*⟩ **lemma** *asciis-of-zero* [*simp*, *code*]:
asciis-of-literal 0 = []
⟨*proof*⟩ **lemma** *asciis-of-Literal* [*simp*, *code*]:
asciis-of-literal (*String.Literal b0 b1 b2 b3 b4 b5 b6 s*) =
of-char (*Char b0 b1 b2 b3 b4 b5 b6 False*) # *asciis-of-literal s*
⟨*proof*⟩ **lift-definition** *literal-of-asciis* :: *integer list* ⇒ *String.literal*
is *map* (*String.ascii-of* ∘ *char-of*)
⟨*proof*⟩ **lemma** *literal-of-asciis-Nil* [*simp*, *code*]:
literal-of-asciis [] = *0*
⟨*proof*⟩ **lemma** *literal-of-asciis-Cons* [*simp*, *code*]:
literal-of-asciis (*k* # *ks*) = (*case char-of k*
of Char b0 b1 b2 b3 b4 b5 b6 b7 ⇒ *String.Literal b0 b1 b2 b3 b4 b5 b6*
(*literal-of-asciis ks*))
⟨*proof*⟩ **lemma** *literal-of-asciis-of-literal* [*simp*]:
literal-of-asciis (*asciis-of-literal s*) = *s*
⟨*proof*⟩ **lemma** *explode-code* [*code*]:
String.explode s = *map char-of* (*asciis-of-literal s*)
⟨*proof*⟩ **lemma** *implode-code* [*code*]:
String.implode cs = *literal-of-asciis* (*map of-char cs*)
⟨*proof*⟩ **lemma** *equal-literal* [*code*]:

HOL.equal (*String.Literal b0 b1 b2 b3 b4 b5 b6 s*)
(*String.Literal a0 a1 a2 a3 a4 a5 a6 r*)
$\longleftrightarrow$ (*b0* $\longleftrightarrow$ *a0*) $\wedge$ (*b1* $\longleftrightarrow$ *a1*) $\wedge$ (*b2* $\longleftrightarrow$ *a2*) $\wedge$ (*b3* $\longleftrightarrow$ *a3*)
$\wedge$ (*b4* $\longleftrightarrow$ *a4*) $\wedge$ (*b5* $\longleftrightarrow$ *a5*) $\wedge$ (*b6* $\longleftrightarrow$ *a6*) $\wedge$ (*s* = *r*)
⟨*proof*⟩

end

74.2.5 Technical code generation setup

Alternative constructor for generated computations

context
begin

qualified definition *Literal′* :: *bool* ⇒ *bool* ⇒ *bool* ⇒ *bool* ⇒ *bool* ⇒ *bool* ⇒ *bool* ⇒ *String.literal* ⇒ *String.literal*
where [*simp*]: *Literal′* = *String.Literal*

lemma [*code*]:
‹*Literal′ b0 b1 b2 b3 b4 b5 b6 s* = *String.literal-of-asciis*
[*foldr* (λ*b k*. *of-bool b* + *k* ∗ *2*) [*b0*, *b1*, *b2*, *b3*, *b4*, *b5*, *b6*] *0*] + *s*›
⟨*proof*⟩

lemma [*code-computation-unfold*]:
String.Literal = *Literal′*
⟨*proof*⟩

end

code-reserved *SML string String Char Str-Literal*
code-reserved *OCaml string String Char Str-Literal*
code-reserved *Haskell Prelude*
code-reserved *Scala string*

code-identifier
code-module *String* ⇀
(*SML*) *Str* **and** (*OCaml*) *Str* **and** (*Haskell*) *Str* **and** (*Scala*) *Str*

code-printing
type-constructor *String.literal* ⇀
(*SML*) *string*
and (*OCaml*) *string*
and (*Haskell*) *String*
and (*Scala*) *String*
| **constant** *STR* "" ⇀
(*SML*)
and (*OCaml*)
and (*Haskell*)
and (*Scala*)

⟨*ML*⟩

```
code-printing
  code-module Str-Literal ⇀
    (SML) ‹structure Str-Literal =
struct

fun map f [] = []
  | map f (x :: xs) = f x :: map f xs; (* deliberate clone not relying on List.- module
*)

fun check-ascii (k : IntInf.int) =
  if 0 <= k andalso k < 128
  then k
  else raise Fail Non−ASCII character in literal;

val char-of-ascii = Char.chr o IntInf.toInt o check-ascii;

val ascii-of-char = check-ascii o IntInf.fromInt o Char.ord;

val literal-of-asciis = String.implode o map char-of-ascii;

val asciis-of-literal = map ascii-of-char o String.explode;

end;› for constant String.literal-of-asciis String.asciis-of-literal
    and (OCaml) ‹module Str-Literal =
struct

let implode f xs =
  let rec length xs = match xs with
      [] -> 0
    | x :: xs -> 1 + length xs in
  let rec nth xs n = match xs with
    (x :: xs) -> if n <= 0 then x else nth xs (n − 1)
  in String.init (length xs) (fun n -> f (nth xs n));;

let explode f s =
  let rec map-range f n =
    if n <= 0 then [] else map-range f (n − 1) @ [f n]
  in map-range (fun n -> f (String.get s n)) (String.length s);;

let z-128 = Z.of-int 128;;

let check-ascii (k : Z.t) =
  if Z.leq Z.zero k && Z.lt k z-128
  then k
  else failwith Non−ASCII character in literal;;
```

let char-of-ascii k = Char.chr (Z.to-int (check-ascii k));;

let ascii-of-char c = check-ascii (Z.of-int (Char.code c));;

let literal-of-asciis ks = implode char-of-ascii ks;;

let asciis-of-literal s = explode ascii-of-char s;;

end;;› **for constant** *String.literal-of-asciis String.asciis-of-literal*
| **constant** (+) :: *String.literal* ⇒ *String.literal* ⇒ *String.literal* ⇀
(*SML*) **infixl** *18* ^
and (*OCaml*) **infixr** *6* ^
and (*Haskell*) **infixr** *5* ++
and (*Scala*) **infixl** *7* +
| **constant** *String.literal-of-asciis* ⇀
(*SML*) *Str'-Literal.literal'-of'-asciis*
and (*OCaml*) *Str'-Literal.literal'-of'-asciis*
and (*Haskell*) *map/* (*let chr k* | (*0* <= *k* && *k* < *128*) = *Prelude.toEnum k* :: *Prelude.Char in chr* . *Prelude.fromInteger*)
and (*Scala*) */ ++/ -.map*((*k*: *BigInt*) => *if* (*BigInt*(*0*) <= *k* && *k* < *BigInt*(*128*)) *k.charValue else sys.error*(*Non−ASCII character in literal*))
| **constant** *String.asciis-of-literal* ⇀
(*SML*) *Str'-Literal.asciis'-of'-literal*
and (*OCaml*) *Str'-Literal.asciis'-of'-literal*
and (*Haskell*) *map/* (*let ord k* | (*k* < *128*) = *Prelude.toInteger k in ord* . (*Prelude.fromEnum* :: *Prelude.Char* −> *Prelude.Int*))
and (*Scala*) !(*-.toList.map*(*c* => { *val k*: *Int* = *c.toInt*; *if* (*k* < *128*) *BigInt*(*k*) *else sys.error*(*Non−ASCII character in literal*) }))
| **class-instance** *String.literal* :: *equal* ⇀
(*Haskell*) −
| **constant** *HOL.equal* :: *String.literal* ⇒ *String.literal* ⇒ *bool* ⇀
(*SML*) !((- : *string*) = -)
and (*OCaml*) !((- : *string*) = -)
and (*Haskell*) **infix** *4* ==
and (*Scala*) **infixl** *5* ==
| **constant** (≤) :: *String.literal* ⇒ *String.literal* ⇒ *bool* ⇀
(*SML*) !((- : *string*) <= -)
and (*OCaml*) !((- : *string*) <= -)
and (*Haskell*) **infix** *4* <=
— Order operations for *String.literal* work in Haskell only if no type class instance needs to be generated, because String = [Char] in Haskell and *char list* need not have the same order as *String.literal*.
and (*Scala*) **infixl** *4* <=
and (*Eval*) **infixl** *6* <=
| **constant** (<) :: *String.literal* ⇒ *String.literal* ⇒ *bool* ⇀
(*SML*) !((- : *string*) < -)
and (*OCaml*) !((- : *string*) < -)
and (*Haskell*) **infix** *4* <
and (*Scala*) **infixl** *4* <

and (*Eval*) **infixl** *6* <

74.2.6 Code generation utility

⟨*ML*⟩

definition *abort* :: *String.literal* ⇒ (*unit* ⇒ *'a*) ⇒ *'a*
where [*simp*]: *abort - f = f ()*

declare [[*code drop*: *Code.abort*]]

lemma *abort-cong*:
msg = msg' ⟹ *Code.abort msg f = Code.abort msg' f*
⟨*proof*⟩

⟨*ML*⟩

code-printing
constant *Code.abort* ⇀
(*SML*) !(*raise*/ *Fail*/ -)
and (*OCaml*) *failwith*
and (*Haskell*) !(*error*/ ::/ *forall a.*/ *String* −> (() −> *a*) −> *a*)
and (*Scala*) !{/ *sys.error*((-));/ ((-)).*apply*(())/ }

74.2.7 Finally

lifting-update *literal.lifting*
lifting-forget *literal.lifting*

end

75 Reflecting Pure types into HOL

theory *Typerep*
imports *String*
begin

datatype *typerep* = *Typerep String.literal typerep list*

class *typerep* =
fixes *typerep* :: *'a itself* ⇒ *typerep*
begin

definition *typerep-of* :: *'a* ⇒ *typerep* **where**
[*simp*]: *typerep-of x* = *typerep TYPE*(*'a*)

end

syntax

-TYPEREP :: type => logic ((1TYPEREP/(1'(-'))))

⟨*ML*⟩

lemma [*code*]:
HOL.equal (Typerep tyco1 tys1) (Typerep tyco2 tys2) ⟷ HOL.equal tyco1 tyco2
∧ list-all2 HOL.equal tys1 tys2
⟨*proof*⟩

lemma [*code nbe*]:
HOL.equal (x :: typerep) x ⟷ True
⟨*proof*⟩

code-printing
type-constructor *typerep* ⇀ (*Eval*) *Term.typ*
| **constant** *Typerep* ⇀ (*Eval*) *Term.Type*/ (-, -)

code-reserved *Eval Term*

hide-const (**open**) *typerep Typerep*

end

76 Predicates as enumerations

theory *Predicate*
imports *String*
begin

76.1 The type of predicate enumerations (a monad)

datatype (*plugins only*: *extraction*) (*dead* *'a*) *pred* = *Pred* (*eval*: *'a* ⇒ *bool*)

lemma *pred-eqI*:
(⋀*w*. *eval P w ⟷ eval Q w*) ⟹ *P* = *Q*
⟨*proof*⟩

lemma *pred-eq-iff*:
P = *Q* ⟹ (⋀*w*. *eval P w ⟷ eval Q w*)
⟨*proof*⟩

instantiation *pred* :: (*type*) *complete-lattice*
begin

definition
P ≤ *Q* ⟷ *eval P* ≤ *eval Q*

definition
P < *Q* ⟷ *eval P* < *eval Q*

definition
$\bot = \mathit{Pred}\ \bot$

lemma *eval-bot* [*simp*]:
$\mathit{eval}\ \bot\ = \bot$
⟨*proof*⟩

definition
$\top = \mathit{Pred}\ \top$

lemma *eval-top* [*simp*]:
$\mathit{eval}\ \top\ = \top$
⟨*proof*⟩

definition
$P \sqcap Q = \mathit{Pred}\ (\mathit{eval}\ P \sqcap \mathit{eval}\ Q)$

lemma *eval-inf* [*simp*]:
$\mathit{eval}\ (P \sqcap Q) = \mathit{eval}\ P \sqcap \mathit{eval}\ Q$
⟨*proof*⟩

definition
$P \sqcup Q = \mathit{Pred}\ (\mathit{eval}\ P \sqcup \mathit{eval}\ Q)$

lemma *eval-sup* [*simp*]:
$\mathit{eval}\ (P \sqcup Q) = \mathit{eval}\ P \sqcup \mathit{eval}\ Q$
⟨*proof*⟩

definition
$\bigsqcap A = \mathit{Pred}\ (\bigsqcap (\mathit{eval}\ `\ A))$

lemma *eval-Inf* [*simp*]:
$\mathit{eval}\ (\bigsqcap A) = \bigsqcap (\mathit{eval}\ `\ A)$
⟨*proof*⟩

definition
$\bigsqcup A = \mathit{Pred}\ (\bigsqcup (\mathit{eval}\ `\ A))$

lemma *eval-Sup* [*simp*]:
$\mathit{eval}\ (\bigsqcup A) = \bigsqcup (\mathit{eval}\ `\ A)$
⟨*proof*⟩

instance ⟨*proof*⟩

end

lemma *eval-INF* [*simp*]:
$\mathit{eval}\ (\bigsqcap (f\ `\ A)) = \bigsqcap ((\mathit{eval} \circ f)\ `\ A)$

⟨*proof*⟩

lemma *eval-SUP* [*simp*]:
eval (⨆ (*f* ‘ *A*)) = ⨆ ((*eval* ∘ *f*) ‘ *A*)
⟨*proof*⟩

instantiation *pred* :: (*type*) *complete-boolean-algebra*
begin

definition
− *P* = *Pred* (− *eval P*)

lemma *eval-compl* [*simp*]:
eval (− *P*) = − *eval P*
⟨*proof*⟩

definition
P − *Q* = *Pred* (*eval P* − *eval Q*)

lemma *eval-minus* [*simp*]:
eval (*P* − *Q*) = *eval P* − *eval Q*
⟨*proof*⟩

instance ⟨*proof*⟩

end

definition *single* :: *'a* ⇒ *'a pred* **where**
single x = *Pred* ((=) *x*)

lemma *eval-single* [*simp*]:
eval (*single x*) = (=) *x*
⟨*proof*⟩

definition *bind* :: *'a pred* ⇒ (*'a* ⇒ *'b pred*) ⇒ *'b pred* (**infixl** ⤜ *70*) **where**
P ⤜ *f* = (⨆ (*f* ‘ {*x*. *eval P x*}))

lemma *eval-bind* [*simp*]:
eval (*P* ⤜ *f*) = *eval* (⨆ (*f* ‘ {*x*. *eval P x*}))
⟨*proof*⟩

lemma *bind-bind*:
(*P* ⤜ *Q*) ⤜ *R* = *P* ⤜ (λ*x*. *Q x* ⤜ *R*)
⟨*proof*⟩

lemma *bind-single*:
P ⤜ *single* = *P*
⟨*proof*⟩

lemma *single-bind*:
single x $\gg\!=$ *P* = *P x*
⟨*proof*⟩

lemma *bottom-bind*:
⊥ $\gg\!=$ *P* = ⊥
⟨*proof*⟩

lemma *sup-bind*:
(*P* ⊔ *Q*) $\gg\!=$ *R* = *P* $\gg\!=$ *R* ⊔ *Q* $\gg\!=$ *R*
⟨*proof*⟩

lemma *Sup-bind*:
(⨆ *A* $\gg\!=$ *f*) = ⨆ ((λ*x*. *x* $\gg\!=$ *f*) ‘ *A*)
⟨*proof*⟩

lemma *pred-iffI*:
assumes ⋀*x*. *eval A x* ⟹ *eval B x*
and ⋀*x*. *eval B x* ⟹ *eval A x*
shows *A* = *B*
⟨*proof*⟩

lemma *singleI*: *eval* (*single x*) *x*
⟨*proof*⟩

lemma *singleI-unit*: *eval* (*single* ()) *x*
⟨*proof*⟩

lemma *singleE*: *eval* (*single x*) *y* ⟹ (*y* = *x* ⟹ *P*) ⟹ *P*
⟨*proof*⟩

lemma *singleE'*: *eval* (*single x*) *y* ⟹ (*x* = *y* ⟹ *P*) ⟹ *P*
⟨*proof*⟩

lemma *bindI*: *eval P x* ⟹ *eval* (*Q x*) *y* ⟹ *eval* (*P* $\gg\!=$ *Q*) *y*
⟨*proof*⟩

lemma *bindE*: *eval* (*R* $\gg\!=$ *Q*) *y* ⟹ (⋀*x*. *eval R x* ⟹ *eval* (*Q x*) *y* ⟹ *P*) ⟹ *P*
⟨*proof*⟩

lemma *botE*: *eval* ⊥ *x* ⟹ *P*
⟨*proof*⟩

lemma *supI1*: *eval A x* ⟹ *eval* (*A* ⊔ *B*) *x*
⟨*proof*⟩

lemma *supI2*: *eval B x* ⟹ *eval* (*A* ⊔ *B*) *x*
⟨*proof*⟩

lemma *supE*: *eval* $(A \sqcup B)$ $x \Longrightarrow (\mathit{eval}\ A\ x \Longrightarrow P) \Longrightarrow (\mathit{eval}\ B\ x \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *single-not-bot* [*simp*]:
$\mathit{single}\ x \neq \bot$
⟨*proof*⟩

lemma *not-bot*:
assumes $A \neq \bot$
obtains x **where** *eval* $A\ x$
⟨*proof*⟩

76.2 Emptiness check and definite choice

definition *is-empty* :: *'a pred* ⇒ *bool* **where**
$\mathit{is\text{-}empty}\ A \longleftrightarrow A = \bot$

lemma *is-empty-bot*:
is-empty $\bot$
⟨*proof*⟩

lemma *not-is-empty-single*:
$\neg\ \mathit{is\text{-}empty}\ (\mathit{single}\ x)$
⟨*proof*⟩

lemma *is-empty-sup*:
$\mathit{is\text{-}empty}\ (A \sqcup B) \longleftrightarrow \mathit{is\text{-}empty}\ A \wedge \mathit{is\text{-}empty}\ B$
⟨*proof*⟩

definition *singleton* :: (*unit* ⇒ *'a*) ⇒ *'a pred* ⇒ *'a* **where**
$\mathit{singleton}\ \mathit{default}\ A = (\mathit{if}\ \exists!x.\ \mathit{eval}\ A\ x\ \mathit{then}\ \mathit{THE}\ x.\ \mathit{eval}\ A\ x\ \mathit{else}\ \mathit{default}\ ())$ **for** *default*

lemma *singleton-eqI*:
$\exists!x.\ \mathit{eval}\ A\ x \Longrightarrow \mathit{eval}\ A\ x \Longrightarrow \mathit{singleton}\ \mathit{default}\ A = x$ **for** *default*
⟨*proof*⟩

lemma *eval-singletonI*:
$\exists!x.\ \mathit{eval}\ A\ x \Longrightarrow \mathit{eval}\ A\ (\mathit{singleton}\ \mathit{default}\ A)$ **for** *default*
⟨*proof*⟩

lemma *single-singleton*:
$\exists!x.\ \mathit{eval}\ A\ x \Longrightarrow \mathit{single}\ (\mathit{singleton}\ \mathit{default}\ A) = A$ **for** *default*
⟨*proof*⟩

lemma *singleton-undefinedI*:
$\neg\ (\exists!x.\ \mathit{eval}\ A\ x) \Longrightarrow \mathit{singleton}\ \mathit{default}\ A = \mathit{default}\ ()$ **for** *default*
⟨*proof*⟩

lemma *singleton-bot*:
singleton default $\bot$ = *default* () **for** *default*
⟨*proof*⟩

lemma *singleton-single*:
singleton default (*single x*) = *x* **for** *default*
⟨*proof*⟩

lemma *singleton-sup-single-single*:
singleton default (*single x* $\sqcup$ *single y*) = (*if x* = *y then x else default* ()) **for** *default*
⟨*proof*⟩

lemma *singleton-sup-aux*:
singleton default (*A* $\sqcup$ *B*) = (*if A* = $\bot$ *then singleton default B*
else if B = $\bot$ *then singleton default A*
else singleton default
(*single* (*singleton default A*) $\sqcup$ *single* (*singleton default B*))) **for** *default*
⟨*proof*⟩

lemma *singleton-sup*:
singleton default (*A* $\sqcup$ *B*) = (*if A* = $\bot$ *then singleton default B*
else if B = $\bot$ *then singleton default A*
else if singleton default A = *singleton default B then singleton default A else default* ()) **for** *default*
⟨*proof*⟩

76.3 Derived operations

definition *if-pred* :: *bool* ⇒ *unit pred* **where**
if-pred-eq: *if-pred b* = (*if b then single* () *else* $\bot$)

definition *holds* :: *unit pred* ⇒ *bool* **where**
holds-eq: *holds P* = *eval P* ()

definition *not-pred* :: *unit pred* ⇒ *unit pred* **where**
not-pred-eq: *not-pred P* = (*if eval P* () *then* $\bot$ *else single* ())

lemma *if-predI*: *P* ⟹ *eval* (*if-pred P*) ()
⟨*proof*⟩

lemma *if-predE*: *eval* (*if-pred b*) *x* ⟹ (*b* ⟹ *x* = () ⟹ *P*) ⟹ *P*
⟨*proof*⟩

lemma *not-predI*: ¬ *P* ⟹ *eval* (*not-pred* (*Pred* (λ*u*. *P*))) ()
⟨*proof*⟩

lemma *not-predI′*: ¬ *eval P* () ⟹ *eval* (*not-pred P*) ()

⟨*proof*⟩

lemma *not-predE*: *eval* (*not-pred* (*Pred* (λ*u*. *P*))) *x* ⟹ (¬ *P* ⟹ *thesis*) ⟹ *thesis*
⟨*proof*⟩

lemma *not-predE'*: *eval* (*not-pred P*) *x* ⟹ (¬ *eval P x* ⟹ *thesis*) ⟹ *thesis*
⟨*proof*⟩
lemma *f* () = *False* ∨ *f* () = *True*
⟨*proof*⟩

lemma *closure-of-bool-cases* [*no-atp*]:
fixes *f* :: *unit* ⇒ *bool*
assumes *f* = (λ*u*. *False*) ⟹ *P f*
assumes *f* = (λ*u*. *True*) ⟹ *P f*
shows *P f*
⟨*proof*⟩

lemma *unit-pred-cases*:
assumes *P* ⊥
assumes *P* (*single* ())
shows *P Q*
⟨*proof*⟩

lemma *holds-if-pred*:
holds (*if-pred b*) = *b*
⟨*proof*⟩

lemma *if-pred-holds*:
if-pred (*holds P*) = *P*
⟨*proof*⟩

lemma *is-empty-holds*:
is-empty P ⟷ ¬ *holds P*
⟨*proof*⟩

definition *map* :: ('*a* ⇒ '*b*) ⇒ '*a pred* ⇒ '*b pred* **where**
map f P = *P* ⤜ (*single* ∘ *f*)

lemma *eval-map* [*simp*]:
eval (*map f P*) = (⨆*x*∈{*x*. *eval P x*}. (λ*y*. *f x* = *y*))
⟨*proof*⟩

functor *map*: *map*
⟨*proof*⟩

76.4 Implementation

datatype (*plugins only*: *code extraction*) (*dead* '*a*) *seq* =

Empty
| *Insert 'a 'a pred*
| *Join 'a pred 'a seq*

primrec *pred-of-seq* :: *'a seq* ⇒ *'a pred* **where**
pred-of-seq Empty = ⊥
| *pred-of-seq* (*Insert x P*) = *single x* ⊔ *P*
| *pred-of-seq* (*Join P xq*) = *P* ⊔ *pred-of-seq xq*

definition *Seq* :: (*unit* ⇒ *'a seq*) ⇒ *'a pred* **where**
Seq f = *pred-of-seq* (*f* ())

code-datatype *Seq*

primrec *member* :: *'a seq* ⇒ *'a* ⇒ *bool* **where**
member Empty x ⟷ *False*
| *member* (*Insert y P*) *x* ⟷ *x* = *y* ∨ *eval P x*
| *member* (*Join P xq*) *x* ⟷ *eval P x* ∨ *member xq x*

lemma *eval-member*:
member xq = *eval* (*pred-of-seq xq*)
⟨*proof*⟩

lemma *eval-code* [*code*]: *eval* (*Seq f*) = *member* (*f* ())
⟨*proof*⟩

lemma *single-code* [*code*]:
single x = *Seq* (λ*u*. *Insert x* ⊥)
⟨*proof*⟩

primrec *apply* :: (*'a* ⇒ *'b pred*) ⇒ *'a seq* ⇒ *'b seq* **where**
apply f Empty = *Empty*
| *apply f* (*Insert x P*) = *Join* (*f x*) (*Join* (*P* ⤜ *f*) *Empty*)
| *apply f* (*Join P xq*) = *Join* (*P* ⤜ *f*) (*apply f xq*)

lemma *apply-bind*:
pred-of-seq (*apply f xq*) = *pred-of-seq xq* ⤜ *f*
⟨*proof*⟩

lemma *bind-code* [*code*]:
Seq g ⤜ *f* = *Seq* (λ*u*. *apply f* (*g* ()))
⟨*proof*⟩

lemma *bot-set-code* [*code*]:
⊥ = *Seq* (λ*u*. *Empty*)
⟨*proof*⟩

primrec *adjunct* :: *'a pred* ⇒ *'a seq* ⇒ *'a seq* **where**
adjunct P Empty = *Join P Empty*

| *adjunct P (Insert x Q) = Insert x (Q ⊔ P)*
| *adjunct P (Join Q xq) = Join Q (adjunct P xq)*

lemma *adjunct-sup*:
pred-of-seq (adjunct P xq) = P ⊔ pred-of-seq xq
⟨*proof*⟩

lemma *sup-code* [*code*]:
Seq f ⊔ Seq g = Seq (λu. case f ()
of Empty ⇒ g ()
| Insert x P ⇒ Insert x (P ⊔ Seq g)
| Join P xq ⇒ adjunct (Seq g) (Join P xq))
⟨*proof*⟩

primrec *contained* :: *'a seq ⇒ 'a pred ⇒ bool* **where**
contained Empty Q ⟷ True
| *contained (Insert x P) Q ⟷ eval Q x ∧ P ≤ Q*
| *contained (Join P xq) Q ⟷ P ≤ Q ∧ contained xq Q*

lemma *single-less-eq-eval*:
single x ≤ P ⟷ eval P x
⟨*proof*⟩

lemma *contained-less-eq*:
contained xq Q ⟷ pred-of-seq xq ≤ Q
⟨*proof*⟩

lemma *less-eq-pred-code* [*code*]:
Seq f ≤ Q = (case f ()
of Empty ⇒ True
| Insert x P ⇒ eval Q x ∧ P ≤ Q
| Join P xq ⇒ P ≤ Q ∧ contained xq Q)
⟨*proof*⟩

instantiation *pred* :: (*type*) *equal*
begin

definition *equal-pred*
where [*simp*]: *HOL.equal P Q ⟷ P = (Q :: 'a pred)*

instance ⟨*proof*⟩

end

lemma [*code*]:
HOL.equal P Q ⟷ P ≤ Q ∧ Q ≤ P **for** *P Q* :: *'a pred*
⟨*proof*⟩

lemma [*code nbe*]:

HOL.equal P P ⟷ *True* **for** *P* :: *'a pred*
⟨*proof*⟩

lemma [*code*]:
case-pred f P = *f* (*eval P*)
⟨*proof*⟩

lemma [*code*]:
rec-pred f P = *f* (*eval P*)
⟨*proof*⟩

inductive *eq* :: *'a* ⇒ *'a* ⇒ *bool* **where** *eq x x*

lemma *eq-is-eq*: *eq x y* ≡ (*x* = *y*)
⟨*proof*⟩

primrec *null* :: *'a seq* ⇒ *bool* **where**
null Empty ⟷ *True*
| *null* (*Insert x P*) ⟷ *False*
| *null* (*Join P xq*) ⟷ *is-empty P* ∧ *null xq*

lemma *null-is-empty*:
null xq ⟷ *is-empty* (*pred-of-seq xq*)
⟨*proof*⟩

lemma *is-empty-code* [*code*]:
is-empty (*Seq f*) ⟷ *null* (*f* ())
⟨*proof*⟩

primrec *the-only* :: (*unit* ⇒ *'a*) ⇒ *'a seq* ⇒ *'a* **where**
the-only default Empty = *default* () **for** *default*
| *the-only default* (*Insert x P*) =
(*if is-empty P then x else let y* = *singleton default P in if x* = *y then x else default* ()) **for** *default*
| *the-only default* (*Join P xq*) =
(*if is-empty P then the-only default xq else if null xq then singleton default P*
else let x = *singleton default P*; *y* = *the-only default xq in*
if x = *y then x else default* ()) **for** *default*

lemma *the-only-singleton*:
the-only default xq = *singleton default* (*pred-of-seq xq*) **for** *default*
⟨*proof*⟩

lemma *singleton-code* [*code*]:
singleton default (*Seq f*) =
(*case f* () *of*
Empty ⇒ *default* ()
| *Insert x P* ⇒ *if is-empty P then x*
else let y = *singleton default P in*

```
      if x = y then x else default ()
  | Join P xq ⇒ if is-empty P then the-only default xq
      else if null xq then singleton default P
      else let x = singleton default P; y = the-only default xq in
        if x = y then x else default ()) for default
⟨proof⟩

definition the :: 'a pred ⇒ 'a where
  the A = (THE x. eval A x)

lemma the-eqI:
  (THE x. eval P x) = x ⟹ the P = x
  ⟨proof⟩

lemma the-eq [code]: the A = singleton (λx. Code.abort (STR ''not-unique'') (λ-. the A)) A
  ⟨proof⟩

code-reflect Predicate
  datatypes pred = Seq and seq = Empty | Insert | Join

⟨ML⟩
```

Conversion from and to sets

```
definition pred-of-set :: 'a set ⇒ 'a pred where
  pred-of-set = Pred ∘ (λA x. x ∈ A)

lemma eval-pred-of-set [simp]:
  eval (pred-of-set A) x ⟷ x ∈A
  ⟨proof⟩

definition set-of-pred :: 'a pred ⇒ 'a set where
  set-of-pred = Collect ∘ eval

lemma member-set-of-pred [simp]:
  x ∈ set-of-pred P ⟷ Predicate.eval P x
  ⟨proof⟩

definition set-of-seq :: 'a seq ⇒ 'a set where
  set-of-seq = set-of-pred ∘ pred-of-seq

lemma member-set-of-seq [simp]:
  x ∈ set-of-seq xq = Predicate.member xq x
  ⟨proof⟩

lemma of-pred-code [code]:
  set-of-pred (Predicate.Seq f) = (case f () of
    Predicate.Empty ⇒ {}
  | Predicate.Insert x P ⇒ insert x (set-of-pred P)
```

| *Predicate.Join P xq ⇒ set-of-pred P ∪ set-of-seq xq*)
⟨*proof*⟩

lemma *of-seq-code* [*code*]:
set-of-seq Predicate.Empty = {}
set-of-seq (Predicate.Insert x P) = insert x (set-of-pred P)
set-of-seq (Predicate.Join P xq) = set-of-pred P ∪ set-of-seq xq
⟨*proof*⟩

Lazy Evaluation of an indexed function

function *iterate-upto* :: (*natural ⇒ 'a*) *⇒ natural ⇒ natural ⇒ 'a Predicate.pred*
where
iterate-upto f n m =
Predicate.Seq (%u. if n > m then Predicate.Empty
else Predicate.Insert (f n) (iterate-upto f (n + 1) m))
⟨*proof*⟩

termination ⟨*proof*⟩

Misc

declare *Inf-set-fold* [**where** *'a = 'a Predicate.pred*, *code*]
declare *Sup-set-fold* [**where** *'a = 'a Predicate.pred*, *code*]

lemma *pred-of-set-fold-sup*:
assumes *finite A*
shows *pred-of-set A = Finite-Set.fold sup bot (Predicate.single ‘ A)* (**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *pred-of-set-set-fold-sup*:
pred-of-set (set xs) = fold sup (List.map Predicate.single xs) bot
⟨*proof*⟩

lemma *pred-of-set-set-foldr-sup* [*code*]:
pred-of-set (set xs) = foldr sup (List.map Predicate.single xs) bot
⟨*proof*⟩

no-notation
bind (**infixl** *⤜ 70*)

hide-type (**open**) *pred seq*
hide-const (**open**) *Pred eval single bind is-empty singleton if-pred not-pred holds Empty Insert Join Seq member pred-of-seq apply adjunct null the-only eq map the iterate-upto*
hide-fact (**open**) *null-def member-def*

end

77 Lazy sequences

theory *Lazy-Sequence*
imports *Predicate*
begin

77.1 Type of lazy sequences

datatype (*plugins only*: *code extraction*) (*dead* 'a) *lazy-sequence* =
  lazy-sequence-of-list 'a *list*

primrec *list-of-lazy-sequence* :: 'a *lazy-sequence* ⇒ 'a *list*
where
  list-of-lazy-sequence (*lazy-sequence-of-list xs*) = *xs*

lemma *lazy-sequence-of-list-of-lazy-sequence* [*simp*]:
  lazy-sequence-of-list (*list-of-lazy-sequence xq*) = *xq*
  ⟨*proof*⟩

lemma *lazy-sequence-eqI*:
  list-of-lazy-sequence xq = *list-of-lazy-sequence yq* ⟹ *xq* = *yq*
  ⟨*proof*⟩

lemma *lazy-sequence-eq-iff*:
  xq = *yq* ⟷ *list-of-lazy-sequence xq* = *list-of-lazy-sequence yq*
  ⟨*proof*⟩

lemma *case-lazy-sequence* [*simp*]:
  case-lazy-sequence f xq = *f* (*list-of-lazy-sequence xq*)
  ⟨*proof*⟩

lemma *rec-lazy-sequence* [*simp*]:
  rec-lazy-sequence f xq = *f* (*list-of-lazy-sequence xq*)
  ⟨*proof*⟩

definition *Lazy-Sequence* :: (*unit* ⇒ ('a × 'a *lazy-sequence*) *option*) ⇒ 'a *lazy-sequence*
where
  Lazy-Sequence f = *lazy-sequence-of-list* (*case f* () *of*
    None ⇒ []
  | *Some* (*x*, *xq*) ⇒ *x* # *list-of-lazy-sequence xq*)

code-datatype *Lazy-Sequence*

declare *list-of-lazy-sequence.simps* [*code del*]
declare *lazy-sequence.case* [*code del*]
declare *lazy-sequence.rec* [*code del*]

lemma *list-of-Lazy-Sequence* [*simp*]:
  list-of-lazy-sequence (*Lazy-Sequence f*) = (*case f* () *of*
    None ⇒ []

| *Some* (*x*, *xq*) ⇒ *x* # *list-of-lazy-sequence xq*)
⟨*proof*⟩

definition *yield* :: *'a lazy-sequence* ⇒ (*'a* × *'a lazy-sequence*) *option*
where
yield xq = (*case list-of-lazy-sequence xq of*
[] ⇒ *None*
| *x* # *xs* ⇒ *Some* (*x*, *lazy-sequence-of-list xs*))

lemma *yield-Seq* [*simp*, *code*]:
yield (*Lazy-Sequence f*) = *f* ()
⟨*proof*⟩

lemma *case-yield-eq* [*simp*]: *case-option g h* (*yield xq*) =
case-list g (λ*x*. *curry h x* ∘ *lazy-sequence-of-list*) (*list-of-lazy-sequence xq*)
⟨*proof*⟩

lemma *equal-lazy-sequence-code* [*code*]:
HOL.equal xq yq = (*case* (*yield xq*, *yield yq*) *of*
(*None*, *None*) ⇒ *True*
| (*Some* (*x*, *xq'*), *Some* (*y*, *yq'*)) ⇒ *HOL.equal x y* ∧ *HOL.equal xq yq*
| - ⇒ *False*)
⟨*proof*⟩

lemma [*code nbe*]:
HOL.equal (*x* :: *'a lazy-sequence*) *x* ⟷ *True*
⟨*proof*⟩

definition *empty* :: *'a lazy-sequence*
where
empty = *lazy-sequence-of-list* []

lemma *list-of-lazy-sequence-empty* [*simp*]:
list-of-lazy-sequence empty = []
⟨*proof*⟩

lemma *empty-code* [*code*]:
empty = *Lazy-Sequence* (λ-. *None*)
⟨*proof*⟩

definition *single* :: *'a* ⇒ *'a lazy-sequence*
where
single x = *lazy-sequence-of-list* [*x*]

lemma *list-of-lazy-sequence-single* [*simp*]:
list-of-lazy-sequence (*single x*) = [*x*]
⟨*proof*⟩

lemma *single-code* [*code*]:

single x = Lazy-Sequence (λ-. Some (x, empty))
⟨*proof*⟩

definition *append* :: *'a lazy-sequence ⇒ 'a lazy-sequence ⇒ 'a lazy-sequence*
where
append xq yq = lazy-sequence-of-list (list-of-lazy-sequence xq @ list-of-lazy-sequence yq)

lemma *list-of-lazy-sequence-append* [*simp*]:
list-of-lazy-sequence (append xq yq) = list-of-lazy-sequence xq @ list-of-lazy-sequence yq
⟨*proof*⟩

lemma *append-code* [*code*]:
append xq yq = Lazy-Sequence (λ-. case yield xq of
None ⇒ yield yq
| Some (x, xq') ⇒ Some (x, append xq' yq))
⟨*proof*⟩

definition *map* :: *('a ⇒ 'b) ⇒ 'a lazy-sequence ⇒ 'b lazy-sequence*
where
map f xq = lazy-sequence-of-list (List.map f (list-of-lazy-sequence xq))

lemma *list-of-lazy-sequence-map* [*simp*]:
list-of-lazy-sequence (map f xq) = List.map f (list-of-lazy-sequence xq)
⟨*proof*⟩

lemma *map-code* [*code*]:
map f xq =
Lazy-Sequence (λ-. map-option (λ(x, xq'). (f x, map f xq')) (yield xq))
⟨*proof*⟩

definition *flat* :: *'a lazy-sequence lazy-sequence ⇒ 'a lazy-sequence*
where
flat xqq = lazy-sequence-of-list (concat (List.map list-of-lazy-sequence (list-of-lazy-sequence xqq)))

lemma *list-of-lazy-sequence-flat* [*simp*]:
list-of-lazy-sequence (flat xqq) = concat (List.map list-of-lazy-sequence (list-of-lazy-sequence xqq))
⟨*proof*⟩

lemma *flat-code* [*code*]:
flat xqq = Lazy-Sequence (λ-. case yield xqq of
None ⇒ None
| Some (xq, xqq') ⇒ yield (append xq (flat xqq')))
⟨*proof*⟩

definition *bind* :: *'a lazy-sequence ⇒ ('a ⇒ 'b lazy-sequence) ⇒ 'b lazy-sequence*

where
bind xq f = flat (map f xq)

definition *if-seq* :: *bool ⇒ unit lazy-sequence*
where
if-seq b = (if b then single () else empty)

definition *those* :: *'a option lazy-sequence ⇒ 'a lazy-sequence option*
where
those xq = map-option lazy-sequence-of-list (List.those (list-of-lazy-sequence xq))

function *iterate-upto* :: *(natural ⇒ 'a) ⇒ natural ⇒ natural ⇒ 'a lazy-sequence*
where
iterate-upto f n m =
Lazy-Sequence (λ-. if n > m then None else Some (f n, iterate-upto f (n + 1) m))
⟨*proof*⟩

termination ⟨*proof*⟩

definition *not-seq* :: *unit lazy-sequence ⇒ unit lazy-sequence*
where
not-seq xq = (case yield xq of
None ⇒ single ()
| Some ((), xq) ⇒ empty)

77.2 Code setup

code-reflect *Lazy-Sequence*
datatypes *lazy-sequence = Lazy-Sequence*

⟨*ML*⟩

77.3 Generator Sequences

77.3.1 General lazy sequence operation

definition *product* :: *'a lazy-sequence ⇒ 'b lazy-sequence ⇒ ('a × 'b) lazy-sequence*
where
product s1 s2 = bind s1 (λa. bind s2 (λb. single (a, b)))

77.3.2 Small lazy typeclasses

class *small-lazy =*
fixes *small-lazy* :: *natural ⇒ 'a lazy-sequence*

instantiation *unit* :: *small-lazy*
begin

definition *small-lazy d = single ()*

instance ⟨*proof*⟩

end

instantiation *int* :: *small-lazy*
begin

maybe optimise this expression -> append (single x) xs == cons x xs Performance difference?

function *small-lazy'* :: *int* ⇒ *int* ⇒ *int lazy-sequence*
where
small-lazy' d i = (*if d* < *i then empty*
else append (*single i*) (*small-lazy' d* (*i* + *1*)))
⟨*proof*⟩

termination
⟨*proof*⟩

definition
small-lazy d = *small-lazy'* (*int* (*nat-of-natural d*)) (− (*int* (*nat-of-natural d*)))

instance ⟨*proof*⟩

end

instantiation *prod* :: (*small-lazy*, *small-lazy*) *small-lazy*
begin

definition
small-lazy d = *product* (*small-lazy d*) (*small-lazy d*)

instance ⟨*proof*⟩

end

instantiation *list* :: (*small-lazy*) *small-lazy*
begin

fun *small-lazy-list* :: *natural* ⇒ *'a list lazy-sequence*
where
small-lazy-list d = *append* (*single* []) (*if d* > *0 then bind* (*product* (*small-lazy* (*d* − *1*)) (*small-lazy* (*d* − *1*))) (λ(*x*, *xs*). *single* (*x* # *xs*)) *else empty*)

instance ⟨*proof*⟩

end

77.4 With Hit Bound Value

assuming in negative context

type-synonym *'a hit-bound-lazy-sequence = 'a option lazy-sequence*

definition *hit-bound* :: *'a hit-bound-lazy-sequence*
where
hit-bound = Lazy-Sequence (λ-. *Some* (*None*, *empty*))

lemma *list-of-lazy-sequence-hit-bound* [*simp*]:
list-of-lazy-sequence hit-bound = [None]
⟨*proof*⟩

definition *hb-single* :: *'a ⇒ 'a hit-bound-lazy-sequence*
where
hb-single x = Lazy-Sequence (λ-. *Some* (*Some x*, *empty*))

definition *hb-map* :: *('a ⇒ 'b) ⇒ 'a hit-bound-lazy-sequence ⇒ 'b hit-bound-lazy-sequence*
where
hb-map f xq = map (*map-option f*) *xq*

lemma *hb-map-code* [*code*]:
hb-map f xq =
Lazy-Sequence (λ-. *map-option* (λ(*x*, *xq'*). (*map-option f x*, *hb-map f xq'*)) (*yield xq*))
⟨*proof*⟩

definition *hb-flat* :: *'a hit-bound-lazy-sequence hit-bound-lazy-sequence ⇒ 'a hit-bound-lazy-sequence*
where
hb-flat xqq = lazy-sequence-of-list (*concat*
(*List.map* ((λ*x*. *case x of None ⇒ [None] | Some xs ⇒ xs*) ∘ *map-option list-of-lazy-sequence*) (*list-of-lazy-sequence xqq*)))

lemma *list-of-lazy-sequence-hb-flat* [*simp*]:
list-of-lazy-sequence (*hb-flat xqq*) =
concat (*List.map* ((λ*x*. *case x of None ⇒ [None] | Some xs ⇒ xs*) ∘ *map-option list-of-lazy-sequence*) (*list-of-lazy-sequence xqq*))
⟨*proof*⟩

lemma *hb-flat-code* [*code*]:
hb-flat xqq = Lazy-Sequence (λ-. *case yield xqq of*
None ⇒ None
| *Some* (*xq*, *xqq'*) ⇒ *yield*
(*append* (*case xq of None ⇒ hit-bound | Some xq ⇒ xq*) (*hb-flat xqq'*)))
⟨*proof*⟩

definition *hb-bind* :: *'a hit-bound-lazy-sequence ⇒ ('a ⇒ 'b hit-bound-lazy-sequence) ⇒ 'b hit-bound-lazy-sequence*
where

hb-bind xq f = hb-flat (hb-map f xq)

definition *hb-if-seq* :: *bool ⇒ unit hit-bound-lazy-sequence*
where
hb-if-seq b = (if b then hb-single () else empty)

definition *hb-not-seq* :: *unit hit-bound-lazy-sequence ⇒ unit lazy-sequence*
where
hb-not-seq xq = (case yield xq of
None ⇒ single ()
| Some (x, xq) ⇒ empty)

hide-const (**open**) *yield empty single append flat map bind*
if-seq those iterate-upto not-seq product

hide-fact (**open**) *yield-def empty-def single-def append-def flat-def map-def bind-def*
if-seq-def those-def not-seq-def product-def

end

78 Depth-Limited Sequences with failure element

theory *Limited-Sequence*
imports *Lazy-Sequence*
begin

78.1 Depth-Limited Sequence

type-synonym *'a dseq = natural ⇒ bool ⇒ 'a lazy-sequence option*

definition *empty* :: *'a dseq*
where
empty = (λ- -. Some Lazy-Sequence.empty)

definition *single* :: *'a ⇒ 'a dseq*
where
single x = (λ- -. Some (Lazy-Sequence.single x))

definition *eval* :: *'a dseq ⇒ natural ⇒ bool ⇒ 'a lazy-sequence option*
where
[*simp*]: *eval f i pol = f i pol*

definition *yield* :: *'a dseq ⇒ natural ⇒ bool ⇒ ('a × 'a dseq) option*
where
yield f i pol = (case eval f i pol of
None ⇒ None
| Some s ⇒ (map-option ∘ apsnd) (λr - -. Some r) (Lazy-Sequence.yield s))

definition *map-seq* :: *('a ⇒ 'b dseq) ⇒ 'a lazy-sequence ⇒ 'b dseq*

where
map-seq f xq i pol = map-option Lazy-Sequence.flat
(Lazy-Sequence.those (Lazy-Sequence.map (λx. f x i pol) xq))

lemma *map-seq-code* [*code*]:
map-seq f xq i pol = (case Lazy-Sequence.yield xq of
None ⇒ Some Lazy-Sequence.empty
| Some (x, xq′) ⇒ (case eval (f x) i pol of
None ⇒ None
| Some yq ⇒ (case map-seq f xq′ i pol of
None ⇒ None
| Some zq ⇒ Some (Lazy-Sequence.append yq zq))))
⟨*proof*⟩

definition *bind* :: *′a dseq ⇒ (′a ⇒ ′b dseq) ⇒ ′b dseq*
where
bind x f = (λi pol.
if i = 0 then
(if pol then Some Lazy-Sequence.empty else None)
else
(case x (i − 1) pol of
None ⇒ None
| Some xq ⇒ map-seq f xq i pol))

definition *union* :: *′a dseq ⇒ ′a dseq ⇒ ′a dseq*
where
union x y = (λi pol. case (x i pol, y i pol) of
(Some xq, Some yq) ⇒ Some (Lazy-Sequence.append xq yq)
| - ⇒ None)

definition *if-seq* :: *bool ⇒ unit dseq*
where
if-seq b = (if b then single () else empty)

definition *not-seq* :: *unit dseq ⇒ unit dseq*
where
not-seq x = (λi pol. case x i (¬ pol) of
None ⇒ Some Lazy-Sequence.empty
| Some xq ⇒ (case Lazy-Sequence.yield xq of
None ⇒ Some (Lazy-Sequence.single ())
| Some - ⇒ Some (Lazy-Sequence.empty)))

definition *map* :: *(′a ⇒ ′b) ⇒ ′a dseq ⇒ ′b dseq*
where
map f g = (λi pol. case g i pol of
None ⇒ None
| Some xq ⇒ Some (Lazy-Sequence.map f xq))

78.2 Positive Depth-Limited Sequence

type-synonym *'a pos-dseq = natural ⇒ 'a Lazy-Sequence.lazy-sequence*

definition *pos-empty :: 'a pos-dseq*
where
pos-empty = (λi. Lazy-Sequence.empty)

definition *pos-single :: 'a ⇒ 'a pos-dseq*
where
pos-single x = (λi. Lazy-Sequence.single x)

definition *pos-bind :: 'a pos-dseq ⇒ ('a ⇒ 'b pos-dseq) ⇒ 'b pos-dseq*
where
pos-bind x f = (λi. Lazy-Sequence.bind (x i) (λa. f a i))

definition *pos-decr-bind :: 'a pos-dseq ⇒ ('a ⇒ 'b pos-dseq) ⇒ 'b pos-dseq*
where
pos-decr-bind x f = (λi.
if i = 0 then
Lazy-Sequence.empty
else
Lazy-Sequence.bind (x (i − 1)) (λa. f a i))

definition *pos-union :: 'a pos-dseq ⇒ 'a pos-dseq ⇒ 'a pos-dseq*
where
pos-union xq yq = (λi. Lazy-Sequence.append (xq i) (yq i))

definition *pos-if-seq :: bool ⇒ unit pos-dseq*
where
pos-if-seq b = (if b then pos-single () else pos-empty)

definition *pos-iterate-upto :: (natural ⇒ 'a) ⇒ natural ⇒ natural ⇒ 'a pos-dseq*
where
pos-iterate-upto f n m = (λi. Lazy-Sequence.iterate-upto f n m)

definition *pos-map :: ('a ⇒ 'b) ⇒ 'a pos-dseq ⇒ 'b pos-dseq*
where
pos-map f xq = (λi. Lazy-Sequence.map f (xq i))

78.3 Negative Depth-Limited Sequence

type-synonym *'a neg-dseq = natural ⇒ 'a Lazy-Sequence.hit-bound-lazy-sequence*

definition *neg-empty :: 'a neg-dseq*
where
neg-empty = (λi. Lazy-Sequence.empty)

definition *neg-single :: 'a ⇒ 'a neg-dseq*
where

neg-single x = (λi. Lazy-Sequence.hb-single x)

definition *neg-bind* :: *'a neg-dseq ⇒ ('a ⇒ 'b neg-dseq) ⇒ 'b neg-dseq*
where
neg-bind x f = (λi. hb-bind (x i) (λa. f a i))

definition *neg-decr-bind* :: *'a neg-dseq ⇒ ('a ⇒ 'b neg-dseq) ⇒ 'b neg-dseq*
where
neg-decr-bind x f = (λi.
if i = 0 then
Lazy-Sequence.hit-bound
else
hb-bind (x (i − 1)) (λa. f a i))

definition *neg-union* :: *'a neg-dseq ⇒ 'a neg-dseq ⇒ 'a neg-dseq*
where
neg-union x y = (λi. Lazy-Sequence.append (x i) (y i))

definition *neg-if-seq* :: *bool ⇒ unit neg-dseq*
where
neg-if-seq b = (if b then neg-single () else neg-empty)

definition *neg-iterate-upto*
where
neg-iterate-upto f n m = (λi. Lazy-Sequence.iterate-upto (λi. Some (f i)) n m)

definition *neg-map* :: *('a ⇒ 'b) ⇒ 'a neg-dseq ⇒ 'b neg-dseq*
where
neg-map f xq = (λi. Lazy-Sequence.hb-map f (xq i))

78.4 Negation

definition *pos-not-seq* :: *unit neg-dseq ⇒ unit pos-dseq*
where
pos-not-seq xq = (λi. Lazy-Sequence.hb-not-seq (xq (3 ∗ i)))

definition *neg-not-seq* :: *unit pos-dseq ⇒ unit neg-dseq*
where
neg-not-seq x = (λi. case Lazy-Sequence.yield (x i) of
None ⇒ Lazy-Sequence.hb-single ()
| Some ((), xq) ⇒ Lazy-Sequence.empty)

⟨*ML*⟩

code-reserved *Eval Limited-Sequence*

hide-const (**open**) *yield empty single eval map-seq bind union if-seq not-seq map*

pos-empty pos-single pos-bind pos-decr-bind pos-union pos-if-seq pos-iterate-upto pos-not-seq pos-map
neg-empty neg-single neg-bind neg-decr-bind neg-union neg-if-seq neg-iterate-upto neg-not-seq neg-map

hide-fact (**open**) *yield-def empty-def single-def eval-def map-seq-def bind-def union-def if-seq-def not-seq-def map-def*
pos-empty-def pos-single-def pos-bind-def pos-union-def pos-if-seq-def pos-iterate-upto-def pos-not-seq-def pos-map-def
neg-empty-def neg-single-def neg-bind-def neg-union-def neg-if-seq-def neg-iterate-upto-def neg-not-seq-def neg-map-def

end

79 Term evaluation using the generic code generator

theory *Code-Evaluation*
imports *Typerep Limited-Sequence*
keywords *value* :: *diag*
begin

79.1 Term representation

79.1.1 Terms and class *term-of*

datatype (*plugins only*: *extraction*) *term* = *dummy-term*

definition *Const* :: *String.literal* ⇒ *typerep* ⇒ *term* **where**
Const - - = *dummy-term*

definition *App* :: *term* ⇒ *term* ⇒ *term* **where**
App - - = *dummy-term*

definition *Abs* :: *String.literal* ⇒ *typerep* ⇒ *term* ⇒ *term* **where**
Abs - - - = *dummy-term*

definition *Free* :: *String.literal* ⇒ *typerep* ⇒ *term* **where**
Free - - = *dummy-term*

code-datatype *Const App Abs Free*

class *term-of* = *typerep* +
fixes *term-of* :: *'a* ⇒ *term*

lemma *term-of-anything*: *term-of x* ≡ *t*
⟨*proof*⟩

definition *valapp* :: (*'a* ⇒ *'b*) × (*unit* ⇒ *term*)

$\Rightarrow$ $'a \times (unit \Rightarrow term) \Rightarrow 'b \times (unit \Rightarrow term)$ **where**
valapp f x = (*fst f* (*fst x*), $\lambda u.$ *App* (*snd f* ()) (*snd x* ()))

lemma *valapp-code* [*code*, *code-unfold*]:
valapp (*f*, *tf*) (*x*, *tx*) = (*f x*, $\lambda u.$ *App* (*tf* ()) (*tx* ()))
⟨*proof*⟩

79.1.2 Syntax

definition *termify* :: $'a \Rightarrow$ *term* **where**
[*code del*]: *termify x* = *dummy-term*

abbreviation *valtermify* :: $'a \Rightarrow 'a \times (unit \Rightarrow term)$ **where**
valtermify x ≡ (*x*, $\lambda u.$ *termify x*)

bundle *term-syntax*
begin

notation *App* (**infixl** <·> *70*)
and *valapp* (**infixl** {·} *70*)

end

79.2 Tools setup and evaluation

context
begin

qualified definition *TERM-OF* :: *'a*::*term-of itself*
where
TERM-OF = *snd* (*Code-Evaluation.term-of* :: $'a \Rightarrow$ -, *TYPE*(*'a*))

qualified definition *TERM-OF-EQUAL* :: *'a*::*term-of itself*
where
TERM-OF-EQUAL = *snd* (λ(*a*::*'a*). (*Code-Evaluation.term-of a*, *HOL.eq a*), *TYPE*(*'a*))

end

lemma *eq-eq-TrueD*:
fixes *x y* :: *'a*::{}
assumes (*x* ≡ *y*) ≡ *Trueprop True*
shows *x* ≡ *y*
⟨*proof*⟩

code-printing
type-constructor *term* ⇀ (*Eval*) *Term.term*
| **constant** *Const* ⇀ (*Eval*) *Term.Const*/ ((-), (-))
| **constant** *App* ⇀ (*Eval*) *Term.*$/ ((-), (-))
| **constant** *Abs* ⇀ (*Eval*) *Term.Abs*/ ((-), (-), (-))

| **constant** *Free* ⇀ (*Eval*) *Term.Free*/ ((-), (-))

⟨*ML*⟩

code-reserved *Eval Code-Evaluation*

⟨*ML*⟩

79.3 Dedicated *term-of* instances

instantiation *fun* :: (*typerep*, *typerep*) *term-of*
begin

definition
term-of (*f* :: *'a* ⇒ *'b*) =
Const (*STR ''Pure.dummy-pattern''*)
(*Typerep.Typerep* (*STR ''fun''*) [*Typerep.typerep TYPE*(*'a*), *Typerep.typerep TYPE*(*'b*)])

instance ⟨*proof*⟩

end

declare [[*code drop*: *rec-term case-term*
term-of :: *typerep* ⇒ - *term-of* :: *term* ⇒ - *term-of* :: *String.literal* ⇒ -
term-of :: - *Predicate.pred* ⇒ *term term-of* :: - *Predicate.seq* ⇒ *term*]]

code-printing
constant *term-of* :: *integer* ⇒ *term* ⇀ (*Eval*) *HOLogic.mk'-number*/ *HOLogic.code'-integerT*
| **constant** *term-of* :: *String.literal* ⇒ *term* ⇀ (*Eval*) *HOLogic.mk'-literal*

declare [[*code drop*: *term-of* :: *integer* ⇒ -]]

lemma *term-of-integer* [*unfolded typerep-fun-def typerep-num-def typerep-integer-def*, *code*]:
term-of (*i* :: *integer*) =
(*if i* > *0 then*
App (*Const* (*STR ''Num.numeral-class.numeral''*) (*TYPEREP*(*num* ⇒ *integer*)))
(*term-of* (*num-of-integer i*))
else if i = *0 then Const* (*STR ''Groups.zero-class.zero''*) *TYPEREP*(*integer*)
else
App (*Const* (*STR ''Groups.uminus-class.uminus''*) *TYPEREP*(*integer* ⇒ *integer*))
(*term-of* (− *i*)))
⟨*proof*⟩

code-reserved *Eval HOLogic*

79.4 Generic reification

⟨*ML*⟩

79.5 Diagnostic

definition *tracing* :: *String.literal* ⇒ *'a* ⇒ *'a* **where**
[*code del*]: *tracing s x* = *x*

code-printing
constant *tracing* :: *String.literal* => *'a* => *'a* ⇀ (*Eval*) *Code'-Evaluation.tracing*

hide-const *dummy-term valapp*
hide-const (**open**) *Const App Abs Free termify valtermify term-of tracing*

end

80 A simple counterexample generator performing random testing

theory *Quickcheck-Random*
imports *Random Code-Evaluation Enum*
begin

⟨*ML*⟩

80.1 Catching Match exceptions

axiomatization *catch-match* :: *'a* => *'a* => *'a*

code-printing
constant *catch-match* ⇀ (*Quickcheck*) ((-) *handle Match* => -)

code-reserved *Quickcheck Match*

80.2 The *random* class

class *random* = *typerep* +
fixes *random* :: *natural* ⇒ *Random.seed* ⇒ (*'a* × (*unit* ⇒ *term*)) × *Random.seed*

80.3 Fundamental and numeric types

instantiation *bool* :: *random*
begin

context
includes *state-combinator-syntax*
begin

definition
random i = Random.range 2 ∘→
(λk. Pair (if k = 0 then Code-Evaluation.valtermify False else Code-Evaluation.valtermify True))

instance ⟨*proof*⟩

end

end

instantiation *itself* :: (*typerep*) *random*
begin

definition
random-itself :: *natural* ⇒ *Random.seed* ⇒ (*'a itself* × (*unit* ⇒ *term*)) × *Random.seed*
where *random-itself - = Pair (Code-Evaluation.valtermify TYPE('a))*

instance ⟨*proof*⟩

end

instantiation *char* :: *random*
begin

context
includes *state-combinator-syntax*
begin

definition
random - = Random.select (Enum.enum :: char list) ∘→ (λc. Pair (c, λu. Code-Evaluation.term-of c))

instance ⟨*proof*⟩

end

end

instantiation *String.literal* :: *random*
begin

definition
random - = Pair (STR ''', λu. Code-Evaluation.term-of (STR '''))

instance ⟨*proof*⟩

end

instantiation *nat* :: *random*
begin

context
includes *state-combinator-syntax*
begin

definition *random-nat* :: *natural* ⇒ *Random.seed*
⇒ (*nat* × (*unit* ⇒ *Code-Evaluation.term*)) × *Random.seed*
where
random-nat i = *Random.range* (*i* + *1*) ∘→ (λ*k*. *Pair* (
let n = *nat-of-natural k*
in (*n*, λ-. *Code-Evaluation.term-of n*)))

instance ⟨*proof*⟩

end

end

instantiation *int* :: *random*
begin

context
includes *state-combinator-syntax*
begin

definition
random i = *Random.range* (*2* ∗ *i* + *1*) ∘→ (λ*k*. *Pair* (
let j = (*if k* ≥ *i then int* (*nat-of-natural* (*k* − *i*)) *else* − (*int* (*nat-of-natural* (*i* − *k*))))
in (*j*, λ-. *Code-Evaluation.term-of j*)))

instance ⟨*proof*⟩

end

end

instantiation *natural* :: *random*
begin

context
includes *state-combinator-syntax*
begin

definition *random-natural* :: *natural* ⇒ *Random.seed*

⇒ (*natural* × (*unit* ⇒ *Code-Evaluation.term*)) × *Random.seed*
where
random-natural i = *Random.range* (*i* + *1*) ∘→ (λ*n*. *Pair* (*n*, λ-. *Code-Evaluation.term-of* *n*))

instance ⟨*proof*⟩

end

end

instantiation *integer* :: *random*
begin

context
includes *state-combinator-syntax*
begin

definition *random-integer* :: *natural* ⇒ *Random.seed*
⇒ (*integer* × (*unit* ⇒ *Code-Evaluation.term*)) × *Random.seed*
where
random-integer i = *Random.range* (*2* ∗ *i* + *1*) ∘→ (λ*k*. *Pair* (
let j = (*if k* ≥ *i then integer-of-natural* (*k* − *i*) *else* − (*integer-of-natural* (*i* − *k*)))
in (*j*, λ-. *Code-Evaluation.term-of j*)))

instance ⟨*proof*⟩

end

end

80.4 Complex generators

Towards *'a* ⇒ *'b*

axiomatization *random-fun-aux* :: *typerep* ⇒ *typerep* ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ (*'a* ⇒ *term*)
⇒ (*Random.seed* ⇒ (*'b* × (*unit* ⇒ *term*)) × *Random.seed*)
⇒ (*Random.seed* ⇒ *Random.seed* × *Random.seed*)
⇒ *Random.seed* ⇒ ((*'a* ⇒ *'b*) × (*unit* ⇒ *term*)) × *Random.seed*

definition *random-fun-lift* :: (*Random.seed* ⇒ (*'b* × (*unit* ⇒ *term*)) × *Random.seed*)
⇒ *Random.seed* ⇒ ((*'a*::*term-of* ⇒ *'b*::*typerep*) × (*unit* ⇒ *term*)) × *Random.seed*
where
random-fun-lift f =
random-fun-aux TYPEREP(*'a*) *TYPEREP*(*'b*) (=) *Code-Evaluation.term-of f* *Random.split-seed*

instantiation *fun* :: ({*equal*, *term-of*}, *random*) *random*
begin

definition
random-fun :: *natural* ⇒ *Random.seed* ⇒ ((*'a* ⇒ *'b*) × (*unit* ⇒ *term*)) × *Random.seed*
where *random i* = *random-fun-lift* (*random i*)

instance ⟨*proof*⟩

end

Towards type copies and datatypes

context
includes *state-combinator-syntax*
begin

definition *collapse* :: (*'a* ⇒ (*'a* ⇒ *'b* × *'a*) × *'a*) ⇒ *'a* ⇒ *'b* × *'a*
where *collapse f* = (*f* ∘→ *id*)

end

definition *beyond* :: *natural* ⇒ *natural* ⇒ *natural*
where *beyond k l* = (*if l* > *k then l else 0*)

lemma *beyond-zero*: *beyond k 0* = *0*
⟨*proof*⟩

context
includes *term-syntax*
begin

definition [*code-unfold*]:
valterm-emptyset = *Code-Evaluation.valtermify* ({} :: (*'a* :: *typerep*) *set*)

definition [*code-unfold*]:
valtermify-insert x s = *Code-Evaluation.valtermify insert* {·} (*x* :: (*'a* :: *typerep* ∗ -)) {·} *s*

end

instantiation *set* :: (*random*) *random*
begin

context
includes *state-combinator-syntax*
begin

fun *random-aux-set*

where
random-aux-set 0 j = collapse (Random.select-weight [(1, Pair valterm-emptyset)])
| random-aux-set (Code-Numeral.Suc i) j =
collapse (Random.select-weight
[(1, Pair valterm-emptyset),
(Code-Numeral.Suc i,
random j ∘→ (%x. random-aux-set i j ∘→ (%s. Pair (valtermify-insert x s))))])

lemma [*code*]:
random-aux-set i j =
collapse (Random.select-weight [(1, Pair valterm-emptyset),
(i, random j ∘→ (%x. random-aux-set (i − 1) j ∘→ (%s. Pair (valtermify-insert x s))))])
⟨*proof*⟩

definition *random-set i = random-aux-set i i*

instance ⟨*proof*⟩

end

end

lemma *random-aux-rec*:
fixes *random-aux :: natural ⇒ 'a*
assumes *random-aux 0 = rhs 0*
and ⋀*k. random-aux (Code-Numeral.Suc k) = rhs (Code-Numeral.Suc k)*
shows *random-aux k = rhs k*
⟨*proof*⟩

80.5 Deriving random generators for datatypes

⟨*ML*⟩

80.6 Code setup

code-printing
constant *random-fun-aux* ⇀ (*Quickcheck*) *Random'-Generators.random'-fun*
— With enough criminal energy this can be abused to derive *False*; for this reason we use a distinguished target *Quickcheck* not spoiling the regular trusted code generation

code-reserved *Quickcheck Random-Generators*

hide-const (**open**) *catch-match random collapse beyond random-fun-aux random-fun-lift*

hide-fact (**open**) *collapse-def beyond-def random-fun-lift-def*

end

81 The Random-Predicate Monad

theory *Random-Pred*
imports *Quickcheck-Random*
begin

fun *iter'* :: *'a itself* ⇒ *natural* ⇒ *natural* ⇒ *Random.seed* ⇒ (*'a*::*random*) *Predicate.pred*
where
iter' T nrandom sz seed = (*if nrandom* = *0 then bot-class.bot else*
let ((*x*, -), *seed'*) = *Quickcheck-Random.random sz seed*
in Predicate.Seq (%*u*. *Predicate.Insert x* (*iter' T* (*nrandom* − *1*) *sz seed'*)))

definition *iter* :: *natural* ⇒ *natural* ⇒ *Random.seed* ⇒ (*'a*::*random*) *Predicate.pred*
where
iter nrandom sz seed = *iter'* (*TYPE*(*'a*)) *nrandom sz seed*

lemma [*code*]:
iter nrandom sz seed = (*if nrandom* = *0 then bot-class.bot else*
let ((*x*, -), *seed'*) = *Quickcheck-Random.random sz seed*
in Predicate.Seq (%*u*. *Predicate.Insert x* (*iter* (*nrandom* − *1*) *sz seed'*)))
⟨*proof*⟩

type-synonym *'a random-pred* = *Random.seed* ⇒ (*'a Predicate.pred* × *Random.seed*)

definition *empty* :: *'a random-pred*
where *empty* = *Pair bot*

definition *single* :: *'a* => *'a random-pred*
where *single x* = *Pair* (*Predicate.single x*)

definition *bind* :: *'a random-pred* ⇒ (*'a* ⇒ *'b random-pred*) ⇒ *'b random-pred*
where
bind R f = (λ*s*. *let*
(*P*, *s'*) = *R s*;
(*s1*, *s2*) = *Random.split-seed s'*
in (*Predicate.bind P* (%*a*. *fst* (*f a s1*)), *s2*))

definition *union* :: *'a random-pred* ⇒ *'a random-pred* ⇒ *'a random-pred*
where
union R1 R2 = (λ*s*. *let*
(*P1*, *s'*) = *R1 s*; (*P2*, *s''*) = *R2 s'*
in (*sup-class.sup P1 P2*, *s''*))

definition *if-randompred* :: *bool* ⇒ *unit random-pred*
where
if-randompred b = (*if b then single* () *else empty*)

definition *iterate-upto* :: (*natural* ⇒ *'a*) => *natural* ⇒ *natural* ⇒ *'a random-pred*
where
iterate-upto f n m = *Pair* (*Predicate.iterate-upto f n m*)

definition *not-randompred* :: *unit random-pred* ⇒ *unit random-pred*
where
not-randompred P = (λ*s*. *let*
(*P'*, *s'*) = *P s*
in if Predicate.eval P' () *then* (*Orderings.bot*, *s'*) *else* (*Predicate.single* (), *s'*))

definition *Random* :: (*Random.seed* ⇒ (*'a* × (*unit* ⇒ *term*)) × *Random.seed*) ⇒ *'a random-pred*
where *Random g* = *scomp g* (*Pair* ∘ (*Predicate.single* ∘ *fst*))

definition *map* :: (*'a* ⇒ *'b*) ⇒ *'a random-pred* ⇒ *'b random-pred*
where *map f P* = *bind P* (*single* ∘ *f*)

hide-const (**open**) *iter'* *iter empty single bind union if-randompred*
iterate-upto not-randompred Random map

hide-fact *iter'.simps*

hide-fact (**open**) *iter-def empty-def single-def bind-def union-def*
if-randompred-def iterate-upto-def not-randompred-def Random-def map-def

end

82 Various kind of sequences inside the random monad

theory *Random-Sequence*
imports *Random-Pred*
begin

type-synonym *'a random-dseq* = *natural* ⇒ *natural* ⇒ *Random.seed* ⇒ (*'a Limited-Sequence.dseq* × *Random.seed*)

definition *empty* :: *'a random-dseq*
where
empty = (%*nrandom size*. *Pair* (*Limited-Sequence.empty*))

definition *single* :: *'a* => *'a random-dseq*
where
single x = (%*nrandom size*. *Pair* (*Limited-Sequence.single x*))

definition *bind* :: *'a random-dseq* => (*'a* ⇒ *'b random-dseq*) ⇒ *'b random-dseq*
where
bind R f = (λ*nrandom size s*. *let*

```
    (P, s') = R nrandom size s;
    (s1, s2) = Random.split-seed s'
  in (Limited-Sequence.bind P (%a. fst (f a nrandom size s1)), s2))

definition union :: 'a random-dseq => 'a random-dseq => 'a random-dseq
where
  union R1 R2 = (λnrandom size s. let
    (S1, s') = R1 nrandom size s; (S2, s'') = R2 nrandom size s'
  in (Limited-Sequence.union S1 S2, s''))

definition if-random-dseq :: bool => unit random-dseq
where
  if-random-dseq b = (if b then single () else empty)

definition not-random-dseq :: unit random-dseq => unit random-dseq
where
  not-random-dseq R = (λnrandom size s. let
    (S, s') = R nrandom size s
   in (Limited-Sequence.not-seq S, s'))

definition map :: ('a => 'b) => 'a random-dseq => 'b random-dseq
where
  map f P = bind P (single ∘ f)

fun Random :: (natural ⇒ Random.seed ⇒ (('a × (unit ⇒ term)) × Random.seed))
⇒ 'a random-dseq
where
  Random g nrandom = (%size. if nrandom <= 0 then (Pair Limited-Sequence.empty)
else
      (scomp (g size) (%r. scomp (Random g (nrandom − 1) size) (%rs. Pair
(Limited-Sequence.union (Limited-Sequence.single (fst r)) rs)))))


type-synonym 'a pos-random-dseq = natural ⇒ natural ⇒ Random.seed ⇒ 'a
Limited-Sequence.pos-dseq

definition pos-empty :: 'a pos-random-dseq
where
  pos-empty = (%nrandom size seed. Limited-Sequence.pos-empty)

definition pos-single :: 'a => 'a pos-random-dseq
where
  pos-single x = (%nrandom size seed. Limited-Sequence.pos-single x)

definition pos-bind :: 'a pos-random-dseq => ('a ⇒ 'b pos-random-dseq) ⇒ 'b
pos-random-dseq
where
  pos-bind R f = (λnrandom size seed. Limited-Sequence.pos-bind (R nrandom size
seed) (%a. f a nrandom size seed))
```

definition *pos-decr-bind* :: *'a pos-random-dseq => ('a ⇒ 'b pos-random-dseq) ⇒ 'b pos-random-dseq*
where
pos-decr-bind R f = (λnrandom size seed. Limited-Sequence.pos-decr-bind (R nrandom size seed) (%a. f a nrandom size seed))

definition *pos-union* :: *'a pos-random-dseq => 'a pos-random-dseq => 'a pos-random-dseq*
where
pos-union R1 R2 = (λnrandom size seed. Limited-Sequence.pos-union (R1 nrandom size seed) (R2 nrandom size seed))

definition *pos-if-random-dseq* :: *bool => unit pos-random-dseq*
where
pos-if-random-dseq b = (if b then pos-single () else pos-empty)

definition *pos-iterate-upto* :: *(natural => 'a) => natural => natural => 'a pos-random-dseq*
where
pos-iterate-upto f n m = (λnrandom size seed. Limited-Sequence.pos-iterate-upto f n m)

definition *pos-map* :: *('a => 'b) => 'a pos-random-dseq => 'b pos-random-dseq*
where
pos-map f P = pos-bind P (pos-single ∘ f)

fun *iter* :: *(Random.seed ⇒ ('a × (unit ⇒ term)) × Random.seed)*
⇒ natural ⇒ Random.seed ⇒ 'a Lazy-Sequence.lazy-sequence
where
iter random nrandom seed =
(if nrandom = 0 then Lazy-Sequence.empty else Lazy-Sequence (%u. let ((x, -), seed') = random seed in Some (x, iter random (nrandom − 1) seed')))

definition *pos-Random* :: *(natural ⇒ Random.seed ⇒ ('a × (unit ⇒ term)) × Random.seed)*
⇒ 'a pos-random-dseq
where
pos-Random g = (%nrandom size seed depth. iter (g size) nrandom seed)

type-synonym *'a neg-random-dseq = natural ⇒ natural ⇒ Random.seed ⇒ 'a Limited-Sequence.neg-dseq*

definition *neg-empty* :: *'a neg-random-dseq*
where
neg-empty = (%nrandom size seed. Limited-Sequence.neg-empty)

definition *neg-single* :: *'a => 'a neg-random-dseq*

where
neg-single x = (%nrandom size seed. Limited-Sequence.neg-single x)

definition *neg-bind :: 'a neg-random-dseq => ('a ⇒ 'b neg-random-dseq) ⇒ 'b neg-random-dseq*
where
neg-bind R f = (λnrandom size seed. Limited-Sequence.neg-bind (R nrandom size seed) (%a. f a nrandom size seed))

definition *neg-decr-bind :: 'a neg-random-dseq => ('a ⇒ 'b neg-random-dseq) ⇒ 'b neg-random-dseq*
where
neg-decr-bind R f = (λnrandom size seed. Limited-Sequence.neg-decr-bind (R nrandom size seed) (%a. f a nrandom size seed))

definition *neg-union :: 'a neg-random-dseq => 'a neg-random-dseq => 'a neg-random-dseq*
where
neg-union R1 R2 = (λnrandom size seed. Limited-Sequence.neg-union (R1 nrandom size seed) (R2 nrandom size seed))

definition *neg-if-random-dseq :: bool => unit neg-random-dseq*
where
neg-if-random-dseq b = (if b then neg-single () else neg-empty)

definition *neg-iterate-upto :: (natural => 'a) => natural => natural => 'a neg-random-dseq*
where
neg-iterate-upto f n m = (λnrandom size seed. Limited-Sequence.neg-iterate-upto f n m)

definition *neg-not-random-dseq :: unit pos-random-dseq => unit neg-random-dseq*
where
neg-not-random-dseq R = (λnrandom size seed. Limited-Sequence.neg-not-seq (R nrandom size seed))

definition *neg-map :: ('a => 'b) => 'a neg-random-dseq => 'b neg-random-dseq*
where
neg-map f P = neg-bind P (neg-single ∘ f)

definition *pos-not-random-dseq :: unit neg-random-dseq => unit pos-random-dseq*
where
pos-not-random-dseq R = (λnrandom size seed. Limited-Sequence.pos-not-seq (R nrandom size seed))

hide-const (**open**)
empty single bind union if-random-dseq not-random-dseq map Random pos-empty pos-single pos-bind pos-decr-bind pos-union pos-if-random-dseq pos-iterate-upto pos-not-random-dseq pos-map iter pos-Random

neg-empty neg-single neg-bind neg-decr-bind neg-union neg-if-random-dseq neg-iterate-upto neg-not-random-dseq neg-map

hide-fact (**open**) *empty-def single-def bind-def union-def if-random-dseq-def not-random-dseq-def map-def Random.simps*
pos-empty-def pos-single-def pos-bind-def pos-decr-bind-def pos-union-def pos-if-random-dseq-def pos-iterate-upto-def pos-not-random-dseq-def pos-map-def iter.simps pos-Random-def neg-empty-def neg-single-def neg-bind-def neg-decr-bind-def neg-union-def neg-if-random-dseq-def neg-iterate-upto-def neg-not-random-dseq-def neg-map-def

end

83 A simple counterexample generator performing exhaustive testing

theory *Quickcheck-Exhaustive*
imports *Quickcheck-Random*
keywords *quickcheck-generator* :: *thy-decl*
begin

83.1 Basic operations for exhaustive generators

definition *orelse* :: *'a option ⇒ 'a option ⇒ 'a option* (**infixr** *orelse 55*)
where [*code-unfold*]: *x orelse y = (case x of Some x' ⇒ Some x' | None ⇒ y)*

83.2 Exhaustive generator type classes

class *exhaustive = term-of +*
fixes *exhaustive* :: *('a ⇒ (bool × term list) option) ⇒ natural ⇒ (bool × term list) option*

class *full-exhaustive = term-of +*
fixes *full-exhaustive* ::
('a × (unit ⇒ term) ⇒ (bool × term list) option) ⇒ natural ⇒ (bool × term list) option

instantiation *natural* :: *full-exhaustive*
begin

function *full-exhaustive-natural'* ::
(natural × (unit ⇒ term) ⇒ (bool × term list) option) ⇒
natural ⇒ natural ⇒ (bool × term list) option
where *full-exhaustive-natural' f d i =*
(if d < i then None
else (f (i, λ-. Code-Evaluation.term-of i)) orelse (full-exhaustive-natural' f d (i + 1)))
⟨*proof*⟩

termination

⟨*proof*⟩

definition *full-exhaustive f d* = *full-exhaustive-natural' f d 0*

instance ⟨*proof*⟩

end

instantiation *natural* :: *exhaustive*
begin

function *exhaustive-natural'* ::
(*natural* ⇒ (*bool* × *term list*) *option*) ⇒ *natural* ⇒ *natural* ⇒ (*bool* × *term list*) *option*
where *exhaustive-natural' f d i* =
(*if d* < *i then None*
else (*f i orelse exhaustive-natural' f d* (*i* + *1*)))
⟨*proof*⟩

termination
⟨*proof*⟩

definition *exhaustive f d* = *exhaustive-natural' f d 0*

instance ⟨*proof*⟩

end

instantiation *integer* :: *exhaustive*
begin

function *exhaustive-integer'* ::
(*integer* ⇒ (*bool* × *term list*) *option*) ⇒ *integer* ⇒ *integer* ⇒ (*bool* × *term list*) *option*
where *exhaustive-integer' f d i* =
(*if d* < *i then None else* (*f i orelse exhaustive-integer' f d* (*i* + *1*)))
⟨*proof*⟩

termination
⟨*proof*⟩

definition *exhaustive f d* = *exhaustive-integer' f* (*integer-of-natural d*) (− (*integer-of-natural d*))

instance ⟨*proof*⟩

end

instantiation *integer* :: *full-exhaustive*

begin

function *full-exhaustive-integer'* ::
(*integer* × (*unit* ⇒ *term*) ⇒ (*bool* × *term list*) *option*) ⇒
integer ⇒ *integer* ⇒ (*bool* × *term list*) *option*
where *full-exhaustive-integer' f d i* =
(*if d* < *i then None*
else
(*case f* (*i*, λ-. *Code-Evaluation.term-of i*) *of*
Some t ⇒ *Some t*
| *None* ⇒ *full-exhaustive-integer' f d* (*i* + *1*)))
⟨*proof*⟩

termination
⟨*proof*⟩

definition *full-exhaustive f d* =
full-exhaustive-integer' f (*integer-of-natural d*) (− (*integer-of-natural d*))

instance ⟨*proof*⟩

end

instantiation *nat* :: *exhaustive*
begin

definition *exhaustive f d* = *exhaustive* (λ*x*. *f* (*nat-of-natural x*)) *d*

instance ⟨*proof*⟩

end

instantiation *nat* :: *full-exhaustive*
begin

definition *full-exhaustive f d* =
full-exhaustive (λ(*x*, *xt*). *f* (*nat-of-natural x*, λ-. *Code-Evaluation.term-of* (*nat-of-natural x*))) *d*

instance ⟨*proof*⟩

end

instantiation *int* :: *exhaustive*
begin

function *exhaustive-int'* ::
(*int* ⇒ (*bool* × *term list*) *option*) ⇒ *int* ⇒ *int* ⇒ (*bool* × *term list*) *option*
where *exhaustive-int' f d i* =

```
    (if d < i then None else (f i orelse exhaustive-int' f d (i + 1)))
⟨proof⟩

termination
  ⟨proof⟩

definition exhaustive f d =
  exhaustive-int' f (int-of-integer (integer-of-natural d))
    (− (int-of-integer (integer-of-natural d)))

instance ⟨proof⟩

end

instantiation int :: full-exhaustive
begin

function full-exhaustive-int' ::
    (int × (unit ⇒ term) ⇒ (bool × term list) option) ⇒
      int ⇒ int ⇒ (bool × term list) option
  where full-exhaustive-int' f d i =
    (if d < i then None
     else
      (case f (i, λ-. Code-Evaluation.term-of i) of
        Some t ⇒ Some t
      | None ⇒ full-exhaustive-int' f d (i + 1)))
⟨proof⟩

termination
  ⟨proof⟩

definition full-exhaustive f d =
  full-exhaustive-int' f (int-of-integer (integer-of-natural d))
    (− (int-of-integer (integer-of-natural d)))

instance ⟨proof⟩

end

instantiation prod :: (exhaustive, exhaustive) exhaustive
begin

definition exhaustive f d = exhaustive (λx. exhaustive (λy. f ((x, y))) d) d

instance ⟨proof⟩

end

context
```

includes *term-syntax*
begin

definition
[*code-unfold*]: *valtermify-pair x y* =
Code-Evaluation.valtermify (*Pair* :: *'a*::*typerep* ⇒ *'b*::*typerep* ⇒ *'a* × *'b*) {·} *x* {·} *y*

end

instantiation *prod* :: (*full-exhaustive*, *full-exhaustive*) *full-exhaustive*
begin

definition *full-exhaustive f d* =
full-exhaustive (λ*x*. *full-exhaustive* (λ*y*. *f* (*valtermify-pair x y*)) *d*) *d*

instance ⟨*proof*⟩

end

instantiation *set* :: (*exhaustive*) *exhaustive*
begin

fun *exhaustive-set*
where
exhaustive-set f i =
(*if i* = *0 then None*
else
f {} *orelse*
exhaustive-set
(λ*A*. *f A orelse exhaustive* (λ*x*. *if x* ∈ *A then None else f* (*insert x A*)) (*i* − *1*)) (*i* − *1*))

instance ⟨*proof*⟩

end

instantiation *set* :: (*full-exhaustive*) *full-exhaustive*
begin

fun *full-exhaustive-set*
where
full-exhaustive-set f i =
(*if i* = *0 then None*
else
f valterm-emptyset orelse
full-exhaustive-set
(λ*A*. *f A orelse Quickcheck-Exhaustive.full-exhaustive*
(λ*x*. *if fst x* ∈ *fst A then None else f* (*valtermify-insert x A*)) (*i* − *1*)) (*i*

− 1))

instance ⟨*proof*⟩

end

instantiation *fun* :: ({*equal*,*exhaustive*}, *exhaustive*) *exhaustive*
begin

fun *exhaustive-fun′* ::
(('*a* ⇒ '*b*) ⇒ (*bool* × *term list*) *option*) ⇒ *natural* ⇒ *natural* ⇒ (*bool* × *term list*) *option*
where
exhaustive-fun′ f i d =
(*exhaustive* (λ*b*. *f* (λ-. *b*)) *d*) *orelse*
(*if i* > *1 then*
exhaustive-fun′
(λ*g*. *exhaustive* (λ*a*. *exhaustive* (λ*b*. *f* (*g*(*a* := *b*))) *d*) *d*) (*i* − *1*) *d else*
None)

definition *exhaustive-fun* ::
(('*a* ⇒ '*b*) ⇒ (*bool* × *term list*) *option*) ⇒ *natural* ⇒ (*bool* × *term list*) *option*
where *exhaustive-fun f d* = *exhaustive-fun′ f d d*

instance ⟨*proof*⟩

end

definition [*code-unfold*]:
valtermify-absdummy =
(λ(*v*, *t*).
(λ-::'*a*. *v*,
λ*u*::*unit*. *Code-Evaluation.Abs* (*STR* ''*x*'') (*Typerep.typerep TYPE*('*a*::*typerep*))
(*t* ())))

context
includes *term-syntax*
begin

definition
[*code-unfold*]: *valtermify-fun-upd g a b* =
Code-Evaluation.valtermify
(*fun-upd* :: ('*a*::*typerep* ⇒ '*b*::*typerep*) ⇒ '*a* ⇒ '*b* ⇒ '*a* ⇒ '*b*) {·} *g* {·} *a* {·} *b*

end

instantiation *fun* :: ({*equal*,*full-exhaustive*}, *full-exhaustive*) *full-exhaustive*
begin

fun *full-exhaustive-fun'* ::
(('*a* ⇒ '*b*) × (*unit* ⇒ *term*) ⇒ (*bool* × *term list*) *option*) ⇒
natural ⇒ *natural* ⇒ (*bool* × *term list*) *option*
where
full-exhaustive-fun' f i d =
full-exhaustive (λ*v*. *f* (*valtermify-absdummy v*)) *d orelse*
(*if i* > *1 then*
full-exhaustive-fun'
(λ*g*. *full-exhaustive*
(λ*a*. *full-exhaustive* (λ*b*. *f* (*valtermify-fun-upd g a b*)) *d*) *d*) (*i* − *1*) *d*
else None)

definition *full-exhaustive-fun* ::
(('*a* ⇒ '*b*) × (*unit* ⇒ *term*) ⇒ (*bool* × *term list*) *option*) ⇒
natural ⇒ (*bool* × *term list*) *option*
where *full-exhaustive-fun f d* = *full-exhaustive-fun' f d d*

instance ⟨*proof*⟩

end

83.2.1 A smarter enumeration scheme for functions over finite datatypes

class *check-all* = *enum* + *term-of* +
fixes *check-all* :: ('*a* × (*unit* ⇒ *term*) ⇒ (*bool* × *term list*) *option*) ⇒ (*bool* ∗ *term list*) *option*
fixes *enum-term-of* :: '*a itself* ⇒ *unit* ⇒ *term list*

fun *check-all-n-lists* :: ('*a*::*check-all list* × (*unit* ⇒ *term list*) ⇒
(*bool* × *term list*) *option*) ⇒ *natural* ⇒ (*bool* ∗ *term list*) *option*
where
check-all-n-lists f n =
(*if n* = *0 then f* ([], (λ-. [])) *else check-all* (λ(*x*, *xt*).
check-all-n-lists (λ(*xs*, *xst*). *f* ((*x* # *xs*), (λ-. (*xt* () # *xst* ()))))) (*n* − *1*)))

context
includes *term-syntax*
begin

definition
[*code-unfold*]: *termify-fun-upd g a b* =
(*Code-Evaluation.termify*
(*fun-upd* :: ('*a*::*typerep* ⇒ '*b*::*typerep*) ⇒ '*a* ⇒ '*b* ⇒ '*a* ⇒ '*b*) <·> *g* <·> *a* <·> *b*)

end

```
definition mk-map-term ::
  (unit ⇒ typerep) ⇒ (unit ⇒ typerep) ⇒
    (unit ⇒ term list) ⇒ (unit ⇒ term list) ⇒ unit ⇒ term
  where mk-map-term T1 T2 domm rng =
    (λ-.
      let
        T1 = T1 ();
        T2 = T2 ();
        update-term =
          (λg (a, b).
            Code-Evaluation.App (Code-Evaluation.App (Code-Evaluation.App
             (Code-Evaluation.Const (STR ''Fun.fun-upd'')
                (Typerep.Typerep (STR ''fun'') [Typerep.Typerep (STR ''fun'') [T1,
T2],
                  Typerep.Typerep (STR ''fun'') [T1,
                   Typerep.Typerep (STR ''fun'') [T2, Typerep.Typerep (STR ''fun'')
[T1, T2]]]]))
                  g) a) b)
      in
        List.foldl update-term
          (Code-Evaluation.Abs (STR ''x'') T1
            (Code-Evaluation.Const (STR ''HOL.undefined'') T2)) (zip (domm ())
(rng ())))

instantiation fun :: ({equal,check-all}, check-all) check-all
begin

definition
  check-all f =
    (let
      mk-term =
        mk-map-term
          (λ-. Typerep.typerep (TYPE('a)))
          (λ-. Typerep.typerep (TYPE('b)))
          (enum-term-of (TYPE('a)));
      enum = (Enum.enum :: 'a list)
    in
      check-all-n-lists
        (λ(ys, yst). f (the ∘ map-of (zip enum ys), mk-term yst))
        (natural-of-nat (length enum)))

definition enum-term-of-fun :: ('a ⇒ 'b) itself ⇒ unit ⇒ term list
  where enum-term-of-fun =
    (λ- -.
      let
        enum-term-of-a = enum-term-of (TYPE('a));
        mk-term =
          mk-map-term
            (λ-. Typerep.typerep (TYPE('a)))
```

```
      (λ-. Typerep.typerep (TYPE('b)))
      enum-term-of-a
  in
    map (λys. mk-term (λ-. ys) ())
     (List.n-lists (length (enum-term-of-a ())) (enum-term-of (TYPE('b)) ()))

instance ⟨proof⟩

end

context
  includes term-syntax
begin

fun check-all-subsets ::
  (('a::typerep) set × (unit ⇒ term) ⇒ (bool × term list) option) ⇒
    ('a × (unit ⇒ term)) list ⇒ (bool × term list) option
where
  check-all-subsets f [] = f valterm-emptyset
| check-all-subsets f (x # xs) =
   check-all-subsets (λs. case f s of Some ts ⇒ Some ts | None ⇒ f (valtermify-insert
x s)) xs

definition
  [code-unfold]: term-emptyset = Code-Evaluation.termify ({} :: ('a::typerep) set)

definition
  [code-unfold]: termify-insert x s =
    Code-Evaluation.termify (insert :: ('a::typerep) ⇒ 'a set ⇒ 'a set)  <·> x <·>
s

definition setify :: ('a::typerep) itself ⇒ term list ⇒ term
where
  setify T ts = foldr (termify-insert T) ts (term-emptyset T)

end

instantiation set :: (check-all) check-all
begin

definition
  check-all-set f =
     check-all-subsets f
      (zip (Enum.enum :: 'a list)
       (map (λa. λu :: unit. a) (Quickcheck-Exhaustive.enum-term-of (TYPE ('a))
())))

definition enum-term-of-set :: 'a set itself ⇒ unit ⇒ term list
  where enum-term-of-set - - =
```

map (*setify* (*TYPE*('*a*))) (*subseqs* (*Quickcheck-Exhaustive.enum-term-of* (*TYPE*('*a*)) ()))

instance ⟨*proof*⟩

end

instantiation *unit* :: *check-all*
begin

definition *check-all f* = *f* (*Code-Evaluation.valtermify* ())

definition *enum-term-of-unit* :: *unit itself* ⇒ *unit* ⇒ *term list*
where *enum-term-of-unit* = (λ- -. [*Code-Evaluation.term-of* ()])

instance ⟨*proof*⟩

end

instantiation *bool* :: *check-all*
begin

definition
check-all f =
(*case f* (*Code-Evaluation.valtermify False*) *of*
Some x' ⇒ *Some x'*
| *None* ⇒ *f* (*Code-Evaluation.valtermify True*))

definition *enum-term-of-bool* :: *bool itself* ⇒ *unit* ⇒ *term list*
where *enum-term-of-bool* = (λ- -. *map Code-Evaluation.term-of* (*Enum.enum* :: *bool list*))

instance ⟨*proof*⟩

end

context
includes *term-syntax*
begin

definition [*code-unfold*]:
termify-pair x y =
Code-Evaluation.termify (*Pair* :: '*a*::*typerep* ⇒ '*b* :: *typerep* ⇒ '*a* ∗ '*b*) <·> *x* <·> *y*

end

instantiation *prod* :: (*check-all*, *check-all*) *check-all*

begin

definition *check-all f* = *check-all* (λ*x*. *check-all* (λ*y*. *f* (*valtermify-pair x y*)))

definition *enum-term-of-prod* :: (*'a* ∗ *'b*) *itself* ⇒ *unit* ⇒ *term list*
where *enum-term-of-prod* =
(λ- -.
map (λ(*x*, *y*). *termify-pair TYPE*(*'a*) *TYPE*(*'b*) *x y*)
(*List.product* (*enum-term-of* (*TYPE*(*'a*)) ()) (*enum-term-of* (*TYPE*(*'b*)) ())))

instance ⟨*proof*⟩

end

context
includes *term-syntax*
begin

definition
[*code-unfold*]: *valtermify-Inl x* =
Code-Evaluation.valtermify (*Inl* :: *'a*::*typerep* ⇒ *'a* + *'b* :: *typerep*) {·} *x*

definition
[*code-unfold*]: *valtermify-Inr x* =
Code-Evaluation.valtermify (*Inr* :: *'b*::*typerep* ⇒ *'a*::*typerep* + *'b*) {·} *x*

end

instantiation *sum* :: (*check-all*, *check-all*) *check-all*
begin

definition
check-all f = *check-all* (λ*a*. *f* (*valtermify-Inl a*)) *orelse check-all* (λ*b*. *f* (*valtermify-Inr b*))

definition *enum-term-of-sum* :: (*'a* + *'b*) *itself* ⇒ *unit* ⇒ *term list*
where *enum-term-of-sum* =
(λ- -.
let
T1 = *Typerep.typerep* (*TYPE*(*'a*));
T2 = *Typerep.typerep* (*TYPE*(*'b*))
in
map
(*Code-Evaluation.App* (*Code-Evaluation.Const* (*STR ''Sum-Type.Inl''*)
(*Typerep.Typerep* (*STR ''fun''*) [*T1*, *Typerep.Typerep* (*STR ''Sum-Type.sum''*) [*T1*, *T2*]])))
(*enum-term-of* (*TYPE*(*'a*)) ()) @
map

```
        (Code-Evaluation.App (Code-Evaluation.Const (STR ''Sum-Type.Inr'')
      (Typerep.Typerep (STR ''fun'') [T2, Typerep.Typerep (STR ''Sum-Type.sum'')
[T1, T2]])))
        (enum-term-of (TYPE('b)) ()))

instance ⟨proof⟩

end

instantiation char :: check-all
begin

primrec check-all-char' ::
  (char × (unit ⇒ term) ⇒ (bool × term list) option) ⇒ char list ⇒ (bool × term
list) option
  where check-all-char' f [] = None
  | check-all-char' f (c # cs) = f (c, λ-. Code-Evaluation.term-of c)
      orelse check-all-char' f cs

definition check-all-char ::
  (char × (unit ⇒ term) ⇒ (bool × term list) option) ⇒ (bool × term list) option
  where check-all f = check-all-char' f Enum.enum

definition enum-term-of-char :: char itself ⇒ unit ⇒ term list
where
  enum-term-of-char = (λ- -. map Code-Evaluation.term-of (Enum.enum :: char
list))

instance ⟨proof⟩

end

instantiation option :: (check-all) check-all
begin

definition
  check-all f =
    f (Code-Evaluation.valtermify (None :: 'a option)) orelse
    check-all
      (λ(x, t).
        f
          (Some x,
            λ-. Code-Evaluation.App
              (Code-Evaluation.Const (STR ''Option.option.Some'')
                (Typerep.Typerep (STR ''fun'')
                [Typerep.typerep TYPE('a),
              Typerep.Typerep (STR ''Option.option'') [Typerep.typerep TYPE('a)]]))
(t ())))
```

definition *enum-term-of-option* :: *'a option itself* ⇒ *unit* ⇒ *term list*
where *enum-term-of-option* =
(λ - -.
Code-Evaluation.term-of (*None* :: *'a option*) #
(*map*
(*Code-Evaluation.App*
(*Code-Evaluation.Const* (*STR ''Option.option.Some''*)
(*Typerep.Typerep* (*STR ''fun''*)
[*Typerep.typerep TYPE*('*a*),
Typerep.Typerep (*STR ''Option.option''*) [*Typerep.typerep TYPE*('*a*)]])))
(*enum-term-of* (*TYPE*('*a*)) ())))

instance ⟨*proof*⟩

end

instantiation *Enum.finite-1* :: *check-all*
begin

definition *check-all f* = *f* (*Code-Evaluation.valtermify Enum.finite-1.a*$_1$)

definition *enum-term-of-finite-1* :: *Enum.finite-1 itself* ⇒ *unit* ⇒ *term list*
where *enum-term-of-finite-1* = (λ- -. [*Code-Evaluation.term-of Enum.finite-1.a*$_1$])

instance ⟨*proof*⟩

end

instantiation *Enum.finite-2* :: *check-all*
begin

definition
check-all f =
(*f* (*Code-Evaluation.valtermify Enum.finite-2.a*$_1$) *orelse*
f (*Code-Evaluation.valtermify Enum.finite-2.a*$_2$))

definition *enum-term-of-finite-2* :: *Enum.finite-2 itself* ⇒ *unit* ⇒ *term list*
where *enum-term-of-finite-2* =
(λ- -. *map Code-Evaluation.term-of* (*Enum.enum* :: *Enum.finite-2 list*))

instance ⟨*proof*⟩

end

instantiation *Enum.finite-3* :: *check-all*
begin

definition

check-all f =
(*f* (*Code-Evaluation.valtermify Enum.finite-3.a*$_1$) *orelse*
f (*Code-Evaluation.valtermify Enum.finite-3.a*$_2$) *orelse*
f (*Code-Evaluation.valtermify Enum.finite-3.a*$_3$))

definition *enum-term-of-finite-3* :: *Enum.finite-3 itself* ⇒ *unit* ⇒ *term list*
where *enum-term-of-finite-3* =
(λ- -. *map Code-Evaluation.term-of* (*Enum.enum* :: *Enum.finite-3 list*))

instance ⟨*proof*⟩

end

instantiation *Enum.finite-4* :: *check-all*
begin

definition
check-all f =
f (*Code-Evaluation.valtermify Enum.finite-4.a*$_1$) *orelse*
f (*Code-Evaluation.valtermify Enum.finite-4.a*$_2$) *orelse*
f (*Code-Evaluation.valtermify Enum.finite-4.a*$_3$) *orelse*
f (*Code-Evaluation.valtermify Enum.finite-4.a*$_4$)

definition *enum-term-of-finite-4* :: *Enum.finite-4 itself* ⇒ *unit* ⇒ *term list*
where *enum-term-of-finite-4* =
(λ- -. *map Code-Evaluation.term-of* (*Enum.enum* :: *Enum.finite-4 list*))

instance ⟨*proof*⟩

end

83.3 Bounded universal quantifiers

class *bounded-forall* =
fixes *bounded-forall* :: ('*a* ⇒ *bool*) ⇒ *natural* ⇒ *bool*

83.4 Fast exhaustive combinators

class *fast-exhaustive* = *term-of* +
fixes *fast-exhaustive* :: ('*a* ⇒ *unit*) ⇒ *natural* ⇒ *unit*

axiomatization *throw-Counterexample* :: *term list* ⇒ *unit*
axiomatization *catch-Counterexample* :: *unit* ⇒ *term list option*

code-printing
constant *throw-Counterexample* ⇀
(*Quickcheck*) *raise* (*Exhaustive'-Generators.Counterexample* -)
| **constant** *catch-Counterexample* ⇀
(*Quickcheck*) (((-); *NONE*) *handle Exhaustive'-Generators.Counterexample ts*
⇒ *SOME ts*)

83.5 Continuation passing style functions as plus monad

type-synonym *'a cps* = (*'a* ⇒ *term list option*) ⇒ *term list option*

definition *cps-empty* :: *'a cps*
where *cps-empty* = (λ*cont*. *None*)

definition *cps-single* :: *'a* ⇒ *'a cps*
where *cps-single v* = (λ*cont*. *cont v*)

definition *cps-bind* :: *'a cps* ⇒ (*'a* ⇒ *'b cps*) ⇒ *'b cps*
where *cps-bind m f* = (λ*cont*. *m* (λ*a*. (*f a*) *cont*))

definition *cps-plus* :: *'a cps* ⇒ *'a cps* ⇒ *'a cps*
where *cps-plus a b* = (λ*c*. *case a c of None* ⇒ *b c* | *Some x* ⇒ *Some x*)

definition *cps-if* :: *bool* ⇒ *unit cps*
where *cps-if b* = (*if b then cps-single* () *else cps-empty*)

definition *cps-not* :: *unit cps* ⇒ *unit cps*
where *cps-not n* = (λ*c*. *case n* (λ*u*. *Some* []) *of None* ⇒ *c* () | *Some* - ⇒ *None*)

type-synonym *'a pos-bound-cps* =
(*'a* ⇒ (*bool* ∗ *term list*) *option*) ⇒ *natural* ⇒ (*bool* ∗ *term list*) *option*

definition *pos-bound-cps-empty* :: *'a pos-bound-cps*
where *pos-bound-cps-empty* = (λ*cont i*. *None*)

definition *pos-bound-cps-single* :: *'a* ⇒ *'a pos-bound-cps*
where *pos-bound-cps-single v* = (λ*cont i*. *cont v*)

definition *pos-bound-cps-bind* :: *'a pos-bound-cps* ⇒ (*'a* ⇒ *'b pos-bound-cps*) ⇒
'b pos-bound-cps
where *pos-bound-cps-bind m f* = (λ*cont i*. *if i* = *0 then None else* (*m* (λ*a*. (*f a*)
cont i) (*i* − *1*)))

definition *pos-bound-cps-plus* :: *'a pos-bound-cps* ⇒ *'a pos-bound-cps* ⇒ *'a pos-bound-cps*
where *pos-bound-cps-plus a b* = (λ*c i*. *case a c i of None* ⇒ *b c i* | *Some x* ⇒
Some x)

definition *pos-bound-cps-if* :: *bool* ⇒ *unit pos-bound-cps*
where *pos-bound-cps-if b* = (*if b then pos-bound-cps-single* () *else pos-bound-cps-empty*)

datatype (*plugins only*: *code extraction*) (*dead 'a*) *unknown* =
Unknown | *Known 'a*

datatype (*plugins only*: *code extraction*) (*dead 'a*) *three-valued* =
Unknown-value | *Value 'a* | *No-value*

type-synonym *'a neg-bound-cps* =

('a unknown ⇒ term list three-valued) ⇒ natural ⇒ term list three-valued

definition *neg-bound-cps-empty* :: *'a neg-bound-cps*
where *neg-bound-cps-empty* = (*λcont i. No-value*)

definition *neg-bound-cps-single* :: *'a ⇒ 'a neg-bound-cps*
where *neg-bound-cps-single v* = (*λcont i. cont* (*Known v*))

definition *neg-bound-cps-bind* :: *'a neg-bound-cps ⇒* (*'a ⇒ 'b neg-bound-cps*) *⇒ 'b neg-bound-cps*
where *neg-bound-cps-bind m f* =
(*λcont i.*
if i = *0 then cont Unknown*
else m (*λa. case a of Unknown ⇒ cont Unknown* | *Known a' ⇒ f a' cont i*) (*i* − *1*))

definition *neg-bound-cps-plus* :: *'a neg-bound-cps ⇒ 'a neg-bound-cps ⇒ 'a neg-bound-cps*
where *neg-bound-cps-plus a b* =
(*λc i.*
case a c i of
No-value ⇒ b c i
| *Value x ⇒ Value x*
| *Unknown-value ⇒*
(*case b c i of*
No-value ⇒ Unknown-value
| *Value x ⇒ Value x*
| *Unknown-value ⇒ Unknown-value*))

definition *neg-bound-cps-if* :: *bool ⇒ unit neg-bound-cps*
where *neg-bound-cps-if b* = (*if b then neg-bound-cps-single* () *else neg-bound-cps-empty*)

definition *neg-bound-cps-not* :: *unit pos-bound-cps ⇒ unit neg-bound-cps*
where *neg-bound-cps-not n* =
(*λc i. case n* (*λu. Some* (*True*, [])) *i of None ⇒ c* (*Known* ()) | *Some* - ⇒ *No-value*)

definition *pos-bound-cps-not* :: *unit neg-bound-cps ⇒ unit pos-bound-cps*
where *pos-bound-cps-not n* =
(*λc i. case n* (*λu. Value* []) *i of No-value ⇒ c* () | *Value* - ⇒ *None* | *Unknown-value ⇒ None*)

83.6 Defining generators for any first-order data type

axiomatization *unknown* :: *'a*

notation (**output**) *unknown* (*?*)

⟨*ML*⟩

declare [[*quickcheck-batch-tester* = *exhaustive*]]

83.7 Defining generators for abstract types

⟨*ML*⟩

hide-fact (**open**) *orelse-def*
no-notation *orelse* (**infixr** *orelse 55*)

hide-const *valtermify-absdummy valtermify-fun-upd*
valterm-emptyset valtermify-insert
valtermify-pair valtermify-Inl valtermify-Inr
termify-fun-upd term-emptyset termify-insert termify-pair setify

hide-const (**open**)
exhaustive full-exhaustive
exhaustive-int′ full-exhaustive-int′
exhaustive-integer′ full-exhaustive-integer′
exhaustive-natural′ full-exhaustive-natural′
throw-Counterexample catch-Counterexample
check-all enum-term-of
orelse unknown mk-map-term check-all-n-lists check-all-subsets

hide-type (**open**) *cps pos-bound-cps neg-bound-cps unknown three-valued*

hide-const (**open**) *cps-empty cps-single cps-bind cps-plus cps-if cps-not*
pos-bound-cps-empty pos-bound-cps-single pos-bound-cps-bind
pos-bound-cps-plus pos-bound-cps-if pos-bound-cps-not
neg-bound-cps-empty neg-bound-cps-single neg-bound-cps-bind
neg-bound-cps-plus neg-bound-cps-if neg-bound-cps-not
Unknown Known Unknown-value Value No-value

end

84 A compiler for predicates defined by introduction rules

theory *Predicate-Compile*
imports *Random-Sequence Quickcheck-Exhaustive*
keywords
code-pred :: *thy-goal* **and**
values :: *diag*
begin

⟨*ML*⟩

84.1 Set membership as a generator predicate

Introduce a new constant for membership to allow fine-grained control in code equations.

definition *contains* :: *'a set => 'a => bool*
where *contains A x* $\longleftrightarrow$ $x \in A$

definition *contains-pred* :: *'a set => 'a => unit Predicate.pred*
where *contains-pred A x = (if x ∈ A then Predicate.single () else bot)*

lemma *pred-of-setE*:
assumes *Predicate.eval (pred-of-set A) x*
obtains *contains A x*
⟨*proof*⟩

lemma *pred-of-setI*: *contains A x ==> Predicate.eval (pred-of-set A) x*
⟨*proof*⟩

lemma *pred-of-set-eq*: *pred-of-set* ≡ *λA. Predicate.Pred (contains A)*
⟨*proof*⟩

lemma *containsI*: *x ∈ A ==> contains A x*
⟨*proof*⟩

lemma *containsE*: **assumes** *contains A x*
obtains *A' x'* **where** *A = A' x = x' x ∈ A*
⟨*proof*⟩

lemma *contains-predI*: *contains A x ==> Predicate.eval (contains-pred A x) ()*
⟨*proof*⟩

lemma *contains-predE*:
assumes *Predicate.eval (contains-pred A x) y*
obtains *contains A x*
⟨*proof*⟩

lemma *contains-pred-eq*: *contains-pred* ≡ *λA x. Predicate.Pred (λy. contains A x)*
⟨*proof*⟩

lemma *contains-pred-notI*:
¬ *contains A x ==> Predicate.eval (Predicate.not-pred (contains-pred A x)) ()*
⟨*proof*⟩

⟨*ML*⟩

hide-const (**open**) *contains contains-pred*
hide-fact (**open**) *pred-of-setE pred-of-setI pred-of-set-eq*
containsI containsE contains-predI contains-predE contains-pred-eq contains-pred-notI

end

85 Counterexample generator performing narrowing-based testing

theory *Quickcheck-Narrowing*
imports *Quickcheck-Random*
keywords *find-unused-assms* :: *diag*
begin

85.1 Counterexample generator

85.1.1 Code generation setup

⟨*ML*⟩

code-printing
  code-module *Typerep* ⇀ (*Haskell-Quickcheck*) ‹
module Typerep(*Typerep*(..)) *where*

data Typerep = *Typerep String* [*Typerep*]
› **for type-constructor** *typerep* **constant** *Typerep.Typerep*
| **type-constructor** *typerep* ⇀ (*Haskell-Quickcheck*) *Typerep.Typerep*
| **constant** *Typerep.Typerep* ⇀ (*Haskell-Quickcheck*) *Typerep.Typerep*

code-reserved *Haskell-Quickcheck Typerep*

code-printing
  type-constructor *integer* ⇀ (*Haskell-Quickcheck*) *Prelude.Int*
| **constant** *0*::*integer* ⇀
    (*Haskell-Quickcheck*) !(*0*// ::// *Prelude.Int*)

⟨*ML*⟩

85.1.2 Narrowing's deep representation of types and terms

datatype (*plugins only*: *code extraction*) *narrowing-type* =
  Narrowing-sum-of-products narrowing-type list list

datatype (*plugins only*: *code extraction*) *narrowing-term* =
  Narrowing-variable integer list narrowing-type
| *Narrowing-constructor integer narrowing-term list*

datatype (*plugins only*: *code extraction*) (*dead* 'a) *narrowing-cons* =
  Narrowing-cons narrowing-type (*narrowing-term list* ⇒ 'a) *list*

primrec *map-cons* :: ('a => 'b) => 'a *narrowing-cons* => 'b *narrowing-cons*
where
  map-cons f (*Narrowing-cons ty cs*) = *Narrowing-cons ty* (*map* (λ*c*. *f* ∘ *c*) *cs*)

85.1.3 From narrowing's deep representation of terms to *HOL.Code-Evaluation*'s terms

class *partial-term-of* = *typerep* +
  fixes *partial-term-of* :: *'a itself* => *narrowing-term* => *Code-Evaluation.term*

lemma *partial-term-of-anything*: *partial-term-of x nt* ≡ *t*
  ⟨*proof*⟩

85.1.4 Auxilary functions for Narrowing

consts *nth* :: *'a list* => *integer* => *'a*

code-printing constant *nth* ⇀ (*Haskell-Quickcheck*) **infixl** *9* !!

consts *error* :: *char list* => *'a*

code-printing constant *error* ⇀ (*Haskell-Quickcheck*) *error*

consts *toEnum* :: *integer* => *char*

code-printing constant *toEnum* ⇀ (*Haskell-Quickcheck*) *Prelude.toEnum*

consts *marker* :: *char*

code-printing constant *marker* ⇀ (*Haskell-Quickcheck*) *''\0'*

85.1.5 Narrowing's basic operations

type-synonym *'a narrowing* = *integer* => *'a narrowing-cons*

definition *cons* :: *'a* => *'a narrowing*
where
  cons a d = (*Narrowing-cons* (*Narrowing-sum-of-products* [[]]) [(λ-. *a*)])

fun *conv* :: (*narrowing-term list* => *'a*) *list* => *narrowing-term* => *'a*
where
  conv cs (*Narrowing-variable p* -) = *error* (*marker* # *map toEnum p*)
| *conv cs* (*Narrowing-constructor i xs*) = (*nth cs i*) *xs*

fun *non-empty* :: *narrowing-type* => *bool*
where
  non-empty (*Narrowing-sum-of-products ps*) = (¬ (*List.null ps*))

definition *apply* :: (*'a* => *'b*) *narrowing* => *'a narrowing* => *'b narrowing*
where
  apply f a d = (*if d* > *0 then*
    (*case f d of Narrowing-cons* (*Narrowing-sum-of-products ps*) *cfs* ⇒
      case a (*d* − *1*) *of Narrowing-cons ta cas* ⇒
      let

```
        shallow = non-empty ta;
        cs = [(λ(x # xs) ⇒ cf xs (conv cas x)). shallow, cf ← cfs]
       in Narrowing-cons (Narrowing-sum-of-products [ta # p. shallow, p ← ps])
cs)
    else Narrowing-cons (Narrowing-sum-of-products []) []
```

definition *sum* :: *'a narrowing => 'a narrowing => 'a narrowing*
where
sum a b d =
(case a d of Narrowing-cons (Narrowing-sum-of-products ssa) ca ⇒
case b d of Narrowing-cons (Narrowing-sum-of-products ssb) cb ⇒
Narrowing-cons (Narrowing-sum-of-products (ssa @ ssb)) (ca @ cb))

lemma [*fundef-cong*]:
assumes *a d = a' d b d = b' d d = d'*
shows *sum a b d = sum a' b' d'*
⟨*proof*⟩

lemma [*fundef-cong*]:
assumes *f d = f' d* (⋀*d'. 0 ≤ d' ∧ d' < d ⟹ a d' = a' d'*)
assumes *d = d'*
shows *apply f a d = apply f' a' d'*
⟨*proof*⟩

85.1.6 Narrowing generator type class

class *narrowing =*
fixes *narrowing :: integer => 'a narrowing-cons*

datatype (*plugins only*: *code extraction*) *property =*
Universal narrowing-type (narrowing-term => property) narrowing-term => Code-Evaluation.term
| Existential narrowing-type (narrowing-term => property) narrowing-term => Code-Evaluation.term
| Property bool

definition *exists* :: (*'a* :: {*narrowing*, *partial-term-of*} => *property*) => *property*
where
exists f = (case narrowing (100 :: integer) of Narrowing-cons ty cs ⇒ Existential ty (λ t. f (conv cs t)) (partial-term-of (TYPE('a))))

definition *all* :: (*'a* :: {*narrowing*, *partial-term-of*} => *property*) => *property*
where
all f = (case narrowing (100 :: integer) of Narrowing-cons ty cs ⇒ Universal ty (λt. f (conv cs t)) (partial-term-of (TYPE('a))))

85.1.7 class *is-testable*

The class *is-testable* ensures that all necessary type instances are generated.

class *is-testable*

instance *bool* :: *is-testable* ⟨*proof*⟩

instance *fun* :: ({*term-of*, *narrowing*, *partial-term-of*}, *is-testable*) *is-testable* ⟨*proof*⟩

definition *ensure-testable* :: *'a* :: *is-testable* => *'a* :: *is-testable*
where
ensure-testable f = *f*

85.1.8 Defining a simple datatype to represent functions in an incomplete and redundant way

datatype (*plugins only*: *code quickcheck-narrowing extraction*) (*dead 'a*, *dead 'b*) *ffun* =
Constant 'b
| *Update 'a 'b* (*'a*, *'b*) *ffun*

primrec *eval-ffun* :: (*'a*, *'b*) *ffun* => *'a* => *'b*
where
eval-ffun (*Constant c*) *x* = *c*
| *eval-ffun* (*Update x' y f*) *x* = (*if x* = *x' then y else eval-ffun f x*)

hide-type (**open**) *ffun*
hide-const (**open**) *Constant Update eval-ffun*

datatype (*plugins only*: *code quickcheck-narrowing extraction*) (*dead 'b*) *cfun* = *Constant 'b*

primrec *eval-cfun* :: *'b cfun* => *'a* => *'b*
where
eval-cfun (*Constant c*) *y* = *c*

hide-type (**open**) *cfun*
hide-const (**open**) *Constant eval-cfun Abs-cfun Rep-cfun*

85.1.9 Setting up the counterexample generator

external-file ‹~~/*src*/*HOL*/*Tools*/*Quickcheck*/*Narrowing-Engine.hs*›
external-file ‹~~/*src*/*HOL*/*Tools*/*Quickcheck*/*PNF-Narrowing-Engine.hs*›
⟨*ML*⟩

definition *narrowing-dummy-partial-term-of* :: (*'a* :: *partial-term-of*) *itself* => *narrowing-term* => *term*
where
narrowing-dummy-partial-term-of = *partial-term-of*

definition *narrowing-dummy-narrowing* :: *integer* => ('*a* :: *narrowing*) *narrowing-cons*
where
narrowing-dummy-narrowing = *narrowing*

lemma [*code*]:
ensure-testable f =
(*let*
x = *narrowing-dummy-narrowing* :: *integer* => *bool narrowing-cons*;
y = *narrowing-dummy-partial-term-of* :: *bool itself* => *narrowing-term* => *term*;
z = (*conv* :: - => - => *unit*) *in f*)
⟨*proof*⟩

85.2 Narrowing for sets

instantiation *set* :: (*narrowing*) *narrowing*
begin

definition *narrowing-set* = *Quickcheck-Narrowing.apply* (*Quickcheck-Narrowing.cons set*) *narrowing*

instance ⟨*proof*⟩

end

85.3 Narrowing for integers

definition *drawn-from* :: '*a list* ⇒ '*a narrowing-cons*
where
drawn-from xs =
Narrowing-cons (*Narrowing-sum-of-products* (*map* (λ-. []) *xs*)) (*map* (λ*x* -. *x*) *xs*)

function *around-zero* :: *int* ⇒ *int list*
where
around-zero i = (*if i* < *0 then* [] *else* (*if i* = *0 then* [*0*] *else around-zero* (*i* − *1*) @ [*i*, −*i*]))
⟨*proof*⟩
termination ⟨*proof*⟩

declare *around-zero.simps* [*simp del*]

lemma *length-around-zero*:
assumes *i* >= *0*
shows *length* (*around-zero i*) = *2* ∗ *nat i* + *1*
⟨*proof*⟩

instantiation *int* :: *narrowing*

begin

definition
narrowing-int d = (*let* (*u* :: - ⇒ - ⇒ *unit*) = *conv*; *i* = *int-of-integer d*
in drawn-from (*around-zero i*))

instance ⟨*proof*⟩

end

declare [[*code drop*: *partial-term-of* :: *int itself* ⇒ -]]

lemma [*code*]:
partial-term-of (*ty* :: *int itself*) (*Narrowing-variable p t*) ≡
Code-Evaluation.Free (*STR ''-''*) (*Typerep.Typerep* (*STR ''Int.int''*) [])
partial-term-of (*ty* :: *int itself*) (*Narrowing-constructor i* []) ≡
(*if i mod 2* = *0*
then Code-Evaluation.term-of (− (*int-of-integer i*) *div 2*)
else Code-Evaluation.term-of ((*int-of-integer i* + *1*) *div 2*))
⟨*proof*⟩

instantiation *integer* :: *narrowing*
begin

definition
narrowing-integer d = (*let* (*u* :: - ⇒ - ⇒ *unit*) = *conv*; *i* = *int-of-integer d*
in drawn-from (*map integer-of-int* (*around-zero i*)))

instance ⟨*proof*⟩

end

declare [[*code drop*: *partial-term-of* :: *integer itself* ⇒ -]]

lemma [*code*]:
partial-term-of (*ty* :: *integer itself*) (*Narrowing-variable p t*) ≡
Code-Evaluation.Free (*STR ''-''*) (*Typerep.Typerep* (*STR ''Code-Numeral.integer''*)
[])
partial-term-of (*ty* :: *integer itself*) (*Narrowing-constructor i* []) ≡
(*if i mod 2* = *0*
then Code-Evaluation.term-of (− *i div 2*)
else Code-Evaluation.term-of ((*i* + *1*) *div 2*))
⟨*proof*⟩

code-printing constant *Code-Evaluation.term-of* :: *integer* ⇒ *term* ⇀ (*Haskell-Quickcheck*)

(*let* { *t* = *Typerep.Typerep Code'-Numeral.integer* [];
mkFunT s t = *Typerep.Typerep fun* [*s*, *t*];
numT = *Typerep.Typerep Num.num* [];

```
mkBit 0 = Generated′-Code.Const Num.num.Bit0 (mkFunT numT numT);
mkBit 1 = Generated′-Code.Const Num.num.Bit1 (mkFunT numT numT);
mkNumeral 1 = Generated′-Code.Const Num.num.One numT;
mkNumeral i = let { q = i `Prelude.div` 2; r = i `Prelude.mod` 2 }
  in Generated′-Code.App (mkBit r) (mkNumeral q);
mkNumber 0 = Generated′-Code.Const Groups.zero′-class.zero t;
mkNumber 1 = Generated′-Code.Const Groups.one′-class.one t;
mkNumber i = if i > 0 then
    Generated′-Code.App
      (Generated′-Code.Const Num.numeral′-class.numeral
         (mkFunT numT t))
      (mkNumeral i)
  else
    Generated′-Code.App
      (Generated′-Code.Const Groups.uminus′-class.uminus (mkFunT t t))
      (mkNumber (− i)); } in mkNumber)
```

85.4 The *find-unused-assms* command

⟨*ML*⟩

85.5 Closing up

hide-type *narrowing-type narrowing-term narrowing-cons property*
hide-const *map-cons nth error toEnum marker empty Narrowing-cons conv non-empty ensure-testable all exists drawn-from around-zero*
hide-const (**open**) *Narrowing-variable Narrowing-constructor apply sum cons*
hide-fact *empty-def cons-def conv.simps non-empty.simps apply-def sum-def ensure-testable-def all-def exists-def*

end

theory *Mirabelle*
imports *Sledgehammer Predicate-Compile Presburger*
begin

⟨*ML*⟩

end

86 Program extraction for HOL

theory *Extraction*
imports *Option*
begin

86.1 Setup

⟨*ML*⟩

lemmas [*extraction-expand*] =
meta-spec atomize-eq atomize-all atomize-imp atomize-conj
allE rev-mp conjE Eq-TrueI Eq-FalseI eqTrueI eqTrueE eq-cong2
notE′ impE′ impE iffE imp-cong simp-thms eq-True eq-False
induct-forall-eq induct-implies-eq induct-equal-eq induct-conj-eq
induct-atomize induct-atomize′ induct-rulify induct-rulify′
induct-rulify-fallback induct-trueI
True-implies-equals implies-True-equals TrueE
False-implies-equals implies-False-swap

lemmas [*extraction-expand-def*] =
HOL.induct-forall-def HOL.induct-implies-def HOL.induct-equal-def HOL.induct-conj-def
HOL.induct-true-def HOL.induct-false-def

datatype (*plugins only*: *code extraction*) *sumbool* = *Left* | *Right*

86.2 Type of extracted program

extract-type
typeof (*Trueprop P*) ≡ *typeof P*

typeof P ≡ *Type* (*TYPE*(*Null*)) ⟹ *typeof Q* ≡ *Type* (*TYPE*('*Q*)) ⟹
typeof (*P* ⟶ *Q*) ≡ *Type* (*TYPE*('*Q*))

typeof Q ≡ *Type* (*TYPE*(*Null*)) ⟹ *typeof* (*P* ⟶ *Q*) ≡ *Type* (*TYPE*(*Null*))

typeof P ≡ *Type* (*TYPE*('*P*)) ⟹ *typeof Q* ≡ *Type* (*TYPE*('*Q*)) ⟹
typeof (*P* ⟶ *Q*) ≡ *Type* (*TYPE*('*P* ⇒ '*Q*))

(λ*x*. *typeof* (*P x*)) ≡ (λ*x*. *Type* (*TYPE*(*Null*))) ⟹
typeof (∀ *x*. *P x*) ≡ *Type* (*TYPE*(*Null*))

(λ*x*. *typeof* (*P x*)) ≡ (λ*x*. *Type* (*TYPE*('*P*))) ⟹
typeof (∀ *x*::'*a*. *P x*) ≡ *Type* (*TYPE*('*a* ⇒ '*P*))

(λ*x*. *typeof* (*P x*)) ≡ (λ*x*. *Type* (*TYPE*(*Null*))) ⟹
typeof (∃ *x*::'*a*. *P x*) ≡ *Type* (*TYPE*('*a*))

(λ*x*. *typeof* (*P x*)) ≡ (λ*x*. *Type* (*TYPE*('*P*))) ⟹
typeof (∃ *x*::'*a*. *P x*) ≡ *Type* (*TYPE*('*a* × '*P*))

typeof P ≡ *Type* (*TYPE*(*Null*)) ⟹ *typeof Q* ≡ *Type* (*TYPE*(*Null*)) ⟹
typeof (*P* ∨ *Q*) ≡ *Type* (*TYPE*(*sumbool*))

typeof P ≡ *Type* (*TYPE*(*Null*)) ⟹ *typeof Q* ≡ *Type* (*TYPE*('*Q*)) ⟹
typeof (*P* ∨ *Q*) ≡ *Type* (*TYPE*('*Q option*))

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE(Null)) \Longrightarrow$
$typeof\ (P \vee Q) \equiv Type\ (TYPE('P\ option))$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE('Q)) \Longrightarrow$
$typeof\ (P \vee Q) \equiv Type\ (TYPE('P + 'Q))$

$typeof\ P \equiv Type\ (TYPE(Null)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE('Q)) \Longrightarrow$
$typeof\ (P \wedge Q) \equiv Type\ (TYPE('Q))$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE(Null)) \Longrightarrow$
$typeof\ (P \wedge Q) \equiv Type\ (TYPE('P))$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE('Q)) \Longrightarrow$
$typeof\ (P \wedge Q) \equiv Type\ (TYPE('P \times 'Q))$

$typeof\ (P = Q) \equiv typeof\ ((P \longrightarrow Q) \wedge (Q \longrightarrow P))$

$typeof\ (x \in P) \equiv typeof\ P$

86.3 Realizability

realizability
$(realizes\ t\ (Trueprop\ P)) \equiv (Trueprop\ (realizes\ t\ P))$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \longrightarrow Q)) \equiv (realizes\ Null\ P \longrightarrow realizes\ t\ Q)$

$(typeof\ P) \equiv (Type\ (TYPE('P))) \Longrightarrow$
$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \longrightarrow Q)) \equiv (\forall x{::}'P.\ realizes\ x\ P \longrightarrow realizes\ Null\ Q)$

$(realizes\ t\ (P \longrightarrow Q)) \equiv (\forall x.\ realizes\ x\ P \longrightarrow realizes\ (t\ x)\ Q)$

$(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (\forall x.\ P\ x)) \equiv (\forall x.\ realizes\ Null\ (P\ x))$

$(realizes\ t\ (\forall x.\ P\ x)) \equiv (\forall x.\ realizes\ (t\ x)\ (P\ x))$

$(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (\exists x.\ P\ x)) \equiv (realizes\ Null\ (P\ t))$

$(realizes\ t\ (\exists x.\ P\ x)) \equiv (realizes\ (snd\ t)\ (P\ (fst\ t)))$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \vee Q)) \equiv$
$(case\ t\ of\ Left \Rightarrow realizes\ Null\ P \mid Right \Rightarrow realizes\ Null\ Q)$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \vee Q)) \equiv$
$(case\ t\ of\ None \Rightarrow realizes\ Null\ P \mid Some\ q \Rightarrow realizes\ q\ Q)$

$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \vee Q)) \equiv$
$(case\ t\ of\ None \Rightarrow realizes\ Null\ Q \mid Some\ p \Rightarrow realizes\ p\ P)$

$(realizes\ t\ (P \vee Q)) \equiv$
$(case\ t\ of\ Inl\ p \Rightarrow realizes\ p\ P \mid Inr\ q \Rightarrow realizes\ q\ Q)$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \wedge Q)) \equiv (realizes\ Null\ P \wedge realizes\ t\ Q)$

$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \wedge Q)) \equiv (realizes\ t\ P \wedge realizes\ Null\ Q)$

$(realizes\ t\ (P \wedge Q)) \equiv (realizes\ (fst\ t)\ P \wedge realizes\ (snd\ t)\ Q)$

$typeof\ P \equiv Type\ (TYPE(Null)) \Longrightarrow$
$realizes\ t\ (\neg\ P) \equiv \neg\ realizes\ Null\ P$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow$
$realizes\ t\ (\neg\ P) \equiv (\forall\, x{::}'P.\ \neg\ realizes\ x\ P)$

$typeof\ (P{::}bool) \equiv Type\ (TYPE(Null)) \Longrightarrow$
$typeof\ Q \equiv Type\ (TYPE(Null)) \Longrightarrow$
$realizes\ t\ (P = Q) \equiv realizes\ Null\ P = realizes\ Null\ Q$

$(realizes\ t\ (P = Q)) \equiv (realizes\ t\ ((P \longrightarrow Q) \wedge (Q \longrightarrow P)))$

86.4 Computational content of basic inference rules

theorem *disjE-realizer*:
assumes *r*: $case\ x\ of\ Inl\ p \Rightarrow P\ p \mid Inr\ q \Rightarrow Q\ q$
and *r1*: $\bigwedge p.\ P\ p \Longrightarrow R\ (f\ p)$ **and** *r2*: $\bigwedge q.\ Q\ q \Longrightarrow R\ (g\ q)$
shows $R\ (case\ x\ of\ Inl\ p \Rightarrow f\ p \mid Inr\ q \Rightarrow g\ q)$
⟨*proof*⟩

theorem *disjE-realizer2*:
assumes *r*: $case\ x\ of\ None \Rightarrow P \mid Some\ q \Rightarrow Q\ q$
and *r1*: $P \Longrightarrow R\ f$ **and** *r2*: $\bigwedge q.\ Q\ q \Longrightarrow R\ (g\ q)$
shows $R\ (case\ x\ of\ None \Rightarrow f \mid Some\ q \Rightarrow g\ q)$
⟨*proof*⟩

theorem *disjE-realizer3*:
assumes *r*: $case\ x\ of\ Left \Rightarrow P \mid Right \Rightarrow Q$
and *r1*: $P \Longrightarrow R\ f$ **and** *r2*: $Q \Longrightarrow R\ g$
shows $R\ (case\ x\ of\ Left \Rightarrow f \mid Right \Rightarrow g)$

⟨*proof*⟩

theorem *conjI-realizer*:
$P\ p \Longrightarrow Q\ q \Longrightarrow P\ (fst\ (p,\ q)) \wedge Q\ (snd\ (p,\ q))$
⟨*proof*⟩

theorem *exI-realizer*:
$P\ y\ x \Longrightarrow P\ (snd\ (x,\ y))\ (fst\ (x,\ y))$ ⟨*proof*⟩

theorem *exE-realizer*: $P\ (snd\ p)\ (fst\ p) \Longrightarrow$
$(\bigwedge x\ y.\ P\ y\ x \Longrightarrow Q\ (f\ x\ y)) \Longrightarrow Q\ (let\ (x,\ y) = p\ in\ f\ x\ y)$
⟨*proof*⟩

theorem *exE-realizer'*: $P\ (snd\ p)\ (fst\ p) \Longrightarrow$
$(\bigwedge x\ y.\ P\ y\ x \Longrightarrow Q) \Longrightarrow Q$ ⟨*proof*⟩

realizers
impI $(P,\ Q)$: $\lambda pq.\ pq$
$\boldsymbol{\lambda}(c\text{: -})\ (d\text{: -})\ P\ Q\ pq\ (h\text{: -}).\ allI \cdot - \cdot c \cdot (\boldsymbol{\lambda}x.\ impI \cdot - \cdot - \cdot - \cdot (h \cdot x))$

impI (P): *Null*
$\boldsymbol{\lambda}(c\text{: -})\ P\ Q\ (h\text{: -}).\ allI \cdot - \cdot c \cdot (\boldsymbol{\lambda}x.\ impI \cdot - \cdot - \cdot - \cdot (h \cdot x))$

impI (Q): $\lambda q.\ q\ \boldsymbol{\lambda}(c\text{: -})\ P\ Q\ q.\ impI \cdot - \cdot - \cdot -$

impI: *Null impI*

mp $(P,\ Q)$: $\lambda pq.\ pq$
$\boldsymbol{\lambda}(c\text{: -})\ (d\text{: -})\ P\ Q\ pq\ (h\text{: -})\ p.\ mp \cdot - \cdot - \cdot - \cdot (spec \cdot - \cdot p \cdot c \cdot h)$

mp (P): *Null*
$\boldsymbol{\lambda}(c\text{: -})\ P\ Q\ (h\text{: -})\ p.\ mp \cdot - \cdot - \cdot - \cdot (spec \cdot - \cdot p \cdot c \cdot h)$

mp (Q): $\lambda q.\ q\ \boldsymbol{\lambda}(c\text{: -})\ P\ Q\ q.\ mp \cdot - \cdot - \cdot -$

mp: *Null mp*

allI (P): $\lambda p.\ p\ \boldsymbol{\lambda}(c\text{: -})\ P\ (d\text{: -})\ p.\ allI \cdot - \cdot d$

allI: *Null allI*

spec (P): $\lambda x\ p.\ p\ x\ \boldsymbol{\lambda}(c\text{: -})\ P\ x\ (d\text{: -})\ p.\ spec \cdot - \cdot x \cdot d$

spec: *Null spec*

exI (P): $\lambda x\ p.\ (x,\ p)\ \boldsymbol{\lambda}(c\text{: -})\ P\ x\ (d\text{: -})\ p.\ exI\text{-}realizer \cdot P \cdot p \cdot x \cdot c \cdot d$

exI: $\lambda x.\ x\ \boldsymbol{\lambda}P\ x\ (c\text{: -})\ (h\text{: -}).\ h$

exE (*P*, *Q*): $\lambda p\ pq.$ *let* $(x, y) = p$ *in* $pq\ x\ y$
$\boldsymbol{\lambda}$(*c*: -) (*d*: -) *P Q* (*e*: -) *p* (*h*: -) *pq*. *exE-realizer* · *P* · *p* · *Q* · *pq* · *c* · *e* · *d* · *h*

exE (*P*): *Null*
$\boldsymbol{\lambda}$(*c*: -) *P Q* (*d*: -) *p*. *exE-realizer*′ · - · - · - · - · *c* · *d*

exE (*Q*): $\lambda x\ pq.\ pq\ x$
$\boldsymbol{\lambda}$(*c*: -) *P Q* (*d*: -) *x* (*h1*: -) *pq* (*h2*: -). *h2* · *x* · *h1*

exE: *Null*
$\boldsymbol{\lambda}$*P Q* (*c*: -) *x* (*h1*: -) (*h2*: -). *h2* · *x* · *h1*

conjI (*P*, *Q*): *Pair*
$\boldsymbol{\lambda}$(*c*: -) (*d*: -) *P Q p* (*h*: -) *q*. *conjI-realizer* · *P* · *p* · *Q* · *q* · *c* · *d* · *h*

conjI (*P*): $\lambda p.\ p$
$\boldsymbol{\lambda}$(*c*: -) *P Q p*. *conjI* · - · -

conjI (*Q*): $\lambda q.\ q$
$\boldsymbol{\lambda}$(*c*: -) *P Q* (*h*: -) *q*. *conjI* · - · - · *h*

conjI: *Null conjI*

conjunct1 (*P*, *Q*): *fst*
$\boldsymbol{\lambda}$(*c*: -) (*d*: -) *P Q pq*. *conjunct1* · - · -

conjunct1 (*P*): $\lambda p.\ p$
$\boldsymbol{\lambda}$(*c*: -) *P Q p*. *conjunct1* · - · -

conjunct1 (*Q*): *Null*
$\boldsymbol{\lambda}$(*c*: -) *P Q q*. *conjunct1* · - · -

conjunct1: *Null conjunct1*

conjunct2 (*P*, *Q*): *snd*
$\boldsymbol{\lambda}$(*c*: -) (*d*: -) *P Q pq*. *conjunct2* · - · -

conjunct2 (*P*): *Null*
$\boldsymbol{\lambda}$(*c*: -) *P Q p*. *conjunct2* · - · -

conjunct2 (*Q*): $\lambda p.\ p$
$\boldsymbol{\lambda}$(*c*: -) *P Q p*. *conjunct2* · - · -

conjunct2: *Null conjunct2*

disjI1 (*P*, *Q*): *Inl*
$\boldsymbol{\lambda}$(*c*: -) (*d*: -) *P Q p*. *iffD2* · - · - · (*sum.case-1* · *P* · - · *p* · *arity-type-bool* · *c* · *d*)

disjI1 (*P*): *Some*
λ(*c*: -) *P Q p*. *iffD2* · - · - · - · (*option.case-2* · - · - *P* · *p* · *arity-type-bool* · *c*)

disjI1 (*Q*): *None*
λ(*c*: -) *P Q*. *iffD2* · - · - · - · (*option.case-1* · - · - · - · *arity-type-bool* · *c*)

disjI1: *Left*
λ*P Q*. *iffD2* · - · - · - · (*sumbool.case-1* · - · - · - · *arity-type-bool*)

disjI2 (*P*, *Q*): *Inr*
λ(*d*: -) (*c*: -) *Q P q*. *iffD2* · - · - · - · (*sum.case-2* · - · - *Q* · *q* · *arity-type-bool* · *c* · *d*)

disjI2 (*P*): *None*
λ(*c*: -) *Q P*. *iffD2* · - · - · - · (*option.case-1* · - · - · - · *arity-type-bool* · *c*)

disjI2 (*Q*): *Some*
λ(*c*: -) *Q P q*. *iffD2* · - · - · - · (*option.case-2* · - · - *Q* · *q* · *arity-type-bool* · *c*)

disjI2: *Right*
λ*Q P*. *iffD2* · - · - · - · (*sumbool.case-2* · - · - · - · *arity-type-bool*)

disjE (*P*, *Q*, *R*): *λpq pr qr*.
(*case pq of Inl p ⇒ pr p* | *Inr q ⇒ qr q*)
λ(*c*: -) (*d*: -) (*e*: -) *P Q R pq* (*h1*: -) *pr* (*h2*: -) *qr*.
disjE-realizer · - · - · - · *pq* · *R* · *pr* · *qr* · *c* · *d* · *e* · *h1* · *h2*

disjE (*Q*, *R*): *λpq pr qr*.
(*case pq of None ⇒ pr* | *Some q ⇒ qr q*)
λ(*c*: -) (*d*: -) *P Q R pq* (*h1*: -) *pr* (*h2*: -) *qr*.
disjE-realizer2 · - · - · - · *pq* · *R* · *pr* · *qr* · *c* · *d* · *h1* · *h2*

disjE (*P*, *R*): *λpq pr qr*.
(*case pq of None ⇒ qr* | *Some p ⇒ pr p*)
λ(*c*: -) (*d*: -) *P Q R pq* (*h1*: -) *pr* (*h2*: -) *qr* (*h3*: -).
disjE-realizer2 · - · - · - · *pq* · *R* · *qr* · *pr* · *c* · *d* · *h1* · *h3* · *h2*

disjE (*R*): *λpq pr qr*.
(*case pq of Left ⇒ pr* | *Right ⇒ qr*)
λ(*c*: -) *P Q R pq* (*h1*: -) *pr* (*h2*: -) *qr*.
disjE-realizer3 · - · - · - · *pq* · *R* · *pr* · *qr* · *c* · *h1* · *h2*

disjE (*P*, *Q*): *Null*
λ(*c*: -) (*d*: -) *P Q R pq*. *disjE-realizer* · - · - · - · *pq* · (*λx*. *R*) · - · - · - · *c* · *d* · *arity-type-bool*

disjE (*Q*): *Null*
λ(*c*: -) *P Q R pq*. *disjE-realizer2* · - · - · - · *pq* · (*λx*. *R*) · - · - · - · *c* · *arity-type-bool*

disjE (*P*): *Null*
λ(*c*: -) *P Q R pq* (*h1*: -) (*h2*: -) (*h3*: -).
disjE-realizer2 · - - - - · *pq* · (λ*x*. *R*) · - - - - · *c* • *arity-type-bool* • *h1* • *h3* • *h2*

disjE: *Null*
λ*P Q R pq*. *disjE-realizer3* · - - - - · *pq* · (λ*x*. *R*) · - - - - • *arity-type-bool*

FalseE (*P*): *default*
λ(*c*: -) *P*. *FalseE* · -

FalseE: *Null FalseE*

notI (*P*): *Null*
λ(*c*: -) *P* (*h*: -). *allI* · - • *c* • (**λ***x*. *notI* · - • (*h* · *x*))

notI: *Null notI*

notE (*P*, *R*): λ*p*. *default*
λ(*c*: -) (*d*: -) *P R* (*h*: -) *p*. *notE* · - - - - • (*spec* · - · *p* • *c* • *h*)

notE (*P*): *Null*
λ(*c*: -) *P R* (*h*: -) *p*. *notE* · - - - - • (*spec* · - · *p* • *c* • *h*)

notE (*R*): *default*
λ(*c*: -) *P R*. *notE* · - - -

notE: *Null notE*

subst (*P*): λ*s t ps*. *ps*
λ(*c*: -) *s t P* (*d*: -) (*h*: -) *ps*. *subst* · *s* · *t* · *P ps* • *d* • *h*

subst: *Null subst*

iffD1 (*P*, *Q*): *fst*
λ(*d*: -) (*c*: -) *Q P pq* (*h*: -) *p*.
mp · - - - - • (*spec* · - · *p* • *d* • (*conjunct1* · - - - - • *h*))

iffD1 (*P*): λ*p*. *p*
λ(*c*: -) *Q P p* (*h*: -). *mp* · - - - - • (*conjunct1* · - - - - • *h*)

iffD1 (*Q*): *Null*
λ(*c*: -) *Q P q1* (*h*: -) *q2*.
mp · - - - - • (*spec* · - · *q2* • *c* • (*conjunct1* · - - - - • *h*))

iffD1: *Null iffD1*

iffD2 (*P*, *Q*): *snd*
λ(*c*: -) (*d*: -) *P Q pq* (*h*: -) *q*.
mp · - - - - • (*spec* · - · *q* • *d* • (*conjunct2* · - - - - • *h*))

iffD2 (*P*): $\lambda p.\ p$
  $\boldsymbol{\lambda}(c\text{: -})\ P\ Q\ p\ (h\text{: -}).\ mp \cdot - \cdot - \cdot (conjunct2 \cdot - \cdot - \cdot h)$

iffD2 (*Q*): *Null*
  $\boldsymbol{\lambda}(c\text{: -})\ P\ Q\ q1\ (h\text{: -})\ q2.$
    $mp \cdot - \cdot - \cdot (spec \cdot - \cdot q2 \cdot c \cdot (conjunct2 \cdot - \cdot - \cdot h))$

iffD2: *Null iffD2*

iffI (*P*, *Q*): *Pair*
  $\boldsymbol{\lambda}(c\text{: -})\ (d\text{: -})\ P\ Q\ pq\ (h1 : \text{-})\ qp\ (h2 : \text{-}).\ conjI\text{-}realizer \cdot$
    $(\lambda pq.\ \forall x.\ P\ x \longrightarrow Q\ (pq\ x)) \cdot pq \cdot$
    $(\lambda qp.\ \forall x.\ Q\ x \longrightarrow P\ (qp\ x)) \cdot qp \cdot$
    $(arity\text{-}type\text{-}fun \cdot c \cdot d) \cdot$
    $(arity\text{-}type\text{-}fun \cdot d \cdot c) \cdot$
    $(allI \cdot - \cdot c \cdot (\boldsymbol{\lambda}x.\ impI \cdot - \cdot - \cdot (h1 \cdot x))) \cdot$
    $(allI \cdot - \cdot d \cdot (\boldsymbol{\lambda}x.\ impI \cdot - \cdot - \cdot (h2 \cdot x)))$

iffI (*P*): $\lambda p.\ p$
  $\boldsymbol{\lambda}(c\text{: -})\ P\ Q\ (h1 : \text{-})\ p\ (h2 : \text{-}).\ conjI \cdot - \cdot - \cdot$
    $(allI \cdot - \cdot c \cdot (\boldsymbol{\lambda}x.\ impI \cdot - \cdot - \cdot (h1 \cdot x))) \cdot$
    $(impI \cdot - \cdot - \cdot h2)$

iffI (*Q*): $\lambda q.\ q$
  $\boldsymbol{\lambda}(c\text{: -})\ P\ Q\ q\ (h1 : \text{-})\ (h2 : \text{-}).\ conjI \cdot - \cdot - \cdot$
    $(impI \cdot - \cdot - \cdot h1) \cdot$
    $(allI \cdot - \cdot c \cdot (\boldsymbol{\lambda}x.\ impI \cdot - \cdot - \cdot (h2 \cdot x)))$

iffI: *Null iffI*

end

87 Extensible records with structural subtyping

theory *Record*
imports *Quickcheck-Exhaustive*
keywords
  record :: *thy-defn* **and**
  print-record :: *diag*
begin

87.1 Introduction

Records are isomorphic to compound tuple types. To implement efficient records, we make this isomorphism explicit. Consider the record access/update simplification *alpha* (*beta-update f rec*) = *alpha rec* for distinct fields alpha and beta of some record rec with n fields. There are $n \mathbin{\widehat{}} 2$ such the-

orems, which prohibits storage of all of them for large n. The rules can be proved on the fly by case decomposition and simplification in O(n) time. By creating O(n) isomorphic-tuple types while defining the record, however, we can prove the access/update simplification in $O(log(n)\hat{}2)$ time.

The O(n) cost of case decomposition is not because O(n) steps are taken, but rather because the resulting rule must contain O(n) new variables and an O(n) size concrete record construction. To sidestep this cost, we would like to avoid case decomposition in proving access/update theorems.

Record types are defined as isomorphic to tuple types. For instance, a record type with fields $'a$, $'b$, $'c$ and $'d$ might be introduced as isomorphic to $'a \times ('b \times ('c \times 'd))$. If we balance the tuple tree to $('a \times 'b) \times ('c \times 'd)$ then accessors can be defined by converting to the underlying type then using O(log(n)) fst or snd operations. Updators can be defined similarly, if we introduce a *fst-update* and *snd-update* function. Furthermore, we can prove the access/update theorem in O(log(n)) steps by using simple rewrites on fst, snd, *fst-update* and *snd-update*.

The catch is that, although O(log(n)) steps were taken, the underlying type we converted to is a tuple tree of size O(n). Processing this term type wastes performance. We avoid this for large n by taking each subtree of size K and defining a new type isomorphic to that tuple subtree. A record can now be defined as isomorphic to a tuple tree of these O(n/K) new types, or, if $n > K*K$, we can repeat the process, until the record can be defined in terms of a tuple tree of complexity less than the constant K.

If we prove the access/update theorem on this type with the analogous steps to the tuple tree, we consume $O(log(n)\hat{}2)$ time as the intermediate terms are $O(log(n))$ in size and the types needed have size bounded by K. To enable this analogous traversal, we define the functions seen below: *iso-tuple-fst*, *iso-tuple-snd*, *iso-tuple-fst-update* and *iso-tuple-snd-update*. These functions generalise tuple operations by taking a parameter that encapsulates a tuple isomorphism. The rewrites needed on these functions now need an additional assumption which is that the isomorphism works.

These rewrites are typically used in a structured way. They are here presented as the introduction rule *isomorphic-tuple.intros* rather than as a rewrite rule set. The introduction form is an optimisation, as net matching can be performed at one term location for each step rather than the simplifier searching the term for possible pattern matches. The rule set is used as it is viewed outside the locale, with the locale assumption (that the isomorphism is valid) left as a rule assumption. All rules are structured to aid net matching, using either a point-free form or an encapsulating predicate.

87.2 Operators and lemmas for types isomorphic to tuples

datatype (*dead* $'a$, *dead* $'b$, *dead* $'c$) *tuple-isomorphism* =

Tuple-Isomorphism $'a \Rightarrow 'b \times 'c\ 'b \times 'c \Rightarrow 'a$

primrec
repr :: $('a, 'b, 'c)$ *tuple-isomorphism* $\Rightarrow 'a \Rightarrow 'b \times 'c$ **where**
repr (*Tuple-Isomorphism r a*) $= r$

primrec
abst :: $('a, 'b, 'c)$ *tuple-isomorphism* $\Rightarrow 'b \times 'c \Rightarrow 'a$ **where**
abst (*Tuple-Isomorphism r a*) $= a$

definition
iso-tuple-fst :: $('a, 'b, 'c)$ *tuple-isomorphism* $\Rightarrow 'a \Rightarrow 'b$ **where**
iso-tuple-fst isom $=$ *fst* $\circ$ *repr isom*

definition
iso-tuple-snd :: $('a, 'b, 'c)$ *tuple-isomorphism* $\Rightarrow 'a \Rightarrow 'c$ **where**
iso-tuple-snd isom $=$ *snd* $\circ$ *repr isom*

definition
iso-tuple-fst-update ::
$('a, 'b, 'c)$ *tuple-isomorphism* $\Rightarrow ('b \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'a)$ **where**
iso-tuple-fst-update isom f $=$ *abst isom* $\circ$ *apfst f* $\circ$ *repr isom*

definition
iso-tuple-snd-update ::
$('a, 'b, 'c)$ *tuple-isomorphism* $\Rightarrow ('c \Rightarrow 'c) \Rightarrow ('a \Rightarrow 'a)$ **where**
iso-tuple-snd-update isom f $=$ *abst isom* $\circ$ *apsnd f* $\circ$ *repr isom*

definition
iso-tuple-cons ::
$('a, 'b, 'c)$ *tuple-isomorphism* $\Rightarrow 'b \Rightarrow 'c \Rightarrow 'a$ **where**
iso-tuple-cons isom $=$ *curry* (*abst isom*)

87.3 Logical infrastructure for records

definition
iso-tuple-surjective-proof-assist :: $'a \Rightarrow 'b \Rightarrow ('a \Rightarrow 'b) \Rightarrow$ *bool* **where**
iso-tuple-surjective-proof-assist x y f $\longleftrightarrow$ *f x* $= y$

definition
iso-tuple-update-accessor-cong-assist ::
$(('b \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'a)) \Rightarrow ('a \Rightarrow 'b) \Rightarrow$ *bool* **where**
iso-tuple-update-accessor-cong-assist upd ac $\longleftrightarrow$
$(\forall f\ v.\ upd\ (\lambda x.\ f\ (ac\ v))\ v = upd\ f\ v) \wedge (\forall v.\ upd\ id\ v = v)$

definition
iso-tuple-update-accessor-eq-assist ::
$(('b \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'a)) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow ('b \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b \Rightarrow$ *bool*
where

iso-tuple-update-accessor-eq-assist upd ac v f v' x ⟷
upd f v = v' ∧ ac v = x ∧ iso-tuple-update-accessor-cong-assist upd ac

lemma *update-accessor-congruence-foldE*:
assumes *uac*: *iso-tuple-update-accessor-cong-assist upd ac*
and *r*: *r = r'* **and** *v*: *ac r' = v'*
and *f*: ⋀*v*. *v' = v* ⟹ *f v = f' v*
shows *upd f r = upd f' r'*
⟨*proof*⟩

lemma *update-accessor-congruence-unfoldE*:
iso-tuple-update-accessor-cong-assist upd ac ⟹
r = r' ⟹ *ac r' = v'* ⟹ (⋀*v*. *v = v'* ⟹ *f v = f' v*) ⟹
upd f r = upd f' r'
⟨*proof*⟩

lemma *iso-tuple-update-accessor-cong-assist-id*:
iso-tuple-update-accessor-cong-assist upd ac ⟹ *upd id = id*
⟨*proof*⟩

lemma *update-accessor-noopE*:
assumes *uac*: *iso-tuple-update-accessor-cong-assist upd ac*
and *ac*: *f (ac x) = ac x*
shows *upd f x = x*
⟨*proof*⟩

lemma *update-accessor-noop-compE*:
assumes *uac*: *iso-tuple-update-accessor-cong-assist upd ac*
and *ac*: *f (ac x) = ac x*
shows *upd (g ∘ f) x = upd g x*
⟨*proof*⟩

lemma *update-accessor-cong-assist-idI*:
iso-tuple-update-accessor-cong-assist id id
⟨*proof*⟩

lemma *update-accessor-cong-assist-triv*:
iso-tuple-update-accessor-cong-assist upd ac ⟹
iso-tuple-update-accessor-cong-assist upd ac
⟨*proof*⟩

lemma *update-accessor-accessor-eqE*:
iso-tuple-update-accessor-eq-assist upd ac v f v' x ⟹ *ac v = x*
⟨*proof*⟩

lemma *update-accessor-updator-eqE*:
iso-tuple-update-accessor-eq-assist upd ac v f v' x ⟹ *upd f v = v'*
⟨*proof*⟩

lemma *iso-tuple-update-accessor-eq-assist-idI*:
$v' = f\ v \Longrightarrow$ *iso-tuple-update-accessor-eq-assist id id v f v' v*
⟨*proof*⟩

lemma *iso-tuple-update-accessor-eq-assist-triv*:
iso-tuple-update-accessor-eq-assist upd ac v f v' x ⟹
iso-tuple-update-accessor-eq-assist upd ac v f v' x
⟨*proof*⟩

lemma *iso-tuple-update-accessor-cong-from-eq*:
iso-tuple-update-accessor-eq-assist upd ac v f v' x ⟹
iso-tuple-update-accessor-cong-assist upd ac
⟨*proof*⟩

lemma *iso-tuple-surjective-proof-assistI*:
$f\ x = y \Longrightarrow$ *iso-tuple-surjective-proof-assist x y f*
⟨*proof*⟩

lemma *iso-tuple-surjective-proof-assist-idE*:
iso-tuple-surjective-proof-assist x y id ⟹ $x = y$
⟨*proof*⟩

locale *isomorphic-tuple* =
fixes *isom* :: (*'a*, *'b*, *'c*) *tuple-isomorphism*
assumes *repr-inv*: $\bigwedge x.$ *abst isom* (*repr isom x*) $= x$
and *abst-inv*: $\bigwedge y.$ *repr isom* (*abst isom y*) $= y$
begin

lemma *repr-inj*: *repr isom x* = *repr isom y* ⟷ $x = y$
⟨*proof*⟩

lemma *abst-inj*: *abst isom x* = *abst isom y* ⟷ $x = y$
⟨*proof*⟩

lemmas *simps* = *Let-def repr-inv abst-inv repr-inj abst-inj*

lemma *iso-tuple-access-update-fst-fst*:
$f \circ h\ g = j \circ f \Longrightarrow$
(*f* ∘ *iso-tuple-fst isom*) ∘ (*iso-tuple-fst-update isom* ∘ *h*) *g* =
j ∘ (*f* ∘ *iso-tuple-fst isom*)
⟨*proof*⟩

lemma *iso-tuple-access-update-snd-snd*:
$f \circ h\ g = j \circ f \Longrightarrow$
(*f* ∘ *iso-tuple-snd isom*) ∘ (*iso-tuple-snd-update isom* ∘ *h*) *g* =
j ∘ (*f* ∘ *iso-tuple-snd isom*)
⟨*proof*⟩

lemma *iso-tuple-access-update-fst-snd*:

$(f \circ \textit{iso-tuple-fst isom}) \circ (\textit{iso-tuple-snd-update isom} \circ h)\ g =$
$\quad \textit{id} \circ (f \circ \textit{iso-tuple-fst isom})$
⟨*proof*⟩

lemma *iso-tuple-access-update-snd-fst*:
$(f \circ \textit{iso-tuple-snd isom}) \circ (\textit{iso-tuple-fst-update isom} \circ h)\ g =$
$\quad \textit{id} \circ (f \circ \textit{iso-tuple-snd isom})$
⟨*proof*⟩

lemma *iso-tuple-update-swap-fst-fst*:
$h\ f \circ j\ g = j\ g \circ h\ f \Longrightarrow$
$\quad (\textit{iso-tuple-fst-update isom} \circ h)\ f \circ (\textit{iso-tuple-fst-update isom} \circ j)\ g =$
$\quad\quad (\textit{iso-tuple-fst-update isom} \circ j)\ g \circ (\textit{iso-tuple-fst-update isom} \circ h)\ f$
⟨*proof*⟩

lemma *iso-tuple-update-swap-snd-snd*:
$h\ f \circ j\ g = j\ g \circ h\ f \Longrightarrow$
$\quad (\textit{iso-tuple-snd-update isom} \circ h)\ f \circ (\textit{iso-tuple-snd-update isom} \circ j)\ g =$
$\quad\quad (\textit{iso-tuple-snd-update isom} \circ j)\ g \circ (\textit{iso-tuple-snd-update isom} \circ h)\ f$
⟨*proof*⟩

lemma *iso-tuple-update-swap-fst-snd*:
$(\textit{iso-tuple-snd-update isom} \circ h)\ f \circ (\textit{iso-tuple-fst-update isom} \circ j)\ g =$
$\quad (\textit{iso-tuple-fst-update isom} \circ j)\ g \circ (\textit{iso-tuple-snd-update isom} \circ h)\ f$
⟨*proof*⟩

lemma *iso-tuple-update-swap-snd-fst*:
$(\textit{iso-tuple-fst-update isom} \circ h)\ f \circ (\textit{iso-tuple-snd-update isom} \circ j)\ g =$
$\quad (\textit{iso-tuple-snd-update isom} \circ j)\ g \circ (\textit{iso-tuple-fst-update isom} \circ h)\ f$
⟨*proof*⟩

lemma *iso-tuple-update-compose-fst-fst*:
$h\ f \circ j\ g = k\ (f \circ g) \Longrightarrow$
$\quad (\textit{iso-tuple-fst-update isom} \circ h)\ f \circ (\textit{iso-tuple-fst-update isom} \circ j)\ g =$
$\quad\quad (\textit{iso-tuple-fst-update isom} \circ k)\ (f \circ g)$
⟨*proof*⟩

lemma *iso-tuple-update-compose-snd-snd*:
$h\ f \circ j\ g = k\ (f \circ g) \Longrightarrow$
$\quad (\textit{iso-tuple-snd-update isom} \circ h)\ f \circ (\textit{iso-tuple-snd-update isom} \circ j)\ g =$
$\quad\quad (\textit{iso-tuple-snd-update isom} \circ k)\ (f \circ g)$
⟨*proof*⟩

lemma *iso-tuple-surjective-proof-assist-step*:
$\textit{iso-tuple-surjective-proof-assist}\ v\ a\ (\textit{iso-tuple-fst isom} \circ f) \Longrightarrow$
$\quad \textit{iso-tuple-surjective-proof-assist}\ v\ b\ (\textit{iso-tuple-snd isom} \circ f) \Longrightarrow$
$\quad \textit{iso-tuple-surjective-proof-assist}\ v\ (\textit{iso-tuple-cons isom}\ a\ b)\ f$
⟨*proof*⟩

lemma *iso-tuple-fst-update-accessor-cong-assist*:
assumes *iso-tuple-update-accessor-cong-assist f g*
shows *iso-tuple-update-accessor-cong-assist*
(*iso-tuple-fst-update isom* ∘ *f*) (*g* ∘ *iso-tuple-fst isom*)
⟨*proof*⟩

lemma *iso-tuple-snd-update-accessor-cong-assist*:
assumes *iso-tuple-update-accessor-cong-assist f g*
shows *iso-tuple-update-accessor-cong-assist*
(*iso-tuple-snd-update isom* ∘ *f*) (*g* ∘ *iso-tuple-snd isom*)
⟨*proof*⟩

lemma *iso-tuple-fst-update-accessor-eq-assist*:
assumes *iso-tuple-update-accessor-eq-assist f g a u a′ v*
shows *iso-tuple-update-accessor-eq-assist*
(*iso-tuple-fst-update isom* ∘ *f*) (*g* ∘ *iso-tuple-fst isom*)
(*iso-tuple-cons isom a b*) *u* (*iso-tuple-cons isom a′ b*) *v*
⟨*proof*⟩

lemma *iso-tuple-snd-update-accessor-eq-assist*:
assumes *iso-tuple-update-accessor-eq-assist f g b u b′ v*
shows *iso-tuple-update-accessor-eq-assist*
(*iso-tuple-snd-update isom* ∘ *f*) (*g* ∘ *iso-tuple-snd isom*)
(*iso-tuple-cons isom a b*) *u* (*iso-tuple-cons isom a b′*) *v*
⟨*proof*⟩

lemma *iso-tuple-cons-conj-eqI*:
a = *c* ∧ *b* = *d* ∧ *P* ⟷ *Q* ⟹
iso-tuple-cons isom a b = *iso-tuple-cons isom c d* ∧ *P* ⟷ *Q*
⟨*proof*⟩

lemmas *intros* =
iso-tuple-access-update-fst-fst
iso-tuple-access-update-snd-snd
iso-tuple-access-update-fst-snd
iso-tuple-access-update-snd-fst
iso-tuple-update-swap-fst-fst
iso-tuple-update-swap-snd-snd
iso-tuple-update-swap-fst-snd
iso-tuple-update-swap-snd-fst
iso-tuple-update-compose-fst-fst
iso-tuple-update-compose-snd-snd
iso-tuple-surjective-proof-assist-step
iso-tuple-fst-update-accessor-eq-assist
iso-tuple-snd-update-accessor-eq-assist
iso-tuple-fst-update-accessor-cong-assist
iso-tuple-snd-update-accessor-cong-assist
iso-tuple-cons-conj-eqI

end

lemma *isomorphic-tuple-intro*:
fixes *repr abst*
assumes *repr-inj*: $\bigwedge x\ y.\ repr\ x = repr\ y \longleftrightarrow x = y$
and *abst-inv*: $\bigwedge z.\ repr\ (abst\ z) = z$
and *v*: $v \equiv Tuple\text{-}Isomorphism\ repr\ abst$
shows *isomorphic-tuple v*
⟨*proof*⟩

definition
tuple-iso-tuple ≡ *Tuple-Isomorphism id id*

lemma *tuple-iso-tuple*:
isomorphic-tuple tuple-iso-tuple
⟨*proof*⟩

lemma *refl-conj-eq*: $Q = R \Longrightarrow P \wedge Q \longleftrightarrow P \wedge R$
⟨*proof*⟩

lemma *iso-tuple-UNIV-I*: $x \in UNIV \equiv True$
⟨*proof*⟩

lemma *iso-tuple-True-simp*: $(True \Longrightarrow PROP\ P) \equiv PROP\ P$
⟨*proof*⟩

lemma *prop-subst*: $s = t \Longrightarrow PROP\ P\ t \Longrightarrow PROP\ P\ s$
⟨*proof*⟩

lemma *K-record-comp*: $(\lambda x.\ c) \circ f = (\lambda x.\ c)$
⟨*proof*⟩

87.4 Concrete record syntax

nonterminal
ident **and**
field-type **and**
field-types **and**
field **and**
fields **and**
field-update **and**
field-updates

syntax
-constify :: *id* => *ident* (-)
-constify :: *longid* => *ident* (-)

-field-type :: *ident* => *type* => *field-type* ((*2*- ::/ -))
:: *field-type* => *field-types* (-)

-field-types :: *field-type* => *field-types* => *field-types* (-,/ -)
-record-type :: *field-types* => *type* ((*3*(|-|)))
-record-type-scheme :: *field-types* => *type* => *type* ((*3*(|-,/ (*2*... ::/ -)|)))

-field :: *ident* => *'a* => *field* ((*2*- =/ -))
:: *field* => *fields* (-)
-fields :: *field* => *fields* => *fields* (-,/ -)
-record :: *fields* => *'a* ((*3*(|-|)))
-record-scheme :: *fields* => *'a* => *'a* ((*3*(|-,/ (*2*... =/ -)|)))

-field-update :: *ident* => *'a* => *field-update* ((*2*- :=/ -))
:: *field-update* => *field-updates* (-)
-field-updates :: *field-update* => *field-updates* => *field-updates* (-,/ -)
-record-update :: *'a* => *field-updates* => *'b* (-/(*3*(|-|)) [*900*, *0*] *900*)

syntax (*ASCII*)
-record-type :: *field-types* => *type* ((*3'*(| - |')))
-record-type-scheme :: *field-types* => *type* => *type* ((*3'*(| -,/ (*2*... ::/ -) |')))
-record :: *fields* => *'a* ((*3'*(| - |')))
-record-scheme :: *fields* => *'a* => *'a* ((*3'*(| -,/ (*2*... =/ -) |')))
-record-update :: *'a* => *field-updates* => *'b* (-/(*3'*(| - |')) [*900*, *0*] *900*)

87.5 Record package

⟨*ML*⟩

hide-const (**open**) *Tuple-Isomorphism repr abst iso-tuple-fst iso-tuple-snd iso-tuple-fst-update iso-tuple-snd-update iso-tuple-cons iso-tuple-surjective-proof-assist iso-tuple-update-accessor-cong-assist iso-tuple-update-accessor-eq-assist tuple-iso-tuple*

end

88 Greatest common divisor and least common multiple

theory *GCD*
imports *Groups-List Code-Numeral*
begin

88.1 Abstract bounded quasi semilattices as common foundation

locale *bounded-quasi-semilattice* = *abel-semigroup* +
fixes *top* :: *'a* (⊤) **and** *bot* :: *'a* (⊥)
and *normalize* :: *'a* ⇒ *'a*
assumes *idem-normalize* [*simp*]: *a* ∗ *a* = *normalize a*
and *normalize-left-idem* [*simp*]: *normalize a* ∗ *b* = *a* ∗ *b*

and *normalize-idem* [*simp*]: *normalize* $(a * b) = a * b$
and *normalize-top* [*simp*]: *normalize* $\top = \top$
and *normalize-bottom* [*simp*]: *normalize* $\bot = \bot$
and *top-left-normalize* [*simp*]: $\top * a =$ *normalize* a
and *bottom-left-bottom* [*simp*]: $\bot * a = \bot$
begin

lemma *left-idem* [*simp*]:
$a * (a * b) = a * b$
⟨*proof*⟩

lemma *right-idem* [*simp*]:
$(a * b) * b = a * b$
⟨*proof*⟩

lemma *comp-fun-idem*: *comp-fun-idem f*
⟨*proof*⟩

interpretation *comp-fun-idem f*
⟨*proof*⟩

lemma *top-right-normalize* [*simp*]:
$a * \top =$ *normalize* a
⟨*proof*⟩

lemma *bottom-right-bottom* [*simp*]:
$a * \bot = \bot$
⟨*proof*⟩

lemma *normalize-right-idem* [*simp*]:
$a *$ *normalize* $b = a * b$
⟨*proof*⟩

end

locale *bounded-quasi-semilattice-set* = *bounded-quasi-semilattice*
begin

interpretation *comp-fun-idem f*
⟨*proof*⟩

definition $F :: 'a\ set \Rightarrow 'a$
where
eq-fold: $F\ A =$ (*if finite* A *then Finite-Set.fold f* $\top$ A *else* $\bot$)

lemma *infinite* [*simp*]:
infinite $A \Longrightarrow F\ A = \bot$
⟨*proof*⟩

lemma *set-eq-fold* [*code*]:
  $F\ (set\ xs) = fold\ f\ xs\ \top$
  ⟨*proof*⟩

lemma *empty* [*simp*]:
  $F\ \{\} = \top$
  ⟨*proof*⟩

lemma *insert* [*simp*]:
  $F\ (insert\ a\ A) = a * F\ A$
  ⟨*proof*⟩

lemma *normalize* [*simp*]:
  $normalize\ (F\ A) = F\ A$
  ⟨*proof*⟩

lemma *in-idem*:
  assumes $a \in A$
  shows $a * F\ A = F\ A$
  ⟨*proof*⟩

lemma *union*:
  $F\ (A \cup B) = F\ A * F\ B$
  ⟨*proof*⟩

lemma *remove*:
  assumes $a \in A$
  shows $F\ A = a * F\ (A - \{a\})$
⟨*proof*⟩

lemma *insert-remove*:
  $F\ (insert\ a\ A) = a * F\ (A - \{a\})$
  ⟨*proof*⟩

lemma *subset*:
  assumes $B \subseteq A$
  shows $F\ B * F\ A = F\ A$
  ⟨*proof*⟩

end

88.2 Abstract GCD and LCM

class *gcd* = *zero* + *one* + *dvd* +
  fixes *gcd* :: $'a \Rightarrow 'a \Rightarrow 'a$
    and *lcm* :: $'a \Rightarrow 'a \Rightarrow 'a$

class *Gcd* = *gcd* +
  fixes *Gcd* :: $'a\ set \Rightarrow 'a$

and *Lcm* :: *'a set* ⇒ *'a*

syntax
-GCD1 :: *pttrns* ⇒ *'b* ⇒ *'b* (*(3GCD -./ -) [0, 10] 10*)
-GCD :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* (*(3GCD -∈-./ -) [0, 0, 10] 10*)
-LCM1 :: *pttrns* ⇒ *'b* ⇒ *'b* (*(3LCM -./ -) [0, 10] 10*)
-LCM :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* (*(3LCM -∈-./ -) [0, 0, 10] 10*)

translations
GCD x y. f ⇌ *GCD x. GCD y. f*
GCD x. f ⇌ *CONST Gcd* (*CONST range* (*λx. f*))
GCD x∈A. f ⇌ *CONST Gcd* ((*λx. f*) ‘ *A*)
LCM x y. f ⇌ *LCM x. LCM y. f*
LCM x. f ⇌ *CONST Lcm* (*CONST range* (*λx. f*))
LCM x∈A. f ⇌ *CONST Lcm* ((*λx. f*) ‘ *A*)

class *semiring-gcd* = *normalization-semidom* + *gcd* +
assumes *gcd-dvd1* [*iff*]: *gcd a b dvd a*
and *gcd-dvd2* [*iff*]: *gcd a b dvd b*
and *gcd-greatest*: *c dvd a* ⟹ *c dvd b* ⟹ *c dvd gcd a b*
and *normalize-gcd* [*simp*]: *normalize* (*gcd a b*) = *gcd a b*
and *lcm-gcd*: *lcm a b* = *normalize* (*a* ∗ *b div gcd a b*)
begin

lemma *gcd-greatest-iff* [*simp*]: *a dvd gcd b c* ⟷ *a dvd b* ∧ *a dvd c*
⟨*proof*⟩

lemma *gcd-dvdI1*: *a dvd c* ⟹ *gcd a b dvd c*
⟨*proof*⟩

lemma *gcd-dvdI2*: *b dvd c* ⟹ *gcd a b dvd c*
⟨*proof*⟩

lemma *dvd-gcdD1*: *a dvd gcd b c* ⟹ *a dvd b*
⟨*proof*⟩

lemma *dvd-gcdD2*: *a dvd gcd b c* ⟹ *a dvd c*
⟨*proof*⟩

lemma *gcd-0-left* [*simp*]: *gcd 0 a* = *normalize a*
⟨*proof*⟩

lemma *gcd-0-right* [*simp*]: *gcd a 0* = *normalize a*
⟨*proof*⟩

lemma *gcd-eq-0-iff* [*simp*]: *gcd a b* = *0* ⟷ *a* = *0* ∧ *b* = *0*
(**is** *?P* ⟷ *?Q*)
⟨*proof*⟩

lemma *unit-factor-gcd*: *unit-factor* (*gcd a b*) = (*if a = 0 ∧ b = 0 then 0 else 1*)
⟨*proof*⟩

lemma *is-unit-gcd-iff* [*simp*]:
is-unit (*gcd a b*) ⟷ *gcd a b = 1*
⟨*proof*⟩

sublocale *gcd*: *abel-semigroup gcd*
⟨*proof*⟩

sublocale *gcd*: *bounded-quasi-semilattice gcd 0 1 normalize*
⟨*proof*⟩

lemma *gcd-self*: *gcd a a = normalize a*
⟨*proof*⟩

lemma *gcd-left-idem*: *gcd a* (*gcd a b*) = *gcd a b*
⟨*proof*⟩

lemma *gcd-right-idem*: *gcd* (*gcd a b*) *b* = *gcd a b*
⟨*proof*⟩

lemma *gcdI*:
assumes *c dvd a* **and** *c dvd b*
and *greatest*: ⋀*d*. *d dvd a* ⟹ *d dvd b* ⟹ *d dvd c*
and *normalize c = c*
shows *c = gcd a b*
⟨*proof*⟩

lemma *gcd-unique*:
d dvd a ∧ d dvd b ∧ normalize d = d ∧ (∀ *e*. *e dvd a ∧ e dvd b* ⟶ *e dvd d*) ⟷ *d = gcd a b*
⟨*proof*⟩

lemma *gcd-dvd-prod*: *gcd a b dvd k ∗ b*
⟨*proof*⟩

lemma *gcd-proj2-if-dvd*: *b dvd a* ⟹ *gcd a b = normalize b*
⟨*proof*⟩

lemma *gcd-proj1-if-dvd*: *a dvd b* ⟹ *gcd a b = normalize a*
⟨*proof*⟩

lemma *gcd-proj1-iff*: *gcd m n = normalize m* ⟷ *m dvd n*
⟨*proof*⟩

lemma *gcd-proj2-iff*: *gcd m n = normalize n* ⟷ *n dvd m*
⟨*proof*⟩

lemma *gcd-mult-left*: *gcd* (*c* ∗ *a*) (*c* ∗ *b*) = *normalize* (*c* ∗ *gcd a b*)
⟨*proof*⟩

lemma *gcd-mult-right*: *gcd* (*a* ∗ *c*) (*b* ∗ *c*) = *normalize* (*gcd b a* ∗ *c*)
⟨*proof*⟩

lemma *dvd-lcm1* [*iff*]: *a dvd lcm a b*
⟨*proof*⟩

lemma *dvd-lcm2* [*iff*]: *b dvd lcm a b*
⟨*proof*⟩

lemma *dvd-lcmI1*: *a dvd b* ⟹ *a dvd lcm b c*
⟨*proof*⟩

lemma *dvd-lcmI2*: *a dvd c* ⟹ *a dvd lcm b c*
⟨*proof*⟩

lemma *lcm-dvdD1*: *lcm a b dvd c* ⟹ *a dvd c*
⟨*proof*⟩

lemma *lcm-dvdD2*: *lcm a b dvd c* ⟹ *b dvd c*
⟨*proof*⟩

lemma *lcm-least*:
assumes *a dvd c* **and** *b dvd c*
shows *lcm a b dvd c*
⟨*proof*⟩

lemma *lcm-least-iff* [*simp*]: *lcm a b dvd c* ⟷ *a dvd c* ∧ *b dvd c*
⟨*proof*⟩

lemma *normalize-lcm* [*simp*]: *normalize* (*lcm a b*) = *lcm a b*
⟨*proof*⟩

lemma *lcm-0-left* [*simp*]: *lcm 0 a* = *0*
⟨*proof*⟩

lemma *lcm-0-right* [*simp*]: *lcm a 0* = *0*
⟨*proof*⟩

lemma *lcm-eq-0-iff*: *lcm a b* = *0* ⟷ *a* = *0* ∨ *b* = *0*
(**is** *?P* ⟷ *?Q*)
⟨*proof*⟩

lemma *zero-eq-lcm-iff*: *0* = *lcm a b* ⟷ *a* = *0* ∨ *b* = *0*
⟨*proof*⟩

lemma *lcm-eq-1-iff* [*simp*]: *lcm a b* = *1* ⟷ *is-unit a* ∧ *is-unit b*

⟨*proof*⟩

lemma *unit-factor-lcm*: *unit-factor* (*lcm a b*) = (*if a = 0 ∨ b = 0 then 0 else 1*)
⟨*proof*⟩

sublocale *lcm*: *abel-semigroup lcm*
⟨*proof*⟩

sublocale *lcm*: *bounded-quasi-semilattice lcm 1 0 normalize*
⟨*proof*⟩

lemma *lcm-self*: *lcm a a* = *normalize a*
⟨*proof*⟩

lemma *lcm-left-idem*: *lcm a* (*lcm a b*) = *lcm a b*
⟨*proof*⟩

lemma *lcm-right-idem*: *lcm* (*lcm a b*) *b* = *lcm a b*
⟨*proof*⟩

lemma *gcd-lcm*:
assumes *a* ≠ *0* **and** *b* ≠ *0*
shows *gcd a b* = *normalize* (*a* ∗ *b div lcm a b*)
⟨*proof*⟩

lemma *lcm-1-left*: *lcm 1 a* = *normalize a*
⟨*proof*⟩

lemma *lcm-1-right*: *lcm a 1* = *normalize a*
⟨*proof*⟩

lemma *lcm-mult-left*: *lcm* (*c* ∗ *a*) (*c* ∗ *b*) = *normalize* (*c* ∗ *lcm a b*)
⟨*proof*⟩

lemma *lcm-mult-right*: *lcm* (*a* ∗ *c*) (*b* ∗ *c*) = *normalize* (*lcm b a* ∗ *c*)
⟨*proof*⟩

lemma *lcm-mult-unit1*: *is-unit a* ⟹ *lcm* (*b* ∗ *a*) *c* = *lcm b c*
⟨*proof*⟩

lemma *lcm-mult-unit2*: *is-unit a* ⟹ *lcm b* (*c* ∗ *a*) = *lcm b c*
⟨*proof*⟩

lemma *lcm-div-unit1*:
is-unit a ⟹ *lcm* (*b div a*) *c* = *lcm b c*
⟨*proof*⟩

lemma *lcm-div-unit2*: *is-unit a* ⟹ *lcm b* (*c div a*) = *lcm b c*
⟨*proof*⟩

lemma *normalize-lcm-left*: *lcm* (*normalize a*) *b* = *lcm a b*
⟨*proof*⟩

lemma *normalize-lcm-right*: *lcm a* (*normalize b*) = *lcm a b*
⟨*proof*⟩

lemma *comp-fun-idem-gcd*: *comp-fun-idem gcd*
⟨*proof*⟩

lemma *comp-fun-idem-lcm*: *comp-fun-idem lcm*
⟨*proof*⟩

lemma *gcd-dvd-antisym*: *gcd a b dvd gcd c d* ⟹ *gcd c d dvd gcd a b* ⟹ *gcd a b* = *gcd c d*
⟨*proof*⟩

declare *unit-factor-lcm* [*simp*]

lemma *lcmI*:
assumes *a dvd c* **and** *b dvd c* **and** ⋀*d*. *a dvd d* ⟹ *b dvd d* ⟹ *c dvd d*
and *normalize c* = *c*
shows *c* = *lcm a b*
⟨*proof*⟩

lemma *gcd-dvd-lcm* [*simp*]: *gcd a b dvd lcm a b*
⟨*proof*⟩

lemmas *lcm-0* = *lcm-0-right*

lemma *lcm-unique*:
a dvd d ∧ *b dvd d* ∧ *normalize d* = *d* ∧ (∀ *e*. *a dvd e* ∧ *b dvd e* ⟶ *d dvd e*) ⟷ *d* = *lcm a b*
⟨*proof*⟩

lemma *lcm-proj1-if-dvd*:
assumes *b dvd a* **shows** *lcm a b* = *normalize a*
⟨*proof*⟩

lemma *lcm-proj2-if-dvd*: *a dvd b* ⟹ *lcm a b* = *normalize b*
⟨*proof*⟩

lemma *lcm-proj1-iff*: *lcm m n* = *normalize m* ⟷ *n dvd m*
⟨*proof*⟩

lemma *lcm-proj2-iff*: *lcm m n* = *normalize n* ⟷ *m dvd n*
⟨*proof*⟩

lemma *gcd-mono*: *a dvd c* ⟹ *b dvd d* ⟹ *gcd a b dvd gcd c d*

⟨*proof*⟩

lemma *lcm-mono*: *a dvd c* ⟹ *b dvd d* ⟹ *lcm a b dvd lcm c d*
⟨*proof*⟩

lemma *dvd-productE*:
assumes *p dvd a* ∗ *b*
obtains *x y* **where** *p* = *x* ∗ *y x dvd a y dvd b*
⟨*proof*⟩

lemma *gcd-mult-unit1*:
assumes *is-unit a* **shows** *gcd* (*b* ∗ *a*) *c* = *gcd b c*
⟨*proof*⟩

lemma *gcd-mult-unit2*: *is-unit a* ⟹ *gcd b* (*c* ∗ *a*) = *gcd b c*
⟨*proof*⟩

lemma *gcd-div-unit1*: *is-unit a* ⟹ *gcd* (*b div a*) *c* = *gcd b c*
⟨*proof*⟩

lemma *gcd-div-unit2*: *is-unit a* ⟹ *gcd b* (*c div a*) = *gcd b c*
⟨*proof*⟩

lemma *normalize-gcd-left*: *gcd* (*normalize a*) *b* = *gcd a b*
⟨*proof*⟩

lemma *normalize-gcd-right*: *gcd a* (*normalize b*) = *gcd a b*
⟨*proof*⟩

lemma *gcd-add1* [*simp*]: *gcd* (*m* + *n*) *n* = *gcd m n*
⟨*proof*⟩

lemma *gcd-add2* [*simp*]: *gcd m* (*m* + *n*) = *gcd m n*
⟨*proof*⟩

lemma *gcd-add-mult*: *gcd m* (*k* ∗ *m* + *n*) = *gcd m n*
⟨*proof*⟩

end

class *ring-gcd* = *comm-ring-1* + *semiring-gcd*
begin

lemma *gcd-neg1* [*simp*]: *gcd* (−*a*) *b* = *gcd a b*
⟨*proof*⟩

lemma *gcd-neg2* [*simp*]: *gcd a* (−*b*) = *gcd a b*
⟨*proof*⟩

lemma *gcd-neg-numeral-1* [*simp*]: *gcd* (− *numeral n*) *a* = *gcd* (*numeral n*) *a*
⟨*proof*⟩

lemma *gcd-neg-numeral-2* [*simp*]: *gcd a* (− *numeral n*) = *gcd a* (*numeral n*)
⟨*proof*⟩

lemma *gcd-diff1*: *gcd* (*m* − *n*) *n* = *gcd m n*
⟨*proof*⟩

lemma *gcd-diff2*: *gcd* (*n* − *m*) *n* = *gcd m n*
⟨*proof*⟩

lemma *lcm-neg1* [*simp*]: *lcm* (−*a*) *b* = *lcm a b*
⟨*proof*⟩

lemma *lcm-neg2* [*simp*]: *lcm a* (−*b*) = *lcm a b*
⟨*proof*⟩

lemma *lcm-neg-numeral-1* [*simp*]: *lcm* (− *numeral n*) *a* = *lcm* (*numeral n*) *a*
⟨*proof*⟩

lemma *lcm-neg-numeral-2* [*simp*]: *lcm a* (− *numeral n*) = *lcm a* (*numeral n*)
⟨*proof*⟩

end

class *semiring-Gcd* = *semiring-gcd* + *Gcd* +
assumes *Gcd-dvd*: *a* ∈ *A* ⟹ *Gcd A dvd a*
and *Gcd-greatest*: (⋀*b*. *b* ∈ *A* ⟹ *a dvd b*) ⟹ *a dvd Gcd A*
and *normalize-Gcd* [*simp*]: *normalize* (*Gcd A*) = *Gcd A*
assumes *dvd-Lcm*: *a* ∈ *A* ⟹ *a dvd Lcm A*
and *Lcm-least*: (⋀*b*. *b* ∈ *A* ⟹ *b dvd a*) ⟹ *Lcm A dvd a*
and *normalize-Lcm* [*simp*]: *normalize* (*Lcm A*) = *Lcm A*
begin

lemma *Lcm-Gcd*: *Lcm A* = *Gcd* {*b*. ∀ *a*∈*A*. *a dvd b*}
⟨*proof*⟩

lemma *Gcd-Lcm*: *Gcd A* = *Lcm* {*b*. ∀ *a*∈*A*. *b dvd a*}
⟨*proof*⟩

lemma *Gcd-empty* [*simp*]: *Gcd* {} = *0*
⟨*proof*⟩

lemma *Lcm-empty* [*simp*]: *Lcm* {} = *1*
⟨*proof*⟩

lemma *Gcd-insert* [*simp*]: *Gcd* (*insert a A*) = *gcd a* (*Gcd A*)
⟨*proof*⟩

lemma *Lcm-insert* [*simp*]: *Lcm* (*insert a A*) = *lcm a* (*Lcm A*)
⟨*proof*⟩

lemma *LcmI*:
assumes ⋀*a*. *a* ∈ *A* ⟹ *a dvd b*
and ⋀*c*. (⋀*a*. *a* ∈ *A* ⟹ *a dvd c*) ⟹ *b dvd c*
and *normalize b* = *b*
shows *b* = *Lcm A*
⟨*proof*⟩

lemma *Lcm-subset*: *A* ⊆ *B* ⟹ *Lcm A dvd Lcm B*
⟨*proof*⟩

lemma *Lcm-Un*: *Lcm* (*A* ∪ *B*) = *lcm* (*Lcm A*) (*Lcm B*)
⟨*proof*⟩

lemma *Gcd-0-iff* [*simp*]: *Gcd A* = *0* ⟷ *A* ⊆ {*0*}
(**is** *?P* ⟷ *?Q*)
⟨*proof*⟩

lemma *Lcm-1-iff* [*simp*]: *Lcm A* = *1* ⟷ (∀ *a*∈*A*. *is-unit a*)
(**is** *?P* ⟷ *?Q*)
⟨*proof*⟩

lemma *unit-factor-Lcm*: *unit-factor* (*Lcm A*) = (*if Lcm A* = *0 then 0 else 1*)
⟨*proof*⟩

lemma *unit-factor-Gcd*: *unit-factor* (*Gcd A*) = (*if Gcd A* = *0 then 0 else 1*)
⟨*proof*⟩

lemma *GcdI*:
assumes ⋀*a*. *a* ∈ *A* ⟹ *b dvd a*
and ⋀*c*. (⋀*a*. *a* ∈ *A* ⟹ *c dvd a*) ⟹ *c dvd b*
and *normalize b* = *b*
shows *b* = *Gcd A*
⟨*proof*⟩

lemma *Gcd-eq-1-I*:
assumes *is-unit a* **and** *a* ∈ *A*
shows *Gcd A* = *1*
⟨*proof*⟩

lemma *Lcm-eq-0-I*:
assumes *0* ∈ *A*
shows *Lcm A* = *0*
⟨*proof*⟩

lemma *Gcd-UNIV* [*simp*]: *Gcd UNIV* = *1*

⟨*proof*⟩

lemma *Lcm-UNIV* [*simp*]: *Lcm UNIV = 0*
⟨*proof*⟩

lemma *Lcm-0-iff*:
assumes *finite A*
shows *Lcm A = 0 ⟷ 0 ∈ A*
⟨*proof*⟩

lemma *Gcd-image-normalize* [*simp*]: *Gcd (normalize ‘ A) = Gcd A*
⟨*proof*⟩

lemma *Gcd-eqI*:
assumes *normalize a = a*
assumes ⋀*b*. *b ∈ A ⟹ a dvd b*
and ⋀*c*. (⋀*b*. *b ∈ A ⟹ c dvd b*) *⟹ c dvd a*
shows *Gcd A = a*
⟨*proof*⟩

lemma *dvd-GcdD*: *x dvd Gcd A ⟹ y ∈ A ⟹ x dvd y*
⟨*proof*⟩

lemma *dvd-Gcd-iff*: *x dvd Gcd A ⟷* (∀ *y*∈*A*. *x dvd y*)
⟨*proof*⟩

lemma *Gcd-mult*: *Gcd* ((∗) *c ‘ A*) = *normalize* (*c ∗ Gcd A*)
⟨*proof*⟩

lemma *Lcm-eqI*:
assumes *normalize a = a*
and ⋀*b*. *b ∈ A ⟹ b dvd a*
and ⋀*c*. (⋀*b*. *b ∈ A ⟹ b dvd c*) *⟹ a dvd c*
shows *Lcm A = a*
⟨*proof*⟩

lemma *Lcm-dvdD*: *Lcm A dvd x ⟹ y ∈ A ⟹ y dvd x*
⟨*proof*⟩

lemma *Lcm-dvd-iff*: *Lcm A dvd x ⟷* (∀ *y*∈*A*. *y dvd x*)
⟨*proof*⟩

lemma *Lcm-mult*:
assumes *A ≠ {}*
shows *Lcm* ((∗) *c ‘ A*) = *normalize* (*c ∗ Lcm A*)
⟨*proof*⟩

lemma *Lcm-no-units*: *Lcm A = Lcm* (*A − {a*. *is-unit a}*)
⟨*proof*⟩

lemma *Lcm-0-iff'*: *Lcm A = 0 ⟷ (∄ l. l ≠ 0 ∧ (∀ a∈A. a dvd l))*
⟨*proof*⟩

lemma *Lcm-no-multiple*: *(∀ m. m ≠ 0 ⟶ (∃ a∈A. ¬ a dvd m)) ⟹ Lcm A = 0*
⟨*proof*⟩

lemma *Lcm-singleton* [*simp*]: *Lcm {a} = normalize a*
⟨*proof*⟩

lemma *Lcm-2* [*simp*]: *Lcm {a, b} = lcm a b*
⟨*proof*⟩

lemma *Gcd-1*: *1 ∈ A ⟹ Gcd A = 1*
⟨*proof*⟩

lemma *Gcd-singleton* [*simp*]: *Gcd {a} = normalize a*
⟨*proof*⟩

lemma *Gcd-2* [*simp*]: *Gcd {a, b} = gcd a b*
⟨*proof*⟩

lemma *Gcd-mono*:
assumes *⋀x. x ∈ A ⟹ f x dvd g x*
shows *(GCD x∈A. f x) dvd (GCD x∈A. g x)*
⟨*proof*⟩

lemma *Lcm-mono*:
assumes *⋀x. x ∈ A ⟹ f x dvd g x*
shows *(LCM x∈A. f x) dvd (LCM x∈A. g x)*
⟨*proof*⟩

end

88.3 An aside: GCD and LCM on finite sets for incomplete gcd rings

context *semiring-gcd*
begin

sublocale *Gcd-fin*: *bounded-quasi-semilattice-set gcd 0 1 normalize*
defines
Gcd-fin (*Gcd_{fin}*) = *Gcd-fin.F* :: *'a set ⇒ 'a* ⟨*proof*⟩

abbreviation *gcd-list* :: *'a list ⇒ 'a*
where *gcd-list xs ≡ Gcd_{fin} (set xs)*

sublocale *Lcm-fin*: *bounded-quasi-semilattice-set lcm 1 0 normalize*
defines

Lcm-fin (Lcm_{fin}) = *Lcm-fin.F* ⟨*proof*⟩

abbreviation *lcm-list* :: *'a list* ⇒ *'a*
where *lcm-list xs* ≡ Lcm_{fin} (*set xs*)

lemma *Gcd-fin-dvd*:
a ∈ *A* ⟹ Gcd_{fin} *A dvd a*
⟨*proof*⟩

lemma *dvd-Lcm-fin*:
a ∈ *A* ⟹ *a dvd* Lcm_{fin} *A*
⟨*proof*⟩

lemma *Gcd-fin-greatest*:
a dvd Gcd_{fin} *A* **if** *finite A* **and** ⋀*b*. *b* ∈ *A* ⟹ *a dvd b*
⟨*proof*⟩

lemma *Lcm-fin-least*:
Lcm_{fin} *A dvd a* **if** *finite A* **and** ⋀*b*. *b* ∈ *A* ⟹ *b dvd a*
⟨*proof*⟩

lemma *gcd-list-greatest*:
a dvd gcd-list bs **if** ⋀*b*. *b* ∈ *set bs* ⟹ *a dvd b*
⟨*proof*⟩

lemma *lcm-list-least*:
lcm-list bs dvd a **if** ⋀*b*. *b* ∈ *set bs* ⟹ *b dvd a*
⟨*proof*⟩

lemma *dvd-Gcd-fin-iff*:
b dvd Gcd_{fin} *A* ⟷ (∀ *a*∈*A*. *b dvd a*) **if** *finite A*
⟨*proof*⟩

lemma *dvd-gcd-list-iff*:
b dvd gcd-list xs ⟷ (∀ *a*∈*set xs*. *b dvd a*)
⟨*proof*⟩

lemma *Lcm-fin-dvd-iff*:
Lcm_{fin} *A dvd b* ⟷ (∀ *a*∈*A*. *a dvd b*) **if** *finite A*
⟨*proof*⟩

lemma *lcm-list-dvd-iff*:
lcm-list xs dvd b ⟷ (∀ *a*∈*set xs*. *a dvd b*)
⟨*proof*⟩

lemma *Gcd-fin-mult*:
Gcd_{fin} (*image* (*times b*) *A*) = *normalize* (*b* ∗ Gcd_{fin} *A*) **if** *finite A*
⟨*proof*⟩

lemma *Lcm-fin-mult*:
$Lcm_{fin}\ (image\ (times\ b)\ A) = normalize\ (b * Lcm_{fin}\ A)$ **if** $A \neq \{\}$
⟨*proof*⟩

lemma *unit-factor-Gcd-fin*:
$unit\text{-}factor\ (Gcd_{fin}\ A) = of\text{-}bool\ (Gcd_{fin}\ A \neq 0)$
⟨*proof*⟩

lemma *unit-factor-Lcm-fin*:
$unit\text{-}factor\ (Lcm_{fin}\ A) = of\text{-}bool\ (Lcm_{fin}\ A \neq 0)$
⟨*proof*⟩

lemma *is-unit-Gcd-fin-iff* [*simp*]:
$is\text{-}unit\ (Gcd_{fin}\ A) \longleftrightarrow Gcd_{fin}\ A = 1$
⟨*proof*⟩

lemma *is-unit-Lcm-fin-iff* [*simp*]:
$is\text{-}unit\ (Lcm_{fin}\ A) \longleftrightarrow Lcm_{fin}\ A = 1$
⟨*proof*⟩

lemma *Gcd-fin-0-iff*:
$Gcd_{fin}\ A = 0 \longleftrightarrow A \subseteq \{0\} \wedge finite\ A$
⟨*proof*⟩

lemma *Lcm-fin-0-iff*:
$Lcm_{fin}\ A = 0 \longleftrightarrow 0 \in A$ **if** *finite A*
⟨*proof*⟩

lemma *Lcm-fin-1-iff*:
$Lcm_{fin}\ A = 1 \longleftrightarrow (\forall\, a{\in}A.\ is\text{-}unit\ a) \wedge finite\ A$
⟨*proof*⟩

end

context *semiring-Gcd*
begin

lemma *Gcd-fin-eq-Gcd* [*simp*]:
$Gcd_{fin}\ A = Gcd\ A$ **if** *finite A* **for** $A :: 'a\ set$
⟨*proof*⟩

lemma *Gcd-set-eq-fold* [*code-unfold*]:
$Gcd\ (set\ xs) = fold\ gcd\ xs\ 0$
⟨*proof*⟩

lemma *Lcm-fin-eq-Lcm* [*simp*]:
$Lcm_{fin}\ A = Lcm\ A$ **if** *finite A* **for** $A :: 'a\ set$
⟨*proof*⟩

lemma *Lcm-set-eq-fold* [*code-unfold*]:
Lcm (*set xs*) = *fold lcm xs 1*
⟨*proof*⟩

end

88.4 Coprimality

context *semiring-gcd*
begin

lemma *coprime-imp-gcd-eq-1* [*simp*]:
gcd a b = *1* **if** *coprime a b*
⟨*proof*⟩

lemma *gcd-eq-1-imp-coprime* [*dest*!]:
coprime a b **if** *gcd a b* = *1*
⟨*proof*⟩

lemma *coprime-iff-gcd-eq-1* [*presburger*, *code*]:
coprime a b ⟷ *gcd a b* = *1*
⟨*proof*⟩

lemma *is-unit-gcd* [*simp*]:
is-unit (*gcd a b*) ⟷ *coprime a b*
⟨*proof*⟩

lemma *coprime-add-one-left* [*simp*]: *coprime* (*a* + *1*) *a*
⟨*proof*⟩

lemma *coprime-add-one-right* [*simp*]: *coprime a* (*a* + *1*)
⟨*proof*⟩

lemma *coprime-mult-left-iff* [*simp*]:
coprime (*a* ∗ *b*) *c* ⟷ *coprime a c* ∧ *coprime b c*
⟨*proof*⟩

lemma *coprime-mult-right-iff* [*simp*]:
coprime c (*a* ∗ *b*) ⟷ *coprime c a* ∧ *coprime c b*
⟨*proof*⟩

lemma *coprime-power-left-iff* [*simp*]:
coprime (*a* ^ *n*) *b* ⟷ *coprime a b* ∨ *n* = *0*
⟨*proof*⟩

lemma *coprime-power-right-iff* [*simp*]:
coprime a (*b* ^ *n*) ⟷ *coprime a b* ∨ *n* = *0*
⟨*proof*⟩

lemma *prod-coprime-left*:
coprime $(\prod i{\in}A.\ f\ i)\ a$ **if** $\bigwedge i.\ i \in A \Longrightarrow$ *coprime* $(f\ i)\ a$
⟨*proof*⟩

lemma *prod-coprime-right*:
coprime $a\ (\prod i{\in}A.\ f\ i)$ **if** $\bigwedge i.\ i \in A \Longrightarrow$ *coprime* $a\ (f\ i)$
⟨*proof*⟩

lemma *prod-list-coprime-left*:
coprime (*prod-list xs*) a **if** $\bigwedge x.\ x \in$ *set xs* $\Longrightarrow$ *coprime* $x\ a$
⟨*proof*⟩

lemma *prod-list-coprime-right*:
coprime a (*prod-list xs*) **if** $\bigwedge x.\ x \in$ *set xs* $\Longrightarrow$ *coprime* $a\ x$
⟨*proof*⟩

lemma *coprime-dvd-mult-left-iff*:
a *dvd* $b * c \longleftrightarrow a$ *dvd* b **if** *coprime* $a\ c$
⟨*proof*⟩

lemma *coprime-dvd-mult-right-iff*:
a *dvd* $c * b \longleftrightarrow a$ *dvd* b **if** *coprime* $a\ c$
⟨*proof*⟩

lemma *divides-mult*:
$a * b$ *dvd* c **if** a *dvd* c **and** b *dvd* c **and** *coprime* $a\ b$
⟨*proof*⟩

lemma *div-gcd-coprime*:
assumes $a \neq 0 \vee b \neq 0$
shows *coprime* $(a\ div\ gcd\ a\ b)\ (b\ div\ gcd\ a\ b)$
⟨*proof*⟩

lemma *gcd-coprime*:
assumes *c*: $gcd\ a\ b \neq 0$
and *a*: $a = a' * gcd\ a\ b$
and *b*: $b = b' * gcd\ a\ b$
shows *coprime* $a'\ b'$
⟨*proof*⟩

lemma *gcd-coprime-exists*:
assumes $gcd\ a\ b \neq 0$
shows $\exists a'\ b'.\ a = a' * gcd\ a\ b \wedge b = b' * gcd\ a\ b \wedge$ *coprime* $a'\ b'$
⟨*proof*⟩

lemma *pow-divides-pow-iff* [*simp*]:
$a \mathbin{\hat{}} n$ *dvd* $b \mathbin{\hat{}} n \longleftrightarrow a$ *dvd* b **if** $n > 0$
⟨*proof*⟩

lemma *coprime-crossproduct*:
fixes *a b c d* :: *'a*
assumes *coprime a d* **and** *coprime b c*
shows *normalize a * normalize c = normalize b * normalize d ⟷*
normalize a = normalize b ∧ normalize c = normalize d
(**is** *?lhs ⟷ ?rhs*)
⟨*proof*⟩

lemma *gcd-mult-left-left-cancel*:
*gcd (c * a) b = gcd a b* **if** *coprime b c*
⟨*proof*⟩

lemma *gcd-mult-left-right-cancel*:
*gcd (a * c) b = gcd a b* **if** *coprime b c*
⟨*proof*⟩

lemma *gcd-mult-right-left-cancel*:
*gcd a (c * b) = gcd a b* **if** *coprime a c*
⟨*proof*⟩

lemma *gcd-mult-right-right-cancel*:
*gcd a (b * c) = gcd a b* **if** *coprime a c*
⟨*proof*⟩

lemma *gcd-exp-weak*:
gcd (a ^ n) (b ^ n) = normalize (gcd a b ^ n)
⟨*proof*⟩

lemma *division-decomp*:
assumes *a dvd b * c*
shows *∃ b' c'. a = b' * c' ∧ b' dvd b ∧ c' dvd c*
⟨*proof*⟩

lemma *lcm-coprime*: *coprime a b ⟹ lcm a b = normalize (a * b)*
⟨*proof*⟩

end

context *ring-gcd*
begin

lemma *coprime-minus-left-iff* [*simp*]:
coprime (− a) b ⟷ coprime a b
⟨*proof*⟩

lemma *coprime-minus-right-iff* [*simp*]:
coprime a (− b) ⟷ coprime a b
⟨*proof*⟩

lemma *coprime-diff-one-left* [*simp*]: *coprime* (*a* − *1*) *a*
⟨*proof*⟩

lemma *coprime-doff-one-right* [*simp*]: *coprime a* (*a* − *1*)
⟨*proof*⟩

end

context *semiring-Gcd*
begin

lemma *Lcm-coprime*:
assumes *finite A*
and $A \neq \{\}$
and $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow a \neq b \Longrightarrow$ *coprime a b*
shows *Lcm A* = *normalize* ($\prod A$)
⟨*proof*⟩

lemma *Lcm-coprime′*:
card A ≠ *0* ⟹
($\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow a \neq b \Longrightarrow$ *coprime a b*) ⟹
Lcm A = *normalize* ($\prod A$)
⟨*proof*⟩

end

And some consequences: cancellation modulo *m*

lemma *mult-mod-cancel-right*:
fixes *m* :: *'a*::{*euclidean-ring-cancel*,*semiring-gcd*}
assumes *eq*: (*a* ∗ *n*) *mod m* = (*b* ∗ *n*) *mod m* **and** *coprime m n*
shows *a mod m* = *b mod m*
⟨*proof*⟩

lemma *mult-mod-cancel-left*:
fixes *m* :: *'a*::{*euclidean-ring-cancel*,*semiring-gcd*}
assumes (*n* ∗ *a*) *mod m* = (*n* ∗ *b*) *mod m* **and** *coprime m n*
shows *a mod m* = *b mod m*
⟨*proof*⟩

88.5 GCD and LCM for multiplicative normalisation functions

class *semiring-gcd-mult-normalize* = *semiring-gcd* + *normalization-semidom-multiplicative*
begin

lemma *mult-gcd-left*: *c* ∗ *gcd a b* = *unit-factor c* ∗ *gcd* (*c* ∗ *a*) (*c* ∗ *b*)
⟨*proof*⟩

lemma *mult-gcd-right*: *gcd a b* ∗ *c* = *gcd* (*a* ∗ *c*) (*b* ∗ *c*) ∗ *unit-factor c*

⟨*proof*⟩

lemma *gcd-mult-distrib'*: *normalize c * gcd a b = gcd (c * a) (c * b)*
⟨*proof*⟩

lemma *gcd-mult-distrib*: *k * gcd a b = gcd (k * a) (k * b) * unit-factor k*
⟨*proof*⟩

lemma *gcd-mult-lcm* [*simp*]: *gcd a b * lcm a b = normalize a * normalize b*
⟨*proof*⟩

lemma *lcm-mult-gcd* [*simp*]: *lcm a b * gcd a b = normalize a * normalize b*
⟨*proof*⟩

lemma *mult-lcm-left*: *c * lcm a b = unit-factor c * lcm (c * a) (c * b)*
⟨*proof*⟩

lemma *mult-lcm-right*: *lcm a b * c = lcm (a * c) (b * c) * unit-factor c*
⟨*proof*⟩

lemma *lcm-gcd-prod*: *lcm a b * gcd a b = normalize (a * b)*
⟨*proof*⟩

lemma *lcm-mult-distrib'*: *normalize c * lcm a b = lcm (c * a) (c * b)*
⟨*proof*⟩

lemma *lcm-mult-distrib*: *k * lcm a b = lcm (k * a) (k * b) * unit-factor k*
⟨*proof*⟩

lemma *coprime-crossproduct'*:
fixes *a b c d*
assumes *b ≠ 0*
assumes *unit-factors*: *unit-factor b = unit-factor d*
assumes *coprime*: *coprime a b coprime c d*
shows *a * d = b * c ⟷ a = c ∧ b = d*
⟨*proof*⟩

lemma *gcd-exp* [*simp*]:
gcd (a ^ n) (b ^ n) = gcd a b ^ n
⟨*proof*⟩

end

88.6 GCD and LCM on *nat* and *int*

instantiation *nat* :: *gcd*
begin

fun *gcd-nat* :: *nat ⇒ nat ⇒ nat*

where *gcd-nat x y* = (*if y* = *0 then x else gcd y* (*x mod y*))

definition *lcm-nat* :: *nat* ⇒ *nat* ⇒ *nat*
where *lcm-nat x y* = *x* ∗ *y div* (*gcd x y*)

instance ⟨*proof*⟩

end

instantiation *int* :: *gcd*
begin

definition *gcd-int* :: *int* ⇒ *int* ⇒ *int*
where *gcd-int x y* = *int* (*gcd* (*nat* |*x*|) (*nat* |*y*|))

definition *lcm-int* :: *int* ⇒ *int* ⇒ *int*
where *lcm-int x y* = *int* (*lcm* (*nat* |*x*|) (*nat* |*y*|))

instance ⟨*proof*⟩

end

lemma *gcd-int-int-eq* [*simp*]:
gcd (*int m*) (*int n*) = *int* (*gcd m n*)
⟨*proof*⟩

lemma *gcd-nat-abs-left-eq* [*simp*]:
gcd (*nat* |*k*|) *n* = *nat* (*gcd k* (*int n*))
⟨*proof*⟩

lemma *gcd-nat-abs-right-eq* [*simp*]:
gcd n (*nat* |*k*|) = *nat* (*gcd* (*int n*) *k*)
⟨*proof*⟩

lemma *abs-gcd-int* [*simp*]:
|*gcd x y*| = *gcd x y*
for *x y* :: *int*
⟨*proof*⟩

lemma *gcd-abs1-int* [*simp*]:
gcd |*x*| *y* = *gcd x y*
for *x y* :: *int*
⟨*proof*⟩

lemma *gcd-abs2-int* [*simp*]:
gcd x |*y*| = *gcd x y*
for *x y* :: *int*
⟨*proof*⟩

lemma *lcm-int-int-eq* [*simp*]:
lcm (*int m*) (*int n*) = *int* (*lcm m n*)
⟨*proof*⟩

lemma *lcm-nat-abs-left-eq* [*simp*]:
lcm (*nat* |*k*|) *n* = *nat* (*lcm k* (*int n*))
⟨*proof*⟩

lemma *lcm-nat-abs-right-eq* [*simp*]:
lcm n (*nat* |*k*|) = *nat* (*lcm* (*int n*) *k*)
⟨*proof*⟩

lemma *lcm-abs1-int* [*simp*]:
lcm |*x*| *y* = *lcm x y*
for *x y* :: *int*
⟨*proof*⟩

lemma *lcm-abs2-int* [*simp*]:
lcm x |*y*| = *lcm x y*
for *x y* :: *int*
⟨*proof*⟩

lemma *abs-lcm-int* [*simp*]: |*lcm i j*| = *lcm i j*
for *i j* :: *int*
⟨*proof*⟩

lemma *gcd-nat-induct* [*case-names base step*]:
fixes *m n* :: *nat*
assumes ⋀*m*. *P m* 0
and ⋀*m n*. 0 < *n* ⟹ *P n* (*m mod n*) ⟹ *P m n*
shows *P m n*
⟨*proof*⟩

lemma *gcd-neg1-int* [*simp*]: *gcd* (− *x*) *y* = *gcd x y*
for *x y* :: *int*
⟨*proof*⟩

lemma *gcd-neg2-int* [*simp*]: *gcd x* (− *y*) = *gcd x y*
for *x y* :: *int*
⟨*proof*⟩

lemma *gcd-cases-int*:
fixes *x y* :: *int*
assumes *x* ≥ 0 ⟹ *y* ≥ 0 ⟹ *P* (*gcd x y*)
and *x* ≥ 0 ⟹ *y* ≤ 0 ⟹ *P* (*gcd x* (− *y*))
and *x* ≤ 0 ⟹ *y* ≥ 0 ⟹ *P* (*gcd* (− *x*) *y*)
and *x* ≤ 0 ⟹ *y* ≤ 0 ⟹ *P* (*gcd* (− *x*) (− *y*))
shows *P* (*gcd x y*)
⟨*proof*⟩

lemma *gcd-ge-0-int* [*simp*]: *gcd* (*x*::*int*) *y* >= *0*
for *x y* :: *int*
⟨*proof*⟩

lemma *lcm-neg1-int*: *lcm* (− *x*) *y* = *lcm x y*
for *x y* :: *int*
⟨*proof*⟩

lemma *lcm-neg2-int*: *lcm x* (− *y*) = *lcm x y*
for *x y* :: *int*
⟨*proof*⟩

lemma *lcm-cases-int*:
fixes *x y* :: *int*
assumes $x \geq 0 \Longrightarrow y \geq 0 \Longrightarrow P\ (lcm\ x\ y)$
and $x \geq 0 \Longrightarrow y \leq 0 \Longrightarrow P\ (lcm\ x\ (-\ y))$
and $x \leq 0 \Longrightarrow y \geq 0 \Longrightarrow P\ (lcm\ (-\ x)\ y)$
and $x \leq 0 \Longrightarrow y \leq 0 \Longrightarrow P\ (lcm\ (-\ x)\ (-\ y))$
shows *P* (*lcm x y*)
⟨*proof*⟩

lemma *lcm-ge-0-int* [*simp*]: $lcm\ x\ y \geq 0$
for *x y* :: *int*
⟨*proof*⟩

lemma *gcd-0-nat*: *gcd x 0* = *x*
for *x* :: *nat*
⟨*proof*⟩

lemma *gcd-0-int* [*simp*]: *gcd x 0* = |*x*|
for *x* :: *int*
⟨*proof*⟩

lemma *gcd-0-left-nat*: *gcd 0 x* = *x*
for *x* :: *nat*
⟨*proof*⟩

lemma *gcd-0-left-int* [*simp*]: *gcd 0 x* = |*x*|
for *x* :: *int*
⟨*proof*⟩

lemma *gcd-red-nat*: *gcd x y* = *gcd y* (*x mod y*)
for *x y* :: *nat*
⟨*proof*⟩

Weaker, but useful for the simplifier.

lemma *gcd-non-0-nat*: $y \neq 0 \Longrightarrow gcd\ x\ y = gcd\ y\ (x\ mod\ y)$
for *x y* :: *nat*

⟨*proof*⟩

lemma *gcd-1-nat* [*simp*]: *gcd m 1 = 1*
for *m* :: *nat*
⟨*proof*⟩

lemma *gcd-Suc-0* [*simp*]: *gcd m* (*Suc 0*) = *Suc 0*
for *m* :: *nat*
⟨*proof*⟩

lemma *gcd-1-int* [*simp*]: *gcd m 1 = 1*
for *m* :: *int*
⟨*proof*⟩

lemma *gcd-idem-nat*: *gcd x x = x*
for *x* :: *nat*
⟨*proof*⟩

lemma *gcd-idem-int*: *gcd x x* = |*x*|
for *x* :: *int*
⟨*proof*⟩

declare *gcd-nat.simps* [*simp del*]

gcd m n divides *m* and *n*. The conjunctions don't seem provable separately.

instance *nat* :: *semiring-gcd*
⟨*proof*⟩

instance *int* :: *ring-gcd*
⟨*proof*⟩

lemma *gcd-le1-nat* [*simp*]: *a* ≠ *0* ⟹ *gcd a b* ≤ *a*
for *a b* :: *nat*
⟨*proof*⟩

lemma *gcd-le2-nat* [*simp*]: *b* ≠ *0* ⟹ *gcd a b* ≤ *b*
for *a b* :: *nat*
⟨*proof*⟩

lemma *gcd-le1-int* [*simp*]: *a* > *0* ⟹ *gcd a b* ≤ *a*
for *a b* :: *int*
⟨*proof*⟩

lemma *gcd-le2-int* [*simp*]: *b* > *0* ⟹ *gcd a b* ≤ *b*
for *a b* :: *int*
⟨*proof*⟩

lemma *gcd-pos-nat* [*simp*]: *gcd m n* > *0* ⟷ *m* ≠ *0* ∨ *n* ≠ *0*
for *m n* :: *nat*

⟨*proof*⟩

lemma *gcd-pos-int* [*simp*]: *gcd m n* > *0* ⟷ *m* ≠ *0* ∨ *n* ≠ *0*
for *m n* :: *int*
⟨*proof*⟩

lemma *gcd-unique-nat*: *d dvd a* ∧ *d dvd b* ∧ (∀ *e*. *e dvd a* ∧ *e dvd b* ⟶ *e dvd d*) ⟷ *d* = *gcd a b*
for *d a* :: *nat*
⟨*proof*⟩

lemma *gcd-unique-int*:
d ≥ *0* ∧ *d dvd a* ∧ *d dvd b* ∧ (∀ *e*. *e dvd a* ∧ *e dvd b* ⟶ *e dvd d*) ⟷ *d* = *gcd a b*
for *d a* :: *int*
⟨*proof*⟩

interpretation *gcd-nat*:
semilattice-neutr-order gcd 0::*nat Rings.dvd* λ*m n*. *m dvd n* ∧ *m* ≠ *n*
⟨*proof*⟩

lemma *gcd-proj1-if-dvd-int* [*simp*]: *x dvd y* ⟹ *gcd x y* = |*x*|
for *x y* :: *int*
⟨*proof*⟩

lemma *gcd-proj2-if-dvd-int* [*simp*]: *y dvd x* ⟹ *gcd x y* = |*y*|
for *x y* :: *int*
⟨*proof*⟩

Multiplication laws.

lemma *gcd-mult-distrib-nat*: *k* ∗ *gcd m n* = *gcd* (*k* ∗ *m*) (*k* ∗ *n*)
for *k m n* :: *nat*
— [1, page 27]
⟨*proof*⟩

lemma *gcd-mult-distrib-int*: |*k*| ∗ *gcd m n* = *gcd* (*k* ∗ *m*) (*k* ∗ *n*)
for *k m n* :: *int*
⟨*proof*⟩

Addition laws.

lemma *gcd-diff1-nat*: *m* ≥ *n* ⟹ *gcd* (*m* − *n*) *n* = *gcd m n*
for *m n* :: *nat*
⟨*proof*⟩

lemma *gcd-diff2-nat*: *n* ≥ *m* ⟹ *gcd* (*n* − *m*) *n* = *gcd m n*
for *m n* :: *nat*
⟨*proof*⟩

lemma *gcd-non-0-int*:
fixes *x y* :: *int*
assumes *y* > *0* **shows** *gcd x y* = *gcd y* (*x mod y*)
⟨*proof*⟩

lemma *gcd-red-int*: *gcd x y* = *gcd y* (*x mod y*)
for *x y* :: *int*
⟨*proof*⟩

lemma *finite-divisors-nat* [*simp*]:
fixes *m* :: *nat*
assumes *m* > *0*
shows *finite* {*d*. *d dvd m*}
⟨*proof*⟩

lemma *finite-divisors-int* [*simp*]:
fixes *i* :: *int*
assumes *i* ≠ *0*
shows *finite* {*d*. *d dvd i*}
⟨*proof*⟩

lemma *Max-divisors-self-nat* [*simp*]: *n* ≠ *0* ⟹ *Max* {*d*::*nat*. *d dvd n*} = *n*
⟨*proof*⟩

lemma *Max-divisors-self-int* [*simp*]:
assumes *n* ≠ *0* **shows** *Max* {*d*::*int*. *d dvd n*} = |*n*|
⟨*proof*⟩

lemma *gcd-is-Max-divisors-nat*:
fixes *m n* :: *nat*
assumes *n* > *0* **shows** *gcd m n* = *Max* {*d*. *d dvd m* ∧ *d dvd n*}
⟨*proof*⟩

lemma *gcd-is-Max-divisors-int*:
fixes *m n* :: *int*
assumes *n* ≠ *0* **shows** *gcd m n* = *Max* {*d*. *d dvd m* ∧ *d dvd n*}
⟨*proof*⟩

lemma *gcd-code-int* [*code*]: *gcd k l* = |*if l* = *0 then k else gcd l* (|*k*| *mod* |*l*|)|
for *k l* :: *int*
⟨*proof*⟩

lemma *coprime-Suc-left-nat* [*simp*]:
coprime (*Suc n*) *n*
⟨*proof*⟩

lemma *coprime-Suc-right-nat* [*simp*]:
coprime n (*Suc n*)
⟨*proof*⟩

lemma *coprime-diff-one-left-nat* [*simp*]:
coprime (*n* − *1*) *n* **if** *n* > *0* **for** *n* :: *nat*
⟨*proof*⟩

lemma *coprime-diff-one-right-nat* [*simp*]:
coprime n (*n* − *1*) **if** *n* > *0* **for** *n* :: *nat*
⟨*proof*⟩

lemma *coprime-crossproduct-nat*:
fixes *a b c d* :: *nat*
assumes *coprime a d* **and** *coprime b c*
shows *a* ∗ *c* = *b* ∗ *d* ⟷ *a* = *b* ∧ *c* = *d*
⟨*proof*⟩

lemma *coprime-crossproduct-int*:
fixes *a b c d* :: *int*
assumes *coprime a d* **and** *coprime b c*
shows |*a*| ∗ |*c*| = |*b*| ∗ |*d*| ⟷ |*a*| = |*b*| ∧ |*c*| = |*d*|
⟨*proof*⟩

88.7 Bezout's theorem

Function *bezw* returns a pair of witnesses to Bezout's theorem – see the theorems that follow the definition.

fun *bezw* :: *nat* ⇒ *nat* ⇒ *int* ∗ *int*
where *bezw x y* =
(*if y* = *0 then* (*1*, *0*)
else
(*snd* (*bezw y* (*x mod y*)),
fst (*bezw y* (*x mod y*)) − *snd* (*bezw y* (*x mod y*)) ∗ *int*(*x div y*)))

lemma *bezw-0* [*simp*]: *bezw x 0* = (*1*, *0*)
⟨*proof*⟩

lemma *bezw-non-0*:
y > *0* ⟹ *bezw x y* =
(*snd* (*bezw y* (*x mod y*)), *fst* (*bezw y* (*x mod y*)) − *snd* (*bezw y* (*x mod y*)) ∗ *int*(*x div y*))
⟨*proof*⟩

declare *bezw.simps* [*simp del*]

lemma *bezw-aux*: *int* (*gcd x y*) = *fst* (*bezw x y*) ∗ *int x* + *snd* (*bezw x y*) ∗ *int y*

⟨*proof*⟩

lemma *bezout-int*: $\exists\, u\ v.\ u * x + v * y = gcd\ x\ y$
  for $x\ y$ :: *int*
⟨*proof*⟩

Versions of Bezout for *nat*, by Amine Chaieb.

lemma *Euclid-induct* [*case-names swap zero add*]:
  fixes P :: $nat \Rightarrow nat \Rightarrow bool$
  assumes *c*: $\bigwedge a\ b.\ P\ a\ b \longleftrightarrow P\ b\ a$
    and *z*: $\bigwedge a.\ P\ a\ 0$
    and *add*: $\bigwedge a\ b.\ P\ a\ b \longrightarrow P\ a\ (a + b)$
  shows $P\ a\ b$
⟨*proof*⟩

lemma *bezout-lemma-nat*:
  fixes *d*::*nat*
  shows $[\![d\ dvd\ a;\ d\ dvd\ b;\ a * x = b * y + d \vee b * x = a * y + d]\!]$
    $\Longrightarrow \exists\, x\ y.\ d\ dvd\ a \wedge d\ dvd\ a + b \wedge (a * x = (a + b) * y + d \vee (a + b) * x = a * y + d)$
  ⟨*proof*⟩

lemma *bezout-add-nat*:
  $\exists (d{::}nat)\ x\ y.\ d\ dvd\ a \wedge d\ dvd\ b \wedge (a * x = b * y + d \vee b * x = a * y + d)$
⟨*proof*⟩

lemma *bezout1-nat*: $\exists (d{::}nat)\ x\ y.\ d\ dvd\ a \wedge d\ dvd\ b \wedge (a * x - b * y = d \vee b * x - a * y = d)$
  ⟨*proof*⟩

lemma *bezout-add-strong-nat*:
  fixes $a\ b$ :: *nat*
  assumes *a*: $a \neq 0$
  shows $\exists\, d\ x\ y.\ d\ dvd\ a \wedge d\ dvd\ b \wedge a * x = b * y + d$
⟨*proof*⟩

lemma *bezout-nat*:
  fixes a :: *nat*
  assumes *a*: $a \neq 0$
  shows $\exists\, x\ y.\ a * x = b * y + gcd\ a\ b$
⟨*proof*⟩

88.8 LCM properties on *nat* and *int*

lemma *lcm-altdef-int* [*code*]: $lcm\ a\ b = |a| * |b|\ div\ gcd\ a\ b$
  for $a\ b$ :: *int*
  ⟨*proof*⟩

lemma *prod-gcd-lcm-nat*: $m * n = gcd\ m\ n * lcm\ m\ n$
for $m\ n :: nat$
⟨*proof*⟩

lemma *prod-gcd-lcm-int*: $|m| * |n| = gcd\ m\ n * lcm\ m\ n$
for $m\ n :: int$
⟨*proof*⟩

lemma *lcm-pos-nat*: $m > 0 \Longrightarrow n > 0 \Longrightarrow lcm\ m\ n > 0$
for $m\ n :: nat$
⟨*proof*⟩

lemma *lcm-pos-int*: $m \neq 0 \Longrightarrow n \neq 0 \Longrightarrow lcm\ m\ n > 0$
for $m\ n :: int$
⟨*proof*⟩

lemma *dvd-pos-nat*: $n > 0 \Longrightarrow m\ dvd\ n \Longrightarrow m > 0$
for $m\ n :: nat$
⟨*proof*⟩

lemma *lcm-unique-nat*:
$a\ dvd\ d \wedge b\ dvd\ d \wedge (\forall e.\ a\ dvd\ e \wedge b\ dvd\ e \longrightarrow d\ dvd\ e) \longleftrightarrow d = lcm\ a\ b$
for $a\ b\ d :: nat$
⟨*proof*⟩

lemma *lcm-unique-int*:
$d \geq 0 \wedge a\ dvd\ d \wedge b\ dvd\ d \wedge (\forall e.\ a\ dvd\ e \wedge b\ dvd\ e \longrightarrow d\ dvd\ e) \longleftrightarrow d = lcm\ a\ b$
for $a\ b\ d :: int$
⟨*proof*⟩

lemma *lcm-proj2-if-dvd-nat* [*simp*]: $x\ dvd\ y \Longrightarrow lcm\ x\ y = y$
for $x\ y :: nat$
⟨*proof*⟩

lemma *lcm-proj2-if-dvd-int* [*simp*]: $x\ dvd\ y \Longrightarrow lcm\ x\ y = |y|$
for $x\ y :: int$
⟨*proof*⟩

lemma *lcm-proj1-if-dvd-nat* [*simp*]: $x\ dvd\ y \Longrightarrow lcm\ y\ x = y$
for $x\ y :: nat$
⟨*proof*⟩

lemma *lcm-proj1-if-dvd-int* [*simp*]: $x\ dvd\ y \Longrightarrow lcm\ y\ x = |y|$
for $x\ y :: int$
⟨*proof*⟩

lemma *lcm-proj1-iff-nat* [*simp*]: $lcm\ m\ n = m \longleftrightarrow n\ dvd\ m$
for $m\ n :: nat$

⟨*proof*⟩

lemma *lcm-proj2-iff-nat* [*simp*]: *lcm m n* = *n* ⟷ *m dvd n*
for *m n* :: *nat*
⟨*proof*⟩

lemma *lcm-proj1-iff-int* [*simp*]: *lcm m n* = |*m*| ⟷ *n dvd m*
for *m n* :: *int*
⟨*proof*⟩

lemma *lcm-proj2-iff-int* [*simp*]: *lcm m n* = |*n*| ⟷ *m dvd n*
for *m n* :: *int*
⟨*proof*⟩

lemma *lcm-1-iff-nat* [*simp*]: *lcm m n* = *Suc 0* ⟷ *m* = *Suc 0* ∧ *n* = *Suc 0*
for *m n* :: *nat*
⟨*proof*⟩

lemma *lcm-1-iff-int* [*simp*]: *lcm m n* = *1* ⟷ (*m* = *1* ∨ *m* = −*1*) ∧ (*n* = *1* ∨ *n* = −*1*)
for *m n* :: *int*
⟨*proof*⟩

88.9 The complete divisibility lattice on *nat* and *int*

Lifting *gcd* and *lcm* to sets (*Gcd* / *Lcm*). *Gcd* is defined via *Lcm* to facilitate the proof that we have a complete lattice.

instantiation *nat* :: *semiring-Gcd*
begin

interpretation *semilattice-neutr-set lcm 1*::*nat*
⟨*proof*⟩

definition *Lcm M* = (*if finite M then F M else 0*) **for** *M* :: *nat set*

lemma *Lcm-nat-empty*: *Lcm* {} = (*1*::*nat*)
⟨*proof*⟩

lemma *Lcm-nat-insert*: *Lcm* (*insert n M*) = *lcm n* (*Lcm M*) **for** *n* :: *nat*
⟨*proof*⟩

lemma *Lcm-nat-infinite*: *infinite M* ⟹ *Lcm M* = *0* **for** *M* :: *nat set*
⟨*proof*⟩

lemma *dvd-Lcm-nat* [*simp*]:
fixes *M* :: *nat set*
assumes *m* ∈ *M*
shows *m dvd Lcm M*
⟨*proof*⟩

lemma *Lcm-dvd-nat* [*simp*]:
fixes M :: *nat set*
assumes $\forall m{\in}M.\ m\ dvd\ n$
shows *Lcm M dvd n*
⟨*proof*⟩

definition *Gcd M* = *Lcm* $\{d.\ \forall m{\in}M.\ d\ dvd\ m\}$ **for** M :: *nat set*

instance
⟨*proof*⟩

end

lemma *Gcd-nat-eq-one*: $1 \in N \Longrightarrow Gcd\ N = 1$
for N :: *nat set*
⟨*proof*⟩

instance *nat* :: *semiring-gcd-mult-normalize*
⟨*proof*⟩

Alternative characterizations of Gcd:

lemma *Gcd-eq-Max*:
fixes M :: *nat set*
assumes *finite* (M::*nat set*) **and** $M \neq \{\}$ **and** $0 \notin M$
shows *Gcd M* = *Max* $(\bigcap m{\in}M.\ \{d.\ d\ dvd\ m\})$
⟨*proof*⟩

lemma *Gcd-remove0-nat*: *finite M* $\Longrightarrow$ *Gcd M* = *Gcd* $(M - \{0\})$
for M :: *nat set*
⟨*proof*⟩

lemma *Lcm-in-lcm-closed-set-nat*:
fixes M :: *nat set*
assumes *finite M* $M \neq \{\}$ $\bigwedge m\ n.\ [\![m \in M;\ n \in M]\!] \Longrightarrow lcm\ m\ n \in M$
shows *Lcm M* $\in M$
⟨*proof*⟩

lemma *Lcm-eq-Max-nat*:
fixes M :: *nat set*
assumes M: *finite M* $M \neq \{\}$ $0 \notin M$ **and** *lcm*: $\bigwedge m\ n.\ [\![m \in M;\ n \in M]\!] \Longrightarrow$ *lcm m n* $\in M$
shows *Lcm M* = *Max M*
⟨*proof*⟩

lemma *mult-inj-if-coprime-nat*:
inj-on f A $\Longrightarrow$ *inj-on g B* $\Longrightarrow$ $(\bigwedge a\ b.\ [\![a{\in}A;\ b{\in}B]\!] \Longrightarrow coprime\ (f\ a)\ (g\ b)) \Longrightarrow$
inj-on $(\lambda(a,\ b).\ f\ a * g\ b)\ (A \times B)$
for f :: $'a \Rightarrow nat$ **and** g :: $'b \Rightarrow nat$

⟨*proof*⟩

88.9.1 Setwise GCD and LCM for integers

instantiation *int* :: *Gcd*
begin

definition *Gcd-int* :: *int set* ⇒ *int*
where *Gcd K* = *int* (*GCD k*∈*K*. (*nat* ∘ *abs*) *k*)

definition *Lcm-int* :: *int set* ⇒ *int*
where *Lcm K* = *int* (*LCM k*∈*K*. (*nat* ∘ *abs*) *k*)

instance ⟨*proof*⟩

end

lemma *Gcd-int-eq* [*simp*]:
(*GCD n*∈*N*. *int n*) = *int* (*Gcd N*)
⟨*proof*⟩

lemma *Gcd-nat-abs-eq* [*simp*]:
(*GCD k*∈*K*. *nat* |*k*|) = *nat* (*Gcd K*)
⟨*proof*⟩

lemma *abs-Gcd-eq* [*simp*]:
|*Gcd K*| = *Gcd K* **for** *K* :: *int set*
⟨*proof*⟩

lemma *Gcd-int-greater-eq-0* [*simp*]:
Gcd K ≥ *0*
for *K* :: *int set*
⟨*proof*⟩

lemma *Gcd-abs-eq* [*simp*]:
(*GCD k*∈*K*. |*k*|) = *Gcd K*
for *K* :: *int set*
⟨*proof*⟩

lemma *Lcm-int-eq* [*simp*]:
(*LCM n*∈*N*. *int n*) = *int* (*Lcm N*)
⟨*proof*⟩

lemma *Lcm-nat-abs-eq* [*simp*]:
(*LCM k*∈*K*. *nat* |*k*|) = *nat* (*Lcm K*)
⟨*proof*⟩

lemma *abs-Lcm-eq* [*simp*]:
|*Lcm K*| = *Lcm K* **for** *K* :: *int set*

⟨*proof*⟩

lemma *Lcm-int-greater-eq-0* [*simp*]:
Lcm K ≥ *0*
for *K* :: *int set*
⟨*proof*⟩

lemma *Lcm-abs-eq* [*simp*]:
(*LCM k*∈*K*. |*k*|) = *Lcm K*
for *K* :: *int set*
⟨*proof*⟩

instance *int* :: *semiring-Gcd*
⟨*proof*⟩

instance *int* :: *semiring-gcd-mult-normalize*
⟨*proof*⟩

88.10 GCD and LCM on *integer*

instantiation *integer* :: *gcd*
begin

context
includes *integer.lifting*
begin

lift-definition *gcd-integer* :: *integer* ⇒ *integer* ⇒ *integer* **is** *gcd* ⟨*proof*⟩

lift-definition *lcm-integer* :: *integer* ⇒ *integer* ⇒ *integer* **is** *lcm* ⟨*proof*⟩

end

instance ⟨*proof*⟩

end

lifting-update *integer.lifting*
lifting-forget *integer.lifting*

context
includes *integer.lifting*
begin

lemma *gcd-code-integer* [*code*]: *gcd k l* = |*if l* = (*0*::*integer*) *then k else gcd l* (|*k*| *mod* |*l*|)|
⟨*proof*⟩

lemma *lcm-code-integer* [*code*]: *lcm a b* = |*a*| ∗ |*b*| *div gcd a b*

for *a b* :: *integer*
⟨*proof*⟩

end

code-printing
constant *gcd* :: *integer* ⇒ - ⇀
(*OCaml*) !(*fun k l -> if Z.equal k Z.zero then/ Z.abs l else if Z.equal/ l Z.zero then Z.abs k else Z.gcd k l*)
and (*Haskell*) *Prelude.gcd*
and (*Scala*) *-.gcd'((-)')*
— There is no gcd operation in the SML standard library, so no code setup for SML

Some code equations

lemmas *Gcd-nat-set-eq-fold* [*code*] = *Gcd-set-eq-fold* [**where** *?'a* = *nat*]
lemmas *Lcm-nat-set-eq-fold* [*code*] = *Lcm-set-eq-fold* [**where** *?'a* = *nat*]
lemmas *Gcd-int-set-eq-fold* [*code*] = *Gcd-set-eq-fold* [**where** *?'a* = *int*]
lemmas *Lcm-int-set-eq-fold* [*code*] = *Lcm-set-eq-fold* [**where** *?'a* = *int*]

Fact aliases.

lemma *lcm-0-iff-nat* [*simp*]: *lcm m n* = *0* ⟷ *m* = *0* ∨ *n* = *0*
for *m n* :: *nat*
⟨*proof*⟩

lemma *lcm-0-iff-int* [*simp*]: *lcm m n* = *0* ⟷ *m* = *0* ∨ *n* = *0*
for *m n* :: *int*
⟨*proof*⟩

lemma *dvd-lcm-I1-nat* [*simp*]: *k dvd m* ⟹ *k dvd lcm m n*
for *k m n* :: *nat*
⟨*proof*⟩

lemma *dvd-lcm-I2-nat* [*simp*]: *k dvd n* ⟹ *k dvd lcm m n*
for *k m n* :: *nat*
⟨*proof*⟩

lemma *dvd-lcm-I1-int* [*simp*]: *i dvd m* ⟹ *i dvd lcm m n*
for *i m n* :: *int*
⟨*proof*⟩

lemma *dvd-lcm-I2-int* [*simp*]: *i dvd n* ⟹ *i dvd lcm m n*
for *i m n* :: *int*
⟨*proof*⟩

lemmas *Gcd-dvd-nat* [*simp*] = *Gcd-dvd* [**where** *?'a* = *nat*]
lemmas *Gcd-dvd-int* [*simp*] = *Gcd-dvd* [**where** *?'a* = *int*]
lemmas *Gcd-greatest-nat* [*simp*] = *Gcd-greatest* [**where** *?'a* = *nat*]
lemmas *Gcd-greatest-int* [*simp*] = *Gcd-greatest* [**where** *?'a* = *int*]

lemma *dvd-Lcm-int* [*simp*]: $m \in M \Longrightarrow m$ *dvd Lcm* M
for M :: *int set*
⟨*proof*⟩

lemma *gcd-neg-numeral-1-int* [*simp*]: *gcd* (− *numeral* n :: *int*) x = *gcd* (*numeral* n) x
⟨*proof*⟩

lemma *gcd-neg-numeral-2-int* [*simp*]: *gcd* x (− *numeral* n :: *int*) = *gcd* x (*numeral* n)
⟨*proof*⟩

lemma *gcd-proj1-if-dvd-nat* [*simp*]: x *dvd* $y \Longrightarrow$ *gcd* x y = x
for x y :: *nat*
⟨*proof*⟩

lemma *gcd-proj2-if-dvd-nat* [*simp*]: y *dvd* $x \Longrightarrow$ *gcd* x y = y
for x y :: *nat*
⟨*proof*⟩

lemma *Gcd-in*:
fixes A :: *nat set*
assumes $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow$ *gcd* a $b \in A$
assumes $A \neq \{\}$
shows *Gcd* $A \in A$
⟨*proof*⟩

lemma *bezout-gcd-nat'*:
fixes a b :: *nat*
shows $\exists x\ y.\ b * y \leq a * x \wedge a * x - b * y =$ *gcd* a $b \vee a * y \leq b * x \wedge b * x - a * y =$ *gcd* a b
⟨*proof*⟩

lemmas *Lcm-eq-0-I-nat* [*simp*] = *Lcm-eq-0-I* [**where** *?'a* = *nat*]
lemmas *Lcm-0-iff-nat* [*simp*] = *Lcm-0-iff* [**where** *?'a* = *nat*]
lemmas *Lcm-least-int* [*simp*] = *Lcm-least* [**where** *?'a* = *int*]

88.11 Characteristic of a semiring

definition (**in** *semiring-1*) *semiring-char* :: *'a itself* ⇒ *nat*
where *semiring-char* - = *Gcd* {*n*. *of-nat* n = (*0* :: *'a*)}

context *semiring-1*
begin

context
fixes *CHAR* :: *nat*
defines *CHAR* ≡ *semiring-char* (*Pure.type* :: *'a itself*)

begin

lemma *of-nat-CHAR* [*simp*]: *of-nat CHAR* = (*0* :: *'a*)
⟨*proof*⟩

lemma *of-nat-eq-0-iff-char-dvd*: *of-nat n* = (*0* :: *'a*) ⟷ *CHAR dvd n*
⟨*proof*⟩

lemma *CHAR-eqI*:
assumes *of-nat c* = (*0* :: *'a*)
assumes ⋀*x*. *of-nat x* = (*0* :: *'a*) ⟹ *c dvd x*
shows *CHAR* = *c*
⟨*proof*⟩

lemma *CHAR-eq0-iff*: *CHAR* = *0* ⟷ (∀ *n*>*0*. *of-nat n* ≠ (*0*::*'a*))
⟨*proof*⟩

lemma *CHAR-pos-iff*: *CHAR* > *0* ⟷ (∃ *n*>*0*. *of-nat n* = (*0*::*'a*))
⟨*proof*⟩

lemma *CHAR-eq-posI*:
assumes *c* > *0 of-nat c* = (*0* :: *'a*) ⋀*x*. *x* > *0* ⟹ *x* < *c* ⟹ *of-nat x* ≠ (*0* :: *'a*)
shows *CHAR* = *c*
⟨*proof*⟩

end
end

lemma (**in** *semiring-char-0*) *CHAR-eq-0* [*simp*]: *semiring-char* (*Pure.type* :: *'a itself*) = *0*
⟨*proof*⟩

syntax *-type-char* :: *type* => *nat* ((*1CHAR*/(*1'*(*-'*))))

translations *CHAR*(*'t*) => *CONST semiring-char* (*CONST Pure.type* :: *'t itself*)

⟨*ML*⟩

end

89 Nitpick: Yet Another Counterexample Generator for Isabelle/HOL

theory *Nitpick*

imports *Record GCD*
keywords
nitpick :: *diag* **and**
nitpick-params :: *thy-decl*
begin

datatype (*plugins only*: *extraction*) (*dead* 'a, *dead* 'b) *fun-box* = *FunBox* 'a ⇒ 'b
datatype (*plugins only*: *extraction*) (*dead* 'a, *dead* 'b) *pair-box* = *PairBox* 'a 'b
datatype (*plugins only*: *extraction*) (*dead* 'a) *word* = *Word* 'a *set*

typedecl *bisim-iterator*
typedecl *unsigned-bit*
typedecl *signed-bit*

consts
unknown :: 'a
is-unknown :: 'a ⇒ *bool*
bisim :: *bisim-iterator* ⇒ 'a ⇒ 'a ⇒ *bool*
bisim-iterator-max :: *bisim-iterator*
Quot :: 'a ⇒ 'b
safe-The :: ('a ⇒ *bool*) ⇒ 'a

Alternative definitions.

lemma *Ex1-unfold*[*nitpick-unfold*]: *Ex1 P* ≡ ∃ *x*. {*x*. *P x*} = {*x*}
⟨*proof*⟩

lemma *rtrancl-unfold*[*nitpick-unfold*]: $r^* \equiv (r^+)^=$
⟨*proof*⟩

lemma *rtranclp-unfold*[*nitpick-unfold*]: *rtranclp r a b* ≡ (*a* = *b* ∨ *tranclp r a b*)
⟨*proof*⟩

lemma *tranclp-unfold*[*nitpick-unfold*]:
tranclp r a b ≡ (*a*, *b*) ∈ *trancl* {(*x*, *y*). *r x y*}
⟨*proof*⟩

lemma [*nitpick-simp*]:
of-nat n = (*if n* = *0 then 0 else 1* + *of-nat* (*n* − *1*))
⟨*proof*⟩

definition *prod* :: 'a *set* ⇒ 'b *set* ⇒ ('a × 'b) *set* **where**
prod A B = {(*a*, *b*). *a* ∈ *A* ∧ *b* ∈ *B*}

definition *refl'* :: ('a × 'a) *set* ⇒ *bool* **where**
refl' r ≡ ∀ *x*. (*x*, *x*) ∈ *r*

definition *wf'* :: ('a × 'a) *set* ⇒ *bool* **where**
wf' r ≡ *acyclic r* ∧ (*finite r* ∨ *unknown*)

definition *card'* :: *'a set* ⇒ *nat* **where**
card' A ≡ *if finite A then length* (*SOME xs. set xs* = *A* ∧ *distinct xs*) *else 0*

definition *sum'* :: (*'a* ⇒ *'b*::*comm-monoid-add*) ⇒ *'a set* ⇒ *'b* **where**
sum' f A ≡ *if finite A then sum-list* (*map f* (*SOME xs. set xs* = *A* ∧ *distinct xs*)) *else 0*

inductive *fold-graph'* :: (*'a* ⇒ *'b* ⇒ *'b*) ⇒ *'b* ⇒ *'a set* ⇒ *'b* ⇒ *bool* **where**
fold-graph' f z {} *z* |
⟦*x* ∈ *A*; *fold-graph' f z* (*A* − {*x*}) *y*⟧ ⟹ *fold-graph' f z A* (*f x y*)

The following lemmas are not strictly necessary but they help the *specialize* optimization.

lemma *The-psimp*[*nitpick-psimp*]: *P* = (=) *x* ⟹ *The P* = *x*
⟨*proof*⟩

lemma *Eps-psimp*[*nitpick-psimp*]:
⟦*P x*; ¬ *P y*; *Eps P* = *y*⟧ ⟹ *Eps P* = *x*
⟨*proof*⟩

lemma *case-unit-unfold*[*nitpick-unfold*]:
case-unit x u ≡ *x*
⟨*proof*⟩

declare *unit.case*[*nitpick-simp del*]

lemma *case-nat-unfold*[*nitpick-unfold*]:
case-nat x f n ≡ *if n* = *0 then x else f* (*n* − *1*)
⟨*proof*⟩

declare *nat.case*[*nitpick-simp del*]

lemma *size-list-simp*[*nitpick-simp*]:
size-list f xs = (*if xs* = [] *then 0 else Suc* (*f* (*hd xs*) + *size-list f* (*tl xs*)))
size xs = (*if xs* = [] *then 0 else Suc* (*size* (*tl xs*)))
⟨*proof*⟩

Auxiliary definitions used to provide an alternative representation for *rat* and *real*.

fun *nat-gcd* :: *nat* ⇒ *nat* ⇒ *nat* **where**
nat-gcd x y = (*if y* = *0 then x else nat-gcd y* (*x mod y*))

declare *nat-gcd.simps* [*simp del*]

definition *nat-lcm* :: *nat* ⇒ *nat* ⇒ *nat* **where**
nat-lcm x y = *x* ∗ *y div* (*nat-gcd x y*)

lemma *gcd-eq-nitpick-gcd* [*nitpick-unfold*]:
gcd x y = *Nitpick.nat-gcd x y*

⟨*proof*⟩

lemma *lcm-eq-nitpick-lcm* [*nitpick-unfold*]:
lcm x y = *Nitpick.nat-lcm x y*
⟨*proof*⟩

definition *Frac* :: *int* × *int* ⇒ *bool* **where**
Frac ≡ λ(*a*, *b*). *b* > *0* ∧ *coprime a b*

consts
Abs-Frac :: *int* × *int* ⇒ *'a*
Rep-Frac :: *'a* ⇒ *int* × *int*

definition *zero-frac* :: *'a* **where**
zero-frac ≡ *Abs-Frac* (*0*, *1*)

definition *one-frac* :: *'a* **where**
one-frac ≡ *Abs-Frac* (*1*, *1*)

definition *num* :: *'a* ⇒ *int* **where**
num ≡ *fst* ∘ *Rep-Frac*

definition *denom* :: *'a* ⇒ *int* **where**
denom ≡ *snd* ∘ *Rep-Frac*

function *norm-frac* :: *int* ⇒ *int* ⇒ *int* × *int* **where**
norm-frac a b =
(*if b* < *0 then norm-frac* (− *a*) (− *b*)
else if a = *0* ∨ *b* = *0 then* (*0*, *1*)
else let c = *gcd a b in* (*a div c*, *b div c*))
⟨*proof*⟩
termination ⟨*proof*⟩

declare *norm-frac.simps*[*simp del*]

definition *frac* :: *int* ⇒ *int* ⇒ *'a* **where**
frac a b ≡ *Abs-Frac* (*norm-frac a b*)

definition *plus-frac* :: *'a* ⇒ *'a* ⇒ *'a* **where**
[*nitpick-simp*]: *plus-frac q r* = (*let d* = *lcm* (*denom q*) (*denom r*) *in*
frac (*num q* ∗ (*d div denom q*) + *num r* ∗ (*d div denom r*)) *d*)

definition *times-frac* :: *'a* ⇒ *'a* ⇒ *'a* **where**
[*nitpick-simp*]: *times-frac q r* = *frac* (*num q* ∗ *num r*) (*denom q* ∗ *denom r*)

definition *uminus-frac* :: *'a* ⇒ *'a* **where**
uminus-frac q ≡ *Abs-Frac* (− *num q*, *denom q*)

definition *number-of-frac* :: *int* ⇒ *'a* **where**

number-of-frac n ≡ *Abs-Frac* (*n*, *1*)

definition *inverse-frac* :: *'a* ⇒ *'a* **where**
inverse-frac q ≡ *frac* (*denom q*) (*num q*)

definition *less-frac* :: *'a* ⇒ *'a* ⇒ *bool* **where**
[*nitpick-simp*]: *less-frac q r* ⟷ *num* (*plus-frac q* (*uminus-frac r*)) < *0*

definition *less-eq-frac* :: *'a* ⇒ *'a* ⇒ *bool* **where**
[*nitpick-simp*]: *less-eq-frac q r* ⟷ *num* (*plus-frac q* (*uminus-frac r*)) ≤ *0*

definition *of-frac* :: *'a* ⇒ *'b*::{*inverse*,*ring-1*} **where**
of-frac q ≡ *of-int* (*num q*) / *of-int* (*denom q*)

axiomatization *wf-wfrec* :: (*'a* × *'a*) *set* ⇒ ((*'a* ⇒ *'b*) ⇒ *'a* ⇒ *'b*) ⇒ *'a* ⇒ *'b*

definition *wf-wfrec'* :: (*'a* × *'a*) *set* ⇒ ((*'a* ⇒ *'b*) ⇒ *'a* ⇒ *'b*) ⇒ *'a* ⇒ *'b* **where**
[*nitpick-simp*]: *wf-wfrec' R F x* = *F* (*cut* (*wf-wfrec R F*) *R x*) *x*

definition *wfrec'* :: (*'a* × *'a*) *set* ⇒ ((*'a* ⇒ *'b*) ⇒ *'a* ⇒ *'b*) ⇒ *'a* ⇒ *'b* **where**
wfrec' R F x ≡ *if wf R then wf-wfrec' R F x else THE y. wfrec-rel R* (λ*f x*. *F* (*cut f R x*) *x*) *x y*

⟨*ML*⟩

hide-const (**open**) *unknown is-unknown bisim bisim-iterator-max Quot safe-The FunBox PairBox Word prod*
refl' wf' card' sum' fold-graph' nat-gcd nat-lcm Frac Abs-Frac Rep-Frac
zero-frac one-frac num denom norm-frac frac plus-frac times-frac uminus-frac number-of-frac
inverse-frac less-frac less-eq-frac of-frac wf-wfrec wf-wfrec wfrec'

hide-type (**open**) *bisim-iterator fun-box pair-box unsigned-bit signed-bit word*

hide-fact (**open**) *Ex1-unfold rtrancl-unfold rtranclp-unfold tranclp-unfold prod-def refl'-def wf'-def*
card'-def sum'-def The-psimp Eps-psimp case-unit-unfold case-nat-unfold
size-list-simp nat-lcm-def Frac-def zero-frac-def one-frac-def
num-def denom-def frac-def plus-frac-def times-frac-def uminus-frac-def
number-of-frac-def inverse-frac-def less-frac-def less-eq-frac-def of-frac-def wf-wfrec'-def
wfrec'-def

end

theory *Nunchaku*
imports *Nitpick*
keywords
nunchaku :: *diag* **and**

nunchaku-params :: *thy-decl*
begin

consts *unreachable* :: *'a*

definition *The-unsafe* :: (*'a* ⇒ *bool*) ⇒ *'a* **where**
The-unsafe = *The*

definition *rmember* :: *'a set* ⇒ *'a* ⇒ *bool* **where**
rmember A x ⟷ *x* ∈ *A*

⟨*ML*⟩

hide-const (**open**) *unreachable The-unsafe rmember*

end

90 Greatest Fixpoint (Codatatype) Operation on Bounded Natural Functors

theory *BNF-Greatest-Fixpoint*
imports *BNF-Fixpoint-Base String*
keywords
codatatype :: *thy-defn* **and**
primcorecursive :: *thy-goal-defn* **and**
primcorec :: *thy-defn*
begin

alias *proj* = *Equiv-Relations.proj*

lemma *one-pointE*: ⟦⋀*x*. *s* = *x* ⟹ *P*⟧ ⟹ *P*
⟨*proof*⟩

lemma *obj-sumE*: ⟦∀ *x*. *s* = *Inl x* ⟶ *P*; ∀ *x*. *s* = *Inr x* ⟶ *P*⟧ ⟹ *P*
⟨*proof*⟩

lemma *not-TrueE*: ¬ *True* ⟹ *P*
⟨*proof*⟩

lemma *neq-eq-eq-contradict*: ⟦*t* ≠ *u*; *s* = *t*; *s* = *u*⟧ ⟹ *P*
⟨*proof*⟩

lemma *converse-Times*: (*A* × *B*)$^{-1}$ = *B* × *A*
⟨*proof*⟩

lemma *equiv-proj*:
assumes *e*: *equiv A R* **and** *m*: *z* ∈ *R*
shows (*proj R* ∘ *fst*) *z* = (*proj R* ∘ *snd*) *z*

⟨*proof*⟩

definition *image2* **where** *image2 A f g* = {(*f a*, *g a*) | *a*. *a* ∈ *A*}

lemma *Id-on-Gr*: *Id-on A* = *Gr A id*
⟨*proof*⟩

lemma *image2-eqI*: ⟦*b* = *f x*; *c* = *g x*; *x* ∈ *A*⟧ ⟹ (*b*, *c*) ∈ *image2 A f g*
⟨*proof*⟩

lemma *IdD*: (*a*, *b*) ∈ *Id* ⟹ *a* = *b*
⟨*proof*⟩

lemma *image2-Gr*: *image2 A f g* = (*Gr A f*)$^{-1}$ *O* (*Gr A g*)
⟨*proof*⟩

lemma *GrD1*: (*x*, *fx*) ∈ *Gr A f* ⟹ *x* ∈ *A*
⟨*proof*⟩

lemma *GrD2*: (*x*, *fx*) ∈ *Gr A f* ⟹ *f x* = *fx*
⟨*proof*⟩

lemma *Gr-incl*: *Gr A f* ⊆ *A* × *B* ⟷ *f* ‘ *A* ⊆ *B*
⟨*proof*⟩

lemma *subset-Collect-iff*: *B* ⊆ *A* ⟹ (*B* ⊆ {*x* ∈ *A*. *P x*}) = (∀ *x* ∈ *B*. *P x*)
⟨*proof*⟩

lemma *subset-CollectI*: *B* ⊆ *A* ⟹ (⋀*x*. *x* ∈ *B* ⟹ *Q x* ⟹ *P x*) ⟹ ({*x* ∈ *B*. *Q x*} ⊆ {*x* ∈ *A*. *P x*})
⟨*proof*⟩

lemma *in-rel-Collect-case-prod-eq*: *in-rel* (*Collect* (*case-prod X*)) = *X*
⟨*proof*⟩

lemma *Collect-case-prod-in-rel-leI*: *X* ⊆ *Y* ⟹ *X* ⊆ *Collect* (*case-prod* (*in-rel Y*))
⟨*proof*⟩

lemma *Collect-case-prod-in-rel-leE*: *X* ⊆ *Collect* (*case-prod* (*in-rel Y*)) ⟹ (*X* ⊆ *Y* ⟹ *R*) ⟹ *R*
⟨*proof*⟩

lemma *conversep-in-rel*: (*in-rel R*)$^{-1-1}$ = *in-rel* (*R*$^{-1}$)
⟨*proof*⟩

lemma *relcompp-in-rel*: *in-rel R OO in-rel S* = *in-rel* (*R O S*)
⟨*proof*⟩

lemma *in-rel-Gr*: *in-rel* (*Gr A f*) = *Grp A f*
⟨*proof*⟩

definition *relImage* **where**
relImage R f ≡ {(*f a1*, *f a2*) | *a1 a2*. (*a1*,*a2*) ∈ *R*}

definition *relInvImage* **where**
relInvImage A R f ≡ {(*a1*, *a2*) | *a1 a2*. *a1* ∈ *A* ∧ *a2* ∈ *A* ∧ (*f a1*, *f a2*) ∈ *R*}

lemma *relImage-Gr*:
⟦*R* ⊆ *A* × *A*⟧ ⟹ *relImage R f* = (*Gr A f*)$^{-1}$ *O R O Gr A f*
⟨*proof*⟩

lemma *relInvImage-Gr*: ⟦*R* ⊆ *B* × *B*⟧ ⟹ *relInvImage A R f* = *Gr A f O R O* (*Gr A f*)$^{-1}$
⟨*proof*⟩

lemma *relImage-mono*:
R1 ⊆ *R2* ⟹ *relImage R1 f* ⊆ *relImage R2 f*
⟨*proof*⟩

lemma *relInvImage-mono*:
R1 ⊆ *R2* ⟹ *relInvImage A R1 f* ⊆ *relInvImage A R2 f*
⟨*proof*⟩

lemma *relInvImage-Id-on*:
(⋀*a1 a2*. *f a1* = *f a2* ⟷ *a1* = *a2*) ⟹ *relInvImage A* (*Id-on B*) *f* ⊆ *Id*
⟨*proof*⟩

lemma *relInvImage-UNIV-relImage*:
R ⊆ *relInvImage UNIV* (*relImage R f*) *f*
⟨*proof*⟩

lemma *relImage-proj*:
assumes *equiv A R*
shows *relImage R* (*proj R*) ⊆ *Id-on* (*A*//*R*)
⟨*proof*⟩

lemma *relImage-relInvImage*:
assumes *R* ⊆ *f* ‘ *A* × *f* ‘ *A*
shows *relImage* (*relInvImage A R f*) *f* = *R*
⟨*proof*⟩

lemma *subst-Pair*: *P x y* ⟹ *a* = (*x*, *y*) ⟹ *P* (*fst a*) (*snd a*)
⟨*proof*⟩

lemma *fst-diag-id*: (*fst* ∘ (λ*x*. (*x*, *x*))) *z* = *id z* ⟨*proof*⟩
lemma *snd-diag-id*: (*snd* ∘ (λ*x*. (*x*, *x*))) *z* = *id z* ⟨*proof*⟩

lemma *fst-diag-fst*: $fst \circ ((\lambda x.\ (x,\ x)) \circ fst) = fst\ \langle proof \rangle$
lemma *snd-diag-fst*: $snd \circ ((\lambda x.\ (x,\ x)) \circ fst) = fst\ \langle proof \rangle$
lemma *fst-diag-snd*: $fst \circ ((\lambda x.\ (x,\ x)) \circ snd) = snd\ \langle proof \rangle$
lemma *snd-diag-snd*: $snd \circ ((\lambda x.\ (x,\ x)) \circ snd) = snd\ \langle proof \rangle$

definition *Succ* **where** $Succ\ Kl\ kl = \{k\ .\ kl\ @\ [k] \in Kl\}$
definition *Shift* **where** $Shift\ Kl\ k = \{kl.\ k\ \#\ kl \in Kl\}$
definition *shift* **where** $shift\ lab\ k = (\lambda kl.\ lab\ (k\ \#\ kl))$

lemma *empty-Shift*: $[\![[] \in Kl;\ k \in Succ\ Kl\ []]\!] \Longrightarrow [] \in Shift\ Kl\ k$
$\langle proof \rangle$

lemma *SuccD*: $k \in Succ\ Kl\ kl \Longrightarrow kl\ @\ [k] \in Kl$
$\langle proof \rangle$

lemmas $SuccE = SuccD[elim\text{-}format]$

lemma *SuccI*: $kl\ @\ [k] \in Kl \Longrightarrow k \in Succ\ Kl\ kl$
$\langle proof \rangle$

lemma *ShiftD*: $kl \in Shift\ Kl\ k \Longrightarrow k\ \#\ kl \in Kl$
$\langle proof \rangle$

lemma *Succ-Shift*: $Succ\ (Shift\ Kl\ k)\ kl = Succ\ Kl\ (k\ \#\ kl)$
$\langle proof \rangle$

lemma *length-Cons*: $length\ (x\ \#\ xs) = Suc\ (length\ xs)$
$\langle proof \rangle$

lemma *length-append-singleton*: $length\ (xs\ @\ [x]) = Suc\ (length\ xs)$
$\langle proof \rangle$

definition $toCard\text{-}pred\ A\ r\ f \equiv inj\text{-}on\ f\ A \wedge f\ `\ A \subseteq Field\ r \wedge Card\text{-}order\ r$
definition $toCard\ A\ r \equiv SOME\ f.\ toCard\text{-}pred\ A\ r\ f$

lemma *ex-toCard-pred*:
$[\![|A| \leq o\ r;\ Card\text{-}order\ r]\!] \Longrightarrow \exists\ f.\ toCard\text{-}pred\ A\ r\ f$
$\langle proof \rangle$

lemma *toCard-pred-toCard*:
$[\![|A| \leq o\ r;\ Card\text{-}order\ r]\!] \Longrightarrow toCard\text{-}pred\ A\ r\ (toCard\ A\ r)$
$\langle proof \rangle$

lemma *toCard-inj*: $[\![|A| \leq o\ r;\ Card\text{-}order\ r;\ x \in A;\ y \in A]\!] \Longrightarrow toCard\ A\ r\ x = toCard\ A\ r\ y \longleftrightarrow x = y$
$\langle proof \rangle$

definition *fromCard A r k* ≡ *SOME b. b* ∈ *A* ∧ *toCard A r b* = *k*

lemma *fromCard-toCard*:
⟦|*A*| ≤*o r*; *Card-order r*; *b* ∈ *A*⟧ ⟹ *fromCard A r* (*toCard A r b*) = *b*
⟨*proof*⟩

lemma *Inl-Field-csum*: *a* ∈ *Field r* ⟹ *Inl a* ∈ *Field* (*r* +*c s*)
⟨*proof*⟩

lemma *Inr-Field-csum*: *a* ∈ *Field s* ⟹ *Inr a* ∈ *Field* (*r* +*c s*)
⟨*proof*⟩

lemma *rec-nat-0-imp*: *f* = *rec-nat f1* (λ*n rec. f2 n rec*) ⟹ *f 0* = *f1*
⟨*proof*⟩

lemma *rec-nat-Suc-imp*: *f* = *rec-nat f1* (λ*n rec. f2 n rec*) ⟹ *f* (*Suc n*) = *f2 n* (*f n*)
⟨*proof*⟩

lemma *rec-list-Nil-imp*: *f* = *rec-list f1* (λ*x xs rec. f2 x xs rec*) ⟹ *f* [] = *f1*
⟨*proof*⟩

lemma *rec-list-Cons-imp*: *f* = *rec-list f1* (λ*x xs rec. f2 x xs rec*) ⟹ *f* (*x* # *xs*) = *f2 x xs* (*f xs*)
⟨*proof*⟩

lemma *not-arg-cong-Inr*: *x* ≠ *y* ⟹ *Inr x* ≠ *Inr y*
⟨*proof*⟩

definition *image2p* **where**
image2p f g R = (λ*x y*. ∃ *x′ y′*. *R x′ y′* ∧ *f x′* = *x* ∧ *g y′* = *y*)

lemma *image2pI*: *R x y* ⟹ *image2p f g R* (*f x*) (*g y*)
⟨*proof*⟩

lemma *image2pE*: ⟦*image2p f g R fx gy*; (⋀*x y. fx* = *f x* ⟹ *gy* = *g y* ⟹ *R x y* ⟹ *P*)⟧ ⟹ *P*
⟨*proof*⟩

lemma *rel-fun-iff-geq-image2p*: *rel-fun R S f g* = (*image2p f g R* ≤ *S*)
⟨*proof*⟩

lemma *rel-fun-image2p*: *rel-fun R* (*image2p f g R*) *f g*
⟨*proof*⟩

90.1 Equivalence relations, quotients, and Hilbert's choice

lemma *equiv-Eps-in*:
⟦*equiv A r*; *X* ∈ *A*//*r*⟧ ⟹ *Eps* (λ*x. x* ∈ *X*) ∈ *X*

⟨*proof*⟩

lemma *equiv-Eps-preserves*:
assumes *ECH*: *equiv A r* **and** *X*: $X \in A//r$
shows *Eps* $(\lambda x.\ x \in X) \in A$
⟨*proof*⟩

lemma *proj-Eps*:
assumes *equiv A r* **and** $X \in A//r$
shows *proj r* (*Eps* $(\lambda x.\ x \in X)) = X$
⟨*proof*⟩

definition *univ* **where** *univ f X* == *f* (*Eps* $(\lambda x.\ x \in X)$)

lemma *univ-commute*:
assumes *ECH*: *equiv A r* **and** *RES*: *f respects r* **and** *x*: $x \in A$
shows (*univ f*) (*proj r x*) = *f x*
⟨*proof*⟩

lemma *univ-preserves*:
assumes *ECH*: *equiv A r* **and** *RES*: *f respects r* **and** *PRES*: $\forall x \in A.\ f\ x \in B$
shows $\forall X \in A//r.$ *univ f X* $\in B$
⟨*proof*⟩

lemma *card-suc-ordLess-imp-ordLeq*:
assumes *ORD*: *Card-order r Card-order r′ card-order r′*
and *LESS*: $r <o$ *card-suc r′*
shows $r \leq o\ r'$
⟨*proof*⟩

lemma *natLeq-ordLess-cinfinite*: ⟦*Cinfinite r*; *card-order r*⟧ ⟹ *natLeq* $<o$ *card-suc r*
⟨*proof*⟩

corollary *natLeq-ordLess-cinfinite′*: ⟦*Cinfinite r′*; *card-order r′*; $r \equiv$ *card-suc r′*⟧ ⟹ *natLeq* $<o$ *r*
⟨*proof*⟩

⟨*ML*⟩

end

91 Filters on predicates

theory *Filter*
imports *Set-Interval Lifting-Set*
begin

91.1 Filters

This definition also allows non-proper filters.

locale *is-filter* =
  fixes $F :: ('a \Rightarrow bool) \Rightarrow bool$
  assumes *True*: $F\ (\lambda x.\ True)$
  assumes *conj*: $F\ (\lambda x.\ P\ x) \Longrightarrow F\ (\lambda x.\ Q\ x) \Longrightarrow F\ (\lambda x.\ P\ x \wedge Q\ x)$
  assumes *mono*: $\forall x.\ P\ x \longrightarrow Q\ x \Longrightarrow F\ (\lambda x.\ P\ x) \Longrightarrow F\ (\lambda x.\ Q\ x)$

typedef *'a filter* = $\{F :: ('a \Rightarrow bool) \Rightarrow bool.\ is\text{-}filter\ F\}$
⟨*proof*⟩

lemma *is-filter-Rep-filter*: *is-filter* (*Rep-filter F*)
  ⟨*proof*⟩

lemma *Abs-filter-inverse'*:
  assumes *is-filter F* **shows** *Rep-filter* (*Abs-filter F*) = *F*
  ⟨*proof*⟩

91.1.1 Eventually

definition *eventually* :: $('a \Rightarrow bool) \Rightarrow 'a\ filter \Rightarrow bool$
  where *eventually P F* $\longleftrightarrow$ *Rep-filter F P*

syntax
  -eventually :: *pttrn => 'a filter => bool => bool* (($3\forall_{F}$ - *in* -./ -) [*0*, *0*, *10*] *10*)
translations
  $\forall_{F} x\ in\ F.\ P == CONST\ eventually\ (\lambda x.\ P)\ F$

lemma *eventually-Abs-filter*:
  assumes *is-filter F* **shows** *eventually P* (*Abs-filter F*) = *F P*
  ⟨*proof*⟩

lemma *filter-eq-iff*:
  shows $F = F' \longleftrightarrow (\forall P.\ eventually\ P\ F = eventually\ P\ F')$
  ⟨*proof*⟩

lemma *eventually-True* [*simp*]: *eventually* $(\lambda x.\ True)\ F$
  ⟨*proof*⟩

lemma *always-eventually*: $\forall x.\ P\ x \Longrightarrow eventually\ P\ F$
⟨*proof*⟩

lemma *eventuallyI*: $(\bigwedge x.\ P\ x) \Longrightarrow eventually\ P\ F$
  ⟨*proof*⟩

lemma *filter-eqI*: $(\bigwedge P.\ eventually\ P\ F \longleftrightarrow eventually\ P\ G) \Longrightarrow F = G$
  ⟨*proof*⟩

lemma *eventually-mono*:
⟦*eventually P F*; ⋀*x*. *P x* ⟹ *Q x*⟧ ⟹ *eventually Q F*
⟨*proof*⟩

lemma *eventually-conj*:
assumes *P*: *eventually* (λ*x*. *P x*) *F*
assumes *Q*: *eventually* (λ*x*. *Q x*) *F*
shows *eventually* (λ*x*. *P x* ∧ *Q x*) *F*
⟨*proof*⟩

lemma *eventually-mp*:
assumes *eventually* (λ*x*. *P x* ⟶ *Q x*) *F*
assumes *eventually* (λ*x*. *P x*) *F*
shows *eventually* (λ*x*. *Q x*) *F*
⟨*proof*⟩

lemma *eventually-rev-mp*:
assumes *eventually* (λ*x*. *P x*) *F*
assumes *eventually* (λ*x*. *P x* ⟶ *Q x*) *F*
shows *eventually* (λ*x*. *Q x*) *F*
⟨*proof*⟩

lemma *eventually-conj-iff*:
eventually (λ*x*. *P x* ∧ *Q x*) *F* ⟷ *eventually P F* ∧ *eventually Q F*
⟨*proof*⟩

lemma *eventually-elim2*:
assumes *eventually* (λ*i*. *P i*) *F*
assumes *eventually* (λ*i*. *Q i*) *F*
assumes ⋀*i*. *P i* ⟹ *Q i* ⟹ *R i*
shows *eventually* (λ*i*. *R i*) *F*
⟨*proof*⟩

lemma *eventually-cong*:
assumes *eventually P F* **and** ⋀*x*. *P x* ⟹ *Q x* ⟷ *R x*
shows *eventually Q F* ⟷ *eventually R F*
⟨*proof*⟩

lemma *eventually-ball-finite-distrib*:
finite A ⟹ (*eventually* (λ*x*. ∀ *y*∈*A*. *P x y*) *net*) ⟷ (∀ *y*∈*A*. *eventually* (λ*x*. *P x y*) *net*)
⟨*proof*⟩

lemma *eventually-ball-finite*:
finite A ⟹ ∀ *y*∈*A*. *eventually* (λ*x*. *P x y*) *net* ⟹ *eventually* (λ*x*. ∀ *y*∈*A*. *P x y*) *net*
⟨*proof*⟩

lemma *eventually-all-finite*:
fixes $P :: 'a \Rightarrow 'b::finite \Rightarrow bool$
assumes $\bigwedge y.\ eventually\ (\lambda x.\ P\ x\ y)\ net$
shows $eventually\ (\lambda x.\ \forall y.\ P\ x\ y)\ net$
⟨*proof*⟩

lemma *eventually-ex*: $(\forall_F x\ in\ F.\ \exists y.\ P\ x\ y) \longleftrightarrow (\exists Y.\ \forall_F x\ in\ F.\ P\ x\ (Y\ x))$
⟨*proof*⟩

lemma *not-eventually-impI*: $eventually\ P\ F \Longrightarrow \neg\ eventually\ Q\ F \Longrightarrow \neg\ eventually\ (\lambda x.\ P\ x \longrightarrow Q\ x)\ F$
⟨*proof*⟩

lemma *not-eventuallyD*: $\neg\ eventually\ P\ F \Longrightarrow \exists x.\ \neg\ P\ x$
⟨*proof*⟩

lemma *eventually-subst*:
assumes $eventually\ (\lambda n.\ P\ n = Q\ n)\ F$
shows $eventually\ P\ F = eventually\ Q\ F$ (**is** *?L = ?R*)
⟨*proof*⟩

91.2 Frequently as dual to eventually

definition $frequently :: ('a \Rightarrow bool) \Rightarrow 'a\ filter \Rightarrow bool$
where $frequently\ P\ F \longleftrightarrow \neg\ eventually\ (\lambda x.\ \neg\ P\ x)\ F$

syntax
$-frequently :: pttrn \Rightarrow 'a\ filter \Rightarrow bool \Rightarrow bool\ \ ((3\exists_F\ -\ in\ -./\ -)\ [0,\ 0,\ 10]\ 10)$
translations
$\exists_F x\ in\ F.\ P == CONST\ frequently\ (\lambda x.\ P)\ F$

lemma *not-frequently-False* [*simp*]: $\neg\ (\exists_F x\ in\ F.\ False)$
⟨*proof*⟩

lemma *frequently-ex*: $\exists_F x\ in\ F.\ P\ x \Longrightarrow \exists x.\ P\ x$
⟨*proof*⟩

lemma *frequentlyE*: **assumes** $frequently\ P\ F$ **obtains** x **where** $P\ x$
⟨*proof*⟩

lemma *frequently-mp*:
assumes *ev*: $\forall_F x\ in\ F.\ P\ x \longrightarrow Q\ x$ **and** *P*: $\exists_F x\ in\ F.\ P\ x$ **shows** $\exists_F x\ in\ F.\ Q\ x$
⟨*proof*⟩

lemma *frequently-rev-mp*:
assumes $\exists_F x\ in\ F.\ P\ x$
assumes $\forall_F x\ in\ F.\ P\ x \longrightarrow Q\ x$
shows $\exists_F x\ in\ F.\ Q\ x$

⟨*proof*⟩

lemma *frequently-mono*: $(\forall x.\ P\ x \longrightarrow Q\ x) \Longrightarrow \mathit{frequently}\ P\ F \Longrightarrow \mathit{frequently}\ Q\ F$
⟨*proof*⟩

lemma *frequently-elim1*: $\exists_F x\ \mathit{in}\ F.\ P\ x \Longrightarrow (\bigwedge i.\ P\ i \Longrightarrow Q\ i) \Longrightarrow \exists_F x\ \mathit{in}\ F.\ Q\ x$
⟨*proof*⟩

lemma *frequently-disj-iff*: $(\exists_F x\ \mathit{in}\ F.\ P\ x \vee Q\ x) \longleftrightarrow (\exists_F x\ \mathit{in}\ F.\ P\ x) \vee (\exists_F x\ \mathit{in}\ F.\ Q\ x)$
⟨*proof*⟩

lemma *frequently-disj*: $\exists_F x\ \mathit{in}\ F.\ P\ x \Longrightarrow \exists_F x\ \mathit{in}\ F.\ Q\ x \Longrightarrow \exists_F x\ \mathit{in}\ F.\ P\ x \vee Q\ x$
⟨*proof*⟩

lemma *frequently-bex-finite-distrib*:
assumes *finite A* **shows** $(\exists_F x\ \mathit{in}\ F.\ \exists y{\in}A.\ P\ x\ y) \longleftrightarrow (\exists y{\in}A.\ \exists_F x\ \mathit{in}\ F.\ P\ x\ y)$
⟨*proof*⟩

lemma *frequently-bex-finite*: *finite A* $\Longrightarrow \exists_F x\ \mathit{in}\ F.\ \exists y{\in}A.\ P\ x\ y \Longrightarrow \exists y{\in}A.\ \exists_F x\ \mathit{in}\ F.\ P\ x\ y$
⟨*proof*⟩

lemma *frequently-all*: $(\exists_F x\ \mathit{in}\ F.\ \forall y.\ P\ x\ y) \longleftrightarrow (\forall Y.\ \exists_F x\ \mathit{in}\ F.\ P\ x\ (Y\ x))$
⟨*proof*⟩

lemma
shows *not-eventually*: $\neg\ \mathit{eventually}\ P\ F \longleftrightarrow (\exists_F x\ \mathit{in}\ F.\ \neg\ P\ x)$
and *not-frequently*: $\neg\ \mathit{frequently}\ P\ F \longleftrightarrow (\forall_F x\ \mathit{in}\ F.\ \neg\ P\ x)$
⟨*proof*⟩

lemma *frequently-imp-iff*:
$(\exists_F x\ \mathit{in}\ F.\ P\ x \longrightarrow Q\ x) \longleftrightarrow (\mathit{eventually}\ P\ F \longrightarrow \mathit{frequently}\ Q\ F)$
⟨*proof*⟩

lemma *frequently-eventually-conj*:
assumes $\exists_F x\ \mathit{in}\ F.\ P\ x$
assumes $\forall_F x\ \mathit{in}\ F.\ Q\ x$
shows $\exists_F x\ \mathit{in}\ F.\ Q\ x \wedge P\ x$
⟨*proof*⟩

lemma *frequently-cong*:
assumes *ev*: *eventually P F* **and** *QR*: $\bigwedge x.\ P\ x \Longrightarrow Q\ x \longleftrightarrow R\ x$
shows *frequently Q F* $\longleftrightarrow$ *frequently R F*
⟨*proof*⟩

lemma *frequently-eventually-frequently*:
frequently P F $\Longrightarrow$ *eventually Q F* $\Longrightarrow$ *frequently* $(\lambda x.\ P\ x \wedge Q\ x)$ *F*
⟨*proof*⟩

lemma *eventually-frequently-const-simps*:
$(\exists_F x\ in\ F.\ P\ x \wedge C) \longleftrightarrow (\exists_F x\ in\ F.\ P\ x) \wedge C$
$(\exists_F x\ in\ F.\ C \wedge P\ x) \longleftrightarrow C \wedge (\exists_F x\ in\ F.\ P\ x)$
$(\forall_F x\ in\ F.\ P\ x \vee C) \longleftrightarrow (\forall_F x\ in\ F.\ P\ x) \vee C$
$(\forall_F x\ in\ F.\ C \vee P\ x) \longleftrightarrow C \vee (\forall_F x\ in\ F.\ P\ x)$
$(\forall_F x\ in\ F.\ P\ x \longrightarrow C) \longleftrightarrow ((\exists_F x\ in\ F.\ P\ x) \longrightarrow C)$
$(\forall_F x\ in\ F.\ C \longrightarrow P\ x) \longleftrightarrow (C \longrightarrow (\forall_F x\ in\ F.\ P\ x))$
⟨*proof*⟩

lemmas *eventually-frequently-simps* =
eventually-frequently-const-simps
not-eventually
eventually-conj-iff
eventually-ball-finite-distrib
eventually-ex
not-frequently
frequently-disj-iff
frequently-bex-finite-distrib
frequently-all
frequently-imp-iff

⟨*ML*⟩

91.2.1 Finer-than relation

$F \leq F'$ means that filter F is finer than filter F'.

instantiation *filter* :: (*type*) *complete-lattice*
begin

definition *le-filter-def*:
$F \leq F' \longleftrightarrow (\forall P.\ eventually\ P\ F' \longrightarrow eventually\ P\ F)$

definition
$(F :: 'a\ filter) < F' \longleftrightarrow F \leq F' \wedge \neg\ F' \leq F$

definition
$top = Abs\text{-}filter\ (\lambda P.\ \forall x.\ P\ x)$

definition
$bot = Abs\text{-}filter\ (\lambda P.\ True)$

definition
$sup\ F\ F' = Abs\text{-}filter\ (\lambda P.\ eventually\ P\ F \wedge eventually\ P\ F')$

definition
inf F F′ = Abs-filter
(λP. ∃ Q R. *eventually Q F* ∧ *eventually R F′* ∧ (∀ x. Q x ∧ R x ⟶ P x))

definition
Sup S = Abs-filter (λP. ∀ F∈S. *eventually P F*)

definition
Inf S = Sup {F::'a *filter*. ∀ F′∈S. F ≤ F′}

lemma *eventually-top* [*simp*]: *eventually P top* ⟷ (∀ x. P x)
⟨*proof*⟩

lemma *eventually-bot* [*simp*]: *eventually P bot*
⟨*proof*⟩

lemma *eventually-sup*:
eventually P (*sup F F′*) ⟷ *eventually P F* ∧ *eventually P F′*
⟨*proof*⟩

lemma *eventually-inf*:
eventually P (*inf F F′*) ⟷
(∃ Q R. *eventually Q F* ∧ *eventually R F′* ∧ (∀ x. Q x ∧ R x ⟶ P x))
⟨*proof*⟩

lemma *eventually-Sup*:
eventually P (*Sup S*) ⟷ (∀ F∈S. *eventually P F*)
⟨*proof*⟩

instance ⟨*proof*⟩

end

instance *filter* :: (*type*) *distrib-lattice*
⟨*proof*⟩

lemma *filter-leD*:
F ≤ F′ ⟹ *eventually P F′* ⟹ *eventually P F*
⟨*proof*⟩

lemma *filter-leI*:
(⋀P. *eventually P F′* ⟹ *eventually P F*) ⟹ F ≤ F′
⟨*proof*⟩

lemma *eventually-False*:
eventually (λx. *False*) F ⟷ F = *bot*
⟨*proof*⟩

lemma *eventually-frequently*: $F \neq bot \Longrightarrow eventually\ P\ F \Longrightarrow frequently\ P\ F$
⟨*proof*⟩

lemma *eventually-frequentlyE*:
assumes *eventually P F*
assumes *eventually* $(\lambda x.\ \neg\ P\ x \vee Q\ x)\ F\ F \neq bot$
shows *frequently Q F*
⟨*proof*⟩

lemma *eventually-const-iff*: *eventually* $(\lambda x.\ P)\ F \longleftrightarrow P \vee F = bot$
⟨*proof*⟩

lemma *eventually-const*[*simp*]: $F \neq bot \Longrightarrow eventually\ (\lambda x.\ P)\ F \longleftrightarrow P$
⟨*proof*⟩

lemma *frequently-const-iff*: *frequently* $(\lambda x.\ P)\ F \longleftrightarrow P \wedge F \neq bot$
⟨*proof*⟩

lemma *frequently-const*[*simp*]: $F \neq bot \Longrightarrow frequently\ (\lambda x.\ P)\ F \longleftrightarrow P$
⟨*proof*⟩

lemma *eventually-happens*: *eventually P net* $\Longrightarrow net = bot \vee (\exists\, x.\ P\ x)$
⟨*proof*⟩

lemma *eventually-happens'*:
assumes $F \neq bot$ *eventually P F*
shows $\exists\, x.\ P\ x$
⟨*proof*⟩

abbreviation (*input*) *trivial-limit* :: *'a filter* ⇒ *bool*
where *trivial-limit F* ≡ $F = bot$

lemma *trivial-limit-def*: *trivial-limit F* $\longleftrightarrow$ *eventually* $(\lambda x.\ False)\ F$
⟨*proof*⟩

lemma *False-imp-not-eventually*: $(\forall\, x.\ \neg\ P\ x\ ) \Longrightarrow \neg$ *trivial-limit net* $\Longrightarrow \neg$ *eventually* $(\lambda x.\ P\ x)$ *net*
⟨*proof*⟩

lemma *trivial-limit-eventually*: *trivial-limit net* $\Longrightarrow$ *eventually P net*
⟨*proof*⟩

lemma *trivial-limit-eq*: *trivial-limit net* $\longleftrightarrow (\forall\, P.$ *eventually P net*$)$
⟨*proof*⟩

lemma *eventually-Inf*: *eventually P* $(Inf\ B) \longleftrightarrow (\exists\, X \subseteq B.$ *finite* $X \wedge$ *eventually P* $(Inf\ X))$
⟨*proof*⟩

lemma *eventually-INF*: *eventually P* ($\prod b{\in}B.\ F\ b$) $\longleftrightarrow$ ($\exists\, X{\subseteq}B.$ *finite X* $\wedge$ *eventually P* ($\prod b{\in}X.\ F\ b$))
⟨*proof*⟩

lemma *Inf-filter-not-bot*:
fixes *B* :: *'a filter set*
shows ($\bigwedge X.\ X \subseteq B \Longrightarrow$ *finite X* $\Longrightarrow$ *Inf X* $\neq$ *bot*) $\Longrightarrow$ *Inf B* $\neq$ *bot*
⟨*proof*⟩

lemma *INF-filter-not-bot*:
fixes *F* :: *'i* $\Rightarrow$ *'a filter*
shows ($\bigwedge X.\ X \subseteq B \Longrightarrow$ *finite X* $\Longrightarrow$ ($\prod b{\in}X.\ F\ b$) $\neq$ *bot*) $\Longrightarrow$ ($\prod b{\in}B.\ F\ b$) $\neq$ *bot*
⟨*proof*⟩

lemma *eventually-Inf-base*:
assumes $B \neq \{\}$ **and** *base*: $\bigwedge F\ G.\ F \in B \Longrightarrow G \in B \Longrightarrow \exists\, x{\in}B.\ x \leq$ *inf F G*
shows *eventually P* (*Inf B*) $\longleftrightarrow$ ($\exists\, b{\in}B.$ *eventually P b*)
⟨*proof*⟩

lemma *eventually-INF-base*:
$B \neq \{\} \Longrightarrow (\bigwedge a\ b.\ a \in B \Longrightarrow b \in B \Longrightarrow \exists\, x{\in}B.\ F\ x \leq$ *inf* (*F a*) (*F b*)) $\Longrightarrow$
eventually P ($\prod b{\in}B.\ F\ b$) $\longleftrightarrow$ ($\exists\, b{\in}B.$ *eventually P* (*F b*))
⟨*proof*⟩

lemma *eventually-INF1*: $i \in I \Longrightarrow$ *eventually P* (*F i*) $\Longrightarrow$ *eventually P* ($\prod i{\in}I.$ *F i*)
⟨*proof*⟩

lemma *eventually-INF-finite*:
assumes *finite A*
shows *eventually P* ($\prod\ x{\in}A.\ F\ x$) $\longleftrightarrow$
($\exists\, Q.\ (\forall\, x{\in}A.$ *eventually* (*Q x*) (*F x*)) $\wedge$ ($\forall\, y.\ (\forall\, x{\in}A.\ Q\ x\ y) \longrightarrow P\ y$))
⟨*proof*⟩

91.2.2 Map function for filters

definition *filtermap* :: (*'a* $\Rightarrow$ *'b*) $\Rightarrow$ *'a filter* $\Rightarrow$ *'b filter*
where *filtermap f F* = *Abs-filter* ($\lambda P.$ *eventually* ($\lambda x.$ *P* (*f x*)) *F*)

lemma *eventually-filtermap*:
eventually P (*filtermap f F*) = *eventually* ($\lambda x.$ *P* (*f x*)) *F*
⟨*proof*⟩

lemma *eventually-comp-filtermap*:
eventually (*P* $\circ$ *f*) *F* $\longleftrightarrow$ *eventually P* (*filtermap f F*)
⟨*proof*⟩

lemma *filtermap-compose*: *filtermap* (*f* $\circ$ *g*) *F* = *filtermap f* (*filtermap g F*)

⟨*proof*⟩

lemma *filtermap-ident*: *filtermap* (λ*x*. *x*) *F* = *F*
⟨*proof*⟩

lemma *filtermap-filtermap*:
filtermap f (*filtermap g F*) = *filtermap* (λ*x*. *f* (*g x*)) *F*
⟨*proof*⟩

lemma *filtermap-mono*: *F* ≤ *F′* ⟹ *filtermap f F* ≤ *filtermap f F′*
⟨*proof*⟩

lemma *filtermap-bot* [*simp*]: *filtermap f bot* = *bot*
⟨*proof*⟩

lemma *filtermap-bot-iff*: *filtermap f F* = *bot* ⟷ *F* = *bot*
⟨*proof*⟩

lemma *filtermap-sup*: *filtermap f* (*sup F1 F2*) = *sup* (*filtermap f F1*) (*filtermap f F2*)
⟨*proof*⟩

lemma *filtermap-SUP*: *filtermap f* (⨆ *b*∈*B*. *F b*) = (⨆ *b*∈*B*. *filtermap f* (*F b*))
⟨*proof*⟩

lemma *filtermap-inf*: *filtermap f* (*inf F1 F2*) ≤ *inf* (*filtermap f F1*) (*filtermap f F2*)
⟨*proof*⟩

lemma *filtermap-INF*: *filtermap f* (⨅ *b*∈*B*. *F b*) ≤ (⨅ *b*∈*B*. *filtermap f* (*F b*))
⟨*proof*⟩

lemma *frequently-filtermap*:
frequently P (*filtermap f F*) = *frequently* (λ*x*. *P* (*f x*)) *F*
⟨*proof*⟩

91.2.3 Contravariant map function for filters

definition *filtercomap* :: (*′a* ⇒ *′b*) ⇒ *′b filter* ⇒ *′a filter* **where**
filtercomap f F = *Abs-filter* (λ*P*. ∃ *Q*. *eventually Q F* ∧ (∀ *x*. *Q* (*f x*) ⟶ *P x*))

lemma *eventually-filtercomap*:
eventually P (*filtercomap f F*) ⟷ (∃ *Q*. *eventually Q F* ∧ (∀ *x*. *Q* (*f x*) ⟶ *P x*))
⟨*proof*⟩

lemma *filtercomap-ident*: *filtercomap* (λ*x*. *x*) *F* = *F*
⟨*proof*⟩

lemma *filtercomap-filtercomap*: *filtercomap f* (*filtercomap g F*) = *filtercomap* (λ*x*. *g* (*f x*)) *F*
⟨*proof*⟩

lemma *filtercomap-mono*: *F* ≤ *F'* ⟹ *filtercomap f F* ≤ *filtercomap f F'*
⟨*proof*⟩

lemma *filtercomap-bot* [*simp*]: *filtercomap f bot* = *bot*
⟨*proof*⟩

lemma *filtercomap-top* [*simp*]: *filtercomap f top* = *top*
⟨*proof*⟩

lemma *filtercomap-inf*: *filtercomap f* (*inf F1 F2*) = *inf* (*filtercomap f F1*) (*filtercomap f F2*)
⟨*proof*⟩

lemma *filtercomap-sup*: *filtercomap f* (*sup F1 F2*) ≥ *sup* (*filtercomap f F1*) (*filtercomap f F2*)
⟨*proof*⟩

lemma *filtercomap-INF*: *filtercomap f* (⨅ *b*∈*B*. *F b*) = (⨅ *b*∈*B*. *filtercomap f* (*F b*))
⟨*proof*⟩

lemma *filtercomap-SUP*:
filtercomap f (⨆ *b*∈*B*. *F b*) ≥ (⨆ *b*∈*B*. *filtercomap f* (*F b*))
⟨*proof*⟩

lemma *filtermap-le-iff-le-filtercomap*: *filtermap f F* ≤ *G* ⟷ *F* ≤ *filtercomap f G*
⟨*proof*⟩

lemma *filtercomap-neq-bot*:
assumes ⋀*P*. *eventually P F* ⟹ ∃ *x*. *P* (*f x*)
shows *filtercomap f F* ≠ *bot*
⟨*proof*⟩

lemma *filtercomap-neq-bot-surj*:
assumes *F* ≠ *bot* **and** *surj f*
shows *filtercomap f F* ≠ *bot*
⟨*proof*⟩

lemma *eventually-filtercomapI* [*intro*]:
assumes *eventually P F*
shows *eventually* (λ*x*. *P* (*f x*)) (*filtercomap f F*)
⟨*proof*⟩

lemma *filtermap-filtercomap*: *filtermap f* (*filtercomap f F*) ≤ *F*
⟨*proof*⟩

lemma *filtercomap-filtermap*: *filtercomap f* (*filtermap f F*) ≥ *F*
⟨*proof*⟩

91.2.4 Standard filters

definition *principal* :: *'a set* ⇒ *'a filter* **where**
principal S = *Abs-filter* (λ*P*. ∀ *x*∈*S*. *P x*)

lemma *eventually-principal*: *eventually P* (*principal S*) ⟷ (∀ *x*∈*S*. *P x*)
⟨*proof*⟩

lemma *eventually-inf-principal*: *eventually P* (*inf F* (*principal s*)) ⟷ *eventually* (λ*x*. *x* ∈ *s* ⟶ *P x*) *F*
⟨*proof*⟩

lemma *principal-UNIV*[*simp*]: *principal UNIV* = *top*
⟨*proof*⟩

lemma *principal-empty*[*simp*]: *principal* {} = *bot*
⟨*proof*⟩

lemma *principal-eq-bot-iff*: *principal X* = *bot* ⟷ *X* = {}
⟨*proof*⟩

lemma *principal-le-iff*[*iff*]: *principal A* ≤ *principal B* ⟷ *A* ⊆ *B*
⟨*proof*⟩

lemma *le-principal*: *F* ≤ *principal A* ⟷ *eventually* (λ*x*. *x* ∈ *A*) *F*
⟨*proof*⟩

lemma *principal-inject*[*iff*]: *principal A* = *principal B* ⟷ *A* = *B*
⟨*proof*⟩

lemma *sup-principal*[*simp*]: *sup* (*principal A*) (*principal B*) = *principal* (*A* ∪ *B*)
⟨*proof*⟩

lemma *inf-principal*[*simp*]: *inf* (*principal A*) (*principal B*) = *principal* (*A* ∩ *B*)
⟨*proof*⟩

lemma *SUP-principal*[*simp*]: (⨆ *i*∈*I*. *principal* (*A i*)) = *principal* (⋃ *i*∈*I*. *A i*)
⟨*proof*⟩

lemma *INF-principal-finite*: *finite X* ⟹ (⨅ *x*∈*X*. *principal* (*f x*)) = *principal* (⋂ *x*∈*X*. *f x*)
⟨*proof*⟩

lemma *filtermap-principal*[*simp*]: *filtermap f* (*principal A*) = *principal* (*f* ‘ *A*)
⟨*proof*⟩

lemma *filtercomap-principal*[*simp*]: *filtercomap f* (*principal A*) = *principal* (*f* −‘ *A*)
⟨*proof*⟩

91.2.5 Order filters

definition *at-top* :: (′*a*::*order*) *filter*
where *at-top* = (⨅ *k*. *principal* {*k* ..})

lemma *at-top-sub*: *at-top* = (⨅ *k*∈{*c*::′*a*::*linorder*..}. *principal* {*k* ..})
⟨*proof*⟩

lemma *eventually-at-top-linorder*: *eventually P at-top* ⟷ (∃ *N*::′*a*::*linorder*. ∀ *n*≥*N*. *P n*)
⟨*proof*⟩

lemma *eventually-filtercomap-at-top-linorder*:
eventually P (*filtercomap f at-top*) ⟷ (∃ *N*::′*a*::*linorder*. ∀ *x*. *f x* ≥ *N* ⟶ *P x*)
⟨*proof*⟩

lemma *eventually-at-top-linorderI*:
fixes *c*::′*a*::*linorder*
assumes ⋀*x*. *c* ≤ *x* ⟹ *P x*
shows *eventually P at-top*
⟨*proof*⟩

lemma *eventually-ge-at-top* [*simp*]:
eventually (λ*x*. (*c*::-::*linorder*) ≤ *x*) *at-top*
⟨*proof*⟩

lemma *eventually-at-top-dense*: *eventually P at-top* ⟷ (∃ *N*::′*a*::{*no-top*, *linorder*}. ∀ *n*>*N*. *P n*)
⟨*proof*⟩

lemma *eventually-filtercomap-at-top-dense*:
eventually P (*filtercomap f at-top*) ⟷ (∃ *N*::′*a*::{*no-top*, *linorder*}. ∀ *x*. *f x* > *N* ⟶ *P x*)
⟨*proof*⟩

lemma *eventually-at-top-not-equal* [*simp*]: *eventually* (λ*x*::′*a*::{*no-top*, *linorder*}. *x* ≠ *c*) *at-top*
⟨*proof*⟩

lemma *eventually-gt-at-top* [*simp*]: *eventually* (λ*x*. (*c*::-::{*no-top*, *linorder*}) < *x*) *at-top*
⟨*proof*⟩

lemma *eventually-all-ge-at-top*:

assumes *eventually P* (*at-top* :: (*'a* :: *linorder*) *filter*)
shows *eventually* (*λx*. *∀ y≥x*. *P y*) *at-top*
⟨*proof*⟩

definition *at-bot* :: (*'a*::*order*) *filter*
where *at-bot* = (*∏ k*. *principal* {.. *k*})

lemma *at-bot-sub*: *at-bot* = (*∏ k∈*{.. *c*::*'a*::*linorder*}. *principal* {.. *k*})
⟨*proof*⟩

lemma *eventually-at-bot-linorder*:
fixes *P* :: *'a*::*linorder* *⇒* *bool* **shows** *eventually P at-bot* *⟷* (*∃ N*. *∀ n≤N*. *P n*)
⟨*proof*⟩

lemma *eventually-filtercomap-at-bot-linorder*:
eventually P (*filtercomap f at-bot*) *⟷* (*∃ N*::*'a*::*linorder*. *∀ x*. *f x* ≤ *N* *⟶* *P x*)
⟨*proof*⟩

lemma *eventually-le-at-bot* [*simp*]:
eventually (*λx*. *x* ≤ (*c*::-::*linorder*)) *at-bot*
⟨*proof*⟩

lemma *eventually-at-bot-dense*: *eventually P at-bot* *⟷* (*∃ N*::*'a*::{*no-bot*, *linorder*}. *∀ n<N*. *P n*)
⟨*proof*⟩

lemma *eventually-filtercomap-at-bot-dense*:
eventually P (*filtercomap f at-bot*) *⟷* (*∃ N*::*'a*::{*no-bot*, *linorder*}. *∀ x*. *f x* < *N* *⟶* *P x*)
⟨*proof*⟩

lemma *eventually-at-bot-not-equal* [*simp*]: *eventually* (*λx*::*'a*::{*no-bot*, *linorder*}. *x* ≠ *c*) *at-bot*
⟨*proof*⟩

lemma *eventually-gt-at-bot* [*simp*]:
eventually (*λx*. *x* < (*c*::-::*unbounded-dense-linorder*)) *at-bot*
⟨*proof*⟩

lemma *trivial-limit-at-bot-linorder* [*simp*]: *¬ trivial-limit* (*at-bot* ::(*'a*::*linorder*) *filter*)
⟨*proof*⟩

lemma *trivial-limit-at-top-linorder* [*simp*]: *¬ trivial-limit* (*at-top* ::(*'a*::*linorder*) *filter*)
⟨*proof*⟩

91.3 Sequentially

abbreviation *sequentially* :: *nat filter*
  where *sequentially* ≡ *at-top*

lemma *eventually-sequentially*:
  eventually P sequentially ⟷ (∃ *N*. ∀ *n*≥*N*. *P n*)
  ⟨*proof*⟩

lemma *sequentially-bot* [*simp*, *intro*]: *sequentially* ≠ *bot*
  ⟨*proof*⟩

lemmas *trivial-limit-sequentially* = *sequentially-bot*

lemma *eventually-False-sequentially* [*simp*]:
  ¬ *eventually* (λ*n*. *False*) *sequentially*
  ⟨*proof*⟩

lemma *le-sequentially*:
  F ≤ *sequentially* ⟷ (∀ *N*. *eventually* (λ*n*. *N* ≤ *n*) *F*)
  ⟨*proof*⟩

lemma *eventually-sequentiallyI* [*intro?*]:
  assumes ⋀*x*. *c* ≤ *x* ⟹ *P x*
  shows *eventually P sequentially*
⟨*proof*⟩

lemma *eventually-sequentially-Suc* [*simp*]: *eventually* (λ*i*. *P* (*Suc i*)) *sequentially* ⟷ *eventually P sequentially*
  ⟨*proof*⟩

lemma *eventually-sequentially-seg* [*simp*]: *eventually* (λ*n*. *P* (*n* + *k*)) *sequentially* ⟷ *eventually P sequentially*
  ⟨*proof*⟩

lemma *filtermap-sequentually-ne-bot*: *filtermap f sequentially* ≠ *bot*
  ⟨*proof*⟩

91.4 Increasing finite subsets

definition *finite-subsets-at-top* **where**
  finite-subsets-at-top A = (⨅ *X*∈{*X*. *finite X* ∧ *X* ⊆ *A*}. *principal* {*Y*. *finite Y* ∧ *X* ⊆ *Y* ∧ *Y* ⊆ *A*})

lemma *eventually-finite-subsets-at-top*:
  eventually P (*finite-subsets-at-top A*) ⟷
    (∃ *X*. *finite X* ∧ *X* ⊆ *A* ∧ (∀ *Y*. *finite Y* ∧ *X* ⊆ *Y* ∧ *Y* ⊆ *A* ⟶ *P Y*))
  ⟨*proof*⟩

lemma *eventually-finite-subsets-at-top-weakI* [*intro*]:

assumes $\bigwedge X.$ *finite* $X \Longrightarrow X \subseteq A \Longrightarrow P\ X$
shows *eventually* P (*finite-subsets-at-top* A)
⟨*proof*⟩

lemma *finite-subsets-at-top-neq-bot* [*simp*]: *finite-subsets-at-top* $A \neq$ *bot*
⟨*proof*⟩

lemma *filtermap-image-finite-subsets-at-top*:
assumes *inj-on* f A
shows *filtermap* ((`) $f$) (*finite-subsets-at-top* $A$) = *finite-subsets-at-top* ($f$ ` A)
⟨*proof*⟩

lemma *eventually-finite-subsets-at-top-finite*:
assumes *finite* A
shows *eventually* P (*finite-subsets-at-top* A) ⟷ $P\ A$
⟨*proof*⟩

lemma *finite-subsets-at-top-finite*: *finite* $A \Longrightarrow$ *finite-subsets-at-top* A = *principal* {A}
⟨*proof*⟩

91.5 The cofinite filter

definition *cofinite* = *Abs-filter* ($\lambda P.$ *finite* {$x.$ ¬ $P\ x$})

abbreviation *Inf-many* :: ('a ⇒ *bool*) ⇒ *bool* (**binder** $\exists_\infty$ *10*)
where *Inf-many* $P \equiv$ *frequently* P *cofinite*

abbreviation *Alm-all* :: ('a ⇒ *bool*) ⇒ *bool* (**binder** $\forall_\infty$ *10*)
where *Alm-all* $P \equiv$ *eventually* P *cofinite*

notation (*ASCII*)
Inf-many (**binder** *INFM* *10*) **and**
Alm-all (**binder** *MOST* *10*)

lemma *eventually-cofinite*: *eventually* P *cofinite* ⟷ *finite* {$x.$ ¬ $P\ x$}
⟨*proof*⟩

lemma *frequently-cofinite*: *frequently* P *cofinite* ⟷ ¬ *finite* {$x.$ $P\ x$}
⟨*proof*⟩

lemma *cofinite-bot*[*simp*]: *cofinite* = (*bot*::'a *filter*) ⟷ *finite* (*UNIV* :: 'a *set*)
⟨*proof*⟩

lemma *cofinite-eq-sequentially*: *cofinite* = *sequentially*
⟨*proof*⟩

91.5.1 Product of filters

definition *prod-filter* :: *'a filter* ⇒ *'b filter* ⇒ (*'a* × *'b*) *filter* (**infixr** $\times_F$ *80*)
where
prod-filter F G =
(⨅(*P*, *Q*)∈{(*P*, *Q*). *eventually P F* ∧ *eventually Q G*}. *principal* {(*x*, *y*). *P x* ∧ *Q y*})

lemma *eventually-prod-filter*: *eventually P* (*F* $\times_F$ *G*) ⟷
(∃ *Pf Pg*. *eventually Pf F* ∧ *eventually Pg G* ∧ (∀ *x y*. *Pf x* ⟶ *Pg y* ⟶ *P* (*x*, *y*)))
⟨*proof*⟩

lemma *eventually-prod1*:
assumes *B* ≠ *bot*
shows (∀$_F$ (*x*, *y*) *in A* $\times_F$ *B*. *P x*) ⟷ (∀$_F$ *x in A*. *P x*)
⟨*proof*⟩

lemma *eventually-prod2*:
assumes *A* ≠ *bot*
shows (∀$_F$ (*x*, *y*) *in A* $\times_F$ *B*. *P y*) ⟷ (∀$_F$ *y in B*. *P y*)
⟨*proof*⟩

lemma *INF-filter-bot-base*:
fixes *F* :: *'a* ⇒ *'b filter*
assumes ∗: ⋀*i j*. *i* ∈ *I* ⟹ *j* ∈ *I* ⟹ ∃ *k*∈*I*. *F k* ≤ *F i* ⊓ *F j*
shows (⨅ *i*∈*I*. *F i*) = *bot* ⟷ (∃ *i*∈*I*. *F i* = *bot*)
⟨*proof*⟩

lemma *Collect-empty-eq-bot*: *Collect P* = {} ⟷ *P* = ⊥
⟨*proof*⟩

lemma *prod-filter-eq-bot*: *A* $\times_F$ *B* = *bot* ⟷ *A* = *bot* ∨ *B* = *bot*
⟨*proof*⟩

lemma *prod-filter-mono*: *F* ≤ *F'* ⟹ *G* ≤ *G'* ⟹ *F* $\times_F$ *G* ≤ *F'* $\times_F$ *G'*
⟨*proof*⟩

lemma *prod-filter-mono-iff*:
assumes *nAB*: *A* ≠ *bot B* ≠ *bot*
shows *A* $\times_F$ *B* ≤ *C* $\times_F$ *D* ⟷ *A* ≤ *C* ∧ *B* ≤ *D*
⟨*proof*⟩

lemma *eventually-prod-same*: *eventually P* (*F* $\times_F$ *F*) ⟷
(∃ *Q*. *eventually Q F* ∧ (∀ *x y*. *Q x* ⟶ *Q y* ⟶ *P* (*x*, *y*)))
⟨*proof*⟩

lemma *eventually-prod-sequentially*:
eventually P (*sequentially* $\times_F$ *sequentially*) ⟷ (∃ *N*. ∀ *m* ≥ *N*. ∀ *n* ≥ *N*. *P* (*n*, *m*))

⟨*proof*⟩

lemma *principal-prod-principal*: *principal A* $\times_F$ *principal B* = *principal* (*A* × *B*)
⟨*proof*⟩

lemma *le-prod-filterI*:
filtermap fst F ≤ *A* ⟹ *filtermap snd F* ≤ *B* ⟹ *F* ≤ *A* $\times_F$ *B*
⟨*proof*⟩

lemma *filtermap-fst-prod-filter*: *filtermap fst* (*A* $\times_F$ *B*) ≤ *A*
⟨*proof*⟩

lemma *filtermap-snd-prod-filter*: *filtermap snd* (*A* $\times_F$ *B*) ≤ *B*
⟨*proof*⟩

lemma *prod-filter-INF*:
assumes *I* ≠ {} **and** *J* ≠ {}
shows (⨅ *i*∈*I*. *A i*) $\times_F$ (⨅ *j*∈*J*. *B j*) = (⨅ *i*∈*I*. ⨅ *j*∈*J*. *A i* $\times_F$ *B j*)
⟨*proof*⟩

lemma *filtermap-Pair*: *filtermap* (λ*x*. (*f x*, *g x*)) *F* ≤ *filtermap f F* $\times_F$ *filtermap g F*
⟨*proof*⟩

lemma *eventually-prodI*: *eventually P F* ⟹ *eventually Q G* ⟹ *eventually* (λ*x*. *P* (*fst x*) ∧ *Q* (*snd x*)) (*F* $\times_F$ *G*)
⟨*proof*⟩

lemma *prod-filter-INF1*: *I* ≠ {} ⟹ (⨅ *i*∈*I*. *A i*) $\times_F$ *B* = (⨅ *i*∈*I*. *A i* $\times_F$ *B*)
⟨*proof*⟩

lemma *prod-filter-INF2*: *J* ≠ {} ⟹ *A* $\times_F$ (⨅ *i*∈*J*. *B i*) = (⨅ *i*∈*J*. *A* $\times_F$ *B i*)
⟨*proof*⟩

lemma *prod-filtermap1*: *prod-filter* (*filtermap f F*) *G* = *filtermap* (*apfst f*) (*prod-filter F G*)
⟨*proof*⟩

lemma *prod-filtermap2*: *prod-filter F* (*filtermap g G*) = *filtermap* (*apsnd g*) (*prod-filter F G*)
⟨*proof*⟩

lemma *prod-filter-assoc*:
prod-filter (*prod-filter F G*) *H* = *filtermap* (λ(*x*, *y*, *z*). ((*x*, *y*), *z*)) (*prod-filter F* (*prod-filter G H*))
⟨*proof*⟩

lemma *prod-filter-principal-singleton*: *prod-filter* (*principal* {*x*}) *F* = *filtermap* (*Pair x*) *F*

⟨*proof*⟩

lemma *prod-filter-principal-singleton2*: *prod-filter F* (*principal* {*x*}) = *filtermap* (λ*a*. (*a*, *x*)) *F*
⟨*proof*⟩

lemma *prod-filter-commute*: *prod-filter F G* = *filtermap prod.swap* (*prod-filter G F*)
⟨*proof*⟩

91.6 Limits

definition *filterlim* :: ('*a* ⇒ '*b*) ⇒ '*b filter* ⇒ '*a filter* ⇒ *bool* **where**
filterlim f F2 F1 ⟷ *filtermap f F1* ≤ *F2*

syntax
-LIM :: *pttrns* ⇒ '*a* ⇒ '*b* ⇒ '*a* ⇒ *bool* ((*3LIM* (-)/ (-)./ (-) :> (-)) [*1000*, *10*, *0*, *10*] *10*)

translations
LIM x F1. *f* :> *F2* == *CONST filterlim* (λ*x*. *f*) *F2 F1*

lemma *filterlim-filtercomapI*: *filterlim f F G* ⟹ *filterlim* (λ*x*. *f* (*g x*)) *F* (*filtercomap g G*)
⟨*proof*⟩

lemma *filterlim-top* [*simp*]: *filterlim f top F*
⟨*proof*⟩

lemma *filterlim-iff*:
(*LIM x F1*. *f x* :> *F2*) ⟷ (∀ *P*. *eventually P F2* ⟶ *eventually* (λ*x*. *P* (*f x*)) *F1*)
⟨*proof*⟩

lemma *filterlim-compose*:
filterlim g F3 F2 ⟹ *filterlim f F2 F1* ⟹ *filterlim* (λ*x*. *g* (*f x*)) *F3 F1*
⟨*proof*⟩

lemma *filterlim-mono*:
filterlim f F2 F1 ⟹ *F2* ≤ *F2'* ⟹ *F1'* ≤ *F1* ⟹ *filterlim f F2' F1'*
⟨*proof*⟩

lemma *filterlim-ident*: *LIM x F*. *x* :> *F*
⟨*proof*⟩

lemma *filterlim-cong*:
F1 = *F1'* ⟹ *F2* = *F2'* ⟹ *eventually* (λ*x*. *f x* = *g x*) *F2* ⟹ *filterlim f F1 F2* = *filterlim g F1' F2'*
⟨*proof*⟩

lemma *filterlim-mono-eventually*:
assumes *filterlim f F G* **and** *ord*: $F \leq F'$ $G' \leq G$
assumes *eq*: *eventually* $(\lambda x.\ f\ x = f'\ x)$ G'
shows *filterlim f' F' G'*
⟨*proof*⟩

lemma *filtermap-mono-strong*: *inj f* $\Longrightarrow$ *filtermap f F* $\leq$ *filtermap f G* $\longleftrightarrow$ $F \leq G$
⟨*proof*⟩

lemma *eventually-compose-filterlim*:
assumes *eventually P F filterlim f F G*
shows *eventually* $(\lambda x.\ P\ (f\ x))$ G
⟨*proof*⟩

lemma *filtermap-eq-strong*: *inj f* $\Longrightarrow$ *filtermap f F* $=$ *filtermap f G* $\longleftrightarrow$ $F = G$
⟨*proof*⟩

lemma *filtermap-fun-inverse*:
assumes *g*: *filterlim g F G*
assumes *f*: *filterlim f G F*
assumes *ev*: *eventually* $(\lambda x.\ f\ (g\ x) = x)$ G
shows *filtermap f F* $= G$
⟨*proof*⟩

lemma *filterlim-principal*:
(*LIM x F. f x :> principal S*) $\longleftrightarrow$ (*eventually* $(\lambda x.\ f\ x \in S)$ F)
⟨*proof*⟩

lemma *filterlim-filtercomap* [*intro*]: *filterlim f F* (*filtercomap f F*)
⟨*proof*⟩

lemma *filterlim-inf*:
(*LIM x F1. f x :> inf F2 F3*) $\longleftrightarrow$ ((*LIM x F1. f x :> F2*) $\wedge$ (*LIM x F1. f x :> F3*))
⟨*proof*⟩

lemma *filterlim-INF*:
(*LIM x F. f x :>* $(\bigsqcap b{\in}B.\ G\ b)$) $\longleftrightarrow$ $(\forall\, b{\in}B.$ *LIM x F. f x :> G b*)
⟨*proof*⟩

lemma *filterlim-INF-INF*:
$(\bigwedge m.\ m \in J \Longrightarrow \exists\, i{\in}I.$ *filtermap f* $(F\ i) \leq G\ m) \Longrightarrow$ *LIM x* $(\bigsqcap i{\in}I.\ F\ i).$ *f x :>* $(\bigsqcap j{\in}J.\ G\ j)$
⟨*proof*⟩

lemma *filterlim-INF'*: $x \in A \Longrightarrow$ *filterlim f F* $(G\ x) \Longrightarrow$ *filterlim f F* $(\bigsqcap\ x{\in}A.\ G\ x)$

⟨*proof*⟩

lemma *filterlim-filtercomap-iff*: *filterlim f* (*filtercomap g G*) *F* ⟷ *filterlim* (*g* ∘ *f*) *G F*
⟨*proof*⟩

lemma *filterlim-iff-le-filtercomap*: *filterlim f F G* ⟷ *G* ≤ *filtercomap f F*
⟨*proof*⟩

lemma *filterlim-base*:
(⋀*m x*. *m* ∈ *J* ⟹ *i m* ∈ *I*) ⟹ (⋀*m x*. *m* ∈ *J* ⟹ *x* ∈ *F* (*i m*) ⟹ *f x* ∈ *G m*) ⟹
LIM x (⨅ *i*∈*I*. *principal* (*F i*)). *f x* :> (⨅ *j*∈*J*. *principal* (*G j*))
⟨*proof*⟩

lemma *filterlim-base-iff*:
assumes *I* ≠ {} **and** *chain*: ⋀*i j*. *i* ∈ *I* ⟹ *j* ∈ *I* ⟹ *F i* ⊆ *F j* ∨ *F j* ⊆ *F i*
shows (*LIM x* (⨅ *i*∈*I*. *principal* (*F i*)). *f x* :> ⨅ *j*∈*J*. *principal* (*G j*)) ⟷
(∀ *j*∈*J*. ∃ *i*∈*I*. ∀ *x*∈*F i*. *f x* ∈ *G j*)
⟨*proof*⟩

lemma *filterlim-filtermap*: *filterlim f F1* (*filtermap g F2*) = *filterlim* (λ*x*. *f* (*g x*)) *F1 F2*
⟨*proof*⟩

lemma *filterlim-sup*:
filterlim f F F1 ⟹ *filterlim f F F2* ⟹ *filterlim f F* (*sup F1 F2*)
⟨*proof*⟩

lemma *filterlim-sequentially-Suc*:
(*LIM x sequentially*. *f* (*Suc x*) :> *F*) ⟷ (*LIM x sequentially*. *f x* :> *F*)
⟨*proof*⟩

lemma *filterlim-Suc*: *filterlim Suc sequentially sequentially*
⟨*proof*⟩

lemma *filterlim-If*:
LIM x inf F (*principal* {*x*. *P x*}). *f x* :> *G* ⟹
LIM x inf F (*principal* {*x*. ¬ *P x*}). *g x* :> *G* ⟹
LIM x F. *if P x then f x else g x* :> *G*
⟨*proof*⟩

lemma *filterlim-Pair*:
LIM x F. *f x* :> *G* ⟹ *LIM x F*. *g x* :> *H* ⟹ *LIM x F*. (*f x*, *g x*) :> *G* ×$_F$ *H*
⟨*proof*⟩

91.7 Limits to *at-top* and *at-bot*

lemma *filterlim-at-top*:

fixes $f :: 'a \Rightarrow ('b::linorder)$
shows $(LIM\ x\ F.\ f\ x :> at\text{-}top) \longleftrightarrow (\forall Z.\ eventually\ (\lambda x.\ Z \leq f\ x)\ F)$
⟨*proof*⟩

lemma *filterlim-at-top-mono*:
$LIM\ x\ F.\ f\ x :> at\text{-}top \Longrightarrow eventually\ (\lambda x.\ f\ x \leq (g\ x::'a::linorder))\ F \Longrightarrow$
$LIM\ x\ F.\ g\ x :> at\text{-}top$
⟨*proof*⟩

lemma *filterlim-at-top-dense*:
fixes $f :: 'a \Rightarrow ('b::unbounded\text{-}dense\text{-}linorder)$
shows $(LIM\ x\ F.\ f\ x :> at\text{-}top) \longleftrightarrow (\forall Z.\ eventually\ (\lambda x.\ Z < f\ x)\ F)$
⟨*proof*⟩

lemma *filterlim-at-top-ge*:
fixes $f :: 'a \Rightarrow ('b::linorder)$ **and** $c :: 'b$
shows $(LIM\ x\ F.\ f\ x :> at\text{-}top) \longleftrightarrow (\forall Z{\geq}c.\ eventually\ (\lambda x.\ Z \leq f\ x)\ F)$
⟨*proof*⟩

lemma *filterlim-at-top-at-top*:
fixes $f :: 'a::linorder \Rightarrow 'b::linorder$
assumes *mono*: $\bigwedge x\ y.\ Q\ x \Longrightarrow Q\ y \Longrightarrow x \leq y \Longrightarrow f\ x \leq f\ y$
assumes *bij*: $\bigwedge x.\ P\ x \Longrightarrow f\ (g\ x) = x\ \bigwedge x.\ P\ x \Longrightarrow Q\ (g\ x)$
assumes Q: *eventually* Q *at-top*
assumes P: *eventually* P *at-top*
shows *filterlim* f *at-top at-top*
⟨*proof*⟩

lemma *filterlim-at-top-gt*:
fixes $f :: 'a \Rightarrow ('b::unbounded\text{-}dense\text{-}linorder)$ **and** $c :: 'b$
shows $(LIM\ x\ F.\ f\ x :> at\text{-}top) \longleftrightarrow (\forall Z{>}c.\ eventually\ (\lambda x.\ Z \leq f\ x)\ F)$
⟨*proof*⟩

lemma *filterlim-at-bot*:
fixes $f :: 'a \Rightarrow ('b::linorder)$
shows $(LIM\ x\ F.\ f\ x :> at\text{-}bot) \longleftrightarrow (\forall Z.\ eventually\ (\lambda x.\ f\ x \leq Z)\ F)$
⟨*proof*⟩

lemma *filterlim-at-bot-dense*:
fixes $f :: 'a \Rightarrow ('b::\{dense\text{-}linorder,\ no\text{-}bot\})$
shows $(LIM\ x\ F.\ f\ x :> at\text{-}bot) \longleftrightarrow (\forall Z.\ eventually\ (\lambda x.\ f\ x < Z)\ F)$
⟨*proof*⟩

lemma *filterlim-at-bot-le*:
fixes $f :: 'a \Rightarrow ('b::linorder)$ **and** $c :: 'b$
shows $(LIM\ x\ F.\ f\ x :> at\text{-}bot) \longleftrightarrow (\forall Z{\leq}c.\ eventually\ (\lambda x.\ Z \geq f\ x)\ F)$
⟨*proof*⟩

lemma *filterlim-at-bot-lt*:

fixes $f :: 'a \Rightarrow ('b::unbounded\text{-}dense\text{-}linorder)$ **and** $c :: 'b$
shows $(LIM\ x\ F.\ f\ x :> at\text{-}bot) \longleftrightarrow (\forall Z<c.\ eventually\ (\lambda x.\ Z \geq f\ x)\ F)$
⟨*proof*⟩

lemma *filterlim-at-top-div-const-nat*:
assumes $c > 0$
shows $filterlim\ (\lambda x::nat.\ x\ div\ c)\ at\text{-}top\ at\text{-}top$
⟨*proof*⟩

lemma *filterlim-finite-subsets-at-top*:
$filterlim\ f\ (finite\text{-}subsets\text{-}at\text{-}top\ A)\ F \longleftrightarrow$
$(\forall X.\ finite\ X \wedge X \subseteq A \longrightarrow eventually\ (\lambda y.\ finite\ (f\ y) \wedge X \subseteq f\ y \wedge f\ y \subseteq A)\ F)$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *filterlim-atMost-at-top*:
$filterlim\ (\lambda n.\ \{..n\})\ (finite\text{-}subsets\text{-}at\text{-}top\ (UNIV :: nat\ set))\ at\text{-}top$
⟨*proof*⟩

lemma *filterlim-lessThan-at-top*:
$filterlim\ (\lambda n.\ \{..<n\})\ (finite\text{-}subsets\text{-}at\text{-}top\ (UNIV :: nat\ set))\ at\text{-}top$
⟨*proof*⟩

lemma *filterlim-minus-const-nat-at-top*:
$filterlim\ (\lambda n.\ n - c)\ sequentially\ sequentially$
⟨*proof*⟩

lemma *filterlim-add-const-nat-at-top*:
$filterlim\ (\lambda n.\ n + c)\ sequentially\ sequentially$
⟨*proof*⟩

91.8 Setup *'a filter* for lifting and transfer

lemma *filtermap-id* [*simp*, *id-simps*]: $filtermap\ id = id$
⟨*proof*⟩

lemma *filtermap-id'* [*simp*]: $filtermap\ (\lambda x.\ x) = (\lambda F.\ F)$
⟨*proof*⟩

context includes *lifting-syntax*
begin

definition *map-filter-on* :: $'a\ set \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a\ filter \Rightarrow 'b\ filter$ **where**
$map\text{-}filter\text{-}on\ X\ f\ F = Abs\text{-}filter\ (\lambda P.\ eventually\ (\lambda x.\ P\ (f\ x) \wedge x \in X)\ F)$

lemma *is-filter-map-filter-on*:
$is\text{-}filter\ (\lambda P.\ \forall_F\ x\ in\ F.\ P\ (f\ x) \wedge x \in X) \longleftrightarrow eventually\ (\lambda x.\ x \in X)\ F$
⟨*proof*⟩

lemma *eventually-map-filter-on*: *eventually P* (*map-filter-on X f F*) = (∀$_F$ *x in F*. *P* (*f x*) ∧ *x* ∈ *X*)
if *eventually* (λ*x*. *x* ∈ *X*) *F*
⟨*proof*⟩

lemma *map-filter-on-UNIV*: *map-filter-on UNIV* = *filtermap*
⟨*proof*⟩

lemma *map-filter-on-comp*: *map-filter-on X f* (*map-filter-on Y g F*) = *map-filter-on Y* (*f* ∘ *g*) *F*
if *g* ‘ *Y* ⊆ *X* **and** *eventually* (λ*x*. *x* ∈ *Y*) *F*
⟨*proof*⟩

inductive *rel-filter* :: ('*a* ⇒ '*b* ⇒ *bool*) ⇒ '*a filter* ⇒ '*b filter* ⇒ *bool* **for** *R F G* **where**
rel-filter R F G **if** *eventually* (*case-prod R*) *Z map-filter-on* {(*x*, *y*). *R x y*} *fst Z* = *F map-filter-on* {(*x*, *y*). *R x y*} *snd Z* = *G*

lemma *rel-filter-eq* [*relator-eq*]: *rel-filter* (=) = (=)
⟨*proof*⟩

lemma *rel-filter-mono* [*relator-mono*]: *rel-filter A* ≤ *rel-filter B* **if** *le*: *A* ≤ *B*
⟨*proof*⟩

lemma *rel-filter-conversep*: *rel-filter* A^{-1-1} = (*rel-filter A*)$^{-1-1}$
⟨*proof*⟩

lemma *rel-filter-distr* [*relator-distr*]:
rel-filter A OO rel-filter B = *rel-filter* (*A OO B*)
⟨*proof*⟩

lemma *filtermap-parametric*: ((*A* ===> *B*) ===> *rel-filter A* ===> *rel-filter B*) *filtermap filtermap*
⟨*proof*⟩

lemma *rel-filter-Grp*: *rel-filter* (*Grp UNIV f*) = *Grp UNIV* (*filtermap f*)
⟨*proof*⟩

lemma *Quotient-filter* [*quot-map*]:
Quotient R Abs Rep T ⟹ *Quotient* (*rel-filter R*) (*filtermap Abs*) (*filtermap Rep*) (*rel-filter T*)
⟨*proof*⟩

lemma *left-total-rel-filter* [*transfer-rule*]: *left-total A* ⟹ *left-total* (*rel-filter A*)
⟨*proof*⟩

lemma *right-total-rel-filter* [*transfer-rule*]: *right-total A* ⟹ *right-total* (*rel-filter A*)

⟨*proof*⟩

lemma *bi-total-rel-filter* [*transfer-rule*]: *bi-total A* ⟹ *bi-total* (*rel-filter A*)
⟨*proof*⟩

lemma *left-unique-rel-filter* [*transfer-rule*]: *left-unique A* ⟹ *left-unique* (*rel-filter A*)
⟨*proof*⟩

lemma *right-unique-rel-filter* [*transfer-rule*]:
right-unique A ⟹ *right-unique* (*rel-filter A*)
⟨*proof*⟩

lemma *bi-unique-rel-filter* [*transfer-rule*]: *bi-unique A* ⟹ *bi-unique* (*rel-filter A*)
⟨*proof*⟩

lemma *eventually-parametric* [*transfer-rule*]:
((*A* ===> (=)) ===> *rel-filter A* ===> (=)) *eventually eventually*
⟨*proof*⟩

lemma *frequently-parametric* [*transfer-rule*]: ((*A* ===> (=)) ===> *rel-filter A* ===> (=)) *frequently frequently*
⟨*proof*⟩

lemma *is-filter-parametric* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-total A*
assumes [*transfer-rule*]: *bi-unique A*
shows (((*A* ===> (=)) ===> (=)) ===> (=)) *is-filter is-filter*
⟨*proof*⟩

lemma *top-filter-parametric* [*transfer-rule*]: *rel-filter A top top* **if** *bi-total A*
⟨*proof*⟩

lemma *bot-filter-parametric* [*transfer-rule*]: *rel-filter A bot bot*
⟨*proof*⟩

lemma *principal-parametric* [*transfer-rule*]: (*rel-set A* ===> *rel-filter A*) *principal principal*
⟨*proof*⟩

lemma *sup-filter-parametric* [*transfer-rule*]:
(*rel-filter A* ===> *rel-filter A* ===> *rel-filter A*) *sup sup*
⟨*proof*⟩

lemma *Sup-filter-parametric* [*transfer-rule*]: (*rel-set* (*rel-filter A*) ===> *rel-filter A*) *Sup Sup*
⟨*proof*⟩

context

fixes $A :: 'a \Rightarrow 'b \Rightarrow bool$
assumes [*transfer-rule*]: *bi-unique A*
begin

lemma *le-filter-parametric* [*transfer-rule*]:
(*rel-filter A* ===> *rel-filter A* ===> (=)) (≤) (≤)
⟨*proof*⟩

lemma *less-filter-parametric* [*transfer-rule*]:
(*rel-filter A* ===> *rel-filter A* ===> (=)) (<) (<)
⟨*proof*⟩

context
assumes [*transfer-rule*]: *bi-total A*
begin

lemma *Inf-filter-parametric* [*transfer-rule*]:
(*rel-set* (*rel-filter A*) ===> *rel-filter A*) *Inf Inf*
⟨*proof*⟩

lemma *inf-filter-parametric* [*transfer-rule*]:
(*rel-filter A* ===> *rel-filter A* ===> *rel-filter A*) *inf inf*
⟨*proof*⟩

end

end

end

context
includes *lifting-syntax*
begin

lemma *prod-filter-parametric* [*transfer-rule*]:
(*rel-filter R* ===> *rel-filter S* ===> *rel-filter* (*rel-prod R S*)) *prod-filter prod-filter*
⟨*proof*⟩

end

Code generation for filters

definition *abstract-filter* :: (*unit* ⇒ *'a filter*) ⇒ *'a filter*
where [*simp*]: *abstract-filter f* = *f* ()

code-datatype *principal abstract-filter*

hide-const (**open**) *abstract-filter*

declare [[*code drop*: *filterlim prod-filter filtermap eventually*

inf :: - *filter* ⇒ - *sup* :: - *filter* ⇒ - *less-eq* :: - *filter* ⇒ - *Abs-filter*]]

declare *filterlim-principal* [*code*]
declare *principal-prod-principal* [*code*]
declare *filtermap-principal* [*code*]
declare *filtercomap-principal* [*code*]
declare *eventually-principal* [*code*]
declare *inf-principal* [*code*]
declare *sup-principal* [*code*]
declare *principal-le-iff* [*code*]

lemma *Rep-filter-iff-eventually* [*simp*, *code*]:
Rep-filter F P ⟷ *eventually P F*
⟨*proof*⟩

lemma *bot-eq-principal-empty* [*code*]:
bot = *principal* {}
⟨*proof*⟩

lemma *top-eq-principal-UNIV* [*code*]:
top = *principal UNIV*
⟨*proof*⟩

instantiation *filter* :: (*equal*) *equal*
begin

definition *equal-filter* :: *'a filter* ⇒ *'a filter* ⇒ *bool*
where *equal-filter F F'* ⟷ *F* = *F'*

lemma *equal-filter* [*code*]:
HOL.equal (*principal A*) (*principal B*) ⟷ *A* = *B*
⟨*proof*⟩

instance
⟨*proof*⟩

end

end

92 Conditionally-complete Lattices

theory *Conditionally-Complete-Lattices*
imports *Finite-Set Lattices-Big Set-Interval*
begin

locale *preordering-bdd* = *preordering*
begin

definition *bdd* :: ‹*'a set* ⇒ *bool*›
  where *unfold*: ‹*bdd A* ⟷ (∃ *M*. ∀ *x* ∈ *A*. *x* ≤ *M*)›

lemma *empty* [*simp*, *intro*]:
  ‹*bdd* {}›
  ⟨*proof*⟩

lemma *I* [*intro*]:
  ‹*bdd A*› **if** ‹⋀*x*. *x* ∈ *A* ⟹ *x* ≤ *M*›
  ⟨*proof*⟩

lemma *E*:
  assumes ‹*bdd A*›
  obtains *M* **where** ‹⋀*x*. *x* ∈ *A* ⟹ *x* ≤ *M*›
  ⟨*proof*⟩

lemma *I2*:
  ‹*bdd* (*f* ‘ *A*)› **if** ‹⋀*x*. *x* ∈ *A* ⟹ *f x* ≤ *M*›
  ⟨*proof*⟩

lemma *mono*:
  ‹*bdd A*› **if** ‹*bdd B*› ‹*A* ⊆ *B*›
  ⟨*proof*⟩

lemma *Int1* [*simp*]:
  ‹*bdd* (*A* ∩ *B*)› **if** ‹*bdd A*›
  ⟨*proof*⟩

lemma *Int2* [*simp*]:
  ‹*bdd* (*A* ∩ *B*)› **if** ‹*bdd B*›
  ⟨*proof*⟩

end

92.1 Preorders

context *preorder*
begin

sublocale *bdd-above*: *preordering-bdd* ‹(≤)› ‹(<)›
  defines *bdd-above-primitive-def*: *bdd-above* = *bdd-above.bdd* ⟨*proof*⟩

sublocale *bdd-below*: *preordering-bdd* ‹(≥)› ‹(>)›
  defines *bdd-below-primitive-def*: *bdd-below* = *bdd-below.bdd* ⟨*proof*⟩

lemma *bdd-above-def*: ‹*bdd-above A* ⟷ (∃ *M*. ∀ *x* ∈ *A*. *x* ≤ *M*)›
  ⟨*proof*⟩

lemma *bdd-below-def*: ‹*bdd-below* $A \longleftrightarrow (\exists M.\ \forall x \in A.\ M \leq x)$›
⟨*proof*⟩

lemma *bdd-aboveI*: $(\bigwedge x.\ x \in A \Longrightarrow x \leq M) \Longrightarrow$ *bdd-above* A
⟨*proof*⟩

lemma *bdd-belowI*: $(\bigwedge x.\ x \in A \Longrightarrow m \leq x) \Longrightarrow$ *bdd-below* A
⟨*proof*⟩

lemma *bdd-aboveI2*: $(\bigwedge x.\ x \in A \Longrightarrow f\ x \leq M) \Longrightarrow$ *bdd-above* $(f`A)$
⟨*proof*⟩

lemma *bdd-belowI2*: $(\bigwedge x.\ x \in A \Longrightarrow m \leq f\ x) \Longrightarrow$ *bdd-below* $(f`A)$
⟨*proof*⟩

lemma *bdd-above-empty*: *bdd-above* $\{\}$
⟨*proof*⟩

lemma *bdd-below-empty*: *bdd-below* $\{\}$
⟨*proof*⟩

lemma *bdd-above-mono*: *bdd-above* $B \Longrightarrow A \subseteq B \Longrightarrow$ *bdd-above* A
⟨*proof*⟩

lemma *bdd-below-mono*: *bdd-below* $B \Longrightarrow A \subseteq B \Longrightarrow$ *bdd-below* A
⟨*proof*⟩

lemma *bdd-above-Int1*: *bdd-above* $A \Longrightarrow$ *bdd-above* $(A \cap B)$
⟨*proof*⟩

lemma *bdd-above-Int2*: *bdd-above* $B \Longrightarrow$ *bdd-above* $(A \cap B)$
⟨*proof*⟩

lemma *bdd-below-Int1*: *bdd-below* $A \Longrightarrow$ *bdd-below* $(A \cap B)$
⟨*proof*⟩

lemma *bdd-below-Int2*: *bdd-below* $B \Longrightarrow$ *bdd-below* $(A \cap B)$
⟨*proof*⟩

lemma *bdd-above-Ioo* [*simp*, *intro*]: *bdd-above* $\{a <..< b\}$
⟨*proof*⟩

lemma *bdd-above-Ico* [*simp*, *intro*]: *bdd-above* $\{a ..< b\}$
⟨*proof*⟩

lemma *bdd-above-Iio* [*simp*, *intro*]: *bdd-above* $\{..< b\}$
⟨*proof*⟩

lemma *bdd-above-Ioc* [*simp*, *intro*]: *bdd-above* $\{a <.. b\}$

⟨*proof*⟩

lemma *bdd-above-Icc* [*simp*, *intro*]: *bdd-above* {*a* .. *b*}
⟨*proof*⟩

lemma *bdd-above-Iic* [*simp*, *intro*]: *bdd-above* {.. *b*}
⟨*proof*⟩

lemma *bdd-below-Ioo* [*simp*, *intro*]: *bdd-below* {*a* <..< *b*}
⟨*proof*⟩

lemma *bdd-below-Ioc* [*simp*, *intro*]: *bdd-below* {*a* <.. *b*}
⟨*proof*⟩

lemma *bdd-below-Ioi* [*simp*, *intro*]: *bdd-below* {*a* <..}
⟨*proof*⟩

lemma *bdd-below-Ico* [*simp*, *intro*]: *bdd-below* {*a* ..< *b*}
⟨*proof*⟩

lemma *bdd-below-Icc* [*simp*, *intro*]: *bdd-below* {*a* .. *b*}
⟨*proof*⟩

lemma *bdd-below-Ici* [*simp*, *intro*]: *bdd-below* {*a* ..}
⟨*proof*⟩

end

context *order-top*
begin

lemma *bdd-above-top* [*simp*, *intro*!]: *bdd-above A*
⟨*proof*⟩

end

context *order-bot*
begin

lemma *bdd-below-bot* [*simp*, *intro*!]: *bdd-below A*
⟨*proof*⟩

end

lemma *bdd-above-image-mono*: *mono f* ⟹ *bdd-above A* ⟹ *bdd-above* (*f*‘*A*)
⟨*proof*⟩

lemma *bdd-below-image-mono*: *mono f* ⟹ *bdd-below A* ⟹ *bdd-below* (*f*‘*A*)
⟨*proof*⟩

lemma *bdd-above-image-antimono*: *antimono f* $\Longrightarrow$ *bdd-below A* $\Longrightarrow$ *bdd-above* (*f*‘*A*)
⟨*proof*⟩

lemma *bdd-below-image-antimono*: *antimono f* $\Longrightarrow$ *bdd-above A* $\Longrightarrow$ *bdd-below* (*f*‘*A*)
⟨*proof*⟩

lemma
fixes *X* :: *'a*::*ordered-ab-group-add set*
shows *bdd-above-uminus*[*simp*]: *bdd-above* (*uminus* ‘ *X*) $\longleftrightarrow$ *bdd-below X*
and *bdd-below-uminus*[*simp*]: *bdd-below* (*uminus* ‘ *X*) $\longleftrightarrow$ *bdd-above X*
⟨*proof*⟩

92.2 Lattices

context *lattice*
begin

lemma *bdd-above-insert* [*simp*]: *bdd-above* (*insert a A*) = *bdd-above A*
⟨*proof*⟩

lemma *bdd-below-insert* [*simp*]: *bdd-below* (*insert a A*) = *bdd-below A*
⟨*proof*⟩

lemma *bdd-finite* [*simp*]:
assumes *finite A* **shows** *bdd-above-finite*: *bdd-above A* **and** *bdd-below-finite*: *bdd-below A*
⟨*proof*⟩

lemma *bdd-above-Un* [*simp*]: *bdd-above* (*A* ∪ *B*) = (*bdd-above A* ∧ *bdd-above B*)
⟨*proof*⟩

lemma *bdd-below-Un* [*simp*]: *bdd-below* (*A* ∪ *B*) = (*bdd-below A* ∧ *bdd-below B*)
⟨*proof*⟩

lemma *bdd-above-image-sup*[*simp*]:
bdd-above ((λ*x*. *sup* (*f x*) (*g x*)) ‘ *A*) $\longleftrightarrow$ *bdd-above* (*f*‘*A*) ∧ *bdd-above* (*g*‘*A*)
⟨*proof*⟩

lemma *bdd-below-image-inf*[*simp*]:
bdd-below ((λ*x*. *inf* (*f x*) (*g x*)) ‘ *A*) $\longleftrightarrow$ *bdd-below* (*f*‘*A*) ∧ *bdd-below* (*g*‘*A*)
⟨*proof*⟩

lemma *bdd-below-UN*[*simp*]: *finite I* $\Longrightarrow$ *bdd-below* ($\bigcup i{\in}I.\ A\ i$) = (∀ *i* ∈ *I*. *bdd-below* (*A i*))
⟨*proof*⟩

lemma *bdd-above-UN*[*simp*]: *finite* $I \Longrightarrow$ *bdd-above* $(\bigcup i{\in}I.\ A\ i) = (\forall\, i \in I.$ *bdd-above* $(A\ i))$
⟨*proof*⟩

end

To avoid name classes with the *complete-lattice*-class we prefix *Sup* and *Inf* in theorem names with c.

92.3 Conditionally complete lattices

class *conditionally-complete-lattice* = *lattice* + *Sup* + *Inf* +
  assumes *cInf-lower*: $x \in X \Longrightarrow$ *bdd-below* $X \Longrightarrow$ *Inf* $X \leq x$
    and *cInf-greatest*: $X \neq \{\} \Longrightarrow (\bigwedge x.\ x \in X \Longrightarrow z \leq x) \Longrightarrow z \leq$ *Inf* X
  assumes *cSup-upper*: $x \in X \Longrightarrow$ *bdd-above* $X \Longrightarrow x \leq$ *Sup* X
    and *cSup-least*: $X \neq \{\} \Longrightarrow (\bigwedge x.\ x \in X \Longrightarrow x \leq z) \Longrightarrow$ *Sup* $X \leq z$
begin

lemma *cSup-upper2*: $x \in X \Longrightarrow y \leq x \Longrightarrow$ *bdd-above* $X \Longrightarrow y \leq$ *Sup* X
  ⟨*proof*⟩

lemma *cInf-lower2*: $x \in X \Longrightarrow x \leq y \Longrightarrow$ *bdd-below* $X \Longrightarrow$ *Inf* $X \leq y$
  ⟨*proof*⟩

lemma *cSup-mono*: $B \neq \{\} \Longrightarrow$ *bdd-above* $A \Longrightarrow (\bigwedge b.\ b \in B \Longrightarrow \exists\, a{\in}A.\ b \leq a)$
$\Longrightarrow$ *Sup* $B \leq$ *Sup* A
  ⟨*proof*⟩

lemma *cInf-mono*: $B \neq \{\} \Longrightarrow$ *bdd-below* $A \Longrightarrow (\bigwedge b.\ b \in B \Longrightarrow \exists\, a{\in}A.\ a \leq b)$
$\Longrightarrow$ *Inf* $A \leq$ *Inf* B
  ⟨*proof*⟩

lemma *cSup-subset-mono*: $A \neq \{\} \Longrightarrow$ *bdd-above* $B \Longrightarrow A \subseteq B \Longrightarrow$ *Sup* $A \leq$ *Sup* B
  ⟨*proof*⟩

lemma *cInf-superset-mono*: $A \neq \{\} \Longrightarrow$ *bdd-below* $B \Longrightarrow A \subseteq B \Longrightarrow$ *Inf* $B \leq$ *Inf* A
  ⟨*proof*⟩

lemma *cSup-eq-maximum*: $z \in X \Longrightarrow (\bigwedge x.\ x \in X \Longrightarrow x \leq z) \Longrightarrow$ *Sup* $X = z$
  ⟨*proof*⟩

lemma *cInf-eq-minimum*: $z \in X \Longrightarrow (\bigwedge x.\ x \in X \Longrightarrow z \leq x) \Longrightarrow$ *Inf* $X = z$
  ⟨*proof*⟩

lemma *cSup-le-iff*: $S \neq \{\} \Longrightarrow$ *bdd-above* $S \Longrightarrow$ *Sup* $S \leq a \longleftrightarrow (\forall\, x{\in}S.\ x \leq a)$
  ⟨*proof*⟩

lemma *le-cInf-iff*: $S \neq \{\} \Longrightarrow$ *bdd-below* $S \Longrightarrow a \leq$ *Inf* $S \longleftrightarrow (\forall\, x{\in}S.\ a \leq x)$
⟨*proof*⟩

lemma *cSup-eq-non-empty*:
assumes *1*: $X \neq \{\}$
assumes *2*: $\bigwedge x.\ x \in X \Longrightarrow x \leq a$
assumes *3*: $\bigwedge y.\ (\bigwedge x.\ x \in X \Longrightarrow x \leq y) \Longrightarrow a \leq y$
shows *Sup* $X = a$
⟨*proof*⟩

lemma *cInf-eq-non-empty*:
assumes *1*: $X \neq \{\}$
assumes *2*: $\bigwedge x.\ x \in X \Longrightarrow a \leq x$
assumes *3*: $\bigwedge y.\ (\bigwedge x.\ x \in X \Longrightarrow y \leq x) \Longrightarrow y \leq a$
shows *Inf* $X = a$
⟨*proof*⟩

lemma *cInf-cSup*: $S \neq \{\} \Longrightarrow$ *bdd-below* $S \Longrightarrow$ *Inf* $S =$ *Sup* $\{x.\ \forall\, s{\in}S.\ x \leq s\}$
⟨*proof*⟩

lemma *cSup-cInf*: $S \neq \{\} \Longrightarrow$ *bdd-above* $S \Longrightarrow$ *Sup* $S =$ *Inf* $\{x.\ \forall\, s{\in}S.\ s \leq x\}$
⟨*proof*⟩

lemma *cSup-insert*: $X \neq \{\} \Longrightarrow$ *bdd-above* $X \Longrightarrow$ *Sup* (*insert* a X) = *sup* a (*Sup* X)
⟨*proof*⟩

lemma *cInf-insert*: $X \neq \{\} \Longrightarrow$ *bdd-below* $X \Longrightarrow$ *Inf* (*insert* a X) = *inf* a (*Inf* X)
⟨*proof*⟩

lemma *cSup-singleton* [*simp*]: *Sup* $\{x\} = x$
⟨*proof*⟩

lemma *cInf-singleton* [*simp*]: *Inf* $\{x\} = x$
⟨*proof*⟩

lemma *cSup-insert-If*: *bdd-above* $X \Longrightarrow$ *Sup* (*insert* a X) = (*if* $X = \{\}$ *then* a *else sup* a (*Sup* X))
⟨*proof*⟩

lemma *cInf-insert-If*: *bdd-below* $X \Longrightarrow$ *Inf* (*insert* a X) = (*if* $X = \{\}$ *then* a *else inf* a (*Inf* X))
⟨*proof*⟩

lemma *le-cSup-finite*: *finite* $X \Longrightarrow x \in X \Longrightarrow x \leq$ *Sup* X
⟨*proof*⟩

lemma *cInf-le-finite*: *finite* $X \Longrightarrow x \in X \Longrightarrow$ *Inf* $X \leq x$
⟨*proof*⟩

lemma *cSup-eq-Sup-fin*: *finite* $X \Longrightarrow X \neq \{\} \Longrightarrow Sup\ X = Sup\text{-}fin\ X$
⟨*proof*⟩

lemma *cInf-eq-Inf-fin*: *finite* $X \Longrightarrow X \neq \{\} \Longrightarrow Inf\ X = Inf\text{-}fin\ X$
⟨*proof*⟩

lemma *cSup-atMost*[*simp*]: $Sup\ \{..x\} = x$
⟨*proof*⟩

lemma *cSup-greaterThanAtMost*[*simp*]: $y < x \Longrightarrow Sup\ \{y<..x\} = x$
⟨*proof*⟩

lemma *cSup-atLeastAtMost*[*simp*]: $y \leq x \Longrightarrow Sup\ \{y..x\} = x$
⟨*proof*⟩

lemma *cInf-atLeast*[*simp*]: $Inf\ \{x..\} = x$
⟨*proof*⟩

lemma *cInf-atLeastLessThan*[*simp*]: $y < x \Longrightarrow Inf\ \{y..<x\} = y$
⟨*proof*⟩

lemma *cInf-atLeastAtMost*[*simp*]: $y \leq x \Longrightarrow Inf\ \{y..x\} = y$
⟨*proof*⟩

lemma *cINF-lower*: *bdd-below* $(f\ `\ A) \Longrightarrow x \in A \Longrightarrow \bigsqcap (f\ `\ A) \leq f\ x$
⟨*proof*⟩

lemma *cINF-greatest*: $A \neq \{\} \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow m \leq f\ x) \Longrightarrow m \leq \bigsqcap (f\ `\ A)$
⟨*proof*⟩

lemma *cSUP-upper*: $x \in A \Longrightarrow$ *bdd-above* $(f\ `\ A) \Longrightarrow f\ x \leq \bigsqcup (f\ `\ A)$
⟨*proof*⟩

lemma *cSUP-least*: $A \neq \{\} \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow f\ x \leq M) \Longrightarrow \bigsqcup (f\ `\ A) \leq M$
⟨*proof*⟩

lemma *cINF-lower2*: *bdd-below* $(f\ `\ A) \Longrightarrow x \in A \Longrightarrow f\ x \leq u \Longrightarrow \bigsqcap (f\ `\ A) \leq u$
⟨*proof*⟩

lemma *cSUP-upper2*: *bdd-above* $(f\ `\ A) \Longrightarrow x \in A \Longrightarrow u \leq f\ x \Longrightarrow u \leq \bigsqcup (f\ `\ A)$
⟨*proof*⟩

lemma *cSUP-const* [*simp*]: $A \neq \{\} \Longrightarrow (\bigsqcup x{\in}A.\ c) = c$
⟨*proof*⟩

lemma *cINF-const* [*simp*]: $A \neq \{\} \Longrightarrow (\bigsqcap x{\in}A.\ c) = c$
⟨*proof*⟩

lemma *le-cINF-iff*: $A \neq \{\} \Longrightarrow$ *bdd-below* $(f \mathbin{`} A) \Longrightarrow u \leq \bigsqcap (f \mathbin{`} A) \longleftrightarrow (\forall x{\in}A.\ u \leq f\ x)$
⟨*proof*⟩

lemma *cSUP-le-iff*: $A \neq \{\} \Longrightarrow$ *bdd-above* $(f \mathbin{`} A) \Longrightarrow \bigsqcup (f \mathbin{`} A) \leq u \longleftrightarrow (\forall x{\in}A.\ f\ x \leq u)$
⟨*proof*⟩

lemma *less-cINF-D*: *bdd-below* $(f`A) \Longrightarrow y < (\bigsqcap i{\in}A.\ f\ i) \Longrightarrow i \in A \Longrightarrow y < f\ i$
⟨*proof*⟩

lemma *cSUP-lessD*: *bdd-above* $(f`A) \Longrightarrow (\bigsqcup i{\in}A.\ f\ i) < y \Longrightarrow i \in A \Longrightarrow f\ i < y$
⟨*proof*⟩

lemma *cINF-insert*: $A \neq \{\} \Longrightarrow$ *bdd-below* $(f \mathbin{`} A) \Longrightarrow \bigsqcap (f \mathbin{`} \mathit{insert}\ a\ A) = \mathit{inf}\ (f\ a)\ (\bigsqcap (f \mathbin{`} A))$
⟨*proof*⟩

lemma *cSUP-insert*: $A \neq \{\} \Longrightarrow$ *bdd-above* $(f \mathbin{`} A) \Longrightarrow \bigsqcup (f \mathbin{`} \mathit{insert}\ a\ A) = \mathit{sup}\ (f\ a)\ (\bigsqcup (f \mathbin{`} A))$
⟨*proof*⟩

lemma *cINF-mono*: $B \neq \{\} \Longrightarrow$ *bdd-below* $(f \mathbin{`} A) \Longrightarrow (\bigwedge m.\ m \in B \Longrightarrow \exists n{\in}A.\ f\ n \leq g\ m) \Longrightarrow \bigsqcap (f \mathbin{`} A) \leq \bigsqcap (g \mathbin{`} B)$
⟨*proof*⟩

lemma *cSUP-mono*: $A \neq \{\} \Longrightarrow$ *bdd-above* $(g \mathbin{`} B) \Longrightarrow (\bigwedge n.\ n \in A \Longrightarrow \exists m{\in}B.\ f\ n \leq g\ m) \Longrightarrow \bigsqcup (f \mathbin{`} A) \leq \bigsqcup (g \mathbin{`} B)$
⟨*proof*⟩

lemma *cINF-superset-mono*: $A \neq \{\} \Longrightarrow$ *bdd-below* $(g \mathbin{`} B) \Longrightarrow A \subseteq B \Longrightarrow (\bigwedge x.\ x \in B \Longrightarrow g\ x \leq f\ x) \Longrightarrow \bigsqcap (g \mathbin{`} B) \leq \bigsqcap (f \mathbin{`} A)$
⟨*proof*⟩

lemma *cSUP-subset-mono*:
$[\![A \neq \{\};$ *bdd-above* $(g \mathbin{`} B);\ A \subseteq B;\ \bigwedge x.\ x \in A \Longrightarrow f\ x \leq g\ x]\!] \Longrightarrow \bigsqcup\ (f \mathbin{`} A) \leq \bigsqcup\ (g \mathbin{`} B)$
⟨*proof*⟩

lemma *less-eq-cInf-inter*: *bdd-below* $A \Longrightarrow$ *bdd-below* $B \Longrightarrow A \cap B \neq \{\} \Longrightarrow \mathit{inf}\ (\mathit{Inf}\ A)\ (\mathit{Inf}\ B) \leq \mathit{Inf}\ (A \cap B)$
⟨*proof*⟩

lemma *cSup-inter-less-eq*: *bdd-above* $A \Longrightarrow$ *bdd-above* $B \Longrightarrow A \cap B \neq \{\} \Longrightarrow \mathit{Sup}\ (A \cap B) \leq \mathit{sup}\ (\mathit{Sup}\ A)\ (\mathit{Sup}\ B)$
⟨*proof*⟩

lemma *cInf-union-distrib*: $A \neq \{\} \Longrightarrow$ *bdd-below* $A \Longrightarrow B \neq \{\} \Longrightarrow$ *bdd-below* $B \Longrightarrow \mathit{Inf}\ (A \cup B) = \mathit{inf}\ (\mathit{Inf}\ A)\ (\mathit{Inf}\ B)$

⟨*proof*⟩

lemma *cINF-union*: $A \neq \{\} \Longrightarrow$ *bdd-below* $(f \mathbin{`} A) \Longrightarrow B \neq \{\} \Longrightarrow$ *bdd-below* $(f \mathbin{`} B) \Longrightarrow \bigsqcap (f \mathbin{`} (A \cup B)) = \bigsqcap (f \mathbin{`} A) \sqcap \bigsqcap (f \mathbin{`} B)$
⟨*proof*⟩

lemma *cSup-union-distrib*: $A \neq \{\} \Longrightarrow$ *bdd-above* $A \Longrightarrow B \neq \{\} \Longrightarrow$ *bdd-above* $B \Longrightarrow$ *Sup* $(A \cup B) =$ *sup* $(Sup\ A)\ (Sup\ B)$
⟨*proof*⟩

lemma *cSUP-union*: $A \neq \{\} \Longrightarrow$ *bdd-above* $(f \mathbin{`} A) \Longrightarrow B \neq \{\} \Longrightarrow$ *bdd-above* $(f \mathbin{`} B) \Longrightarrow \bigsqcup (f \mathbin{`} (A \cup B)) = \bigsqcup (f \mathbin{`} A) \sqcup \bigsqcup (f \mathbin{`} B)$
⟨*proof*⟩

lemma *cINF-inf-distrib*: $A \neq \{\} \Longrightarrow$ *bdd-below* $(f`A) \Longrightarrow$ *bdd-below* $(g`A) \Longrightarrow \bigsqcap (f \mathbin{`} A) \sqcap \bigsqcap (g \mathbin{`} A) = (\bigsqcap a{\in}A.\ inf\ (f\ a)\ (g\ a))$
⟨*proof*⟩

lemma *SUP-sup-distrib*: $A \neq \{\} \Longrightarrow$ *bdd-above* $(f`A) \Longrightarrow$ *bdd-above* $(g`A) \Longrightarrow \bigsqcup (f \mathbin{`} A) \sqcup \bigsqcup (g \mathbin{`} A) = (\bigsqcup a{\in}A.\ sup\ (f\ a)\ (g\ a))$
⟨*proof*⟩

lemma *cInf-le-cSup*:
$A \neq \{\} \Longrightarrow$ *bdd-above* $A \Longrightarrow$ *bdd-below* $A \Longrightarrow$ *Inf* $A \leq$ *Sup* A
⟨*proof*⟩

context
fixes $f :: 'a \Rightarrow 'b$::*conditionally-complete-lattice*
assumes *mono* f
begin

lemma *mono-cInf*: $[\![$*bdd-below* A; $A{\neq}\{\}]\!] \Longrightarrow f\ (Inf\ A) \leq (INF\ x{\in}A.\ f\ x)$
⟨*proof*⟩

lemma *mono-cSup*: $[\![$*bdd-above* A; $A{\neq}\{\}]\!] \Longrightarrow (SUP\ x{\in}A.\ f\ x) \leq f\ (Sup\ A)$
⟨*proof*⟩

lemma *mono-cINF*: $[\![$*bdd-below* $(A`I)$; $I{\neq}\{\}]\!] \Longrightarrow f\ (INF\ i{\in}I.\ A\ i) \leq (INF\ x{\in}I.\ f\ (A\ x))$
⟨*proof*⟩

lemma *mono-cSUP*: $[\![$*bdd-above* $(A`I)$; $I{\neq}\{\}]\!] \Longrightarrow (SUP\ x{\in}I.\ f\ (A\ x)) \leq f\ (SUP\ i{\in}I.\ A\ i)$
⟨*proof*⟩

end

end

The special case of well-orderings

lemma *wellorder-InfI*:
fixes $k :: 'a::\{wellorder, conditionally\text{-}complete\text{-}lattice\}$
assumes $k \in A$ **shows** $Inf\ A \in A$
⟨*proof*⟩

lemma *wellorder-Inf-le1*:
fixes $k :: 'a::\{wellorder, conditionally\text{-}complete\text{-}lattice\}$
assumes $k \in A$ **shows** $Inf\ A \leq k$
⟨*proof*⟩

92.4 Complete lattices

instance *complete-lattice* ⊆ *conditionally-complete-lattice*
⟨*proof*⟩

lemma *cSup-eq*:
fixes $a :: 'a :: \{conditionally\text{-}complete\text{-}lattice, no\text{-}bot\}$
assumes *upper*: $\bigwedge x.\ x \in X \Longrightarrow x \leq a$
assumes *least*: $\bigwedge y.\ (\bigwedge x.\ x \in X \Longrightarrow x \leq y) \Longrightarrow a \leq y$
shows $Sup\ X = a$
⟨*proof*⟩

lemma *cSup-unique*:
fixes $b :: 'a :: \{conditionally\text{-}complete\text{-}lattice, no\text{-}bot\}$
assumes $\bigwedge c.\ (\forall x \in s.\ x \leq c) \longleftrightarrow b \leq c$
shows $Sup\ s = b$
⟨*proof*⟩

lemma *cInf-eq*:
fixes $a :: 'a :: \{conditionally\text{-}complete\text{-}lattice, no\text{-}top\}$
assumes *upper*: $\bigwedge x.\ x \in X \Longrightarrow a \leq x$
assumes *least*: $\bigwedge y.\ (\bigwedge x.\ x \in X \Longrightarrow y \leq x) \Longrightarrow y \leq a$
shows $Inf\ X = a$
⟨*proof*⟩

lemma *cInf-unique*:
fixes $b :: 'a :: \{conditionally\text{-}complete\text{-}lattice, no\text{-}top\}$
assumes $\bigwedge c.\ (\forall x \in s.\ x \geq c) \longleftrightarrow b \geq c$
shows $Inf\ s = b$
⟨*proof*⟩

class *conditionally-complete-linorder* = *conditionally-complete-lattice* + *linorder*
begin

lemma *less-cSup-iff*:
$X \neq \{\} \Longrightarrow bdd\text{-}above\ X \Longrightarrow y < Sup\ X \longleftrightarrow (\exists x{\in}X.\ y < x)$
⟨*proof*⟩

lemma *cInf-less-iff*: $X \neq \{\} \Longrightarrow bdd\text{-}below\ X \Longrightarrow Inf\ X < y \longleftrightarrow (\exists x{\in}X.\ x < y)$

⟨*proof*⟩

lemma *cINF-less-iff*: $A \neq \{\} \Longrightarrow$ *bdd-below* $(f`A) \Longrightarrow (\bigsqcap i{\in}A.\ f\ i) < a \longleftrightarrow (\exists\, x{\in}A.\ f\ x < a)$
⟨*proof*⟩

lemma *less-cSUP-iff*: $A \neq \{\} \Longrightarrow$ *bdd-above* $(f`A) \Longrightarrow a < (\bigsqcup i{\in}A.\ f\ i) \longleftrightarrow (\exists\, x{\in}A.\ a < f\ x)$
⟨*proof*⟩

lemma *less-cSupE*:
assumes $y <$ *Sup* X $X \neq \{\}$ **obtains** x **where** $x \in X$ $y < x$
⟨*proof*⟩

lemma *less-cSupD*:
$X \neq \{\} \Longrightarrow z <$ *Sup* $X \Longrightarrow \exists\, x{\in}X.\ z < x$
⟨*proof*⟩

lemma *cInf-lessD*:
$X \neq \{\} \Longrightarrow$ *Inf* $X < z \Longrightarrow \exists\, x{\in}X.\ x < z$
⟨*proof*⟩

lemma *complete-interval*:
assumes $a < b$ **and** $P\ a$ **and** $\neg\ P\ b$
shows $\exists\, c.\ a \leq c \wedge c \leq b \wedge (\forall x.\ a \leq x \wedge x < c \longrightarrow P\ x) \wedge$
$(\forall\, d.\ (\forall x.\ a \leq x \wedge x < d \longrightarrow P\ x) \longrightarrow d \leq c)$
⟨*proof*⟩

end

92.5 Instances

instance *complete-linorder* < *conditionally-complete-linorder*
⟨*proof*⟩

lemma *cSup-eq-Max*: *finite* $(X{::}'a{::}$*conditionally-complete-linorder set*$) \Longrightarrow X \neq \{\} \Longrightarrow$ *Sup* $X =$ *Max* X
⟨*proof*⟩

lemma *cInf-eq-Min*: *finite* $(X{::}'a{::}$*conditionally-complete-linorder set*$) \Longrightarrow X \neq \{\} \Longrightarrow$ *Inf* $X =$ *Min* X
⟨*proof*⟩

lemma *cSup-lessThan*[*simp*]: *Sup* $\{..{<}x{::}'a{::}\{$*conditionally-complete-linorder*, *no-bot*, *dense-linorder*$\}\} = x$
⟨*proof*⟩

lemma *cSup-greaterThanLessThan*[*simp*]: $y < x \Longrightarrow$ *Sup* $\{y{<}..{<}x{::}'a{::}\{$*conditionally-complete-linorder*, *dense-linorder*$\}\} = x$

⟨*proof*⟩

lemma *cSup-atLeastLessThan*[*simp*]: $y < x \Longrightarrow Sup\ \{y..<x::'a::\{conditionally\text{-}complete\text{-}linorder, dense\text{-}linorder\}\} = x$
⟨*proof*⟩

lemma *cInf-greaterThan*[*simp*]: $Inf\ \{x::'a::\{conditionally\text{-}complete\text{-}linorder, no\text{-}top, dense\text{-}linorder\}\ <..\} = x$
⟨*proof*⟩

lemma *cInf-greaterThanAtMost*[*simp*]: $y < x \Longrightarrow Inf\ \{y<..x::'a::\{conditionally\text{-}complete\text{-}linorder, dense\text{-}linorder\}\} = y$
⟨*proof*⟩

lemma *cInf-greaterThanLessThan*[*simp*]: $y < x \Longrightarrow Inf\ \{y<..<x::'a::\{conditionally\text{-}complete\text{-}linorder, dense\text{-}linorder\}\} = y$
⟨*proof*⟩

lemma *Inf-insert-finite*:
fixes $S :: 'a::conditionally\text{-}complete\text{-}linorder\ set$
shows $finite\ S \Longrightarrow Inf\ (insert\ x\ S) = (if\ S = \{\}\ then\ x\ else\ min\ x\ (Inf\ S))$
⟨*proof*⟩

lemma *Sup-insert-finite*:
fixes $S :: 'a::conditionally\text{-}complete\text{-}linorder\ set$
shows $finite\ S \Longrightarrow Sup\ (insert\ x\ S) = (if\ S = \{\}\ then\ x\ else\ max\ x\ (Sup\ S))$
⟨*proof*⟩

lemma *finite-imp-less-Inf*:
fixes $a :: 'a::conditionally\text{-}complete\text{-}linorder$
shows $[\![finite\ X;\ x \in X;\ \bigwedge x.\ x{\in}X \Longrightarrow a < x]\!] \Longrightarrow a < Inf\ X$
⟨*proof*⟩

lemma *finite-less-Inf-iff*:
fixes $a :: 'a :: conditionally\text{-}complete\text{-}linorder$
shows $[\![finite\ X;\ X \neq \{\}]\!] \Longrightarrow a < Inf\ X \longleftrightarrow (\forall x \in X.\ a < x)$
⟨*proof*⟩

lemma *finite-imp-Sup-less*:
fixes $a :: 'a::conditionally\text{-}complete\text{-}linorder$
shows $[\![finite\ X;\ x \in X;\ \bigwedge x.\ x{\in}X \Longrightarrow a > x]\!] \Longrightarrow a > Sup\ X$
⟨*proof*⟩

lemma *finite-Sup-less-iff*:
fixes $a :: 'a :: conditionally\text{-}complete\text{-}linorder$
shows $[\![finite\ X;\ X \neq \{\}]\!] \Longrightarrow a > Sup\ X \longleftrightarrow (\forall x \in X.\ a > x)$
⟨*proof*⟩

class *linear-continuum* = *conditionally-complete-linorder* + *dense-linorder* +

assumes *UNIV-not-singleton*: $\exists\, a\ b{::}'a.\ a \neq b$
begin

lemma *ex-gt-or-lt*: $\exists\, b.\ a < b \vee b < a$
⟨*proof*⟩

end

context
fixes $f{::}'a \Rightarrow 'b{::}\{$*conditionally-complete-linorder*,*ordered-ab-group-add*$\}$
begin

lemma *bdd-above-uminus-image*: *bdd-above* $((\lambda x.\ -\ f\ x)\ `\ A) \longleftrightarrow$ *bdd-below* $(f\ `\ A)$
⟨*proof*⟩

lemma *bdd-below-uminus-image*: *bdd-below* $((\lambda x.\ -\ f\ x)\ `\ A) \longleftrightarrow$ *bdd-above* $(f\ `\ A)$
⟨*proof*⟩

lemma *uminus-cSUP*:
assumes *bdd-above* $(f\ `\ A)$ $A \neq \{\}$
shows $-\ (SUP\ x{\in}A.\ f\ x) = (INF\ x{\in}A.\ -\ f\ x)$
⟨*proof*⟩

end

context
fixes $f{::}'a \Rightarrow 'b{::}\{$*conditionally-complete-linorder*,*ordered-ab-group-add*$\}$
begin

lemma *uminus-cINF*:
assumes *bdd-below* $(f\ `\ A)$ $A \neq \{\}$
shows $-\ (INF\ x{\in}A.\ f\ x) = (SUP\ x{\in}A.\ -\ f\ x)$
⟨*proof*⟩

lemma *Sup-add-eq*:
assumes *bdd-above* $(f\ `\ A)$ $A \neq \{\}$
shows $(SUP\ x{\in}A.\ a + f\ x) = a + (SUP\ x{\in}A.\ f\ x)$ (**is** *?L=?R*)
⟨*proof*⟩

lemma *Inf-add-eq*: — you don't get a shorter proof by duality
assumes *bdd-below* $(f\ `\ A)$ $A \neq \{\}$
shows $(INF\ x{\in}A.\ a + f\ x) = a + (INF\ x{\in}A.\ f\ x)$ (**is** *?L=?R*)
⟨*proof*⟩

end

instantiation *nat* :: *conditionally-complete-linorder*
begin

definition *Sup* (*X*::*nat set*) = (*if X*={} *then 0 else Max X*)
definition *Inf* (*X*::*nat set*) = (*LEAST n. n* ∈ *X*)

lemma *bdd-above-nat*: *bdd-above X* ⟷ *finite* (*X*::*nat set*)
⟨*proof*⟩

instance
⟨*proof*⟩

end

lemma *Inf-nat-def1*:
fixes *K*::*nat set*
assumes *K* ≠ {}
shows *Inf K* ∈ *K*
⟨*proof*⟩

lemma *Sup-nat-empty* [*simp*]: *Sup* {} = (*0*::*nat*)
⟨*proof*⟩

instantiation *int* :: *conditionally-complete-linorder*
begin

definition *Sup* (*X*::*int set*) = (*THE x. x* ∈ *X* ∧ (∀ *y*∈*X. y* ≤ *x*))
definition *Inf* (*X*::*int set*) = − (*Sup* (*uminus* ‘ *X*))

instance
⟨*proof*⟩
end

lemma *interval-cases*:
fixes *S* :: *'a* :: *conditionally-complete-linorder set*
assumes *ivl*: ⋀*a b x. a* ∈ *S* ⟹ *b* ∈ *S* ⟹ *a* ≤ *x* ⟹ *x* ≤ *b* ⟹ *x* ∈ *S*
shows ∃ *a b. S* = {} ∨
S = *UNIV* ∨
S = {..<*b*} ∨
S = {..*b*} ∨
S = {*a*<..} ∨
S = {*a*..} ∨
S = {*a*<..<*b*} ∨
S = {*a*<..*b*} ∨
S = {*a*..<*b*} ∨
S = {*a*..*b*}
⟨*proof*⟩

lemma *cSUP-eq-cINF-D*:
fixes f :: - ⇒ 'b::*conditionally-complete-lattice*
assumes *eq*: $(\bigsqcup x{\in}A.\ f\ x) = (\bigsqcap x{\in}A.\ f\ x)$
and *bdd*: *bdd-above* $(f\ `\ A)$ *bdd-below* $(f\ `\ A)$
and *a*: $a \in A$
shows $f\ a = (\bigsqcap x{\in}A.\ f\ x)$
⟨*proof*⟩

lemma *cSUP-UNION*:
fixes f :: - ⇒ 'b::*conditionally-complete-lattice*
assumes *ne*: $A \neq \{\}$ $\bigwedge x.\ x \in A \Longrightarrow B(x) \neq \{\}$
and *bdd-UN*: *bdd-above* $(\bigcup x{\in}A.\ f\ `\ B\ x)$
shows $(\bigsqcup z \in \bigcup x{\in}A.\ B\ x.\ f\ z) = (\bigsqcup x{\in}A.\ \bigsqcup z{\in}B\ x.\ f\ z)$
⟨*proof*⟩

lemma *cINF-UNION*:
fixes f :: - ⇒ 'b::*conditionally-complete-lattice*
assumes *ne*: $A \neq \{\}$ $\bigwedge x.\ x \in A \Longrightarrow B(x) \neq \{\}$
and *bdd-UN*: *bdd-below* $(\bigcup x{\in}A.\ f\ `\ B\ x)$
shows $(\bigsqcap z \in \bigcup x{\in}A.\ B\ x.\ f\ z) = (\bigsqcap x{\in}A.\ \bigsqcap z{\in}B\ x.\ f\ z)$
⟨*proof*⟩

lemma *cSup-abs-le*:
fixes S :: ('a::{*linordered-idom*,*conditionally-complete-linorder*}) *set*
shows $S \neq \{\} \Longrightarrow (\bigwedge x.\ x{\in}S \Longrightarrow |x| \leq a) \Longrightarrow |Sup\ S| \leq a$
⟨*proof*⟩

end

93 Factorial Function, Rising Factorials

theory *Factorial*
imports *Groups-List*
begin

93.1 Factorial Function

context *semiring-char-0*
begin

definition *fact* :: *nat* ⇒ 'a
where *fact-prod*: $fact\ n = of\text{-}nat\ (\prod \{1..n\})$

lemma *fact-prod-Suc*: $fact\ n = of\text{-}nat\ (prod\ Suc\ \{0..<n\})$
⟨*proof*⟩

lemma *fact-prod-rev*: $fact\ n = of\text{-}nat\ (\prod i = 0..<n.\ n - i)$
⟨*proof*⟩

lemma *fact-0* [*simp*]: *fact 0 = 1*
⟨*proof*⟩

lemma *fact-1* [*simp*]: *fact 1 = 1*
⟨*proof*⟩

lemma *fact-Suc-0* [*simp*]: *fact (Suc 0) = 1*
⟨*proof*⟩

lemma *fact-Suc* [*simp*]: *fact (Suc n) = of-nat (Suc n) * fact n*
⟨*proof*⟩

lemma *fact-2* [*simp*]: *fact 2 = 2*
⟨*proof*⟩

lemma *fact-split*: $k \leq n \Longrightarrow$ *fact n = of-nat (prod Suc {n − k..<n}) * fact (n − k)*
⟨*proof*⟩

end

lemma *of-nat-fact* [*simp*]: *of-nat (fact n) = fact n*
⟨*proof*⟩

lemma *of-int-fact* [*simp*]: *of-int (fact n) = fact n*
⟨*proof*⟩

lemma *fact-reduce*: $n > 0 \Longrightarrow$ *fact n = of-nat n * fact (n − 1)*
⟨*proof*⟩

lemma *fact-nonzero* [*simp*]: *fact n* $\neq$ *(0::'a::{semiring-char-0,semiring-no-zero-divisors})*
⟨*proof*⟩

lemma *fact-mono-nat*: $m \leq n \Longrightarrow$ *fact m ≤ (fact n :: nat)*
⟨*proof*⟩

lemma *fact-in-Nats*: *fact n* $\in \mathbb{N}$
⟨*proof*⟩

lemma *fact-in-Ints*: *fact n* $\in \mathbb{Z}$
⟨*proof*⟩

context
assumes *SORT-CONSTRAINT('a::linordered-semidom)*
begin

lemma *fact-mono*: $m \leq n \Longrightarrow$ *fact m ≤ (fact n :: 'a)*
⟨*proof*⟩

lemma *fact-ge-1* [*simp*]: *fact n* $\geq$ (*1* :: *'a*)
⟨*proof*⟩

lemma *fact-gt-zero* [*simp*]: *fact n* $>$ (*0* :: *'a*)
⟨*proof*⟩

lemma *fact-ge-zero* [*simp*]: *fact n* $\geq$ (*0* :: *'a*)
⟨*proof*⟩

lemma *fact-not-neg* [*simp*]: $\neg$ *fact n* $<$ (*0* :: *'a*)
⟨*proof*⟩

lemma *fact-le-power*: *fact n* $\leq$ (*of-nat* (*n*^*n*) :: *'a*)
⟨*proof*⟩

end

lemma *fact-less-mono-nat*: *0* $<$ *m* $\Longrightarrow$ *m* $<$ *n* $\Longrightarrow$ *fact m* $<$ (*fact n* :: *nat*)
⟨*proof*⟩

lemma *fact-less-mono*: *0* $<$ *m* $\Longrightarrow$ *m* $<$ *n* $\Longrightarrow$ *fact m* $<$ (*fact n* :: *'a*::*linordered-semidom*)
⟨*proof*⟩

lemma *fact-ge-Suc-0-nat* [*simp*]: *fact n* $\geq$ *Suc 0*
⟨*proof*⟩

lemma *dvd-fact*: *1* $\leq$ *m* $\Longrightarrow$ *m* $\leq$ *n* $\Longrightarrow$ *m dvd fact n*
⟨*proof*⟩

lemma *fact-ge-self*: *fact n* $\geq$ *n*
⟨*proof*⟩

lemma *fact-dvd*: *n* $\leq$ *m* $\Longrightarrow$ *fact n dvd* (*fact m* :: *'a*::*linordered-semidom*)
⟨*proof*⟩

lemma *fact-mod*: *m* $\leq$ *n* $\Longrightarrow$ *fact n mod* (*fact m* :: *'a*::{*semidom-modulo*, *linordered-semidom*})
= *0*
⟨*proof*⟩

lemma *fact-eq-fact-times*:
assumes *m* $\geq$ *n*
shows *fact m* = *fact n* $*$ $\prod$ {*Suc n*..*m*}
⟨*proof*⟩

lemma *fact-div-fact*:
assumes *m* $\geq$ *n*
shows *fact m div fact n* = $\prod$ {*n* + *1*..*m*}
⟨*proof*⟩

lemma *fact-num-eq-if*: *fact m* = (*if m* = *0 then 1 else of-nat m* ∗ *fact* (*m* − *1*))
⟨*proof*⟩

lemma *fact-div-fact-le-pow*:
assumes *r* ≤ *n*
shows *fact n div fact* (*n* − *r*) ≤ *n* ^ *r*
⟨*proof*⟩

lemma *prod-Suc-fact*: *prod Suc* {*0*..<*n*} = *fact n*
⟨*proof*⟩

lemma *prod-Suc-Suc-fact*: *prod Suc* {*Suc 0*..<*n*} = *fact n*
⟨*proof*⟩

lemma *fact-numeral*: *fact* (*numeral k*) = *numeral k* ∗ *fact* (*pred-numeral k*)
— Evaluation for specific numerals
⟨*proof*⟩

93.2 Pochhammer's symbol: generalized rising factorial

See https://en.wikipedia.org/wiki/Pochhammer_symbol.

context *comm-semiring-1*
begin

definition *pochhammer* :: *'a* ⇒ *nat* ⇒ *'a*
where *pochhammer-prod*: *pochhammer a n* = *prod* (λ*i*. *a* + *of-nat i*) {*0*..<*n*}

lemma *pochhammer-prod-rev*: *pochhammer a n* = *prod* (λ*i*. *a* + *of-nat* (*n* − *i*))
{*1*..*n*}
⟨*proof*⟩

lemma *pochhammer-Suc-prod*: *pochhammer a* (*Suc n*) = *prod* (λ*i*. *a* + *of-nat i*)
{*0*..*n*}
⟨*proof*⟩

lemma *pochhammer-Suc-prod-rev*: *pochhammer a* (*Suc n*) = *prod* (λ*i*. *a* + *of-nat*
(*n* − *i*)) {*0*..*n*}
⟨*proof*⟩

lemma *pochhammer-0* [*simp*]: *pochhammer a 0* = *1*
⟨*proof*⟩

lemma *pochhammer-1* [*simp*]: *pochhammer a 1* = *a*
⟨*proof*⟩

lemma *pochhammer-Suc0* [*simp*]: *pochhammer a* (*Suc 0*) = *a*
⟨*proof*⟩

lemma *pochhammer-Suc*: *pochhammer a (Suc n) = pochhammer a n * (a + of-nat n)*
⟨*proof*⟩

end

lemma *pochhammer-nonneg*:
fixes *x :: 'a :: linordered-semidom*
shows *x > 0 ⟹ pochhammer x n ≥ 0*
⟨*proof*⟩

lemma *pochhammer-pos*:
fixes *x :: 'a :: linordered-semidom*
shows *x > 0 ⟹ pochhammer x n > 0*
⟨*proof*⟩

context *comm-semiring-1*
begin

lemma *pochhammer-of-nat*: *pochhammer (of-nat x) n = of-nat (pochhammer x n)*
⟨*proof*⟩

end

context *comm-ring-1*
begin

lemma *pochhammer-of-int*: *pochhammer (of-int x) n = of-int (pochhammer x n)*
⟨*proof*⟩

end

lemma *pochhammer-rec*: *pochhammer a (Suc n) = a * pochhammer (a + 1) n*
⟨*proof*⟩

lemma *pochhammer-rec'*: *pochhammer z (Suc n) = (z + of-nat n) * pochhammer z n*
⟨*proof*⟩

lemma *pochhammer-fact*: *fact n = pochhammer 1 n*
⟨*proof*⟩

lemma *pochhammer-of-nat-eq-0-lemma*: *k > n ⟹ pochhammer (− (of-nat n :: 'a:: idom)) k = 0*
⟨*proof*⟩

lemma *pochhammer-of-nat-eq-0-lemma'*:
assumes *kn*: *k ≤ n*
shows *pochhammer (− (of-nat n :: 'a::{idom,ring-char-0})) k ≠ 0*

⟨*proof*⟩

lemma *pochhammer-of-nat-eq-0-iff*:
pochhammer (− (*of-nat n* :: 'a::{*idom*,*ring-char-0*})) *k* = *0* ⟷ *k* > *n*
(**is** *?l* = *?r*)
⟨*proof*⟩

lemma *pochhammer-0-left*:
pochhammer 0 n = (*if n* = *0 then 1 else 0*)
⟨*proof*⟩

lemma *pochhammer-eq-0-iff*: *pochhammer a n* = (*0*::'a::*field-char-0*) ⟷ (∃ *k* < *n*. *a* = − *of-nat k*)
⟨*proof*⟩

lemma *pochhammer-eq-0-mono*:
pochhammer a n = (*0*::'a::*field-char-0*) ⟹ *m* ≥ *n* ⟹ *pochhammer a m* = *0*
⟨*proof*⟩

lemma *pochhammer-neq-0-mono*:
pochhammer a m ≠ (*0*::'a::*field-char-0*) ⟹ *m* ≥ *n* ⟹ *pochhammer a n* ≠ *0*
⟨*proof*⟩

lemma *pochhammer-minus*:
pochhammer (− *b*) *k* = ((− *1*) ^ *k* :: 'a::*comm-ring-1*) ∗ *pochhammer* (*b* − *of-nat k* + *1*) *k*
⟨*proof*⟩

lemma *pochhammer-minus'*:
pochhammer (*b* − *of-nat k* + *1*) *k* = ((− *1*) ^ *k* :: 'a::*comm-ring-1*) ∗ *pochhammer* (− *b*) *k*
⟨*proof*⟩

lemma *pochhammer-same*: *pochhammer* (− *of-nat n*) *n* =
((− *1*) ^ *n* :: 'a::{*semiring-char-0*,*comm-ring-1*,*semiring-no-zero-divisors*}) ∗ *fact n*
⟨*proof*⟩

lemma *pochhammer-product'*: *pochhammer z* (*n* + *m*) = *pochhammer z n* ∗ *pochhammer* (*z* + *of-nat n*) *m*
⟨*proof*⟩

lemma *pochhammer-product*:
m ≤ *n* ⟹ *pochhammer z n* = *pochhammer z m* ∗ *pochhammer* (*z* + *of-nat m*) (*n* − *m*)
⟨*proof*⟩

lemma *pochhammer-times-pochhammer-half*:
fixes *z* :: 'a::*field-char-0*

shows *pochhammer z (Suc n) * pochhammer (z + 1/2) (Suc n) = (∏ k=0..2*n+1. z + of-nat k / 2)*
⟨*proof*⟩

lemma *pochhammer-double*:
fixes *z :: 'a::field-char-0*
shows *pochhammer (2 * z) (2 * n) = of-nat (2^(2*n)) * pochhammer z n * pochhammer (z+1/2) n*
⟨*proof*⟩

lemma *fact-double*:
*fact (2 * n) = (2 ^ (2 * n) * pochhammer (1 / 2) n * fact n :: 'a::field-char-0)*
⟨*proof*⟩

lemma *pochhammer-absorb-comp*: *(r − of-nat k) * pochhammer (− r) k = r * pochhammer (−r + 1) k*
(**is** *?lhs = ?rhs*)
for *r :: 'a::comm-ring-1*
⟨*proof*⟩

93.3 Misc

lemma *fact-code* [*code*]:
fact n = (of-nat (fold-atLeastAtMost-nat ((*)) 2 n 1) :: 'a::semiring-char-0)
⟨*proof*⟩

lemma *pochhammer-code* [*code*]:
pochhammer a n =
(if n = 0 then 1
*else fold-atLeastAtMost-nat (λn acc. (a + of-nat n) * acc) 0 (n − 1) 1)*
⟨*proof*⟩

lemma *mult-less-iff1*: *0 < z ⟹ x * z < y * z ⟷ x < y*
for *x y z :: 'a::linordered-idom*
⟨*proof*⟩

lemma *mult-le-cancel-iff1*: *0 < z ⟹ x * z ≤ y * z ⟷ x ≤ y*
for *x y z :: 'a::linordered-idom*
⟨*proof*⟩

lemma *mult-le-cancel-iff2*: *0 < z ⟹ z * x ≤ z * y ⟷ x ≤ y*
for *x y z :: 'a::linordered-idom*
⟨*proof*⟩

end

94 Binomial Coefficients and Binomial Theorem

theory *Binomial*
imports *Presburger Factorial*
begin

94.1 Binomial coefficients

This development is based on the work of Andy Gordon and Florian Kammueller.

Combinatorial definition

definition *binomial* :: *nat* ⇒ *nat* ⇒ *nat* (**infixl** *choose* 65)
where *n choose k* = *card* {*K*∈*Pow* {*0*..<*n*}. *card K* = *k*}

theorem *n-subsets*:
assumes *finite A*
shows *card* {*B*. *B* ⊆ *A* ∧ *card B* = *k*} = *card A choose k*
⟨*proof*⟩

Recursive characterization

lemma *binomial-n-0* [*simp*]: *n choose 0* = *1*
⟨*proof*⟩

lemma *binomial-0-Suc* [*simp*]: *0 choose Suc k* = *0*
⟨*proof*⟩

lemma *binomial-Suc-Suc* [*simp*]: *Suc n choose Suc k* = (*n choose k*) + (*n choose Suc k*)
⟨*proof*⟩

lemma *binomial-eq-0*: *n* < *k* ⟹ *n choose k* = *0*
⟨*proof*⟩

lemma *zero-less-binomial*: *k* ≤ *n* ⟹ *n choose k* > *0*
⟨*proof*⟩

lemma *binomial-eq-0-iff* [*simp*]: *n choose k* = *0* ⟷ *n* < *k*
⟨*proof*⟩

lemma *zero-less-binomial-iff* [*simp*]: *n choose k* > *0* ⟷ *k* ≤ *n*
⟨*proof*⟩

lemma *binomial-n-n* [*simp*]: *n choose n* = *1*
⟨*proof*⟩

lemma *binomial-Suc-n* [*simp*]: *Suc n choose n* = *Suc n*
⟨*proof*⟩

lemma *binomial-1* [*simp*]: *n choose Suc 0 = n*
⟨*proof*⟩

lemma *choose-reduce-nat*:
0 < n ⟹ 0 < k ⟹
n choose k = ((n − 1) choose (k − 1)) + ((n − 1) choose k)
⟨*proof*⟩

lemma *Suc-times-binomial-eq*: *Suc n ∗ (n choose k) = (Suc n choose Suc k) ∗ Suc k*
⟨*proof*⟩

lemma *binomial-le-pow2*: *n choose k ≤ 2^n*
⟨*proof*⟩

The absorption property.

lemma *Suc-times-binomial*: *Suc k ∗ (Suc n choose Suc k) = Suc n ∗ (n choose k)*
⟨*proof*⟩

This is the well-known version of absorption, but it's harder to use because of the need to reason about division.

lemma *binomial-Suc-Suc-eq-times*: *(Suc n choose Suc k) = (Suc n ∗ (n choose k)) div Suc k*
⟨*proof*⟩

Another version of absorption, with *−1* instead of *Suc*.

lemma *times-binomial-minus1-eq*: *0 < k ⟹ k ∗ (n choose k) = n ∗ ((n − 1) choose (k − 1))*
⟨*proof*⟩

94.2 The binomial theorem (courtesy of Tobias Nipkow):

Avigad's version, generalized to any commutative ring

theorem *binomial-ring*: *(a + b :: 'a::comm-semiring-1)^n =*
(∑ k≤n. (of-nat (n choose k)) ∗ a^k ∗ b^(n−k))
⟨*proof*⟩

Original version for the naturals.

corollary *binomial*: *(a + b :: nat)^n = (∑ k≤n. (of-nat (n choose k)) ∗ a^k ∗ b^(n − k))*
⟨*proof*⟩

lemma *binomial-fact-lemma*: *k ≤ n ⟹ fact k ∗ fact (n − k) ∗ (n choose k) = fact n*
⟨*proof*⟩

lemma *binomial-fact'*:
assumes *k ≤ n*

shows *n choose k = fact n div (fact k * fact (n − k))*
⟨*proof*⟩

lemma *binomial-fact*:
assumes *kn*: *k ≤ n*
shows (*of-nat* (*n choose k*) :: *'a*::*field-char-0*) = *fact n* / (*fact k * fact* (*n − k*))
⟨*proof*⟩

lemma *fact-binomial*:
assumes *k ≤ n*
shows *fact k * of-nat* (*n choose k*) = (*fact n* / *fact* (*n − k*) :: *'a*::*field-char-0*)
⟨*proof*⟩

lemma *choose-two*: *n choose 2 = n * (n − 1) div 2*
⟨*proof*⟩

lemma *choose-row-sum*: (∑ *k*≤*n*. *n choose k*) = *2^n*
⟨*proof*⟩

lemma *sum-choose-lower*: (∑ *k*≤*n*. (*r*+*k*) *choose k*) = *Suc* (*r*+*n*) *choose n*
⟨*proof*⟩

lemma *sum-choose-upper*: (∑ *k*≤*n*. *k choose m*) = *Suc n choose Suc m*
⟨*proof*⟩

lemma *choose-alternating-sum*:
n > 0 ⟹ (∑ *i*≤*n*. (*−1*)^*i* * *of-nat* (*n choose i*)) = (*0* :: *'a*::*comm-ring-1*)
⟨*proof*⟩

lemma *choose-even-sum*:
assumes *n > 0*
shows *2 * (∑ i≤n. if even i then of-nat (n choose i) else 0) = (2 ^ n :: 'a::comm-ring-1)*
⟨*proof*⟩

lemma *choose-odd-sum*:
assumes *n > 0*
shows *2 * (∑ i≤n. if odd i then of-nat (n choose i) else 0) = (2 ^ n :: 'a::comm-ring-1)*
⟨*proof*⟩

NW diagonal sum property

lemma *sum-choose-diagonal*:
assumes *m ≤ n*
shows (∑ *k*≤*m*. (*n − k*) *choose* (*m − k*)) = *Suc n choose m*
⟨*proof*⟩

94.3 Generalized binomial coefficients

definition *gbinomial* :: *'a*::{*semidom-divide*,*semiring-char-0*} ⇒ *nat* ⇒ *'a* (**infixl** *gchoose 65*)
where *gbinomial-prod-rev*: *a gchoose k* = *prod* (λ*i*. *a* − *of-nat i*) {*0*..<*k*} *div fact k*

lemma *gbinomial-0* [*simp*]:
a gchoose 0 = *1*
0 gchoose (*Suc k*) = *0*
⟨*proof*⟩

lemma *gbinomial-Suc*: *a gchoose* (*Suc k*) = *prod* (λ*i*. *a* − *of-nat i*) {*0*..*k*} *div fact* (*Suc k*)
⟨*proof*⟩

lemma *gbinomial-1* [*simp*]: *a gchoose 1* = *a*
⟨*proof*⟩

lemma *gbinomial-Suc0* [*simp*]: *a gchoose Suc 0* = *a*
⟨*proof*⟩

lemma *gbinomial-mult-fact*: *fact k* ∗ (*a gchoose k*) = (∏ *i* = *0*..<*k*. *a* − *of-nat i*)
for *a* :: *'a*::*field-char-0*
⟨*proof*⟩

lemma *gbinomial-mult-fact'*: (*a gchoose k*) ∗ *fact k* = (∏ *i* = *0*..<*k*. *a* − *of-nat i*)
for *a* :: *'a*::*field-char-0*
⟨*proof*⟩

lemma *gbinomial-pochhammer*: *a gchoose k* = (− *1*) ^ *k* ∗ *pochhammer* (− *a*) *k* / *fact k*
for *a* :: *'a*::*field-char-0*
⟨*proof*⟩

lemma *gbinomial-pochhammer'*: *a gchoose k* = *pochhammer* (*a* − *of-nat k* + *1*) *k* / *fact k*
for *a* :: *'a*::*field-char-0*
⟨*proof*⟩

lemma *gbinomial-binomial*: *n gchoose k* = *n choose k*
⟨*proof*⟩

lemma *of-nat-gbinomial*: *of-nat* (*n gchoose k*) = (*of-nat n gchoose k* :: *'a*::*field-char-0*)
⟨*proof*⟩

lemma *binomial-gbinomial*: *of-nat* (*n choose k*) = (*of-nat n gchoose k* :: *'a*::*field-char-0*)
⟨*proof*⟩

⟨*ML*⟩

lemma *gbinomial-mult-1*:
fixes $a :: 'a$::*field-char-0*
shows $a * (a\ gchoose\ k) = of\text{-}nat\ k * (a\ gchoose\ k) + of\text{-}nat\ (Suc\ k) * (a\ gchoose\ (Suc\ k))$
(**is** $?l = ?r$)
⟨*proof*⟩

lemma *gbinomial-mult-1'*:
$(a\ gchoose\ k) * a = of\text{-}nat\ k * (a\ gchoose\ k) + of\text{-}nat\ (Suc\ k) * (a\ gchoose\ (Suc\ k))$
for $a :: 'a$::*field-char-0*
⟨*proof*⟩

lemma *gbinomial-Suc-Suc*: $(a + 1)\ gchoose\ (Suc\ k) = a\ gchoose\ k + (a\ gchoose\ (Suc\ k))$
for $a :: 'a$::*field-char-0*
⟨*proof*⟩

lemma *gbinomial-reduce-nat*: $0 < k \Longrightarrow a\ gchoose\ k = (a - 1)\ gchoose\ (k - 1) + ((a - 1)\ gchoose\ k)$
for $a :: 'a$::*field-char-0*
⟨*proof*⟩

lemma *gchoose-row-sum-weighted*:
$(\sum k = 0..m.\ (r\ gchoose\ k) * (r/2 - of\text{-}nat\ k)) = of\text{-}nat(Suc\ m)\ /\ 2 * (r\ gchoose\ (Suc\ m))$
for $r :: 'a$::*field-char-0*
⟨*proof*⟩

lemma *binomial-symmetric*:
assumes *kn*: $k \leq n$
shows $n\ choose\ k = n\ choose\ (n - k)$
⟨*proof*⟩

lemma *choose-rising-sum*:
$(\sum j \leq m.\ ((n + j)\ choose\ n)) = ((n + m + 1)\ choose\ (n + 1))$
$(\sum j \leq m.\ ((n + j)\ choose\ n)) = ((n + m + 1)\ choose\ m)$
⟨*proof*⟩

lemma *choose-linear-sum*: $(\sum i \leq n.\ i * (n\ choose\ i)) = n * 2\ \hat{}\ (n - 1)$
⟨*proof*⟩

lemma *choose-alternating-linear-sum*:
assumes $n \neq 1$
shows $(\sum i \leq n.\ (-1)\hat{}i * of\text{-}nat\ i * of\text{-}nat\ (n\ choose\ i) :: 'a$::*comm-ring-1*$) = 0$
⟨*proof*⟩

lemma *vandermonde*: $(\sum k \leq r.\ (m\ choose\ k) * (n\ choose\ (r - k))) = (m + n)$

choose r
⟨*proof*⟩

lemma *choose-square-sum*: $(\sum k{\leq}n.\ (n\ choose\ k)\hat{}2) = ((2{*}n)\ choose\ n)$
⟨*proof*⟩

lemma *pochhammer-binomial-sum*:
fixes *a b* :: *'a*::*comm-ring-1*
shows *pochhammer* $(a + b)\ n = (\sum k{\leq}n.\ of\text{-}nat\ (n\ choose\ k) * pochhammer\ a\ k * pochhammer\ b\ (n - k))$
⟨*proof*⟩

Contributed by Manuel Eberl, generalised by LCP. Alternative definition of the binomial coefficient as $\prod i{<}k.\ (n - i)\ /\ (k - i)$.

lemma *gbinomial-altdef-of-nat*: *a gchoose k* = $(\prod i = 0..{<}k.\ (a - of\text{-}nat\ i)\ /\ of\text{-}nat\ (k - i) :: 'a)$
for *k* :: *nat* **and** *a* :: *'a*::*field-char-0*
⟨*proof*⟩

lemma *gbinomial-ge-n-over-k-pow-k*:
fixes *k* :: *nat*
and *a* :: *'a*::*linordered-field*
assumes *of-nat k* ≤ *a*
shows $(a\ /\ of\text{-}nat\ k :: 'a)\ \hat{}\ k \leq a\ gchoose\ k$
⟨*proof*⟩

lemma *gbinomial-negated-upper*: $(a\ gchoose\ k) = (-1)\ \hat{}\ k * ((of\text{-}nat\ k - a - 1)\ gchoose\ k)$
⟨*proof*⟩

lemma *gbinomial-minus*: $((-a)\ gchoose\ k) = (-1)\ \hat{}\ k * ((a + of\text{-}nat\ k - 1)\ gchoose\ k)$
⟨*proof*⟩

lemma *Suc-times-gbinomial*: $of\text{-}nat\ (Suc\ k) * ((a + 1)\ gchoose\ (Suc\ k)) = (a + 1) * (a\ gchoose\ k)$
⟨*proof*⟩

lemma *gbinomial-factors*: $((a + 1)\ gchoose\ (Suc\ k)) = (a + 1)\ /\ of\text{-}nat\ (Suc\ k) * (a\ gchoose\ k)$
⟨*proof*⟩

lemma *gbinomial-rec*: $((a + 1)\ gchoose\ (Suc\ k)) = (a\ gchoose\ k) * ((a + 1)\ /\ of\text{-}nat\ (Suc\ k))$
⟨*proof*⟩

lemma *gbinomial-of-nat-symmetric*: $k \leq n \Longrightarrow (of\text{-}nat\ n)\ gchoose\ k = (of\text{-}nat\ n)\ gchoose\ (n - k)$
⟨*proof*⟩

The absorption identity (equation 5.5 [3, p. 157]):

$$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}, \quad \text{integer } k \neq 0.$$

lemma *gbinomial-absorption'*: $k > 0 \Longrightarrow a$ *gchoose* $k = (a$ */ of-nat* $k) * (a - 1$ *gchoose* $(k - 1))$
⟨*proof*⟩

The absorption identity is written in the following form to avoid division by k (the lower index) and therefore remove the $k \neq 0$ restriction [3, p. 157]:

$$k\binom{r}{k} = r\binom{r-1}{k-1}, \quad \text{integer } k.$$

lemma *gbinomial-absorption*: *of-nat* $(Suc\ k) * (a$ *gchoose* $Suc\ k) = a * ((a - 1)$ *gchoose* $k)$
⟨*proof*⟩

The absorption identity for natural number binomial coefficients:

lemma *binomial-absorption*: $Suc\ k * (n$ *choose* $Suc\ k) = n * ((n - 1)$ *choose* $k)$
⟨*proof*⟩

The absorption companion identity for natural number coefficients, following the proof by GKP [3, p. 157]:

lemma *binomial-absorb-comp*: $(n - k) * (n$ *choose* $k) = n * ((n - 1)$ *choose* $k)$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

The generalised absorption companion identity:

lemma *gbinomial-absorb-comp*: $(a -$ *of-nat* $k) * (a$ *gchoose* $k) = a * ((a - 1)$ *gchoose* $k)$
⟨*proof*⟩

lemma *gbinomial-addition-formula*:
a *gchoose* $(Suc\ k) = ((a - 1)$ *gchoose* $(Suc\ k)) + ((a - 1)$ *gchoose* $k)$
⟨*proof*⟩

lemma *binomial-addition-formula*:
$0 < n \Longrightarrow n$ *choose* $(Suc\ k) = ((n - 1)$ *choose* $(Suc\ k)) + ((n - 1)$ *choose* $k)$
⟨*proof*⟩

Equation 5.9 of the reference material [3, p. 159] is a useful summation formula, operating on both indices:

$$\sum_{k \leq n} \binom{r+k}{k} = \binom{r+n+1}{n}, \quad \text{integer } n.$$

lemma *gbinomial-parallel-sum*: $(\sum k{\leq}n.\ (a +$ *of-nat* $k)$ *gchoose* $k) = (a +$ *of-nat* $n + 1)$ *gchoose* n
⟨*proof*⟩

94.3.1 Summation on the upper index

Another summation formula is equation 5.10 of the reference material [3, p. 160], aptly named *summation on the upper index*:

$$\sum_{0 \le k \le n} \binom{k}{m} = \binom{n+1}{m+1}, \quad \text{integers } m, n \ge 0.$$

lemma *gbinomial-sum-up-index*:
(∑ *j* = *0*..*n*. (*of-nat j gchoose k*) :: *'a*::*field-char-0*) = (*of-nat n* + *1*) *gchoose* (*k* + *1*)
⟨*proof*⟩

lemma *gbinomial-index-swap*:
((−*1*) ^ *k*) ∗ ((− (*of-nat n*) − *1*) *gchoose k*) = ((−*1*) ^ *n*) ∗ ((− (*of-nat k*) − *1*) *gchoose n*)
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *gbinomial-sum-lower-neg*: (∑ *k*≤*m*. (*a gchoose k*) ∗ (− *1*) ^ *k*) = (− *1*) ^ *m* ∗ (*a* − *1 gchoose m*)
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *gbinomial-partial-row-sum*:
(∑ *k*≤*m*. (*a gchoose k*) ∗ ((*a* / *2*) − *of-nat k*)) = ((*of-nat m* + *1*)/*2*) ∗ (*a gchoose* (*m* + *1*))
⟨*proof*⟩

lemma *sum-bounds-lt-plus1*: (∑ *k*<*mm*. *f* (*Suc k*)) = (∑ *k*=*1*..*mm*. *f k*)
⟨*proof*⟩

lemma *gbinomial-partial-sum-poly*:
(∑ *k*≤*m*. (*of-nat m* + *a gchoose k*) ∗ *x*^*k* ∗ *y*^(*m*−*k*)) =
(∑ *k*≤*m*. (−*a gchoose k*) ∗ (−*x*)^*k* ∗ (*x* + *y*)^(*m*−*k*))
(**is** *?lhs m* = *?rhs m*)
⟨*proof*⟩

lemma *gbinomial-partial-sum-poly-xpos*:
(∑ *k*≤*m*. (*of-nat m* + *a gchoose k*) ∗ *x*^*k* ∗ *y*^(*m*−*k*)) =
(∑ *k*≤*m*. (*of-nat k* + *a* − *1 gchoose k*) ∗ *x*^*k* ∗ (*x* + *y*)^(*m*−*k*)) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *binomial-r-part-sum*: (∑ *k*≤*m*. (*2* ∗ *m* + *1 choose k*)) = *2* ^ (*2* ∗ *m*)
⟨*proof*⟩

lemma *gbinomial-r-part-sum*: (∑ *k*≤*m*. (*2* ∗ (*of-nat m*) + *1 gchoose k*)) = *2* ^ (*2* ∗ *m*)

(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *gbinomial-sum-nat-pow2*:
(∑ *k*≤*m*. (*of-nat* (*m* + *k*) *gchoose k* :: *'a*::*field-char-0*) / *2* ^ *k*) = *2* ^ *m*
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *gbinomial-trinomial-revision*:
assumes *k* ≤ *m*
shows (*a gchoose m*) * (*of-nat m gchoose k*) = (*a gchoose k*) * (*a* − *of-nat k gchoose* (*m* − *k*))
⟨*proof*⟩

Versions of the theorems above for the natural-number version of "choose"

lemma *binomial-altdef-of-nat*:
k ≤ *n* ⟹ *of-nat* (*n choose k*) = (∏ *i* = *0*..<*k*. *of-nat* (*n* − *i*) / *of-nat* (*k* − *i*) :: *'a*)
for *n k* :: *nat* **and** *x* :: *'a*::*field-char-0*
⟨*proof*⟩

lemma *binomial-ge-n-over-k-pow-k*: *k* ≤ *n* ⟹ (*of-nat n* / *of-nat k* :: *'a*) ^ *k* ≤ *of-nat* (*n choose k*)
for *k n* :: *nat* **and** *x* :: *'a*::*linordered-field*
⟨*proof*⟩

lemma *binomial-le-pow*:
assumes *r* ≤ *n*
shows *n choose r* ≤ *n* ^ *r*
⟨*proof*⟩

lemma *binomial-altdef-nat*: *k* ≤ *n* ⟹ *n choose k* = *fact n div* (*fact k* * *fact* (*n* − *k*))
for *k n* :: *nat*
⟨*proof*⟩

lemma *choose-dvd*:
assumes *k* ≤ *n* **shows** *fact k* * *fact* (*n* − *k*) *dvd* (*fact n* :: *'a*::*linordered-semidom*)
⟨*proof*⟩

lemma *fact-fact-dvd-fact*:
fact k * *fact n dvd* (*fact* (*k* + *n*) :: *'a*::*linordered-semidom*)
⟨*proof*⟩

lemma *choose-mult-lemma*:
((*m* + *r* + *k*) *choose* (*m* + *k*)) * ((*m* + *k*) *choose k*) = ((*m* + *r* + *k*) *choose k*) * ((*m* + *r*) *choose m*)
(**is** *?lhs* = -)
⟨*proof*⟩

The "Subset of a Subset" identity.

lemma *choose-mult*:
$k \leq m \Longrightarrow m \leq n \Longrightarrow (n \text{ choose } m) * (m \text{ choose } k) = (n \text{ choose } k) * ((n - k) \text{ choose } (m - k))$
⟨*proof*⟩

lemma *of-nat-binomial-eq-mult-binomial-Suc*:
assumes $k \leq n$
shows $(\textit{of-nat} :: (\textit{nat} \Rightarrow ('a :: \textit{field-char-0}))) (n \text{ choose } k) = \textit{of-nat } (n + 1 - k) / \textit{of-nat } (n + 1) * \textit{of-nat } (\textit{Suc } n \text{ choose } k)$
⟨*proof*⟩

94.4 More on Binomial Coefficients

lemma *choose-one*: $n \text{ choose } 1 = n$ **for** $n :: \textit{nat}$
⟨*proof*⟩

The famous inclusion-exclusion formula for the cardinality of a union

lemma *int-card-UNION*:
assumes *finite A*
and $\forall k \in A.$ *finite k*
shows $\textit{int } (\textit{card } (\bigcup A)) = (\sum I \mid I \subseteq A \wedge I \neq \{\}. (-1) \hat{} (\textit{card } I + 1) * \textit{int } (\textit{card } (\bigcap I)))$
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *card-UNION*:
assumes *finite A*
and $\forall k \in A.$ *finite k*
shows $\textit{card } (\bigcup A) = \textit{nat } (\sum I \mid I \subseteq A \wedge I \neq \{\}. (-1) \hat{} (\textit{card } I + 1) * \textit{int } (\textit{card } (\bigcap I)))$
⟨*proof*⟩

lemma *card-UNION-nonneg*:
assumes *finite A*
and $\forall k \in A.$ *finite k*
shows $(\sum I \mid I \subseteq A \wedge I \neq \{\}. (-1) \hat{} (\textit{card } I + 1) * \textit{int } (\textit{card } (\bigcap I))) \geq 0$
⟨*proof*⟩

The number of nat lists of length m summing to N is $N + m - 1 \text{ choose } N$:

lemma *card-length-sum-list-rec*:
assumes $m \geq 1$
shows $\textit{card } \{l::\textit{nat list}. \textit{length } l = m \wedge \textit{sum-list } l = N\} =$
$\textit{card } \{l. \textit{length } l = (m - 1) \wedge \textit{sum-list } l = N\} +$
$\textit{card } \{l. \textit{length } l = m \wedge \textit{sum-list } l + 1 = N\}$
(**is** *card ?C = card ?A + card ?B*)
⟨*proof*⟩

lemma *card-length-sum-list*: *card* {*l*::*nat list*. *size l* = *m* ∧ *sum-list l* = *N*} = (*N* + *m* − *1*) *choose N*
— by Holden Lee, tidied by Tobias Nipkow
⟨*proof*⟩

lemma *card-disjoint-shuffles*:
assumes *set xs* ∩ *set ys* = {}
shows *card* (*shuffles xs ys*) = (*length xs* + *length ys*) *choose length xs*
⟨*proof*⟩

lemma *Suc-times-binomial-add*: *Suc a* ∗ (*Suc* (*a* + *b*) *choose Suc a*) = *Suc b* ∗ (*Suc* (*a* + *b*) *choose a*)
— by Lukas Bulwahn
⟨*proof*⟩

94.5 Executable code

lemma *gbinomial-code* [*code*]:
a gchoose k =
(*if k* = *0 then 1*
else fold-atLeastAtMost-nat (λ*k acc*. (*a* − *of-nat k*) ∗ *acc*) *0* (*k* − *1*) *1* / *fact k*)
⟨*proof*⟩

lemma *binomial-code* [*code*]:
n choose k =
(*if k* > *n then 0*
else if 2 ∗ *k* > *n then n choose* (*n* − *k*)
else (*fold-atLeastAtMost-nat* (∗) (*n* − *k* + *1*) *n 1 div fact k*))
⟨*proof*⟩

end

95 Main HOL

Classical Higher-order Logic – only "Main", excluding real and complex numbers etc.

theory *Main*
imports
Predicate-Compile
Quickcheck-Narrowing
Mirabelle
Extraction
Nunchaku
BNF-Greatest-Fixpoint
Filter
Conditionally-Complete-Lattices

Binomial
GCD
Divides
begin

95.1 Namespace cleanup

hide-const (**open**)
czero cinfinite cfinite csum cone ctwo Csum cprod cexp image2 image2p vimage2p Gr Grp collect
fsts snds setl setr convol pick-middlep fstOp sndOp csquare relImage relInvImage Succ Shift
shift proj id-bnf

hide-fact (**open**) *id-bnf-def type-definition-id-bnf-UNIV*

95.2 Syntax cleanup

no-notation
ordLeq2 (**infix** $<=o$ *50*) **and**
ordLeq3 (**infix** $\leq o$ *50*) **and**
ordLess2 (**infix** $<o$ *50*) **and**
ordIso2 (**infix** $=o$ *50*) **and**
card-of (|-|) **and**
BNF-Cardinal-Arithmetic.csum (**infixr** $+c$ *65*) **and**
BNF-Cardinal-Arithmetic.cprod (**infixr** $*c$ *80*) **and**
BNF-Cardinal-Arithmetic.cexp (**infixr** $\hat{}c$ *90*) **and**
BNF-Def.convol ($\langle$(-,/ -)$\rangle$)

bundle *cardinal-syntax*
begin

notation
ordLeq2 (**infix** $<=o$ *50*) **and**
ordLeq3 (**infix** $\leq o$ *50*) **and**
ordLess2 (**infix** $<o$ *50*) **and**
ordIso2 (**infix** $=o$ *50*) **and**
card-of (|-|) **and**
BNF-Cardinal-Arithmetic.csum (**infixr** $+c$ *65*) **and**
BNF-Cardinal-Arithmetic.cprod (**infixr** $*c$ *80*) **and**
BNF-Cardinal-Arithmetic.cexp (**infixr** $\hat{}c$ *90*)

alias *cinfinite* = *BNF-Cardinal-Arithmetic.cinfinite*
alias *czero* = *BNF-Cardinal-Arithmetic.czero*
alias *cone* = *BNF-Cardinal-Arithmetic.cone*
alias *ctwo* = *BNF-Cardinal-Arithmetic.ctwo*

end

95.3 Lattice syntax

bundle *lattice-syntax*
begin

notation
  bot (⊥) **and**
  top (⊤) **and**
  inf (**infixl** ⊓ *70*) **and**
  sup (**infixl** ⊔ *65*) **and**
  Inf (⨅ - [*900*] *900*) **and**
  Sup (⨆ - [*900*] *900*)

syntax
  -INF1 :: *pttrns* ⇒ *'b* ⇒ *'b* ((*3*⨅ -./ -) [*0*, *10*] *10*)
  -INF :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* ((*3*⨅ -∈-./ -) [*0*, *0*, *10*] *10*)
  -SUP1 :: *pttrns* ⇒ *'b* ⇒ *'b* ((*3*⨆ -./ -) [*0*, *10*] *10*)
  -SUP :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* ((*3*⨆ -∈-./ -) [*0*, *0*, *10*] *10*)

end

bundle *no-lattice-syntax*
begin

no-notation
  bot (⊥) **and**
  top (⊤) **and**
  inf (**infixl** ⊓ *70*) **and**
  sup (**infixl** ⊔ *65*) **and**
  Inf (⨅ - [*900*] *900*) **and**
  Sup (⨆ - [*900*] *900*)

no-syntax
  -INF1 :: *pttrns* ⇒ *'b* ⇒ *'b* ((*3*⨅ -./ -) [*0*, *10*] *10*)
  -INF :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* ((*3*⨅ -∈-./ -) [*0*, *0*, *10*] *10*)
  -SUP1 :: *pttrns* ⇒ *'b* ⇒ *'b* ((*3*⨆ -./ -) [*0*, *10*] *10*)
  -SUP :: *pttrn* ⇒ *'a set* ⇒ *'b* ⇒ *'b* ((*3*⨆ -∈-./ -) [*0*, *0*, *10*] *10*)

end

unbundle *no-lattice-syntax*

end

96 Archimedean Fields, Floor and Ceiling Functions

theory *Archimedean-Field*
imports *Main*

begin

lemma *cInf-abs-ge*:
fixes S :: $'a$::{*linordered-idom*,*conditionally-complete-linorder*} *set*
assumes $S \neq \{\}$
and *bdd*: $\bigwedge x.\ x \in S \Longrightarrow |x| \leq a$
shows $|Inf\ S| \leq a$
⟨*proof*⟩

lemma *cSup-asclose*:
fixes S :: $'a$::{*linordered-idom*,*conditionally-complete-linorder*} *set*
assumes *S*: $S \neq \{\}$
and *b*: $\forall x \in S.\ |x - l| \leq e$
shows $|Sup\ S - l| \leq e$
⟨*proof*⟩

lemma *cInf-asclose*:
fixes S :: $'a$::{*linordered-idom*,*conditionally-complete-linorder*} *set*
assumes *S*: $S \neq \{\}$
and *b*: $\forall x \in S.\ |x - l| \leq e$
shows $|Inf\ S - l| \leq e$
⟨*proof*⟩

96.1 Class of Archimedean fields

Archimedean fields have no infinite elements.

class *archimedean-field* = *linordered-field* +
assumes *ex-le-of-int*: $\exists z.\ x \leq of\text{-}int\ z$

lemma *ex-less-of-int*: $\exists z.\ x < of\text{-}int\ z$
for x :: $'a$::*archimedean-field*
⟨*proof*⟩

lemma *ex-of-int-less*: $\exists z.\ of\text{-}int\ z < x$
for x :: $'a$::*archimedean-field*
⟨*proof*⟩

lemma *reals-Archimedean2*: $\exists n.\ x < of\text{-}nat\ n$
for x :: $'a$::*archimedean-field*
⟨*proof*⟩

lemma *real-arch-simple*: $\exists n.\ x \leq of\text{-}nat\ n$
for x :: $'a$::*archimedean-field*
⟨*proof*⟩

Archimedean fields have no infinitesimal elements.

lemma *reals-Archimedean*:
fixes x :: $'a$::*archimedean-field*
assumes $0 < x$

shows $\exists n.\ inverse\ (of\text{-}nat\ (Suc\ n)) < x$
⟨*proof*⟩

lemma *ex-inverse-of-nat-less*:
fixes $x :: 'a{::}archimedean\text{-}field$
assumes $0 < x$
shows $\exists n{>}0.\ inverse\ (of\text{-}nat\ n) < x$
⟨*proof*⟩

lemma *ex-less-of-nat-mult*:
fixes $x :: 'a{::}archimedean\text{-}field$
assumes $0 < x$
shows $\exists n.\ y < of\text{-}nat\ n * x$
⟨*proof*⟩

96.2 Existence and uniqueness of floor function

lemma *exists-least-lemma*:
assumes $\neg\ P\ 0$ **and** $\exists n.\ P\ n$
shows $\exists n.\ \neg\ P\ n \wedge P\ (Suc\ n)$
⟨*proof*⟩

lemma *floor-exists*:
fixes $x :: 'a{::}archimedean\text{-}field$
shows $\exists z.\ of\text{-}int\ z \leq x \wedge x < of\text{-}int\ (z + 1)$
⟨*proof*⟩

lemma *floor-exists1*: $\exists!z.\ of\text{-}int\ z \leq x \wedge x < of\text{-}int\ (z + 1)$
for $x :: 'a{::}archimedean\text{-}field$
⟨*proof*⟩

96.3 Floor function

class *floor-ceiling* = *archimedean-field* +
fixes *floor* :: $'a \Rightarrow int$ $(\lfloor\text{-}\rfloor)$
assumes *floor-correct*: $of\text{-}int\ \lfloor x \rfloor \leq x \wedge x < of\text{-}int\ (\lfloor x \rfloor + 1)$

lemma *floor-unique*: $of\text{-}int\ z \leq x \Longrightarrow x < of\text{-}int\ z + 1 \Longrightarrow \lfloor x \rfloor = z$
⟨*proof*⟩

lemma *floor-eq-iff*: $\lfloor x \rfloor = a \longleftrightarrow of\text{-}int\ a \leq x \wedge x < of\text{-}int\ a + 1$
⟨*proof*⟩

lemma *of-int-floor-le* [*simp*]: $of\text{-}int\ \lfloor x \rfloor \leq x$
⟨*proof*⟩

lemma *le-floor-iff*: $z \leq \lfloor x \rfloor \longleftrightarrow of\text{-}int\ z \leq x$
⟨*proof*⟩

lemma *floor-less-iff*: $\lfloor x \rfloor < z \longleftrightarrow x < of\text{-}int\ z$

⟨*proof*⟩

lemma *less-floor-iff*: $z < \lfloor x \rfloor \longleftrightarrow$ *of-int* $z + 1 \leq x$
⟨*proof*⟩

lemma *floor-le-iff*: $\lfloor x \rfloor \leq z \longleftrightarrow x <$ *of-int* $z + 1$
⟨*proof*⟩

lemma *floor-split*[*linarith-split*]: $P\ \lfloor t \rfloor \longleftrightarrow (\forall i.$ *of-int* $i \leq t \wedge t <$ *of-int* $i + 1 \longrightarrow P\ i)$
⟨*proof*⟩

lemma *floor-mono*:
assumes $x \leq y$
shows $\lfloor x \rfloor \leq \lfloor y \rfloor$
⟨*proof*⟩

lemma *floor-less-cancel*: $\lfloor x \rfloor < \lfloor y \rfloor \Longrightarrow x < y$
⟨*proof*⟩

lemma *floor-of-int* [*simp*]: $\lfloor$*of-int* $z \rfloor = z$
⟨*proof*⟩

lemma *floor-of-nat* [*simp*]: $\lfloor$*of-nat* $n \rfloor =$ *int* n
⟨*proof*⟩

lemma *le-floor-add*: $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$
⟨*proof*⟩

Floor with numerals.

lemma *floor-zero* [*simp*]: $\lfloor 0 \rfloor = 0$
⟨*proof*⟩

lemma *floor-one* [*simp*]: $\lfloor 1 \rfloor = 1$
⟨*proof*⟩

lemma *floor-numeral* [*simp*]: $\lfloor$*numeral* $v \rfloor =$ *numeral* v
⟨*proof*⟩

lemma *floor-neg-numeral* [*simp*]: $\lfloor -$ *numeral* $v \rfloor = -$ *numeral* v
⟨*proof*⟩

lemma *zero-le-floor* [*simp*]: $0 \leq \lfloor x \rfloor \longleftrightarrow 0 \leq x$
⟨*proof*⟩

lemma *one-le-floor* [*simp*]: $1 \leq \lfloor x \rfloor \longleftrightarrow 1 \leq x$
⟨*proof*⟩

lemma *numeral-le-floor* [*simp*]: *numeral* $v \leq \lfloor x \rfloor \longleftrightarrow$ *numeral* $v \leq x$

⟨*proof*⟩

lemma *neg-numeral-le-floor* [*simp*]: − *numeral v* ≤ ⌊*x*⌋ ⟷ − *numeral v* ≤ *x*
⟨*proof*⟩

lemma *zero-less-floor* [*simp*]: *0* < ⌊*x*⌋ ⟷ *1* ≤ *x*
⟨*proof*⟩

lemma *one-less-floor* [*simp*]: *1* < ⌊*x*⌋ ⟷ *2* ≤ *x*
⟨*proof*⟩

lemma *numeral-less-floor* [*simp*]: *numeral v* < ⌊*x*⌋ ⟷ *numeral v* + *1* ≤ *x*
⟨*proof*⟩

lemma *neg-numeral-less-floor* [*simp*]: − *numeral v* < ⌊*x*⌋ ⟷ − *numeral v* + *1* ≤ *x*
⟨*proof*⟩

lemma *floor-le-zero* [*simp*]: ⌊*x*⌋ ≤ *0* ⟷ *x* < *1*
⟨*proof*⟩

lemma *floor-le-one* [*simp*]: ⌊*x*⌋ ≤ *1* ⟷ *x* < *2*
⟨*proof*⟩

lemma *floor-le-numeral* [*simp*]: ⌊*x*⌋ ≤ *numeral v* ⟷ *x* < *numeral v* + *1*
⟨*proof*⟩

lemma *floor-le-neg-numeral* [*simp*]: ⌊*x*⌋ ≤ − *numeral v* ⟷ *x* < − *numeral v* + *1*
⟨*proof*⟩

lemma *floor-less-zero* [*simp*]: ⌊*x*⌋ < *0* ⟷ *x* < *0*
⟨*proof*⟩

lemma *floor-less-one* [*simp*]: ⌊*x*⌋ < *1* ⟷ *x* < *1*
⟨*proof*⟩

lemma *floor-less-numeral* [*simp*]: ⌊*x*⌋ < *numeral v* ⟷ *x* < *numeral v*
⟨*proof*⟩

lemma *floor-less-neg-numeral* [*simp*]: ⌊*x*⌋ < − *numeral v* ⟷ *x* < − *numeral v*
⟨*proof*⟩

lemma *le-mult-floor-Ints*:
assumes *0* ≤ *a a* ∈ *Ints*
shows *of-int* (⌊*a*⌋ ∗ ⌊*b*⌋) ≤ (*of-int*⌊*a* ∗ *b*⌋ :: *'a* :: *linordered-idom*)
⟨*proof*⟩

Addition and subtraction of integers.

lemma *floor-add-int*: $\lfloor x \rfloor + z = \lfloor x + \textit{of-int}\ z \rfloor$
⟨*proof*⟩

lemma *int-add-floor*: $z + \lfloor x \rfloor = \lfloor \textit{of-int}\ z + x \rfloor$
⟨*proof*⟩

lemma *one-add-floor*: $\lfloor x \rfloor + 1 = \lfloor x + 1 \rfloor$
⟨*proof*⟩

lemma *floor-diff-of-int* [*simp*]: $\lfloor x - \textit{of-int}\ z \rfloor = \lfloor x \rfloor - z$
⟨*proof*⟩

lemma *floor-uminus-of-int* [*simp*]: $\lfloor - (\textit{of-int}\ z) \rfloor = - z$
⟨*proof*⟩

lemma *floor-diff-numeral* [*simp*]: $\lfloor x - \textit{numeral}\ v \rfloor = \lfloor x \rfloor - \textit{numeral}\ v$
⟨*proof*⟩

lemma *floor-diff-one* [*simp*]: $\lfloor x - 1 \rfloor = \lfloor x \rfloor - 1$
⟨*proof*⟩

lemma *le-mult-floor*:
assumes $0 \leq a$ **and** $0 \leq b$
shows $\lfloor a \rfloor * \lfloor b \rfloor \leq \lfloor a * b \rfloor$
⟨*proof*⟩

lemma *floor-divide-of-int-eq*: $\lfloor \textit{of-int}\ k\ /\ \textit{of-int}\ l \rfloor = k\ \textit{div}\ l$
for $k\ l :: \textit{int}$
⟨*proof*⟩

lemma *floor-divide-of-nat-eq*: $\lfloor \textit{of-nat}\ m\ /\ \textit{of-nat}\ n \rfloor = \textit{of-nat}\ (m\ \textit{div}\ n)$
for $m\ n :: \textit{nat}$
⟨*proof*⟩

lemma *floor-divide-lower*:
fixes $q :: 'a{::}\textit{floor-ceiling}$
shows $q > 0 \Longrightarrow \textit{of-int} \lfloor p\ /\ q \rfloor * q \leq p$
⟨*proof*⟩

lemma *floor-divide-upper*:
fixes $q :: 'a{::}\textit{floor-ceiling}$
shows $q > 0 \Longrightarrow p < (\textit{of-int} \lfloor p\ /\ q \rfloor + 1) * q$
⟨*proof*⟩

96.4 Ceiling function

definition *ceiling* :: $'a{::}\textit{floor-ceiling} \Rightarrow \textit{int}$ ($\lceil\text{-}\rceil$)
where $\lceil x \rceil = - \lfloor - x \rfloor$

lemma *ceiling-correct*: *of-int* $\lceil x \rceil - 1 < x \wedge x \leq$ *of-int* $\lceil x \rceil$
⟨*proof*⟩

lemma *ceiling-unique*: *of-int* $z - 1 < x \Longrightarrow x \leq$ *of-int* $z \Longrightarrow \lceil x \rceil = z$
⟨*proof*⟩

lemma *ceiling-eq-iff*: $\lceil x \rceil = a \longleftrightarrow$ *of-int* $a - 1 < x \wedge x \leq$ *of-int* a
⟨*proof*⟩

lemma *le-of-int-ceiling* [*simp*]: $x \leq$ *of-int* $\lceil x \rceil$
⟨*proof*⟩

lemma *ceiling-le-iff*: $\lceil x \rceil \leq z \longleftrightarrow x \leq$ *of-int* z
⟨*proof*⟩

lemma *less-ceiling-iff*: $z < \lceil x \rceil \longleftrightarrow$ *of-int* $z < x$
⟨*proof*⟩

lemma *ceiling-less-iff*: $\lceil x \rceil < z \longleftrightarrow x \leq$ *of-int* $z - 1$
⟨*proof*⟩

lemma *le-ceiling-iff*: $z \leq \lceil x \rceil \longleftrightarrow$ *of-int* $z - 1 < x$
⟨*proof*⟩

lemma *ceiling-mono*: $x \geq y \Longrightarrow \lceil x \rceil \geq \lceil y \rceil$
⟨*proof*⟩

lemma *ceiling-less-cancel*: $\lceil x \rceil < \lceil y \rceil \Longrightarrow x < y$
⟨*proof*⟩

lemma *ceiling-of-int* [*simp*]: $\lceil$*of-int* $z\rceil = z$
⟨*proof*⟩

lemma *ceiling-of-nat* [*simp*]: $\lceil$*of-nat* $n\rceil =$ *int* n
⟨*proof*⟩

lemma *ceiling-add-le*: $\lceil x + y \rceil \leq \lceil x \rceil + \lceil y \rceil$
⟨*proof*⟩

lemma *mult-ceiling-le*:
assumes $0 \leq a$ **and** $0 \leq b$
shows $\lceil a * b \rceil \leq \lceil a \rceil * \lceil b \rceil$
⟨*proof*⟩

lemma *mult-ceiling-le-Ints*:
assumes $0 \leq a$ $a \in$ *Ints*
shows (*of-int* $\lceil a * b \rceil :: {'a} ::$ *linordered-idom*) $\leq$ *of-int*$(\lceil a \rceil * \lceil b \rceil)$
⟨*proof*⟩

lemma *finite-int-segment*:
fixes a :: *'a*::*floor-ceiling*
shows *finite* $\{x \in \mathbb{Z}.\ a \leq x \wedge x \leq b\}$
⟨*proof*⟩

corollary *finite-abs-int-segment*:
fixes a :: *'a*::*floor-ceiling*
shows *finite* $\{k \in \mathbb{Z}.\ |k| \leq a\}$
⟨*proof*⟩

96.4.1 Ceiling with numerals.

lemma *ceiling-zero* [*simp*]: $\lceil 0 \rceil = 0$
⟨*proof*⟩

lemma *ceiling-one* [*simp*]: $\lceil 1 \rceil = 1$
⟨*proof*⟩

lemma *ceiling-numeral* [*simp*]: $\lceil numeral\ v \rceil = numeral\ v$
⟨*proof*⟩

lemma *ceiling-neg-numeral* [*simp*]: $\lceil - numeral\ v \rceil = - numeral\ v$
⟨*proof*⟩

lemma *ceiling-le-zero* [*simp*]: $\lceil x \rceil \leq 0 \longleftrightarrow x \leq 0$
⟨*proof*⟩

lemma *ceiling-le-one* [*simp*]: $\lceil x \rceil \leq 1 \longleftrightarrow x \leq 1$
⟨*proof*⟩

lemma *ceiling-le-numeral* [*simp*]: $\lceil x \rceil \leq numeral\ v \longleftrightarrow x \leq numeral\ v$
⟨*proof*⟩

lemma *ceiling-le-neg-numeral* [*simp*]: $\lceil x \rceil \leq - numeral\ v \longleftrightarrow x \leq - numeral\ v$
⟨*proof*⟩

lemma *ceiling-less-zero* [*simp*]: $\lceil x \rceil < 0 \longleftrightarrow x \leq -1$
⟨*proof*⟩

lemma *ceiling-less-one* [*simp*]: $\lceil x \rceil < 1 \longleftrightarrow x \leq 0$
⟨*proof*⟩

lemma *ceiling-less-numeral* [*simp*]: $\lceil x \rceil < numeral\ v \longleftrightarrow x \leq numeral\ v - 1$
⟨*proof*⟩

lemma *ceiling-less-neg-numeral* [*simp*]: $\lceil x \rceil < - numeral\ v \longleftrightarrow x \leq - numeral\ v - 1$
⟨*proof*⟩

lemma *zero-le-ceiling* [*simp*]: $0 \leq \lceil x \rceil \longleftrightarrow -1 < x$
⟨*proof*⟩

lemma *one-le-ceiling* [*simp*]: $1 \leq \lceil x \rceil \longleftrightarrow 0 < x$
⟨*proof*⟩

lemma *numeral-le-ceiling* [*simp*]: *numeral* $v \leq \lceil x \rceil \longleftrightarrow$ *numeral* $v - 1 < x$
⟨*proof*⟩

lemma *neg-numeral-le-ceiling* [*simp*]: $-$ *numeral* $v \leq \lceil x \rceil \longleftrightarrow -$ *numeral* $v - 1 < x$
⟨*proof*⟩

lemma *zero-less-ceiling* [*simp*]: $0 < \lceil x \rceil \longleftrightarrow 0 < x$
⟨*proof*⟩

lemma *one-less-ceiling* [*simp*]: $1 < \lceil x \rceil \longleftrightarrow 1 < x$
⟨*proof*⟩

lemma *numeral-less-ceiling* [*simp*]: *numeral* $v < \lceil x \rceil \longleftrightarrow$ *numeral* $v < x$
⟨*proof*⟩

lemma *neg-numeral-less-ceiling* [*simp*]: $-$ *numeral* $v < \lceil x \rceil \longleftrightarrow -$ *numeral* $v < x$
⟨*proof*⟩

lemma *ceiling-altdef*: $\lceil x \rceil =$ (*if* $x =$ *of-int* $\lfloor x \rfloor$ *then* $\lfloor x \rfloor$ *else* $\lfloor x \rfloor + 1$)
⟨*proof*⟩

lemma *floor-le-ceiling* [*simp*]: $\lfloor x \rfloor \leq \lceil x \rceil$
⟨*proof*⟩

96.4.2 Addition and subtraction of integers.

lemma *ceiling-add-of-int* [*simp*]: $\lceil x +$ *of-int* $z \rceil = \lceil x \rceil + z$
⟨*proof*⟩

lemma *ceiling-add-numeral* [*simp*]: $\lceil x +$ *numeral* $v \rceil = \lceil x \rceil +$ *numeral* v
⟨*proof*⟩

lemma *ceiling-add-one* [*simp*]: $\lceil x + 1 \rceil = \lceil x \rceil + 1$
⟨*proof*⟩

lemma *ceiling-diff-of-int* [*simp*]: $\lceil x -$ *of-int* $z \rceil = \lceil x \rceil - z$
⟨*proof*⟩

lemma *ceiling-diff-numeral* [*simp*]: $\lceil x -$ *numeral* $v \rceil = \lceil x \rceil -$ *numeral* v
⟨*proof*⟩

lemma *ceiling-diff-one* [*simp*]: $\lceil x - 1 \rceil = \lceil x \rceil - 1$

⟨*proof*⟩

lemma *ceiling-split*[*linarith-split*]: $P\ \lceil t \rceil \longleftrightarrow (\forall i.\ \textit{of-int}\ i - 1 < t \wedge t \leq \textit{of-int}\ i \longrightarrow P\ i)$
⟨*proof*⟩

lemma *ceiling-diff-floor-le-1*: $\lceil x \rceil - \lfloor x \rfloor \leq 1$
⟨*proof*⟩

lemma *nat-approx-posE*:
fixes *e*:: *'a*::{*archimedean-field*,*floor-ceiling*}
assumes $0 < e$
obtains *n* :: *nat* **where** $1\ /\ \textit{of-nat}(\textit{Suc}\ n) < e$
⟨*proof*⟩

lemma *ceiling-divide-upper*:
fixes *q* :: *'a*::*floor-ceiling*
shows $q > 0 \Longrightarrow p \leq \textit{of-int}\ (\textit{ceiling}\ (p\ /\ q)) * q$
⟨*proof*⟩

lemma *ceiling-divide-lower*:
fixes *q* :: *'a*::*floor-ceiling*
shows $q > 0 \Longrightarrow (\textit{of-int}\ \lceil p\ /\ q \rceil - 1) * q < p$
⟨*proof*⟩

96.5 Negation

lemma *floor-minus*: $\lfloor - x \rfloor = - \lceil x \rceil$
⟨*proof*⟩

lemma *ceiling-minus*: $\lceil - x \rceil = - \lfloor x \rfloor$
⟨*proof*⟩

96.6 Natural numbers

lemma *of-nat-floor*: $r \geq 0 \Longrightarrow \textit{of-nat}\ (\textit{nat}\ \lfloor r \rfloor) \leq r$
⟨*proof*⟩

lemma *of-nat-ceiling*: $\textit{of-nat}\ (\textit{nat}\ \lceil r \rceil) \geq r$
⟨*proof*⟩

96.7 Frac Function

definition *frac* :: *'a* ⇒ *'a*::*floor-ceiling*
where $\textit{frac}\ x \equiv x - \textit{of-int}\ \lfloor x \rfloor$

lemma *frac-lt-1*: $\textit{frac}\ x < 1$
⟨*proof*⟩

lemma *frac-eq-0-iff* [*simp*]: $\textit{frac}\ x = 0 \longleftrightarrow x \in \mathbb{Z}$

⟨*proof*⟩

lemma *frac-ge-0* [*simp*]: *frac x* ≥ *0*
⟨*proof*⟩

lemma *frac-gt-0-iff* [*simp*]: *frac x* > *0* ⟷ *x* ∉ ℤ
⟨*proof*⟩

lemma *frac-of-int* [*simp*]: *frac* (*of-int z*) = *0*
⟨*proof*⟩

lemma *frac-frac* [*simp*]: *frac* (*frac x*) = *frac x*
⟨*proof*⟩

lemma *floor-add*: ⌊*x* + *y*⌋ = (*if frac x* + *frac y* < *1 then* ⌊*x*⌋ + ⌊*y*⌋ *else* (⌊*x*⌋ + ⌊*y*⌋) + *1*)
⟨*proof*⟩

lemma *floor-add2*[*simp*]: *x* ∈ ℤ ∨ *y* ∈ ℤ ⟹ ⌊*x* + *y*⌋ = ⌊*x*⌋ + ⌊*y*⌋
⟨*proof*⟩

lemma *frac-add*:
frac (*x* + *y*) = (*if frac x* + *frac y* < *1 then frac x* + *frac y else* (*frac x* + *frac y*) − *1*)
⟨*proof*⟩

lemma *frac-unique-iff*: *frac x* = *a* ⟷ *x* − *a* ∈ ℤ ∧ *0* ≤ *a* ∧ *a* < *1*
for *x* :: *'a*::*floor-ceiling*
⟨*proof*⟩

lemma *frac-eq*: *frac x* = *x* ⟷ *0* ≤ *x* ∧ *x* < *1*
⟨*proof*⟩

lemma *frac-neg*: *frac* (− *x*) = (*if x* ∈ ℤ *then 0 else 1* − *frac x*)
for *x* :: *'a*::*floor-ceiling*
⟨*proof*⟩

lemma *frac-in-Ints-iff* [*simp*]: *frac x* ∈ ℤ ⟷ *x* ∈ ℤ
⟨*proof*⟩

lemma *frac-1-eq*: *frac* (*x*+*1*) = *frac x*
⟨*proof*⟩

96.8 Rounding to the nearest integer

definition *round* :: *'a*::*floor-ceiling* ⇒ *int*
where *round x* = ⌊*x* + *1*/*2*⌋

lemma *of-int-round-ge*: *of-int* (*round x*) ≥ *x* − *1*/*2*

and *of-int-round-le*: $\mathit{of\text{-}int}\ (\mathit{round}\ x) \leq x + 1/2$
and *of-int-round-abs-le*: $|\mathit{of\text{-}int}\ (\mathit{round}\ x) - x| \leq 1/2$
and *of-int-round-gt*: $\mathit{of\text{-}int}\ (\mathit{round}\ x) > x - 1/2$
⟨*proof*⟩

lemma *round-of-int* [*simp*]: $\mathit{round}\ (\mathit{of\text{-}int}\ n) = n$
⟨*proof*⟩

lemma *round-0* [*simp*]: $\mathit{round}\ 0 = 0$
⟨*proof*⟩

lemma *round-1* [*simp*]: $\mathit{round}\ 1 = 1$
⟨*proof*⟩

lemma *round-numeral* [*simp*]: $\mathit{round}\ (\mathit{numeral}\ n) = \mathit{numeral}\ n$
⟨*proof*⟩

lemma *round-neg-numeral* [*simp*]: $\mathit{round}\ (-\mathit{numeral}\ n) = -\mathit{numeral}\ n$
⟨*proof*⟩

lemma *round-of-nat* [*simp*]: $\mathit{round}\ (\mathit{of\text{-}nat}\ n) = \mathit{of\text{-}nat}\ n$
⟨*proof*⟩

lemma *round-mono*: $x \leq y \Longrightarrow \mathit{round}\ x \leq \mathit{round}\ y$
⟨*proof*⟩

lemma *round-unique*: $\mathit{of\text{-}int}\ y > x - 1/2 \Longrightarrow \mathit{of\text{-}int}\ y \leq x + 1/2 \Longrightarrow \mathit{round}\ x = y$
⟨*proof*⟩

lemma *round-unique'*: $|x - \mathit{of\text{-}int}\ n| < 1/2 \Longrightarrow \mathit{round}\ x = n$
⟨*proof*⟩

lemma *round-altdef*: $\mathit{round}\ x = (\mathit{if}\ \mathit{frac}\ x \geq 1/2\ \mathit{then}\ \lceil x \rceil\ \mathit{else}\ \lfloor x \rfloor)$
⟨*proof*⟩

lemma *floor-le-round*: $\lfloor x \rfloor \leq \mathit{round}\ x$
⟨*proof*⟩

lemma *ceiling-ge-round*: $\lceil x \rceil \geq \mathit{round}\ x$
⟨*proof*⟩

lemma *round-diff-minimal*: $|z - \mathit{of\text{-}int}\ (\mathit{round}\ z)| \leq |z - \mathit{of\text{-}int}\ m|$
for $z :: 'a$::*floor-ceiling*
⟨*proof*⟩

end

97 Rational numbers

theory *Rat*
imports *Archimedean-Field*
begin

97.1 Rational numbers as quotient

97.1.1 Construction of the type of rational numbers

definition *ratrel* :: $(int \times int) \Rightarrow (int \times int) \Rightarrow bool$
where $ratrel = (\lambda x\ y.\ snd\ x \neq 0 \wedge snd\ y \neq 0 \wedge fst\ x * snd\ y = fst\ y * snd\ x)$

lemma *ratrel-iff* [*simp*]: $ratrel\ x\ y \longleftrightarrow snd\ x \neq 0 \wedge snd\ y \neq 0 \wedge fst\ x * snd\ y = fst\ y * snd\ x$
⟨*proof*⟩

lemma *exists-ratrel-refl*: $\exists x.\ ratrel\ x\ x$
⟨*proof*⟩

lemma *symp-ratrel*: *symp ratrel*
⟨*proof*⟩

lemma *transp-ratrel*: *transp ratrel*
⟨*proof*⟩

lemma *part-equivp-ratrel*: *part-equivp ratrel*
⟨*proof*⟩

quotient-type *rat* = $int \times int$ / *partial*: *ratrel*
morphisms *Rep-Rat Abs-Rat*
⟨*proof*⟩

lemma *Domainp-cr-rat* [*transfer-domain-rule*]: *Domainp pcr-rat* = $(\lambda x.\ snd\ x \neq 0)$
⟨*proof*⟩

97.1.2 Representation and basic operations

lift-definition *Fract* :: $int \Rightarrow int \Rightarrow rat$
is $\lambda a\ b.\ if\ b = 0\ then\ (0,\ 1)\ else\ (a,\ b)$
⟨*proof*⟩

lemma *eq-rat*:
$\bigwedge a\ b\ c\ d.\ b \neq 0 \Longrightarrow d \neq 0 \Longrightarrow Fract\ a\ b = Fract\ c\ d \longleftrightarrow a * d = c * b$
$\bigwedge a.\ Fract\ a\ 0 = Fract\ 0\ 1$
$\bigwedge a\ c.\ Fract\ 0\ a = Fract\ 0\ c$
⟨*proof*⟩

lemma *Rat-cases* [*case-names Fract*, *cases type*: *rat*]:

assumes *that*: $\bigwedge a\ b.\ q = Fract\ a\ b \Longrightarrow b > 0 \Longrightarrow coprime\ a\ b \Longrightarrow C$
shows C
⟨*proof*⟩

lemma *Rat-induct* [*case-names Fract*, *induct type*: *rat*]:
assumes $\bigwedge a\ b.\ b > 0 \Longrightarrow coprime\ a\ b \Longrightarrow P\ (Fract\ a\ b)$
shows $P\ q$
⟨*proof*⟩

instantiation *rat* :: *field*
begin

lift-definition *zero-rat* :: *rat* **is** $(0, 1)$
⟨*proof*⟩

lift-definition *one-rat* :: *rat* **is** $(1, 1)$
⟨*proof*⟩

lemma *Zero-rat-def*: $0 = Fract\ 0\ 1$
⟨*proof*⟩

lemma *One-rat-def*: $1 = Fract\ 1\ 1$
⟨*proof*⟩

lift-definition *plus-rat* :: *rat* ⇒ *rat* ⇒ *rat*
is $\lambda x\ y.\ (fst\ x * snd\ y + fst\ y * snd\ x,\ snd\ x * snd\ y)$
⟨*proof*⟩

lemma *add-rat* [*simp*]:
assumes $b \neq 0$ **and** $d \neq 0$
shows $Fract\ a\ b + Fract\ c\ d = Fract\ (a * d + c * b)\ (b * d)$
⟨*proof*⟩

lift-definition *uminus-rat* :: *rat* ⇒ *rat* **is** $\lambda x.\ (-\ fst\ x,\ snd\ x)$
⟨*proof*⟩

lemma *minus-rat* [*simp*]: $-\ Fract\ a\ b = Fract\ (-\ a)\ b$
⟨*proof*⟩

lemma *minus-rat-cancel* [*simp*]: $Fract\ (-\ a)\ (-\ b) = Fract\ a\ b$
⟨*proof*⟩

definition *diff-rat-def*: $q - r = q + -\ r$ **for** $q\ r$ :: *rat*

lemma *diff-rat* [*simp*]:
$b \neq 0 \Longrightarrow d \neq 0 \Longrightarrow Fract\ a\ b - Fract\ c\ d = Fract\ (a * d - c * b)\ (b * d)$
⟨*proof*⟩

lift-definition *times-rat* :: *rat* ⇒ *rat* ⇒ *rat*

is $\lambda x\ y.\ (fst\ x * fst\ y,\ snd\ x * snd\ y)$
⟨*proof*⟩

lemma *mult-rat* [*simp*]: $Fract\ a\ b * Fract\ c\ d = Fract\ (a * c)\ (b * d)$
⟨*proof*⟩

lemma *mult-rat-cancel*: $c \neq 0 \Longrightarrow Fract\ (c * a)\ (c * b) = Fract\ a\ b$
⟨*proof*⟩

lift-definition *inverse-rat* :: $rat \Rightarrow rat$
is $\lambda x.\ if\ fst\ x = 0\ then\ (0,\ 1)\ else\ (snd\ x,\ fst\ x)$
⟨*proof*⟩

lemma *inverse-rat* [*simp*]: $inverse\ (Fract\ a\ b) = Fract\ b\ a$
⟨*proof*⟩

definition *divide-rat-def*: $q\ div\ r = q * inverse\ r$ **for** $q\ r :: rat$

lemma *divide-rat* [*simp*]: $Fract\ a\ b\ div\ Fract\ c\ d = Fract\ (a * d)\ (b * c)$
⟨*proof*⟩

instance
⟨*proof*⟩

end

lemma *div-add-self1-no-field* [*simp*]:
assumes *NO-MATCH* $(x :: 'b :: field)\ b\ (b :: 'a :: euclidean\text{-}semiring\text{-}cancel) \neq 0$
shows $(b + a)\ div\ b = a\ div\ b + 1$
⟨*proof*⟩

lemma *div-add-self2-no-field* [*simp*]:
assumes *NO-MATCH* $(x :: 'b :: field)\ b\ (b :: 'a :: euclidean\text{-}semiring\text{-}cancel) \neq 0$
shows $(a + b)\ div\ b = a\ div\ b + 1$
⟨*proof*⟩

lemma *of-nat-rat*: $of\text{-}nat\ k = Fract\ (of\text{-}nat\ k)\ 1$
⟨*proof*⟩

lemma *of-int-rat*: $of\text{-}int\ k = Fract\ k\ 1$
⟨*proof*⟩

lemma *Fract-of-nat-eq*: $Fract\ (of\text{-}nat\ k)\ 1 = of\text{-}nat\ k$
⟨*proof*⟩

lemma *Fract-of-int-eq*: $Fract\ k\ 1 = of\text{-}int\ k$

⟨*proof*⟩

lemma *rat-number-collapse*:
Fract 0 k = 0
Fract 1 1 = 1
Fract (numeral w) 1 = numeral w
Fract (− numeral w) 1 = − numeral w
Fract (− 1) 1 = − 1
Fract k 0 = 0
⟨*proof*⟩

lemma *rat-number-expand*:
0 = Fract 0 1
1 = Fract 1 1
numeral k = Fract (numeral k) 1
− 1 = Fract (− 1) 1
− numeral k = Fract (− numeral k) 1
⟨*proof*⟩

lemma *Rat-cases-nonzero* [*case-names Fract 0*]:
assumes *Fract*: $\bigwedge a\ b.\ q = Fract\ a\ b \Longrightarrow b > 0 \Longrightarrow a \neq 0 \Longrightarrow coprime\ a\ b \Longrightarrow C$
and *0*: $q = 0 \Longrightarrow C$
shows *C*
⟨*proof*⟩

97.1.3 Function *normalize*

lemma *Fract-coprime*: *Fract (a div gcd a b) (b div gcd a b) = Fract a b*
⟨*proof*⟩

definition *normalize* :: *int × int ⇒ int × int*
where *normalize p =*
(if snd p > 0 then (let a = gcd (fst p) (snd p) in (fst p div a, snd p div a))
else if snd p = 0 then (0, 1)
else (let a = − gcd (fst p) (snd p) in (fst p div a, snd p div a)))

lemma *normalize-crossproduct*:
assumes *q ≠ 0 s ≠ 0*
assumes *normalize (p, q) = normalize (r, s)*
shows *p * s = r * q*
⟨*proof*⟩

lemma *normalize-eq*: *normalize (a, b) = (p, q) ⟹ Fract p q = Fract a b*
⟨*proof*⟩

lemma *normalize-denom-pos*: *normalize r = (p, q) ⟹ q > 0*
⟨*proof*⟩

lemma *normalize-coprime*: *normalize r = (p, q) ⟹ coprime p q*
⟨*proof*⟩

lemma *normalize-stable* [*simp*]: *q > 0 ⟹ coprime p q ⟹ normalize (p, q) = (p, q)*
⟨*proof*⟩

lemma *normalize-denom-zero* [*simp*]: *normalize (p, 0) = (0, 1)*
⟨*proof*⟩

lemma *normalize-negative* [*simp*]: *q < 0 ⟹ normalize (p, q) = normalize (− p, − q)*
⟨*proof*⟩

Decompose a fraction into normalized, i.e. coprime numerator and denominator:

definition *quotient-of* :: *rat ⇒ int × int*
where *quotient-of x =*
(THE pair. x = Fract (fst pair) (snd pair) ∧ snd pair > 0 ∧ coprime (fst pair) (snd pair))

lemma *quotient-of-unique*: *∃!p. r = Fract (fst p) (snd p) ∧ snd p > 0 ∧ coprime (fst p) (snd p)*
⟨*proof*⟩

lemma *quotient-of-Fract* [*code*]: *quotient-of (Fract a b) = normalize (a, b)*
⟨*proof*⟩

lemma *quotient-of-number* [*simp*]:
quotient-of 0 = (0, 1)
quotient-of 1 = (1, 1)
quotient-of (numeral k) = (numeral k, 1)
quotient-of (− 1) = (− 1, 1)
quotient-of (− numeral k) = (− numeral k, 1)
⟨*proof*⟩

lemma *quotient-of-eq*: *quotient-of (Fract a b) = (p, q) ⟹ Fract p q = Fract a b*
⟨*proof*⟩

lemma *quotient-of-denom-pos*: *quotient-of r = (p, q) ⟹ q > 0*
⟨*proof*⟩

lemma *quotient-of-denom-pos′*: *snd (quotient-of r) > 0*
⟨*proof*⟩

lemma *quotient-of-coprime*: *quotient-of r = (p, q) ⟹ coprime p q*
⟨*proof*⟩

lemma *quotient-of-inject*:

assumes *quotient-of a = quotient-of b*
shows *a = b*
⟨*proof*⟩

lemma *quotient-of-inject-eq*: *quotient-of a = quotient-of b ⟷ a = b*
⟨*proof*⟩

97.1.4 Various

lemma *Fract-of-int-quotient*: *Fract k l = of-int k / of-int l*
⟨*proof*⟩

lemma *Fract-add-one*: *n ≠ 0 ⟹ Fract (m + n) n = Fract m n + 1*
⟨*proof*⟩

lemma *quotient-of-div*:
assumes *r*: *quotient-of r = (n,d)*
shows *r = of-int n / of-int d*
⟨*proof*⟩

97.1.5 The ordered field of rational numbers

lift-definition *positive* :: *rat ⇒ bool*
is *λx. 0 < fst x ∗ snd x*
⟨*proof*⟩

lemma *positive-zero*: *¬ positive 0*
⟨*proof*⟩

lemma *positive-add*: *positive x ⟹ positive y ⟹ positive (x + y)*
⟨*proof*⟩

lemma *positive-mult*: *positive x ⟹ positive y ⟹ positive (x ∗ y)*
⟨*proof*⟩

lemma *positive-minus*: *¬ positive x ⟹ x ≠ 0 ⟹ positive (− x)*
⟨*proof*⟩

instantiation *rat* :: *linordered-field*
begin

definition *x < y ⟷ positive (y − x)*

definition *x ≤ y ⟷ x < y ∨ x = y* **for** *x y* :: *rat*

definition *|a| = (if a < 0 then − a else a)* **for** *a* :: *rat*

definition *sgn a = (if a = 0 then 0 else if 0 < a then 1 else − 1)* **for** *a* :: *rat*

instance

⟨*proof*⟩

end

instantiation *rat* :: *distrib-lattice*
begin

definition (*inf* :: *rat* ⇒ *rat* ⇒ *rat*) = *min*

definition (*sup* :: *rat* ⇒ *rat* ⇒ *rat*) = *max*

instance
⟨*proof*⟩

end

lemma *positive-rat*: *positive* (*Fract a b*) ⟷ *0* < *a* ∗ *b*
⟨*proof*⟩

lemma *less-rat* [*simp*]:
b ≠ *0* ⟹ *d* ≠ *0* ⟹ *Fract a b* < *Fract c d* ⟷ (*a* ∗ *d*) ∗ (*b* ∗ *d*) < (*c* ∗ *b*) ∗ (*b* ∗ *d*)
⟨*proof*⟩

lemma *le-rat* [*simp*]:
b ≠ *0* ⟹ *d* ≠ *0* ⟹ *Fract a b* ≤ *Fract c d* ⟷ (*a* ∗ *d*) ∗ (*b* ∗ *d*) ≤ (*c* ∗ *b*) ∗ (*b* ∗ *d*)
⟨*proof*⟩

lemma *abs-rat* [*simp*, *code*]: |*Fract a b*| = *Fract* |*a*| |*b*|
⟨*proof*⟩

lemma *sgn-rat* [*simp*, *code*]: *sgn* (*Fract a b*) = *of-int* (*sgn a* ∗ *sgn b*)
⟨*proof*⟩

lemma *Rat-induct-pos* [*case-names Fract*, *induct type*: *rat*]:
assumes *step*: ⋀*a b*. *0* < *b* ⟹ *P* (*Fract a b*)
shows *P q*
⟨*proof*⟩

lemma *zero-less-Fract-iff*: *0* < *b* ⟹ *0* < *Fract a b* ⟷ *0* < *a*
⟨*proof*⟩

lemma *Fract-less-zero-iff*: *0* < *b* ⟹ *Fract a b* < *0* ⟷ *a* < *0*
⟨*proof*⟩

lemma *zero-le-Fract-iff*: *0* < *b* ⟹ *0* ≤ *Fract a b* ⟷ *0* ≤ *a*
⟨*proof*⟩

lemma *Fract-le-zero-iff*: $0 < b \Longrightarrow Fract\ a\ b \leq 0 \longleftrightarrow a \leq 0$
⟨*proof*⟩

lemma *one-less-Fract-iff*: $0 < b \Longrightarrow 1 < Fract\ a\ b \longleftrightarrow b < a$
⟨*proof*⟩

lemma *Fract-less-one-iff*: $0 < b \Longrightarrow Fract\ a\ b < 1 \longleftrightarrow a < b$
⟨*proof*⟩

lemma *one-le-Fract-iff*: $0 < b \Longrightarrow 1 \leq Fract\ a\ b \longleftrightarrow b \leq a$
⟨*proof*⟩

lemma *Fract-le-one-iff*: $0 < b \Longrightarrow Fract\ a\ b \leq 1 \longleftrightarrow a \leq b$
⟨*proof*⟩

97.1.6 Rationals are an Archimedean field

lemma *rat-floor-lemma*: $of\text{-}int\ (a\ div\ b) \leq Fract\ a\ b \wedge Fract\ a\ b < of\text{-}int\ (a\ div\ b + 1)$
⟨*proof*⟩

instance *rat* :: *archimedean-field*
⟨*proof*⟩

instantiation *rat* :: *floor-ceiling*
begin

definition *floor-rat* :: $rat \Rightarrow int$
where $\lfloor x \rfloor = (THE\ z.\ of\text{-}int\ z \leq x \wedge x < of\text{-}int\ (z + 1))$ **for** x :: *rat*

instance
⟨*proof*⟩

end

lemma *floor-Fract* [*simp*]: $\lfloor Fract\ a\ b \rfloor = a\ div\ b$
⟨*proof*⟩

97.2 Linear arithmetic setup

⟨*ML*⟩

97.3 Embedding from Rationals to other Fields

context *field-char-0*
begin

lift-definition *of-rat* :: $rat \Rightarrow 'a$
is $\lambda x.\ of\text{-}int\ (fst\ x)\ /\ of\text{-}int\ (snd\ x)$
⟨*proof*⟩

end

lemma *of-rat-rat*: $b \neq 0 \Longrightarrow$ *of-rat* (*Fract a b*) = *of-int a* / *of-int b*
⟨*proof*⟩

lemma *of-rat-0* [*simp*]: *of-rat 0* = *0*
⟨*proof*⟩

lemma *of-rat-1* [*simp*]: *of-rat 1* = *1*
⟨*proof*⟩

lemma *of-rat-add*: *of-rat* (*a* + *b*) = *of-rat a* + *of-rat b*
⟨*proof*⟩

lemma *of-rat-minus*: *of-rat* (− *a*) = − *of-rat a*
⟨*proof*⟩

lemma *of-rat-neg-one* [*simp*]: *of-rat* (− *1*) = − *1*
⟨*proof*⟩

lemma *of-rat-diff*: *of-rat* (*a* − *b*) = *of-rat a* − *of-rat b*
⟨*proof*⟩

lemma *of-rat-mult*: *of-rat* (*a* ∗ *b*) = *of-rat a* ∗ *of-rat b*
⟨*proof*⟩

lemma *of-rat-sum*: *of-rat* ($\sum$ *a*∈*A*. *f a*) = ($\sum$ *a*∈*A*. *of-rat* (*f a*))
⟨*proof*⟩

lemma *of-rat-prod*: *of-rat* ($\prod$ *a*∈*A*. *f a*) = ($\prod$ *a*∈*A*. *of-rat* (*f a*))
⟨*proof*⟩

lemma *nonzero-of-rat-inverse*: *a* ≠ *0* ⟹ *of-rat* (*inverse a*) = *inverse* (*of-rat a*)
⟨*proof*⟩

lemma *of-rat-inverse*: (*of-rat* (*inverse a*) :: *'a*::*field-char-0*) = *inverse* (*of-rat a*)
⟨*proof*⟩

lemma *nonzero-of-rat-divide*: *b* ≠ *0* ⟹ *of-rat* (*a* / *b*) = *of-rat a* / *of-rat b*
⟨*proof*⟩

lemma *of-rat-divide*: (*of-rat* (*a* / *b*) :: *'a*::*field-char-0*) = *of-rat a* / *of-rat b*
⟨*proof*⟩

lemma *of-rat-power*: (*of-rat* (*a* ^ *n*) :: *'a*::*field-char-0*) = *of-rat a* ^ *n*
⟨*proof*⟩

lemma *of-rat-eq-iff* [*simp*]: *of-rat a* = *of-rat b* ⟷ *a* = *b*

⟨*proof*⟩

lemma *of-rat-eq-0-iff* [*simp*]: *of-rat a* = *0* ⟷ *a* = *0*
⟨*proof*⟩

lemma *zero-eq-of-rat-iff* [*simp*]: *0* = *of-rat a* ⟷ *0* = *a*
⟨*proof*⟩

lemma *of-rat-eq-1-iff* [*simp*]: *of-rat a* = *1* ⟷ *a* = *1*
⟨*proof*⟩

lemma *one-eq-of-rat-iff* [*simp*]: *1* = *of-rat a* ⟷ *1* = *a*
⟨*proof*⟩

lemma *of-rat-less*: (*of-rat r* :: *'a*::*linordered-field*) < *of-rat s* ⟷ *r* < *s*
⟨*proof*⟩

lemma *of-rat-less-eq*: (*of-rat r* :: *'a*::*linordered-field*) ≤ *of-rat s* ⟷ *r* ≤ *s*
⟨*proof*⟩

lemma *of-rat-le-0-iff* [*simp*]: (*of-rat r* :: *'a*::*linordered-field*) ≤ *0* ⟷ *r* ≤ *0*
⟨*proof*⟩

lemma *zero-le-of-rat-iff* [*simp*]: *0* ≤ (*of-rat r* :: *'a*::*linordered-field*) ⟷ *0* ≤ *r*
⟨*proof*⟩

lemma *of-rat-le-1-iff* [*simp*]: (*of-rat r* :: *'a*::*linordered-field*) ≤ *1* ⟷ *r* ≤ *1*
⟨*proof*⟩

lemma *one-le-of-rat-iff* [*simp*]: *1* ≤ (*of-rat r* :: *'a*::*linordered-field*) ⟷ *1* ≤ *r*
⟨*proof*⟩

lemma *of-rat-less-0-iff* [*simp*]: (*of-rat r* :: *'a*::*linordered-field*) < *0* ⟷ *r* < *0*
⟨*proof*⟩

lemma *zero-less-of-rat-iff* [*simp*]: *0* < (*of-rat r* :: *'a*::*linordered-field*) ⟷ *0* < *r*
⟨*proof*⟩

lemma *of-rat-less-1-iff* [*simp*]: (*of-rat r* :: *'a*::*linordered-field*) < *1* ⟷ *r* < *1*
⟨*proof*⟩

lemma *one-less-of-rat-iff* [*simp*]: *1* < (*of-rat r* :: *'a*::*linordered-field*) ⟷ *1* < *r*
⟨*proof*⟩

lemma *of-rat-eq-id* [*simp*]: *of-rat* = *id*
⟨*proof*⟩

lemma *abs-of-rat* [*simp*]:
|*of-rat r*| = (*of-rat* |*r*| :: *'a* :: *linordered-field*)

⟨*proof*⟩

Collapse nested embeddings.

lemma *of-rat-of-nat-eq* [*simp*]: *of-rat* (*of-nat n*) = *of-nat n*
⟨*proof*⟩

lemma *of-rat-of-int-eq* [*simp*]: *of-rat* (*of-int z*) = *of-int z*
⟨*proof*⟩

lemma *of-rat-numeral-eq* [*simp*]: *of-rat* (*numeral w*) = *numeral w*
⟨*proof*⟩

lemma *of-rat-neg-numeral-eq* [*simp*]: *of-rat* (− *numeral w*) = − *numeral w*
⟨*proof*⟩

lemma *of-rat-floor* [*simp*]:
⌊*of-rat r*⌋ = ⌊*r*⌋
⟨*proof*⟩

lemma *of-rat-ceiling* [*simp*]:
⌈*of-rat r*⌉ = ⌈*r*⌉
⟨*proof*⟩

lemmas *zero-rat* = *Zero-rat-def*
lemmas *one-rat* = *One-rat-def*

abbreviation *rat-of-nat* :: *nat* ⇒ *rat*
where *rat-of-nat* ≡ *of-nat*

abbreviation *rat-of-int* :: *int* ⇒ *rat*
where *rat-of-int* ≡ *of-int*

97.4 The Set of Rational Numbers

context *field-char-0*
begin

definition *Rats* :: *'a set* (ℚ)
where ℚ = *range of-rat*

end

lemma *Rats-cases* [*cases set*: *Rats*]:
assumes *q* ∈ ℚ
obtains (*of-rat*) *r* **where** *q* = *of-rat r*
⟨*proof*⟩

lemma *Rats-cases'*:
assumes (*x* :: *'a* :: *field-char-0*) ∈ ℚ

obtains *a b* **where** *b > 0 coprime a b x = of-int a / of-int b*
⟨*proof*⟩

lemma *Rats-of-rat* [*simp*]: *of-rat r* ∈ ℚ
⟨*proof*⟩

lemma *Rats-of-int* [*simp*]: *of-int z* ∈ ℚ
⟨*proof*⟩

lemma *Ints-subset-Rats*: ℤ ⊆ ℚ
⟨*proof*⟩

lemma *Rats-of-nat* [*simp*]: *of-nat n* ∈ ℚ
⟨*proof*⟩

lemma *Nats-subset-Rats*: ℕ ⊆ ℚ
⟨*proof*⟩

lemma *Rats-number-of* [*simp*]: *numeral w* ∈ ℚ
⟨*proof*⟩

lemma *Rats-0* [*simp*]: *0* ∈ ℚ
⟨*proof*⟩

lemma *Rats-1* [*simp*]: *1* ∈ ℚ
⟨*proof*⟩

lemma *Rats-add* [*simp*]: *a* ∈ ℚ ⟹ *b* ∈ ℚ ⟹ *a + b* ∈ ℚ
⟨*proof*⟩

lemma *Rats-minus-iff* [*simp*]: *− a* ∈ ℚ ⟷ *a* ∈ ℚ
⟨*proof*⟩

lemma *Rats-diff* [*simp*]: *a* ∈ ℚ ⟹ *b* ∈ ℚ ⟹ *a − b* ∈ ℚ
⟨*proof*⟩

lemma *Rats-mult* [*simp*]: *a* ∈ ℚ ⟹ *b* ∈ ℚ ⟹ *a * b* ∈ ℚ
⟨*proof*⟩

lemma *Rats-inverse* [*simp*]: *a* ∈ ℚ ⟹ *inverse a* ∈ ℚ
for *a* :: *'a*::*field-char-0*
⟨*proof*⟩

lemma *Rats-divide* [*simp*]: *a* ∈ ℚ ⟹ *b* ∈ ℚ ⟹ *a / b* ∈ ℚ
for *a b* :: *'a*::*field-char-0*
⟨*proof*⟩

lemma *Rats-power* [*simp*]: *a* ∈ ℚ ⟹ *a ^ n* ∈ ℚ
for *a* :: *'a*::*field-char-0*

⟨*proof*⟩

lemma *Rats-induct* [*case-names of-rat*, *induct set*: *Rats*]: $q \in \mathbb{Q} \Longrightarrow (\bigwedge r.\ P\ (\mathit{of\text{-}rat}\ r)) \Longrightarrow P\ q$
⟨*proof*⟩

lemma *Rats-infinite*: $\neg\ \mathit{finite}\ \mathbb{Q}$
⟨*proof*⟩

lemma *Rats-add-iff*: $a \in \mathbb{Q} \vee b \in \mathbb{Q} \Longrightarrow a+b \in \mathbb{Q} \longleftrightarrow a \in \mathbb{Q} \wedge b \in \mathbb{Q}$
⟨*proof*⟩

lemma *Rats-diff-iff*: $a \in \mathbb{Q} \vee b \in \mathbb{Q} \Longrightarrow a-b \in \mathbb{Q} \longleftrightarrow a \in \mathbb{Q} \wedge b \in \mathbb{Q}$
⟨*proof*⟩

lemma *Rats-mult-iff*: $a \in \mathbb{Q}-\{0\} \vee b \in \mathbb{Q}-\{0\} \Longrightarrow a*b \in \mathbb{Q} \longleftrightarrow a \in \mathbb{Q} \wedge b \in \mathbb{Q}$
⟨*proof*⟩

lemma *Rats-inverse-iff* [*simp*]: $\mathit{inverse}\ a \in \mathbb{Q} \longleftrightarrow a \in \mathbb{Q}$
⟨*proof*⟩

lemma *Rats-divide-iff*: $a \in \mathbb{Q}-\{0\} \vee b \in \mathbb{Q}-\{0\} \Longrightarrow a/b \in \mathbb{Q} \longleftrightarrow a \in \mathbb{Q} \wedge b \in \mathbb{Q}$
⟨*proof*⟩

97.5 Implementation of rational numbers as pairs of integers

Formal constructor

definition *Frct* :: $\mathit{int} \times \mathit{int} \Rightarrow \mathit{rat}$
where [*simp*]: $\mathit{Frct}\ p = \mathit{Fract}\ (\mathit{fst}\ p)\ (\mathit{snd}\ p)$

lemma [*code abstype*]: $\mathit{Frct}\ (\mathit{quotient\text{-}of}\ q) = q$
⟨*proof*⟩

Numerals

declare *quotient-of-Fract* [*code abstract*]

definition *of-int* :: $\mathit{int} \Rightarrow \mathit{rat}$
where [*code-abbrev*]: $\mathit{of\text{-}int} = \mathit{Int.of\text{-}int}$

hide-const (**open**) *of-int*

lemma *quotient-of-int* [*code abstract*]: $\mathit{quotient\text{-}of}\ (\mathit{Rat.of\text{-}int}\ a) = (a,\ 1)$
⟨*proof*⟩

lemma [*code-unfold*]: $\mathit{numeral}\ k = \mathit{Rat.of\text{-}int}\ (\mathit{numeral}\ k)$
⟨*proof*⟩

lemma [*code-unfold*]: $-\ \mathit{numeral}\ k = \mathit{Rat.of\text{-}int}\ (-\ \mathit{numeral}\ k)$

⟨*proof*⟩

lemma *Frct-code-post* [*code-post*]:
Frct (*0*, *a*) = *0*
Frct (*a*, *0*) = *0*
Frct (*1*, *1*) = *1*
Frct (*numeral k*, *1*) = *numeral k*
Frct (*1*, *numeral k*) = *1* / *numeral k*
Frct (*numeral k*, *numeral l*) = *numeral k* / *numeral l*
Frct (− *a*, *b*) = − *Frct* (*a*, *b*)
Frct (*a*, − *b*) = − *Frct* (*a*, *b*)
− (− *Frct q*) = *Frct q*
⟨*proof*⟩

Operations

lemma *rat-zero-code* [*code abstract*]: *quotient-of 0* = (*0*, *1*)
⟨*proof*⟩

lemma *rat-one-code* [*code abstract*]: *quotient-of 1* = (*1*, *1*)
⟨*proof*⟩

lemma *rat-plus-code* [*code abstract*]:
quotient-of (*p* + *q*) = (*let* (*a*, *c*) = *quotient-of p*; (*b*, *d*) = *quotient-of q*
in normalize (*a* ∗ *d* + *b* ∗ *c*, *c* ∗ *d*))
⟨*proof*⟩

lemma *rat-uminus-code* [*code abstract*]:
quotient-of (− *p*) = (*let* (*a*, *b*) = *quotient-of p in* (− *a*, *b*))
⟨*proof*⟩

lemma *rat-minus-code* [*code abstract*]:
quotient-of (*p* − *q*) =
(*let* (*a*, *c*) = *quotient-of p*; (*b*, *d*) = *quotient-of q*
in normalize (*a* ∗ *d* − *b* ∗ *c*, *c* ∗ *d*))
⟨*proof*⟩

lemma *rat-times-code* [*code abstract*]:
quotient-of (*p* ∗ *q*) =
(*let* (*a*, *c*) = *quotient-of p*; (*b*, *d*) = *quotient-of q*
in normalize (*a* ∗ *b*, *c* ∗ *d*))
⟨*proof*⟩

lemma *rat-inverse-code* [*code abstract*]:
quotient-of (*inverse p*) =
(*let* (*a*, *b*) = *quotient-of p*
in if a = *0 then* (*0*, *1*) *else* (*sgn a* ∗ *b*, |*a*|))
⟨*proof*⟩

lemma *rat-divide-code* [*code abstract*]:

quotient-of (*p* / *q*) =
(*let* (*a*, *c*) = *quotient-of p*; (*b*, *d*) = *quotient-of q*
in normalize (*a* ∗ *d*, *c* ∗ *b*))
⟨*proof*⟩

lemma *rat-abs-code* [*code abstract*]:
quotient-of |*p*| = (*let* (*a*, *b*) = *quotient-of p in* (|*a*|, *b*))
⟨*proof*⟩

lemma *rat-sgn-code* [*code abstract*]: *quotient-of* (*sgn p*) = (*sgn* (*fst* (*quotient-of p*)), *1*)
⟨*proof*⟩

lemma *rat-floor-code* [*code*]: ⌊*p*⌋ = (*let* (*a*, *b*) = *quotient-of p in a div b*)
⟨*proof*⟩

instantiation *rat* :: *equal*
begin

definition [*code*]: *HOL.equal a b* ⟷ *quotient-of a* = *quotient-of b*

instance
⟨*proof*⟩

lemma *rat-eq-refl* [*code nbe*]: *HOL.equal* (*r*::*rat*) *r* ⟷ *True*
⟨*proof*⟩

end

lemma *rat-less-eq-code* [*code*]:
p ≤ *q* ⟷ (*let* (*a*, *c*) = *quotient-of p*; (*b*, *d*) = *quotient-of q in a* ∗ *d* ≤ *c* ∗ *b*)
⟨*proof*⟩

lemma *rat-less-code* [*code*]:
p < *q* ⟷ (*let* (*a*, *c*) = *quotient-of p*; (*b*, *d*) = *quotient-of q in a* ∗ *d* < *c* ∗ *b*)
⟨*proof*⟩

lemma [*code*]: *of-rat p* = (*let* (*a*, *b*) = *quotient-of p in of-int a* / *of-int b*)
⟨*proof*⟩

Quickcheck

context
includes *term-syntax*
begin

definition
valterm-fract :: *int* × (*unit* ⇒ *Code-Evaluation.term*) ⇒
int × (*unit* ⇒ *Code-Evaluation.term*) ⇒
rat × (*unit* ⇒ *Code-Evaluation.term*)

where [*code-unfold*]: *valterm-fract k l = Code-Evaluation.valtermify Fract* {·} *k* {·} *l*

end

instantiation *rat* :: *random*
begin

context
includes *state-combinator-syntax*
begin

definition
Quickcheck-Random.random i =
Quickcheck-Random.random i ∘→ (λ*num. Random.range i* ∘→ (λ*denom. Pair*
(*let j = int-of-integer* (*integer-of-natural* (*denom + 1*))
in valterm-fract num (*j*, λ*u. Code-Evaluation.term-of j*))))

instance ⟨*proof*⟩

end

end

instantiation *rat* :: *exhaustive*
begin

definition
exhaustive-rat f d =
Quickcheck-Exhaustive.exhaustive
(λ*l. Quickcheck-Exhaustive.exhaustive*
(λ*k. f* (*Fract k* (*int-of-integer* (*integer-of-natural l*) + *1*))) *d*) *d*

instance ⟨*proof*⟩

end

instantiation *rat* :: *full-exhaustive*
begin

definition
full-exhaustive-rat f d =
Quickcheck-Exhaustive.full-exhaustive
(λ(*l*, -). *Quickcheck-Exhaustive.full-exhaustive*
(λ*k. f*
(*let j = int-of-integer* (*integer-of-natural l*) + *1*
in valterm-fract k (*j*, λ-. *Code-Evaluation.term-of j*))) *d*) *d*

instance ⟨*proof*⟩

end

instance *rat* :: *partial-term-of* ⟨*proof*⟩

lemma [*code*]:
partial-term-of (*ty* :: *rat itself*) (*Quickcheck-Narrowing.Narrowing-variable p tt*)
≡
Code-Evaluation.Free (*STR ''-''*) (*Typerep.Typerep* (*STR ''Rat.rat''*) [])
partial-term-of (*ty* :: *rat itself*) (*Quickcheck-Narrowing.Narrowing-constructor 0*
[*l*, *k*]) ≡
Code-Evaluation.App
(*Code-Evaluation.Const* (*STR ''Rat.Frct''*)
(*Typerep.Typerep* (*STR ''fun''*)
[*Typerep.Typerep* (*STR ''Product-Type.prod''*)
[*Typerep.Typerep* (*STR ''Int.int''*) [], *Typerep.Typerep* (*STR ''Int.int''*) []],
Typerep.Typerep (*STR ''Rat.rat''*) []]))
(*Code-Evaluation.App*
(*Code-Evaluation.App*
(*Code-Evaluation.Const* (*STR ''Product-Type.Pair''*)
(*Typerep.Typerep* (*STR ''fun''*)
[*Typerep.Typerep* (*STR ''Int.int''*) [],
Typerep.Typerep (*STR ''fun''*)
[*Typerep.Typerep* (*STR ''Int.int''*) [],
Typerep.Typerep (*STR ''Product-Type.prod''*)
[*Typerep.Typerep* (*STR ''Int.int''*) [], *Typerep.Typerep* (*STR ''Int.int''*)
[]]]]))
(*partial-term-of* (*TYPE*(*int*)) *l*)) (*partial-term-of* (*TYPE*(*int*)) *k*))
⟨*proof*⟩

instantiation *rat* :: *narrowing*
begin

definition
narrowing =
Quickcheck-Narrowing.apply
(*Quickcheck-Narrowing.apply*
(*Quickcheck-Narrowing.cons* (λ*nom denom*. *Fract nom denom*)) *narrowing*)
narrowing

instance ⟨*proof*⟩

end

97.6 Setup for Nitpick

⟨*ML*⟩

lemmas [*nitpick-unfold*] =

inverse-rat-inst.inverse-rat
one-rat-inst.one-rat ord-rat-inst.less-rat
ord-rat-inst.less-eq-rat plus-rat-inst.plus-rat times-rat-inst.times-rat
uminus-rat-inst.uminus-rat zero-rat-inst.zero-rat

97.7 Float syntax

syntax -*Float* :: *float-const* ⇒ *'a* (-)

⟨*ML*⟩

Test:

lemma *123.456* = *−111.111* + *200* + *30* + *4* + *5/10* + *6/100* + (*7/1000*::*rat*)
⟨*proof*⟩

97.8 Hiding implementation details

hide-const (**open**) *normalize positive*

lifting-update *rat.lifting*
lifting-forget *rat.lifting*

end

98 Development of the Reals using Cauchy Sequences

theory *Real*
imports *Rat*
begin

This theory contains a formalization of the real numbers as equivalence classes of Cauchy sequences of rationals. See the AFP entry *Dedekind-Real* for an alternative construction using Dedekind cuts.

98.1 Preliminary lemmas

Useful in convergence arguments

lemma *inverse-of-nat-le*:
fixes *n*::*nat* **shows** ⟦*n* ≤ *m*; *n*≠*0*⟧ ⟹ *1* / *of-nat m* ≤ (*1*::*'a*::*linordered-field*) / *of-nat n*
⟨*proof*⟩

lemma *add-diff-add*: (*a* + *c*) − (*b* + *d*) = (*a* − *b*) + (*c* − *d*)
for *a b c d* :: *'a*::*ab-group-add*
⟨*proof*⟩

lemma *minus-diff-minus*: $- a - - b = - (a - b)$
for $a\ b :: 'a::ab\text{-}group\text{-}add$
⟨*proof*⟩

lemma *mult-diff-mult*: $(x * y - a * b) = x * (y - b) + (x - a) * b$
for $x\ y\ a\ b :: 'a::ring$
⟨*proof*⟩

lemma *inverse-diff-inverse*:
fixes $a\ b :: 'a::division\text{-}ring$
assumes $a \neq 0$ **and** $b \neq 0$
shows $inverse\ a - inverse\ b = - (inverse\ a * (a - b) * inverse\ b)$
⟨*proof*⟩

lemma *obtain-pos-sum*:
fixes $r :: rat$ **assumes** r: $0 < r$
obtains $s\ t$ **where** $0 < s$ **and** $0 < t$ **and** $r = s + t$
⟨*proof*⟩

98.2 Sequences that converge to zero

definition *vanishes* :: $(nat \Rightarrow rat) \Rightarrow bool$
where $vanishes\ X \longleftrightarrow (\forall r{>}0.\ \exists k.\ \forall n{\geq}k.\ |X\ n| < r)$

lemma *vanishesI*: $(\bigwedge r.\ 0 < r \Longrightarrow \exists k.\ \forall n{\geq}k.\ |X\ n| < r) \Longrightarrow vanishes\ X$
⟨*proof*⟩

lemma *vanishesD*: $vanishes\ X \Longrightarrow 0 < r \Longrightarrow \exists k.\ \forall n{\geq}k.\ |X\ n| < r$
⟨*proof*⟩

lemma *vanishes-const* [*simp*]: $vanishes\ (\lambda n.\ c) \longleftrightarrow c = 0$
⟨*proof*⟩

lemma *vanishes-minus*: $vanishes\ X \Longrightarrow vanishes\ (\lambda n.\ -\ X\ n)$
⟨*proof*⟩

lemma *vanishes-add*:
assumes X: $vanishes\ X$
and Y: $vanishes\ Y$
shows $vanishes\ (\lambda n.\ X\ n + Y\ n)$
⟨*proof*⟩

lemma *vanishes-diff*:
assumes $vanishes\ X\ vanishes\ Y$
shows $vanishes\ (\lambda n.\ X\ n - Y\ n)$
⟨*proof*⟩

lemma *vanishes-mult-bounded*:
assumes X: $\exists a{>}0.\ \forall n.\ |X\ n| < a$

assumes Y: *vanishes* $(\lambda n.\ Y\ n)$
shows *vanishes* $(\lambda n.\ X\ n * Y\ n)$
⟨*proof*⟩

98.3 Cauchy sequences

definition *cauchy* :: $(nat \Rightarrow rat) \Rightarrow bool$
where *cauchy* $X \longleftrightarrow (\forall r{>}0.\ \exists k.\ \forall m{\geq}k.\ \forall n{\geq}k.\ |X\ m - X\ n| < r)$

lemma *cauchyI*: $(\bigwedge r.\ 0 < r \Longrightarrow \exists k.\ \forall m{\geq}k.\ \forall n{\geq}k.\ |X\ m - X\ n| < r) \Longrightarrow$ *cauchy* X
⟨*proof*⟩

lemma *cauchyD*: *cauchy* $X \Longrightarrow 0 < r \Longrightarrow \exists k.\ \forall m{\geq}k.\ \forall n{\geq}k.\ |X\ m - X\ n| < r$
⟨*proof*⟩

lemma *cauchy-const* [*simp*]: *cauchy* $(\lambda n.\ x)$
⟨*proof*⟩

lemma *cauchy-add* [*simp*]:
assumes X: *cauchy* X **and** Y: *cauchy* Y
shows *cauchy* $(\lambda n.\ X\ n + Y\ n)$
⟨*proof*⟩

lemma *cauchy-minus* [*simp*]:
assumes X: *cauchy* X
shows *cauchy* $(\lambda n.\ -\ X\ n)$
⟨*proof*⟩

lemma *cauchy-diff* [*simp*]:
assumes *cauchy* X *cauchy* Y
shows *cauchy* $(\lambda n.\ X\ n - Y\ n)$
⟨*proof*⟩

lemma *cauchy-imp-bounded*:
assumes *cauchy* X
shows $\exists b{>}0.\ \forall n.\ |X\ n| < b$
⟨*proof*⟩

lemma *cauchy-mult* [*simp*]:
assumes X: *cauchy* X **and** Y: *cauchy* Y
shows *cauchy* $(\lambda n.\ X\ n * Y\ n)$
⟨*proof*⟩

lemma *cauchy-not-vanishes-cases*:
assumes X: *cauchy* X
assumes nz: $\neg$ *vanishes* X
shows $\exists b{>}0.\ \exists k.\ (\forall n{\geq}k.\ b < -\ X\ n) \vee (\forall n{\geq}k.\ b < X\ n)$
⟨*proof*⟩

lemma *cauchy-not-vanishes*:
assumes X: *cauchy* X
and *nz*: $\neg$ *vanishes* X
shows $\exists\, b{>}0.\ \exists\, k.\ \forall\, n{\geq}k.\ b < |X\ n|$
⟨*proof*⟩

lemma *cauchy-inverse* [*simp*]:
assumes X: *cauchy* X
and *nz*: $\neg$ *vanishes* X
shows *cauchy* $(\lambda n.\ \mathit{inverse}\ (X\ n))$
⟨*proof*⟩

lemma *vanishes-diff-inverse*:
assumes X: *cauchy* X $\neg$ *vanishes* X
and Y: *cauchy* Y $\neg$ *vanishes* Y
and XY: *vanishes* $(\lambda n.\ X\ n - Y\ n)$
shows *vanishes* $(\lambda n.\ \mathit{inverse}\ (X\ n) - \mathit{inverse}\ (Y\ n))$
⟨*proof*⟩

98.4 Equivalence relation on Cauchy sequences

definition *realrel* :: $(nat \Rightarrow rat) \Rightarrow (nat \Rightarrow rat) \Rightarrow bool$
where *realrel* $= (\lambda X\ Y.\ \mathit{cauchy}\ X \wedge \mathit{cauchy}\ Y \wedge \mathit{vanishes}\ (\lambda n.\ X\ n - Y\ n))$

lemma *realrelI* [*intro?*]: *cauchy* $X \Longrightarrow$ *cauchy* $Y \Longrightarrow$ *vanishes* $(\lambda n.\ X\ n - Y\ n)$
$\Longrightarrow$ *realrel* $X\ Y$
⟨*proof*⟩

lemma *realrel-refl*: *cauchy* $X \Longrightarrow$ *realrel* $X\ X$
⟨*proof*⟩

lemma *symp-realrel*: *symp realrel*
⟨*proof*⟩

lemma *transp-realrel*: *transp realrel*
⟨*proof*⟩

lemma *part-equivp-realrel*: *part-equivp realrel*
⟨*proof*⟩

98.5 The field of real numbers

quotient-type *real* $=$ *nat* $\Rightarrow$ *rat* / *partial*: *realrel*
morphisms *rep-real Real*
⟨*proof*⟩

lemma *cr-real-eq*: *pcr-real* $= (\lambda x\ y.\ \mathit{cauchy}\ x \wedge \mathit{Real}\ x = y)$
⟨*proof*⟩

lemma *Real-induct* [*induct type*: *real*]:
  assumes $\bigwedge X.$ *cauchy* $X \Longrightarrow P$ (*Real* X)
  shows $P\ x$
⟨*proof*⟩

lemma *eq-Real*: *cauchy* $X \Longrightarrow$ *cauchy* $Y \Longrightarrow$ *Real* $X =$ *Real* $Y \longleftrightarrow$ *vanishes* ($\lambda n.$ $X\ n - Y\ n$)
  ⟨*proof*⟩

lemma *Domainp-pcr-real* [*transfer-domain-rule*]: *Domainp pcr-real* = *cauchy*
  ⟨*proof*⟩

instantiation *real* :: *field*
begin

lift-definition *zero-real* :: *real* **is** $\lambda n.$ *0*
  ⟨*proof*⟩

lift-definition *one-real* :: *real* **is** $\lambda n.$ *1*
  ⟨*proof*⟩

lift-definition *plus-real* :: *real* ⇒ *real* ⇒ *real* **is** $\lambda X\ Y\ n.\ X\ n + Y\ n$
  ⟨*proof*⟩

lift-definition *uminus-real* :: *real* ⇒ *real* **is** $\lambda X\ n.\ -\ X\ n$
  ⟨*proof*⟩

lift-definition *times-real* :: *real* ⇒ *real* ⇒ *real* **is** $\lambda X\ Y\ n.\ X\ n * Y\ n$
⟨*proof*⟩

lift-definition *inverse-real* :: *real* ⇒ *real*
  is $\lambda X.$ *if vanishes* X *then* ($\lambda n.$ *0*) *else* ($\lambda n.$ *inverse* ($X\ n$))
⟨*proof*⟩

definition $x - y = x + - y$ **for** $x\ y$ :: *real*

definition x *div* $y = x *$ *inverse* y **for** $x\ y$ :: *real*

lemma *add-Real*: *cauchy* $X \Longrightarrow$ *cauchy* $Y \Longrightarrow$ *Real* $X +$ *Real* $Y =$ *Real* ($\lambda n.$ $X\ n + Y\ n$)
  ⟨*proof*⟩

lemma *minus-Real*: *cauchy* $X \Longrightarrow -$ *Real* $X =$ *Real* ($\lambda n.$ $-\ X\ n$)
  ⟨*proof*⟩

lemma *diff-Real*: *cauchy* $X \Longrightarrow$ *cauchy* $Y \Longrightarrow$ *Real* $X -$ *Real* $Y =$ *Real* ($\lambda n.$ $X\ n - Y\ n$)
  ⟨*proof*⟩

lemma *mult-Real*: *cauchy X* $\Longrightarrow$ *cauchy Y* $\Longrightarrow$ *Real X* $*$ *Real Y* $=$ *Real* ($\lambda n.$ *X n* $*$ *Y n*)
⟨*proof*⟩

lemma *inverse-Real*:
cauchy X $\Longrightarrow$ *inverse* (*Real X*) $=$ (*if vanishes X then 0 else Real* ($\lambda n.$ *inverse* (*X n*)))
⟨*proof*⟩

instance
⟨*proof*⟩

end

98.6 Positive reals

lift-definition *positive* :: *real* $\Rightarrow$ *bool*
is $\lambda X.\ \exists\, r{>}0.\ \exists\, k.\ \forall\, n{\geq}k.\ r < X\ n$
⟨*proof*⟩

lemma *positive-Real*: *cauchy X* $\Longrightarrow$ *positive* (*Real X*) $\longleftrightarrow$ ($\exists\, r{>}0.\ \exists\, k.\ \forall\, n{\geq}k.\ r < X\ n$)
⟨*proof*⟩

lemma *positive-zero*: $\neg$ *positive 0*
⟨*proof*⟩

lemma *positive-add*:
assumes *positive x positive y* **shows** *positive* (*x* $+$ *y*)
⟨*proof*⟩

lemma *positive-mult*:
assumes *positive x positive y* **shows** *positive* (*x* $*$ *y*)
⟨*proof*⟩

lemma *positive-minus*: $\neg$ *positive x* $\Longrightarrow$ *x* $\neq$ *0* $\Longrightarrow$ *positive* ($-$ *x*)
⟨*proof*⟩

instantiation *real* :: *linordered-field*
begin

definition *x* $<$ *y* $\longleftrightarrow$ *positive* (*y* $-$ *x*)

definition *x* $\leq$ *y* $\longleftrightarrow$ *x* $<$ *y* $\vee$ *x* $=$ *y* **for** *x y* :: *real*

definition $|a|$ $=$ (*if a* $<$ *0 then* $-$ *a else a*) **for** *a* :: *real*

definition *sgn a* $=$ (*if a* $=$ *0 then 0 else if 0* $<$ *a then 1 else* $-$ *1*) **for** *a* :: *real*

instance
⟨*proof*⟩

end

instantiation *real* :: *distrib-lattice*
begin

definition (*inf* :: *real* ⇒ *real* ⇒ *real*) = *min*

definition (*sup* :: *real* ⇒ *real* ⇒ *real*) = *max*

instance
⟨*proof*⟩

end

lemma *of-nat-Real*: *of-nat x* = *Real* (λ*n*. *of-nat x*)
⟨*proof*⟩

lemma *of-int-Real*: *of-int x* = *Real* (λ*n*. *of-int x*)
⟨*proof*⟩

lemma *of-rat-Real*: *of-rat x* = *Real* (λ*n*. *x*)
⟨*proof*⟩

instance *real* :: *archimedean-field*
⟨*proof*⟩

instantiation *real* :: *floor-ceiling*
begin

definition [*code del*]: ⌊*x*::*real*⌋ = (*THE z*. *of-int z* ≤ *x* ∧ *x* < *of-int* (*z* + *1*))

instance
⟨*proof*⟩

end

98.7 Completeness

lemma *not-positive-Real*:
assumes *cauchy X* **shows** ¬ *positive* (*Real X*) ⟷ (∀ *r*>*0*. ∃ *k*. ∀ *n*≥*k*. *X n* ≤ *r*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *le-Real*:
assumes *cauchy X cauchy Y*
shows *Real X* ≤ *Real Y* = (∀ *r*>*0*. ∃ *k*. ∀ *n*≥*k*. *X n* ≤ *Y n* + *r*)

⟨*proof*⟩

lemma *le-RealI*:
assumes *Y*: *cauchy Y*
shows ∀ *n*. *x* ≤ *of-rat* (*Y n*) ⟹ *x* ≤ *Real Y*
⟨*proof*⟩

lemma *Real-leI*:
assumes *X*: *cauchy X*
assumes *le*: ∀ *n*. *of-rat* (*X n*) ≤ *y*
shows *Real X* ≤ *y*
⟨*proof*⟩

lemma *less-RealD*:
assumes *cauchy Y*
shows *x* < *Real Y* ⟹ ∃ *n*. *x* < *of-rat* (*Y n*)
⟨*proof*⟩

lemma *of-nat-less-two-power* [*simp*]: *of-nat n* < (*2*::*'a*::*linordered-idom*) ^ *n*
⟨*proof*⟩

lemma *complete-real*:
fixes *S* :: *real set*
assumes ∃ *x*. *x* ∈ *S* **and** ∃ *z*. ∀ *x*∈*S*. *x* ≤ *z*
shows ∃ *y*. (∀ *x*∈*S*. *x* ≤ *y*) ∧ (∀ *z*. (∀ *x*∈*S*. *x* ≤ *z*) ⟶ *y* ≤ *z*)
⟨*proof*⟩

instantiation *real* :: *linear-continuum*
begin

98.8 Supremum of a set of reals

definition *Sup X* = (*LEAST z*::*real*. ∀ *x*∈*X*. *x* ≤ *z*)
definition *Inf X* = − *Sup* (*uminus* ‘ *X*) **for** *X* :: *real set*

instance
⟨*proof*⟩

end

98.9 Hiding implementation details

hide-const (**open**) *vanishes cauchy positive Real*

declare *Real-induct* [*induct del*]
declare *Abs-real-induct* [*induct del*]
declare *Abs-real-cases* [*cases del*]

lifting-update *real.lifting*
lifting-forget *real.lifting*

98.10 Embedding numbers into the Reals

abbreviation *real-of-nat* :: *nat* ⇒ *real*
  where *real-of-nat* ≡ *of-nat*

abbreviation *real* :: *nat* ⇒ *real*
  where *real* ≡ *of-nat*

abbreviation *real-of-int* :: *int* ⇒ *real*
  where *real-of-int* ≡ *of-int*

abbreviation *real-of-rat* :: *rat* ⇒ *real*
  where *real-of-rat* ≡ *of-rat*

declare [[*coercion-enabled*]]

declare [[*coercion of-nat* :: *nat* ⇒ *int*]]
declare [[*coercion of-nat* :: *nat* ⇒ *real*]]
declare [[*coercion of-int* :: *int* ⇒ *real*]]

declare [[*coercion-map map*]]
declare [[*coercion-map* λ*f g h x*. *g* (*h* (*f x*))]]
declare [[*coercion-map* λ*f g* (*x*,*y*). (*f x*, *g y*)]]

declare *of-int-eq-0-iff* [*algebra*, *presburger*]
declare *of-int-eq-1-iff* [*algebra*, *presburger*]
declare *of-int-eq-iff* [*algebra*, *presburger*]
declare *of-int-less-0-iff* [*algebra*, *presburger*]
declare *of-int-less-1-iff* [*algebra*, *presburger*]
declare *of-int-less-iff* [*algebra*, *presburger*]
declare *of-int-le-0-iff* [*algebra*, *presburger*]
declare *of-int-le-1-iff* [*algebra*, *presburger*]
declare *of-int-le-iff* [*algebra*, *presburger*]
declare *of-int-0-less-iff* [*algebra*, *presburger*]
declare *of-int-0-le-iff* [*algebra*, *presburger*]
declare *of-int-1-less-iff* [*algebra*, *presburger*]
declare *of-int-1-le-iff* [*algebra*, *presburger*]

lemma *int-less-real-le*: *n* < *m* ⟷ *real-of-int n* + *1* ≤ *real-of-int m*
⟨*proof*⟩

lemma *int-le-real-less*: *n* ≤ *m* ⟷ *real-of-int n* < *real-of-int m* + *1*
  ⟨*proof*⟩

lemma *real-of-int-div-aux*:
  (*real-of-int x*) / (*real-of-int d*) =
    real-of-int (*x div d*) + (*real-of-int* (*x mod d*)) / (*real-of-int d*)
⟨*proof*⟩

lemma *real-of-int-div*:
d dvd n $\Longrightarrow$ *real-of-int* (*n div d*) = *real-of-int n* / *real-of-int d* **for** *d n* :: *int*
⟨*proof*⟩

lemma *real-of-int-div2*: *0* ≤ *real-of-int n* / *real-of-int x* − *real-of-int* (*n div x*)
⟨*proof*⟩

lemma *real-of-int-div3*: *real-of-int n* / *real-of-int x* − *real-of-int* (*n div x*) ≤ *1*
⟨*proof*⟩

lemma *real-of-int-div4*: *real-of-int* (*n div x*) ≤ *real-of-int n* / *real-of-int x*
⟨*proof*⟩

98.11 Embedding the Naturals into the Reals

lemma *real-of-card*: *real* (*card A*) = *sum* (λ*x*. *1*) *A*
⟨*proof*⟩

lemma *nat-less-real-le*: *n* < *m* ⟷ *real n* + *1* ≤ *real m*
⟨*proof*⟩

lemma *nat-le-real-less*: *n* ≤ *m* ⟷ *real n* < *real m* + *1*
for *m n* :: *nat*
⟨*proof*⟩

lemma *real-of-nat-div-aux*: *real x* / *real d* = *real* (*x div d*) + *real* (*x mod d*) / *real d*
⟨*proof*⟩

lemma *real-of-nat-div*: *d dvd n* $\Longrightarrow$ *real*(*n div d*) = *real n* / *real d*
⟨*proof*⟩

lemma *real-of-nat-div2*: *0* ≤ *real n* / *real x* − *real* (*n div x*) **for** *n x* :: *nat*
⟨*proof*⟩

lemma *real-of-nat-div3*: *real n* / *real x* − *real* (*n div x*) ≤ *1* **for** *n x* :: *nat*
⟨*proof*⟩

lemma *real-of-nat-div4*: *real* (*n div x*) ≤ *real n* / *real x* **for** *n x* :: *nat*
⟨*proof*⟩

lemma *real-binomial-eq-mult-binomial-Suc*:
assumes *k* ≤ *n*
shows *real*(*n choose k*) = (*n* + *1* − *k*) / (*n* + *1*) ∗ (*Suc n choose k*)
⟨*proof*⟩

98.12 The Archimedean Property of the Reals

lemma *real-arch-inverse*: $0 < e \longleftrightarrow (\exists\, n{::}nat.\ n \neq 0 \wedge 0 < inverse\ (real\ n) \wedge inverse\ (real\ n) < e)$
⟨*proof*⟩

lemma *reals-Archimedean3*: $0 < x \Longrightarrow \forall\, y.\ \exists\, n.\ y < real\ n * x$
⟨*proof*⟩

lemma *real-archimedian-rdiv-eq-0*:
assumes *x0*: $x \geq 0$
and *c*: $c \geq 0$
and *xc*: $\bigwedge m{::}nat.\ m > 0 \Longrightarrow real\ m * x \leq c$
shows $x = 0$
⟨*proof*⟩

lemma *inverse-Suc*: $inverse\ (Suc\ n) > 0$
⟨*proof*⟩

lemma *Archimedean-eventually-inverse*:
fixes $\varepsilon{::}real$ **shows** $(\forall_F\ n\ in\ sequentially.\ inverse\ (real\ (Suc\ n)) < \varepsilon) \longleftrightarrow 0 < \varepsilon$
(**is** *?lhs=?rhs*)
⟨*proof*⟩

On the relationship between two different ways of converting to 0

lemma *Inter-eq-Inter-inverse-Suc*:
assumes $\bigwedge r'\ r.\ r' < r \Longrightarrow A\ r' \subseteq A\ r$
shows $\bigcap\ (A\ `\ \{0<..\}) = (\bigcap n.\ A(inverse(Suc\ n)))$
⟨*proof*⟩

98.13 Rationals

lemma *Rats-abs-iff*[*simp*]:
$|(x{::}real)| \in \mathbb{Q} \longleftrightarrow x \in \mathbb{Q}$
⟨*proof*⟩

lemma *Rats-eq-int-div-int*: $\mathbb{Q} = \{real\text{-}of\text{-}int\ i\ /\ real\text{-}of\text{-}int\ j \mid i\ j.\ j \neq 0\}$ (**is** - = *?S*)
⟨*proof*⟩

lemma *Rats-eq-int-div-nat*: $\mathbb{Q} = \{\ real\text{-}of\text{-}int\ i\ /\ real\ n \mid i\ n.\ n \neq 0\}$
⟨*proof*⟩

lemma *Rats-abs-nat-div-natE*:
assumes $x \in \mathbb{Q}$
obtains $m\ n :: nat$ **where** $n \neq 0$ **and** $|x| = real\ m\ /\ real\ n$ **and** $coprime\ m\ n$
⟨*proof*⟩

98.14 Density of the Rational Reals in the Reals

This density proof is due to Stefan Richter and was ported by TN. The original source is *Real Analysis* by H.L. Royden. It employs the Archimedean property of the reals.

lemma *Rats-dense-in-real*:
fixes x :: *real*
assumes $x < y$
shows $\exists\, r{\in}\mathbb{Q}.\ x < r \wedge r < y$
⟨*proof*⟩

lemma *of-rat-dense*:
fixes $x\ y$ :: *real*
assumes $x < y$
shows $\exists\, q$ :: *rat*. $x <$ *of-rat* $q \wedge$ *of-rat* $q < y$
⟨*proof*⟩

98.15 Numerals and Arithmetic

⟨*ML*⟩

98.16 Simprules combining $x + y$ and 0

lemma *real-add-minus-iff* [*simp*]: $x + - a = 0 \longleftrightarrow x = a$
for $x\ a$ :: *real*
⟨*proof*⟩

lemma *real-add-less-0-iff*: $x + y < 0 \longleftrightarrow y < - x$
for $x\ y$ :: *real*
⟨*proof*⟩

lemma *real-0-less-add-iff*: $0 < x + y \longleftrightarrow - x < y$
for $x\ y$ :: *real*
⟨*proof*⟩

lemma *real-add-le-0-iff*: $x + y \leq 0 \longleftrightarrow y \leq - x$
for $x\ y$ :: *real*
⟨*proof*⟩

lemma *real-0-le-add-iff*: $0 \leq x + y \longleftrightarrow - x \leq y$
for $x\ y$ :: *real*
⟨*proof*⟩

98.17 Lemmas about powers

lemma *two-realpow-ge-one*: $(1{::}real) \leq 2\ \hat{}\ n$
⟨*proof*⟩

declare *sum-squares-eq-zero-iff* [*simp*] *sum-power2-eq-zero-iff* [*simp*]

lemma *real-minus-mult-self-le* [*simp*]: $-(u * u) \leq x * x$
for *u x* :: *real*
⟨*proof*⟩

lemma *realpow-square-minus-le* [*simp*]: $-u^2 \leq x^2$
for *u x* :: *real*
⟨*proof*⟩

98.18 Density of the Reals

lemma *field-lbound-gt-zero*: $0 < d1 \Longrightarrow 0 < d2 \Longrightarrow \exists e.\ 0 < e \wedge e < d1 \wedge e < d2$
for *d1 d2* :: *'a*::*linordered-field*
⟨*proof*⟩

lemma *field-less-half-sum*: $x < y \Longrightarrow x < (x + y) / 2$
for *x y* :: *'a*::*linordered-field*
⟨*proof*⟩

lemma *field-sum-of-halves*: $x / 2 + x / 2 = x$
for *x* :: *'a*::*linordered-field*
⟨*proof*⟩

98.19 Archimedean properties and useful consequences

Bernoulli's inequality

proposition *Bernoulli-inequality*:
fixes *x* :: *real*
assumes $-1 \leq x$
shows $1 + n * x \leq (1 + x) \hat{} n$
⟨*proof*⟩

corollary *Bernoulli-inequality-even*:
fixes *x* :: *real*
assumes *even n*
shows $1 + n * x \leq (1 + x) \hat{} n$
⟨*proof*⟩

corollary *real-arch-pow*:
fixes *x* :: *real*
assumes *x*: $1 < x$
shows $\exists n.\ y < x \hat{} n$
⟨*proof*⟩

corollary *real-arch-pow-inv*:
fixes *x y* :: *real*
assumes *y*: $y > 0$

and *x1*: $x < 1$
shows $\exists\, n.\ x \hat{} n < y$
⟨*proof*⟩

lemma *forall-pos-mono*:
$(\bigwedge d\ e{::}real.\ d < e \Longrightarrow P\ d \Longrightarrow P\ e) \Longrightarrow$
$(\bigwedge n{::}nat.\ n \neq 0 \Longrightarrow P\ (inverse\ (real\ n))) \Longrightarrow (\bigwedge e.\ 0 < e \Longrightarrow P\ e)$
⟨*proof*⟩

lemma *forall-pos-mono-1*:
$(\bigwedge d\ e{::}real.\ d < e \Longrightarrow P\ d \Longrightarrow P\ e) \Longrightarrow$
$(\bigwedge n.\ P\ (inverse\ (real\ (Suc\ n)))) \Longrightarrow 0 < e \Longrightarrow P\ e$
⟨*proof*⟩

lemma *Archimedean-eventually-pow*:
fixes $x{::}real$
assumes $1 < x$
shows $\forall_F\ n\ in\ sequentially.\ b < x \hat{}\ n$
⟨*proof*⟩

lemma *Archimedean-eventually-pow-inverse*:
fixes $x{::}real$
assumes $|x| < 1\ \varepsilon > 0$
shows $\forall_F\ n\ in\ sequentially.\ |x \hat{} n| < \varepsilon$
⟨*proof*⟩

98.20 Floor and Ceiling Functions from the Reals to the Integers

lemma *real-of-nat-less-numeral-iff* [*simp*]: $real\ n < numeral\ w \longleftrightarrow n < numeral\ w$
for $n :: nat$
⟨*proof*⟩

lemma *numeral-less-real-of-nat-iff* [*simp*]: $numeral\ w < real\ n \longleftrightarrow numeral\ w < n$
for $n :: nat$
⟨*proof*⟩

lemma *numeral-le-real-of-nat-iff* [*simp*]: $numeral\ n \leq real\ m \longleftrightarrow numeral\ n \leq m$
for $m :: nat$
⟨*proof*⟩

lemma *of-int-floor-cancel* [*simp*]: $of\text{-}int\ \lfloor x \rfloor = x \longleftrightarrow (\exists\, n{::}int.\ x = of\text{-}int\ n)$
⟨*proof*⟩

lemma *of-int-floor* [*simp*]: $a \in \mathbb{Z} \Longrightarrow of\text{-}int\ (floor\ a) = a$
⟨*proof*⟩

lemma *floor-frac* [*simp*]: $\lfloor frac\ r \rfloor = 0$
⟨*proof*⟩

lemma *frac-1* [*simp*]: $frac\ 1 = 0$
⟨*proof*⟩

lemma *frac-in-Rats-iff* [*simp*]:
fixes $r::'a::\{floor\text{-}ceiling, field\text{-}char\text{-}0\}$
shows $frac\ r \in \mathbb{Q} \longleftrightarrow r \in \mathbb{Q}$
⟨*proof*⟩

lemma *floor-eq*: $real\text{-}of\text{-}int\ n < x \Longrightarrow x < real\text{-}of\text{-}int\ n + 1 \Longrightarrow \lfloor x \rfloor = n$
⟨*proof*⟩

lemma *floor-eq2*: $real\text{-}of\text{-}int\ n \leq x \Longrightarrow x < real\text{-}of\text{-}int\ n + 1 \Longrightarrow \lfloor x \rfloor = n$
⟨*proof*⟩

lemma *floor-eq3*: $real\ n < x \Longrightarrow x < real\ (Suc\ n) \Longrightarrow nat\ \lfloor x \rfloor = n$
⟨*proof*⟩

lemma *floor-eq4*: $real\ n \leq x \Longrightarrow x < real\ (Suc\ n) \Longrightarrow nat\ \lfloor x \rfloor = n$
⟨*proof*⟩

lemma *real-of-int-floor-ge-diff-one* [*simp*]: $r - 1 \leq real\text{-}of\text{-}int\ \lfloor r \rfloor$
⟨*proof*⟩

lemma *real-of-int-floor-gt-diff-one* [*simp*]: $r - 1 < real\text{-}of\text{-}int\ \lfloor r \rfloor$
⟨*proof*⟩

lemma *real-of-int-floor-add-one-ge* [*simp*]: $r \leq real\text{-}of\text{-}int\ \lfloor r \rfloor + 1$
⟨*proof*⟩

lemma *real-of-int-floor-add-one-gt* [*simp*]: $r < real\text{-}of\text{-}int\ \lfloor r \rfloor + 1$
⟨*proof*⟩

lemma *floor-divide-real-eq-div*:
assumes $0 \leq b$
shows $\lfloor a\ /\ real\text{-}of\text{-}int\ b \rfloor = \lfloor a \rfloor\ div\ b$
⟨*proof*⟩

lemma *floor-one-divide-eq-div-numeral* [*simp*]:
$\lfloor 1\ /\ numeral\ b::real \rfloor = 1\ div\ numeral\ b$
⟨*proof*⟩

lemma *floor-minus-one-divide-eq-div-numeral* [*simp*]:
$\lfloor -\ (1\ /\ numeral\ b)::real \rfloor = -\ 1\ div\ numeral\ b$
⟨*proof*⟩

lemma *floor-divide-eq-div-numeral* [*simp*]:

$\lfloor numeral\ a\ /\ numeral\ b{::}real \rfloor = numeral\ a\ div\ numeral\ b$
⟨*proof*⟩

lemma *floor-minus-divide-eq-div-numeral* [*simp*]:
$\lfloor - (numeral\ a\ /\ numeral\ b){::}real \rfloor = -\ numeral\ a\ div\ numeral\ b$
⟨*proof*⟩

lemma *of-int-ceiling-cancel* [*simp*]: $of\text{-}int\ \lceil x \rceil = x \longleftrightarrow (\exists\, n{::}int.\ x = of\text{-}int\ n)$
⟨*proof*⟩

lemma *of-int-ceiling* [*simp*]: $a \in \mathbb{Z} \Longrightarrow of\text{-}int\ (ceiling\ a) = a$
⟨*proof*⟩

lemma *ceiling-eq*: $of\text{-}int\ n < x \Longrightarrow x \leq of\text{-}int\ n + 1 \Longrightarrow \lceil x \rceil = n + 1$
⟨*proof*⟩

lemma *of-int-ceiling-diff-one-le* [*simp*]: $of\text{-}int\ \lceil r \rceil - 1 \leq r$
⟨*proof*⟩

lemma *of-int-ceiling-le-add-one* [*simp*]: $of\text{-}int\ \lceil r \rceil \leq r + 1$
⟨*proof*⟩

lemma *ceiling-le*: $x \leq of\text{-}int\ a \Longrightarrow \lceil x \rceil \leq a$
⟨*proof*⟩

lemma *ceiling-divide-eq-div*: $\lceil of\text{-}int\ a\ /\ of\text{-}int\ b \rceil = -\ (-\ a\ div\ b)$
⟨*proof*⟩

lemma *ceiling-divide-eq-div-numeral* [*simp*]:
$\lceil numeral\ a\ /\ numeral\ b :: real \rceil = -\ (-\ numeral\ a\ div\ numeral\ b)$
⟨*proof*⟩

lemma *ceiling-minus-divide-eq-div-numeral* [*simp*]:
$\lceil -\ (numeral\ a\ /\ numeral\ b :: real) \rceil = -\ (numeral\ a\ div\ numeral\ b)$
⟨*proof*⟩

The following lemmas are remnants of the erstwhile functions natfloor and natceiling.

lemma *nat-floor-neg*: $x \leq 0 \Longrightarrow nat\ \lfloor x \rfloor = 0$
for $x :: real$
⟨*proof*⟩

lemma *le-nat-floor*: $real\ x \leq a \Longrightarrow x \leq nat\ \lfloor a \rfloor$
⟨*proof*⟩

lemma *le-mult-nat-floor*: $nat\ \lfloor a \rfloor * nat\ \lfloor b \rfloor \leq nat\ \lfloor a * b \rfloor$
⟨*proof*⟩

lemma *nat-ceiling-le-eq* [*simp*]: $nat\ \lceil x \rceil \leq a \longleftrightarrow x \leq real\ a$

⟨*proof*⟩

lemma *real-nat-ceiling-ge*: $x \leq \mathit{real}\ (\mathit{nat}\ \lceil x \rceil)$
⟨*proof*⟩

lemma *Rats-no-top-le*: $\exists\, q \in \mathbb{Q}.\ x \leq q$
for $x :: \mathit{real}$
⟨*proof*⟩

lemma *Rats-no-bot-less*: $\exists\, q \in \mathbb{Q}.\ q < x$ **for** $x :: \mathit{real}$
⟨*proof*⟩

98.21 Exponentiation with floor

lemma *floor-power*:
assumes $x = \mathit{of\text{-}int}\ \lfloor x \rfloor$
shows $\lfloor x \mathbin{\hat{}} n \rfloor = \lfloor x \rfloor \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *floor-numeral-power* [*simp*]: $\lfloor \mathit{numeral}\ x \mathbin{\hat{}} n \rfloor = \mathit{numeral}\ x \mathbin{\hat{}} n$
⟨*proof*⟩

lemma *ceiling-numeral-power* [*simp*]: $\lceil \mathit{numeral}\ x \mathbin{\hat{}} n \rceil = \mathit{numeral}\ x \mathbin{\hat{}} n$
⟨*proof*⟩

98.22 Implementation of rational real numbers

Formal constructor

definition *Ratreal* :: *rat* ⇒ *real*
where [*code-abbrev*, *simp*]: *Ratreal* = *real-of-rat*

code-datatype *Ratreal*

Quasi-Numerals

lemma [*code-abbrev*]:
$\mathit{real\text{-}of\text{-}rat}\ (\mathit{numeral}\ k) = \mathit{numeral}\ k$
$\mathit{real\text{-}of\text{-}rat}\ (-\ \mathit{numeral}\ k) = -\ \mathit{numeral}\ k$
$\mathit{real\text{-}of\text{-}rat}\ (\mathit{rat\text{-}of\text{-}int}\ a) = \mathit{real\text{-}of\text{-}int}\ a$
⟨*proof*⟩

lemma [*code-post*]:
$\mathit{real\text{-}of\text{-}rat}\ 0 = 0$
$\mathit{real\text{-}of\text{-}rat}\ 1 = 1$
$\mathit{real\text{-}of\text{-}rat}\ (-\ 1) = -\ 1$
$\mathit{real\text{-}of\text{-}rat}\ (1\ /\ \mathit{numeral}\ k) = 1\ /\ \mathit{numeral}\ k$
$\mathit{real\text{-}of\text{-}rat}\ (\mathit{numeral}\ k\ /\ \mathit{numeral}\ l) = \mathit{numeral}\ k\ /\ \mathit{numeral}\ l$
$\mathit{real\text{-}of\text{-}rat}\ (-\ (1\ /\ \mathit{numeral}\ k)) = -\ (1\ /\ \mathit{numeral}\ k)$
$\mathit{real\text{-}of\text{-}rat}\ (-\ (\mathit{numeral}\ k\ /\ \mathit{numeral}\ l)) = -\ (\mathit{numeral}\ k\ /\ \mathit{numeral}\ l)$
⟨*proof*⟩

Operations

lemma *zero-real-code* [*code*]: *0 = Ratreal 0*
⟨*proof*⟩

lemma *one-real-code* [*code*]: *1 = Ratreal 1*
⟨*proof*⟩

instantiation *real* :: *equal*
begin

definition *HOL.equal x y* ⟷ *x − y = 0* **for** *x* :: *real*

instance ⟨*proof*⟩

lemma *real-equal-code* [*code*]: *HOL.equal* (*Ratreal x*) (*Ratreal y*) ⟷ *HOL.equal x y*
⟨*proof*⟩

lemma [*code nbe*]: *HOL.equal x x* ⟷ *True*
for *x* :: *real*
⟨*proof*⟩

end

lemma *real-less-eq-code* [*code*]: *Ratreal x* ≤ *Ratreal y* ⟷ *x* ≤ *y*
⟨*proof*⟩

lemma *real-less-code* [*code*]: *Ratreal x* < *Ratreal y* ⟷ *x* < *y*
⟨*proof*⟩

lemma *real-plus-code* [*code*]: *Ratreal x + Ratreal y = Ratreal* (*x + y*)
⟨*proof*⟩

lemma *real-times-code* [*code*]: *Ratreal x ∗ Ratreal y = Ratreal* (*x ∗ y*)
⟨*proof*⟩

lemma *real-uminus-code* [*code*]: *− Ratreal x = Ratreal* (*− x*)
⟨*proof*⟩

lemma *real-minus-code* [*code*]: *Ratreal x − Ratreal y = Ratreal* (*x − y*)
⟨*proof*⟩

lemma *real-inverse-code* [*code*]: *inverse* (*Ratreal x*) = *Ratreal* (*inverse x*)
⟨*proof*⟩

lemma *real-divide-code* [*code*]: *Ratreal x / Ratreal y = Ratreal* (*x / y*)
⟨*proof*⟩

lemma *real-floor-code* [*code*]: ⌊*Ratreal x*⌋ = ⌊*x*⌋

⟨*proof*⟩

Quickcheck

context
includes *term-syntax*
begin

definition
valterm-ratreal :: *rat* × (*unit* ⇒ *Code-Evaluation.term*) ⇒ *real* × (*unit* ⇒ *Code-Evaluation.term*)
where [*code-unfold*]: *valterm-ratreal k* = *Code-Evaluation.valtermify Ratreal* {·} *k*

end

instantiation *real* :: *random*
begin

context
includes *state-combinator-syntax*
begin

definition
Quickcheck-Random.random i = *Quickcheck-Random.random i* ∘→ (λ*r*. *Pair* (*valterm-ratreal r*))

instance ⟨*proof*⟩

end

end

instantiation *real* :: *exhaustive*
begin

definition
exhaustive-real f d = *Quickcheck-Exhaustive.exhaustive* (λ*r*. *f* (*Ratreal r*)) *d*

instance ⟨*proof*⟩

end

instantiation *real* :: *full-exhaustive*
begin

definition
full-exhaustive-real f d = *Quickcheck-Exhaustive.full-exhaustive* (λ*r*. *f* (*valterm-ratreal r*)) *d*

instance ⟨*proof*⟩

end

instantiation *real* :: *narrowing*
begin

definition
narrowing-real = *Quickcheck-Narrowing.apply* (*Quickcheck-Narrowing.cons Ratreal*) *narrowing*

instance ⟨*proof*⟩

end

98.23 Setup for Nitpick

⟨*ML*⟩

lemmas [*nitpick-unfold*] = *inverse-real-inst.inverse-real one-real-inst.one-real ord-real-inst.less-real ord-real-inst.less-eq-real plus-real-inst.plus-real times-real-inst.times-real uminus-real-inst.uminus-real zero-real-inst.zero-real*

98.24 Setup for SMT

⟨*ML*⟩

lemma [*z3-rule*]:
$0 + x = x$
$x + 0 = x$
$0 * x = 0$
$1 * x = x$
$-x = -1 * x$
$x + y = y + x$
for x y :: *real*
⟨*proof*⟩

lemma [*smt-arith-multiplication*]:
fixes A B :: *real* **and** p n :: *real*
assumes $A \leq B$ $0 < n$ $p > 0$
shows $(A \mathbin{/} n) * p \leq (B \mathbin{/} n) * p$
⟨*proof*⟩

lemma [*smt-arith-multiplication*]:
fixes A B :: *real* **and** p n :: *real*
assumes $A < B$ $0 < n$ $p > 0$
shows $(A \mathbin{/} n) * p < (B \mathbin{/} n) * p$
⟨*proof*⟩

lemma [*smt-arith-multiplication*]:
fixes A B :: *real* **and** p n :: *int*
assumes $A \leq B$ $0 < n$ $p > 0$
shows $(A \mathbin{/} n) * p \leq (B \mathbin{/} n) * p$
⟨*proof*⟩

lemma [*smt-arith-multiplication*]:
fixes A B :: *real* **and** p n :: *int*
assumes $A < B$ $0 < n$ $p > 0$
shows $(A \mathbin{/} n) * p < (B \mathbin{/} n) * p$
⟨*proof*⟩

lemmas [*smt-arith-multiplication*] =
verit-le-mono-div[*THEN mult-left-mono*, *unfolded int-distrib*, *of* - - ‹*nat* (*floor* (- :: *real*))› ‹*nat* (*floor* (- :: *real*))›]
div-le-mono[*THEN mult-left-mono*, *unfolded int-distrib*, *of* - - ‹*nat* (*floor* (- :: *real*))› ‹*nat* (*floor* (- :: *real*))›]
verit-le-mono-div-int[*THEN mult-left-mono*, *unfolded int-distrib*, *of* - - ‹*floor* (- :: *real*)› ‹*floor* (- :: *real*)›]
zdiv-mono1[*THEN mult-left-mono*, *unfolded int-distrib*, *of* - - ‹*floor* (- :: *real*)› ‹*floor* (- :: *real*)›]
arg-cong[*of* - - ‹λ*a* :: *real*. *a* / *real* (*n*::*nat*) * *real* (*p*::*nat*)› **for** *n p* :: *nat*, *THEN sym*]
arg-cong[*of* - - ‹λ*a* :: *real*. *a* / *real-of-int n* * *real-of-int p*› **for** *n p* :: *int*, *THEN sym*]
arg-cong[*of* - - ‹λ*a* :: *real*. *a* / *n* * *p*› **for** *n p* :: *real*, *THEN sym*]

lemmas [*smt-arith-simplify*] =
floor-one floor-numeral div-by-1 times-divide-eq-right
nonzero-mult-div-cancel-left division-ring-divide-zero div-0
divide-minus-left zero-less-divide-iff

98.25 Setup for Argo

⟨*ML*⟩

end

99 Topological Spaces

theory *Topological-Spaces*
imports *Main*
begin

named-theorems *continuous-intros structural introduction rules for continuity*

99.1 Topological space

class *open* =

fixes *open* :: *'a set ⇒ bool*

class *topological-space* = *open* +
assumes *open-UNIV* [*simp*, *intro*]: *open UNIV*
assumes *open-Int* [*intro*]: *open S ⟹ open T ⟹ open* (*S* ∩ *T*)
assumes *open-Union* [*intro*]: ∀ *S*∈*K*. *open S ⟹ open* (⋃ *K*)
begin

definition *closed* :: *'a set ⇒ bool*
where *closed S* ⟷ *open* (− *S*)

lemma *open-empty* [*continuous-intros*, *intro*, *simp*]: *open* {}
⟨*proof*⟩

lemma *open-Un* [*continuous-intros*, *intro*]: *open S ⟹ open T ⟹ open* (*S* ∪ *T*)
⟨*proof*⟩

lemma *open-UN* [*continuous-intros*, *intro*]: ∀ *x*∈*A*. *open* (*B x*) ⟹ *open* (⋃ *x*∈*A*. *B x*)
⟨*proof*⟩

lemma *open-Inter* [*continuous-intros*, *intro*]: *finite S ⟹* ∀ *T*∈*S*. *open T ⟹ open* (⋂ *S*)
⟨*proof*⟩

lemma *open-INT* [*continuous-intros*, *intro*]: *finite A ⟹* ∀ *x*∈*A*. *open* (*B x*) ⟹ *open* (⋂ *x*∈*A*. *B x*)
⟨*proof*⟩

lemma *openI*:
assumes ⋀*x*. *x* ∈ *S ⟹* ∃ *T*. *open T* ∧ *x* ∈ *T* ∧ *T* ⊆ *S*
shows *open S*
⟨*proof*⟩

lemma *open-subopen*: *open S* ⟷ (∀ *x*∈*S*. ∃ *T*. *open T* ∧ *x* ∈ *T* ∧ *T* ⊆ *S*)
⟨*proof*⟩

lemma *closed-empty* [*continuous-intros*, *intro*, *simp*]: *closed* {}
⟨*proof*⟩

lemma *closed-Un* [*continuous-intros*, *intro*]: *closed S ⟹ closed T ⟹ closed* (*S* ∪ *T*)
⟨*proof*⟩

lemma *closed-UNIV* [*continuous-intros*, *intro*, *simp*]: *closed UNIV*
⟨*proof*⟩

lemma *closed-Int* [*continuous-intros*, *intro*]: *closed S ⟹ closed T ⟹ closed* (*S* ∩ *T*)

⟨*proof*⟩

lemma *closed-INT* [*continuous-intros*, *intro*]: $\forall x{\in}A.\ closed\ (B\ x) \Longrightarrow closed\ (\bigcap x{\in}A.\ B\ x)$
⟨*proof*⟩

lemma *closed-Inter* [*continuous-intros*, *intro*]: $\forall S{\in}K.\ closed\ S \Longrightarrow closed\ (\bigcap K)$
⟨*proof*⟩

lemma *closed-Union* [*continuous-intros*, *intro*]: *finite* $S \Longrightarrow \forall T{\in}S.\ closed\ T \Longrightarrow closed\ (\bigcup S)$
⟨*proof*⟩

lemma *closed-UN* [*continuous-intros*, *intro*]:
finite $A \Longrightarrow \forall x{\in}A.\ closed\ (B\ x) \Longrightarrow closed\ (\bigcup x{\in}A.\ B\ x)$
⟨*proof*⟩

lemma *open-closed*: $open\ S \longleftrightarrow closed\ (-\ S)$
⟨*proof*⟩

lemma *closed-open*: $closed\ S \longleftrightarrow open\ (-\ S)$
⟨*proof*⟩

lemma *open-Diff* [*continuous-intros*, *intro*]: $open\ S \Longrightarrow closed\ T \Longrightarrow open\ (S - T)$
⟨*proof*⟩

lemma *closed-Diff* [*continuous-intros*, *intro*]: $closed\ S \Longrightarrow open\ T \Longrightarrow closed\ (S - T)$
⟨*proof*⟩

lemma *open-Compl* [*continuous-intros*, *intro*]: $closed\ S \Longrightarrow open\ (-\ S)$
⟨*proof*⟩

lemma *closed-Compl* [*continuous-intros*, *intro*]: $open\ S \Longrightarrow closed\ (-\ S)$
⟨*proof*⟩

lemma *open-Collect-neg*: $closed\ \{x.\ P\ x\} \Longrightarrow open\ \{x.\ \neg\ P\ x\}$
⟨*proof*⟩

lemma *open-Collect-conj*:
assumes $open\ \{x.\ P\ x\}$ $open\ \{x.\ Q\ x\}$
shows $open\ \{x.\ P\ x \wedge Q\ x\}$
⟨*proof*⟩

lemma *open-Collect-disj*:
assumes $open\ \{x.\ P\ x\}$ $open\ \{x.\ Q\ x\}$
shows $open\ \{x.\ P\ x \vee Q\ x\}$
⟨*proof*⟩

lemma *open-Collect-ex*: $(\bigwedge i.\ open\ \{x.\ P\ i\ x\}) \Longrightarrow open\ \{x.\ \exists\, i.\ P\ i\ x\}$
⟨*proof*⟩

lemma *open-Collect-imp*: $closed\ \{x.\ P\ x\} \Longrightarrow open\ \{x.\ Q\ x\} \Longrightarrow open\ \{x.\ P\ x \longrightarrow Q\ x\}$
⟨*proof*⟩

lemma *open-Collect-const*: $open\ \{x.\ P\}$
⟨*proof*⟩

lemma *closed-Collect-neg*: $open\ \{x.\ P\ x\} \Longrightarrow closed\ \{x.\ \neg\ P\ x\}$
⟨*proof*⟩

lemma *closed-Collect-conj*:
assumes $closed\ \{x.\ P\ x\}\ closed\ \{x.\ Q\ x\}$
shows $closed\ \{x.\ P\ x \wedge Q\ x\}$
⟨*proof*⟩

lemma *closed-Collect-disj*:
assumes $closed\ \{x.\ P\ x\}\ closed\ \{x.\ Q\ x\}$
shows $closed\ \{x.\ P\ x \vee Q\ x\}$
⟨*proof*⟩

lemma *closed-Collect-all*: $(\bigwedge i.\ closed\ \{x.\ P\ i\ x\}) \Longrightarrow closed\ \{x.\ \forall\, i.\ P\ i\ x\}$
⟨*proof*⟩

lemma *closed-Collect-imp*: $open\ \{x.\ P\ x\} \Longrightarrow closed\ \{x.\ Q\ x\} \Longrightarrow closed\ \{x.\ P\ x \longrightarrow Q\ x\}$
⟨*proof*⟩

lemma *closed-Collect-const*: $closed\ \{x.\ P\}$
⟨*proof*⟩

end

99.2 Hausdorff and other separation properties

class *t0-space* = *topological-space* +
assumes *t0-space*: $x \neq y \Longrightarrow \exists\, U.\ open\ U \wedge \neg\ (x \in U \longleftrightarrow y \in U)$

class *t1-space* = *topological-space* +
assumes *t1-space*: $x \neq y \Longrightarrow \exists\, U.\ open\ U \wedge x \in U \wedge y \notin U$

instance *t1-space* ⊆ *t0-space*
⟨*proof*⟩

context *t1-space* **begin**

lemma *separation-t1*: $x \neq y \longleftrightarrow (\exists\, U.\ \mathit{open}\ U \wedge x \in U \wedge y \notin U)$
⟨*proof*⟩

lemma *closed-singleton* [*iff*]: *closed* $\{a\}$
⟨*proof*⟩

lemma *closed-insert* [*continuous-intros*, *simp*]:
assumes *closed S*
shows *closed* (*insert a S*)
⟨*proof*⟩

lemma *finite-imp-closed*: *finite* $S \Longrightarrow$ *closed S*
⟨*proof*⟩

end

T2 spaces are also known as Hausdorff spaces.

class *t2-space* = *topological-space* +
assumes *hausdorff*: $x \neq y \Longrightarrow \exists\, U\ V.\ \mathit{open}\ U \wedge \mathit{open}\ V \wedge x \in U \wedge y \in V \wedge U \cap V = \{\}$

instance *t2-space* ⊆ *t1-space*
⟨*proof*⟩

lemma (**in** *t2-space*) *separation-t2*: $x \neq y \longleftrightarrow (\exists\, U\ V.\ \mathit{open}\ U \wedge \mathit{open}\ V \wedge x \in U \wedge y \in V \wedge U \cap V = \{\})$
⟨*proof*⟩

lemma (**in** *t0-space*) *separation-t0*: $x \neq y \longleftrightarrow (\exists\, U.\ \mathit{open}\ U \wedge \neg\ (x \in U \longleftrightarrow y \in U))$
⟨*proof*⟩

A classical separation axiom for topological space, the T3 axiom – also called regularity: if a point is not in a closed set, then there are open sets separating them.

class *t3-space* = *t2-space* +
assumes *t3-space*: *closed* $S \Longrightarrow y \notin S \Longrightarrow \exists\, U\ V.\ \mathit{open}\ U \wedge \mathit{open}\ V \wedge y \in U \wedge S \subseteq V \wedge U \cap V = \{\}$

A classical separation axiom for topological space, the T4 axiom – also called normality: if two closed sets are disjoint, then there are open sets separating them.

class *t4-space* = *t2-space* +
assumes *t4-space*: *closed* $S \Longrightarrow$ *closed* $T \Longrightarrow S \cap T = \{\} \Longrightarrow \exists\, U\ V.\ \mathit{open}\ U \wedge \mathit{open}\ V \wedge S \subseteq U \wedge T \subseteq V \wedge U \cap V = \{\}$

T4 is stronger than T3, and weaker than metric.

instance *t4-space* ⊆ *t3-space*

⟨*proof*⟩

A perfect space is a topological space with no isolated points.

class *perfect-space* = *topological-space* +
  assumes *not-open-singleton*: ¬ *open* {*x*}

lemma (**in** *perfect-space*) *UNIV-not-singleton*: *UNIV* ≠ {*x*}
  for *x*::′*a*
  ⟨*proof*⟩

99.3 Generators for toplogies

inductive *generate-topology* :: ′*a set set* ⇒ ′*a set* ⇒ *bool* **for** *S* :: ′*a set set*
  where
    UNIV: *generate-topology S UNIV*
  | *Int*: *generate-topology S* (*a* ∩ *b*) **if** *generate-topology S a* **and** *generate-topology S b*
  | *UN*: *generate-topology S* (⋃ *K*) **if** (⋀*k*. *k* ∈ *K* ⟹ *generate-topology S k*)
  | *Basis*: *generate-topology S s* **if** *s* ∈ *S*

hide-fact (**open**) *UNIV Int UN Basis*

lemma *generate-topology-Union*:
  (⋀*k*. *k* ∈ *I* ⟹ *generate-topology S* (*K k*)) ⟹ *generate-topology S* (⋃ *k*∈*I*. *K k*)
  ⟨*proof*⟩

lemma *topological-space-generate-topology*: *class.topological-space* (*generate-topology S*)
  ⟨*proof*⟩

99.4 Order topologies

class *order-topology* = *order* + *open* +
  assumes *open-generated-order*: *open* = *generate-topology* (*range* (λ*a*. {..< *a*}) ∪ *range* (λ*a*. {*a* <..}))
begin

subclass *topological-space*
  ⟨*proof*⟩

lemma *open-greaterThan* [*continuous-intros*, *simp*]: *open* {*a* <..}
  ⟨*proof*⟩

lemma *open-lessThan* [*continuous-intros*, *simp*]: *open* {..< *a*}
  ⟨*proof*⟩

lemma *open-greaterThanLessThan* [*continuous-intros*, *simp*]: *open* {*a* <..< *b*}
  ⟨*proof*⟩

end

class *linorder-topology* = *linorder* + *order-topology*

lemma *closed-atMost* [*continuous-intros*, *simp*]: *closed* {..*a*}
for *a* :: *'a*::*linorder-topology*
⟨*proof*⟩

lemma *closed-atLeast* [*continuous-intros*, *simp*]: *closed* {*a*..}
for *a* :: *'a*::*linorder-topology*
⟨*proof*⟩

lemma *closed-atLeastAtMost* [*continuous-intros*, *simp*]: *closed* {*a*..*b*}
for *a b* :: *'a*::*linorder-topology*
⟨*proof*⟩

lemma (**in** *order*) *less-separate*:
assumes $x < y$
shows $\exists\, a\ b.\ x \in \{..< a\} \wedge y \in \{b <..\} \wedge \{..< a\} \cap \{b <..\} = \{\}$
⟨*proof*⟩

instance *linorder-topology* ⊆ *t2-space*
⟨*proof*⟩

lemma (**in** *linorder-topology*) *open-right*:
assumes *open S x* ∈ *S*
and *gt-ex*: $x < y$
shows $\exists\, b{>}x.\ \{x\ ..<\ b\} \subseteq S$
⟨*proof*⟩

lemma (**in** *linorder-topology*) *open-left*:
assumes *open S x* ∈ *S*
and *lt-ex*: $y < x$
shows $\exists\, b{<}x.\ \{b <..\ x\} \subseteq S$
⟨*proof*⟩

99.5 Setup some topologies

99.5.1 Boolean is an order topology

class *discrete-topology* = *topological-space* +
assumes *open-discrete*: $\bigwedge A.$ *open A*

instance *discrete-topology* < *t2-space*
⟨*proof*⟩

instantiation *bool* :: *linorder-topology*
begin

definition *open-bool* :: *bool set* ⇒ *bool*

where *open-bool* = *generate-topology* (*range* (λ*a*. {..< *a*}) ∪ *range* (λ*a*. {*a* <..}))

instance
⟨*proof*⟩

end

instance *bool* :: *discrete-topology*
⟨*proof*⟩

instantiation *nat* :: *linorder-topology*
begin

definition *open-nat* :: *nat set* ⇒ *bool*
where *open-nat* = *generate-topology* (*range* (λ*a*. {..< *a*}) ∪ *range* (λ*a*. {*a* <..}))

instance
⟨*proof*⟩

end

instance *nat* :: *discrete-topology*
⟨*proof*⟩

instantiation *int* :: *linorder-topology*
begin

definition *open-int* :: *int set* ⇒ *bool*
where *open-int* = *generate-topology* (*range* (λ*a*. {..< *a*}) ∪ *range* (λ*a*. {*a* <..}))

instance
⟨*proof*⟩

end

instance *int* :: *discrete-topology*
⟨*proof*⟩

99.5.2 Topological filters

definition (**in** *topological-space*) *nhds* :: *'a* ⇒ *'a filter*
where *nhds a* = (*INF S*∈{*S*. *open S* ∧ *a* ∈ *S*}. *principal S*)

definition (**in** *topological-space*) *at-within* :: *'a* ⇒ *'a set* ⇒ *'a filter*
(*at* (-)/ *within* (-) [*1000*, *60*] *60*)
where *at a within s* = *inf* (*nhds a*) (*principal* (*s* − {*a*}))

abbreviation (**in** *topological-space*) *at* :: *'a* ⇒ *'a filter* (*at*)
where *at x* ≡ *at x within* (*CONST UNIV*)

abbreviation (**in** *order-topology*) *at-right* :: *'a* ⇒ *'a filter*
where *at-right x* ≡ *at x within* {*x* <..}

abbreviation (**in** *order-topology*) *at-left* :: *'a* ⇒ *'a filter*
where *at-left x* ≡ *at x within* {..< *x*}

lemma (**in** *topological-space*) *nhds-generated-topology*:
open = *generate-topology T* ⟹ *nhds x* = (*INF S*∈{*S*∈*T*. *x* ∈ *S*}. *principal S*)
⟨*proof*⟩

lemma (**in** *topological-space*) *eventually-nhds*:
eventually P (*nhds a*) ⟷ (∃ *S*. *open S* ∧ *a* ∈ *S* ∧ (∀ *x*∈*S*. *P x*))
⟨*proof*⟩

lemma *eventually-eventually*:
eventually (λ*y*. *eventually P* (*nhds y*)) (*nhds x*) = *eventually P* (*nhds x*)
⟨*proof*⟩

lemma (**in** *topological-space*) *eventually-nhds-in-open*:
open s ⟹ *x* ∈ *s* ⟹ *eventually* (λ*y*. *y* ∈ *s*) (*nhds x*)
⟨*proof*⟩

lemma (**in** *topological-space*) *eventually-nhds-x-imp-x*: *eventually P* (*nhds x*) ⟹ *P x*
⟨*proof*⟩

lemma (**in** *topological-space*) *nhds-neq-bot* [*simp*]: *nhds a* ≠ *bot*
⟨*proof*⟩

lemma (**in** *t1-space*) *t1-space-nhds*: *x* ≠ *y* ⟹ (∀ *F x in nhds x*. *x* ≠ *y*)
⟨*proof*⟩

lemma (**in** *topological-space*) *nhds-discrete-open*: *open* {*x*} ⟹ *nhds x* = *principal* {*x*}
⟨*proof*⟩

lemma (**in** *discrete-topology*) *nhds-discrete*: *nhds x* = *principal* {*x*}
⟨*proof*⟩

lemma (**in** *discrete-topology*) *at-discrete*: *at x within S* = *bot*
⟨*proof*⟩

lemma (**in** *discrete-topology*) *tendsto-discrete*:
filterlim (*f* :: *'b* ⇒ *'a*) (*nhds y*) *F* ⟷ *eventually* (λ*x*. *f x* = *y*) *F*
⟨*proof*⟩

lemma (**in** *topological-space*) *at-within-eq*:
at x within s = (*INF S*∈{*S*. *open S* ∧ *x* ∈ *S*}. *principal* (*S* ∩ *s* − {*x*}))

⟨*proof*⟩

lemma (**in** *topological-space*) *eventually-at-filter*:
eventually P (*at a within s*) ⟷ *eventually* (λ*x*. *x* ≠ *a* ⟶ *x* ∈ *s* ⟶ *P x*) (*nhds a*)
⟨*proof*⟩

lemma (**in** *topological-space*) *at-le*: *s* ⊆ *t* ⟹ *at x within s* ≤ *at x within t*
⟨*proof*⟩

lemma (**in** *topological-space*) *eventually-at-topological*:
eventually P (*at a within s*) ⟷ (∃ *S*. *open S* ∧ *a* ∈ *S* ∧ (∀ *x*∈*S*. *x* ≠ *a* ⟶ *x* ∈ *s* ⟶ *P x*))
⟨*proof*⟩

lemma *eventually-at-in-open*:
assumes *open A x* ∈ *A*
shows *eventually* (λ*y*. *y* ∈ *A* − {*x*}) (*at x*)
⟨*proof*⟩

lemma *eventually-at-in-open′*:
assumes *open A x* ∈ *A*
shows *eventually* (λ*y*. *y* ∈ *A*) (*at x*)
⟨*proof*⟩

lemma (**in** *topological-space*) *at-within-open*: *a* ∈ *S* ⟹ *open S* ⟹ *at a within S* = *at a*
⟨*proof*⟩

lemma (**in** *topological-space*) *at-within-open-NO-MATCH*:
a ∈ *s* ⟹ *open s* ⟹ *NO-MATCH UNIV s* ⟹ *at a within s* = *at a*
⟨*proof*⟩

lemma (**in** *topological-space*) *at-within-open-subset*:
a ∈ *S* ⟹ *open S* ⟹ *S* ⊆ *T* ⟹ *at a within T* = *at a*
⟨*proof*⟩

lemma (**in** *topological-space*) *at-within-nhd*:
assumes *x* ∈ *S open S T* ∩ *S* − {*x*} = *U* ∩ *S* − {*x*}
shows *at x within T* = *at x within U*
⟨*proof*⟩

lemma (**in** *topological-space*) *at-within-empty* [*simp*]: *at a within* {} = *bot*
⟨*proof*⟩

lemma (**in** *topological-space*) *at-within-union*:
at x within (*S* ∪ *T*) = *sup* (*at x within S*) (*at x within T*)
⟨*proof*⟩

lemma (**in** *topological-space*) *at-eq-bot-iff*: *at a = bot* ⟷ *open* {*a*}
⟨*proof*⟩

lemma (**in** *t1-space*) *eventually-neq-at-within*:
eventually (λ*w*. *w* ≠ *x*) (*at z within A*)
⟨*proof*⟩

lemma (**in** *perfect-space*) *at-neq-bot* [*simp*]: *at a* ≠ *bot*
⟨*proof*⟩

lemma (**in** *order-topology*) *nhds-order*:
nhds x = inf (*INF a*∈{*x* <..}. *principal* {..< *a*}) (*INF a*∈{..< *x*}. *principal* {*a* <..})
⟨*proof*⟩

lemma (**in** *topological-space*) *filterlim-at-within-If*:
assumes *filterlim f G* (*at x within* (*A* ∩ {*x*. *P x*}))
and *filterlim g G* (*at x within* (*A* ∩ {*x*. ¬*P x*}))
shows *filterlim* (λ*x*. *if P x then f x else g x*) *G* (*at x within A*)
⟨*proof*⟩

lemma (**in** *topological-space*) *filterlim-at-If*:
assumes *filterlim f G* (*at x within* {*x*. *P x*})
and *filterlim g G* (*at x within* {*x*. ¬*P x*})
shows *filterlim* (λ*x*. *if P x then f x else g x*) *G* (*at x*)
⟨*proof*⟩
lemma (**in** *linorder-topology*) *at-within-order*:
assumes *UNIV* ≠ {*x*}
shows *at x within s* =
inf (*INF a*∈{*x* <..}. *principal* ({..< *a*} ∩ *s* − {*x*}))
(*INF a*∈{..< *x*}. *principal* ({*a* <..} ∩ *s* − {*x*}))
⟨*proof*⟩

lemma (**in** *linorder-topology*) *at-left-eq*:
y < *x* ⟹ *at-left x* = (*INF a*∈{..< *x*}. *principal* {*a* <..< *x*})
⟨*proof*⟩

lemma (**in** *linorder-topology*) *eventually-at-left*:
y < *x* ⟹ *eventually P* (*at-left x*) ⟷ (∃ *b*<*x*. ∀ *y*>*b*. *y* < *x* ⟶ *P y*)
⟨*proof*⟩

lemma (**in** *linorder-topology*) *at-right-eq*:
x < *y* ⟹ *at-right x* = (*INF a*∈{*x* <..}. *principal* {*x* <..< *a*})
⟨*proof*⟩

lemma (**in** *linorder-topology*) *eventually-at-right*:
x < *y* ⟹ *eventually P* (*at-right x*) ⟷ (∃ *b*>*x*. ∀ *y*>*x*. *y* < *b* ⟶ *P y*)
⟨*proof*⟩

lemma *eventually-at-right-less*: $\forall_F\ y$ *in at-right* (x::'a::{*linorder-topology*, *no-top*}). $x < y$
⟨*proof*⟩

lemma *trivial-limit-at-right-top*: *at-right* (*top*::-::{*order-top*,*linorder-topology*}) = *bot*
⟨*proof*⟩

lemma *trivial-limit-at-left-bot*: *at-left* (*bot*::-::{*order-bot*,*linorder-topology*}) = *bot*
⟨*proof*⟩

lemma *trivial-limit-at-left-real* [*simp*]: ¬ *trivial-limit* (*at-left x*)
for x :: 'a::{*no-bot*,*dense-order*,*linorder-topology*}
⟨*proof*⟩

lemma *trivial-limit-at-right-real* [*simp*]: ¬ *trivial-limit* (*at-right x*)
for x :: 'a::{*no-top*,*dense-order*,*linorder-topology*}
⟨*proof*⟩

lemma (**in** *linorder-topology*) *at-eq-sup-left-right*: *at x* = *sup* (*at-left x*) (*at-right x*)
⟨*proof*⟩

lemma (**in** *linorder-topology*) *eventually-at-split*:
eventually P (*at x*) ⟷ *eventually P* (*at-left x*) ∧ *eventually P* (*at-right x*)
⟨*proof*⟩

lemma (**in** *order-topology*) *eventually-at-leftI*:
assumes ⋀x. $x \in \{a<..<b\} \Longrightarrow P\ x$
shows *eventually P* (*at-left b*)
⟨*proof*⟩

lemma (**in** *order-topology*) *eventually-at-rightI*:
assumes ⋀x. $x \in \{a<..<b\} \Longrightarrow P\ x$
shows *eventually P* (*at-right a*)
⟨*proof*⟩

lemma *eventually-filtercomap-nhds*:
eventually P (*filtercomap f* (*nhds x*)) ⟷ (∃ S. *open S* ∧ $x \in S$ ∧ (∀ x. $f\ x \in S$ ⟶ *P x*))
⟨*proof*⟩

lemma *eventually-filtercomap-at-topological*:
eventually P (*filtercomap f* (*at A within B*)) ⟷
(∃ S. *open S* ∧ $A \in S$ ∧ (∀ x. $f\ x \in S \cap B - \{A\}$ ⟶ *P x*)) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *eventually-at-right-field*:
eventually P (*at-right x*) ⟷ (∃ b>x. ∀ y>x. $y < b$ ⟶ *P y*)

for $x :: 'a$::{*linordered-field*, *linorder-topology*}
⟨*proof*⟩

lemma *eventually-at-left-field*:
eventually P (*at-left* x) $\longleftrightarrow$ $(\exists\, b{<}x.\ \forall\, y{>}b.\ y < x \longrightarrow P\ y)$
for $x :: 'a$::{*linordered-field*, *linorder-topology*}
⟨*proof*⟩

lemma *filtermap-nhds-eq-imp-filtermap-at-eq*:
assumes *filtermap* f (*nhds* z) = *nhds* ($f\ z$)
assumes *eventually* $(\lambda x.\ f\ x = f\ z \longrightarrow x = z)$ (*at* z)
shows *filtermap* f (*at* z) = *at* ($f\ z$)
⟨*proof*⟩

99.5.3 Tendsto

abbreviation (**in** *topological-space*)
tendsto :: $('b \Rightarrow 'a) \Rightarrow 'a \Rightarrow 'b$ *filter* $\Rightarrow$ *bool* (**infixr** $\longrightarrow$ 55)
where $(f \longrightarrow l)\ F \equiv$ *filterlim* f (*nhds* l) F

definition (**in** *t2-space*) *Lim* :: $'f$ *filter* $\Rightarrow$ $('f \Rightarrow 'a) \Rightarrow 'a$
where *Lim* $A\ f$ = (*THE* l. $(f \longrightarrow l)\ A$)

lemma (**in** *topological-space*) *tendsto-eq-rhs*: $(f \longrightarrow x)\ F \Longrightarrow x = y \Longrightarrow (f \longrightarrow y)\ F$
⟨*proof*⟩

named-theorems *tendsto-intros introduction rules for tendsto*
⟨*ML*⟩

context *topological-space* **begin**

lemma *tendsto-def*:
$(f \longrightarrow l)\ F \longleftrightarrow (\forall\, S.\ \textit{open}\ S \longrightarrow l \in S \longrightarrow \textit{eventually}\ (\lambda x.\ f\ x \in S)\ F)$
⟨*proof*⟩

lemma *tendsto-cong*: $(f \longrightarrow c)\ F \longleftrightarrow (g \longrightarrow c)\ F$ **if** *eventually* $(\lambda x.\ f\ x = g\ x)\ F$
⟨*proof*⟩

lemma *tendsto-mono*: $F \leq F' \Longrightarrow (f \longrightarrow l)\ F' \Longrightarrow (f \longrightarrow l)\ F$
⟨*proof*⟩

lemma *tendsto-ident-at* [*tendsto-intros*, *simp*, *intro*]: $((\lambda x.\ x) \longrightarrow a)$ (*at a within s*)
⟨*proof*⟩

lemma *tendsto-const* [*tendsto-intros*, *simp*, *intro*]: $((\lambda x.\ k) \longrightarrow k)\ F$
⟨*proof*⟩

lemma *filterlim-at*:
(*LIM x F. f x :> at b within s*) $\longleftrightarrow$ *eventually* ($\lambda x.\ f\ x \in s \wedge f\ x \neq b$) *F* $\wedge$ (*f* $\longrightarrow$ *b*) *F*
⟨*proof*⟩

lemma (**in** −)
assumes *filterlim f* (*nhds L*) *F*
shows *tendsto-imp-filterlim-at-right*:
eventually ($\lambda x.\ f\ x > L$) *F* $\Longrightarrow$ *filterlim f* (*at-right L*) *F*
and *tendsto-imp-filterlim-at-left*:
eventually ($\lambda x.\ f\ x < L$) *F* $\Longrightarrow$ *filterlim f* (*at-left L*) *F*
⟨*proof*⟩

lemma *filterlim-at-withinI*:
assumes *filterlim f* (*nhds c*) *F*
assumes *eventually* ($\lambda x.\ f\ x \in A - \{c\}$) *F*
shows *filterlim f* (*at c within A*) *F*
⟨*proof*⟩

lemma *filterlim-atI*:
assumes *filterlim f* (*nhds c*) *F*
assumes *eventually* ($\lambda x.\ f\ x \neq c$) *F*
shows *filterlim f* (*at c*) *F*
⟨*proof*⟩

lemma *topological-tendstoI*:
($\bigwedge S.$ *open S* $\Longrightarrow$ *l* $\in$ *S* $\Longrightarrow$ *eventually* ($\lambda x.\ f\ x \in S$) *F*) $\Longrightarrow$ (*f* $\longrightarrow$ *l*) *F*
⟨*proof*⟩

lemma *topological-tendstoD*:
(*f* $\longrightarrow$ *l*) *F* $\Longrightarrow$ *open S* $\Longrightarrow$ *l* $\in$ *S* $\Longrightarrow$ *eventually* ($\lambda x.\ f\ x \in S$) *F*
⟨*proof*⟩

lemma *tendsto-bot* [*simp*]: (*f* $\longrightarrow$ *a*) *bot*
⟨*proof*⟩

lemma *tendsto-eventually*: *eventually* ($\lambda x.\ f\ x = l$) *net* $\Longrightarrow$ (($\lambda x.\ f\ x$) $\longrightarrow$ *l*) *net*
⟨*proof*⟩

lemma *tendsto-principal-singleton*[*simp*]:
shows (*f* $\longrightarrow$ *f x*) (*principal* {*x*})
⟨*proof*⟩

end

lemma (**in** *topological-space*) *filterlim-within-subset*:
filterlim f l (*at x within S*) $\Longrightarrow$ *T* $\subseteq$ *S* $\Longrightarrow$ *filterlim f l* (*at x within T*)

⟨*proof*⟩

lemmas *tendsto-within-subset* = *filterlim-within-subset*

lemma (**in** *order-topology*) *order-tendsto-iff*:
(*f* ⟶ *x*) *F* ⟷ (∀ *l*<*x*. *eventually* (λ*x*. *l* < *f x*) *F*) ∧ (∀ *u*>*x*. *eventually* (λ*x*. *f x* < *u*) *F*)
⟨*proof*⟩

lemma (**in** *order-topology*) *order-tendstoI*:
(⋀*a*. *a* < *y* ⟹ *eventually* (λ*x*. *a* < *f x*) *F*) ⟹ (⋀*a*. *y* < *a* ⟹ *eventually* (λ*x*. *f x* < *a*) *F*) ⟹
(*f* ⟶ *y*) *F*
⟨*proof*⟩

lemma (**in** *order-topology*) *order-tendstoD*:
assumes (*f* ⟶ *y*) *F*
shows *a* < *y* ⟹ *eventually* (λ*x*. *a* < *f x*) *F*
and *y* < *a* ⟹ *eventually* (λ*x*. *f x* < *a*) *F*
⟨*proof*⟩

lemma (**in** *linorder-topology*) *tendsto-max*[*tendsto-intros*]:
assumes *X*: (*X* ⟶ *x*) *net*
and *Y*: (*Y* ⟶ *y*) *net*
shows ((λ*x*. *max* (*X x*) (*Y x*)) ⟶ *max x y*) *net*
⟨*proof*⟩

lemma (**in** *linorder-topology*) *tendsto-min*[*tendsto-intros*]:
assumes *X*: (*X* ⟶ *x*) *net*
and *Y*: (*Y* ⟶ *y*) *net*
shows ((λ*x*. *min* (*X x*) (*Y x*)) ⟶ *min x y*) *net*
⟨*proof*⟩

lemma (**in** *order-topology*)
assumes *a* < *b*
shows *at-within-Icc-at-right*: *at a within* {*a*..*b*} = *at-right a*
and *at-within-Icc-at-left*: *at b within* {*a*..*b*} = *at-left b*
⟨*proof*⟩

lemma (**in** *order-topology*) *at-within-Icc-at*: *a* < *x* ⟹ *x* < *b* ⟹ *at x within* {*a*..*b*} = *at x*
⟨*proof*⟩

lemma (**in** *t2-space*) *tendsto-unique*:
assumes *F* ≠ *bot*
and (*f* ⟶ *a*) *F*
and (*f* ⟶ *b*) *F*
shows *a* = *b*
⟨*proof*⟩

lemma (**in** *t2-space*) *tendsto-const-iff*:
fixes a b :: $'a$
assumes $\neg$ *trivial-limit* F
shows $((\lambda x.\ a) \longrightarrow b)\ F \longleftrightarrow a = b$
⟨*proof*⟩

lemma (**in** *t2-space*) *tendsto-unique'*:
assumes $F \neq bot$
shows $\exists_{\leq 1} l.\ (f \longrightarrow l)\ F$
⟨*proof*⟩

lemma *Lim-in-closed-set*:
assumes *closed* S *eventually* $(\lambda x.\ f(x) \in S)\ F$ $F \neq bot$ $(f \longrightarrow l)\ F$
shows $l \in S$
⟨*proof*⟩

lemma (**in** *t3-space*) *nhds-closed*:
assumes $x \in A$ **and** *open* A
shows $\exists A'.\ x \in A' \wedge$ *closed* $A' \wedge A' \subseteq A \wedge$ *eventually* $(\lambda y.\ y \in A')$ (*nhds* x)
⟨*proof*⟩

lemma (**in** *order-topology*) *increasing-tendsto*:
assumes *bdd*: *eventually* $(\lambda n.\ f\ n \leq l)\ F$
and *en*: $\bigwedge x.\ x < l \Longrightarrow$ *eventually* $(\lambda n.\ x < f\ n)\ F$
shows $(f \longrightarrow l)\ F$
⟨*proof*⟩

lemma (**in** *order-topology*) *decreasing-tendsto*:
assumes *bdd*: *eventually* $(\lambda n.\ l \leq f\ n)\ F$
and *en*: $\bigwedge x.\ l < x \Longrightarrow$ *eventually* $(\lambda n.\ f\ n < x)\ F$
shows $(f \longrightarrow l)\ F$
⟨*proof*⟩

lemma (**in** *order-topology*) *tendsto-sandwich*:
assumes *ev*: *eventually* $(\lambda n.\ f\ n \leq g\ n)$ *net eventually* $(\lambda n.\ g\ n \leq h\ n)$ *net*
assumes *lim*: $(f \longrightarrow c)$ *net* $(h \longrightarrow c)$ *net*
shows $(g \longrightarrow c)$ *net*
⟨*proof*⟩

lemma (**in** *t1-space*) *limit-frequently-eq*:
assumes $F \neq bot$
and *frequently* $(\lambda x.\ f\ x = c)\ F$
and $(f \longrightarrow d)\ F$
shows $d = c$
⟨*proof*⟩

lemma (**in** *t1-space*) *tendsto-imp-eventually-ne*:
assumes $(f \longrightarrow c)\ F$ $c \neq c'$

shows *eventually* $(\lambda z.\ f\ z \neq c')\ F$
⟨*proof*⟩

lemma (**in** *linorder-topology*) *tendsto-le*:
assumes F: ¬ *trivial-limit* F
and x: $(f \longrightarrow x)\ F$
and y: $(g \longrightarrow y)\ F$
and *ev*: *eventually* $(\lambda x.\ g\ x \leq f\ x)\ F$
shows $y \leq x$
⟨*proof*⟩

lemma (**in** *linorder-topology*) *tendsto-lowerbound*:
assumes x: $(f \longrightarrow x)\ F$
and *ev*: *eventually* $(\lambda i.\ a \leq f\ i)\ F$
and F: ¬ *trivial-limit* F
shows $a \leq x$
⟨*proof*⟩

lemma (**in** *linorder-topology*) *tendsto-upperbound*:
assumes x: $(f \longrightarrow x)\ F$
and *ev*: *eventually* $(\lambda i.\ a \geq f\ i)\ F$
and F: ¬ *trivial-limit* F
shows $a \geq x$
⟨*proof*⟩

lemma *filterlim-at-within-not-equal*:
fixes f::$'a \Rightarrow 'b$::*t2-space*
assumes *filterlim* f (*at* a *within* s) F
shows *eventually* $(\lambda w.\ f\ w \in s \wedge f\ w \neq b)\ F$
⟨*proof*⟩

99.5.4 Rules about *Lim*

lemma *tendsto-Lim*: ¬ *trivial-limit net* $\Longrightarrow$ $(f \longrightarrow l)$ *net* $\Longrightarrow$ *Lim net* $f = l$
⟨*proof*⟩

lemma *Lim-ident-at*: ¬ *trivial-limit* (*at* x *within* s) $\Longrightarrow$ *Lim* (*at* x *within* s) $(\lambda x.\ x) = x$
⟨*proof*⟩

lemma *Lim-cong*:
assumes *eventually* $(\lambda x.\ f\ x = g\ x)\ F$ $F = G$
shows *Lim* $F\ f$ = *Lim* $G\ g$
⟨*proof*⟩

lemma *eventually-Lim-ident-at*:
$(\forall_F\ y$ *in at* x *within* $X.\ P$ (*Lim* (*at* x *within* X) $(\lambda x.\ x))\ y) \longleftrightarrow$
$(\forall_F\ y$ *in at* x *within* $X.\ P\ x\ y)$ **for** x::$'a$::*t2-space*
⟨*proof*⟩

lemma *filterlim-at-bot-at-right*:
fixes $f :: 'a::linorder\text{-}topology \Rightarrow 'b::linorder$
assumes *mono*: $\bigwedge x\ y.\ Q\ x \Longrightarrow Q\ y \Longrightarrow x \leq y \Longrightarrow f\ x \leq f\ y$
and *bij*: $\bigwedge x.\ P\ x \Longrightarrow f\ (g\ x) = x\ \bigwedge x.\ P\ x \Longrightarrow Q\ (g\ x)$
and *Q*: *eventually Q* (*at-right a*)
and *bound*: $\bigwedge b.\ Q\ b \Longrightarrow a < b$
and *P*: *eventually P at-bot*
shows *filterlim f at-bot* (*at-right a*)
⟨*proof*⟩

lemma *filterlim-at-top-at-left*:
fixes $f :: 'a::linorder\text{-}topology \Rightarrow 'b::linorder$
assumes *mono*: $\bigwedge x\ y.\ Q\ x \Longrightarrow Q\ y \Longrightarrow x \leq y \Longrightarrow f\ x \leq f\ y$
and *bij*: $\bigwedge x.\ P\ x \Longrightarrow f\ (g\ x) = x\ \bigwedge x.\ P\ x \Longrightarrow Q\ (g\ x)$
and *Q*: *eventually Q* (*at-left a*)
and *bound*: $\bigwedge b.\ Q\ b \Longrightarrow b < a$
and *P*: *eventually P at-top*
shows *filterlim f at-top* (*at-left a*)
⟨*proof*⟩

lemma *filterlim-split-at*:
filterlim f F (*at-left x*) $\Longrightarrow$ *filterlim f F* (*at-right x*) $\Longrightarrow$
filterlim f F (*at x*)
for $x :: 'a::linorder\text{-}topology$
⟨*proof*⟩

lemma *filterlim-at-split*:
filterlim f F (*at x*) $\longleftrightarrow$ *filterlim f F* (*at-left x*) $\wedge$ *filterlim f F* (*at-right x*)
for $x :: 'a::linorder\text{-}topology$
⟨*proof*⟩

lemma *eventually-nhds-top*:
fixes $P :: 'a :: \{order\text{-}top, linorder\text{-}topology\} \Rightarrow bool$
and $b :: 'a$
assumes $b < top$
shows *eventually P* (*nhds top*) $\longleftrightarrow$ $(\exists\, b{<}top.\ (\forall\, z.\ b < z \longrightarrow P\ z))$
⟨*proof*⟩

lemma *tendsto-at-within-iff-tendsto-nhds*:
$(g \longrightarrow g\ l)$ (*at l within S*) $\longleftrightarrow$ $(g \longrightarrow g\ l)$ (*inf* (*nhds l*) (*principal S*))
⟨*proof*⟩

99.6 Limits on sequences

abbreviation (**in** *topological-space*)
LIMSEQ :: $[nat \Rightarrow 'a, 'a] \Rightarrow bool$ $(((\text{-})/ \longrightarrow (\text{-}))\ [60,\ 60]\ 60)$
where $X \longrightarrow L \equiv (X \longrightarrow L)$ *sequentially*

abbreviation (**in** *t2-space*) *lim* :: (*nat* ⇒ *'a*) ⇒ *'a*
where *lim X* ≡ *Lim sequentially X*

definition (**in** *topological-space*) *convergent* :: (*nat* ⇒ *'a*) ⇒ *bool*
where *convergent X* = (∃ *L*. *X* ⟶ *L*)

lemma *lim-def*: *lim X* = (*THE L*. *X* ⟶ *L*)
⟨*proof*⟩

lemma *lim-explicit*:
f ⟶ *f0* ⟷ (∀ *S*. *open S* ⟶ *f0* ∈ *S* ⟶ (∃ *N*. ∀ *n*≥*N*. *f n* ∈ *S*))
⟨*proof*⟩

99.7 Monotone sequences and subsequences

Definition of monotonicity. The use of disjunction here complicates proofs considerably. One alternative is to add a Boolean argument to indicate the direction. Another is to develop the notions of increasing and decreasing first.

definition *monoseq* :: (*nat* ⇒ *'a*::*order*) ⇒ *bool*
where *monoseq X* ⟷ (∀ *m*. ∀ *n*≥*m*. *X m* ≤ *X n*) ∨ (∀ *m*. ∀ *n*≥*m*. *X n* ≤ *X m*)

abbreviation *incseq* :: (*nat* ⇒ *'a*::*order*) ⇒ *bool*
where *incseq X* ≡ *mono X*

lemma *incseq-def*: *incseq X* ⟷ (∀ *m*. ∀ *n*≥*m*. *X n* ≥ *X m*)
⟨*proof*⟩

abbreviation *decseq* :: (*nat* ⇒ *'a*::*order*) ⇒ *bool*
where *decseq X* ≡ *antimono X*

lemma *decseq-def*: *decseq X* ⟷ (∀ *m*. ∀ *n*≥*m*. *X n* ≤ *X m*)
⟨*proof*⟩

99.7.1 Definition of subsequence.

lemma *strict-mono-leD*: *strict-mono r* ⟹ *m* ≤ *n* ⟹ *r m* ≤ *r n*
⟨*proof*⟩

lemma *strict-mono-id*: *strict-mono id*
⟨*proof*⟩

lemma *incseq-SucI*: (⋀*n*. *X n* ≤ *X* (*Suc n*)) ⟹ *incseq X*
⟨*proof*⟩

lemma *incseqD*: *incseq f* ⟹ *i* ≤ *j* ⟹ *f i* ≤ *f j*
⟨*proof*⟩

lemma *incseq-SucD*: *incseq A* ⟹ *A i* ≤ *A* (*Suc i*)

⟨*proof*⟩

lemma *incseq-Suc-iff*: *incseq f* ⟷ (∀ *n*. *f n* ≤ *f* (*Suc n*))
⟨*proof*⟩

lemma *incseq-const*[*simp*, *intro*]: *incseq* (λ*x*. *k*)
⟨*proof*⟩

lemma *decseq-SucI*: (⋀*n*. *X* (*Suc n*) ≤ *X n*) ⟹ *decseq X*
⟨*proof*⟩

lemma *decseqD*: *decseq f* ⟹ *i* ≤ *j* ⟹ *f j* ≤ *f i*
⟨*proof*⟩

lemma *decseq-SucD*: *decseq A* ⟹ *A* (*Suc i*) ≤ *A i*
⟨*proof*⟩

lemma *decseq-Suc-iff*: *decseq f* ⟷ (∀ *n*. *f* (*Suc n*) ≤ *f n*)
⟨*proof*⟩

lemma *decseq-const*[*simp*, *intro*]: *decseq* (λ*x*. *k*)
⟨*proof*⟩

lemma *monoseq-iff*: *monoseq X* ⟷ *incseq X* ∨ *decseq X*
⟨*proof*⟩

lemma *monoseq-Suc*: *monoseq X* ⟷ (∀ *n*. *X n* ≤ *X* (*Suc n*)) ∨ (∀ *n*. *X* (*Suc n*) ≤ *X n*)
⟨*proof*⟩

lemma *monoI1*: ∀ *m*. ∀ *n* ≥ *m*. *X m* ≤ *X n* ⟹ *monoseq X*
⟨*proof*⟩

lemma *monoI2*: ∀ *m*. ∀ *n* ≥ *m*. *X n* ≤ *X m* ⟹ *monoseq X*
⟨*proof*⟩

lemma *mono-SucI1*: ∀ *n*. *X n* ≤ *X* (*Suc n*) ⟹ *monoseq X*
⟨*proof*⟩

lemma *mono-SucI2*: ∀ *n*. *X* (*Suc n*) ≤ *X n* ⟹ *monoseq X*
⟨*proof*⟩

lemma *monoseq-minus*:
fixes *a* :: *nat* ⇒ *'a*::*ordered-ab-group-add*
assumes *monoseq a*
shows *monoseq* (λ *n*. − *a n*)
⟨*proof*⟩

99.7.2 Subsequence (alternative definition, (e.g. Hoskins)

For any sequence, there is a monotonic subsequence.

lemma *seq-monosub*:
fixes $s :: nat \Rightarrow 'a::linorder$
shows $\exists f.\ strict\text{-}mono\ f \wedge monoseq\ (\lambda n.\ (s\ (f\ n)))$
⟨*proof*⟩

lemma *seq-suble*:
assumes *sf*: $strict\text{-}mono\ (f :: nat \Rightarrow nat)$
shows $n \leq f\ n$
⟨*proof*⟩

lemma *eventually-subseq*:
$strict\text{-}mono\ r \Longrightarrow eventually\ P\ sequentially \Longrightarrow eventually\ (\lambda n.\ P\ (r\ n))\ sequentially$
⟨*proof*⟩

lemma *not-eventually-sequentiallyD*:
assumes $\neg\ eventually\ P\ sequentially$
shows $\exists\, r::nat \Rightarrow nat.\ strict\text{-}mono\ r \wedge (\forall\, n.\ \neg\ P\ (r\ n))$
⟨*proof*⟩

lemma *sequentially-offset*:
assumes $eventually\ (\lambda i.\ P\ i)\ sequentially$
shows $eventually\ (\lambda i.\ P\ (i + k))\ sequentially$
⟨*proof*⟩

lemma *seq-offset-neg*:
$(f \longrightarrow l)\ sequentially \Longrightarrow ((\lambda i.\ f(i - k)) \longrightarrow l)\ sequentially$
⟨*proof*⟩

lemma *filterlim-subseq*: $strict\text{-}mono\ f \Longrightarrow filterlim\ f\ sequentially\ sequentially$
⟨*proof*⟩

lemma *strict-mono-o*: $strict\text{-}mono\ r \Longrightarrow strict\text{-}mono\ s \Longrightarrow strict\text{-}mono\ (r \circ s)$
⟨*proof*⟩

lemma *strict-mono-compose*: $strict\text{-}mono\ r \Longrightarrow strict\text{-}mono\ s \Longrightarrow strict\text{-}mono\ (\lambda x.\ r\ (s\ x))$
⟨*proof*⟩

lemma *incseq-imp-monoseq*: $incseq\ X \Longrightarrow monoseq\ X$
⟨*proof*⟩

lemma *decseq-imp-monoseq*: $decseq\ X \Longrightarrow monoseq\ X$
⟨*proof*⟩

lemma *decseq-eq-incseq*: $decseq\ X = incseq\ (\lambda n.\ -\ X\ n)$

for $X :: nat \Rightarrow 'a::ordered\text{-}ab\text{-}group\text{-}add$
⟨*proof*⟩

lemma *INT-decseq-offset*:
assumes *decseq F*
shows $(\bigcap i.\ F\ i) = (\bigcap i{\in}\{n..\}.\ F\ i)$
⟨*proof*⟩

lemma *LIMSEQ-const-iff*: $(\lambda n.\ k) \longrightarrow l \longleftrightarrow k = l$
for $k\ l :: 'a::t2\text{-}space$
⟨*proof*⟩

lemma *LIMSEQ-SUP*: $incseq\ X \Longrightarrow X \longrightarrow (SUP\ i.\ X\ i :: 'a::\{complete\text{-}linorder,linorder\text{-}topology\})$
⟨*proof*⟩

lemma *LIMSEQ-INF*: $decseq\ X \Longrightarrow X \longrightarrow (INF\ i.\ X\ i :: 'a::\{complete\text{-}linorder,linorder\text{-}topology\})$
⟨*proof*⟩

lemma *LIMSEQ-ignore-initial-segment*: $f \longrightarrow a \Longrightarrow (\lambda n.\ f\ (n + k)) \longrightarrow a$
⟨*proof*⟩

lemma *LIMSEQ-offset*: $(\lambda n.\ f\ (n + k)) \longrightarrow a \Longrightarrow f \longrightarrow a$
⟨*proof*⟩

lemma *LIMSEQ-Suc*: $f \longrightarrow l \Longrightarrow (\lambda n.\ f\ (Suc\ n)) \longrightarrow l$
⟨*proof*⟩

lemma *LIMSEQ-imp-Suc*: $(\lambda n.\ f\ (Suc\ n)) \longrightarrow l \Longrightarrow f \longrightarrow l$
⟨*proof*⟩

lemma *LIMSEQ-lessThan-iff-atMost*:
shows $(\lambda n.\ f\ \{..<n\}) \longrightarrow x \longleftrightarrow (\lambda n.\ f\ \{..n\}) \longrightarrow x$
⟨*proof*⟩

lemma (**in** *t2-space*) *LIMSEQ-Uniq*: $\exists_{\le 1} l.\ X \longrightarrow l$
⟨*proof*⟩

lemma (**in** *t2-space*) *LIMSEQ-unique*: $X \longrightarrow a \Longrightarrow X \longrightarrow b \Longrightarrow a = b$
⟨*proof*⟩

lemma *LIMSEQ-le-const*: $X \longrightarrow x \Longrightarrow \exists N.\ \forall n{\ge}N.\ a \le X\ n \Longrightarrow a \le x$
for $a\ x :: 'a::linorder\text{-}topology$
⟨*proof*⟩

lemma *LIMSEQ-le*: $X \longrightarrow x \Longrightarrow Y \longrightarrow y \Longrightarrow \exists N.\ \forall n{\ge}N.\ X\ n \le Y\ n$
$\Longrightarrow x \le y$
for $x\ y :: 'a::linorder\text{-}topology$
⟨*proof*⟩

lemma *LIMSEQ-le-const2*: $X \longrightarrow x \Longrightarrow \exists N.\ \forall n{\geq}N.\ X\ n \leq a \Longrightarrow x \leq a$
for $a\ x :: 'a{::}$*linorder-topology*
⟨*proof*⟩

lemma *Lim-bounded*: $f \longrightarrow l \Longrightarrow \forall n{\geq}M.\ f\ n \leq C \Longrightarrow l \leq C$
for $l :: 'a{::}$*linorder-topology*
⟨*proof*⟩

lemma *Lim-bounded2*:
fixes $f :: nat \Rightarrow 'a{::}$*linorder-topology*
assumes *lim*:$f \longrightarrow l$ **and** *ge*: $\forall n{\geq}N.\ f\ n \geq C$
shows $l \geq C$
⟨*proof*⟩

lemma *lim-mono*:
fixes $X\ Y :: nat \Rightarrow 'a{::}$*linorder-topology*
assumes $\bigwedge n.\ N \leq n \Longrightarrow X\ n \leq Y\ n$
and $X \longrightarrow x$
and $Y \longrightarrow y$
shows $x \leq y$
⟨*proof*⟩

lemma *Sup-lim*:
fixes $a :: 'a{::}\{$*complete-linorder*,*linorder-topology*$\}$
assumes $\bigwedge n.\ b\ n \in s$
and $b \longrightarrow a$
shows $a \leq Sup\ s$
⟨*proof*⟩

lemma *Inf-lim*:
fixes $a :: 'a{::}\{$*complete-linorder*,*linorder-topology*$\}$
assumes $\bigwedge n.\ b\ n \in s$
and $b \longrightarrow a$
shows $Inf\ s \leq a$
⟨*proof*⟩

lemma *SUP-Lim*:
fixes $X :: nat \Rightarrow 'a{::}\{$*complete-linorder*,*linorder-topology*$\}$
assumes *inc*: *incseq* X
and l: $X \longrightarrow l$
shows $(SUP\ n.\ X\ n) = l$
⟨*proof*⟩

lemma *INF-Lim*:
fixes $X :: nat \Rightarrow 'a{::}\{$*complete-linorder*,*linorder-topology*$\}$
assumes *dec*: *decseq* X
and l: $X \longrightarrow l$
shows $(INF\ n.\ X\ n) = l$
⟨*proof*⟩

lemma *convergentD*: *convergent X* $\Longrightarrow$ $\exists\, L.\ X \longrightarrow L$
⟨*proof*⟩

lemma *convergentI*: $X \longrightarrow L \Longrightarrow$ *convergent X*
⟨*proof*⟩

lemma *convergent-LIMSEQ-iff*: *convergent X* $\longleftrightarrow$ $X \longrightarrow$ *lim X*
⟨*proof*⟩

lemma *convergent-const*: *convergent* $(\lambda n.\ c)$
⟨*proof*⟩

lemma *monoseq-le*:
monoseq a $\Longrightarrow$ $a \longrightarrow x \Longrightarrow$
$(\forall\, n.\ a\ n \leq x) \wedge (\forall\, m.\ \forall\, n{\geq}m.\ a\ m \leq a\ n) \vee$
$(\forall\, n.\ x \leq a\ n) \wedge (\forall\, m.\ \forall\, n{\geq}m.\ a\ n \leq a\ m)$
for x :: *'a*::*linorder-topology*
⟨*proof*⟩

lemma *LIMSEQ-subseq-LIMSEQ*: $X \longrightarrow L \Longrightarrow$ *strict-mono f* $\Longrightarrow$ $(X \circ f) \longrightarrow L$
⟨*proof*⟩

lemma *convergent-subseq-convergent*: *convergent X* $\Longrightarrow$ *strict-mono f* $\Longrightarrow$ *convergent* $(X \circ f)$
⟨*proof*⟩

lemma *limI*: $X \longrightarrow L \Longrightarrow$ *lim X* $= L$
⟨*proof*⟩

lemma *lim-le*: *convergent f* $\Longrightarrow$ $(\bigwedge n.\ f\ n \leq x) \Longrightarrow$ *lim f* $\leq x$
for x :: *'a*::*linorder-topology*
⟨*proof*⟩

lemma *lim-const* [*simp*]: *lim* $(\lambda m.\ a) = a$
⟨*proof*⟩

99.7.3 Increasing and Decreasing Series

lemma *incseq-le*: *incseq X* $\Longrightarrow$ $X \longrightarrow L \Longrightarrow X\ n \leq L$
for L :: *'a*::*linorder-topology*
⟨*proof*⟩

lemma *decseq-ge*: *decseq X* $\Longrightarrow$ $X \longrightarrow L \Longrightarrow L \leq X\ n$
for L :: *'a*::*linorder-topology*
⟨*proof*⟩

99.8 First countable topologies

class *first-countable-topology* = *topological-space* +
  assumes *first-countable-basis*:
    $\exists A{::}nat \Rightarrow 'a\ set.\ (\forall i.\ x \in A\ i \wedge open\ (A\ i)) \wedge (\forall S.\ open\ S \wedge x \in S \longrightarrow (\exists i.\ A\ i \subseteq S))$

lemma (**in** *first-countable-topology*) *countable-basis-at-decseq*:
  obtains $A :: nat \Rightarrow 'a\ set$ **where**
    $\bigwedge i.\ open\ (A\ i)\ \bigwedge i.\ x \in (A\ i)$
    $\bigwedge S.\ open\ S \Longrightarrow x \in S \Longrightarrow eventually\ (\lambda i.\ A\ i \subseteq S)\ sequentially$
⟨*proof*⟩

lemma (**in** *first-countable-topology*) *nhds-countable*:
  obtains $X :: nat \Rightarrow 'a\ set$
  where $decseq\ X\ \bigwedge n.\ open\ (X\ n)\ \bigwedge n.\ x \in X\ n\ nhds\ x = (INF\ n.\ principal\ (X\ n))$
⟨*proof*⟩

lemma (**in** *first-countable-topology*) *countable-basis*:
  obtains $A :: nat \Rightarrow 'a\ set$ **where**
    $\bigwedge i.\ open\ (A\ i)\ \bigwedge i.\ x \in A\ i$
    $\bigwedge F.\ (\forall n.\ F\ n \in A\ n) \Longrightarrow F \longrightarrow x$
⟨*proof*⟩

lemma (**in** *first-countable-topology*) *sequentially-imp-eventually-nhds-within*:
  assumes $\forall f.\ (\forall n.\ f\ n \in s) \wedge f \longrightarrow a \longrightarrow eventually\ (\lambda n.\ P\ (f\ n))\ sequentially$
  shows $eventually\ P\ (inf\ (nhds\ a)\ (principal\ s))$
⟨*proof*⟩

lemma (**in** *first-countable-topology*) *eventually-nhds-within-iff-sequentially*:
  $eventually\ P\ (inf\ (nhds\ a)\ (principal\ s)) \longleftrightarrow$
    $(\forall f.\ (\forall n.\ f\ n \in s) \wedge f \longrightarrow a \longrightarrow eventually\ (\lambda n.\ P\ (f\ n))\ sequentially)$
⟨*proof*⟩

lemma (**in** *first-countable-topology*) *eventually-nhds-iff-sequentially*:
  $eventually\ P\ (nhds\ a) \longleftrightarrow (\forall f.\ f \longrightarrow a \longrightarrow eventually\ (\lambda n.\ P\ (f\ n))\ sequentially)$
  ⟨*proof*⟩

lemma *Inf-as-limit*:
  fixes $A{::}'a{::}\{linorder\text{-}topology, first\text{-}countable\text{-}topology, complete\text{-}linorder\}\ set$
  assumes $A \neq \{\}$
  shows $\exists u.\ (\forall n.\ u\ n \in A) \wedge u \longrightarrow Inf\ A$
⟨*proof*⟩

lemma *tendsto-at-iff-sequentially*:
  $(f \longrightarrow a)\ (at\ x\ within\ s) \longleftrightarrow (\forall X.\ (\forall i.\ X\ i \in s - \{x\}) \longrightarrow X \longrightarrow x \longrightarrow ((f \circ X) \longrightarrow a))$

for $f :: 'a$::*first-countable-topology* $\Rightarrow$ -
⟨*proof*⟩

lemma *approx-from-above-dense-linorder*:
fixes x::$'a$::{*dense-linorder*, *linorder-topology*, *first-countable-topology*}
assumes $x < y$
shows $\exists u.\ (\forall n.\ u\ n > x) \wedge (u \longrightarrow x)$
⟨*proof*⟩

lemma *approx-from-below-dense-linorder*:
fixes x::$'a$::{*dense-linorder*, *linorder-topology*, *first-countable-topology*}
assumes $x > y$
shows $\exists u.\ (\forall n.\ u\ n < x) \wedge (u \longrightarrow x)$
⟨*proof*⟩

99.9 Function limit at a point

abbreviation *LIM* :: ($'a$::*topological-space* $\Rightarrow$ $'b$::*topological-space*) $\Rightarrow$ $'a \Rightarrow 'b \Rightarrow$ *bool*
(((-)/ −(-)/→ (-)) [*60*, *0*, *60*] *60*)
where $f\ -a\rightarrow\ L \equiv (f \longrightarrow L)\ (at\ a)$

lemma *tendsto-within-open*: $a \in S \Longrightarrow$ *open* $S \Longrightarrow (f \longrightarrow l)\ (at\ a\ within\ S) \longleftrightarrow (f\ -a\rightarrow\ l)$
⟨*proof*⟩

lemma *tendsto-within-open-NO-MATCH*:
$a \in S \Longrightarrow$ *NO-MATCH UNIV* $S \Longrightarrow$ *open* $S \Longrightarrow (f \longrightarrow l)(at\ a\ within\ S) \longleftrightarrow (f \longrightarrow l)(at\ a)$
for $f :: 'a$::*topological-space* $\Rightarrow$ $'b$::*topological-space*
⟨*proof*⟩

lemma *LIM-const-not-eq*[*tendsto-intros*]: $k \neq L \Longrightarrow \neg\ (\lambda x.\ k)\ -a\rightarrow\ L$
for $a :: 'a$::*perfect-space* **and** $k\ L :: 'b$::*t2-space*
⟨*proof*⟩

lemmas *LIM-not-zero* = *LIM-const-not-eq* [**where** $L = 0$]

lemma *LIM-const-eq*: $(\lambda x.\ k)\ -a\rightarrow\ L \Longrightarrow k = L$
for $a :: 'a$::*perfect-space* **and** $k\ L :: 'b$::*t2-space*
⟨*proof*⟩

lemma *LIM-unique*: $f\ -a\rightarrow\ L \Longrightarrow f\ -a\rightarrow\ M \Longrightarrow L = M$
for $a :: 'a$::*perfect-space* **and** $L\ M :: 'b$::*t2-space*
⟨*proof*⟩

lemma *LIM-Uniq*: $\exists_{\leq 1} L$::$'b$::*t2-space*. $f\ -a\rightarrow\ L$
for $a :: 'a$::*perfect-space*
⟨*proof*⟩

Limits are equal for functions equal except at limit point.

lemma *LIM-equal*: $\forall x.\ x \neq a \longrightarrow f\ x = g\ x \Longrightarrow (f \ -a\rightarrow\ l) \longleftrightarrow (g \ -a\rightarrow\ l)$
⟨*proof*⟩

lemma *LIM-cong*: $a = b \Longrightarrow (\bigwedge x.\ x \neq b \Longrightarrow f\ x = g\ x) \Longrightarrow l = m \Longrightarrow (f \ -a\rightarrow\ l) \longleftrightarrow (g \ -b\rightarrow\ m)$
⟨*proof*⟩

lemma *tendsto-cong-limit*: $(f \longrightarrow l)\ F \Longrightarrow k = l \Longrightarrow (f \longrightarrow k)\ F$
⟨*proof*⟩

lemma *tendsto-at-iff-tendsto-nhds*: $g \ -l\rightarrow\ g\ l \longleftrightarrow (g \longrightarrow g\ l)\ (nhds\ l)$
⟨*proof*⟩

lemma *tendsto-compose*: $g \ -l\rightarrow\ g\ l \Longrightarrow (f \longrightarrow l)\ F \Longrightarrow ((\lambda x.\ g\ (f\ x)) \longrightarrow g\ l)\ F$
⟨*proof*⟩

lemma *tendsto-compose-eventually*:
$g \ -l\rightarrow\ m \Longrightarrow (f \longrightarrow l)\ F \Longrightarrow eventually\ (\lambda x.\ f\ x \neq l)\ F \Longrightarrow ((\lambda x.\ g\ (f\ x)) \longrightarrow m)\ F$
⟨*proof*⟩

lemma *LIM-compose-eventually*:
assumes $f \ -a\rightarrow\ b$
and $g \ -b\rightarrow\ c$
and $eventually\ (\lambda x.\ f\ x \neq b)\ (at\ a)$
shows $(\lambda x.\ g\ (f\ x)) \ -a\rightarrow\ c$
⟨*proof*⟩

lemma *tendsto-compose-filtermap*: $((g \circ f) \longrightarrow T)\ F \longleftrightarrow (g \longrightarrow T)\ (filtermap\ f\ F)$
⟨*proof*⟩

lemma *tendsto-compose-at*:
assumes *f*: $(f \longrightarrow y)\ F$ **and** *g*: $(g \longrightarrow z)\ (at\ y)$ **and** *fg*: $eventually\ (\lambda w.\ f\ w = y \longrightarrow g\ y = z)\ F$
shows $((g \circ f) \longrightarrow z)\ F$
⟨*proof*⟩

lemma *tendsto-nhds-iff*: $(f \longrightarrow (c :: 'a :: t1\text{-}space))\ (nhds\ x) \longleftrightarrow f \ -x\rightarrow\ c \wedge f\ x = c$
⟨*proof*⟩

99.9.1 Relation of *LIM* and *LIMSEQ*

lemma (**in** *first-countable-topology*) *sequentially-imp-eventually-within*:
$(\forall f.\ (\forall n.\ f\ n \in s \wedge f\ n \neq a) \wedge f \longrightarrow a \longrightarrow eventually\ (\lambda n.\ P\ (f\ n))\ sequentially) \Longrightarrow$

eventually P (at a within s)
⟨*proof*⟩

lemma (**in** *first-countable-topology*) *sequentially-imp-eventually-at*:
(∀ *f*. (∀ *n*. *f n* ≠ *a*) ∧ *f* ⟶ *a* ⟶ *eventually* (λ*n*. *P* (*f n*)) *sequentially*) ⟹ *eventually P* (*at a*)
⟨*proof*⟩

lemma *LIMSEQ-SEQ-conv*:
(∀ *S*. (∀ *n*. *S n* ≠ *a*) ∧ *S* ⟶ *a* ⟶ (λ*n*. *X* (*S n*)) ⟶ *L*) ⟷ *X* −*a*→ *L* (**is** *?lhs*=*?rhs*)
for *a* :: *'a*::*first-countable-topology* **and** *L* :: *'b*::*topological-space*
⟨*proof*⟩

lemma *sequentially-imp-eventually-at-left*:
fixes *a* :: *'a*::{*linorder-topology*,*first-countable-topology*}
assumes *b*[*simp*]: *b* < *a*
and ∗: ⋀*f*. (⋀*n*. *b* < *f n*) ⟹ (⋀*n*. *f n* < *a*) ⟹ *incseq f* ⟹ *f* ⟶ *a* ⟹ *eventually* (λ*n*. *P* (*f n*)) *sequentially*
shows *eventually P* (*at-left a*)
⟨*proof*⟩

lemma *tendsto-at-left-sequentially*:
fixes *a b* :: *'b*::{*linorder-topology*,*first-countable-topology*}
assumes *b* < *a*
assumes ∗: ⋀*S*. (⋀*n*. *S n* < *a*) ⟹ (⋀*n*. *b* < *S n*) ⟹ *incseq S* ⟹ *S* ⟶ *a* ⟹
(λ*n*. *X* (*S n*)) ⟶ *L*
shows (*X* ⟶ *L*) (*at-left a*)
⟨*proof*⟩

lemma *sequentially-imp-eventually-at-right*:
fixes *a b* :: *'a*::{*linorder-topology*,*first-countable-topology*}
assumes *b*[*simp*]: *a* < *b*
assumes ∗: ⋀*f*. (⋀*n*. *a* < *f n*) ⟹ (⋀*n*. *f n* < *b*) ⟹ *decseq f* ⟹ *f* ⟶ *a* ⟹
eventually (λ*n*. *P* (*f n*)) *sequentially*
shows *eventually P* (*at-right a*)
⟨*proof*⟩

lemma *tendsto-at-right-sequentially*:
fixes *a* :: - :: {*linorder-topology*, *first-countable-topology*}
assumes *a* < *b*
and ∗: ⋀*S*. (⋀*n*. *a* < *S n*) ⟹ (⋀*n*. *S n* < *b*) ⟹ *decseq S* ⟹ *S* ⟶ *a* ⟹
(λ*n*. *X* (*S n*)) ⟶ *L*
shows (*X* ⟶ *L*) (*at-right a*)
⟨*proof*⟩

99.10 Continuity

99.10.1 Continuity on a set

definition *continuous-on* :: *'a set* ⇒ (*'a*::*topological-space* ⇒ *'b*::*topological-space*) ⇒ *bool*
  where *continuous-on s f* ⟷ (∀ *x*∈*s*. (*f* ⟶ *f x*) (*at x within s*))

lemma *continuous-on-cong* [*cong*]:
  s = *t* ⟹ (⋀*x*. *x* ∈ *t* ⟹ *f x* = *g x*) ⟹ *continuous-on s f* ⟷ *continuous-on t g*
  ⟨*proof*⟩

lemma *continuous-on-cong-simp*:
  s = *t* ⟹ (⋀*x*. *x* ∈ *t* =*simp*=> *f x* = *g x*) ⟹ *continuous-on s f* ⟷ *continuous-on t g*
  ⟨*proof*⟩

lemma *continuous-on-topological*:
  continuous-on s f ⟷
    (∀ *x*∈*s*. ∀ *B*. *open B* ⟶ *f x* ∈ *B* ⟶ (∃ *A*. *open A* ∧ *x* ∈ *A* ∧ (∀ *y*∈*s*. *y* ∈ *A* ⟶ *f y* ∈ *B*)))
  ⟨*proof*⟩

lemma *continuous-on-open-invariant*:
  continuous-on s f ⟷ (∀ *B*. *open B* ⟶ (∃ *A*. *open A* ∧ *A* ∩ *s* = *f* −‘ *B* ∩ *s*))
⟨*proof*⟩

lemma *continuous-on-open-vimage*:
  open s ⟹ *continuous-on s f* ⟷ (∀ *B*. *open B* ⟶ *open* (*f* −‘ *B* ∩ *s*))
  ⟨*proof*⟩

corollary *continuous-imp-open-vimage*:
  assumes *continuous-on s f open s open B f* −‘ *B* ⊆ *s*
  shows *open* (*f* −‘ *B*)
  ⟨*proof*⟩

corollary *open-vimage*[*continuous-intros*]:
  assumes *open s*
    and *continuous-on UNIV f*
  shows *open* (*f* −‘ *s*)
  ⟨*proof*⟩

lemma *continuous-on-closed-invariant*:
  continuous-on s f ⟷ (∀ *B*. *closed B* ⟶ (∃ *A*. *closed A* ∧ *A* ∩ *s* = *f* −‘ *B* ∩ *s*))
⟨*proof*⟩

lemma *continuous-on-closed-vimage*:
  closed s ⟹ *continuous-on s f* ⟷ (∀ *B*. *closed B* ⟶ *closed* (*f* −‘ *B* ∩ *s*))
  ⟨*proof*⟩

corollary *closed-vimage-Int*[*continuous-intros*]:
assumes *closed s*
and *continuous-on t f*
and *t*: *closed t*
shows *closed* (*f* −‘ *s* ∩ *t*)
⟨*proof*⟩

corollary *closed-vimage*[*continuous-intros*]:
assumes *closed s*
and *continuous-on UNIV f*
shows *closed* (*f* −‘ *s*)
⟨*proof*⟩

lemma *continuous-on-empty* [*simp*]: *continuous-on* {} *f*
⟨*proof*⟩

lemma *continuous-on-sing* [*simp*]: *continuous-on* {*x*} *f*
⟨*proof*⟩

lemma *continuous-on-open-Union*:
(⋀*s*. *s* ∈ *S* ⟹ *open s*) ⟹ (⋀*s*. *s* ∈ *S* ⟹ *continuous-on s f*) ⟹ *continuous-on* (⋃ *S*) *f*
⟨*proof*⟩

lemma *continuous-on-open-UN*:
(⋀*s*. *s* ∈ *S* ⟹ *open* (*A s*)) ⟹ (⋀*s*. *s* ∈ *S* ⟹ *continuous-on* (*A s*) *f*) ⟹
continuous-on (⋃ *s*∈*S*. *A s*) *f*
⟨*proof*⟩

lemma *continuous-on-open-Un*:
open s ⟹ *open t* ⟹ *continuous-on s f* ⟹ *continuous-on t f* ⟹ *continuous-on* (*s* ∪ *t*) *f*
⟨*proof*⟩

lemma *continuous-on-closed-Un*:
closed s ⟹ *closed t* ⟹ *continuous-on s f* ⟹ *continuous-on t f* ⟹ *continuous-on* (*s* ∪ *t*) *f*
⟨*proof*⟩

lemma *continuous-on-closed-Union*:
assumes *finite I*
⋀*i*. *i* ∈ *I* ⟹ *closed* (*U i*)
⋀*i*. *i* ∈ *I* ⟹ *continuous-on* (*U i*) *f*
shows *continuous-on* (⋃ *i* ∈ *I*. *U i*) *f*
⟨*proof*⟩

lemma *continuous-on-If*:
assumes *closed*: *closed s closed t*

and *cont*: *continuous-on s f continuous-on t g*
and *P*: $\bigwedge x.\ x \in s \Longrightarrow \neg\ P\ x \Longrightarrow f\ x = g\ x\ \bigwedge x.\ x \in t \Longrightarrow P\ x \Longrightarrow f\ x = g\ x$
shows *continuous-on* $(s \cup t)$ $(\lambda x.$ *if P x then f x else g x*$)$
(**is** *continuous-on - ?h*)
⟨*proof*⟩

lemma *continuous-on-cases*:
closed s $\Longrightarrow$ *closed t* $\Longrightarrow$ *continuous-on s f* $\Longrightarrow$ *continuous-on t g* $\Longrightarrow$
$\forall x.\ (x{\in}s \wedge \neg\ P\ x) \vee (x \in t \wedge P\ x) \longrightarrow f\ x = g\ x \Longrightarrow$
continuous-on $(s \cup t)$ $(\lambda x.$ *if P x then f x else g x*$)$
⟨*proof*⟩

lemma *continuous-on-id*[*continuous-intros*,*simp*]: *continuous-on s* $(\lambda x.\ x)$
⟨*proof*⟩

lemma *continuous-on-id'*[*continuous-intros*,*simp*]: *continuous-on s id*
⟨*proof*⟩

lemma *continuous-on-const*[*continuous-intros*,*simp*]: *continuous-on s* $(\lambda x.\ c)$
⟨*proof*⟩

lemma *continuous-on-subset*: *continuous-on s f* $\Longrightarrow$ $t \subseteq s$ $\Longrightarrow$ *continuous-on t f*
⟨*proof*⟩

lemma *continuous-on-compose*[*continuous-intros*]:
continuous-on s f $\Longrightarrow$ *continuous-on* $(f\ `\ s)$ *g* $\Longrightarrow$ *continuous-on s* $(g \circ f)$
⟨*proof*⟩

lemma *continuous-on-compose2*:
continuous-on t g $\Longrightarrow$ *continuous-on s f* $\Longrightarrow$ $f\ `\ s \subseteq t \Longrightarrow$ *continuous-on s* $(\lambda x.$ *g* $(f\ x))$
⟨*proof*⟩

lemma *continuous-on-generate-topology*:
assumes $*$: *open* $=$ *generate-topology X*
and $**$: $\bigwedge B.\ B \in X \Longrightarrow \exists\, C.$ *open* $C \wedge C \cap A = f\ -`\ B \cap A$
shows *continuous-on A f*
⟨*proof*⟩

lemma *continuous-onI-mono*:
fixes *f* :: *'a*::*linorder-topology* $\Rightarrow$ *'b*::{*dense-order*,*linorder-topology*}
assumes *open* $(f`A)$
and *mono*: $\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow x \leq y \Longrightarrow f\ x \leq f\ y$
shows *continuous-on A f*
⟨*proof*⟩

lemma *continuous-on-IccI*:
$[\![(f \longrightarrow f\ a)$ $(at\text{-}right\ a)$;
$(f \longrightarrow f\ b)$ $(at\text{-}left\ b)$;

$(\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow f \ -x\to\ f\ x);\ a < b]\!] \Longrightarrow$
continuous-on $\{a\ ..\ b\}$ *f*
for *a*::*'a*::*linorder-topology*
⟨*proof*⟩

lemma
fixes *a b*::*'a*::*linorder-topology*
assumes *continuous-on* $\{a\ ..\ b\}$ *f a* < *b*
shows *continuous-on-Icc-at-rightD*: (*f* ⟶ *f a*) (*at-right a*)
and *continuous-on-Icc-at-leftD*: (*f* ⟶ *f b*) (*at-left b*)
⟨*proof*⟩

lemma *continuous-on-discrete* [*simp*]:
continuous-on A (*f* :: *'a* :: *discrete-topology* ⇒ -)
⟨*proof*⟩

lemma *continuous-on-of-nat* [*continuous-intros*]:
assumes *continuous-on A f*
shows *continuous-on A* (λ*n*. *of-nat* (*f n*))
⟨*proof*⟩

lemma *continuous-on-of-int* [*continuous-intros*]:
assumes *continuous-on A f*
shows *continuous-on A* (λ*n*. *of-int* (*f n*))
⟨*proof*⟩

99.10.2 Continuity at a point

definition *continuous* :: *'a*::*t2-space filter* ⇒ (*'a* ⇒ *'b*::*topological-space*) ⇒ *bool*
where *continuous F f* ⟷ (*f* ⟶ *f* (*Lim F* (λ*x*. *x*))) *F*

lemma *continuous-bot*[*continuous-intros*, *simp*]: *continuous bot f*
⟨*proof*⟩

lemma *continuous-trivial-limit*: *trivial-limit net* ⟹ *continuous net f*
⟨*proof*⟩

lemma *continuous-within*: *continuous* (*at x within s*) *f* ⟷ (*f* ⟶ *f x*) (*at x within s*)
⟨*proof*⟩

lemma *continuous-within-topological*:
continuous (*at x within s*) *f* ⟷
(∀ *B*. *open B* ⟶ *f x* ∈ *B* ⟶ (∃ *A*. *open A* ∧ *x* ∈ *A* ∧ (∀ *y*∈*s*. *y* ∈ *A* ⟶ *f y* ∈ *B*)))
⟨*proof*⟩

lemma *continuous-within-compose*[*continuous-intros*]:
continuous (*at x within s*) *f* ⟹ *continuous* (*at* (*f x*) *within f* ‘ *s*) *g* ⟹

continuous (*at x within s*) (*g* ∘ *f*)
⟨*proof*⟩

lemma *continuous-within-compose2*:
continuous (*at x within s*) *f* ⟹ *continuous* (*at* (*f x*) *within f* ' *s*) *g* ⟹
continuous (*at x within s*) (λ*x*. *g* (*f x*))
⟨*proof*⟩

lemma *continuous-at*: *continuous* (*at x*) *f* ⟷ *f* −*x*→ *f x*
⟨*proof*⟩

lemma *continuous-ident*[*continuous-intros*, *simp*]: *continuous* (*at x within S*) (λ*x*. *x*)
⟨*proof*⟩

lemma *continuous-id*[*continuous-intros*, *simp*]: *continuous* (*at x within S*) *id*
⟨*proof*⟩

lemma *continuous-const*[*continuous-intros*, *simp*]: *continuous F* (λ*x*. *c*)
⟨*proof*⟩

lemma *continuous-on-eq-continuous-within*:
continuous-on s f ⟷ (∀ *x*∈*s*. *continuous* (*at x within s*) *f*)
⟨*proof*⟩

lemma *continuous-discrete* [*simp*]:
continuous (*at x within A*) (*f* :: *'a* :: *discrete-topology* ⇒ -)
⟨*proof*⟩

abbreviation *isCont* :: (*'a*::*t2-space* ⇒ *'b*::*topological-space*) ⇒ *'a* ⇒ *bool*
where *isCont f a* ≡ *continuous* (*at a*) *f*

lemma *isCont-def*: *isCont f a* ⟷ *f* −*a*→ *f a*
⟨*proof*⟩

lemma *isContD*: *isCont f x* ⟹ *f* −*x*→ *f x*
⟨*proof*⟩

lemma *isCont-cong*:
assumes *eventually* (λ*x*. *f x* = *g x*) (*nhds x*)
shows *isCont f x* ⟷ *isCont g x*
⟨*proof*⟩

lemma *continuous-at-imp-continuous-at-within*: *isCont f x* ⟹ *continuous* (*at x within s*) *f*
⟨*proof*⟩

lemma *continuous-on-eq-continuous-at*: *open s* ⟹ *continuous-on s f* ⟷ (∀ *x*∈*s*. *isCont f x*)

⟨*proof*⟩

lemma *continuous-within-open*: $a \in A \Longrightarrow$ *open* $A \Longrightarrow$ *continuous* (*at a within A*) $f \longleftrightarrow$ *isCont f a*
⟨*proof*⟩

lemma *continuous-at-imp-continuous-on*: $\forall x{\in}s.$ *isCont f x* $\Longrightarrow$ *continuous-on s f*
⟨*proof*⟩

lemma *isCont-o2*: *isCont f a* $\Longrightarrow$ *isCont g* (*f a*) $\Longrightarrow$ *isCont* ($\lambda x.$ *g* (*f x*)) *a*
⟨*proof*⟩

lemma *continuous-at-compose*[*continuous-intros*]: *isCont f a* $\Longrightarrow$ *isCont g* (*f a*) $\Longrightarrow$ *isCont* (*g* ∘ *f*) *a*
⟨*proof*⟩

lemma *isCont-tendsto-compose*: *isCont g l* $\Longrightarrow$ (*f* ⟶ *l*) *F* $\Longrightarrow$ ((λ*x*. *g* (*f x*)) ⟶ *g l*) *F*
⟨*proof*⟩

lemma *continuous-on-tendsto-compose*:
assumes *f-cont*: *continuous-on s f*
and *g*: (*g* ⟶ *l*) *F*
and *l*: *l* ∈ *s*
and *ev*: $\forall_F x$ *in F*. *g x* ∈ *s*
shows ((λ*x*. *f* (*g x*)) ⟶ *f l*) *F*
⟨*proof*⟩

lemma *continuous-within-compose3*:
isCont g (*f x*) $\Longrightarrow$ *continuous* (*at x within s*) *f* $\Longrightarrow$ *continuous* (*at x within s*) (λ*x*. *g* (*f x*))
⟨*proof*⟩

lemma *at-within-isCont-imp-nhds*:
fixes *f*:: *'a*:: {*t2-space,perfect-space*} ⇒ *'b*:: *t2-space*
assumes $\forall_F$ *w in at z*. *f w* = *g w isCont f z isCont g z*
shows $\forall_F$ *w in nhds z*. *f w* = *g w*
⟨*proof*⟩

lemma *filtermap-nhds-open-map'*:
assumes *cont*: *isCont f a*
and *open A a* ∈ *A*
and *open-map*: $\bigwedge S.$ *open S* $\Longrightarrow$ *S* ⊆ *A* $\Longrightarrow$ *open* (*f* ‘ *S*)
shows *filtermap f* (*nhds a*) = *nhds* (*f a*)
⟨*proof*⟩

lemma *filtermap-nhds-open-map*:
assumes *cont*: *isCont f a*
and *open-map*: $\bigwedge S.$ *open S* $\Longrightarrow$ *open* (*f*‘*S*)

shows *filtermap f* (*nhds a*) = *nhds* (*f a*)
⟨*proof*⟩

lemma *continuous-at-split*:
continuous (*at x*) *f* ⟷ *continuous* (*at-left x*) *f* ∧ *continuous* (*at-right x*) *f*
for *x* :: *'a*::*linorder-topology*
⟨*proof*⟩

lemma *continuous-on-max* [*continuous-intros*]:
fixes *f g* :: *'a*::*topological-space* ⇒ *'b*::*linorder-topology*
shows *continuous-on A f* ⟹ *continuous-on A g* ⟹ *continuous-on A* (λ*x*. *max* (*f x*) (*g x*))
⟨*proof*⟩

lemma *continuous-on-min* [*continuous-intros*]:
fixes *f g* :: *'a*::*topological-space* ⇒ *'b*::*linorder-topology*
shows *continuous-on A f* ⟹ *continuous-on A g* ⟹ *continuous-on A* (λ*x*. *min* (*f x*) (*g x*))
⟨*proof*⟩

lemma *continuous-max* [*continuous-intros*]:
fixes *f* :: *'a*::*t2-space* ⇒ *'b*::*linorder-topology*
shows ⟦*continuous F f*; *continuous F g*⟧ ⟹ *continuous F* (λ*x*. (*max* (*f x*) (*g x*)))
⟨*proof*⟩

lemma *continuous-min* [*continuous-intros*]:
fixes *f* :: *'a*::*t2-space* ⇒ *'b*::*linorder-topology*
shows ⟦*continuous F f*; *continuous F g*⟧ ⟹ *continuous F* (λ*x*. (*min* (*f x*) (*g x*)))
⟨*proof*⟩

The following open/closed Collect lemmas are ported from Sébastien Gouëzel's *Ergodic-Theory*.

lemma *open-Collect-neq*:
fixes *f g* :: *'a*::*topological-space* ⇒ *'b*::*t2-space*
assumes *f*: *continuous-on UNIV f* **and** *g*: *continuous-on UNIV g*
shows *open* {*x*. *f x* ≠ *g x*}
⟨*proof*⟩

lemma *closed-Collect-eq*:
fixes *f g* :: *'a*::*topological-space* ⇒ *'b*::*t2-space*
assumes *f*: *continuous-on UNIV f* **and** *g*: *continuous-on UNIV g*
shows *closed* {*x*. *f x* = *g x*}
⟨*proof*⟩

lemma *open-Collect-less*:
fixes *f g* :: *'a*::*topological-space* ⇒ *'b*::*linorder-topology*
assumes *f*: *continuous-on UNIV f* **and** *g*: *continuous-on UNIV g*
shows *open* {*x*. *f x* < *g x*}

⟨*proof*⟩

lemma *closed-Collect-le*:
fixes $f\ g :: 'a :: topological\text{-}space \Rightarrow 'b::linorder\text{-}topology$
assumes f: *continuous-on UNIV f*
and g: *continuous-on UNIV g*
shows *closed* $\{x.\ f\ x \leq g\ x\}$
⟨*proof*⟩

99.10.3 Open-cover compactness

context *topological-space*
begin

definition *compact* :: $'a\ set \Rightarrow bool$ **where**
compact-eq-Heine-Borel:
$compact\ S \longleftrightarrow (\forall C.\ (\forall c{\in}C.\ open\ c) \wedge S \subseteq \bigcup C \longrightarrow (\exists D{\subseteq}C.\ finite\ D \wedge S \subseteq \bigcup D))$

lemma *compactI*:
assumes $\bigwedge C.\ \forall t{\in}C.\ open\ t \Longrightarrow s \subseteq \bigcup C \Longrightarrow \exists C'.\ C' \subseteq C \wedge finite\ C' \wedge s \subseteq \bigcup C'$
shows *compact s*
⟨*proof*⟩

lemma *compact-empty*[*simp*]: *compact* {}
⟨*proof*⟩

lemma *compactE*:
assumes *compact S* $S \subseteq \bigcup \mathcal{T}$ $\bigwedge B.\ B \in \mathcal{T} \Longrightarrow open\ B$
obtains $\mathcal{T}'$ **where** $\mathcal{T}' \subseteq \mathcal{T}$ *finite* $\mathcal{T}'$ $S \subseteq \bigcup \mathcal{T}'$
⟨*proof*⟩

lemma *compactE-image*:
assumes *compact S*
and *opn*: $\bigwedge T.\ T \in C \Longrightarrow open\ (f\ T)$
and S: $S \subseteq (\bigcup c{\in}C.\ f\ c)$
obtains C' **where** $C' \subseteq C$ **and** *finite* C' **and** $S \subseteq (\bigcup c{\in}C'.\ f\ c)$
⟨*proof*⟩

lemma *compact-Int-closed* [*intro*]:
assumes *compact S*
and *closed T*
shows *compact* $(S \cap T)$
⟨*proof*⟩

lemma *compact-diff*: $[\![compact\ S;\ open\ T]\!] \Longrightarrow compact(S - T)$
⟨*proof*⟩

lemma *inj-setminus*: *inj-on uminus* (*A*::*'a set set*)
⟨*proof*⟩

99.11 Finite intersection property

lemma *compact-fip*:
compact U ⟷
(∀ *A*. (∀ *a*∈*A*. *closed a*) ⟶ (∀ *B* ⊆ *A*. *finite B* ⟶ *U* ∩ ⋂ *B* ≠ {}) ⟶ *U* ∩ ⋂ *A* ≠ {})
(**is** - ⟷ *?R*)
⟨*proof*⟩

lemma *compact-imp-fip*:
assumes *compact S*
and ⋀*T*. *T* ∈ *F* ⟹ *closed T*
and ⋀*F'*. *finite F'* ⟹ *F'* ⊆ *F* ⟹ *S* ∩ (⋂ *F'*) ≠ {}
shows *S* ∩ (⋂ *F*) ≠ {}
⟨*proof*⟩

lemma *compact-imp-fip-image*:
assumes *compact s*
and *P*: ⋀*i*. *i* ∈ *I* ⟹ *closed* (*f i*)
and *Q*: ⋀*I'*. *finite I'* ⟹ *I'* ⊆ *I* ⟹ (*s* ∩ (⋂ *i*∈*I'*. *f i*) ≠ {})
shows *s* ∩ (⋂ *i*∈*I*. *f i*) ≠ {}
⟨*proof*⟩

end

lemma (**in** *t2-space*) *compact-imp-closed*:
assumes *compact s*
shows *closed s*
⟨*proof*⟩

lemma *compact-continuous-image*:
assumes *f*: *continuous-on s f*
and *s*: *compact s*
shows *compact* (*f* ' *s*)
⟨*proof*⟩

lemma *continuous-on-inv*:
fixes *f* :: *'a*::*topological-space* ⇒ *'b*::*t2-space*
assumes *continuous-on s f*
and *compact s*
and ∀ *x*∈*s*. *g* (*f x*) = *x*
shows *continuous-on* (*f* ' *s*) *g*
⟨*proof*⟩

lemma *continuous-on-inv-into*:
fixes *f* :: *'a*::*topological-space* ⇒ *'b*::*t2-space*

assumes *s*: *continuous-on s f compact s*
and *f*: *inj-on f s*
shows *continuous-on* (*f ' s*) (*the-inv-into s f*)
⟨*proof*⟩

lemma (**in** *linorder-topology*) *compact-attains-sup*:
assumes *compact S S* ≠ {}
shows ∃ *s*∈*S*. ∀ *t*∈*S*. *t* ≤ *s*
⟨*proof*⟩

lemma (**in** *linorder-topology*) *compact-attains-inf*:
assumes *compact S S* ≠ {}
shows ∃ *s*∈*S*. ∀ *t*∈*S*. *s* ≤ *t*
⟨*proof*⟩

lemma *continuous-attains-sup*:
fixes *f* :: *'a*::*topological-space* ⇒ *'b*::*linorder-topology*
shows *compact s* ⟹ *s* ≠ {} ⟹ *continuous-on s f* ⟹ (∃ *x*∈*s*. ∀ *y*∈*s*. *f y* ≤ *f x*)
⟨*proof*⟩

lemma *continuous-attains-inf*:
fixes *f* :: *'a*::*topological-space* ⇒ *'b*::*linorder-topology*
shows *compact s* ⟹ *s* ≠ {} ⟹ *continuous-on s f* ⟹ (∃ *x*∈*s*. ∀ *y*∈*s*. *f x* ≤ *f y*)
⟨*proof*⟩

99.12 Connectedness

context *topological-space*
begin

definition *connected S* ⟷
¬ (∃ *A B*. *open A* ∧ *open B* ∧ *S* ⊆ *A* ∪ *B* ∧ *A* ∩ *B* ∩ *S* = {} ∧ *A* ∩ *S* ≠ {} ∧ *B* ∩ *S* ≠ {})

lemma *connectedI*:
(⋀*A B*. *open A* ⟹ *open B* ⟹ *A* ∩ *U* ≠ {} ⟹ *B* ∩ *U* ≠ {} ⟹ *A* ∩ *B* ∩ *U* = {} ⟹ *U* ⊆ *A* ∪ *B* ⟹ *False*)
⟹ *connected U*
⟨*proof*⟩

lemma *connected-empty* [*simp*]: *connected* {}
⟨*proof*⟩

lemma *connected-sing* [*simp*]: *connected* {*x*}
⟨*proof*⟩

lemma *connectedD*:
connected A ⟹ *open U* ⟹ *open V* ⟹ *U* ∩ *V* ∩ *A* = {} ⟹ *A* ⊆ *U* ∪ *V*

$\Longrightarrow U \cap A = \{\} \vee V \cap A = \{\}$
⟨*proof*⟩

end

lemma *connected-closed*:
$\mathit{connected}\ s \longleftrightarrow$
$\neg\ (\exists A\ B.\ \mathit{closed}\ A \wedge \mathit{closed}\ B \wedge s \subseteq A \cup B \wedge A \cap B \cap s = \{\} \wedge A \cap s \neq \{\} \wedge B \cap s \neq \{\})$
⟨*proof*⟩

lemma *connected-closedD*:
$[\![\mathit{connected}\ s;\ A \cap B \cap s = \{\};\ s \subseteq A \cup B;\ \mathit{closed}\ A;\ \mathit{closed}\ B]\!] \Longrightarrow A \cap s = \{\} \vee B \cap s = \{\}$
⟨*proof*⟩

lemma *connected-Union*:
assumes *cs*: $\bigwedge s.\ s \in S \Longrightarrow \mathit{connected}\ s$
and *ne*: $\bigcap S \neq \{\}$
shows $\mathit{connected}(\bigcup S)$
⟨*proof*⟩

lemma *connected-Un*: $\mathit{connected}\ s \Longrightarrow \mathit{connected}\ t \Longrightarrow s \cap t \neq \{\} \Longrightarrow \mathit{connected}\ (s \cup t)$
⟨*proof*⟩

lemma *connected-diff-open-from-closed*:
assumes *st*: $s \subseteq t$
and *tu*: $t \subseteq u$
and *s*: *open s*
and *t*: *closed t*
and *u*: *connected u*
and *ts*: $\mathit{connected}\ (t - s)$
shows $\mathit{connected}(u - s)$
⟨*proof*⟩

lemma *connected-iff-const*:
fixes S :: *'a::topological-space set*
shows $\mathit{connected}\ S \longleftrightarrow (\forall P{::}'a \Rightarrow \mathit{bool}.\ \mathit{continuous\text{-}on}\ S\ P \longrightarrow (\exists c.\ \forall s{\in}S.\ P\ s = c))$
⟨*proof*⟩

lemma *connectedD-const*: $\mathit{connected}\ S \Longrightarrow \mathit{continuous\text{-}on}\ S\ P \Longrightarrow \exists c.\ \forall s{\in}S.\ P\ s = c$
for P :: *'a::topological-space* $\Rightarrow$ *bool*
⟨*proof*⟩

lemma *connectedI-const*:
$(\bigwedge P{::}'a{::}\mathit{topological\text{-}space} \Rightarrow \mathit{bool}.\ \mathit{continuous\text{-}on}\ S\ P \Longrightarrow \exists c.\ \forall s{\in}S.\ P\ s = c)$

$\Longrightarrow$ *connected S*
⟨*proof*⟩

lemma *connected-local-const*:
assumes *connected A a* ∈ *A b* ∈ *A*
and ∗: ∀ *a*∈*A*. *eventually* (λ*b*. *f a* = *f b*) (*at a within A*)
shows *f a* = *f b*
⟨*proof*⟩

lemma (**in** *linorder-topology*) *connectedD-interval*:
assumes *connected U*
and *xy*: *x* ∈ *U y* ∈ *U*
and *x* ≤ *z z* ≤ *y*
shows *z* ∈ *U*
⟨*proof*⟩

lemma (**in** *linorder-topology*) *not-in-connected-cases*:
assumes *conn*: *connected S*
assumes *nbdd*: *x* ∉ *S*
assumes *ne*: *S* ≠ {}
obtains *bdd-above S* ⋀*y*. *y* ∈ *S* $\Longrightarrow$ *x* ≥ *y* | *bdd-below S* ⋀*y*. *y* ∈ *S* $\Longrightarrow$ *x* ≤ *y*
⟨*proof*⟩

lemma *connected-continuous-image*:
assumes ∗: *continuous-on s f*
and *connected s*
shows *connected* (*f* ‘ *s*)
⟨*proof*⟩

lemma *connected-Un-UN*:
assumes *connected A* ⋀*X*. *X* ∈ *B* $\Longrightarrow$ *connected X* ⋀*X*. *X* ∈ *B* $\Longrightarrow$ *A* ∩ *X* ≠ {}
shows *connected* (*A* ∪ ⋃ *B*)
⟨*proof*⟩

100 Linear Continuum Topologies

class *linear-continuum-topology* = *linorder-topology* + *linear-continuum*
begin

lemma *Inf-notin-open*:
assumes *A*: *open A*
and *bnd*: ∀ *a*∈*A*. *x* < *a*
shows *Inf A* ∉ *A*
⟨*proof*⟩

lemma *Sup-notin-open*:
assumes *A*: *open A*
and *bnd*: ∀ *a*∈*A*. *a* < *x*

shows *Sup A* $\notin$ *A*
⟨*proof*⟩

end

instance *linear-continuum-topology* $\subseteq$ *perfect-space*
⟨*proof*⟩

lemma *connectedI-interval*:
fixes *U* :: *'a* :: *linear-continuum-topology set*
assumes $*$: $\bigwedge x\ y\ z.\ x \in U \Longrightarrow y \in U \Longrightarrow x \leq z \Longrightarrow z \leq y \Longrightarrow z \in U$
shows *connected U*
⟨*proof*⟩

lemma *connected-iff-interval*: *connected U* $\longleftrightarrow$ ($\forall x{\in}U.\ \forall y{\in}U.\ \forall z.\ x \leq z \longrightarrow z \leq y \longrightarrow z \in U$)
for *U* :: *'a*::*linear-continuum-topology set*
⟨*proof*⟩

lemma *connected-UNIV*[*simp*]: *connected* (*UNIV*::*'a*::*linear-continuum-topology set*)
⟨*proof*⟩

lemma *connected-Ioi*[*simp*]: *connected* {*a*<..}
for *a* :: *'a*::*linear-continuum-topology*
⟨*proof*⟩

lemma *connected-Ici*[*simp*]: *connected* {*a*..}
for *a* :: *'a*::*linear-continuum-topology*
⟨*proof*⟩

lemma *connected-Iio*[*simp*]: *connected* {..<*a*}
for *a* :: *'a*::*linear-continuum-topology*
⟨*proof*⟩

lemma *connected-Iic*[*simp*]: *connected* {..*a*}
for *a* :: *'a*::*linear-continuum-topology*
⟨*proof*⟩

lemma *connected-Ioo*[*simp*]: *connected* {*a*<..<*b*}
for *a b* :: *'a*::*linear-continuum-topology*
⟨*proof*⟩

lemma *connected-Ioc*[*simp*]: *connected* {*a*<..*b*}
for *a b* :: *'a*::*linear-continuum-topology*
⟨*proof*⟩

lemma *connected-Ico*[*simp*]: *connected* {*a*..<*b*}
for *a b* :: *'a*::*linear-continuum-topology*
⟨*proof*⟩

lemma *connected-Icc*[*simp*]: *connected* $\{a..b\}$
for a b :: *'a*::*linear-continuum-topology*
⟨*proof*⟩

lemma *connected-contains-Ioo*:
fixes A :: *'a* :: *linorder-topology set*
assumes *connected* A $a \in A$ $b \in A$ **shows** $\{a <..< b\} \subseteq A$
⟨*proof*⟩

lemma *connected-contains-Icc*:
fixes A :: *'a*::*linorder-topology set*
assumes *connected* A $a \in A$ $b \in A$
shows $\{a..b\} \subseteq A$
⟨*proof*⟩

100.1 Intermediate Value Theorem

lemma *IVT'*:
fixes f :: *'a*::*linear-continuum-topology* ⇒ *'b*::*linorder-topology*
assumes y: $f\ a \leq y$ $y \leq f\ b$ $a \leq b$
and ∗: *continuous-on* $\{a\ ..\ b\}$ f
shows $\exists x.\ a \leq x \wedge x \leq b \wedge f\ x = y$
⟨*proof*⟩

lemma *IVT2'*:
fixes f :: *'a* :: *linear-continuum-topology* ⇒ *'b* :: *linorder-topology*
assumes y: $f\ b \leq y$ $y \leq f\ a$ $a \leq b$
and ∗: *continuous-on* $\{a\ ..\ b\}$ f
shows $\exists x.\ a \leq x \wedge x \leq b \wedge f\ x = y$
⟨*proof*⟩

lemma *IVT*:
fixes f :: *'a*::*linear-continuum-topology* ⇒ *'b*::*linorder-topology*
shows $f\ a \leq y \Longrightarrow y \leq f\ b \Longrightarrow a \leq b \Longrightarrow (\forall x.\ a \leq x \wedge x \leq b \longrightarrow \mathit{isCont}\ f\ x)$
$\Longrightarrow$
$\exists x.\ a \leq x \wedge x \leq b \wedge f\ x = y$
⟨*proof*⟩

lemma *IVT2*:
fixes f :: *'a*::*linear-continuum-topology* ⇒ *'b*::*linorder-topology*
shows $f\ b \leq y \Longrightarrow y \leq f\ a \Longrightarrow a \leq b \Longrightarrow (\forall x.\ a \leq x \wedge x \leq b \longrightarrow \mathit{isCont}\ f\ x)$
$\Longrightarrow$
$\exists x.\ a \leq x \wedge x \leq b \wedge f\ x = y$
⟨*proof*⟩

lemma *continuous-inj-imp-mono*:
fixes f :: *'a*::*linear-continuum-topology* ⇒ *'b*::*linorder-topology*
assumes x: $a < x$ $x < b$

and *cont*: *continuous-on* $\{a..b\}$ f
and *inj*: *inj-on* f $\{a..b\}$
shows $(f\ a < f\ x \wedge f\ x < f\ b) \vee (f\ b < f\ x \wedge f\ x < f\ a)$
⟨*proof*⟩

lemma *continuous-at-Sup-mono*:
fixes f :: $'a$::{*linorder-topology*,*conditionally-complete-linorder*} ⇒
$'b$::{*linorder-topology*,*conditionally-complete-linorder*}
assumes *mono f*
and *cont*: *continuous* (*at-left* (*Sup S*)) *f*
and *S*: $S \neq \{\}$ *bdd-above S*
shows *f* (*Sup S*) = (*SUP s*∈*S*. *f s*)
⟨*proof*⟩

lemma *continuous-at-Sup-antimono*:
fixes f :: $'a$::{*linorder-topology*,*conditionally-complete-linorder*} ⇒
$'b$::{*linorder-topology*,*conditionally-complete-linorder*}
assumes *antimono f*
and *cont*: *continuous* (*at-left* (*Sup S*)) *f*
and *S*: $S \neq \{\}$ *bdd-above S*
shows *f* (*Sup S*) = (*INF s*∈*S*. *f s*)
⟨*proof*⟩

lemma *continuous-at-Inf-mono*:
fixes f :: $'a$::{*linorder-topology*,*conditionally-complete-linorder*} ⇒
$'b$::{*linorder-topology*,*conditionally-complete-linorder*}
assumes *mono f*
and *cont*: *continuous* (*at-right* (*Inf S*)) *f*
and *S*: $S \neq \{\}$ *bdd-below S*
shows *f* (*Inf S*) = (*INF s*∈*S*. *f s*)
⟨*proof*⟩

lemma *continuous-at-Inf-antimono*:
fixes f :: $'a$::{*linorder-topology*,*conditionally-complete-linorder*} ⇒
$'b$::{*linorder-topology*,*conditionally-complete-linorder*}
assumes *antimono f*
and *cont*: *continuous* (*at-right* (*Inf S*)) *f*
and *S*: $S \neq \{\}$ *bdd-below S*
shows *f* (*Inf S*) = (*SUP s*∈*S*. *f s*)
⟨*proof*⟩

100.2 Uniform spaces

class *uniformity* =
fixes *uniformity* :: $('a \times 'a)$ *filter*
begin

abbreviation *uniformity-on* :: $'a$ *set* ⇒ $('a \times 'a)$ *filter*
where *uniformity-on s* ≡ *inf uniformity* (*principal* (*s*×*s*))

end

lemma *uniformity-Abort*:
uniformity =
Filter.abstract-filter (λ*u*. *Code.abort* (*STR* ''*uniformity is not executable*'') (λ*u*. *uniformity*))
⟨*proof*⟩

class *open-uniformity* = *open* + *uniformity* +
assumes *open-uniformity*:
⋀*U*. *open U* ⟷ (∀ *x*∈*U*. *eventually* (λ(*x*′, *y*). *x*′ = *x* ⟶ *y* ∈ *U*) *uniformity*)
begin

subclass *topological-space*
⟨*proof*⟩

end

class *uniform-space* = *open-uniformity* +
assumes *uniformity-refl*: *eventually E uniformity* ⟹ *E* (*x*, *x*)
and *uniformity-sym*: *eventually E uniformity* ⟹ *eventually* (λ(*x*, *y*). *E* (*y*, *x*)) *uniformity*
and *uniformity-trans*:
eventually E uniformity ⟹
∃ *D*. *eventually D uniformity* ∧ (∀ *x y z*. *D* (*x*, *y*) ⟶ *D* (*y*, *z*) ⟶ *E* (*x*, *z*))
begin

lemma *uniformity-bot*: *uniformity* ≠ *bot*
⟨*proof*⟩

lemma *uniformity-trans*′:
eventually E uniformity ⟹
eventually (λ((*x*, *y*), (*y*′, *z*)). *y* = *y*′ ⟶ *E* (*x*, *z*)) (*uniformity* ×$_F$ *uniformity*)
⟨*proof*⟩

lemma *uniformity-transE*:
assumes *eventually E uniformity*
obtains *D* **where** *eventually D uniformity* ⋀*x y z*. *D* (*x*, *y*) ⟹ *D* (*y*, *z*) ⟹ *E* (*x*, *z*)
⟨*proof*⟩

lemma *eventually-nhds-uniformity*:
eventually P (*nhds x*) ⟷ *eventually* (λ(*x*′, *y*). *x*′ = *x* ⟶ *P y*) *uniformity*
(**is** - ⟷ *?N P x*)
⟨*proof*⟩

100.2.1 Totally bounded sets

definition *totally-bounded* :: *'a set ⇒ bool*
where *totally-bounded S ⟷*
(∀ *E*. *eventually E uniformity ⟶* (∃ *X*. *finite X* ∧ (∀ *s*∈*S*. ∃ *x*∈*X*. *E* (*x*, *s*))))

lemma *totally-bounded-empty*[*iff*]: *totally-bounded* {}
⟨*proof*⟩

lemma *totally-bounded-subset*: *totally-bounded S ⟹ T* ⊆ *S ⟹ totally-bounded T*
⟨*proof*⟩

lemma *totally-bounded-Union*[*intro*]:
assumes *M*: *finite M* ⋀*S*. *S* ∈ *M ⟹ totally-bounded S*
shows *totally-bounded* (⋃ *M*)
⟨*proof*⟩

100.2.2 Cauchy filter

definition *cauchy-filter* :: *'a filter ⇒ bool*
where *cauchy-filter F ⟷ F* ×$_F$ *F* ≤ *uniformity*

definition *Cauchy* :: (*nat ⇒ 'a*) *⇒ bool*
where *Cauchy-uniform*: *Cauchy X* = *cauchy-filter* (*filtermap X sequentially*)

lemma *Cauchy-uniform-iff*:
Cauchy X ⟷ (∀ *P*. *eventually P uniformity ⟶* (∃ *N*. ∀ *n*≥*N*. ∀ *m*≥*N*. *P* (*X n*, *X m*)))
⟨*proof*⟩

lemma *nhds-imp-cauchy-filter*:
assumes ∗: *F* ≤ *nhds x*
shows *cauchy-filter F*
⟨*proof*⟩

lemma *LIMSEQ-imp-Cauchy*: *X* ⟶ *x ⟹ Cauchy X*
⟨*proof*⟩

lemma *Cauchy-subseq-Cauchy*:
assumes *Cauchy X strict-mono f*
shows *Cauchy* (*X* ∘ *f*)
⟨*proof*⟩

lemma *convergent-Cauchy*: *convergent X ⟹ Cauchy X*
⟨*proof*⟩

definition *complete* :: *'a set ⇒ bool*
where *complete-uniform*: *complete S ⟷*
(∀ *F* ≤ *principal S*. *F* ≠ *bot ⟶ cauchy-filter F ⟶* (∃ *x*∈*S*. *F* ≤ *nhds x*))

lemma (**in** *uniform-space*) *cauchy-filter-complete-converges*:
assumes *cauchy-filter F complete A F* $\leq$ *principal A F* $\neq$ *bot*
shows $\exists\, c.\ F \leq nhds\ c$
⟨*proof*⟩

end

100.2.3 Uniformly continuous functions

definition *uniformly-continuous-on* :: *'a set* ⇒ (*'a*::*uniform-space* ⇒ *'b*::*uniform-space*) ⇒ *bool*
where *uniformly-continuous-on-uniformity*: *uniformly-continuous-on s f* ⟷
(*LIM* (*x*, *y*) (*uniformity-on s*). (*f x*, *f y*) :> *uniformity*)

lemma *uniformly-continuous-onD*:
uniformly-continuous-on s f ⟹ *eventually E uniformity* ⟹
eventually (λ(*x*, *y*). *x* ∈ *s* ⟶ *y* ∈ *s* ⟶ *E* (*f x*, *f y*)) *uniformity*
⟨*proof*⟩

lemma *uniformly-continuous-on-const*[*continuous-intros*]: *uniformly-continuous-on s* (λ*x*. *c*)
⟨*proof*⟩

lemma *uniformly-continuous-on-id*[*continuous-intros*]: *uniformly-continuous-on s* (λ*x*. *x*)
⟨*proof*⟩

lemma *uniformly-continuous-on-compose*:
uniformly-continuous-on s g ⟹ *uniformly-continuous-on* (*g*‘*s*) *f* ⟹
uniformly-continuous-on s (λ*x*. *f* (*g x*))
⟨*proof*⟩

lemma *uniformly-continuous-imp-continuous*:
assumes *f*: *uniformly-continuous-on s f*
shows *continuous-on s f*
⟨*proof*⟩

101 Product Topology

101.1 Product is a topological space

instantiation *prod* :: (*topological-space*, *topological-space*) *topological-space*
begin

definition *open-prod-def*[*code del*]:
open (*S* :: (*'a* × *'b*) *set*) ⟷
(∀ *x*∈*S*. ∃ *A B*. *open A* ∧ *open B* ∧ *x* ∈ *A* × *B* ∧ *A* × *B* ⊆ *S*)

lemma *open-prod-elim*:
assumes *open S* **and** $x \in S$
obtains *A B* **where** *open A* **and** *open B* **and** $x \in A \times B$ **and** $A \times B \subseteq S$
⟨*proof*⟩

lemma *open-prod-intro*:
assumes $\bigwedge x.\ x \in S \Longrightarrow \exists\, A\ B.\ open\ A \wedge open\ B \wedge x \in A \times B \wedge A \times B \subseteq S$
shows *open S*
⟨*proof*⟩

instance
⟨*proof*⟩

end

declare [[*code abort*: *open* :: ('*a*::*topological-space* × '*b*::*topological-space*) *set* ⇒ *bool*]]

lemma *open-Times*: *open S* ⟹ *open T* ⟹ *open* (*S* × *T*)
⟨*proof*⟩

lemma *fst-vimage-eq-Times*: *fst* −‘ *S* = *S* × *UNIV*
⟨*proof*⟩

lemma *snd-vimage-eq-Times*: *snd* −‘ *S* = *UNIV* × *S*
⟨*proof*⟩

lemma *open-vimage-fst*: *open S* ⟹ *open* (*fst* −‘ *S*)
⟨*proof*⟩

lemma *open-vimage-snd*: *open S* ⟹ *open* (*snd* −‘ *S*)
⟨*proof*⟩

lemma *closed-vimage-fst*: *closed S* ⟹ *closed* (*fst* −‘ *S*)
⟨*proof*⟩

lemma *closed-vimage-snd*: *closed S* ⟹ *closed* (*snd* −‘ *S*)
⟨*proof*⟩

lemma *closed-Times*: *closed S* ⟹ *closed T* ⟹ *closed* (*S* × *T*)
⟨*proof*⟩

lemma *subset-fst-imageI*: *A* × *B* ⊆ *S* ⟹ *y* ∈ *B* ⟹ *A* ⊆ *fst* ‘ *S*
⟨*proof*⟩

lemma *subset-snd-imageI*: *A* × *B* ⊆ *S* ⟹ *x* ∈ *A* ⟹ *B* ⊆ *snd* ‘ *S*
⟨*proof*⟩

lemma *open-image-fst*:

assumes *open S*
shows *open* (*fst* ' *S*)
⟨*proof*⟩

lemma *open-image-snd*:
assumes *open S*
shows *open* (*snd* ' *S*)
⟨*proof*⟩

lemma *nhds-prod*: *nhds* (*a*, *b*) = *nhds a* ×$_F$ *nhds b*
⟨*proof*⟩

101.1.1 Continuity of operations

lemma *tendsto-fst* [*tendsto-intros*]:
assumes (*f* ⟶ *a*) *F*
shows ((λ*x*. *fst* (*f x*)) ⟶ *fst a*) *F*
⟨*proof*⟩

lemma *tendsto-snd* [*tendsto-intros*]:
assumes (*f* ⟶ *a*) *F*
shows ((λ*x*. *snd* (*f x*)) ⟶ *snd a*) *F*
⟨*proof*⟩

lemma *tendsto-Pair* [*tendsto-intros*]:
assumes (*f* ⟶ *a*) *F* **and** (*g* ⟶ *b*) *F*
shows ((λ*x*. (*f x*, *g x*)) ⟶ (*a*, *b*)) *F*
⟨*proof*⟩

lemma *continuous-fst*[*continuous-intros*]: *continuous F f* ⟹ *continuous F* (λ*x*. *fst* (*f x*))
⟨*proof*⟩

lemma *continuous-snd*[*continuous-intros*]: *continuous F f* ⟹ *continuous F* (λ*x*. *snd* (*f x*))
⟨*proof*⟩

lemma *continuous-Pair*[*continuous-intros*]:
continuous F f ⟹ *continuous F g* ⟹ *continuous F* (λ*x*. (*f x*, *g x*))
⟨*proof*⟩

lemma *continuous-on-fst*[*continuous-intros*]:
continuous-on s f ⟹ *continuous-on s* (λ*x*. *fst* (*f x*))
⟨*proof*⟩

lemma *continuous-on-snd*[*continuous-intros*]:
continuous-on s f ⟹ *continuous-on s* (λ*x*. *snd* (*f x*))
⟨*proof*⟩

lemma *continuous-on-Pair*[*continuous-intros*]:
continuous-on s f ⟹ *continuous-on s g* ⟹ *continuous-on s* (λ*x*. (*f x*, *g x*))
⟨*proof*⟩

lemma *continuous-on-swap*[*continuous-intros*]: *continuous-on A prod.swap*
⟨*proof*⟩

lemma *continuous-on-swap-args*:
assumes *continuous-on* (*A*×*B*) (λ(*x*,*y*). *d x y*)
shows *continuous-on* (*B*×*A*) (λ(*x*,*y*). *d y x*)
⟨*proof*⟩

lemma *isCont-fst* [*simp*]: *isCont f a* ⟹ *isCont* (λ*x*. *fst* (*f x*)) *a*
⟨*proof*⟩

lemma *isCont-snd* [*simp*]: *isCont f a* ⟹ *isCont* (λ*x*. *snd* (*f x*)) *a*
⟨*proof*⟩

lemma *isCont-Pair* [*simp*]: ⟦*isCont f a*; *isCont g a*⟧ ⟹ *isCont* (λ*x*. (*f x*, *g x*)) *a*
⟨*proof*⟩

lemma *continuous-on-compose-Pair*:
assumes *f*: *continuous-on* (*Sigma A B*) (λ(*a*, *b*). *f a b*)
assumes *g*: *continuous-on C g*
assumes *h*: *continuous-on C h*
assumes *subset*: ⋀*c*. *c* ∈ *C* ⟹ *g c* ∈ *A* ⋀*c*. *c* ∈ *C* ⟹ *h c* ∈ *B* (*g c*)
shows *continuous-on C* (λ*c*. *f* (*g c*) (*h c*))
⟨*proof*⟩

101.1.2 Connectedness of products

proposition *connected-Times*:
assumes *S*: *connected S* **and** *T*: *connected T*
shows *connected* (*S* × *T*)
⟨*proof*⟩

corollary *connected-Times-eq* [*simp*]:
connected (*S* × *T*) ⟷ *S* = {} ∨ *T* = {} ∨ *connected S* ∧ *connected T* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

101.1.3 Separation axioms

instance *prod* :: (*t0-space*, *t0-space*) *t0-space*
⟨*proof*⟩

instance *prod* :: (*t1-space*, *t1-space*) *t1-space*
⟨*proof*⟩

instance *prod* :: (*t2-space*, *t2-space*) *t2-space*

⟨*proof*⟩

lemma *isCont-swap*[*continuous-intros*]: *isCont prod.swap a*
⟨*proof*⟩

lemma *open-diagonal-complement*:
open {(*x*,*y*) |*x y*. *x* ≠ (*y*::('*a*::*t2-space*))}
⟨*proof*⟩

lemma *closed-diagonal*:
closed {*y*. ∃ *x*::('*a*::*t2-space*). *y* = (*x*,*x*)}
⟨*proof*⟩

lemma *open-superdiagonal*:
open {(*x*,*y*) | *x y*. *x* > (*y*::'*a*::{*linorder-topology*})}
⟨*proof*⟩

lemma *closed-subdiagonal*:
closed {(*x*,*y*) | *x y*. *x* ≤ (*y*::'*a*::{*linorder-topology*})}
⟨*proof*⟩

lemma *open-subdiagonal*:
open {(*x*,*y*) | *x y*. *x* < (*y*::'*a*::{*linorder-topology*})}
⟨*proof*⟩

lemma *closed-superdiagonal*:
closed {(*x*,*y*) | *x y*. *x* ≥ (*y*::('*a*::{*linorder-topology*}))}
⟨*proof*⟩

end

theory *Hull*
imports *Main*
begin

101.2 A generic notion of the convex, affine, conic hull, or closed "hull".

definition *hull* :: ('*a set* ⇒ *bool*) ⇒ '*a set* ⇒ '*a set* (**infixl** *hull* 75)
where *S hull s* = ⋂{*t*. *S t* ∧ *s* ⊆ *t*}

lemma *hull-same*: *S s* ⟹ *S hull s* = *s*
⟨*proof*⟩

lemma *hull-in*: (⋀*T*. *Ball T S* ⟹ *S* (⋂ *T*)) ⟹ *S* (*S hull s*)
⟨*proof*⟩

lemma *hull-eq*: (⋀*T*. *Ball T S* ⟹ *S* (⋂ *T*)) ⟹ (*S hull s*) = *s* ⟷ *S s*

⟨*proof*⟩

lemma *hull-hull* [*simp*]: S *hull* (S *hull* s) = S *hull* s
⟨*proof*⟩

lemma *hull-subset*[*intro*]: $s \subseteq (S \text{ hull } s)$
⟨*proof*⟩

lemma *hull-mono*: $s \subseteq t \Longrightarrow (S \text{ hull } s) \subseteq (S \text{ hull } t)$
⟨*proof*⟩

lemma *hull-antimono*: $\forall x.\ S\ x \longrightarrow T\ x \Longrightarrow (T \text{ hull } s) \subseteq (S \text{ hull } s)$
⟨*proof*⟩

lemma *hull-minimal*: $s \subseteq t \Longrightarrow S\ t \Longrightarrow (S \text{ hull } s) \subseteq t$
⟨*proof*⟩

lemma *subset-hull*: $S\ t \Longrightarrow S \text{ hull } s \subseteq t \longleftrightarrow s \subseteq t$
⟨*proof*⟩

lemma *hull-UNIV* [*simp*]: S *hull* *UNIV* = *UNIV*
⟨*proof*⟩

lemma *hull-unique*: $s \subseteq t \Longrightarrow S\ t \Longrightarrow (\bigwedge t'.\ s \subseteq t' \Longrightarrow S\ t' \Longrightarrow t \subseteq t') \Longrightarrow (S$ *hull* $s = t)$
⟨*proof*⟩

lemma *hull-induct*: $[\![a \in Q \text{ hull } S;\ \bigwedge x.\ x \in S \Longrightarrow P\ x;\ Q\ \{x.\ P\ x\}]\!] \Longrightarrow P\ a$
⟨*proof*⟩

lemma *hull-inc*: $x \in S \Longrightarrow x \in P \text{ hull } S$
⟨*proof*⟩

lemma *hull-Un-subset*: $(S \text{ hull } s) \cup (S \text{ hull } t) \subseteq (S \text{ hull } (s \cup t))$
⟨*proof*⟩

lemma *hull-Un*:
assumes T: $\bigwedge T.$ *Ball* $T\ S \Longrightarrow S\ (\bigcap T)$
shows S *hull* $(s \cup t) = S$ *hull* $(S$ *hull* $s \cup S$ *hull* $t)$
⟨*proof*⟩

lemma *hull-Un-left*: P *hull* $(S \cup T) = P$ *hull* $(P$ *hull* $S \cup T)$
⟨*proof*⟩

lemma *hull-Un-right*: P *hull* $(S \cup T) = P$ *hull* $(S \cup P$ *hull* $T)$
⟨*proof*⟩

lemma *hull-insert*:
P *hull* (*insert* $a\ S) = P$ *hull* (*insert* a (P *hull* S))

⟨*proof*⟩

lemma *hull-redundant-eq*: $a \in (S\ hull\ s) \longleftrightarrow S\ hull\ (insert\ a\ s) = S\ hull\ s$
⟨*proof*⟩

lemma *hull-redundant*: $a \in (S\ hull\ s) \Longrightarrow S\ hull\ (insert\ a\ s) = S\ hull\ s$
⟨*proof*⟩

end

102 Modules

Bases of a linear algebra based on modules (i.e. vector spaces of rings).

theory *Modules*
imports *Hull*
begin

102.1 Locale for additive functions

locale *additive* =
fixes $f :: 'a::ab\text{-}group\text{-}add \Rightarrow 'b::ab\text{-}group\text{-}add$
assumes *add*: $f\ (x + y) = f\ x + f\ y$
begin

lemma *zero*: $f\ 0 = 0$
⟨*proof*⟩

lemma *minus*: $f\ (-\ x) = -\ f\ x$
⟨*proof*⟩

lemma *diff*: $f\ (x - y) = f\ x - f\ y$
⟨*proof*⟩

lemma *sum*: $f\ (sum\ g\ A) = (\sum x{\in}A.\ f\ (g\ x))$
⟨*proof*⟩

end

Modules form the central spaces in linear algebra. They are a generalization from vector spaces by replacing the scalar field by a scalar ring.

locale *module* =
fixes *scale* :: $'a::comm\text{-}ring\text{-}1 \Rightarrow 'b::ab\text{-}group\text{-}add \Rightarrow 'b$ (**infixr** $*s$ *75*)
assumes *scale-right-distrib* [*algebra-simps*, *algebra-split-simps*]:
$a *s (x + y) = a *s x + a *s y$
and *scale-left-distrib* [*algebra-simps*, *algebra-split-simps*]:
$(a + b) *s x = a *s x + b *s x$
and *scale-scale* [*simp*]: $a *s (b *s x) = (a * b) *s x$
and *scale-one* [*simp*]: $1 *s x = x$

begin

lemma *scale-left-commute*: $a *s (b *s x) = b *s (a *s x)$
⟨*proof*⟩

lemma *scale-zero-left* [*simp*]: $0 *s x = 0$
and *scale-minus-left* [*simp*]: $(- a) *s x = - (a *s x)$
and *scale-left-diff-distrib* [*algebra-simps*, *algebra-split-simps*]:
$(a - b) *s x = a *s x - b *s x$
and *scale-sum-left*: $(sum\ f\ A) *s x = (\sum a{\in}A.\ (f\ a) *s x)$
⟨*proof*⟩

lemma *scale-zero-right* [*simp*]: $a *s 0 = 0$
and *scale-minus-right* [*simp*]: $a *s (- x) = - (a *s x)$
and *scale-right-diff-distrib* [*algebra-simps*, *algebra-split-simps*]:
$a *s (x - y) = a *s x - a *s y$
and *scale-sum-right*: $a *s (sum\ f\ A) = (\sum x{\in}A.\ a *s (f\ x))$
⟨*proof*⟩

lemma *sum-constant-scale*: $(\sum x{\in}A.\ y) = scale\ (of\text{-}nat\ (card\ A))\ y$
⟨*proof*⟩

end

⟨*ML*⟩

context *module*
begin

lemma [*field-simps*, *field-split-simps*]:
shows *scale-left-distrib-NO-MATCH*: $NO\text{-}MATCH\ (x\ div\ y)\ c \Longrightarrow (a + b) *s x = a *s x + b *s x$
and *scale-right-distrib-NO-MATCH*: $NO\text{-}MATCH\ (x\ div\ y)\ a \Longrightarrow a *s (x + y) = a *s x + a *s y$
and *scale-left-diff-distrib-NO-MATCH*: $NO\text{-}MATCH\ (x\ div\ y)\ c \Longrightarrow (a - b) *s x = a *s x - b *s x$
and *scale-right-diff-distrib-NO-MATCH*: $NO\text{-}MATCH\ (x\ div\ y)\ a \Longrightarrow a *s (x - y) = a *s x - a *s y$
⟨*proof*⟩

end

⟨*ML*⟩

103 Subspace

context *module*
begin

definition *subspace* :: *'b set ⇒ bool*
where *subspace S ⟷ 0 ∈ S ∧ (∀ x∈S. ∀ y∈S. x + y ∈ S) ∧ (∀ c. ∀ x∈S. c ∗s x ∈ S)*

lemma *subspaceI*:
0 ∈ S ⟹ (⋀x y. x ∈ S ⟹ y ∈ S ⟹ x + y ∈ S) ⟹ (⋀c x. x ∈ S ⟹ c ∗s x ∈ S) ⟹ subspace S
⟨*proof*⟩

lemma *subspace-UNIV*[*simp*]: *subspace UNIV*
⟨*proof*⟩

lemma *subspace-single-0*[*simp*]: *subspace {0}*
⟨*proof*⟩

lemma *subspace-0*: *subspace S ⟹ 0 ∈ S*
⟨*proof*⟩

lemma *subspace-add*: *subspace S ⟹ x ∈ S ⟹ y ∈ S ⟹ x + y ∈ S*
⟨*proof*⟩

lemma *subspace-scale*: *subspace S ⟹ x ∈ S ⟹ c ∗s x ∈ S*
⟨*proof*⟩

lemma *subspace-neg*: *subspace S ⟹ x ∈ S ⟹ − x ∈ S*
⟨*proof*⟩

lemma *subspace-diff*: *subspace S ⟹ x ∈ S ⟹ y ∈ S ⟹ x − y ∈ S*
⟨*proof*⟩

lemma *subspace-sum*: *subspace A ⟹ (⋀x. x ∈ B ⟹ f x ∈ A) ⟹ sum f B ∈ A*
⟨*proof*⟩

lemma *subspace-Int*: *(⋀i. i ∈ I ⟹ subspace (s i)) ⟹ subspace (⋂ i∈I. s i)*
⟨*proof*⟩

lemma *subspace-Inter*: *∀ s ∈ f. subspace s ⟹ subspace (⋂ f)*
⟨*proof*⟩

lemma *subspace-inter*: *subspace A ⟹ subspace B ⟹ subspace (A ∩ B)*
⟨*proof*⟩

104 Span: subspace generated by a set

definition *span* :: *'b set ⇒ 'b set*
where *span-explicit*: *span b = {(∑ a∈t. r a ∗s a) | t r. finite t ∧ t ⊆ b}*

lemma *span-explicit'*:
span b = {(∑ v | f v ≠ 0. f v ∗s v) | f. finite {v. f v ≠ 0}} ∧ (∀ v. f v ≠ 0 ⟶ v

$\in b)\}$
⟨*proof*⟩

lemma *span-alt*:
$span\ B = \{(\sum x \mid f\ x \neq 0.\ f\ x *s\ x) \mid f.\ \{x.\ f\ x \neq 0\} \subseteq B \wedge finite\ \{x.\ f\ x \neq 0\}\}$
⟨*proof*⟩

lemma *span-finite*:
assumes *fS*: *finite S*
shows $span\ S = range\ (\lambda u.\ \sum v{\in}S.\ u\ v *s\ v)$
⟨*proof*⟩

lemma *span-induct-alt* [*consumes 1*, *case-names base step*, *induct set*: *span*]:
assumes *x*: $x \in span\ S$
assumes *h0*: $h\ 0$ **and** *hS*: $\bigwedge c\ x\ y.\ x \in S \Longrightarrow h\ y \Longrightarrow h\ (c *s\ x + y)$
shows $h\ x$
⟨*proof*⟩

lemma *span-mono*: $A \subseteq B \Longrightarrow span\ A \subseteq span\ B$
⟨*proof*⟩

lemma *span-base*: $a \in S \Longrightarrow a \in span\ S$
⟨*proof*⟩

lemma *span-superset*: $S \subseteq span\ S$
⟨*proof*⟩

lemma *span-zero*: $0 \in span\ S$
⟨*proof*⟩

lemma *span-UNIV*[*simp*]: $span\ UNIV = UNIV$
⟨*proof*⟩

lemma *span-add*: $x \in span\ S \Longrightarrow y \in span\ S \Longrightarrow x + y \in span\ S$
⟨*proof*⟩

lemma *span-scale*: $x \in span\ S \Longrightarrow c *s\ x \in span\ S$
⟨*proof*⟩

lemma *subspace-span* [*iff*]: $subspace\ (span\ S)$
⟨*proof*⟩

lemma *span-neg*: $x \in span\ S \Longrightarrow - x \in span\ S$
⟨*proof*⟩

lemma *span-diff*: $x \in span\ S \Longrightarrow y \in span\ S \Longrightarrow x - y \in span\ S$
⟨*proof*⟩

lemma *span-sum*: $(\bigwedge x.\ x \in A \Longrightarrow f\ x \in span\ S) \Longrightarrow sum\ f\ A \in span\ S$

⟨*proof*⟩

lemma *span-minimal*: $S \subseteq T \Longrightarrow$ *subspace* $T \Longrightarrow$ *span* $S \subseteq T$
⟨*proof*⟩

lemma *span-def*: *span* S = *subspace hull* S
⟨*proof*⟩

lemma *span-unique*:
$S \subseteq T \Longrightarrow$ *subspace* $T \Longrightarrow (\bigwedge T'.\ S \subseteq T' \Longrightarrow$ *subspace* $T' \Longrightarrow T \subseteq T') \Longrightarrow$ *span* $S = T$
⟨*proof*⟩

lemma *span-subspace-induct*[*consumes 2*]:
assumes x: $x \in$ *span* S
and P: *subspace* P
and *SP*: $\bigwedge x.\ x \in S \Longrightarrow x \in P$
shows $x \in P$
⟨*proof*⟩

lemma (**in** *module*) *span-induct*[*consumes 1*, *case-names base step*, *induct set*: *span*]:
assumes x: $x \in$ *span* S
and P: *subspace* (*Collect* P)
and *SP*: $\bigwedge x.\ x \in S \Longrightarrow P\ x$
shows $P\ x$
⟨*proof*⟩

lemma *span-empty*[*simp*]: *span* $\{\} = \{0\}$
⟨*proof*⟩

lemma *span-subspace*: $A \subseteq B \Longrightarrow B \subseteq$ *span* $A \Longrightarrow$ *subspace* $B \Longrightarrow$ *span* $A = B$
⟨*proof*⟩

lemma *span-span*: *span* (*span* A) = *span* A
⟨*proof*⟩

lemma *span-add-eq*: **assumes** x: $x \in$ *span* S **shows** $x + y \in$ *span* $S \longleftrightarrow y \in$ *span* S
⟨*proof*⟩

lemma *span-add-eq2*: **assumes** y: $y \in$ *span* S **shows** $x + y \in$ *span* $S \longleftrightarrow x \in$ *span* S
⟨*proof*⟩

lemma *span-singleton*: *span* $\{x\}$ = *range* $(\lambda k.\ k *s x)$
⟨*proof*⟩

lemma *span-Un*: *span* $(S \cup T) = \{x + y \mid x\ y.\ x \in span\ S \wedge y \in span\ T\}$
⟨*proof*⟩

lemma *span-insert*: *span* (*insert a S*) = $\{x.\ \exists k.\ (x - k *s a) \in span\ S\}$
⟨*proof*⟩

lemma *span-breakdown*:
assumes *bS*: $b \in S$
and *aS*: $a \in span\ S$
shows $\exists k.\ a - k *s b \in span\ (S - \{b\})$
⟨*proof*⟩

lemma *span-breakdown-eq*: $x \in span\ (insert\ a\ S) \longleftrightarrow (\exists k.\ x - k *s a \in span\ S)$
⟨*proof*⟩

lemmas *span-clauses* = *span-base span-zero span-add span-scale*

lemma *span-eq-iff*[*simp*]: *span s* = *s* ⟷ *subspace s*
⟨*proof*⟩

lemma *span-eq*: *span S* = *span T* ⟷ $S \subseteq span\ T \wedge T \subseteq span\ S$
⟨*proof*⟩

lemma *eq-span-insert-eq*:
assumes $(x - y) \in span\ S$
shows *span*(*insert x S*) = *span*(*insert y S*)
⟨*proof*⟩

105 Dependent and independent sets

definition *dependent* :: *'b set* ⇒ *bool*
where *dependent-explicit*: *dependent s* ⟷ (∃ *t u*. *finite t* ∧ *t* ⊆ *s* ∧ (∑ *v*∈*t*. *u v* ∗*s v*) = *0* ∧ (∃ *v*∈*t*. *u v* ≠ *0*))

abbreviation *independent s* ≡ ¬ *dependent s*

lemma *dependent-mono*: *dependent B* ⟹ *B* ⊆ *A* ⟹ *dependent A*
⟨*proof*⟩

lemma *independent-mono*: *independent A* ⟹ *B* ⊆ *A* ⟹ *independent B*
⟨*proof*⟩

lemma *dependent-zero*: *0* ∈ *A* ⟹ *dependent A*
⟨*proof*⟩

lemma *independent-empty*[*intro*]: *independent* {}
⟨*proof*⟩

lemma *independent-explicit-module*:

independent s $\longleftrightarrow$ $(\forall t\ u\ v.$ *finite t* $\longrightarrow$ $t \subseteq s \longrightarrow (\sum v{\in}t.\ u\ v *s\ v) = 0 \longrightarrow v \in t \longrightarrow u\ v = 0)$
⟨*proof*⟩

lemma *independentD*: *independent s* $\Longrightarrow$ *finite t* $\Longrightarrow$ $t \subseteq s \Longrightarrow (\sum v{\in}t.\ u\ v *s\ v) = 0 \Longrightarrow v \in t \Longrightarrow u\ v = 0$
⟨*proof*⟩

lemma *independent-Union-directed*:
assumes *directed*: $\bigwedge c\ d.\ c \in C \Longrightarrow d \in C \Longrightarrow c \subseteq d \vee d \subseteq c$
assumes *indep*: $\bigwedge c.\ c \in C \Longrightarrow$ *independent c*
shows *independent* $(\bigcup C)$
⟨*proof*⟩

lemma *dependent-finite*:
assumes *finite S*
shows *dependent S* $\longleftrightarrow$ $(\exists u.\ (\exists v \in S.\ u\ v \neq 0) \wedge (\sum v{\in}S.\ u\ v *s\ v) = 0)$
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *dependent-alt*:
dependent B $\longleftrightarrow$
$(\exists X.$ *finite* $\{x.\ X\ x \neq 0\} \wedge \{x.\ X\ x \neq 0\} \subseteq B \wedge (\sum x|X\ x \neq 0.\ X\ x *s\ x) = 0 \wedge (\exists x.\ X\ x \neq 0))$
⟨*proof*⟩

lemma *independent-alt*:
independent B $\longleftrightarrow$
$(\forall X.$ *finite* $\{x.\ X\ x \neq 0\} \longrightarrow \{x.\ X\ x \neq 0\} \subseteq B \longrightarrow (\sum x|X\ x \neq 0.\ X\ x *s\ x) = 0 \longrightarrow (\forall x.\ X\ x = 0))$
⟨*proof*⟩

lemma *independentD-alt*:
independent B $\Longrightarrow$ *finite* $\{x.\ X\ x \neq 0\} \Longrightarrow \{x.\ X\ x \neq 0\} \subseteq B \Longrightarrow (\sum x|X\ x \neq 0.\ X\ x *s\ x) = 0 \Longrightarrow X\ x = 0$
⟨*proof*⟩

lemma *independentD-unique*:
assumes *B*: *independent B*
and *X*: *finite* $\{x.\ X\ x \neq 0\}$ $\{x.\ X\ x \neq 0\} \subseteq B$
and *Y*: *finite* $\{x.\ Y\ x \neq 0\}$ $\{x.\ Y\ x \neq 0\} \subseteq B$
and $(\sum x \mid X\ x \neq 0.\ X\ x *s\ x) = (\sum x|\ Y\ x \neq 0.\ Y\ x *s\ x)$
shows $X = Y$
⟨*proof*⟩

106 Representation of a vector on a specific basis

definition *representation* :: *'b set* $\Rightarrow$ *'b* $\Rightarrow$ *'b* $\Rightarrow$ *'a*
where *representation basis v* =

(*if independent basis* $\wedge$ $v \in$ *span basis then*
SOME f. $(\forall v.\ f\ v \neq 0 \longrightarrow v \in basis) \wedge$ *finite* $\{v.\ f\ v \neq 0\} \wedge (\sum v{\in}\{v.\ f\ v \neq 0\}.\ f\ v *s\ v) = v$
else $(\lambda b.\ 0)$)

lemma *unique-representation*:
assumes *basis*: *independent basis*
and *in-basis*: $\bigwedge v.\ f\ v \neq 0 \Longrightarrow v \in basis\ \bigwedge v.\ g\ v \neq 0 \Longrightarrow v \in basis$
and [*simp*]: *finite* $\{v.\ f\ v \neq 0\}$ *finite* $\{v.\ g\ v \neq 0\}$
and *eq*: $(\sum v{\in}\{v.\ f\ v \neq 0\}.\ f\ v *s\ v) = (\sum v{\in}\{v.\ g\ v \neq 0\}.\ g\ v *s\ v)$
shows $f = g$
⟨*proof*⟩

lemma
shows *representation-ne-zero*: $\bigwedge b.$ *representation basis v b* $\neq 0 \Longrightarrow b \in basis$
and *finite-representation*: *finite* $\{b.$ *representation basis v b* $\neq 0\}$
and *sum-nonzero-representation-eq*:
independent basis $\Longrightarrow v \in$ *span basis* $\Longrightarrow (\sum b \mid$ *representation basis v b* $\neq 0.$ *representation basis v b* $*s\ b) = v$
⟨*proof*⟩

lemma *sum-representation-eq*:
$(\sum b{\in}B.$ *representation basis v b* $*s\ b) = v$
if *independent basis* $v \in$ *span basis finite B basis* $\subseteq B$
⟨*proof*⟩

lemma *representation-eqI*:
assumes *basis*: *independent basis* **and** *b*: $v \in$ *span basis*
and *ne-zero*: $\bigwedge b.\ f\ b \neq 0 \Longrightarrow b \in basis$
and *finite*: *finite* $\{b.\ f\ b \neq 0\}$
and *eq*: $(\sum b \mid f\ b \neq 0.\ f\ b *s\ b) = v$
shows *representation basis v* $= f$
⟨*proof*⟩

lemma *representation-basis*:
assumes *basis*: *independent basis* **and** *b*: $b \in basis$
shows *representation basis b* $= (\lambda v.$ *if* $v = b$ *then 1 else 0*$)$
⟨*proof*⟩

lemma *representation-zero*: *representation basis 0* $= (\lambda b.\ 0)$
⟨*proof*⟩

lemma *representation-diff*:
assumes *basis*: *independent basis* **and** *v*: $v \in$ *span basis* **and** *u*: $u \in$ *span basis*
shows *representation basis* $(u - v) = (\lambda b.$ *representation basis u b* $-$ *representation basis v b*$)$
⟨*proof*⟩

lemma *representation-neg*:

independent basis $\Longrightarrow$ $v \in$ *span basis* $\Longrightarrow$ *representation basis* $(-\ v) = (\lambda b.\ -$ *representation basis v b*)
⟨*proof*⟩

lemma *representation-add*:
independent basis $\Longrightarrow$ $v \in$ *span basis* $\Longrightarrow$ $u \in$ *span basis* $\Longrightarrow$
representation basis $(u + v) = (\lambda b.$ *representation basis u b* $+$ *representation basis v b*)
⟨*proof*⟩

lemma *representation-sum*:
independent basis $\Longrightarrow$ $(\bigwedge i.\ i \in I \Longrightarrow v\ i \in$ *span basis*$) \Longrightarrow$
representation basis $(sum\ v\ I) = (\lambda b.\ \sum i{\in}I.$ *representation basis* $(v\ i)\ b)$
⟨*proof*⟩

lemma *representation-scale*:
assumes *basis*: *independent basis* **and** *v*: $v \in$ *span basis*
shows *representation basis* $(r *s v) = (\lambda b.\ r *$ *representation basis v b*)
⟨*proof*⟩

lemma *representation-extend*:
assumes *basis*: *independent basis* **and** *v*: $v \in$ *span basis'* **and** *basis'*: *basis'* $\subseteq$ *basis*
shows *representation basis v* $=$ *representation basis' v*
⟨*proof*⟩

The set B is the maximal independent set for *span* B, or A is the minimal spanning set

lemma *spanning-subset-independent*:
assumes *BA*: $B \subseteq A$
and *iA*: *independent A*
and *AsB*: $A \subseteq$ *span B*
shows $A = B$
⟨*proof*⟩

end

A linear function is a mapping between two modules over the same ring.

locale *module-hom* $=$ *m1*: *module s1* $+$ *m2*: *module s2*
for *s1* :: *'a*::*comm-ring-1* $\Rightarrow$ *'b*::*ab-group-add* $\Rightarrow$ *'b* (**infixr** $*a$ *75*)
and *s2* :: *'a*::*comm-ring-1* $\Rightarrow$ *'c*::*ab-group-add* $\Rightarrow$ *'c* (**infixr** $*b$ *75*) $+$
fixes *f* :: *'b* $\Rightarrow$ *'c*
assumes *add*: *f* $(b1 + b2) = f\ b1 + f\ b2$
and *scale*: *f* $(r *a\ b) = r *b\ f\ b$
begin

lemma *zero*[*simp*]: *f 0* $=$ *0*
⟨*proof*⟩

lemma *neg*: $f\ (-\ x) = -\ f\ x$
⟨*proof*⟩

lemma *diff*: $f\ (x\ -\ y) = f\ x\ -\ f\ y$
⟨*proof*⟩

lemma *sum*: $f\ (sum\ g\ S) = (\sum a{\in}S.\ f\ (g\ a))$
⟨*proof*⟩

lemma *inj-on-iff-eq-0*:
assumes *s*: *m1.subspace s*
shows *inj-on f s* $\longleftrightarrow$ $(\forall x{\in}s.\ f\ x = 0 \longrightarrow x = 0)$
⟨*proof*⟩

lemma *inj-iff-eq-0*: *inj f* $= (\forall x.\ f\ x = 0 \longrightarrow x = 0)$
⟨*proof*⟩

lemma *subspace-image*: **assumes** *S*: *m1.subspace S* **shows** *m2.subspace* $(f\ `\ S)$
⟨*proof*⟩

lemma *subspace-vimage*: *m2.subspace S* $\Longrightarrow$ *m1.subspace* $(f\ -`\ S)$
⟨*proof*⟩

lemma *subspace-kernel*: *m1.subspace* $\{x.\ f\ x = 0\}$
⟨*proof*⟩

lemma *span-image*: *m2.span* $(f\ `\ S) = f\ `\ (m1.span\ S)$
⟨*proof*⟩

lemma *dependent-inj-imageD*:
assumes *d*: *m2.dependent* $(f\ `\ s)$ **and** *i*: *inj-on f* $(m1.span\ s)$
shows *m1.dependent s*
⟨*proof*⟩

lemma *eq-0-on-span*:
assumes *f0*: $\bigwedge x.\ x \in b \Longrightarrow f\ x = 0$ **and** *x*: $x \in m1.span\ b$ **shows** $f\ x = 0$
⟨*proof*⟩

lemma *independent-injective-image*: *m1.independent s* $\Longrightarrow$ *inj-on f* $(m1.span\ s)$ $\Longrightarrow$ *m2.independent* $(f\ `\ s)$
⟨*proof*⟩

lemma *inj-on-span-independent-image*:
assumes *ifB*: *m2.independent* $(f\ `\ B)$ **and** *f*: *inj-on f B* **shows** *inj-on f* $(m1.span\ B)$
⟨*proof*⟩

lemma *inj-on-span-iff-independent-image*: *m2.independent* $(f\ `\ B)$ $\Longrightarrow$ *inj-on f* $(m1.span\ B)$ $\longleftrightarrow$ *inj-on f B*

⟨*proof*⟩

lemma *subspace-linear-preimage*: *m2.subspace S* ⟹ *m1.subspace* {*x*. *f x* ∈ *S*}
⟨*proof*⟩

lemma *spans-image*: *V* ⊆ *m1.span B* ⟹ *f* ‘ *V* ⊆ *m2.span* (*f* ‘ *B*)
⟨*proof*⟩

Relation between bases and injectivity/surjectivity of map.

lemma *spanning-surjective-image*:
assumes *us*: *UNIV* ⊆ *m1.span S*
and *sf*: *surj f*
shows *UNIV* ⊆ *m2.span* (*f* ‘ *S*)
⟨*proof*⟩

lemmas *independent-inj-on-image* = *independent-injective-image*

lemma *independent-inj-image*:
m1.independent S ⟹ *inj f* ⟹ *m2.independent* (*f* ‘ *S*)
⟨*proof*⟩

end

lemma *module-hom-iff*:
module-hom s1 s2 f ⟷
module s1 ∧ *module s2* ∧
(∀ *x y*. *f* (*x* + *y*) = *f x* + *f y*) ∧ (∀ *c x*. *f* (*s1 c x*) = *s2 c* (*f x*))
⟨*proof*⟩

locale *module-pair* = *m1*: *module s1* + *m2*: *module s2*
for *s1* :: ′*a* :: *comm-ring-1* ⇒ ′*b* ⇒ ′*b* :: *ab-group-add*
and *s2* :: ′*a* :: *comm-ring-1* ⇒ ′*c* ⇒ ′*c* :: *ab-group-add*
begin

lemma *module-hom-zero*: *module-hom s1 s2* (λ*x*. *0*)
⟨*proof*⟩

lemma *module-hom-add*: *module-hom s1 s2 f* ⟹ *module-hom s1 s2 g* ⟹ *module-hom s1 s2* (λ*x*. *f x* + *g x*)
⟨*proof*⟩

lemma *module-hom-sub*: *module-hom s1 s2 f* ⟹ *module-hom s1 s2 g* ⟹ *module-hom s1 s2* (λ*x*. *f x* − *g x*)
⟨*proof*⟩

lemma *module-hom-neg*: *module-hom s1 s2 f* ⟹ *module-hom s1 s2* (λ*x*. − *f x*)
⟨*proof*⟩

lemma *module-hom-scale*: *module-hom s1 s2 f* ⟹ *module-hom s1 s2* (λ*x*. *s2 c* (*f*

x))
⟨*proof*⟩

lemma *module-hom-compose-scale*:
module-hom s1 s2 (λ*x*. *s2* (*f x*) (*c*))
if *module-hom s1* (∗) *f*
⟨*proof*⟩

lemma *bij-module-hom-imp-inv-module-hom*: *module-hom scale1 scale2 f* ⟹ *bij f* ⟹
module-hom scale2 scale1 (*inv f*)
⟨*proof*⟩

lemma *module-hom-sum*: (⋀*i*. *i* ∈ *I* ⟹ *module-hom s1 s2* (*f i*)) ⟹ (*I* = {} ⟹ *module s1* ∧ *module s2*) ⟹ *module-hom s1 s2* (λ*x*. ∑ *i*∈*I*. *f i x*)
⟨*proof*⟩

lemma *module-hom-eq-on-span*: *f x* = *g x*
if *module-hom s1 s2 f module-hom s1 s2 g*
and (⋀*x*. *x* ∈ *B* ⟹ *f x* = *g x*) *x* ∈ *m1.span B*
⟨*proof*⟩

end

context *module* **begin**

lemma *module-hom-scale-self*[*simp*]:
module-hom scale scale (λ*x*. *scale c x*)
⟨*proof*⟩

lemma *module-hom-scale-left*[*simp*]:
module-hom (∗) *scale* (λ*r*. *scale r x*)
⟨*proof*⟩

lemma *module-hom-id*: *module-hom scale scale id*
⟨*proof*⟩

lemma *module-hom-ident*: *module-hom scale scale* (λ*x*. *x*)
⟨*proof*⟩

lemma *module-hom-uminus*: *module-hom scale scale uminus*
⟨*proof*⟩

end

lemma *module-hom-compose*: *module-hom s1 s2 f* ⟹ *module-hom s2 s3 g* ⟹ *module-hom s1 s3* (*g o f*)
⟨*proof*⟩

end

107 Vector Spaces

theory *Vector-Spaces*
imports *Modules*
begin

lemma *isomorphism-expand*:
$f \circ g = id \wedge g \circ f = id \longleftrightarrow (\forall x.\ f\ (g\ x) = x) \wedge (\forall x.\ g\ (f\ x) = x)$
⟨*proof*⟩

lemma *left-right-inverse-eq*:
assumes *fg*: $f \circ g = id$
and *gh*: $g \circ h = id$
shows $f = h$
⟨*proof*⟩

lemma *ordLeq3-finite-infinite*:
assumes *A*: *finite A* **and** *B*: *infinite B* **shows** *ordLeq3* (*card-of A*) (*card-of B*)
⟨*proof*⟩

locale *vector-space* =
fixes *scale* :: *'a*::*field* ⇒ *'b*::*ab-group-add* ⇒ *'b* (**infixr** ∗*s* *75*)
assumes *vector-space-assms*:— re-stating the assumptions of *module* instead of extending *module* allows us to rewrite in the sublocale.
$a *s (x + y) = a *s x + a *s y$
$(a + b) *s x = a *s x + b *s x$
$a *s (b *s x) = (a * b) *s x$
$1 *s x = x$

lemma *module-iff-vector-space*: *module s* ⟷ *vector-space s*
⟨*proof*⟩

locale *linear* = *vs1*: *vector-space s1* + *vs2*: *vector-space s2* + *module-hom s1 s2 f*
for *s1* :: *'a*::*field* ⇒ *'b*::*ab-group-add* ⇒ *'b* (**infixr** ∗*a* *75*)
and *s2* :: *'a*::*field* ⇒ *'c*::*ab-group-add* ⇒ *'c* (**infixr** ∗*b* *75*)
and *f* :: *'b* ⇒ *'c*

lemma *module-hom-iff-linear*: *module-hom s1 s2 f* ⟷ *linear s1 s2 f*
⟨*proof*⟩
lemmas *module-hom-eq-linear* = *module-hom-iff-linear*[*abs-def*, *THEN meta-eq-to-obj-eq*]
lemmas *linear-iff-module-hom* = *module-hom-iff-linear*[*symmetric*]
lemmas *linear-module-homI* = *module-hom-iff-linear*[*THEN iffD1*]
and *module-hom-linearI* = *module-hom-iff-linear*[*THEN iffD2*]

context *vector-space* **begin**

sublocale *module scale* **rewrites** *module-hom* = *linear*

⟨*proof*⟩

lemmas— from *module*
linear-id = *module-hom-id*
and *linear-ident* = *module-hom-ident*
and *linear-scale-self* = *module-hom-scale-self*
and *linear-scale-left* = *module-hom-scale-left*
and *linear-uminus* = *module-hom-uminus*

lemma *linear-imp-scale*:
fixes $D::'a \Rightarrow 'b$
assumes *linear* (∗) *scale D*
obtains *d* **where** $D = (\lambda x.\ scale\ x\ d)$
⟨*proof*⟩

lemma *scale-eq-0-iff* [*simp*]: $scale\ a\ x = 0 \longleftrightarrow a = 0 \vee x = 0$
⟨*proof*⟩

lemma *scale-left-imp-eq*:
assumes *nonzero*: $a \neq 0$
and *scale*: $scale\ a\ x = scale\ a\ y$
shows $x = y$
⟨*proof*⟩

lemma *scale-right-imp-eq*:
assumes *nonzero*: $x \neq 0$
and *scale*: $scale\ a\ x = scale\ b\ x$
shows $a = b$
⟨*proof*⟩

lemma *scale-cancel-left* [*simp*]: $scale\ a\ x = scale\ a\ y \longleftrightarrow x = y \vee a = 0$
⟨*proof*⟩

lemma *scale-cancel-right* [*simp*]: $scale\ a\ x = scale\ b\ x \longleftrightarrow a = b \vee x = 0$
⟨*proof*⟩

lemma *injective-scale*: $c \neq 0 \Longrightarrow inj\ (\lambda x.\ scale\ c\ x)$
⟨*proof*⟩

lemma *dependent-def*: $dependent\ P \longleftrightarrow (\exists a \in P.\ a \in span\ (P - \{a\}))$
⟨*proof*⟩

lemma *dependent-single*[*simp*]: $dependent\ \{x\} \longleftrightarrow x = 0$
⟨*proof*⟩

lemma *in-span-insert*:
assumes *a*: $a \in span\ (insert\ b\ S)$
and *na*: $a \notin span\ S$
shows $b \in span\ (insert\ a\ S)$

⟨*proof*⟩

lemma *dependent-insertD*: **assumes** *a*: $a \notin span\ S$ **and** *S*: *dependent* (*insert a S*) **shows** *dependent S*
⟨*proof*⟩

lemma *independent-insertI*: $a \notin span\ S \Longrightarrow independent\ S \Longrightarrow independent\ (insert\ a\ S)$
⟨*proof*⟩

lemma *independent-insert*:
$independent\ (insert\ a\ S) \longleftrightarrow (if\ a \in S\ then\ independent\ S\ else\ independent\ S \wedge a \notin span\ S)$
⟨*proof*⟩

lemma *maximal-independent-subset-extend*:
assumes $S \subseteq V$ *independent S*
obtains *B* **where** $S \subseteq B\ B \subseteq V\ independent\ B\ V \subseteq span\ B$
⟨*proof*⟩

lemma *maximal-independent-subset*:
obtains *B* **where** $B \subseteq V\ independent\ B\ V \subseteq span\ B$
⟨*proof*⟩

Extends a basis from B to a basis of the entire space.

definition *extend-basis* :: *'b set* ⇒ *'b set*
where $extend\text{-}basis\ B = (SOME\ B'.\ B \subseteq B' \wedge independent\ B' \wedge span\ B' = UNIV)$

lemma
assumes *B*: *independent B*
shows *extend-basis-superset*: $B \subseteq extend\text{-}basis\ B$
and *independent-extend-basis*: *independent* (*extend-basis B*)
and *span-extend-basis*[*simp*]: $span\ (extend\text{-}basis\ B) = UNIV$
⟨*proof*⟩

lemma *in-span-delete*:
assumes *a*: $a \in span\ S$ **and** *na*: $a \notin span\ (S - \{b\})$
shows $b \in span\ (insert\ a\ (S - \{b\}))$
⟨*proof*⟩

lemma *span-redundant*: $x \in span\ S \Longrightarrow span\ (insert\ x\ S) = span\ S$
⟨*proof*⟩

lemma *span-trans*: $x \in span\ S \Longrightarrow y \in span\ (insert\ x\ S) \Longrightarrow y \in span\ S$
⟨*proof*⟩

lemma *span-insert-0*[*simp*]: $span\ (insert\ 0\ S) = span\ S$
⟨*proof*⟩

lemma *span-delete-0* [*simp*]: $span(S - \{0\}) = span\ S$
⟨*proof*⟩

lemma *span-image-scale*:
assumes *finite S* **and** *nz*: $\bigwedge x.\ x \in S \Longrightarrow c\ x \neq 0$
shows $span\ ((\lambda x.\ c\ x\ *s\ x)\ `\ S) = span\ S$
⟨*proof*⟩

lemma *exchange-lemma*:
assumes *f*: *finite T*
and *i*: *independent S*
and *sp*: $S \subseteq span\ T$
shows $\exists t'.\ card\ t' = card\ T \wedge finite\ t' \wedge S \subseteq t' \wedge t' \subseteq S \cup T \wedge S \subseteq span\ t'$
⟨*proof*⟩

lemma *independent-span-bound*:
assumes *f*: *finite T*
and *i*: *independent S*
and *sp*: $S \subseteq span\ T$
shows $finite\ S \wedge card\ S \leq card\ T$
⟨*proof*⟩

lemma *independent-explicit-finite-subsets*:
$independent\ A \longleftrightarrow (\forall S \subseteq A.\ finite\ S \longrightarrow (\forall u.\ (\sum v{\in}S.\ u\ v\ *s\ v) = 0 \longrightarrow (\forall v{\in}S.\ u\ v = 0)))$
⟨*proof*⟩

lemma *independent-if-scalars-zero*:
assumes *fin-A*: *finite A*
and *sum*: $\bigwedge f\ x.\ (\sum x{\in}A.\ f\ x\ *s\ x) = 0 \Longrightarrow x \in A \Longrightarrow f\ x = 0$
shows *independent A*
⟨*proof*⟩

lemma *bij-if-span-eq-span-bases*:
assumes *B*: *independent B* **and** *C*: *independent C*
and *eq*: *span B* = *span C*
shows $\exists f.\ bij\text{-}betw\ f\ B\ C$
⟨*proof*⟩

definition *dim* :: *'b set* ⇒ *nat*
where *dim V* = (*if* ∃ *b*. *independent b* ∧ *span b* = *span V then*
card (*SOME b*. *independent b* ∧ *span b* = *span V*) *else 0*)

lemma *dim-eq-card*:
assumes *BV*: *span B* = *span V* **and** *B*: *independent B*
shows *dim V* = *card B*
⟨*proof*⟩

lemma *basis-card-eq-dim*: $B \subseteq V \Longrightarrow V \subseteq span\ B \Longrightarrow independent\ B \Longrightarrow card\ B = dim\ V$
⟨*proof*⟩

lemma *basis-exists*:
obtains B **where** $B \subseteq V\ independent\ B\ V \subseteq span\ B\ card\ B = dim\ V$
⟨*proof*⟩

lemma *dim-eq-card-independent*: $independent\ B \Longrightarrow dim\ B = card\ B$
⟨*proof*⟩

lemma *dim-span*[*simp*]: $dim\ (span\ S) = dim\ S$
⟨*proof*⟩

lemma *dim-span-eq-card-independent*: $independent\ B \Longrightarrow dim\ (span\ B) = card\ B$
⟨*proof*⟩

lemma *dim-le-card*: **assumes** $V \subseteq span\ W\ finite\ W$ **shows** $dim\ V \leq card\ W$
⟨*proof*⟩

lemma *span-eq-dim*: $span\ S = span\ T \Longrightarrow dim\ S = dim\ T$
⟨*proof*⟩

corollary *dim-le-card'*:
$finite\ s \Longrightarrow dim\ s \leq card\ s$
⟨*proof*⟩

lemma *span-card-ge-dim*:
$B \subseteq V \Longrightarrow V \subseteq span\ B \Longrightarrow finite\ B \Longrightarrow dim\ V \leq card\ B$
⟨*proof*⟩

lemma *dim-unique*:
$B \subseteq V \Longrightarrow V \subseteq span\ B \Longrightarrow independent\ B \Longrightarrow card\ B = n \Longrightarrow dim\ V = n$
⟨*proof*⟩

lemma *subspace-sums*: $[\![subspace\ S;\ subspace\ T]\!] \Longrightarrow subspace\ \{x + y | x\ y.\ x \in S \wedge y \in T\}$
⟨*proof*⟩

end

lemma *linear-iff*: $linear\ s1\ s2\ f \longleftrightarrow$
$(vector\text{-}space\ s1 \wedge vector\text{-}space\ s2 \wedge (\forall x\ y.\ f\ (x + y) = f\ x + f\ y) \wedge (\forall c\ x.\ f\ (s1\ c\ x) = s2\ c\ (f\ x)))$
⟨*proof*⟩

context begin
qualified lemma *linear-compose*: $linear\ s1\ s2\ f \Longrightarrow linear\ s2\ s3\ g \Longrightarrow linear\ s1\ s3\ (g\ o\ f)$

⟨*proof*⟩
end

locale *vector-space-pair* = *vs1*: *vector-space s1* + *vs2*: *vector-space s2*
for *s1* :: *'a*::*field* ⇒ *'b*::*ab-group-add* ⇒ *'b* (**infixr** ∗*a* 75)
and *s2* :: *'a*::*field* ⇒ *'c*::*ab-group-add* ⇒ *'c* (**infixr** ∗*b* 75)
begin

context fixes *f* **assumes** *linear s1 s2 f* **begin**
interpretation *linear s1 s2 f* ⟨*proof*⟩
lemmas— from locale *module-hom*
linear-0 = *zero*
and *linear-add* = *add*
and *linear-scale* = *scale*
and *linear-neg* = *neg*
and *linear-diff* = *diff*
and *linear-sum* = *sum*
and *linear-inj-on-iff-eq-0* = *inj-on-iff-eq-0*
and *linear-inj-iff-eq-0* = *inj-iff-eq-0*
and *linear-subspace-image* = *subspace-image*
and *linear-subspace-vimage* = *subspace-vimage*
and *linear-subspace-kernel* = *subspace-kernel*
and *linear-span-image* = *span-image*
and *linear-dependent-inj-imageD* = *dependent-inj-imageD*
and *linear-eq-0-on-span* = *eq-0-on-span*
and *linear-independent-injective-image* = *independent-injective-image*
and *linear-inj-on-span-independent-image* = *inj-on-span-independent-image*
and *linear-inj-on-span-iff-independent-image* = *inj-on-span-iff-independent-image*
and *linear-subspace-linear-preimage* = *subspace-linear-preimage*
and *linear-spans-image* = *spans-image*
and *linear-spanning-surjective-image* = *spanning-surjective-image*
end

sublocale *module-pair*
rewrites *module-hom* = *linear*
⟨*proof*⟩

lemmas— from locale *module-pair*
linear-eq-on-span = *module-hom-eq-on-span*
and *linear-compose-scale-right* = *module-hom-scale*
and *linear-compose-add* = *module-hom-add*
and *linear-zero* = *module-hom-zero*
and *linear-compose-sub* = *module-hom-sub*
and *linear-compose-neg* = *module-hom-neg*
and *linear-compose-scale* = *module-hom-compose-scale*

lemma *linear-indep-image-lemma*:
assumes *lf*: *linear s1 s2 f*
and *fB*: *finite B*

and *ifB*: *vs2.independent* (*f ' B*)
and *fi*: *inj-on f B*
and *xsB*: $x \in$ *vs1.span B*
and *fx*: $f\ x = 0$
shows $x = 0$
⟨*proof*⟩

lemma *linear-eq-on*:
assumes *l*: *linear s1 s2 f linear s1 s2 g*
assumes *x*: $x \in$ *vs1.span B* **and** *eq*: $\bigwedge b.\ b \in B \Longrightarrow f\ b = g\ b$
shows $f\ x = g\ x$
⟨*proof*⟩

definition *construct* :: $'b$ *set* $\Rightarrow$ ($'b \Rightarrow 'c$) $\Rightarrow$ ($'b \Rightarrow 'c$)
where *construct B g v* = ($\sum b \mid$ *vs1.representation* (*vs1.extend-basis B*) *v b* $\neq$ *0*.
vs1.representation (*vs1.extend-basis B*) *v b* $*b$ (*if b* $\in$ *B then g b else 0*))

lemma *construct-cong*: ($\bigwedge b.\ b \in B \Longrightarrow f\ b = g\ b$) $\Longrightarrow$ *construct B f* = *construct B g*
⟨*proof*⟩

lemma *linear-construct*:
assumes *B*[*simp*]: *vs1.independent B*
shows *linear s1 s2* (*construct B f*)
⟨*proof*⟩

lemma *construct-basis*:
assumes *B*[*simp*]: *vs1.independent B* **and** *b*: $b \in B$
shows *construct B f b* = *f b*
⟨*proof*⟩

lemma *construct-outside*:
assumes *B*: *vs1.independent B* **and** *v*: $v \in$ *vs1.span* (*vs1.extend-basis B* − *B*)
shows *construct B f v* = *0*
⟨*proof*⟩

lemma *construct-add*:
assumes *B*[*simp*]: *vs1.independent B*
shows *construct B* ($\lambda x.\ f\ x + g\ x$) *v* = *construct B f v* + *construct B g v*
⟨*proof*⟩

lemma *construct-scale*:
assumes *B*[*simp*]: *vs1.independent B*
shows *construct B* ($\lambda x.\ c\ *b\ f\ x$) *v* = *c* $*b$ *construct B f v*
⟨*proof*⟩

lemma *construct-in-span*:
assumes *B*[*simp*]: *vs1.independent B*

shows *construct B f v* ∈ *vs2.span* (*f* ‘ *B*)
⟨*proof*⟩

lemma *linear-compose-sum*:
assumes *lS*: ∀ *a* ∈ *S*. *linear s1 s2* (*f a*)
shows *linear s1 s2* (λ*x*. *sum* (λ*a*. *f a x*) *S*)
⟨*proof*⟩

lemma *in-span-in-range-construct*:
x ∈ *range* (*construct B f*) **if** *i*: *vs1.independent B* **and** *x*: *x* ∈ *vs2.span* (*f* ‘ *B*)
⟨*proof*⟩

lemma *range-construct-eq-span*:
range (*construct B f*) = *vs2.span* (*f* ‘ *B*)
if *vs1.independent B*
⟨*proof*⟩

lemma *linear-independent-extend-subspace*:
— legacy: use *construct* instead
assumes *vs1.independent B*
shows ∃ *g*. *linear s1 s2 g* ∧ (∀ *x*∈*B*. *g x* = *f x*) ∧ *range g* = *vs2.span* (*f*‘*B*)
⟨*proof*⟩

lemma *linear-independent-extend*:
vs1.independent B ⟹ ∃ *g*. *linear s1 s2 g* ∧ (∀ *x*∈*B*. *g x* = *f x*)
⟨*proof*⟩

lemma *linear-exists-left-inverse-on*:
assumes *lf*: *linear s1 s2 f*
assumes *V*: *vs1.subspace V* **and** *f*: *inj-on f V*
shows ∃ *g*. *g* ‘ *UNIV* ⊆ *V* ∧ *linear s2 s1 g* ∧ (∀ *v*∈*V*. *g* (*f v*) = *v*)
⟨*proof*⟩

lemma *linear-exists-right-inverse-on*:
assumes *lf*: *linear s1 s2 f*
assumes *vs1.subspace V*
shows ∃ *g*. *g* ‘ *UNIV* ⊆ *V* ∧ *linear s2 s1 g* ∧ (∀ *v*∈*f* ‘ *V*. *f* (*g v*) = *v*)
⟨*proof*⟩

lemma *linear-inj-on-left-inverse*:
assumes *lf*: *linear s1 s2 f*
assumes *fi*: *inj-on f* (*vs1.span S*)
shows ∃ *g*. *range g* ⊆ *vs1.span S* ∧ *linear s2 s1 g* ∧ (∀ *x*∈*vs1.span S*. *g* (*f x*) = *x*)
⟨*proof*⟩

lemma *linear-injective-left-inverse*: *linear s1 s2 f* ⟹ *inj f* ⟹ ∃ *g*. *linear s2 s1 g* ∧ *g* ∘ *f* = *id*
⟨*proof*⟩

lemma *linear-surj-right-inverse*:
assumes *lf*: *linear s1 s2 f*
assumes *sf*: *vs2.span T ⊆ f‘vs1.span S*
shows ∃ *g*. *range g ⊆ vs1.span S ∧ linear s2 s1 g ∧* (∀ *x*∈*vs2.span T*. *f* (*g x*) = *x*)
⟨*proof*⟩

lemma *linear-surjective-right-inverse*: *linear s1 s2 f ⟹ surj f ⟹* ∃ *g*. *linear s2 s1 g ∧ f* ∘ *g* = *id*
⟨*proof*⟩

lemma *finite-basis-to-basis-subspace-isomorphism*:
assumes *s*: *vs1.subspace S*
and *t*: *vs2.subspace T*
and *d*: *vs1.dim S* = *vs2.dim T*
and *fB*: *finite B*
and *B*: *B ⊆ S vs1.independent B S ⊆ vs1.span B card B* = *vs1.dim S*
and *fC*: *finite C*
and *C*: *C ⊆ T vs2.independent C T ⊆ vs2.span C card C* = *vs2.dim T*
shows ∃ *f*. *linear s1 s2 f ∧ f ‘ B* = *C ∧ f ‘ S* = *T ∧ inj-on f S*
⟨*proof*⟩

end

locale *finite-dimensional-vector-space* = *vector-space* +
fixes *Basis* :: *'b set*
assumes *finite-Basis*: *finite Basis*
and *independent-Basis*: *independent Basis*
and *span-Basis*: *span Basis* = *UNIV*
begin

definition *dimension* = *card Basis*

lemma *finiteI-independent*: *independent B ⟹ finite B*
⟨*proof*⟩

lemma *dim-empty* [*simp*]: *dim* {} = *0*
⟨*proof*⟩

lemma *dim-insert*:
dim (*insert x S*) = (*if x* ∈ *span S then dim S else dim S* + *1*)
⟨*proof*⟩

lemma *dim-singleton* [*simp*]: *dim*{*x*} = (*if x* = *0 then 0 else 1*)
⟨*proof*⟩

proposition *choose-subspace-of-subspace*:

assumes $n \leq dim\ S$
obtains T **where** $subspace\ T\ T \subseteq span\ S\ dim\ T = n$
⟨*proof*⟩

lemma *basis-subspace-exists*:
assumes *subspace S*
obtains B **where** *finite B* $B \subseteq S$ *independent B span B* $= S$ *card B* $= dim\ S$
⟨*proof*⟩

lemma *dim-mono*: **assumes** $V \subseteq span\ W$ **shows** $dim\ V \leq dim\ W$
⟨*proof*⟩

lemma *dim-subset*: $S \subseteq T \Longrightarrow dim\ S \leq dim\ T$
⟨*proof*⟩

lemma *dim-eq-0* [*simp*]:
$dim\ S = 0 \longleftrightarrow S \subseteq \{0\}$
⟨*proof*⟩

lemma *dim-UNIV*[*simp*]: $dim\ UNIV = card\ Basis$
⟨*proof*⟩

lemma *independent-card-le-dim*: **assumes** $B \subseteq V$ **and** *independent B* **shows** *card* $B \leq dim\ V$
⟨*proof*⟩

lemma *dim-subset-UNIV*: $dim\ S \leq dimension$
⟨*proof*⟩

lemma *card-ge-dim-independent*:
assumes BV: $B \subseteq V$
and *iB*: *independent B*
and *dVB*: $dim\ V \leq card\ B$
shows $V \subseteq span\ B$
⟨*proof*⟩

lemma *card-le-dim-spanning*:
assumes BV: $B \subseteq V$
and *VB*: $V \subseteq span\ B$
and *fB*: *finite B*
and *dVB*: $dim\ V \geq card\ B$
shows *independent B*
⟨*proof*⟩

lemma *card-eq-dim*: $B \subseteq V \Longrightarrow card\ B = dim\ V \Longrightarrow finite\ B \Longrightarrow independent\ B \longleftrightarrow V \subseteq span\ B$
⟨*proof*⟩

lemma *subspace-dim-equal*:

assumes *subspace S*
and *subspace T*
and $S \subseteq T$
and $dim\ S \geq dim\ T$
shows $S = T$
⟨*proof*⟩

corollary *dim-eq-span*:
shows $[\![S \subseteq T; dim\ T \leq dim\ S]\!] \Longrightarrow span\ S = span\ T$
⟨*proof*⟩

lemma *dim-psubset*:
$span\ S \subset span\ T \Longrightarrow dim\ S < dim\ T$
⟨*proof*⟩

lemma *dim-eq-full*:
shows $dim\ S = dimension \longleftrightarrow span\ S = UNIV$
⟨*proof*⟩

lemma *indep-card-eq-dim-span*:
assumes *independent B*
shows *finite B* $\wedge$ $card\ B = dim\ (span\ B)$
⟨*proof*⟩

More general size bound lemmas.

lemma *independent-bound-general*:
independent S $\Longrightarrow$ *finite S* $\wedge$ $card\ S \leq dim\ S$
⟨*proof*⟩

lemma *independent-explicit*:
shows *independent B* $\longleftrightarrow$ *finite B* $\wedge$ $(\forall c.\ (\sum v{\in}B.\ c\ v *s\ v) = 0 \longrightarrow (\forall v \in B.\ c\ v = 0))$
⟨*proof*⟩

proposition *dim-sums-Int*:
assumes *subspace S subspace T*
shows $dim\ \{x + y\ |x\ y.\ x \in S \wedge y \in T\} + dim(S \cap T) = dim\ S + dim\ T$ (**is** *dim ?ST + - = -*)
⟨*proof*⟩

lemma *dependent-biggerset-general*:
(*finite S* $\Longrightarrow$ $card\ S > dim\ S$) $\Longrightarrow$ *dependent S*
⟨*proof*⟩

lemma *subset-le-dim*:
$S \subseteq span\ T \Longrightarrow dim\ S \leq dim\ T$
⟨*proof*⟩

lemma *linear-inj-imp-surj*:

assumes *lf*: *linear scale scale f*
and *fi*: *inj f*
shows *surj f*
⟨*proof*⟩

end

locale *finite-dimensional-vector-space-pair-1* =
vs1: *finite-dimensional-vector-space s1 B1* + *vs2*: *vector-space s2*
for *s1* :: *'a*::*field* ⇒ *'b*::*ab-group-add* ⇒ *'b* (**infixr** ∗*a* 75)
and *B1* :: *'b set*
and *s2* :: *'a*::*field* ⇒ *'c*::*ab-group-add* ⇒ *'c* (**infixr** ∗*b* 75)
begin

sublocale *vector-space-pair s1 s2* ⟨*proof*⟩

lemma *dim-image-eq*:
assumes *lf*: *linear s1 s2 f*
and *fi*: *inj-on f* (*vs1.span S*)
shows *vs2.dim* (*f* ‘ *S*) = *vs1.dim S*
⟨*proof*⟩

lemma *dim-image-le*:
assumes *lf*: *linear s1 s2 f*
shows *vs2.dim* (*f* ‘ *S*) ≤ *vs1.dim* (*S*)
⟨*proof*⟩

end

locale *finite-dimensional-vector-space-pair* =
vs1: *finite-dimensional-vector-space s1 B1* + *vs2*: *finite-dimensional-vector-space s2 B2*
for *s1* :: *'a*::*field* ⇒ *'b*::*ab-group-add* ⇒ *'b* (**infixr** ∗*a* 75)
and *B1* :: *'b set*
and *s2* :: *'a*::*field* ⇒ *'c*::*ab-group-add* ⇒ *'c* (**infixr** ∗*b* 75)
and *B2* :: *'c set*
begin

sublocale *finite-dimensional-vector-space-pair-1* ⟨*proof*⟩

lemma *linear-surjective-imp-injective*:
assumes *lf*: *linear s1 s2 f* **and** *sf*: *surj f* **and** *eq*: *vs2.dim UNIV* = *vs1.dim UNIV*
shows *inj f*
⟨*proof*⟩

lemma *linear-injective-imp-surjective*:
assumes *lf*: *linear s1 s2 f* **and** *sf*: *inj f* **and** *eq*: *vs2.dim UNIV* = *vs1.dim UNIV*
shows *surj f*

⟨*proof*⟩

lemma *linear-injective-isomorphism*:
assumes *lf*: *linear s1 s2 f*
and *fi*: *inj f*
and *dims*: *vs2.dim UNIV = vs1.dim UNIV*
shows $\exists f'.$ *linear s2 s1* $f' \wedge (\forall x.\ f'\ (f\ x) = x) \wedge (\forall x.\ f\ (f'\ x) = x)$
⟨*proof*⟩

lemma *linear-surjective-isomorphism*:
assumes *lf*: *linear s1 s2 f*
and *sf*: *surj f*
and *dims*: *vs2.dim UNIV = vs1.dim UNIV*
shows $\exists f'.$ *linear s2 s1* $f' \wedge (\forall x.\ f'\ (f\ x) = x) \wedge (\forall x.\ f\ (f'\ x) = x)$
⟨*proof*⟩

lemma *basis-to-basis-subspace-isomorphism*:
assumes *s*: *vs1.subspace S*
and *t*: *vs2.subspace T*
and *d*: *vs1.dim S = vs2.dim T*
and *B*: $B \subseteq S$ *vs1.independent B* $S \subseteq$ *vs1.span B card B = vs1.dim S*
and *C*: $C \subseteq T$ *vs2.independent C* $T \subseteq$ *vs2.span C card C = vs2.dim T*
shows $\exists f.$ *linear s1 s2* $f \wedge f\ `\ B = C \wedge f\ `\ S = T \wedge$ *inj-on f S*
⟨*proof*⟩

end

context *finite-dimensional-vector-space* **begin**

lemma *linear-surj-imp-inj*:
assumes *lf*: *linear scale scale f* **and** *sf*: *surj f*
shows *inj f*
⟨*proof*⟩

lemma *linear-inverse-left*:
assumes *lf*: *linear scale scale f*
and *lf'*: *linear scale scale f'*
shows $f \circ f' = id \longleftrightarrow f' \circ f = id$
⟨*proof*⟩

lemma *left-inverse-linear*:
assumes *lf*: *linear scale scale f*
and *gf*: $g \circ f = id$
shows *linear scale scale g*
⟨*proof*⟩

lemma *inj-linear-imp-inv-linear*:
assumes *linear scale scale f inj f* **shows** *linear scale scale* (*inv f*)
⟨*proof*⟩

lemma *right-inverse-linear*:
  assumes *lf*: *linear scale scale f*
    and *gf*: $f \circ g = id$
  shows *linear scale scale g*
⟨*proof*⟩

end

context *finite-dimensional-vector-space-pair* **begin**

lemma *subspace-isomorphism*:
  assumes *s*: *vs1.subspace S*
    and *t*: *vs2.subspace T*
    and *d*: *vs1.dim S* = *vs2.dim T*
  shows $\exists f.$ *linear s1 s2 f* $\wedge$ *f ' S* = *T* $\wedge$ *inj-on f S*
⟨*proof*⟩

end

hide-const (**open**) *linear*

end

108 Vector Spaces and Algebras over the Reals

theory *Real-Vector-Spaces*
imports *Real Topological-Spaces Vector-Spaces*
begin

108.1 Real vector spaces

class *scaleR* =
  fixes *scaleR* :: *real* ⇒ *'a* ⇒ *'a* (**infixr** $*_R$ *75*)
begin

abbreviation *divideR* :: *'a* ⇒ *real* ⇒ *'a* (**infixl** $'/_R$ *70*)
  where $x /_R r \equiv inverse\ r *_R x$

end

class *real-vector* = *scaleR* + *ab-group-add* +
  assumes *scaleR-add-right*: $a *_R (x + y) = a *_R x + a *_R y$
  and *scaleR-add-left*: $(a + b) *_R x = a *_R x + b *_R x$
  and *scaleR-scaleR*: $a *_R b *_R x = (a * b) *_R x$
  and *scaleR-one*: $1 *_R x = x$

class *real-algebra* = *real-vector* + *ring* +
  assumes *mult-scaleR-left* [*simp*]: $a *_R x * y = a *_R (x * y)$

and *mult-scaleR-right* [*simp*]: $x * a *_R y = a *_R (x * y)$

class *real-algebra-1* = *real-algebra* + *ring-1*

class *real-div-algebra* = *real-algebra-1* + *division-ring*

class *real-field* = *real-div-algebra* + *field*

instantiation *real* :: *real-field*
begin

definition *real-scaleR-def* [*simp*]: *scaleR a x* = *a* * *x*

instance
⟨*proof*⟩

end

locale *linear* = *Vector-Spaces.linear scaleR*::-⇒-⇒'*a*::*real-vector scaleR*::-⇒-⇒'*b*::*real-vector*
begin

lemmas *scaleR* = *scale*

end

global-interpretation *real-vector?*: *vector-space scaleR* :: *real* ⇒ 'a ⇒ 'a :: *real-vector*
rewrites *Vector-Spaces.linear* (*$*_R*) (*$*_R*) = *linear*
and *Vector-Spaces.linear* (*) (*$*_R*) = *linear*
defines *dependent-raw-def*: *dependent* = *real-vector.dependent*
and *representation-raw-def*: *representation* = *real-vector.representation*
and *subspace-raw-def*: *subspace* = *real-vector.subspace*
and *span-raw-def*: *span* = *real-vector.span*
and *extend-basis-raw-def*: *extend-basis* = *real-vector.extend-basis*
and *dim-raw-def*: *dim* = *real-vector.dim*
⟨*proof*⟩

hide-const (**open**)— locale constants
real-vector.dependent
real-vector.independent
real-vector.representation
real-vector.subspace
real-vector.span
real-vector.extend-basis
real-vector.dim

abbreviation *independent x* ≡ ¬ *dependent x*

global-interpretation *real-vector?*: *vector-space-pair scaleR*::-⇒-⇒'*a*::*real-vector*
scaleR::-⇒-⇒'*b*::*real-vector*

rewrites *Vector-Spaces.linear* $(*_R)$ $(*_R)$ = *linear*
and *Vector-Spaces.linear* $(*)$ $(*_R)$ = *linear*
defines *construct-raw-def*: *construct* = *real-vector.construct*
⟨*proof*⟩

hide-const (**open**)— locale constants
real-vector.construct

lemma *linear-compose*: *linear f* $\Longrightarrow$ *linear g* $\Longrightarrow$ *linear* $(g \circ f)$
⟨*proof*⟩

Recover original theorem names

lemmas *scaleR-left-commute* = *real-vector.scale-left-commute*
lemmas *scaleR-zero-left* = *real-vector.scale-zero-left*
lemmas *scaleR-minus-left* = *real-vector.scale-minus-left*
lemmas *scaleR-diff-left* = *real-vector.scale-left-diff-distrib*
lemmas *scaleR-sum-left* = *real-vector.scale-sum-left*
lemmas *scaleR-zero-right* = *real-vector.scale-zero-right*
lemmas *scaleR-minus-right* = *real-vector.scale-minus-right*
lemmas *scaleR-diff-right* = *real-vector.scale-right-diff-distrib*
lemmas *scaleR-sum-right* = *real-vector.scale-sum-right*
lemmas *scaleR-eq-0-iff* = *real-vector.scale-eq-0-iff*
lemmas *scaleR-left-imp-eq* = *real-vector.scale-left-imp-eq*
lemmas *scaleR-right-imp-eq* = *real-vector.scale-right-imp-eq*
lemmas *scaleR-cancel-left* = *real-vector.scale-cancel-left*
lemmas *scaleR-cancel-right* = *real-vector.scale-cancel-right*

lemma [*field-simps*]:
$c \neq 0 \Longrightarrow a = b \mathbin{/_R} c \longleftrightarrow c *_R a = b$
$c \neq 0 \Longrightarrow b \mathbin{/_R} c = a \longleftrightarrow b = c *_R a$
$c \neq 0 \Longrightarrow a + b \mathbin{/_R} c = (c *_R a + b) \mathbin{/_R} c$
$c \neq 0 \Longrightarrow a \mathbin{/_R} c + b = (a + c *_R b) \mathbin{/_R} c$
$c \neq 0 \Longrightarrow a - b \mathbin{/_R} c = (c *_R a - b) \mathbin{/_R} c$
$c \neq 0 \Longrightarrow a \mathbin{/_R} c - b = (a - c *_R b) \mathbin{/_R} c$
$c \neq 0 \Longrightarrow - (a \mathbin{/_R} c) + b = (- a + c *_R b) \mathbin{/_R} c$
$c \neq 0 \Longrightarrow - (a \mathbin{/_R} c) - b = (- a - c *_R b) \mathbin{/_R} c$
for $a\ b :: 'a ::$ *real-vector*
⟨*proof*⟩

Legacy names

lemmas *scaleR-left-distrib* = *scaleR-add-left*
lemmas *scaleR-right-distrib* = *scaleR-add-right*
lemmas *scaleR-left-diff-distrib* = *scaleR-diff-left*
lemmas *scaleR-right-diff-distrib* = *scaleR-diff-right*

lemmas *linear-injective-0* = *linear-inj-iff-eq-0*
and *linear-injective-on-subspace-0* = *linear-inj-on-iff-eq-0*
and *linear-cmul* = *linear-scale*
and *linear-scaleR* = *linear-scale-self*

and *subspace-mul* = *subspace-scale*
and *span-linear-image* = *linear-span-image*
and *span-0* = *span-zero*
and *span-mul* = *span-scale*
and *injective-scaleR* = *injective-scale*

lemma *scaleR-minus1-left* [*simp*]: *scaleR* (−*1*) *x* = − *x*
for *x* :: *'a*::*real-vector*
⟨*proof*⟩

lemma *scaleR-2*:
fixes *x* :: *'a*::*real-vector*
shows *scaleR 2 x* = *x* + *x*
⟨*proof*⟩

lemma *scaleR-half-double* [*simp*]:
fixes *a* :: *'a*::*real-vector*
shows (*1* / *2*) *∗*$_R$ (*a* + *a*) = *a*
⟨*proof*⟩

lemma *shift-zero-ident* [*simp*]:
fixes *f* :: *'a* ⇒ *'b*::*real-vector*
shows (+)*0* ∘ *f* = *f*
⟨*proof*⟩

lemma *linear-scale-real*:
fixes *r*::*real* **shows** *linear f* ⟹ *f* (*r* ∗ *b*) = *r* ∗ *f b*
⟨*proof*⟩

interpretation *scaleR-left*: *additive* (λ*a*. *scaleR a x* :: *'a*::*real-vector*)
⟨*proof*⟩

interpretation *scaleR-right*: *additive* (λ*x*. *scaleR a x* :: *'a*::*real-vector*)
⟨*proof*⟩

lemma *nonzero-inverse-scaleR-distrib*:
a ≠ *0* ⟹ *x* ≠ *0* ⟹ *inverse* (*scaleR a x*) = *scaleR* (*inverse a*) (*inverse x*)
for *x* :: *'a*::*real-div-algebra*
⟨*proof*⟩

lemma *inverse-scaleR-distrib*: *inverse* (*scaleR a x*) = *scaleR* (*inverse a*) (*inverse x*)
for *x* :: *'a*::{*real-div-algebra*,*division-ring*}
⟨*proof*⟩

lemmas *sum-constant-scaleR* = *real-vector.sum-constant-scale*— legacy name

named-theorems *vector-add-divide-simps to simplify sums of scaled vectors*

lemma [*vector-add-divide-simps*]:
*v + (b / z) *R w = (if z = 0 then v else (z *R v + b *R w) /R z)*
*a *R v + (b / z) *R w = (if z = 0 then a *R v else ((a * z) *R v + b *R w) /R z)*
*(a / z) *R v + w = (if z = 0 then w else (a *R v + z *R w) /R z)*
*(a / z) *R v + b *R w = (if z = 0 then b *R w else (a *R v + (b * z) *R w) /R z)*
*v − (b / z) *R w = (if z = 0 then v else (z *R v − b *R w) /R z)*
*a *R v − (b / z) *R w = (if z = 0 then a *R v else ((a * z) *R v − b *R w) /R z)*
*(a / z) *R v − w = (if z = 0 then −w else (a *R v − z *R w) /R z)*
*(a / z) *R v − b *R w = (if z = 0 then −b *R w else (a *R v − (b * z) *R w) /R z)*
for *v* :: *'a* :: *real-vector*
⟨*proof*⟩

lemma *eq-vector-fraction-iff* [*vector-add-divide-simps*]:
fixes *x* :: *'a* :: *real-vector*
shows *(x = (u / v) *R a) ⟷ (if v=0 then x = 0 else v *R x = u *R a)*
⟨*proof*⟩

lemma *vector-fraction-eq-iff* [*vector-add-divide-simps*]:
fixes *x* :: *'a* :: *real-vector*
shows *((u / v) *R a = x) ⟷ (if v=0 then x = 0 else u *R a = v *R x)*
⟨*proof*⟩

lemma *real-vector-affinity-eq*:
fixes *x* :: *'a* :: *real-vector*
assumes *m0*: *m ≠ 0*
shows *m *R x + c = y ⟷ x = inverse m *R y − (inverse m *R c)*
(**is** *?lhs ⟷ ?rhs*)
⟨*proof*⟩

lemma *real-vector-eq-affinity*: *m ≠ 0 ⟹ y = m *R x + c ⟷ inverse m *R y − (inverse m *R c) = x*
for *x* :: *'a*::*real-vector*
⟨*proof*⟩

lemma *scaleR-eq-iff* [*simp*]: *b + u *R a = a + u *R b ⟷ a = b ∨ u = 1*
for *a* :: *'a*::*real-vector*
⟨*proof*⟩

lemma *scaleR-collapse* [*simp*]: *(1 − u) *R a + u *R a = a*
for *a* :: *'a*::*real-vector*
⟨*proof*⟩

108.2 Embedding of the Reals into any *real-algebra-1*: *of-real*

definition *of-real* :: *real* ⇒ *'a*::*real-algebra-1*
where *of-real r* = *scaleR r 1*

lemma *scaleR-conv-of-real*: *scaleR r x* = *of-real r* ∗ *x*
⟨*proof*⟩

lemma *of-real-0* [*simp*]: *of-real 0* = *0*
⟨*proof*⟩

lemma *of-real-1* [*simp*]: *of-real 1* = *1*
⟨*proof*⟩

lemma *of-real-add* [*simp*]: *of-real* (*x* + *y*) = *of-real x* + *of-real y*
⟨*proof*⟩

lemma *of-real-minus* [*simp*]: *of-real* (− *x*) = − *of-real x*
⟨*proof*⟩

lemma *of-real-diff* [*simp*]: *of-real* (*x* − *y*) = *of-real x* − *of-real y*
⟨*proof*⟩

lemma *of-real-mult* [*simp*]: *of-real* (*x* ∗ *y*) = *of-real x* ∗ *of-real y*
⟨*proof*⟩

lemma *of-real-sum*[*simp*]: *of-real* (*sum f s*) = (∑ *x*∈*s*. *of-real* (*f x*))
⟨*proof*⟩

lemma *of-real-prod*[*simp*]: *of-real* (*prod f s*) = (∏ *x*∈*s*. *of-real* (*f x*))
⟨*proof*⟩

lemma *nonzero-of-real-inverse*:
x ≠ *0* ⟹ *of-real* (*inverse x*) = *inverse* (*of-real x* :: *'a*::*real-div-algebra*)
⟨*proof*⟩

lemma *of-real-inverse* [*simp*]:
of-real (*inverse x*) = *inverse* (*of-real x* :: *'a*::{*real-div-algebra*,*division-ring*})
⟨*proof*⟩

lemma *nonzero-of-real-divide*:
y ≠ *0* ⟹ *of-real* (*x* / *y*) = (*of-real x* / *of-real y* :: *'a*::*real-field*)
⟨*proof*⟩

lemma *of-real-divide* [*simp*]:
of-real (*x* / *y*) = (*of-real x* / *of-real y* :: *'a*::*real-div-algebra*)
⟨*proof*⟩

lemma *of-real-power* [*simp*]:
of-real (*x* ^ *n*) = (*of-real x* :: *'a*::{*real-algebra-1*}) ^ *n*

⟨*proof*⟩

lemma *of-real-power-int* [*simp*]:
of-real (*power-int x n*) = *power-int* (*of-real x* :: *'a* :: {*real-div-algebra*,*division-ring*}) *n*
⟨*proof*⟩

lemma *of-real-eq-iff* [*simp*]: *of-real x* = *of-real y* ⟷ *x* = *y*
⟨*proof*⟩

lemma *inj-of-real*: *inj of-real*
⟨*proof*⟩

lemmas *of-real-eq-0-iff* [*simp*] = *of-real-eq-iff* [*of - 0*, *simplified*]
lemmas *of-real-eq-1-iff* [*simp*] = *of-real-eq-iff* [*of - 1*, *simplified*]

lemma *minus-of-real-eq-of-real-iff* [*simp*]: −*of-real x* = *of-real y* ⟷ −*x* = *y*
⟨*proof*⟩

lemma *of-real-eq-minus-of-real-iff* [*simp*]: *of-real x* = −*of-real y* ⟷ *x* = −*y*
⟨*proof*⟩

lemma *of-real-eq-id* [*simp*]: *of-real* = (*id* :: *real* ⇒ *real*)
⟨*proof*⟩

Collapse nested embeddings.

lemma *of-real-of-nat-eq* [*simp*]: *of-real* (*of-nat n*) = *of-nat n*
⟨*proof*⟩

lemma *of-real-of-int-eq* [*simp*]: *of-real* (*of-int z*) = *of-int z*
⟨*proof*⟩

lemma *of-real-numeral* [*simp*]: *of-real* (*numeral w*) = *numeral w*
⟨*proof*⟩

lemma *of-real-neg-numeral* [*simp*]: *of-real* (− *numeral w*) = − *numeral w*
⟨*proof*⟩

lemma *numeral-power-int-eq-of-real-cancel-iff* [*simp*]:
power-int (*numeral x*) *n* = (*of-real y* :: *'a* :: {*real-div-algebra*, *division-ring*}) ⟷
power-int (*numeral x*) *n* = *y*
⟨*proof*⟩

lemma *of-real-eq-numeral-power-int-cancel-iff* [*simp*]:
(*of-real y* :: *'a* :: {*real-div-algebra*, *division-ring*}) = *power-int* (*numeral x*) *n* ⟷
y = *power-int* (*numeral x*) *n*
⟨*proof*⟩

lemma *of-real-eq-of-real-power-int-cancel-iff* [*simp*]:

power-int (*of-real b* :: *'a* :: {*real-div-algebra*, *division-ring*}) *w* = *of-real x* ⟷
power-int b w = *x*
⟨*proof*⟩

lemma *of-real-in-Ints-iff* [*simp*]: *of-real x* ∈ ℤ ⟷ *x* ∈ ℤ
⟨*proof*⟩

lemma *Ints-of-real* [*intro*]: *x* ∈ ℤ ⟹ *of-real x* ∈ ℤ
⟨*proof*⟩

Every real algebra has characteristic zero.

instance *real-algebra-1* < *ring-char-0*
⟨*proof*⟩

lemma *fraction-scaleR-times* [*simp*]:
fixes *a* :: *'a*::*real-algebra-1*
shows (*numeral u* / *numeral v*) *R* (*numeral w* ∗ *a*) = (*numeral u* ∗ *numeral w* / *numeral v*) *R a*
⟨*proof*⟩

lemma *inverse-scaleR-times* [*simp*]:
fixes *a* :: *'a*::*real-algebra-1*
shows (*1* / *numeral v*) *R* (*numeral w* ∗ *a*) = (*numeral w* / *numeral v*) *R a*
⟨*proof*⟩

lemma *scaleR-times* [*simp*]:
fixes *a* :: *'a*::*real-algebra-1*
shows (*numeral u*) *R* (*numeral w* ∗ *a*) = (*numeral u* ∗ *numeral w*) *R a*
⟨*proof*⟩

instance *real-field* < *field-char-0* ⟨*proof*⟩

108.3 The Set of Real Numbers

definition *Reals* :: *'a*::*real-algebra-1 set* (ℝ)
where ℝ = *range of-real*

lemma *Reals-of-real* [*simp*]: *of-real r* ∈ ℝ
⟨*proof*⟩

lemma *Reals-of-int* [*simp*]: *of-int z* ∈ ℝ
⟨*proof*⟩

lemma *Reals-of-nat* [*simp*]: *of-nat n* ∈ ℝ
⟨*proof*⟩

lemma *Reals-numeral* [*simp*]: *numeral w* ∈ ℝ
⟨*proof*⟩

lemma *Reals-0* [*simp*]: $0 \in \mathbb{R}$ **and** *Reals-1* [*simp*]: $1 \in \mathbb{R}$
⟨*proof*⟩

lemma *Reals-add* [*simp*]: $a \in \mathbb{R} \Longrightarrow b \in \mathbb{R} \Longrightarrow a + b \in \mathbb{R}$
⟨*proof*⟩

lemma *Reals-minus* [*simp*]: $a \in \mathbb{R} \Longrightarrow - a \in \mathbb{R}$
⟨*proof*⟩

lemma *Reals-minus-iff* [*simp*]: $- a \in \mathbb{R} \longleftrightarrow a \in \mathbb{R}$
⟨*proof*⟩

lemma *Reals-diff* [*simp*]: $a \in \mathbb{R} \Longrightarrow b \in \mathbb{R} \Longrightarrow a - b \in \mathbb{R}$
⟨*proof*⟩

lemma *Reals-mult* [*simp*]: $a \in \mathbb{R} \Longrightarrow b \in \mathbb{R} \Longrightarrow a * b \in \mathbb{R}$
⟨*proof*⟩

lemma *nonzero-Reals-inverse*: $a \in \mathbb{R} \Longrightarrow a \neq 0 \Longrightarrow inverse\ a \in \mathbb{R}$
for a :: $'a$::*real-div-algebra*
⟨*proof*⟩

lemma *Reals-inverse*: $a \in \mathbb{R} \Longrightarrow inverse\ a \in \mathbb{R}$
for a :: $'a$::{*real-div-algebra*,*division-ring*}
⟨*proof*⟩

lemma *Reals-inverse-iff* [*simp*]: $inverse\ x \in \mathbb{R} \longleftrightarrow x \in \mathbb{R}$
for x :: $'a$::{*real-div-algebra*,*division-ring*}
⟨*proof*⟩

lemma *nonzero-Reals-divide*: $a \in \mathbb{R} \Longrightarrow b \in \mathbb{R} \Longrightarrow b \neq 0 \Longrightarrow a\ /\ b \in \mathbb{R}$
for $a\ b$ :: $'a$::*real-field*
⟨*proof*⟩

lemma *Reals-divide* [*simp*]: $a \in \mathbb{R} \Longrightarrow b \in \mathbb{R} \Longrightarrow a\ /\ b \in \mathbb{R}$
for $a\ b$ :: $'a$::{*real-field*,*field*}
⟨*proof*⟩

lemma *Reals-power* [*simp*]: $a \in \mathbb{R} \Longrightarrow a\ \hat{}\ n \in \mathbb{R}$
for a :: $'a$::*real-algebra-1*
⟨*proof*⟩

lemma *Reals-cases* [*cases set*: *Reals*]:
assumes $q \in \mathbb{R}$
obtains (*of-real*) r **where** $q = of\text{-}real\ r$
⟨*proof*⟩

lemma *sum-in-Reals* [*intro*,*simp*]: $(\bigwedge i.\ i \in s \Longrightarrow f\ i \in \mathbb{R}) \Longrightarrow sum\ f\ s \in \mathbb{R}$
⟨*proof*⟩

lemma *prod-in-Reals* [*intro*,*simp*]: $(\bigwedge i.\ i \in s \Longrightarrow f\ i \in \mathbb{R}) \Longrightarrow prod\ f\ s \in \mathbb{R}$
⟨*proof*⟩

lemma *Reals-induct* [*case-names of-real*, *induct set*: *Reals*]:
$q \in \mathbb{R} \Longrightarrow (\bigwedge r.\ P\ (of\text{-}real\ r)) \Longrightarrow P\ q$
⟨*proof*⟩

108.4 Ordered real vector spaces

class *ordered-real-vector* = *real-vector* + *ordered-ab-group-add* +
assumes *scaleR-left-mono*: $x \leq y \Longrightarrow 0 \leq a \Longrightarrow a *_R x \leq a *_R y$
and *scaleR-right-mono*: $a \leq b \Longrightarrow 0 \leq x \Longrightarrow a *_R x \leq b *_R x$
begin

lemma *scaleR-mono*:
$a \leq b \Longrightarrow x \leq y \Longrightarrow 0 \leq b \Longrightarrow 0 \leq x \Longrightarrow a *_R x \leq b *_R y$
⟨*proof*⟩

lemma *scaleR-mono'*:
$a \leq b \Longrightarrow c \leq d \Longrightarrow 0 \leq a \Longrightarrow 0 \leq c \Longrightarrow a *_R c \leq b *_R d$
⟨*proof*⟩

lemma *pos-le-divideR-eq* [*field-simps*]:
$a \leq b /_R c \longleftrightarrow c *_R a \leq b$ (**is** *?P* $\longleftrightarrow$ *?Q*) **if** $0 < c$
⟨*proof*⟩

lemma *pos-less-divideR-eq* [*field-simps*]:
$a < b /_R c \longleftrightarrow c *_R a < b$ **if** $c > 0$
⟨*proof*⟩

lemma *pos-divideR-le-eq* [*field-simps*]:
$b /_R c \leq a \longleftrightarrow b \leq c *_R a$ **if** $c > 0$
⟨*proof*⟩

lemma *pos-divideR-less-eq* [*field-simps*]:
$b /_R c < a \longleftrightarrow b < c *_R a$ **if** $c > 0$
⟨*proof*⟩

lemma *pos-le-minus-divideR-eq* [*field-simps*]:
$a \leq - (b /_R c) \longleftrightarrow c *_R a \leq - b$ **if** $c > 0$
⟨*proof*⟩

lemma *pos-less-minus-divideR-eq* [*field-simps*]:
$a < - (b /_R c) \longleftrightarrow c *_R a < - b$ **if** $c > 0$
⟨*proof*⟩

lemma *pos-minus-divideR-le-eq* [*field-simps*]:
$- (b /_R c) \leq a \longleftrightarrow - b \leq c *_R a$ **if** $c > 0$

⟨*proof*⟩

lemma *pos-minus-divideR-less-eq* [*field-simps*]:
$-(b /_R c) < a \longleftrightarrow -b < c *_R a$ **if** $c > 0$
⟨*proof*⟩

lemma *scaleR-image-atLeastAtMost*: $c > 0 \Longrightarrow$ *scaleR* c ‘ $\{x..y\} = \{c *_R x..c *_R y\}$
⟨*proof*⟩

end

lemma *neg-le-divideR-eq* [*field-simps*]:
$a \leq b /_R c \longleftrightarrow b \leq c *_R a$ (**is** *?P* $\longleftrightarrow$ *?Q*) **if** $c < 0$
for $a\ b :: 'a ::$ *ordered-real-vector*
⟨*proof*⟩

lemma *neg-less-divideR-eq* [*field-simps*]:
$a < b /_R c \longleftrightarrow b < c *_R a$ **if** $c < 0$
for $a\ b :: 'a ::$ *ordered-real-vector*
⟨*proof*⟩

lemma *neg-divideR-le-eq* [*field-simps*]:
$b /_R c \leq a \longleftrightarrow c *_R a \leq b$ **if** $c < 0$
for $a\ b :: 'a ::$ *ordered-real-vector*
⟨*proof*⟩

lemma *neg-divideR-less-eq* [*field-simps*]:
$b /_R c < a \longleftrightarrow c *_R a < b$ **if** $c < 0$
for $a\ b :: 'a ::$ *ordered-real-vector*
⟨*proof*⟩

lemma *neg-le-minus-divideR-eq* [*field-simps*]:
$a \leq -(b /_R c) \longleftrightarrow -b \leq c *_R a$ **if** $c < 0$
for $a\ b :: 'a ::$ *ordered-real-vector*
⟨*proof*⟩

lemma *neg-less-minus-divideR-eq* [*field-simps*]:
$a < -(b /_R c) \longleftrightarrow -b < c *_R a$ **if** $c < 0$
for $a\ b :: 'a ::$ *ordered-real-vector*
⟨*proof*⟩

lemma *neg-minus-divideR-le-eq* [*field-simps*]:
$-(b /_R c) \leq a \longleftrightarrow c *_R a \leq -b$ **if** $c < 0$
for $a\ b :: 'a ::$ *ordered-real-vector*
⟨*proof*⟩

lemma *neg-minus-divideR-less-eq* [*field-simps*]:
$-(b /_R c) < a \longleftrightarrow c *_R a < -b$ **if** $c < 0$

for $a\ b :: 'a ::$ *ordered-real-vector*
⟨*proof*⟩

lemma [*field-split-simps*]:
$a = b\ /_R\ c \longleftrightarrow (\textit{if } c = 0 \textit{ then } a = 0 \textit{ else } c *_R a = b)$
$b\ /_R\ c = a \longleftrightarrow (\textit{if } c = 0 \textit{ then } a = 0 \textit{ else } b = c *_R a)$
$a + b\ /_R\ c = (\textit{if } c = 0 \textit{ then } a \textit{ else } (c *_R a + b)\ /_R\ c)$
$a\ /_R\ c + b = (\textit{if } c = 0 \textit{ then } b \textit{ else } (a + c *_R b)\ /_R\ c)$
$a - b\ /_R\ c = (\textit{if } c = 0 \textit{ then } a \textit{ else } (c *_R a - b)\ /_R\ c)$
$a\ /_R\ c - b = (\textit{if } c = 0 \textit{ then } -b \textit{ else } (a - c *_R b)\ /_R\ c)$
$-(a\ /_R\ c) + b = (\textit{if } c = 0 \textit{ then } b \textit{ else } (-a + c *_R b)\ /_R\ c)$
$-(a\ /_R\ c) - b = (\textit{if } c = 0 \textit{ then } -b \textit{ else } (-a - c *_R b)\ /_R\ c)$
for $a\ b :: 'a ::$ *real-vector*
⟨*proof*⟩

lemma [*field-split-simps*]:
$0 < c \Longrightarrow a \le b\ /_R\ c \longleftrightarrow (\textit{if } c > 0 \textit{ then } c *_R a \le b \textit{ else if } c < 0 \textit{ then } b \le c *_R a \textit{ else } a \le 0)$
$0 < c \Longrightarrow a < b\ /_R\ c \longleftrightarrow (\textit{if } c > 0 \textit{ then } c *_R a < b \textit{ else if } c < 0 \textit{ then } b < c *_R a \textit{ else } a < 0)$
$0 < c \Longrightarrow b\ /_R\ c \le a \longleftrightarrow (\textit{if } c > 0 \textit{ then } b \le c *_R a \textit{ else if } c < 0 \textit{ then } c *_R a \le b \textit{ else } a \ge 0)$
$0 < c \Longrightarrow b\ /_R\ c < a \longleftrightarrow (\textit{if } c > 0 \textit{ then } b < c *_R a \textit{ else if } c < 0 \textit{ then } c *_R a < b \textit{ else } a > 0)$
$0 < c \Longrightarrow a \le -(b\ /_R\ c) \longleftrightarrow (\textit{if } c > 0 \textit{ then } c *_R a \le -b \textit{ else if } c < 0 \textit{ then } -b \le c *_R a \textit{ else } a \le 0)$
$0 < c \Longrightarrow a < -(b\ /_R\ c) \longleftrightarrow (\textit{if } c > 0 \textit{ then } c *_R a < -b \textit{ else if } c < 0 \textit{ then } -b < c *_R a \textit{ else } a < 0)$
$0 < c \Longrightarrow -(b\ /_R\ c) \le a \longleftrightarrow (\textit{if } c > 0 \textit{ then } -b \le c *_R a \textit{ else if } c < 0 \textit{ then } c *_R a \le -b \textit{ else } a \ge 0)$
$0 < c \Longrightarrow -(b\ /_R\ c) < a \longleftrightarrow (\textit{if } c > 0 \textit{ then } -b < c *_R a \textit{ else if } c < 0 \textit{ then } c *_R a < -b \textit{ else } a > 0)$
for $a\ b :: 'a ::$ *ordered-real-vector*
⟨*proof*⟩

lemma *scaleR-nonneg-nonneg*: $0 \le a \Longrightarrow 0 \le x \Longrightarrow 0 \le a *_R x$
for $x :: 'a$::*ordered-real-vector*
⟨*proof*⟩

lemma *scaleR-nonneg-nonpos*: $0 \le a \Longrightarrow x \le 0 \Longrightarrow a *_R x \le 0$
for $x :: 'a$::*ordered-real-vector*
⟨*proof*⟩

lemma *scaleR-nonpos-nonneg*: $a \le 0 \Longrightarrow 0 \le x \Longrightarrow a *_R x \le 0$
for $x :: 'a$::*ordered-real-vector*
⟨*proof*⟩

lemma *split-scaleR-neg-le*: $(0 \le a \wedge x \le 0) \vee (a \le 0 \wedge 0 \le x) \Longrightarrow a *_R x \le 0$
for $x :: 'a$::*ordered-real-vector*

⟨*proof*⟩

lemma *le-add-iff1*: $a *_R e + c \leq b *_R e + d \longleftrightarrow (a - b) *_R e + c \leq d$
for c d e :: *'a*::*ordered-real-vector*
⟨*proof*⟩

lemma *le-add-iff2*: $a *_R e + c \leq b *_R e + d \longleftrightarrow c \leq (b - a) *_R e + d$
for c d e :: *'a*::*ordered-real-vector*
⟨*proof*⟩

lemma *scaleR-left-mono-neg*: $b \leq a \Longrightarrow c \leq 0 \Longrightarrow c *_R a \leq c *_R b$
for a b :: *'a*::*ordered-real-vector*
⟨*proof*⟩

lemma *scaleR-right-mono-neg*: $b \leq a \Longrightarrow c \leq 0 \Longrightarrow a *_R c \leq b *_R c$
for c :: *'a*::*ordered-real-vector*
⟨*proof*⟩

lemma *scaleR-nonpos-nonpos*: $a \leq 0 \Longrightarrow b \leq 0 \Longrightarrow 0 \leq a *_R b$
for b :: *'a*::*ordered-real-vector*
⟨*proof*⟩

lemma *split-scaleR-pos-le*: $(0 \leq a \wedge 0 \leq b) \vee (a \leq 0 \wedge b \leq 0) \Longrightarrow 0 \leq a *_R b$
for b :: *'a*::*ordered-real-vector*
⟨*proof*⟩

lemma *zero-le-scaleR-iff*:
fixes b :: *'a*::*ordered-real-vector*
shows $0 \leq a *_R b \longleftrightarrow 0 < a \wedge 0 \leq b \vee a < 0 \wedge b \leq 0 \vee a = 0$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *scaleR-le-0-iff*: $a *_R b \leq 0 \longleftrightarrow 0 < a \wedge b \leq 0 \vee a < 0 \wedge 0 \leq b \vee a = 0$
for *b*::*'a*::*ordered-real-vector*
⟨*proof*⟩

lemma *scaleR-le-cancel-left*: $c *_R a \leq c *_R b \longleftrightarrow (0 < c \longrightarrow a \leq b) \wedge (c < 0 \longrightarrow b \leq a)$
for b :: *'a*::*ordered-real-vector*
⟨*proof*⟩

lemma *scaleR-le-cancel-left-pos*: $0 < c \Longrightarrow c *_R a \leq c *_R b \longleftrightarrow a \leq b$
for b :: *'a*::*ordered-real-vector*
⟨*proof*⟩

lemma *scaleR-le-cancel-left-neg*: $c < 0 \Longrightarrow c *_R a \leq c *_R b \longleftrightarrow b \leq a$
for b :: *'a*::*ordered-real-vector*
⟨*proof*⟩

lemma *scaleR-left-le-one-le*: $0 \leq x \Longrightarrow a \leq 1 \Longrightarrow a *_R x \leq x$
for x :: $'a$::*ordered-real-vector* **and** a :: *real*
⟨*proof*⟩

108.5 Real normed vector spaces

class *dist* =
fixes *dist* :: $'a \Rightarrow 'a \Rightarrow$ *real*

class *norm* =
fixes *norm* :: $'a \Rightarrow$ *real*

class *sgn-div-norm* = *scaleR* + *norm* + *sgn* +
assumes *sgn-div-norm*: *sgn* x = x /$_R$ *norm* x

class *dist-norm* = *dist* + *norm* + *minus* +
assumes *dist-norm*: *dist* x y = *norm* (x − y)

class *uniformity-dist* = *dist* + *uniformity* +
assumes *uniformity-dist*: *uniformity* = (*INF* e∈{0 <..}. *principal* {(x, y). *dist* x y < e}))
begin

lemma *eventually-uniformity-metric*:
eventually P *uniformity* ⟷ (∃ e>0. ∀ x y. *dist* x y < e ⟶ P (x, y))
⟨*proof*⟩

end

class *real-normed-vector* = *real-vector* + *sgn-div-norm* + *dist-norm* + *uniformity-dist* + *open-uniformity* +
assumes *norm-eq-zero* [*simp*]: *norm* x = 0 ⟷ x = 0
and *norm-triangle-ineq*: *norm* (x + y) ≤ *norm* x + *norm* y
and *norm-scaleR* [*simp*]: *norm* (*scaleR* a x) = |a| * *norm* x
begin

lemma *norm-ge-zero* [*simp*]: 0 ≤ *norm* x
⟨*proof*⟩

lemma *bdd-below-norm-image*: *bdd-below* (*norm* ‘ A)
⟨*proof*⟩

end

class *real-normed-algebra* = *real-algebra* + *real-normed-vector* +
assumes *norm-mult-ineq*: *norm* (x * y) ≤ *norm* x * *norm* y

class *real-normed-algebra-1* = *real-algebra-1* + *real-normed-algebra* +
assumes *norm-one* [*simp*]: *norm* 1 = 1

lemma (**in** *real-normed-algebra-1*) *scaleR-power* [*simp*]: (*scaleR x y*) ^ *n* = *scaleR* (*x*^*n*) (*y*^*n*)
⟨*proof*⟩

class *real-normed-div-algebra* = *real-div-algebra* + *real-normed-vector* +
assumes *norm-mult*: *norm* (*x* * *y*) = *norm x* * *norm y*

class *real-normed-field* = *real-field* + *real-normed-div-algebra*

instance *real-normed-div-algebra* < *real-normed-algebra-1*
⟨*proof*⟩

context *real-normed-vector* **begin**

lemma *norm-zero* [*simp*]: *norm* (*0*::'*a*) = *0*
⟨*proof*⟩

lemma *zero-less-norm-iff* [*simp*]: *norm x* > *0* ⟷ *x* ≠ *0*
⟨*proof*⟩

lemma *norm-not-less-zero* [*simp*]: ¬ *norm x* < *0*
⟨*proof*⟩

lemma *norm-le-zero-iff* [*simp*]: *norm x* ≤ *0* ⟷ *x* = *0*
⟨*proof*⟩

lemma *norm-minus-cancel* [*simp*]: *norm* (− *x*) = *norm x*
⟨*proof*⟩

lemma *norm-minus-commute*: *norm* (*a* − *b*) = *norm* (*b* − *a*)
⟨*proof*⟩

lemma *dist-add-cancel* [*simp*]: *dist* (*a* + *b*) (*a* + *c*) = *dist b c*
⟨*proof*⟩

lemma *dist-add-cancel2* [*simp*]: *dist* (*b* + *a*) (*c* + *a*) = *dist b c*
⟨*proof*⟩

lemma *norm-uminus-minus*: *norm* (− *x* − *y*) = *norm* (*x* + *y*)
⟨*proof*⟩

lemma *norm-triangle-ineq2*: *norm a* − *norm b* ≤ *norm* (*a* − *b*)
⟨*proof*⟩

lemma *norm-triangle-ineq3*: |*norm a* − *norm b*| ≤ *norm* (*a* − *b*)
⟨*proof*⟩

lemma *norm-triangle-ineq4*: *norm* (*a* − *b*) ≤ *norm a* + *norm b*

⟨*proof*⟩

lemma *norm-triangle-le-diff*: $norm\ x + norm\ y \leq e \Longrightarrow norm\ (x - y) \leq e$
⟨*proof*⟩

lemma *norm-diff-ineq*: $norm\ a - norm\ b \leq norm\ (a + b)$
⟨*proof*⟩

lemma *norm-triangle-sub*: $norm\ x \leq norm\ y + norm\ (x - y)$
⟨*proof*⟩

lemma *norm-triangle-le*: $norm\ x + norm\ y \leq e \Longrightarrow norm\ (x + y) \leq e$
⟨*proof*⟩

lemma *norm-triangle-lt*: $norm\ x + norm\ y < e \Longrightarrow norm\ (x + y) < e$
⟨*proof*⟩

lemma *norm-add-leD*: $norm\ (a + b) \leq c \Longrightarrow norm\ b \leq norm\ a + c$
⟨*proof*⟩

lemma *norm-diff-triangle-ineq*: $norm\ ((a + b) - (c + d)) \leq norm\ (a - c) + norm\ (b - d)$
⟨*proof*⟩

lemma *norm-diff-triangle-le*: $norm\ (x - z) \leq e1 + e2$
if $norm\ (x - y) \leq e1$ $norm\ (y - z) \leq e2$
⟨*proof*⟩

lemma *norm-diff-triangle-less*: $norm\ (x - z) < e1 + e2$
if $norm\ (x - y) < e1$ $norm\ (y - z) < e2$
⟨*proof*⟩

lemma *norm-triangle-mono*:
$norm\ a \leq r \Longrightarrow norm\ b \leq s \Longrightarrow norm\ (a + b) \leq r + s$
⟨*proof*⟩

lemma *norm-sum*: $norm\ (sum\ f\ A) \leq (\sum i{\in}A.\ norm\ (f\ i))$
for $f{::}'b \Rightarrow 'a$
⟨*proof*⟩

lemma *sum-norm-le*: $norm\ (sum\ f\ S) \leq sum\ g\ S$
if $\bigwedge x.\ x \in S \Longrightarrow norm\ (f\ x) \leq g\ x$
for $f{::}'b \Rightarrow 'a$
⟨*proof*⟩

lemma *abs-norm-cancel* [*simp*]: $|norm\ a| = norm\ a$
⟨*proof*⟩

lemma *sum-norm-bound*:

norm (*sum f S*) ≤ *of-nat* (*card S*)∗*K*
if ⋀*x*. *x* ∈ *S* ⟹ *norm* (*f x*) ≤ *K*
for *f* :: *'b* ⇒ *'a*
⟨*proof*⟩

lemma *norm-add-less*: *norm x* < *r* ⟹ *norm y* < *s* ⟹ *norm* (*x* + *y*) < *r* + *s*
⟨*proof*⟩

end

lemma *dist-scaleR* [*simp*]: *dist* (*x* ∗*R a*) (*y* ∗*R a*) = |*x* − *y*| ∗ *norm a*
for *a* :: *'a*::*real-normed-vector*
⟨*proof*⟩

lemma *norm-mult-less*: *norm x* < *r* ⟹ *norm y* < *s* ⟹ *norm* (*x* ∗ *y*) < *r* ∗ *s*
for *x y* :: *'a*::*real-normed-algebra*
⟨*proof*⟩

lemma *norm-of-real* [*simp*]: *norm* (*of-real r* :: *'a*::*real-normed-algebra-1*) = |*r*|
⟨*proof*⟩

lemma *norm-numeral* [*simp*]: *norm* (*numeral w*::*'a*::*real-normed-algebra-1*) = *numeral w*
⟨*proof*⟩

lemma *norm-neg-numeral* [*simp*]: *norm* (− *numeral w*::*'a*::*real-normed-algebra-1*) = *numeral w*
⟨*proof*⟩

lemma *norm-of-real-add1* [*simp*]: *norm* (*of-real x* + *1* :: *'a* :: *real-normed-div-algebra*) = |*x* + *1*|
⟨*proof*⟩

lemma *norm-of-real-addn* [*simp*]:
norm (*of-real x* + *numeral b* :: *'a* :: *real-normed-div-algebra*) = |*x* + *numeral b*|
⟨*proof*⟩

lemma *norm-of-int* [*simp*]: *norm* (*of-int z*::*'a*::*real-normed-algebra-1*) = |*of-int z*|
⟨*proof*⟩

lemma *norm-of-nat* [*simp*]: *norm* (*of-nat n*::*'a*::*real-normed-algebra-1*) = *of-nat n*
⟨*proof*⟩

lemma *nonzero-norm-inverse*: *a* ≠ *0* ⟹ *norm* (*inverse a*) = *inverse* (*norm a*)
for *a* :: *'a*::*real-normed-div-algebra*
⟨*proof*⟩

lemma *norm-inverse*: *norm* (*inverse a*) = *inverse* (*norm a*)
for *a* :: *'a*::{*real-normed-div-algebra*,*division-ring*}

⟨*proof*⟩

lemma *nonzero-norm-divide*: $b \neq 0 \Longrightarrow norm\ (a\ /\ b) = norm\ a\ /\ norm\ b$
for $a\ b :: 'a$::*real-normed-field*
⟨*proof*⟩

lemma *norm-divide*: $norm\ (a\ /\ b) = norm\ a\ /\ norm\ b$
for $a\ b :: 'a$::{*real-normed-field*,*field*}
⟨*proof*⟩

lemma *dist-divide-right*: $dist\ (a/c)\ (b/c) = dist\ a\ b\ /\ norm\ c$ **for** $c :: 'a ::$ *real-normed-field*
⟨*proof*⟩

lemma *norm-inverse-le-norm*:
fixes $x :: 'a$::*real-normed-div-algebra*
shows $r \leq norm\ x \Longrightarrow 0 < r \Longrightarrow norm\ (inverse\ x) \leq inverse\ r$
⟨*proof*⟩

lemma *norm-power-ineq*: $norm\ (x\ \hat{}\ n) \leq norm\ x\ \hat{}\ n$
for $x :: 'a$::*real-normed-algebra-1*
⟨*proof*⟩

lemma *norm-power*: $norm\ (x\ \hat{}\ n) = norm\ x\ \hat{}\ n$
for $x :: 'a$::*real-normed-div-algebra*
⟨*proof*⟩

lemma *norm-power-int*: $norm\ (power\text{-}int\ x\ n) = power\text{-}int\ (norm\ x)\ n$
for $x :: 'a$::*real-normed-div-algebra*
⟨*proof*⟩

lemma *power-eq-imp-eq-norm*:
fixes $w :: 'a$::*real-normed-div-algebra*
assumes *eq*: $w\ \hat{}\ n = z\ \hat{}\ n$ **and** $n > 0$
shows $norm\ w = norm\ z$
⟨*proof*⟩

lemma *power-eq-1-iff*:
fixes $w :: 'a$::*real-normed-div-algebra*
shows $w\ \hat{}\ n = 1 \Longrightarrow norm\ w = 1 \vee n = 0$
⟨*proof*⟩

lemma *norm-mult-numeral1* [*simp*]: $norm\ (numeral\ w * a) = numeral\ w * norm\ a$
for $a\ b :: 'a$::{*real-normed-field*,*field*}
⟨*proof*⟩

lemma *norm-mult-numeral2* [*simp*]: $norm\ (a * numeral\ w) = norm\ a * numeral\ w$

for *a b* :: *'a*::{*real-normed-field*,*field*}
⟨*proof*⟩

lemma *norm-divide-numeral* [*simp*]: *norm* (*a* / *numeral w*) = *norm a* / *numeral w*
for *a b* :: *'a*::{*real-normed-field*,*field*}
⟨*proof*⟩

lemma *norm-of-real-diff* [*simp*]:
norm (*of-real b* − *of-real a* :: *'a*::*real-normed-algebra-1*) ≤ |*b* − *a*|
⟨*proof*⟩

Despite a superficial resemblance, *norm-eq-1* is not relevant.

lemma *square-norm-one*:
fixes *x* :: *'a*::*real-normed-div-algebra*
assumes $x^2 = 1$
shows *norm x* = *1*
⟨*proof*⟩

lemma *norm-less-p1*: *norm x* < *norm* (*of-real* (*norm x*) + *1* :: *'a*)
for *x* :: *'a*::*real-normed-algebra-1*
⟨*proof*⟩

lemma *prod-norm*: *prod* (λ*x*. *norm* (*f x*)) *A* = *norm* (*prod f A*)
for *f* :: *'a* ⇒ *'b*::{*comm-semiring-1*,*real-normed-div-algebra*}
⟨*proof*⟩

lemma *norm-prod-le*:
norm (*prod f A*) ≤ (∏ *a*∈*A*. *norm* (*f a* :: *'a* :: {*real-normed-algebra-1*,*comm-monoid-mult*}))
⟨*proof*⟩

lemma *norm-prod-diff*:
fixes *z w* :: *'i* ⇒ *'a*::{*real-normed-algebra-1*, *comm-monoid-mult*}
shows (⋀*i*. *i* ∈ *I* ⟹ *norm* (*z i*) ≤ *1*) ⟹ (⋀*i*. *i* ∈ *I* ⟹ *norm* (*w i*) ≤ *1*) ⟹
norm ((∏ *i*∈*I*. *z i*) − (∏ *i*∈*I*. *w i*)) ≤ (∑ *i*∈*I*. *norm* (*z i* − *w i*))
⟨*proof*⟩

lemma *norm-power-diff*:
fixes *z w* :: *'a*::{*real-normed-algebra-1*, *comm-monoid-mult*}
assumes *norm z* ≤ *1 norm w* ≤ *1*
shows *norm* (*z*^*m* − *w*^*m*) ≤ *m* ∗ *norm* (*z* − *w*)
⟨*proof*⟩

108.6 Metric spaces

class *metric-space* = *uniformity-dist* + *open-uniformity* +
assumes *dist-eq-0-iff* [*simp*]: *dist x y* = *0* ⟷ *x* = *y*
and *dist-triangle2*: *dist x y* ≤ *dist x z* + *dist y z*
begin

lemma *dist-self* [*simp*]: *dist x x = 0*
⟨*proof*⟩

lemma *zero-le-dist* [*simp*]: *0 ≤ dist x y*
⟨*proof*⟩

lemma *zero-less-dist-iff*: *0 < dist x y ⟷ x ≠ y*
⟨*proof*⟩

lemma *dist-not-less-zero* [*simp*]: *¬ dist x y < 0*
⟨*proof*⟩

lemma *dist-le-zero-iff* [*simp*]: *dist x y ≤ 0 ⟷ x = y*
⟨*proof*⟩

lemma *dist-commute*: *dist x y = dist y x*
⟨*proof*⟩

lemma *dist-commute-lessI*: *dist y x < e ⟹ dist x y < e*
⟨*proof*⟩

lemma *dist-triangle*: *dist x z ≤ dist x y + dist y z*
⟨*proof*⟩

lemma *dist-triangle3*: *dist x y ≤ dist a x + dist a y*
⟨*proof*⟩

lemma *abs-dist-diff-le*: *|dist a b − dist b c| ≤ dist a c*
⟨*proof*⟩

lemma *dist-pos-lt*: *x ≠ y ⟹ 0 < dist x y*
⟨*proof*⟩

lemma *dist-nz*: *x ≠ y ⟷ 0 < dist x y*
⟨*proof*⟩

declare *dist-nz* [*symmetric*, *simp*]

lemma *dist-triangle-le*: *dist x z + dist y z ≤ e ⟹ dist x y ≤ e*
⟨*proof*⟩

lemma *dist-triangle-lt*: *dist x z + dist y z < e ⟹ dist x y < e*
⟨*proof*⟩

lemma *dist-triangle-less-add*: *dist x1 y < e1 ⟹ dist x2 y < e2 ⟹ dist x1 x2 < e1 + e2*
⟨*proof*⟩

lemma *dist-triangle-half-l*: *dist x1 y* $<$ *e* / *2* $\Longrightarrow$ *dist x2 y* $<$ *e* / *2* $\Longrightarrow$ *dist x1 x2* $<$ *e*
⟨*proof*⟩

lemma *dist-triangle-half-r*: *dist y x1* $<$ *e* / *2* $\Longrightarrow$ *dist y x2* $<$ *e* / *2* $\Longrightarrow$ *dist x1 x2* $<$ *e*
⟨*proof*⟩

lemma *dist-triangle-third*:
assumes *dist x1 x2* $<$ *e/3 dist x2 x3* $<$ *e/3 dist x3 x4* $<$ *e/3*
shows *dist x1 x4* $<$ *e*
⟨*proof*⟩

subclass *uniform-space*
⟨*proof*⟩

lemma *open-dist*: *open S* $\longleftrightarrow$ ($\forall$ *x*$\in$*S*. $\exists$ *e*$>$*0*. $\forall$ *y*. *dist y x* $<$ *e* $\longrightarrow$ *y* $\in$ *S*)
⟨*proof*⟩

lemma *open-ball*: *open* {*y*. *dist x y* $<$ *d*}
⟨*proof*⟩

subclass *first-countable-topology*
⟨*proof*⟩

end

instance *metric-space* $\subseteq$ *t2-space*
⟨*proof*⟩

Every normed vector space is a metric space.

instance *real-normed-vector* $<$ *metric-space*
⟨*proof*⟩

108.7 Class instances for real numbers

instantiation *real* :: *real-normed-field*
begin

definition *dist-real-def*: *dist x y* = |*x* − *y*|

definition *uniformity-real-def* [*code del*]:
(*uniformity* :: (*real* × *real*) *filter*) = (*INF e*$\in${*0* $<$..}. *principal* {(*x*, *y*). *dist x y* $<$ *e*})

definition *open-real-def* [*code del*]:
open (*U* :: *real set*) $\longleftrightarrow$ ($\forall$ *x*$\in$*U*. *eventually* (λ(*x*′, *y*). *x*′ = *x* $\longrightarrow$ *y* $\in$ *U*) *uniformity*)

definition *real-norm-def* [*simp*]: *norm r* = |*r*|

instance
⟨*proof*⟩

end

declare *uniformity-Abort*[**where** *'a*=*real*, *code*]

lemma *dist-of-real* [*simp*]: *dist* (*of-real x* :: *'a*) (*of-real y*) = *dist x y*
for *a* :: *'a*::*real-normed-div-algebra*
⟨*proof*⟩

declare [[*code abort*: *open* :: *real set* ⇒ *bool*]]

instance *real* :: *linorder-topology*
⟨*proof*⟩

instance *real* :: *linear-continuum-topology* ⟨*proof*⟩

lemmas *open-real-greaterThan* = *open-greaterThan*[**where** *'a*=*real*]
lemmas *open-real-lessThan* = *open-lessThan*[**where** *'a*=*real*]
lemmas *open-real-greaterThanLessThan* = *open-greaterThanLessThan*[**where** *'a*=*real*]
lemmas *closed-real-atMost* = *closed-atMost*[**where** *'a*=*real*]
lemmas *closed-real-atLeast* = *closed-atLeast*[**where** *'a*=*real*]
lemmas *closed-real-atLeastAtMost* = *closed-atLeastAtMost*[**where** *'a*=*real*]

instance *real* :: *ordered-real-vector*
⟨*proof*⟩

108.8 Extra type constraints

Only allow *open* in class *topological-space*.

⟨*ML*⟩

Only allow *uniformity* in class *uniform-space*.

⟨*ML*⟩

Only allow *dist* in class *metric-space*.

⟨*ML*⟩

Only allow *norm* in class *real-normed-vector*.

⟨*ML*⟩

108.9 Sign function

lemma *norm-sgn*: *norm* (*sgn x*) = (*if x* = *0 then 0 else 1*)
for *x* :: *'a*::*real-normed-vector*
⟨*proof*⟩

lemma *sgn-zero* [*simp*]: *sgn* (*0*::*'a*::*real-normed-vector*) = *0*
⟨*proof*⟩

lemma *sgn-zero-iff*: *sgn x* = *0* ⟷ *x* = *0*
for *x* :: *'a*::*real-normed-vector*
⟨*proof*⟩

lemma *sgn-minus*: *sgn* (− *x*) = − *sgn x*
for *x* :: *'a*::*real-normed-vector*
⟨*proof*⟩

lemma *sgn-scaleR*: *sgn* (*scaleR r x*) = *scaleR* (*sgn r*) (*sgn x*)
for *x* :: *'a*::*real-normed-vector*
⟨*proof*⟩

lemma *sgn-one* [*simp*]: *sgn* (*1*::*'a*::*real-normed-algebra-1*) = *1*
⟨*proof*⟩

lemma *sgn-of-real*: *sgn* (*of-real r* :: *'a*::*real-normed-algebra-1*) = *of-real* (*sgn r*)
⟨*proof*⟩

lemma *sgn-mult*: *sgn* (*x* ∗ *y*) = *sgn x* ∗ *sgn y*
for *x y* :: *'a*::*real-normed-div-algebra*
⟨*proof*⟩

hide-fact (**open**) *sgn-mult*

lemma *real-sgn-eq*: *sgn x* = *x* / |*x*|
for *x* :: *real*
⟨*proof*⟩

lemma *zero-le-sgn-iff* [*simp*]: *0* ≤ *sgn x* ⟷ *0* ≤ *x*
for *x* :: *real*
⟨*proof*⟩

lemma *sgn-le-0-iff* [*simp*]: *sgn x* ≤ *0* ⟷ *x* ≤ *0*
for *x* :: *real*
⟨*proof*⟩

lemma *norm-conv-dist*: *norm x* = *dist x 0*
⟨*proof*⟩

declare *norm-conv-dist* [*symmetric*, *simp*]

lemma *dist-0-norm* [*simp*]: *dist 0 x* = *norm x*
for *x* :: *'a*::*real-normed-vector*
⟨*proof*⟩

lemma *dist-diff* [*simp*]: *dist a* (*a* − *b*) = *norm b* *dist* (*a* − *b*) *a* = *norm b*

⟨*proof*⟩

lemma *dist-of-int*: *dist* (*of-int m*) (*of-int n* :: *'a* :: *real-normed-algebra-1*) = *of-int* |*m* − *n*|
⟨*proof*⟩

lemma *dist-of-nat*:
dist (*of-nat m*) (*of-nat n* :: *'a* :: *real-normed-algebra-1*) = *of-int* |*int m* − *int n*|
⟨*proof*⟩

108.10 Bounded Linear and Bilinear Operators

lemma *linearI*: *linear f*
if ⋀*b1 b2*. *f* (*b1* + *b2*) = *f b1* + *f b2*
⋀*r b*. *f* (*r* ∗*R b*) = *r* ∗*R f b*
⟨*proof*⟩

lemma *linear-iff*:
linear f ⟷ (∀ *x y*. *f* (*x* + *y*) = *f x* + *f y*) ∧ (∀ *c x*. *f* (*c* ∗*R x*) = *c* ∗*R f x*)
(**is** *linear f* ⟷ *?rhs*)
⟨*proof*⟩

lemmas *linear-scaleR-left* = *linear-scale-left*
lemmas *linear-imp-scaleR* = *linear-imp-scale*

corollary *real-linearD*:
fixes *f* :: *real* ⇒ *real*
assumes *linear f* **obtains** *c* **where** *f* = (∗) *c*
⟨*proof*⟩

lemma *linear-times-of-real*: *linear* (λ*x*. *a* ∗ *of-real x*)
⟨*proof*⟩

locale *bounded-linear* = *linear f* **for** *f* :: *'a*::*real-normed-vector* ⇒ *'b*::*real-normed-vector*
+
assumes *bounded*: ∃ *K*. ∀ *x*. *norm* (*f x*) ≤ *norm x* ∗ *K*
begin

lemma *pos-bounded*: ∃ *K*>*0*. ∀ *x*. *norm* (*f x*) ≤ *norm x* ∗ *K*
⟨*proof*⟩

lemma *nonneg-bounded*: ∃ *K*≥*0*. ∀ *x*. *norm* (*f x*) ≤ *norm x* ∗ *K*
⟨*proof*⟩

lemma *linear*: *linear f*
⟨*proof*⟩

end

lemma *bounded-linear-intro*:
assumes $\bigwedge x\ y.\ f\ (x + y) = f\ x + f\ y$
and $\bigwedge r\ x.\ f\ (scaleR\ r\ x) = scaleR\ r\ (f\ x)$
and $\bigwedge x.\ norm\ (f\ x) \leq norm\ x * K$
shows *bounded-linear f*
⟨*proof*⟩

locale *bounded-bilinear* =
fixes *prod* :: *'a*::*real-normed-vector* ⇒ *'b*::*real-normed-vector* ⇒ *'c*::*real-normed-vector*
(**infixl** ** 70*)
assumes *add-left*: $prod\ (a + a')\ b = prod\ a\ b + prod\ a'\ b$
and *add-right*: $prod\ a\ (b + b') = prod\ a\ b + prod\ a\ b'$
and *scaleR-left*: $prod\ (scaleR\ r\ a)\ b = scaleR\ r\ (prod\ a\ b)$
and *scaleR-right*: $prod\ a\ (scaleR\ r\ b) = scaleR\ r\ (prod\ a\ b)$
and *bounded*: $\exists K.\ \forall a\ b.\ norm\ (prod\ a\ b) \leq norm\ a * norm\ b * K$
begin

lemma *pos-bounded*: $\exists K{>}0.\ \forall a\ b.\ norm\ (a ** b) \leq norm\ a * norm\ b * K$
⟨*proof*⟩

lemma *nonneg-bounded*: $\exists K{\geq}0.\ \forall a\ b.\ norm\ (a ** b) \leq norm\ a * norm\ b * K$
⟨*proof*⟩

lemma *additive-right*: *additive* $(\lambda b.\ prod\ a\ b)$
⟨*proof*⟩

lemma *additive-left*: *additive* $(\lambda a.\ prod\ a\ b)$
⟨*proof*⟩

lemma *zero-left*: $prod\ 0\ b = 0$
⟨*proof*⟩

lemma *zero-right*: $prod\ a\ 0 = 0$
⟨*proof*⟩

lemma *minus-left*: $prod\ (-\ a)\ b = -\ prod\ a\ b$
⟨*proof*⟩

lemma *minus-right*: $prod\ a\ (-\ b) = -\ prod\ a\ b$
⟨*proof*⟩

lemma *diff-left*: $prod\ (a - a')\ b = prod\ a\ b - prod\ a'\ b$
⟨*proof*⟩

lemma *diff-right*: $prod\ a\ (b - b') = prod\ a\ b - prod\ a\ b'$
⟨*proof*⟩

lemma *sum-left*: $prod\ (sum\ g\ S)\ x = sum\ ((\lambda i.\ prod\ (g\ i)\ x))\ S$
⟨*proof*⟩

lemma *sum-right*: *prod x* (*sum g S*) = *sum* ((λ*i*. (*prod x* (*g i*)))) *S*
⟨*proof*⟩

lemma *bounded-linear-left*: *bounded-linear* (λ*a*. *a* ** *b*)
⟨*proof*⟩

lemma *bounded-linear-right*: *bounded-linear* (λ*b*. *a* ** *b*)
⟨*proof*⟩

lemma *prod-diff-prod*: (*x* ** *y* − *a* ** *b*) = (*x* − *a*) ** (*y* − *b*) + (*x* − *a*) ** *b* + *a* ** (*y* − *b*)
⟨*proof*⟩

lemma *flip*: *bounded-bilinear* (λ*x y*. *y* ** *x*)
⟨*proof*⟩

lemma *comp1*:
assumes *bounded-linear g*
shows *bounded-bilinear* (λ*x*. (**) (*g x*))
⟨*proof*⟩

lemma *comp*: *bounded-linear f* ⟹ *bounded-linear g* ⟹ *bounded-bilinear* (λ*x y*. *f x* ** *g y*)
⟨*proof*⟩

end

lemma *bounded-linear-ident*[*simp*]: *bounded-linear* (λ*x*. *x*)
⟨*proof*⟩

lemma *bounded-linear-zero*[*simp*]: *bounded-linear* (λ*x*. *0*)
⟨*proof*⟩

lemma *bounded-linear-add*:
assumes *bounded-linear f*
and *bounded-linear g*
shows *bounded-linear* (λ*x*. *f x* + *g x*)
⟨*proof*⟩

lemma *bounded-linear-minus*:
assumes *bounded-linear f*
shows *bounded-linear* (λ*x*. − *f x*)
⟨*proof*⟩

lemma *bounded-linear-sub*: *bounded-linear f* ⟹ *bounded-linear g* ⟹ *bounded-linear* (λ*x*. *f x* − *g x*)
⟨*proof*⟩

lemma *bounded-linear-sum*:
fixes $f :: 'i \Rightarrow 'a::real\text{-}normed\text{-}vector \Rightarrow 'b::real\text{-}normed\text{-}vector$
shows $(\bigwedge i.\ i \in I \Longrightarrow bounded\text{-}linear\ (f\ i)) \Longrightarrow bounded\text{-}linear\ (\lambda x.\ \sum i{\in}I.\ f\ i\ x)$
⟨*proof*⟩

lemma *bounded-linear-compose*:
assumes *bounded-linear f*
and *bounded-linear g*
shows *bounded-linear* $(\lambda x.\ f\ (g\ x))$
⟨*proof*⟩

lemma *bounded-bilinear-mult*: *bounded-bilinear* $((*) :: 'a \Rightarrow 'a \Rightarrow 'a::real\text{-}normed\text{-}algebra)$
⟨*proof*⟩

lemma *bounded-linear-mult-left*: *bounded-linear* $(\lambda x::'a::real\text{-}normed\text{-}algebra.\ x * y)$
⟨*proof*⟩

lemma *bounded-linear-mult-right*: *bounded-linear* $(\lambda y::'a::real\text{-}normed\text{-}algebra.\ x * y)$
⟨*proof*⟩

lemmas *bounded-linear-mult-const* =
bounded-linear-mult-left [*THEN bounded-linear-compose*]

lemmas *bounded-linear-const-mult* =
bounded-linear-mult-right [*THEN bounded-linear-compose*]

lemma *bounded-linear-divide*: *bounded-linear* $(\lambda x.\ x\ /\ y)$
for $y :: 'a::real\text{-}normed\text{-}field$
⟨*proof*⟩

lemma *bounded-bilinear-scaleR*: *bounded-bilinear scaleR*
⟨*proof*⟩

lemma *bounded-linear-scaleR-left*: *bounded-linear* $(\lambda r.\ scaleR\ r\ x)$
⟨*proof*⟩

lemma *bounded-linear-scaleR-right*: *bounded-linear* $(\lambda x.\ scaleR\ r\ x)$
⟨*proof*⟩

lemmas *bounded-linear-scaleR-const* =
bounded-linear-scaleR-left[*THEN bounded-linear-compose*]

lemmas *bounded-linear-const-scaleR* =
bounded-linear-scaleR-right[*THEN bounded-linear-compose*]

lemma *bounded-linear-of-real*: *bounded-linear* ($\lambda r.$ *of-real* r)
⟨*proof*⟩

lemma *real-bounded-linear*: *bounded-linear* $f \longleftrightarrow (\exists\, c{::}\mathit{real}.\ f = (\lambda x.\ x * c))$
for $f :: \mathit{real} \Rightarrow \mathit{real}$
⟨*proof*⟩

instance *real-normed-algebra-1* $\subseteq$ *perfect-space*
⟨*proof*⟩

108.11 Filters and Limits on Metric Space

lemma (**in** *metric-space*) *nhds-metric*: *nhds* x = (*INF* $e{\in}\{0 <..\}$. *principal* $\{y.$ *dist* $y\ x < e\}$)
⟨*proof*⟩

lemma *tendsto-iff-uniformity*:
— More general analogus of *tendsto-iff* below. Applies to all uniform spaces, not just metric ones.
fixes $l :: \langle 'b :: \mathit{uniform\text{-}space} \rangle$
shows ‹(*f* ⟶ *l*) *F* ⟷ (∀ *E*. *eventually E uniformity* ⟶ (∀$_F$ *x in F*. *E* (*f x*, *l*)))›
⟨*proof*⟩

lemma (**in** *metric-space*) *tendsto-iff*: (*f* ⟶ *l*) *F* ⟷ (∀ *e*>*0*. *eventually* (λ*x*. *dist* (*f x*) *l* < *e*) *F*)
⟨*proof*⟩

lemma *tendsto-dist-iff*:
((*f* ⟶ *l*) *F*) ⟷ (((λ*x*. *dist* (*f x*) *l*) ⟶ *0*) *F*)
⟨*proof*⟩

lemma (**in** *metric-space*) *tendstoI* [*intro?*]:
(⋀*e*. *0* < *e* ⟹ *eventually* (λ*x*. *dist* (*f x*) *l* < *e*) *F*) ⟹ (*f* ⟶ *l*) *F*
⟨*proof*⟩

lemma (**in** *metric-space*) *tendstoD*: (*f* ⟶ *l*) *F* ⟹ *0* < *e* ⟹ *eventually* (λ*x*. *dist* (*f x*) *l* < *e*) *F*
⟨*proof*⟩

lemma (**in** *metric-space*) *eventually-nhds-metric*:
eventually P (*nhds a*) ⟷ (∃ *d*>*0*. ∀ *x*. *dist x a* < *d* ⟶ *P x*)
⟨*proof*⟩

lemma *eventually-at*: *eventually P* (*at a within S*) ⟷ (∃ *d*>*0*. ∀ *x*∈*S*. *x* ≠ *a* ∧ *dist x a* < *d* ⟶ *P x*)
for $a :: 'a :: \mathit{metric\text{-}space}$
⟨*proof*⟩

lemma *frequently-at*: *frequently P (at a within S)* $\longleftrightarrow$ $(\forall\, d{>}0.\ \exists\, x{\in}S.\ x \neq a \wedge$ *dist x a* $<$ *d* $\wedge$ *P x*)
for *a* :: *'a* :: *metric-space*
⟨*proof*⟩

lemma *eventually-at-le*: *eventually P (at a within S)* $\longleftrightarrow$ $(\exists\, d{>}0.\ \forall\, x{\in}S.\ x \neq a$ $\wedge$ *dist x a* $\leq$ *d* $\longrightarrow$ *P x*)
for *a* :: *'a*::*metric-space*
⟨*proof*⟩

lemma *eventually-at-left-real*: *a* $>$ (*b* :: *real*) $\Longrightarrow$ *eventually* $(\lambda x.\ x \in \{b{<}..{<}a\})$ (*at-left a*)
⟨*proof*⟩

lemma *eventually-at-right-real*: *a* $<$ (*b* :: *real*) $\Longrightarrow$ *eventually* $(\lambda x.\ x \in \{a{<}..{<}b\})$ (*at-right a*)
⟨*proof*⟩

lemma *metric-tendsto-imp-tendsto*:
fixes *a* :: *'a* :: *metric-space*
and *b* :: *'b* :: *metric-space*
assumes *f*: (*f* $\longrightarrow$ *a*) *F*
and *le*: *eventually* $(\lambda x.$ *dist* (*g x*) *b* $\leq$ *dist* (*f x*) *a*) *F*
shows (*g* $\longrightarrow$ *b*) *F*
⟨*proof*⟩

lemma *filterlim-real-sequentially*: *LIM x sequentially. real x :> at-top*
⟨*proof*⟩

lemma *filterlim-nat-sequentially*: *filterlim nat sequentially at-top*
⟨*proof*⟩

lemma *filterlim-floor-sequentially*: *filterlim floor at-top at-top*
⟨*proof*⟩

lemma *filterlim-sequentially-iff-filterlim-real*:
filterlim f sequentially F $\longleftrightarrow$ *filterlim* $(\lambda x.$ *real* (*f x*)) *at-top F* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

108.11.1 Limits of Sequences

lemma *lim-sequentially*: *X* $\longrightarrow$ *L* $\longleftrightarrow$ $(\forall\, r{>}0.\ \exists\, no.\ \forall\, n{\geq}no.$ *dist* (*X n*) *L* $<$ *r*)
for *L* :: *'a*::*metric-space*
⟨*proof*⟩

lemmas *LIMSEQ-def* = *lim-sequentially*

lemma *LIMSEQ-iff-nz*: $X \longrightarrow L \longleftrightarrow (\forall r{>}0.\ \exists no{>}0.\ \forall n{\geq}no.\ dist\ (X\ n)\ L < r)$
for L :: *'a*::*metric-space*
⟨*proof*⟩

lemma *metric-LIMSEQ-I*: $(\bigwedge r.\ 0 < r \Longrightarrow \exists no.\ \forall n{\geq}no.\ dist\ (X\ n)\ L < r) \Longrightarrow X \longrightarrow L$
for L :: *'a*::*metric-space*
⟨*proof*⟩

lemma *metric-LIMSEQ-D*: $X \longrightarrow L \Longrightarrow 0 < r \Longrightarrow \exists no.\ \forall n{\geq}no.\ dist\ (X\ n)\ L < r$
for L :: *'a*::*metric-space*
⟨*proof*⟩

lemma *LIMSEQ-norm-0*:
assumes $\bigwedge n$::*nat*. *norm* $(f\ n) < 1\ /\ real\ (Suc\ n)$
shows $f \longrightarrow 0$
⟨*proof*⟩

108.11.2 Limits of Functions

lemma *LIM-def*: $f\ {-}a{\rightarrow}\ L \longleftrightarrow (\forall r > 0.\ \exists s > 0.\ \forall x.\ x \neq a \wedge dist\ x\ a < s \longrightarrow dist\ (f\ x)\ L < r)$
for a :: *'a*::*metric-space* **and** L :: *'b*::*metric-space*
⟨*proof*⟩

lemma *metric-LIM-I*:
$(\bigwedge r.\ 0 < r \Longrightarrow \exists s{>}0.\ \forall x.\ x \neq a \wedge dist\ x\ a < s \longrightarrow dist\ (f\ x)\ L < r) \Longrightarrow f\ {-}a{\rightarrow}\ L$
for a :: *'a*::*metric-space* **and** L :: *'b*::*metric-space*
⟨*proof*⟩

lemma *metric-LIM-D*: $f\ {-}a{\rightarrow}\ L \Longrightarrow 0 < r \Longrightarrow \exists s{>}0.\ \forall x.\ x \neq a \wedge dist\ x\ a < s \longrightarrow dist\ (f\ x)\ L < r$
for a :: *'a*::*metric-space* **and** L :: *'b*::*metric-space*
⟨*proof*⟩

lemma *metric-LIM-imp-LIM*:
fixes l :: *'a*::*metric-space*
and m :: *'b*::*metric-space*
assumes f: $f\ {-}a{\rightarrow}\ l$
and *le*: $\bigwedge x.\ x \neq a \Longrightarrow dist\ (g\ x)\ m \leq dist\ (f\ x)\ l$
shows $g\ {-}a{\rightarrow}\ m$
⟨*proof*⟩

lemma *metric-LIM-equal2*:
fixes a :: *'a*::*metric-space*
assumes $g\ {-}a{\rightarrow}\ l\ 0 < R$

and $\bigwedge x.\ x \neq a \Longrightarrow dist\ x\ a < R \Longrightarrow f\ x = g\ x$
shows $f \ -a\rightarrow\ l$
⟨*proof*⟩

lemma *metric-LIM-compose2*:
fixes $a :: 'a$::*metric-space*
assumes *f*: $f \ -a\rightarrow\ b$
and *g*: $g \ -b\rightarrow\ c$
and *inj*: $\exists\, d{>}0.\ \forall x.\ x \neq a \wedge dist\ x\ a < d \longrightarrow f\ x \neq b$
shows $(\lambda x.\ g\ (f\ x)) \ -a\rightarrow\ c$
⟨*proof*⟩

lemma *metric-isCont-LIM-compose2*:
fixes $f :: 'a :: metric\text{-}space \Rightarrow$ -
assumes *f* [*unfolded isCont-def*]: *isCont f a*
and *g*: $g \ -f\ a\rightarrow\ l$
and *inj*: $\exists\, d{>}0.\ \forall x.\ x \neq a \wedge dist\ x\ a < d \longrightarrow f\ x \neq f\ a$
shows $(\lambda x.\ g\ (f\ x)) \ -a\rightarrow\ l$
⟨*proof*⟩

108.12 Complete metric spaces

108.13 Cauchy sequences

lemma (**in** *metric-space*) *Cauchy-def*: $Cauchy\ X = (\forall\, e{>}0.\ \exists\, M.\ \forall\, m{\geq}M.\ \forall\, n{\geq}M.\ dist\ (X\ m)\ (X\ n) < e)$
⟨*proof*⟩

lemma (**in** *metric-space*) *Cauchy-altdef*: $Cauchy\ f \longleftrightarrow (\forall\, e{>}0.\ \exists\, M.\ \forall\, m{\geq}M.\ \forall\, n{>}m.\ dist\ (f\ m)\ (f\ n) < e)$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma (**in** *metric-space*) *Cauchy-altdef2*: $Cauchy\ s \longleftrightarrow (\forall\, e{>}0.\ \exists\, N$::*nat*$.\ \forall\, n{\geq}N.\ dist(s\ n)(s\ N) < e)$ (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma (**in** *metric-space*) *metric-CauchyI*:
$(\bigwedge e.\ 0 < e \Longrightarrow \exists\, M.\ \forall\, m{\geq}M.\ \forall\, n{\geq}M.\ dist\ (X\ m)\ (X\ n) < e) \Longrightarrow Cauchy\ X$
⟨*proof*⟩

lemma (**in** *metric-space*) *CauchyI'*:
$(\bigwedge e.\ 0 < e \Longrightarrow \exists\, M.\ \forall\, m{\geq}M.\ \forall\, n{>}m.\ dist\ (X\ m)\ (X\ n) < e) \Longrightarrow Cauchy\ X$
⟨*proof*⟩

lemma (**in** *metric-space*) *metric-CauchyD*:
$Cauchy\ X \Longrightarrow 0 < e \Longrightarrow \exists\, M.\ \forall\, m{\geq}M.\ \forall\, n{\geq}M.\ dist\ (X\ m)\ (X\ n) < e$
⟨*proof*⟩

lemma (**in** *metric-space*) *metric-Cauchy-iff2*:

Cauchy X = (∀ *j*. (∃ *M*. ∀ *m* ≥ *M*. ∀ *n* ≥ *M*. *dist* (*X m*) (*X n*) < *inverse*(*real* (*Suc j*))))
⟨*proof*⟩

lemma *Cauchy-iff2*: *Cauchy X* ⟷ (∀ *j*. (∃ *M*. ∀ *m* ≥ *M*. ∀ *n* ≥ *M*. |*X m* − *X n*| < *inverse* (*real* (*Suc j*))))
⟨*proof*⟩

lemma *lim-1-over-n* [*tendsto-intros*]: ((λ*n*. *1* / *of-nat n*) ⟶ (*0*::'*a*::*real-normed-field*)) *sequentially*
⟨*proof*⟩

lemma (**in** *metric-space*) *complete-def*:
shows *complete S* = (∀ *f*. (∀ *n*. *f n* ∈ *S*) ∧ *Cauchy f* ⟶ (∃ *l*∈*S*. *f* ⟶ *l*))
⟨*proof*⟩

apparently unused

lemma (**in** *metric-space*) *totally-bounded-metric*:
totally-bounded S ⟷ (∀ *e*>*0*. ∃ *k*. *finite k* ∧ *S* ⊆ (⋃ *x*∈*k*. {*y*. *dist x y* < *e*}))
⟨*proof*⟩

⟨*ML*⟩

lemma *cauchy-filter-metric*:
fixes *F* :: '*a*::{*uniformity-dist*,*uniform-space*} *filter*
shows *cauchy-filter F* ⟷ (∀ *e*. *e*>*0* ⟶ (∃ *P*. *eventually P F* ∧ (∀ *x y*. *P x* ∧ *P y* ⟶ *dist x y* < *e*)))
⟨*proof*⟩

lemma *cauchy-filter-metric-filtermap*:
fixes *f* :: '*a* ⇒ '*b*::{*uniformity-dist*,*uniform-space*}
shows *cauchy-filter* (*filtermap f F*) ⟷ (∀ *e*. *e*>*0* ⟶ (∃ *P*. *eventually P F* ∧ (∀ *x y*. *P x* ∧ *P y* ⟶ *dist* (*f x*) (*f y*) < *e*)))
⟨*proof*⟩

⟨*ML*⟩

108.13.1 Cauchy Sequences are Convergent

class *complete-space* = *metric-space* +
assumes *Cauchy-convergent*: *Cauchy X* ⟹ *convergent X*

lemma *Cauchy-convergent-iff*: *Cauchy X* ⟷ *convergent X*
for *X* :: *nat* ⇒ '*a*::*complete-space*
⟨*proof*⟩

To prove that a Cauchy sequence converges, it suffices to show that a sub-

sequence converges.

lemma *Cauchy-converges-subseq*:
fixes u::*nat* $\Rightarrow$ $'a$::*metric-space*
assumes *Cauchy* u
strict-mono r
$(u \circ r) \longrightarrow l$
shows $u \longrightarrow l$
⟨*proof*⟩

108.14 The set of real numbers is a complete metric space

Proof that Cauchy sequences converge based on the one from http://pirate.shu.edu/~wachsmut/ira/numseq/proofs/cauconv.html

If sequence X is Cauchy, then its limit is the lub of $\{r.\ \exists N.\ \forall n{\geq}N.\ r < X\ n\}$

lemma *increasing-LIMSEQ*:
fixes f :: *nat* $\Rightarrow$ *real*
assumes *inc*: $\bigwedge n.\ f\ n \leq f\ (Suc\ n)$
and *bdd*: $\bigwedge n.\ f\ n \leq l$
and *en*: $\bigwedge e.\ 0 < e \Longrightarrow \exists n.\ l \leq f\ n + e$
shows $f \longrightarrow l$
⟨*proof*⟩

lemma *real-Cauchy-convergent*:
fixes X :: *nat* $\Rightarrow$ *real*
assumes X: *Cauchy* X
shows *convergent* X
⟨*proof*⟩

instance *real* :: *complete-space*
⟨*proof*⟩

class *banach* = *real-normed-vector* + *complete-space*

instance *real* :: *banach* ⟨*proof*⟩

lemma *tendsto-at-topI-sequentially*:
fixes f :: *real* $\Rightarrow$ $'b$::*first-countable-topology*
assumes $*$: $\bigwedge X.$ *filterlim* X *at-top sequentially* $\Longrightarrow$ $(\lambda n.\ f\ (X\ n)) \longrightarrow y$
shows $(f \longrightarrow y)$ *at-top*
⟨*proof*⟩

lemma *tendsto-at-topI-sequentially-real*:
fixes f :: *real* $\Rightarrow$ *real*
assumes *mono*: *mono* f
and *limseq*: $(\lambda n.\ f\ (real\ n)) \longrightarrow y$
shows $(f \longrightarrow y)$ *at-top*

⟨*proof*⟩

end

109 Limits on Real Vector Spaces

theory *Limits*
imports *Real-Vector-Spaces*
begin

lemma *range-mult* [*simp*]:
fixes *a*::*real* **shows** *range* ((∗) *a*) = (*if a*=*0 then* {*0*} *else UNIV*)
⟨*proof*⟩

109.1 Filter going to infinity norm

definition *at-infinity* :: *'a*::*real-normed-vector filter*
where *at-infinity* = (*INF r*. *principal* {*x*. *r* ≤ *norm x*})

lemma *eventually-at-infinity*: *eventually P at-infinity* ⟷ (∃ *b*. ∀ *x*. *b* ≤ *norm x* ⟶ *P x*)
⟨*proof*⟩

corollary *eventually-at-infinity-pos*:
eventually p at-infinity ⟷ (∃ *b*. *0* < *b* ∧ (∀ *x*. *norm x* ≥ *b* ⟶ *p x*))
⟨*proof*⟩

lemma *at-infinity-eq-at-top-bot*: (*at-infinity* :: *real filter*) = *sup at-top at-bot*
⟨*proof*⟩

lemma *at-top-le-at-infinity*: *at-top* ≤ (*at-infinity* :: *real filter*)
⟨*proof*⟩

lemma *at-bot-le-at-infinity*: *at-bot* ≤ (*at-infinity* :: *real filter*)
⟨*proof*⟩

lemma *filterlim-at-top-imp-at-infinity*: *filterlim f at-top F* ⟹ *filterlim f at-infinity F*
for *f* :: - ⇒ *real*
⟨*proof*⟩

lemma *filterlim-real-at-infinity-sequentially*: *filterlim real at-infinity sequentially*
⟨*proof*⟩

lemma *lim-infinity-imp-sequentially*: (*f* ⟶ *l*) *at-infinity* ⟹ ((λ*n*. *f*(*n*)) ⟶ *l*) *sequentially*
⟨*proof*⟩

109.1.1 Boundedness

definition *Bfun* :: $('a \Rightarrow 'b{::}\textit{metric-space}) \Rightarrow 'a\ \textit{filter} \Rightarrow \textit{bool}$
where *Bfun-metric-def*: $\textit{Bfun}\ f\ F = (\exists y.\ \exists K{>}0.\ \textit{eventually}\ (\lambda x.\ \textit{dist}\ (f\ x)\ y \leq K)\ F)$

abbreviation *Bseq* :: $(\textit{nat} \Rightarrow 'a{::}\textit{metric-space}) \Rightarrow \textit{bool}$
where $\textit{Bseq}\ X \equiv \textit{Bfun}\ X\ \textit{sequentially}$

lemma *Bseq-conv-Bfun*: $\textit{Bseq}\ X \longleftrightarrow \textit{Bfun}\ X\ \textit{sequentially}$ ⟨*proof*⟩

lemma *Bseq-ignore-initial-segment*: $\textit{Bseq}\ X \Longrightarrow \textit{Bseq}\ (\lambda n.\ X\ (n + k))$
⟨*proof*⟩

lemma *Bseq-offset*: $\textit{Bseq}\ (\lambda n.\ X\ (n + k)) \Longrightarrow \textit{Bseq}\ X$
⟨*proof*⟩

lemma *Bfun-def*: $\textit{Bfun}\ f\ F \longleftrightarrow (\exists K{>}0.\ \textit{eventually}\ (\lambda x.\ \textit{norm}\ (f\ x) \leq K)\ F)$
⟨*proof*⟩

lemma *BfunI*:
assumes *K*: $\textit{eventually}\ (\lambda x.\ \textit{norm}\ (f\ x) \leq K)\ F$
shows $\textit{Bfun}\ f\ F$
⟨*proof*⟩

lemma *BfunE*:
assumes $\textit{Bfun}\ f\ F$
obtains B **where** $0 < B$ **and** $\textit{eventually}\ (\lambda x.\ \textit{norm}\ (f\ x) \leq B)\ F$
⟨*proof*⟩

lemma *Cauchy-Bseq*:
assumes $\textit{Cauchy}\ X$ **shows** $\textit{Bseq}\ X$
⟨*proof*⟩

109.1.2 Bounded Sequences

lemma *BseqI'*: $(\bigwedge n.\ \textit{norm}\ (X\ n) \leq K) \Longrightarrow \textit{Bseq}\ X$
⟨*proof*⟩

lemma *Bseq-def*: $\textit{Bseq}\ X \longleftrightarrow (\exists K{>}0.\ \forall n.\ \textit{norm}\ (X\ n) \leq K)$
⟨*proof*⟩

lemma *BseqE*: $\textit{Bseq}\ X \Longrightarrow (\bigwedge K.\ 0 < K \Longrightarrow \forall n.\ \textit{norm}\ (X\ n) \leq K \Longrightarrow Q) \Longrightarrow Q$
⟨*proof*⟩

lemma *BseqD*: $\textit{Bseq}\ X \Longrightarrow \exists K.\ 0 < K \wedge (\forall n.\ \textit{norm}\ (X\ n) \leq K)$
⟨*proof*⟩

lemma *BseqI*: $0 < K \Longrightarrow \forall n.\ \textit{norm}\ (X\ n) \leq K \Longrightarrow \textit{Bseq}\ X$

⟨*proof*⟩

lemma *Bseq-bdd-above*: *Bseq X* ⟹ *bdd-above* (*range X*)
for *X* :: *nat* ⇒ *real*
⟨*proof*⟩

lemma *Bseq-bdd-above'*: *Bseq X* ⟹ *bdd-above* (*range* (λ*n*. *norm* (*X n*)))
for *X* :: *nat* ⇒ *'a* :: *real-normed-vector*
⟨*proof*⟩

lemma *Bseq-bdd-below*: *Bseq X* ⟹ *bdd-below* (*range X*)
for *X* :: *nat* ⇒ *real*
⟨*proof*⟩

lemma *Bseq-eventually-mono*:
assumes *eventually* (λ*n*. *norm* (*f n*) ≤ *norm* (*g n*)) *sequentially Bseq g*
shows *Bseq f*
⟨*proof*⟩

lemma *lemma-NBseq-def*: (∃ *K* > *0*. ∀ *n*. *norm* (*X n*) ≤ *K*) ⟷ (∃ *N*. ∀ *n*. *norm* (*X n*) ≤ *real*(*Suc N*))
⟨*proof*⟩

Alternative definition for *Bseq*.

lemma *Bseq-iff*: *Bseq X* ⟷ (∃ *N*. ∀ *n*. *norm* (*X n*) ≤ *real*(*Suc N*))
⟨*proof*⟩

lemma *lemma-NBseq-def2*: (∃ *K* > *0*. ∀ *n*. *norm* (*X n*) ≤ *K*) = (∃ *N*. ∀ *n*. *norm* (*X n*) < *real*(*Suc N*))
⟨*proof*⟩

Yet another definition for Bseq.

lemma *Bseq-iff1a*: *Bseq X* ⟷ (∃ *N*. ∀ *n*. *norm* (*X n*) < *real* (*Suc N*))
⟨*proof*⟩

109.1.3 A Few More Equivalence Theorems for Boundedness

Alternative formulation for boundedness.

lemma *Bseq-iff2*: *Bseq X* ⟷ (∃ *k* > *0*. ∃ *x*. ∀ *n*. *norm* (*X n* + − *x*) ≤ *k*)
⟨*proof*⟩

Alternative formulation for boundedness.

lemma *Bseq-iff3*: *Bseq X* ⟷ (∃ *k*>*0*. ∃ *N*. ∀ *n*. *norm* (*X n* + − *X N*) ≤ *k*)
(**is** *?P* ⟷ *?Q*)
⟨*proof*⟩

109.1.4 Upper Bounds and Lubs of Bounded Sequences

lemma *Bseq-minus-iff*: *Bseq* (λ*n*. − (*X n*) :: *'a*::*real-normed-vector*) ⟷ *Bseq X*

⟨*proof*⟩

lemma *Bseq-add*:
fixes *f* :: *nat* ⇒ *'a*::*real-normed-vector*
assumes *Bseq f*
shows *Bseq* (λ*x*. *f x* + *c*)
⟨*proof*⟩

lemma *Bseq-add-iff*: *Bseq* (λ*x*. *f x* + *c*) ⟷ *Bseq f*
for *f* :: *nat* ⇒ *'a*::*real-normed-vector*
⟨*proof*⟩

lemma *Bseq-mult*:
fixes *f g* :: *nat* ⇒ *'a*::*real-normed-field*
assumes *Bseq f* **and** *Bseq g*
shows *Bseq* (λ*x*. *f x* ∗ *g x*)
⟨*proof*⟩

lemma *Bfun-const* [*simp*]: *Bfun* (λ-. *c*) *F*
⟨*proof*⟩

lemma *Bseq-cmult-iff*:
fixes *c* :: *'a*::*real-normed-field*
assumes *c* ≠ *0*
shows *Bseq* (λ*x*. *c* ∗ *f x*) ⟷ *Bseq f*
⟨*proof*⟩

lemma *Bseq-subseq*: *Bseq f* ⟹ *Bseq* (λ*x*. *f* (*g x*))
for *f* :: *nat* ⇒ *'a*::*real-normed-vector*
⟨*proof*⟩

lemma *Bseq-Suc-iff*: *Bseq* (λ*n*. *f* (*Suc n*)) ⟷ *Bseq f*
for *f* :: *nat* ⇒ *'a*::*real-normed-vector*
⟨*proof*⟩

lemma *increasing-Bseq-subseq-iff*:
assumes ⋀*x y*. *x* ≤ *y* ⟹ *norm* (*f x* :: *'a*::*real-normed-vector*) ≤ *norm* (*f y*)
strict-mono g
shows *Bseq* (λ*x*. *f* (*g x*)) ⟷ *Bseq f*
⟨*proof*⟩

lemma *nonneg-incseq-Bseq-subseq-iff*:
fixes *f* :: *nat* ⇒ *real*
and *g* :: *nat* ⇒ *nat*
assumes ⋀*x*. *f x* ≥ *0* *incseq f* *strict-mono g*
shows *Bseq* (λ*x*. *f* (*g x*)) ⟷ *Bseq f*
⟨*proof*⟩

lemma *Bseq-eq-bounded*: *range f* ⊆ {*a*..*b*} ⟹ *Bseq f*

for *a b* :: *real*
⟨*proof*⟩

lemma *incseq-bounded*: *incseq X* ⟹ ∀ *i*. *X i* ≤ *B* ⟹ *Bseq X*
for *B* :: *real*
⟨*proof*⟩

lemma *decseq-bounded*: *decseq X* ⟹ ∀ *i*. *B* ≤ *X i* ⟹ *Bseq X*
for *B* :: *real*
⟨*proof*⟩

109.1.5 Polynomal function extremal theorem, from HOL Light

lemma *polyfun-extremal-lemma*:
fixes *c* :: *nat* ⇒ *'a*::*real-normed-div-algebra*
assumes *0* < *e*
shows ∃ *M*. ∀ *z*. *M* ≤ *norm*(*z*) ⟶ *norm* (∑ *i*≤*n*. *c*(*i*) ∗ *z*^*i*) ≤ *e* ∗ *norm*(*z*) ^ (*Suc n*)
⟨*proof*⟩

lemma *polyfun-extremal*:
fixes *c* :: *nat* ⇒ *'a*::*real-normed-div-algebra*
assumes *k*: *c k* ≠ *0* *1*≤*k* **and** *kn*: *k*≤*n*
shows *eventually* (λ*z*. *norm* (∑ *i*≤*n*. *c*(*i*) ∗ *z*^*i*) ≥ *B*) *at-infinity*
⟨*proof*⟩

109.2 Convergence to Zero

definition *Zfun* :: (*'a* ⇒ *'b*::*real-normed-vector*) ⇒ *'a filter* ⇒ *bool*
where *Zfun f F* = (∀ *r*>*0*. *eventually* (λ*x*. *norm* (*f x*) < *r*) *F*)

lemma *ZfunI*: (⋀*r*. *0* < *r* ⟹ *eventually* (λ*x*. *norm* (*f x*) < *r*) *F*) ⟹ *Zfun f F*
⟨*proof*⟩

lemma *ZfunD*: *Zfun f F* ⟹ *0* < *r* ⟹ *eventually* (λ*x*. *norm* (*f x*) < *r*) *F*
⟨*proof*⟩

lemma *Zfun-ssubst*: *eventually* (λ*x*. *f x* = *g x*) *F* ⟹ *Zfun g F* ⟹ *Zfun f F*
⟨*proof*⟩

lemma *Zfun-zero*: *Zfun* (λ*x*. *0*) *F*
⟨*proof*⟩

lemma *Zfun-norm-iff*: *Zfun* (λ*x*. *norm* (*f x*)) *F* = *Zfun* (λ*x*. *f x*) *F*
⟨*proof*⟩

lemma *Zfun-imp-Zfun*:
assumes *f*: *Zfun f F*
and *g*: *eventually* (λ*x*. *norm* (*g x*) ≤ *norm* (*f x*) ∗ *K*) *F*
shows *Zfun* (λ*x*. *g x*) *F*

⟨*proof*⟩

lemma *Zfun-le*: *Zfun g F* ⟹ ∀ *x*. *norm* (*f x*) ≤ *norm* (*g x*) ⟹ *Zfun f F*
⟨*proof*⟩

lemma *Zfun-add*:
assumes *f*: *Zfun f F*
and *g*: *Zfun g F*
shows *Zfun* (λ*x*. *f x* + *g x*) *F*
⟨*proof*⟩

lemma *Zfun-minus*: *Zfun f F* ⟹ *Zfun* (λ*x*. − *f x*) *F*
⟨*proof*⟩

lemma *Zfun-diff*: *Zfun f F* ⟹ *Zfun g F* ⟹ *Zfun* (λ*x*. *f x* − *g x*) *F*
⟨*proof*⟩

lemma (**in** *bounded-linear*) *Zfun*:
assumes *g*: *Zfun g F*
shows *Zfun* (λ*x*. *f* (*g x*)) *F*
⟨*proof*⟩

lemma (**in** *bounded-bilinear*) *Zfun*:
assumes *f*: *Zfun f F*
and *g*: *Zfun g F*
shows *Zfun* (λ*x*. *f x* ** *g x*) *F*
⟨*proof*⟩

lemma (**in** *bounded-bilinear*) *Zfun-left*: *Zfun f F* ⟹ *Zfun* (λ*x*. *f x* ** *a*) *F*
⟨*proof*⟩

lemma (**in** *bounded-bilinear*) *Zfun-right*: *Zfun f F* ⟹ *Zfun* (λ*x*. *a* ** *f x*) *F*
⟨*proof*⟩

lemmas *Zfun-mult* = *bounded-bilinear.Zfun* [*OF bounded-bilinear-mult*]
lemmas *Zfun-mult-right* = *bounded-bilinear.Zfun-right* [*OF bounded-bilinear-mult*]
lemmas *Zfun-mult-left* = *bounded-bilinear.Zfun-left* [*OF bounded-bilinear-mult*]

lemma *tendsto-Zfun-iff*: (*f* ⟶ *a*) *F* = *Zfun* (λ*x*. *f x* − *a*) *F*
⟨*proof*⟩

lemma *tendsto-0-le*:
(*f* ⟶ 0) *F* ⟹ *eventually* (λ*x*. *norm* (*g x*) ≤ *norm* (*f x*) ∗ *K*) *F* ⟹ (*g* ⟶ 0) *F*
⟨*proof*⟩

109.2.1 Distance and norms

lemma *tendsto-dist* [*tendsto-intros*]:

fixes $l\ m :: 'a$::*metric-space*
assumes f: $(f \longrightarrow l)\ F$
and g: $(g \longrightarrow m)\ F$
shows $((\lambda x.\ dist\ (f\ x)\ (g\ x)) \longrightarrow dist\ l\ m)\ F$
⟨*proof*⟩

lemma *continuous-dist*[*continuous-intros*]:
fixes $f\ g :: - \Rightarrow 'a ::$ *metric-space*
shows *continuous* $F\ f \Longrightarrow$ *continuous* $F\ g \Longrightarrow$ *continuous* $F\ (\lambda x.\ dist\ (f\ x)\ (g\ x))$
⟨*proof*⟩

lemma *continuous-on-dist*[*continuous-intros*]:
fixes $f\ g :: - \Rightarrow 'a ::$ *metric-space*
shows *continuous-on* $s\ f \Longrightarrow$ *continuous-on* $s\ g \Longrightarrow$ *continuous-on* $s\ (\lambda x.\ dist\ (f\ x)\ (g\ x))$
⟨*proof*⟩

lemma *continuous-at-dist*: *isCont* $(dist\ a)\ b$
⟨*proof*⟩

lemma *tendsto-norm* [*tendsto-intros*]: $(f \longrightarrow a)\ F \Longrightarrow ((\lambda x.\ norm\ (f\ x)) \longrightarrow norm\ a)\ F$
⟨*proof*⟩

lemma *continuous-norm* [*continuous-intros*]: *continuous* $F\ f \Longrightarrow$ *continuous* $F\ (\lambda x.\ norm\ (f\ x))$
⟨*proof*⟩

lemma *continuous-on-norm* [*continuous-intros*]:
continuous-on $s\ f \Longrightarrow$ *continuous-on* $s\ (\lambda x.\ norm\ (f\ x))$
⟨*proof*⟩

lemma *continuous-on-norm-id* [*continuous-intros*]: *continuous-on* S *norm*
⟨*proof*⟩

lemma *tendsto-norm-zero*: $(f \longrightarrow 0)\ F \Longrightarrow ((\lambda x.\ norm\ (f\ x)) \longrightarrow 0)\ F$
⟨*proof*⟩

lemma *tendsto-norm-zero-cancel*: $((\lambda x.\ norm\ (f\ x)) \longrightarrow 0)\ F \Longrightarrow (f \longrightarrow 0)\ F$
⟨*proof*⟩

lemma *tendsto-norm-zero-iff*: $((\lambda x.\ norm\ (f\ x)) \longrightarrow 0)\ F \longleftrightarrow (f \longrightarrow 0)\ F$
⟨*proof*⟩

lemma *tendsto-rabs* [*tendsto-intros*]: $(f \longrightarrow l)\ F \Longrightarrow ((\lambda x.\ |f\ x|) \longrightarrow |l|)\ F$
for $l ::$ *real*
⟨*proof*⟩

lemma *continuous-rabs* [*continuous-intros*]:
continuous F f $\Longrightarrow$ *continuous F* ($\lambda x.$ $|f\ x :: real|$)
⟨*proof*⟩

lemma *continuous-on-rabs* [*continuous-intros*]:
continuous-on s f $\Longrightarrow$ *continuous-on s* ($\lambda x.$ $|f\ x :: real|$)
⟨*proof*⟩

lemma *tendsto-rabs-zero*: $(f \longrightarrow (0::real))\ F \Longrightarrow ((\lambda x.\ |f\ x|) \longrightarrow 0)\ F$
⟨*proof*⟩

lemma *tendsto-rabs-zero-cancel*: $((\lambda x.\ |f\ x|) \longrightarrow (0::real))\ F \Longrightarrow (f \longrightarrow 0)\ F$
⟨*proof*⟩

lemma *tendsto-rabs-zero-iff*: $((\lambda x.\ |f\ x|) \longrightarrow (0::real))\ F \longleftrightarrow (f \longrightarrow 0)\ F$
⟨*proof*⟩

109.3 Topological Monoid

class *topological-monoid-add* = *topological-space* + *monoid-add* +
assumes *tendsto-add-Pair*: *LIM x* ($nhds\ a \times_F nhds\ b$). *fst x* + *snd x* :> *nhds* (*a* + *b*)

class *topological-comm-monoid-add* = *topological-monoid-add* + *comm-monoid-add*

lemma *tendsto-add* [*tendsto-intros*]:
fixes *a b* :: *'a*::*topological-monoid-add*
shows $(f \longrightarrow a)\ F \Longrightarrow (g \longrightarrow b)\ F \Longrightarrow ((\lambda x.\ f\ x + g\ x) \longrightarrow a + b)\ F$
⟨*proof*⟩

lemma *continuous-add* [*continuous-intros*]:
fixes *f g* :: - ⇒ *'b*::*topological-monoid-add*
shows *continuous F f* $\Longrightarrow$ *continuous F g* $\Longrightarrow$ *continuous F* ($\lambda x.$ *f x* + *g x*)
⟨*proof*⟩

lemma *continuous-on-add* [*continuous-intros*]:
fixes *f g* :: - ⇒ *'b*::*topological-monoid-add*
shows *continuous-on s f* $\Longrightarrow$ *continuous-on s g* $\Longrightarrow$ *continuous-on s* ($\lambda x.$ *f x* + *g x*)
⟨*proof*⟩

lemma *tendsto-add-zero*:
fixes *f g* :: - ⇒ *'b*::*topological-monoid-add*
shows $(f \longrightarrow 0)\ F \Longrightarrow (g \longrightarrow 0)\ F \Longrightarrow ((\lambda x.\ f\ x + g\ x) \longrightarrow 0)\ F$
⟨*proof*⟩

lemma *tendsto-sum* [*tendsto-intros*]:
fixes *f* :: *'a* ⇒ *'b* ⇒ *'c*::*topological-comm-monoid-add*

shows $(\bigwedge i.\ i \in I \Longrightarrow (f\ i \longrightarrow a\ i)\ F) \Longrightarrow ((\lambda x.\ \sum i{\in}I.\ f\ i\ x) \longrightarrow (\sum i{\in}I.\ a\ i))\ F$
⟨*proof*⟩

lemma *tendsto-null-sum*:
fixes $f :: 'a \Rightarrow 'b \Rightarrow 'c$::*topological-comm-monoid-add*
assumes $\bigwedge i.\ i \in I \Longrightarrow ((\lambda x.\ f\ x\ i) \longrightarrow 0)\ F$
shows $((\lambda i.\ sum\ (f\ i)\ I) \longrightarrow 0)\ F$
⟨*proof*⟩

lemma *continuous-sum* [*continuous-intros*]:
fixes $f :: 'a \Rightarrow 'b$::*t2-space* $\Rightarrow 'c$::*topological-comm-monoid-add*
shows $(\bigwedge i.\ i \in I \Longrightarrow continuous\ F\ (f\ i)) \Longrightarrow continuous\ F\ (\lambda x.\ \sum i{\in}I.\ f\ i\ x)$
⟨*proof*⟩

lemma *continuous-on-sum* [*continuous-intros*]:
fixes $f :: 'a \Rightarrow 'b$::*topological-space* $\Rightarrow 'c$::*topological-comm-monoid-add*
shows $(\bigwedge i.\ i \in I \Longrightarrow continuous\text{-}on\ S\ (f\ i)) \Longrightarrow continuous\text{-}on\ S\ (\lambda x.\ \sum i{\in}I.\ f\ i\ x)$
⟨*proof*⟩

instance *nat* :: *topological-comm-monoid-add*
⟨*proof*⟩

instance *int* :: *topological-comm-monoid-add*
⟨*proof*⟩

109.3.1 Topological group

class *topological-group-add* = *topological-monoid-add* + *group-add* +
assumes *tendsto-uminus-nhds*: $(uminus \longrightarrow -\ a)\ (nhds\ a)$
begin

lemma *tendsto-minus* [*tendsto-intros*]: $(f \longrightarrow a)\ F \Longrightarrow ((\lambda x.\ -\ f\ x) \longrightarrow -\ a)\ F$
⟨*proof*⟩

end

class *topological-ab-group-add* = *topological-group-add* + *ab-group-add*

instance *topological-ab-group-add* < *topological-comm-monoid-add* ⟨*proof*⟩

lemma *continuous-minus* [*continuous-intros*]: *continuous F f* ⟹ *continuous F* $(\lambda x.\ -\ f\ x)$
for $f :: 'a$::*t2-space* $\Rightarrow 'b$::*topological-group-add*
⟨*proof*⟩

lemma *continuous-on-minus* [*continuous-intros*]: *continuous-on s f* ⟹ *continu-*

ous-on s (λ*x*. − *f x*)
for *f* :: - ⇒ *'b*::*topological-group-add*
⟨*proof*⟩

lemma *tendsto-minus-cancel*: ((λ*x*. − *f x*) ⟶ − *a*) *F* ⟹ (*f* ⟶ *a*) *F*
for *a* :: *'a*::*topological-group-add*
⟨*proof*⟩

lemma *tendsto-minus-cancel-left*:
(*f* ⟶ − (*y*::-::*topological-group-add*)) *F* ⟷ ((λ*x*. − *f x*) ⟶ *y*) *F*
⟨*proof*⟩

lemma *tendsto-diff* [*tendsto-intros*]:
fixes *a b* :: *'a*::*topological-group-add*
shows (*f* ⟶ *a*) *F* ⟹ (*g* ⟶ *b*) *F* ⟹ ((λ*x*. *f x* − *g x*) ⟶ *a* − *b*) *F*
⟨*proof*⟩

lemma *continuous-diff* [*continuous-intros*]:
fixes *f g* :: *'a*::*t2-space* ⇒ *'b*::*topological-group-add*
shows *continuous F f* ⟹ *continuous F g* ⟹ *continuous F* (λ*x*. *f x* − *g x*)
⟨*proof*⟩

lemma *continuous-on-diff* [*continuous-intros*]:
fixes *f g* :: - ⇒ *'b*::*topological-group-add*
shows *continuous-on s f* ⟹ *continuous-on s g* ⟹ *continuous-on s* (λ*x*. *f x* − *g x*)
⟨*proof*⟩

lemma *continuous-on-op-minus*: *continuous-on* (*s*::*'a*::*topological-group-add set*) ((−) *x*)
⟨*proof*⟩

instance *real-normed-vector* < *topological-ab-group-add*
⟨*proof*⟩

lemmas *real-tendsto-sandwich* = *tendsto-sandwich*[**where** *'a*=*real*]

109.3.2 Linear operators and multiplication

lemma *linear-times* [*simp*]: *linear* (λ*x*. *c* ∗ *x*)
for *c* :: *'a*::*real-algebra*
⟨*proof*⟩

lemma (**in** *bounded-linear*) *tendsto*: (*g* ⟶ *a*) *F* ⟹ ((λ*x*. *f* (*g x*)) ⟶ *f a*) *F*
⟨*proof*⟩

lemma (**in** *bounded-linear*) *continuous*: *continuous F g* ⟹ *continuous F* (λ*x*. *f* (*g x*))
⟨*proof*⟩

lemma (**in** *bounded-linear*) *continuous-on*: *continuous-on s g* $\Longrightarrow$ *continuous-on s* $(\lambda x.\ f\ (g\ x))$
⟨*proof*⟩

lemma (**in** *bounded-linear*) *tendsto-zero*: $(g \longrightarrow 0)\ F \Longrightarrow ((\lambda x.\ f\ (g\ x)) \longrightarrow 0)\ F$
⟨*proof*⟩

lemma (**in** *bounded-bilinear*) *tendsto*:
$(f \longrightarrow a)\ F \Longrightarrow (g \longrightarrow b)\ F \Longrightarrow ((\lambda x.\ f\ x\ **\ g\ x) \longrightarrow a\ **\ b)\ F$
⟨*proof*⟩

lemma (**in** *bounded-bilinear*) *continuous*:
continuous F f $\Longrightarrow$ *continuous F g* $\Longrightarrow$ *continuous F* $(\lambda x.\ f\ x\ **\ g\ x)$
⟨*proof*⟩

lemma (**in** *bounded-bilinear*) *continuous-on*:
continuous-on s f $\Longrightarrow$ *continuous-on s g* $\Longrightarrow$ *continuous-on s* $(\lambda x.\ f\ x\ **\ g\ x)$
⟨*proof*⟩

lemma (**in** *bounded-bilinear*) *tendsto-zero*:
assumes *f*: $(f \longrightarrow 0)\ F$
and *g*: $(g \longrightarrow 0)\ F$
shows $((\lambda x.\ f\ x\ **\ g\ x) \longrightarrow 0)\ F$
⟨*proof*⟩

lemma (**in** *bounded-bilinear*) *tendsto-left-zero*:
$(f \longrightarrow 0)\ F \Longrightarrow ((\lambda x.\ f\ x\ **\ c) \longrightarrow 0)\ F$
⟨*proof*⟩

lemma (**in** *bounded-bilinear*) *tendsto-right-zero*:
$(f \longrightarrow 0)\ F \Longrightarrow ((\lambda x.\ c\ **\ f\ x) \longrightarrow 0)\ F$
⟨*proof*⟩

lemmas *tendsto-of-real* [*tendsto-intros*] =
bounded-linear.tendsto [*OF bounded-linear-of-real*]

lemmas *tendsto-scaleR* [*tendsto-intros*] =
bounded-bilinear.tendsto [*OF bounded-bilinear-scaleR*]

Analogous type class for multiplication

class *topological-semigroup-mult* = *topological-space* + *semigroup-mult* +
assumes *tendsto-mult-Pair*: *LIM x* $(nhds\ a \times_F nhds\ b)$. *fst x* $*$ *snd x* :> *nhds* $(a * b)$

instance *real-normed-algebra* < *topological-semigroup-mult*
⟨*proof*⟩

lemma *tendsto-mult* [*tendsto-intros*]:
fixes *a b* :: *'a*::*topological-semigroup-mult*
shows $(f \longrightarrow a)\ F \Longrightarrow (g \longrightarrow b)\ F \Longrightarrow ((\lambda x.\ f\ x * g\ x) \longrightarrow a * b)\ F$
⟨*proof*⟩

lemma *tendsto-mult-left*: $(f \longrightarrow l)\ F \Longrightarrow ((\lambda x.\ c * (f\ x)) \longrightarrow c * l)\ F$
for *c* :: *'a*::*topological-semigroup-mult*
⟨*proof*⟩

lemma *tendsto-mult-right*: $(f \longrightarrow l)\ F \Longrightarrow ((\lambda x.\ (f\ x) * c) \longrightarrow l * c)\ F$
for *c* :: *'a*::*topological-semigroup-mult*
⟨*proof*⟩

lemma *tendsto-mult-left-iff* [*simp*]:
$c \neq 0 \Longrightarrow tendsto(\lambda x.\ c * f\ x)\ (c * l)\ F \longleftrightarrow tendsto\ f\ l\ F$ **for** *c* :: *'a*::{*topological-semigroup-mult*,*field*}
⟨*proof*⟩

lemma *tendsto-mult-right-iff* [*simp*]:
$c \neq 0 \Longrightarrow tendsto(\lambda x.\ f\ x * c)\ (l * c)\ F \longleftrightarrow tendsto\ f\ l\ F$ **for** *c* :: *'a*::{*topological-semigroup-mult*,*field*}
⟨*proof*⟩

lemma *tendsto-zero-mult-left-iff* [*simp*]:
fixes *c*::*'a*::{*topological-semigroup-mult*,*field*} **assumes** $c \neq 0$ **shows** $(\lambda n.\ c * a\ n) \longrightarrow 0 \longleftrightarrow a \longrightarrow 0$
⟨*proof*⟩

lemma *tendsto-zero-mult-right-iff* [*simp*]:
fixes *c*::*'a*::{*topological-semigroup-mult*,*field*} **assumes** $c \neq 0$ **shows** $(\lambda n.\ a\ n * c) \longrightarrow 0 \longleftrightarrow a \longrightarrow 0$
⟨*proof*⟩

lemma *tendsto-zero-divide-iff* [*simp*]:
fixes *c*::*'a*::{*topological-semigroup-mult*,*field*} **assumes** $c \neq 0$ **shows** $(\lambda n.\ a\ n\ /\ c) \longrightarrow 0 \longleftrightarrow a \longrightarrow 0$
⟨*proof*⟩

lemma *lim-const-over-n* [*tendsto-intros*]:
fixes *a* :: *'a*::*real-normed-field*
shows $(\lambda n.\ a\ /\ of\text{-}nat\ n) \longrightarrow 0$
⟨*proof*⟩

lemmas *continuous-of-real* [*continuous-intros*] =
bounded-linear.continuous [*OF bounded-linear-of-real*]

lemmas *continuous-scaleR* [*continuous-intros*] =
bounded-bilinear.continuous [*OF bounded-bilinear-scaleR*]

lemmas *continuous-mult* [*continuous-intros*] =
bounded-bilinear.continuous [*OF bounded-bilinear-mult*]

lemmas *continuous-on-of-real* [*continuous-intros*] =
bounded-linear.continuous-on [*OF bounded-linear-of-real*]

lemmas *continuous-on-scaleR* [*continuous-intros*] =
bounded-bilinear.continuous-on [*OF bounded-bilinear-scaleR*]

lemmas *continuous-on-mult* [*continuous-intros*] =
bounded-bilinear.continuous-on [*OF bounded-bilinear-mult*]

lemmas *tendsto-mult-zero* =
bounded-bilinear.tendsto-zero [*OF bounded-bilinear-mult*]

lemmas *tendsto-mult-left-zero* =
bounded-bilinear.tendsto-left-zero [*OF bounded-bilinear-mult*]

lemmas *tendsto-mult-right-zero* =
bounded-bilinear.tendsto-right-zero [*OF bounded-bilinear-mult*]

lemma *continuous-mult-left*:
fixes *c*::*'a*::*real-normed-algebra*
shows *continuous F f* $\Longrightarrow$ *continuous F* ($\lambda x.$ *c* $*$ *f x*)
⟨*proof*⟩

lemma *continuous-mult-right*:
fixes *c*::*'a*::*real-normed-algebra*
shows *continuous F f* $\Longrightarrow$ *continuous F* ($\lambda x.$ *f x* $*$ *c*)
⟨*proof*⟩

lemma *continuous-on-mult-left*:
fixes *c*::*'a*::*real-normed-algebra*
shows *continuous-on s f* $\Longrightarrow$ *continuous-on s* ($\lambda x.$ *c* $*$ *f x*)
⟨*proof*⟩

lemma *continuous-on-mult-right*:
fixes *c*::*'a*::*real-normed-algebra*
shows *continuous-on s f* $\Longrightarrow$ *continuous-on s* ($\lambda x.$ *f x* $*$ *c*)
⟨*proof*⟩

lemma *continuous-on-mult-const* [*simp*]:
fixes *c*::*'a*::*real-normed-algebra*
shows *continuous-on s* (($*$) *c*)
⟨*proof*⟩

lemma *tendsto-divide-zero*:
fixes *c* :: *'a*::*real-normed-field*
shows (*f* $\longrightarrow$ *0*) *F* $\Longrightarrow$ (($\lambda x.$ *f x* / *c*) $\longrightarrow$ *0*) *F*
⟨*proof*⟩

lemma *tendsto-power* [*tendsto-intros*]: $(f \longrightarrow a)\ F \Longrightarrow ((\lambda x.\ f\ x\ \hat{}\ n) \longrightarrow a\ \hat{}\ n)\ F$
for $f :: 'a \Rightarrow 'b$::{*power*,*real-normed-algebra*}
⟨*proof*⟩

lemma *tendsto-null-power*: $[\![(f \longrightarrow 0)\ F;\ 0 < n]\!] \Longrightarrow ((\lambda x.\ f\ x\ \hat{}\ n) \longrightarrow 0)\ F$
for $f :: 'a \Rightarrow 'b$::{*power*,*real-normed-algebra-1*}
⟨*proof*⟩

lemma *continuous-power* [*continuous-intros*]: *continuous F f* ⟹ *continuous F* $(\lambda x.\ (f\ x)\hat{}n)$
for $f :: 'a$::*t2-space* ⇒ $'b$::{*power*,*real-normed-algebra*}
⟨*proof*⟩

lemma *continuous-on-power* [*continuous-intros*]:
fixes $f :: - \Rightarrow 'b$::{*power*,*real-normed-algebra*}
shows *continuous-on s f* ⟹ *continuous-on s* $(\lambda x.\ (f\ x)\hat{}n)$
⟨*proof*⟩

lemma *tendsto-prod* [*tendsto-intros*]:
fixes $f :: 'a \Rightarrow 'b \Rightarrow 'c$::{*real-normed-algebra*,*comm-ring-1*}
shows $(\bigwedge i.\ i \in S \Longrightarrow (f\ i \longrightarrow L\ i)\ F) \Longrightarrow ((\lambda x.\ \prod i{\in}S.\ f\ i\ x) \longrightarrow (\prod i{\in}S.\ L\ i))\ F$
⟨*proof*⟩

lemma *continuous-prod* [*continuous-intros*]:
fixes $f :: 'a \Rightarrow 'b$::*t2-space* ⇒ $'c$::{*real-normed-algebra*,*comm-ring-1*}
shows $(\bigwedge i.\ i \in S \Longrightarrow$ *continuous F* $(f\ i)) \Longrightarrow$ *continuous F* $(\lambda x.\ \prod i{\in}S.\ f\ i\ x)$
⟨*proof*⟩

lemma *continuous-on-prod* [*continuous-intros*]:
fixes $f :: 'a \Rightarrow - \Rightarrow 'c$::{*real-normed-algebra*,*comm-ring-1*}
shows $(\bigwedge i.\ i \in S \Longrightarrow$ *continuous-on s* $(f\ i)) \Longrightarrow$ *continuous-on s* $(\lambda x.\ \prod i{\in}S.\ f\ i\ x)$
⟨*proof*⟩

lemma *tendsto-of-real-iff*:
$((\lambda x.\ of\text{-}real\ (f\ x) :: 'a$::*real-normed-div-algebra*$) \longrightarrow of\text{-}real\ c)\ F \longleftrightarrow (f \longrightarrow c)\ F$
⟨*proof*⟩

lemma *tendsto-add-const-iff*:
$((\lambda x.\ c + f\ x :: 'a$::*topological-group-add*$) \longrightarrow c + d)\ F \longleftrightarrow (f \longrightarrow d)\ F$
⟨*proof*⟩

class *topological-monoid-mult* = *topological-semigroup-mult* + *monoid-mult*
class *topological-comm-monoid-mult* = *topological-monoid-mult* + *comm-monoid-mult*

lemma *tendsto-power-strong* [*tendsto-intros*]:
fixes $f :: - \Rightarrow 'b :: topological\text{-}monoid\text{-}mult$
assumes $(f \longrightarrow a)\ F\ (g \longrightarrow b)\ F$
shows $((\lambda x.\ f\ x\ \hat{}\ g\ x) \longrightarrow a\ \hat{}\ b)\ F$
⟨*proof*⟩

lemma *continuous-mult'* [*continuous-intros*]:
fixes $f\ g :: - \Rightarrow 'b :: topological\text{-}semigroup\text{-}mult$
shows $continuous\ F\ f \Longrightarrow continuous\ F\ g \Longrightarrow continuous\ F\ (\lambda x.\ f\ x * g\ x)$
⟨*proof*⟩

lemma *continuous-power'* [*continuous-intros*]:
fixes $f :: - \Rightarrow 'b :: topological\text{-}monoid\text{-}mult$
shows $continuous\ F\ f \Longrightarrow continuous\ F\ g \Longrightarrow continuous\ F\ (\lambda x.\ f\ x\ \hat{}\ g\ x)$
⟨*proof*⟩

lemma *continuous-on-mult'* [*continuous-intros*]:
fixes $f\ g :: - \Rightarrow 'b :: topological\text{-}semigroup\text{-}mult$
shows $continuous\text{-}on\ A\ f \Longrightarrow continuous\text{-}on\ A\ g \Longrightarrow continuous\text{-}on\ A\ (\lambda x.\ f\ x * g\ x)$
⟨*proof*⟩

lemma *continuous-on-power'* [*continuous-intros*]:
fixes $f :: - \Rightarrow 'b :: topological\text{-}monoid\text{-}mult$
shows $continuous\text{-}on\ A\ f \Longrightarrow continuous\text{-}on\ A\ g \Longrightarrow continuous\text{-}on\ A\ (\lambda x.\ f\ x\ \hat{}\ g\ x)$
⟨*proof*⟩

lemma *tendsto-mult-one*:
fixes $f\ g :: - \Rightarrow 'b :: topological\text{-}monoid\text{-}mult$
shows $(f \longrightarrow 1)\ F \Longrightarrow (g \longrightarrow 1)\ F \Longrightarrow ((\lambda x.\ f\ x * g\ x) \longrightarrow 1)\ F$
⟨*proof*⟩

lemma *tendsto-prod'* [*tendsto-intros*]:
fixes $f :: 'a \Rightarrow 'b \Rightarrow 'c :: topological\text{-}comm\text{-}monoid\text{-}mult$
shows $(\bigwedge i.\ i \in I \Longrightarrow (f\ i \longrightarrow a\ i)\ F) \Longrightarrow ((\lambda x.\ \prod i \in I.\ f\ i\ x) \longrightarrow (\prod i \in I.\ a\ i))\ F$
⟨*proof*⟩

lemma *tendsto-one-prod'*:
fixes $f :: 'a \Rightarrow 'b \Rightarrow 'c :: topological\text{-}comm\text{-}monoid\text{-}mult$
assumes $\bigwedge i.\ i \in I \Longrightarrow ((\lambda x.\ f\ x\ i) \longrightarrow 1)\ F$
shows $((\lambda i.\ prod\ (f\ i)\ I) \longrightarrow 1)\ F$
⟨*proof*⟩

lemma *LIMSEQ-prod-0*:
fixes $f :: nat \Rightarrow 'a :: \{semidom, topological\text{-}space\}$
assumes $f\ i = 0$

shows $(\lambda n.\ prod\ f\ \{..n\}) \longrightarrow 0$
⟨*proof*⟩

lemma *LIMSEQ-prod-nonneg*:
fixes $f :: nat \Rightarrow 'a::\{linordered\text{-}semidom, linorder\text{-}topology\}$
assumes *0*: $\bigwedge n.\ 0 \leq f\ n$ **and** *a*: $(\lambda n.\ prod\ f\ \{..n\}) \longrightarrow a$
shows $a \geq 0$
⟨*proof*⟩

lemma *continuous-prod'* [*continuous-intros*]:
fixes $f :: 'a \Rightarrow 'b::t2\text{-}space \Rightarrow 'c::topological\text{-}comm\text{-}monoid\text{-}mult$
shows $(\bigwedge i.\ i \in I \Longrightarrow continuous\ F\ (f\ i)) \Longrightarrow continuous\ F\ (\lambda x.\ \prod i{\in}I.\ f\ i\ x)$
⟨*proof*⟩

lemma *continuous-on-prod'* [*continuous-intros*]:
fixes $f :: 'a \Rightarrow 'b::topological\text{-}space \Rightarrow 'c::topological\text{-}comm\text{-}monoid\text{-}mult$
shows $(\bigwedge i.\ i \in I \Longrightarrow continuous\text{-}on\ S\ (f\ i)) \Longrightarrow continuous\text{-}on\ S\ (\lambda x.\ \prod i{\in}I.\ f\ i\ x)$
⟨*proof*⟩

instance *nat* :: *topological-comm-monoid-mult*
⟨*proof*⟩

instance *int* :: *topological-comm-monoid-mult*
⟨*proof*⟩

class *comm-real-normed-algebra-1* = *real-normed-algebra-1* + *comm-monoid-mult*

context *real-normed-field*
begin

subclass *comm-real-normed-algebra-1*
⟨*proof*⟩

end

109.3.3 Inverse and division

lemma (**in** *bounded-bilinear*) *Zfun-prod-Bfun*:
assumes *f*: *Zfun f F*
and *g*: *Bfun g F*
shows *Zfun* $(\lambda x.\ f\ x **\ g\ x)$ *F*
⟨*proof*⟩

lemma (**in** *bounded-bilinear*) *Bfun-prod-Zfun*:
assumes *f*: *Bfun f F*
and *g*: *Zfun g F*
shows *Zfun* $(\lambda x.\ f\ x **\ g\ x)$ *F*
⟨*proof*⟩

lemma *Bfun-inverse*:
fixes a :: *'a*::*real-normed-div-algebra*
assumes f: $(f \longrightarrow a)\ F$
assumes a: $a \neq 0$
shows *Bfun* $(\lambda x.\ inverse\ (f\ x))\ F$
⟨*proof*⟩

lemma *tendsto-inverse* [*tendsto-intros*]:
fixes a :: *'a*::*real-normed-div-algebra*
assumes f: $(f \longrightarrow a)\ F$
and a: $a \neq 0$
shows $((\lambda x.\ inverse\ (f\ x)) \longrightarrow inverse\ a)\ F$
⟨*proof*⟩

lemma *continuous-inverse*:
fixes f :: *'a*::*t2-space* ⇒ *'b*::*real-normed-div-algebra*
assumes *continuous* $F\ f$
and $f\ (Lim\ F\ (\lambda x.\ x)) \neq 0$
shows *continuous* $F\ (\lambda x.\ inverse\ (f\ x))$
⟨*proof*⟩

lemma *continuous-at-within-inverse*[*continuous-intros*]:
fixes f :: *'a*::*t2-space* ⇒ *'b*::*real-normed-div-algebra*
assumes *continuous* $(at\ a\ within\ s)\ f$
and $f\ a \neq 0$
shows *continuous* $(at\ a\ within\ s)\ (\lambda x.\ inverse\ (f\ x))$
⟨*proof*⟩

lemma *continuous-on-inverse*[*continuous-intros*]:
fixes f :: *'a*::*topological-space* ⇒ *'b*::*real-normed-div-algebra*
assumes *continuous-on* $s\ f$
and $\forall x{\in}s.\ f\ x \neq 0$
shows *continuous-on* $s\ (\lambda x.\ inverse\ (f\ x))$
⟨*proof*⟩

lemma *tendsto-divide* [*tendsto-intros*]:
fixes $a\ b$ :: *'a*::*real-normed-field*
shows $(f \longrightarrow a)\ F \Longrightarrow (g \longrightarrow b)\ F \Longrightarrow b \neq 0 \Longrightarrow ((\lambda x.\ f\ x\ /\ g\ x) \longrightarrow a\ /\ b)\ F$
⟨*proof*⟩

lemma *continuous-divide*:
fixes $f\ g$ :: *'a*::*t2-space* ⇒ *'b*::*real-normed-field*
assumes *continuous* $F\ f$
and *continuous* $F\ g$
and $g\ (Lim\ F\ (\lambda x.\ x)) \neq 0$
shows *continuous* $F\ (\lambda x.\ (f\ x)\ /\ (g\ x))$
⟨*proof*⟩

lemma *continuous-at-within-divide*[*continuous-intros*]:
fixes *f g* :: *'a*::*t2-space* ⇒ *'b*::*real-normed-field*
assumes *continuous* (*at a within s*) *f continuous* (*at a within s*) *g*
and *g a* ≠ *0*
shows *continuous* (*at a within s*) (λ*x*. (*f x*) / (*g x*))
⟨*proof*⟩

lemma *isCont-divide*[*continuous-intros*, *simp*]:
fixes *f g* :: *'a*::*t2-space* ⇒ *'b*::*real-normed-field*
assumes *isCont f a isCont g a g a* ≠ *0*
shows *isCont* (λ*x*. (*f x*) / *g x*) *a*
⟨*proof*⟩

lemma *continuous-on-divide*[*continuous-intros*]:
fixes *f* :: *'a*::*topological-space* ⇒ *'b*::*real-normed-field*
assumes *continuous-on s f continuous-on s g*
and ∀ *x*∈*s*. *g x* ≠ *0*
shows *continuous-on s* (λ*x*. (*f x*) / (*g x*))
⟨*proof*⟩

lemma *tendsto-power-int* [*tendsto-intros*]:
fixes *a* :: *'a*::*real-normed-div-algebra*
assumes *f*: (*f* ⟶ *a*) *F*
and *a*: *a* ≠ *0*
shows ((λ*x*. *power-int* (*f x*) *n*) ⟶ *power-int a n*) *F*
⟨*proof*⟩

lemma *continuous-power-int*:
fixes *f* :: *'a*::*t2-space* ⇒ *'b*::*real-normed-div-algebra*
assumes *continuous F f*
and *f* (*Lim F* (λ*x*. *x*)) ≠ *0*
shows *continuous F* (λ*x*. *power-int* (*f x*) *n*)
⟨*proof*⟩

lemma *continuous-at-within-power-int*[*continuous-intros*]:
fixes *f* :: *'a*::*t2-space* ⇒ *'b*::*real-normed-div-algebra*
assumes *continuous* (*at a within s*) *f*
and *f a* ≠ *0*
shows *continuous* (*at a within s*) (λ*x*. *power-int* (*f x*) *n*)
⟨*proof*⟩

lemma *continuous-on-power-int* [*continuous-intros*]:
fixes *f* :: *'a*::*topological-space* ⇒ *'b*::*real-normed-div-algebra*
assumes *continuous-on s f* **and** ∀ *x*∈*s*. *f x* ≠ *0*
shows *continuous-on s* (λ*x*. *power-int* (*f x*) *n*)
⟨*proof*⟩

lemma *tendsto-power-int'* [*tendsto-intros*]:

fixes $a :: 'a$::*real-normed-div-algebra*
assumes f: $(f \longrightarrow a)\ F$
and a: $a \neq 0 \vee n \geq 0$
shows $((\lambda x.\ power\text{-}int\ (f\ x)\ n) \longrightarrow power\text{-}int\ a\ n)\ F$
⟨*proof*⟩

lemma *tendsto-sgn* [*tendsto-intros*]: $(f \longrightarrow l)\ F \Longrightarrow l \neq 0 \Longrightarrow ((\lambda x.\ sgn\ (f\ x)) \longrightarrow sgn\ l)\ F$
for $l :: 'a$::*real-normed-vector*
⟨*proof*⟩

lemma *continuous-sgn*:
fixes $f :: 'a$::*t2-space* $\Rightarrow$ $'b$::*real-normed-vector*
assumes *continuous* $F\ f$
and $f\ (Lim\ F\ (\lambda x.\ x)) \neq 0$
shows *continuous* $F\ (\lambda x.\ sgn\ (f\ x))$
⟨*proof*⟩

lemma *continuous-at-within-sgn*[*continuous-intros*]:
fixes $f :: 'a$::*t2-space* $\Rightarrow$ $'b$::*real-normed-vector*
assumes *continuous* $(at\ a\ within\ s)\ f$
and $f\ a \neq 0$
shows *continuous* $(at\ a\ within\ s)\ (\lambda x.\ sgn\ (f\ x))$
⟨*proof*⟩

lemma *isCont-sgn*[*continuous-intros*]:
fixes $f :: 'a$::*t2-space* $\Rightarrow$ $'b$::*real-normed-vector*
assumes *isCont* $f\ a$
and $f\ a \neq 0$
shows *isCont* $(\lambda x.\ sgn\ (f\ x))\ a$
⟨*proof*⟩

lemma *continuous-on-sgn*[*continuous-intros*]:
fixes $f :: 'a$::*topological-space* $\Rightarrow$ $'b$::*real-normed-vector*
assumes *continuous-on* $s\ f$
and $\forall x{\in}s.\ f\ x \neq 0$
shows *continuous-on* $s\ (\lambda x.\ sgn\ (f\ x))$
⟨*proof*⟩

lemma *filterlim-at-infinity*:
fixes $f :: - \Rightarrow 'a$::*real-normed-vector*
assumes $0 \leq c$
shows $(LIM\ x\ F.\ f\ x :> at\text{-}infinity) \longleftrightarrow (\forall r{>}c.\ eventually\ (\lambda x.\ r \leq norm\ (f\ x))\ F)$
⟨*proof*⟩

lemma *filterlim-at-infinity-imp-norm-at-top*:
fixes F
assumes *filterlim* f *at-infinity* F

shows *filterlim* (λ*x*. *norm* (*f x*)) *at-top F*
⟨*proof*⟩

lemma *filterlim-norm-at-top-imp-at-infinity*:
fixes *F*
assumes *filterlim* (λ*x*. *norm* (*f x*)) *at-top F*
shows *filterlim f at-infinity F*
⟨*proof*⟩

lemma *filterlim-norm-at-top*: *filterlim norm at-top at-infinity*
⟨*proof*⟩

lemma *filterlim-at-infinity-conv-norm-at-top*:
filterlim f at-infinity G ⟷ *filterlim* (λ*x*. *norm* (*f x*)) *at-top G*
⟨*proof*⟩

lemma *eventually-not-equal-at-infinity*:
eventually (λ*x*. *x* ≠ (*a* :: *'a* :: {*real-normed-vector*})) *at-infinity*
⟨*proof*⟩

lemma *filterlim-int-of-nat-at-topD*:
fixes *F*
assumes *filterlim* (λ*x*. *f* (*int x*)) *F at-top*
shows *filterlim f F at-top*
⟨*proof*⟩

lemma *filterlim-int-sequentially* [*tendsto-intros*]:
filterlim int at-top sequentially
⟨*proof*⟩

lemma *filterlim-real-of-int-at-top* [*tendsto-intros*]:
filterlim real-of-int at-top at-top
⟨*proof*⟩

lemma *filterlim-abs-real*: *filterlim* (*abs*::*real* ⇒ *real*) *at-top at-top*
⟨*proof*⟩

lemma *filterlim-of-real-at-infinity* [*tendsto-intros*]:
filterlim (*of-real* :: *real* ⇒ *'a* :: *real-normed-algebra-1*) *at-infinity at-top*
⟨*proof*⟩

lemma *not-tendsto-and-filterlim-at-infinity*:
fixes *c* :: *'a*::*real-normed-vector*
assumes *F* ≠ *bot*
and (*f* ⟶ *c*) *F*
and *filterlim f at-infinity F*
shows *False*
⟨*proof*⟩

lemma *filterlim-at-infinity-imp-not-convergent*:
assumes *filterlim f at-infinity sequentially*
shows ¬ *convergent f*
⟨*proof*⟩

lemma *filterlim-at-infinity-imp-eventually-ne*:
assumes *filterlim f at-infinity F*
shows *eventually* (λ*z*. *f z* ≠ *c*) *F*
⟨*proof*⟩

lemma *tendsto-of-nat* [*tendsto-intros*]:
filterlim (*of-nat* :: *nat* ⇒ *'a*::*real-normed-algebra-1*) *at-infinity sequentially*
⟨*proof*⟩

109.4 Relate *at*, *at-left* and *at-right*

This lemmas are useful for conversion between *at x* to *at-left x* and *at-right x* and also *at-right* (*0*::*'a*).

lemmas *filterlim-split-at-real* = *filterlim-split-at*[**where** *'a*=*real*]

lemma *filtermap-nhds-shift*: *filtermap* (λ*x*. *x* − *d*) (*nhds a*) = *nhds* (*a* − *d*)
for *a d* :: *'a*::*real-normed-vector*
⟨*proof*⟩

lemma *filtermap-nhds-minus*: *filtermap* (λ*x*. − *x*) (*nhds a*) = *nhds* (− *a*)
for *a* :: *'a*::*real-normed-vector*
⟨*proof*⟩

lemma *filtermap-at-shift*: *filtermap* (λ*x*. *x* − *d*) (*at a*) = *at* (*a* − *d*)
for *a d* :: *'a*::*real-normed-vector*
⟨*proof*⟩

lemma *filtermap-at-right-shift*: *filtermap* (λ*x*. *x* − *d*) (*at-right a*) = *at-right* (*a* − *d*)
for *a d* :: *real*
⟨*proof*⟩

lemma *filterlim-shift*:
fixes *d* :: *'a*::*real-normed-vector*
assumes *filterlim f F* (*at a*)
shows *filterlim* (*f* ∘ (+) *d*) *F* (*at* (*a* − *d*))
⟨*proof*⟩

lemma *filterlim-shift-iff*:
fixes *d* :: *'a*::*real-normed-vector*
shows *filterlim* (*f* ∘ (+) *d*) *F* (*at* (*a* − *d*)) = *filterlim f F* (*at a*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *at-right-to-0*: *at-right a* = *filtermap* (λ*x*. *x* + *a*) (*at-right 0*)
for *a* :: *real*
⟨*proof*⟩

lemma *filterlim-at-right-to-0*:
filterlim f F (*at-right a*) ⟷ *filterlim* (λ*x*. *f* (*x* + *a*)) *F* (*at-right 0*)
for *a* :: *real*
⟨*proof*⟩

lemma *eventually-at-right-to-0*:
eventually P (*at-right a*) ⟷ *eventually* (λ*x*. *P* (*x* + *a*)) (*at-right 0*)
for *a* :: *real*
⟨*proof*⟩

lemma *at-to-0*: *at a* = *filtermap* (λ*x*. *x* + *a*) (*at 0*)
for *a* :: *'a*::*real-normed-vector*
⟨*proof*⟩

lemma *filterlim-at-to-0*:
filterlim f F (*at a*) ⟷ *filterlim* (λ*x*. *f* (*x* + *a*)) *F* (*at 0*)
for *a* :: *'a*::*real-normed-vector*
⟨*proof*⟩

lemma *eventually-at-to-0*:
eventually P (*at a*) ⟷ *eventually* (λ*x*. *P* (*x* + *a*)) (*at 0*)
for *a* :: *'a*::*real-normed-vector*
⟨*proof*⟩

lemma *filtermap-at-minus*: *filtermap* (λ*x*. − *x*) (*at a*) = *at* (− *a*)
for *a* :: *'a*::*real-normed-vector*
⟨*proof*⟩

lemma *at-left-minus*: *at-left a* = *filtermap* (λ*x*. − *x*) (*at-right* (− *a*))
for *a* :: *real*
⟨*proof*⟩

lemma *at-right-minus*: *at-right a* = *filtermap* (λ*x*. − *x*) (*at-left* (− *a*))
for *a* :: *real*
⟨*proof*⟩

lemma *filterlim-at-left-to-right*:
filterlim f F (*at-left a*) ⟷ *filterlim* (λ*x*. *f* (− *x*)) *F* (*at-right* (−*a*))
for *a* :: *real*
⟨*proof*⟩

lemma *eventually-at-left-to-right*:
eventually P (*at-left a*) ⟷ *eventually* (λ*x*. *P* (− *x*)) (*at-right* (−*a*))
for *a* :: *real*

⟨*proof*⟩

lemma *filterlim-uminus-at-top-at-bot*: *LIM x at-bot. − x :: real :> at-top*
⟨*proof*⟩

lemma *filterlim-uminus-at-bot-at-top*: *LIM x at-top. − x :: real :> at-bot*
⟨*proof*⟩

lemma *at-bot-mirror* :
shows (*at-bot*::('*a*::{*ordered-ab-group-add*,*linorder*} *filter*)) = *filtermap uminus at-top*
⟨*proof*⟩

lemma *at-top-mirror* :
shows (*at-top*::('*a*::{*ordered-ab-group-add*,*linorder*} *filter*)) = *filtermap uminus at-bot*
⟨*proof*⟩

lemma *filterlim-at-top-mirror*: (*LIM x at-top. f x :> F*) ⟷ (*LIM x at-bot. f* (−*x*::*real*) *:> F*)
⟨*proof*⟩

lemma *filterlim-at-bot-mirror*: (*LIM x at-bot. f x :> F*) ⟷ (*LIM x at-top. f* (−*x*::*real*) *:> F*)
⟨*proof*⟩

lemma *filterlim-uminus-at-top*: (*LIM x F. f x :> at-top*) ⟷ (*LIM x F. − (f x)* :: *real :> at-bot*)
⟨*proof*⟩

lemma *tendsto-at-botI-sequentially*:
fixes *f* :: *real* ⇒ '*b*::*first-countable-topology*
assumes ∗: ⋀*X*. *filterlim X at-bot sequentially* ⟹ (λ*n*. *f* (*X n*)) ⟶ *y*
shows (*f* ⟶ *y*) *at-bot*
⟨*proof*⟩

lemma *filterlim-at-infinity-imp-filterlim-at-top*:
assumes *filterlim* (*f* :: '*a* ⇒ *real*) *at-infinity F*
assumes *eventually* (λ*x*. *f x* > *0*) *F*
shows *filterlim f at-top F*
⟨*proof*⟩

lemma *filterlim-at-infinity-imp-filterlim-at-bot*:
assumes *filterlim* (*f* :: '*a* ⇒ *real*) *at-infinity F*
assumes *eventually* (λ*x*. *f x* < *0*) *F*
shows *filterlim f at-bot F*
⟨*proof*⟩

lemma *filterlim-uminus-at-bot*: (*LIM x F. f x :> at-bot*) ⟷ (*LIM x F. − (f x)* ::

real :> *at-top*)
⟨*proof*⟩

lemma *filterlim-inverse-at-top-right*: *LIM x at-right* (*0*::*real*). *inverse x* :> *at-top*
⟨*proof*⟩

lemma *tendsto-inverse-0*:
fixes *x* :: - ⇒ *'a*::*real-normed-div-algebra*
shows (*inverse* ⟶ (*0*::*'a*)) *at-infinity*
⟨*proof*⟩

lemma *tendsto-add-filterlim-at-infinity*:
fixes *c* :: *'b*::*real-normed-vector*
and *F* :: *'a filter*
assumes (*f* ⟶ *c*) *F*
and *filterlim g at-infinity F*
shows *filterlim* (λ*x*. *f x* + *g x*) *at-infinity F*
⟨*proof*⟩

lemma *tendsto-add-filterlim-at-infinity'*:
fixes *c* :: *'b*::*real-normed-vector*
and *F* :: *'a filter*
assumes *filterlim f at-infinity F*
and (*g* ⟶ *c*) *F*
shows *filterlim* (λ*x*. *f x* + *g x*) *at-infinity F*
⟨*proof*⟩

lemma *filterlim-inverse-at-right-top*: *LIM x at-top*. *inverse x* :> *at-right* (*0*::*real*)
⟨*proof*⟩

lemma *filterlim-inverse-at-top*:
(*f* ⟶ (*0* :: *real*)) *F* ⟹ *eventually* (λ*x*. *0* < *f x*) *F* ⟹ *LIM x F*. *inverse* (*f x*) :> *at-top*
⟨*proof*⟩

lemma *filterlim-inverse-at-bot-neg*:
LIM x (*at-left* (*0*::*real*)). *inverse x* :> *at-bot*
⟨*proof*⟩

lemma *filterlim-inverse-at-bot*:
(*f* ⟶ (*0* :: *real*)) *F* ⟹ *eventually* (λ*x*. *f x* < *0*) *F* ⟹ *LIM x F*. *inverse* (*f x*) :> *at-bot*
⟨*proof*⟩

lemma *at-right-to-top*: (*at-right* (*0*::*real*)) = *filtermap inverse at-top*
⟨*proof*⟩

lemma *eventually-at-right-to-top*:
eventually P (*at-right* (*0*::*real*)) ⟷ *eventually* (λ*x*. *P* (*inverse x*)) *at-top*

⟨*proof*⟩

lemma *filterlim-at-right-to-top*:
filterlim f F (*at-right* (*0*::*real*)) ⟷ (*LIM x at-top. f* (*inverse x*) :> *F*)
⟨*proof*⟩

lemma *at-top-to-right*: *at-top* = *filtermap inverse* (*at-right* (*0*::*real*))
⟨*proof*⟩

lemma *eventually-at-top-to-right*:
eventually P at-top ⟷ *eventually* (λ*x*. *P* (*inverse x*)) (*at-right* (*0*::*real*))
⟨*proof*⟩

lemma *filterlim-at-top-to-right*:
filterlim f F at-top ⟷ (*LIM x* (*at-right* (*0*::*real*)). *f* (*inverse x*) :> *F*)
⟨*proof*⟩

lemma *filterlim-inverse-at-infinity*:
fixes *x* :: - ⇒ *'a*::{*real-normed-div-algebra*, *division-ring*}
shows *filterlim inverse at-infinity* (*at* (*0*::*'a*))
⟨*proof*⟩

lemma *filterlim-inverse-at-iff*:
fixes *g* :: *'a* ⇒ *'b*::{*real-normed-div-algebra*, *division-ring*}
shows (*LIM x F. inverse* (*g x*) :> *at 0*) ⟷ (*LIM x F. g x* :> *at-infinity*)
⟨*proof*⟩

lemma *tendsto-mult-filterlim-at-infinity*:
fixes *c* :: *'a*::*real-normed-field*
assumes (*f* ⟶ *c*) *F c* ≠ *0*
assumes *filterlim g at-infinity F*
shows *filterlim* (λ*x*. *f x* ∗ *g x*) *at-infinity F*
⟨*proof*⟩

lemma *filterlim-power-int-neg-at-infinity*:
fixes *f* :: - ⇒ *'a*::{*real-normed-div-algebra*, *division-ring*}
assumes *n* < *0* **and** *lim*: (*f* ⟶ *0*) *F* **and** *ev*: *eventually* (λ*x*. *f x* ≠ *0*) *F*
shows *filterlim* (λ*x*. *f x powi n*) *at-infinity F*
⟨*proof*⟩

lemma *tendsto-inverse-0-at-top*: *LIM x F. f x* :> *at-top* ⟹ ((λ*x*. *inverse* (*f x*) :: *real*) ⟶ *0*) *F*
⟨*proof*⟩

lemma *filterlim-inverse-at-top-iff*:
eventually (λ*x*. *0* < *f x*) *F* ⟹ (*LIM x F. inverse* (*f x*) :> *at-top*) ⟷ (*f* ⟶ (*0* :: *real*)) *F*
⟨*proof*⟩

lemma *filterlim-at-top-iff-inverse-0*:
eventually $(\lambda x.\ 0 < f\ x)\ F \Longrightarrow (LIM\ x\ F.\ f\ x :> at\text{-}top) \longleftrightarrow ((inverse \circ f) \longrightarrow (0 :: real))\ F$
⟨*proof*⟩

lemma *real-tendsto-divide-at-top*:
fixes *c*::*real*
assumes $(f \longrightarrow c)\ F$
assumes *filterlim g at-top F*
shows $((\lambda x.\ f\ x\ /\ g\ x) \longrightarrow 0)\ F$
⟨*proof*⟩

lemma *mult-nat-left-at-top*: $c > 0 \Longrightarrow$ *filterlim* $(\lambda x.\ c * x)$ *at-top sequentially*
for *c* :: *nat*
⟨*proof*⟩

lemma *mult-nat-right-at-top*: $c > 0 \Longrightarrow$ *filterlim* $(\lambda x.\ x * c)$ *at-top sequentially*
for *c* :: *nat*
⟨*proof*⟩

lemma *filterlim-times-pos*:
$LIM\ x\ F1.\ c * f\ x :> at\text{-}right\ l$
if *filterlim f* $(at\text{-}right\ p)$ *F1* $0 < c\ l = c * p$
for *c*::*'a*::{*linordered-field*, *linorder-topology*}
⟨*proof*⟩

lemma *filtermap-nhds-times*: $c \neq 0 \Longrightarrow$ *filtermap* $(times\ c)$ $(nhds\ a) = nhds\ (c * a)$
for *a c* :: *'a*::*real-normed-field*
⟨*proof*⟩

lemma *filtermap-times-pos-at-right*:
fixes *c*::*'a*::{*linordered-field*, *linorder-topology*}
assumes $c > 0$
shows *filtermap* $(times\ c)$ $(at\text{-}right\ p) = at\text{-}right\ (c * p)$
⟨*proof*⟩

lemma *at-to-infinity*: $(at\ (0::'a::\{real\text{-}normed\text{-}field,field\})) = filtermap\ inverse\ at\text{-}infinity$
⟨*proof*⟩

lemma *lim-at-infinity-0*:
fixes *l* :: *'a*::{*real-normed-field*,*field*}
shows $(f \longrightarrow l)\ at\text{-}infinity \longleftrightarrow ((f \circ inverse) \longrightarrow l)\ (at\ (0::'a))$
⟨*proof*⟩

lemma *lim-zero-infinity*:
fixes *l* :: *'a*::{*real-normed-field*,*field*}
shows $((\lambda x.\ f(1\ /\ x)) \longrightarrow l)\ (at\ (0::'a)) \Longrightarrow (f \longrightarrow l)\ at\text{-}infinity$
⟨*proof*⟩

We only show rules for multiplication and addition when the functions are either against a real value or against infinity. Further rules are easy to derive by using *filterlim ?f at-top ?F* = (*LIM x ?F. − ?f x :> at-bot*).

lemma *filterlim-tendsto-pos-mult-at-top*:
assumes *f*: (*f* ⟶ *c*) *F*
and *c*: *0* < *c*
and *g*: *LIM x F. g x :> at-top*
shows *LIM x F.* (*f x* ∗ *g x* :: *real*) :> *at-top*
⟨*proof*⟩

lemma *filterlim-at-top-mult-at-top*:
assumes *f*: *LIM x F. f x :> at-top*
and *g*: *LIM x F. g x :> at-top*
shows *LIM x F.* (*f x* ∗ *g x* :: *real*) :> *at-top*
⟨*proof*⟩

lemma *filterlim-at-top-mult-tendsto-pos*:
assumes *f*: (*f* ⟶ *c*) *F*
and *c*: *0* < *c*
and *g*: *LIM x F. g x :> at-top*
shows *LIM x F.* (*g x* ∗ *f x*:: *real*) :> *at-top*
⟨*proof*⟩

lemma *filterlim-tendsto-pos-mult-at-bot*:
fixes *c* :: *real*
assumes (*f* ⟶ *c*) *F 0* < *c filterlim g at-bot F*
shows *LIM x F. f x* ∗ *g x* :> *at-bot*
⟨*proof*⟩

lemma *filterlim-tendsto-neg-mult-at-bot*:
fixes *c* :: *real*
assumes *c*: (*f* ⟶ *c*) *F c* < *0* **and** *g*: *filterlim g at-top F*
shows *LIM x F. f x* ∗ *g x* :> *at-bot*
⟨*proof*⟩

lemma *filterlim-cmult-at-bot-at-top*:
assumes *filterlim* (*h* :: - ⇒ *real*) *at-top F c* ≠ *0 G* = (*if c* > *0 then at-top else at-bot*)
shows *filterlim* (λ*x*. *c* ∗ *h x*) *G F*
⟨*proof*⟩

lemma *filterlim-pow-at-top*:
fixes *f* :: *'a* ⇒ *real*
assumes *0* < *n*
and *f*: *LIM x F. f x :> at-top*
shows *LIM x F.* (*f x*)^*n* :: *real* :> *at-top*
⟨*proof*⟩

lemma *filterlim-pow-at-bot-even*:
fixes $f :: real \Rightarrow real$
shows $0 < n \Longrightarrow LIM\ x\ F.\ f\ x :> at\text{-}bot \Longrightarrow even\ n \Longrightarrow LIM\ x\ F.\ (f\ x)\hat{}n :> at\text{-}top$
⟨*proof*⟩

lemma *filterlim-pow-at-bot-odd*:
fixes $f :: real \Rightarrow real$
shows $0 < n \Longrightarrow LIM\ x\ F.\ f\ x :> at\text{-}bot \Longrightarrow odd\ n \Longrightarrow LIM\ x\ F.\ (f\ x)\hat{}n :> at\text{-}bot$
⟨*proof*⟩

lemma *filterlim-power-at-infinity* [*tendsto-intros*]:
fixes F **and** $f :: 'a \Rightarrow 'b :: real\text{-}normed\text{-}div\text{-}algebra$
assumes $filterlim\ f\ at\text{-}infinity\ F\ n > 0$
shows $filterlim\ (\lambda x.\ f\ x\ \hat{}\ n)\ at\text{-}infinity\ F$
⟨*proof*⟩

lemma *filterlim-tendsto-add-at-top*:
assumes f: $(f \longrightarrow c)\ F$
and g: $LIM\ x\ F.\ g\ x :> at\text{-}top$
shows $LIM\ x\ F.\ (f\ x + g\ x :: real) :> at\text{-}top$
⟨*proof*⟩

lemma *LIM-at-top-divide*:
fixes $f\ g :: 'a \Rightarrow real$
assumes f: $(f \longrightarrow a)\ F\ 0 < a$
and g: $(g \longrightarrow 0)\ F\ eventually\ (\lambda x.\ 0 < g\ x)\ F$
shows $LIM\ x\ F.\ f\ x\ /\ g\ x :> at\text{-}top$
⟨*proof*⟩

lemma *filterlim-at-top-add-at-top*:
assumes f: $LIM\ x\ F.\ f\ x :> at\text{-}top$
and g: $LIM\ x\ F.\ g\ x :> at\text{-}top$
shows $LIM\ x\ F.\ (f\ x + g\ x :: real) :> at\text{-}top$
⟨*proof*⟩

lemma *tendsto-divide-0*:
fixes $f :: - \Rightarrow 'a :: \{real\text{-}normed\text{-}div\text{-}algebra,\ division\text{-}ring\}$
assumes f: $(f \longrightarrow c)\ F$
and g: $LIM\ x\ F.\ g\ x :> at\text{-}infinity$
shows $((\lambda x.\ f\ x\ /\ g\ x) \longrightarrow 0)\ F$
⟨*proof*⟩

lemma *linear-plus-1-le-power*:
fixes $x :: real$
assumes x: $0 \le x$
shows $real\ n * x + 1 \le (x + 1)\ \hat{}\ n$
⟨*proof*⟩

lemma *filterlim-realpow-sequentially-gt1*:
fixes $x :: 'a :: real\text{-}normed\text{-}div\text{-}algebra$
assumes $x[arith]$: $1 < norm\ x$
shows $LIM\ n\ sequentially.\ x\ \hat{}\ n :> at\text{-}infinity$
⟨*proof*⟩

lemma *filterlim-divide-at-infinity*:
fixes $f\ g :: 'a \Rightarrow 'a :: real\text{-}normed\text{-}field$
assumes $filterlim\ f\ (nhds\ c)\ F\ filterlim\ g\ (at\ 0)\ F\ c \neq 0$
shows $filterlim\ (\lambda x.\ f\ x\ /\ g\ x)\ at\text{-}infinity\ F$
⟨*proof*⟩

109.5 Floor and Ceiling

lemma *eventually-floor-less*:
fixes $f :: 'a \Rightarrow 'b::\{order\text{-}topology, floor\text{-}ceiling\}$
assumes f: $(f \longrightarrow l)\ F$
and l: $l \notin \mathbb{Z}$
shows $\forall_F\ x\ in\ F.\ of\text{-}int\ (floor\ l) < f\ x$
⟨*proof*⟩

lemma *eventually-less-ceiling*:
fixes $f :: 'a \Rightarrow 'b::\{order\text{-}topology, floor\text{-}ceiling\}$
assumes f: $(f \longrightarrow l)\ F$
and l: $l \notin \mathbb{Z}$
shows $\forall_F\ x\ in\ F.\ f\ x < of\text{-}int\ (ceiling\ l)$
⟨*proof*⟩

lemma *eventually-floor-eq*:
fixes $f::'a \Rightarrow 'b::\{order\text{-}topology, floor\text{-}ceiling\}$
assumes f: $(f \longrightarrow l)\ F$
and l: $l \notin \mathbb{Z}$
shows $\forall_F\ x\ in\ F.\ floor\ (f\ x) = floor\ l$
⟨*proof*⟩

lemma *eventually-ceiling-eq*:
fixes $f::'a \Rightarrow 'b::\{order\text{-}topology, floor\text{-}ceiling\}$
assumes f: $(f \longrightarrow l)\ F$
and l: $l \notin \mathbb{Z}$
shows $\forall_F\ x\ in\ F.\ ceiling\ (f\ x) = ceiling\ l$
⟨*proof*⟩

lemma *tendsto-of-int-floor*:
fixes $f::'a \Rightarrow 'b::\{order\text{-}topology, floor\text{-}ceiling\}$
assumes $(f \longrightarrow l)\ F$
and $l \notin \mathbb{Z}$
shows $((\lambda x.\ of\text{-}int\ (floor\ (f\ x))) :: 'c::\{ring\text{-}1, topological\text{-}space\}) \longrightarrow of\text{-}int\ (floor$

l)) *F*
⟨*proof*⟩

lemma *tendsto-of-int-ceiling*:
fixes *f*::′*a* ⇒ ′*b*::{*order-topology*,*floor-ceiling*}
assumes (*f* ⟶ *l*) *F*
and *l* ∉ ℤ
shows ((λ*x*. *of-int* (*ceiling* (*f x*))::′*c*::{*ring-1*,*topological-space*}) ⟶ *of-int* (*ceiling l*)) *F*
⟨*proof*⟩

lemma *continuous-on-of-int-floor*:
continuous-on (*UNIV* − ℤ::′*a*::{*order-topology*, *floor-ceiling*} *set*)
(λ*x*. *of-int* (*floor x*)::′*b*::{*ring-1*, *topological-space*})
⟨*proof*⟩

lemma *continuous-on-of-int-ceiling*:
continuous-on (*UNIV* − ℤ::′*a*::{*order-topology*, *floor-ceiling*} *set*)
(λ*x*. *of-int* (*ceiling x*)::′*b*::{*ring-1*, *topological-space*})
⟨*proof*⟩

109.6 Limits of Sequences

lemma [*trans*]: *X* = *Y* ⟹ *Y* ⟶ *z* ⟹ *X* ⟶ *z*
⟨*proof*⟩

lemma *LIMSEQ-iff*:
fixes *L* :: ′*a*::*real-normed-vector*
shows (*X* ⟶ *L*) = (∀ *r*>*0*. ∃ *no*. ∀ *n* ≥ *no*. *norm* (*X n* − *L*) < *r*)
⟨*proof*⟩

lemma *LIMSEQ-I*: (⋀*r*. *0* < *r* ⟹ ∃ *no*. ∀ *n*≥*no*. *norm* (*X n* − *L*) < *r*) ⟹ *X* ⟶ *L*
for *L* :: ′*a*::*real-normed-vector*
⟨*proof*⟩

lemma *LIMSEQ-D*: *X* ⟶ *L* ⟹ *0* < *r* ⟹ ∃ *no*. ∀ *n*≥*no*. *norm* (*X n* − *L*) < *r*
for *L* :: ′*a*::*real-normed-vector*
⟨*proof*⟩

lemma *LIMSEQ-linear*: *X* ⟶ *x* ⟹ *l* > *0* ⟹ (λ *n*. *X* (*n* ∗ *l*)) ⟶ *x*
⟨*proof*⟩

Transformation of limit.

lemma *Lim-transform*: (*g* ⟶ *a*) *F* ⟹ ((λ*x*. *f x* − *g x*) ⟶ *0*) *F* ⟹ (*f* ⟶ *a*) *F*
for *a b* :: ′*a*::*real-normed-vector*
⟨*proof*⟩

lemma *Lim-transform2*: $(f \longrightarrow a)\ F \Longrightarrow ((\lambda x.\ f\ x - g\ x) \longrightarrow 0)\ F \Longrightarrow (g \longrightarrow a)\ F$
for $a\ b :: 'a$::*real-normed-vector*
⟨*proof*⟩

proposition *Lim-transform-eq*: $((\lambda x.\ f\ x - g\ x) \longrightarrow 0)\ F \Longrightarrow (f \longrightarrow a)\ F \longleftrightarrow (g \longrightarrow a)\ F$
for $a :: 'a$::*real-normed-vector*
⟨*proof*⟩

lemma *Lim-transform-eventually*:
$[\![(f \longrightarrow l)\ F;\ \textit{eventually}\ (\lambda x.\ f\ x = g\ x)\ F]\!] \Longrightarrow (g \longrightarrow l)\ F$
⟨*proof*⟩

lemma *Lim-transform-within*:
assumes $(f \longrightarrow l)\ (\textit{at}\ x\ \textit{within}\ S)$
and $0 < d$
and $\bigwedge x'.\ x' \in S \Longrightarrow 0 < \textit{dist}\ x'\ x \Longrightarrow \textit{dist}\ x'\ x < d \Longrightarrow f\ x' = g\ x'$
shows $(g \longrightarrow l)\ (\textit{at}\ x\ \textit{within}\ S)$
⟨*proof*⟩

lemma *filterlim-transform-within*:
assumes *filterlim* $g\ G\ (\textit{at}\ x\ \textit{within}\ S)$
assumes $G \leq F\ 0<d\ (\bigwedge x'.\ x' \in S \Longrightarrow 0 < \textit{dist}\ x'\ x \Longrightarrow \textit{dist}\ x'\ x < d \Longrightarrow f\ x' = g\ x')$
shows *filterlim* $f\ F\ (\textit{at}\ x\ \textit{within}\ S)$
⟨*proof*⟩

Common case assuming being away from some crucial point like 0.

lemma *Lim-transform-away-within*:
fixes $a\ b :: 'a$::*t1-space*
assumes $a \neq b$
and $\forall x \in S.\ x \neq a \wedge x \neq b \longrightarrow f\ x = g\ x$
and $(f \longrightarrow l)\ (\textit{at}\ a\ \textit{within}\ S)$
shows $(g \longrightarrow l)\ (\textit{at}\ a\ \textit{within}\ S)$
⟨*proof*⟩

lemma *Lim-transform-away-at*:
fixes $a\ b :: 'a$::*t1-space*
assumes *ab*: $a \neq b$
and *fg*: $\forall x.\ x \neq a \wedge x \neq b \longrightarrow f\ x = g\ x$
and *fl*: $(f \longrightarrow l)\ (\textit{at}\ a)$
shows $(g \longrightarrow l)\ (\textit{at}\ a)$
⟨*proof*⟩

Alternatively, within an open set.

lemma *Lim-transform-within-open*:
assumes $(f \longrightarrow l)\ (\textit{at}\ a\ \textit{within}\ T)$

and *open s* **and** $a \in s$
and $\bigwedge x.\ x{\in}s \Longrightarrow x \neq a \Longrightarrow f\ x = g\ x$
shows $(g \longrightarrow l)\ (at\ a\ within\ T)$
⟨*proof*⟩

A congruence rule allowing us to transform limits assuming not at point.

lemma *Lim-cong-within*:
assumes $a = b$
and $x = y$
and $S = T$
and $\bigwedge x.\ x \neq b \Longrightarrow x \in T \Longrightarrow f\ x = g\ x$
shows $(f \longrightarrow x)\ (at\ a\ within\ S) \longleftrightarrow (g \longrightarrow y)\ (at\ b\ within\ T)$
⟨*proof*⟩

An unbounded sequence's inverse tends to 0.

lemma *LIMSEQ-inverse-zero*:
assumes $\bigwedge r{::}real.\ \exists N.\ \forall n{\geq}N.\ r < X\ n$
shows $(\lambda n.\ inverse\ (X\ n)) \longrightarrow 0$
⟨*proof*⟩

The sequence $(1::'a)\ /\ n$ tends to 0 as n tends to infinity.

lemma *LIMSEQ-inverse-real-of-nat*: $(\lambda n.\ inverse\ (real\ (Suc\ n))) \longrightarrow 0$
⟨*proof*⟩

The sequence $r + (1::'a)\ /\ n$ tends to r as n tends to infinity is now easily proved.

lemma *LIMSEQ-inverse-real-of-nat-add*: $(\lambda n.\ r + inverse\ (real\ (Suc\ n))) \longrightarrow r$
⟨*proof*⟩

lemma *LIMSEQ-inverse-real-of-nat-add-minus*: $(\lambda n.\ r + -inverse\ (real\ (Suc\ n))) \longrightarrow r$
⟨*proof*⟩

lemma *LIMSEQ-inverse-real-of-nat-add-minus-mult*: $(\lambda n.\ r * (1 + - inverse\ (real\ (Suc\ n)))) \longrightarrow r$
⟨*proof*⟩

lemma *lim-inverse-n*: $((\lambda n.\ inverse(of\text{-}nat\ n)) \longrightarrow (0::'a{::}real\text{-}normed\text{-}field))\ sequentially$
⟨*proof*⟩

lemma *lim-inverse-n'*: $((\lambda n.\ 1\ /\ n) \longrightarrow 0)\ sequentially$
⟨*proof*⟩

lemma *LIMSEQ-Suc-n-over-n*: $(\lambda n.\ of\text{-}nat\ (Suc\ n)\ /\ of\text{-}nat\ n :: 'a :: real\text{-}normed\text{-}field) \longrightarrow 1$
⟨*proof*⟩

lemma *LIMSEQ-n-over-Suc-n*: $(\lambda n.\ of\text{-}nat\ n\ /\ of\text{-}nat\ (Suc\ n) :: 'a :: real\text{-}normed\text{-}field) \longrightarrow 1$
⟨*proof*⟩

109.7 Convergence on sequences

lemma *convergent-cong*:
assumes *eventually* $(\lambda x.\ f\ x = g\ x)$ *sequentially*
shows *convergent* $f \longleftrightarrow$ *convergent* g
⟨*proof*⟩

lemma *convergent-Suc-iff*: *convergent* $(\lambda n.\ f\ (Suc\ n)) \longleftrightarrow$ *convergent* f
⟨*proof*⟩

lemma *convergent-ignore-initial-segment*: *convergent* $(\lambda n.\ f\ (n + m)) =$ *convergent* f
⟨*proof*⟩

lemma *convergent-add*:
fixes $X\ Y :: nat \Rightarrow 'a$::*topological-monoid-add*
assumes *convergent* $(\lambda n.\ X\ n)$
and *convergent* $(\lambda n.\ Y\ n)$
shows *convergent* $(\lambda n.\ X\ n + Y\ n)$
⟨*proof*⟩

lemma *convergent-sum*:
fixes $X :: 'a \Rightarrow nat \Rightarrow 'b$::*topological-comm-monoid-add*
shows $(\bigwedge i.\ i \in A \Longrightarrow$ *convergent* $(\lambda n.\ X\ i\ n)) \Longrightarrow$ *convergent* $(\lambda n.\ \sum i{\in}A.\ X\ i\ n)$
⟨*proof*⟩

lemma (**in** *bounded-linear*) *convergent*:
assumes *convergent* $(\lambda n.\ X\ n)$
shows *convergent* $(\lambda n.\ f\ (X\ n))$
⟨*proof*⟩

lemma (**in** *bounded-bilinear*) *convergent*:
assumes *convergent* $(\lambda n.\ X\ n)$
and *convergent* $(\lambda n.\ Y\ n)$
shows *convergent* $(\lambda n.\ X\ n\ **\ Y\ n)$
⟨*proof*⟩

lemma *convergent-minus-iff*:
fixes $X :: nat \Rightarrow 'a$::*topological-group-add*
shows *convergent* $X \longleftrightarrow$ *convergent* $(\lambda n.\ -\ X\ n)$
⟨*proof*⟩

lemma *convergent-diff*:

fixes *X Y* :: *nat* ⇒ *'a*::*topological-group-add*
assumes *convergent* (λ*n*. *X n*)
assumes *convergent* (λ*n*. *Y n*)
shows *convergent* (λ*n*. *X n* − *Y n*)
⟨*proof*⟩

lemma *convergent-norm*:
assumes *convergent f*
shows *convergent* (λ*n*. *norm* (*f n*))
⟨*proof*⟩

lemma *convergent-of-real*:
convergent f ⟹ *convergent* (λ*n*. *of-real* (*f n*) :: *'a*::*real-normed-algebra-1*)
⟨*proof*⟩

lemma *convergent-add-const-iff*:
convergent (λ*n*. *c* + *f n* :: *'a*::*topological-ab-group-add*) ⟷ *convergent f*
⟨*proof*⟩

lemma *convergent-add-const-right-iff*:
convergent (λ*n*. *f n* + *c* :: *'a*::*topological-ab-group-add*) ⟷ *convergent f*
⟨*proof*⟩

lemma *convergent-diff-const-right-iff*:
convergent (λ*n*. *f n* − *c* :: *'a*::*topological-ab-group-add*) ⟷ *convergent f*
⟨*proof*⟩

lemma *convergent-mult*:
fixes *X Y* :: *nat* ⇒ *'a*::*topological-semigroup-mult*
assumes *convergent* (λ*n*. *X n*)
and *convergent* (λ*n*. *Y n*)
shows *convergent* (λ*n*. *X n* ∗ *Y n*)
⟨*proof*⟩

lemma *convergent-mult-const-iff*:
assumes *c* ≠ *0*
shows *convergent* (λ*n*. *c* ∗ *f n* :: *'a*::{*field*,*topological-semigroup-mult*}) ⟷ *convergent f*
⟨*proof*⟩

lemma *convergent-mult-const-right-iff*:
fixes *c* :: *'a*::{*field*,*topological-semigroup-mult*}
assumes *c* ≠ *0*
shows *convergent* (λ*n*. *f n* ∗ *c*) ⟷ *convergent f*
⟨*proof*⟩

lemma *convergent-imp-Bseq*: *convergent f* ⟹ *Bseq f*
⟨*proof*⟩

A monotone sequence converges to its least upper bound.

lemma *LIMSEQ-incseq-SUP*:
fixes $X :: nat \Rightarrow 'a::\{conditionally\text{-}complete\text{-}linorder,linorder\text{-}topology\}$
assumes u: *bdd-above* (*range* X)
and X: *incseq* X
shows $X \longrightarrow (SUP\ i.\ X\ i)$
⟨*proof*⟩

lemma *LIMSEQ-decseq-INF*:
fixes $X :: nat \Rightarrow 'a::\{conditionally\text{-}complete\text{-}linorder,\ linorder\text{-}topology\}$
assumes u: *bdd-below* (*range* X)
and X: *decseq* X
shows $X \longrightarrow (INF\ i.\ X\ i)$
⟨*proof*⟩

Main monotonicity theorem.

lemma *Bseq-monoseq-convergent*: *Bseq* $X \Longrightarrow$ *monoseq* $X \Longrightarrow$ *convergent* X
for $X :: nat \Rightarrow real$
⟨*proof*⟩

lemma *Bseq-mono-convergent*: *Bseq* $X \Longrightarrow (\forall m\ n.\ m \leq n \longrightarrow X\ m \leq X\ n) \Longrightarrow$ *convergent* X
for $X :: nat \Rightarrow real$
⟨*proof*⟩

lemma *monoseq-imp-convergent-iff-Bseq*: *monoseq* $f \Longrightarrow$ *convergent* $f \longleftrightarrow$ *Bseq* f
for $f :: nat \Rightarrow real$
⟨*proof*⟩

lemma *Bseq-monoseq-convergent'-inc*:
fixes $f :: nat \Rightarrow real$
shows *Bseq* $(\lambda n.\ f\ (n + M)) \Longrightarrow (\bigwedge m\ n.\ M \leq m \Longrightarrow m \leq n \Longrightarrow f\ m \leq f\ n) \Longrightarrow$ *convergent* f
⟨*proof*⟩

lemma *Bseq-monoseq-convergent'-dec*:
fixes $f :: nat \Rightarrow real$
shows *Bseq* $(\lambda n.\ f\ (n + M)) \Longrightarrow (\bigwedge m\ n.\ M \leq m \Longrightarrow m \leq n \Longrightarrow f\ m \geq f\ n) \Longrightarrow$ *convergent* f
⟨*proof*⟩

lemma *Cauchy-iff*: *Cauchy* $X \longleftrightarrow (\forall e{>}0.\ \exists M.\ \forall m{\geq}M.\ \forall n{\geq}M.\ norm\ (X\ m - X\ n) < e)$
for $X :: nat \Rightarrow 'a::real\text{-}normed\text{-}vector$
⟨*proof*⟩

lemma *CauchyI*: $(\bigwedge e.\ 0 < e \Longrightarrow \exists M.\ \forall m{\geq}M.\ \forall n{\geq}M.\ norm\ (X\ m - X\ n) < e) \Longrightarrow$ *Cauchy* X
for $X :: nat \Rightarrow 'a::real\text{-}normed\text{-}vector$
⟨*proof*⟩

lemma *CauchyD*: *Cauchy X* $\Longrightarrow$ *0* $<$ *e* $\Longrightarrow$ $\exists M.\ \forall m{\geq}M.\ \forall n{\geq}M.$ *norm* (*X m* $-$ *X n*) $<$ *e*
for *X* :: *nat* $\Rightarrow$ *'a*::*real-normed-vector*
⟨*proof*⟩

lemma *incseq-convergent*:
fixes *X* :: *nat* $\Rightarrow$ *real*
assumes *incseq X*
and $\forall i.$ *X i* $\leq$ *B*
obtains *L* **where** *X* $\longrightarrow$ *L* $\forall i.$ *X i* $\leq$ *L*
⟨*proof*⟩

lemma *decseq-convergent*:
fixes *X* :: *nat* $\Rightarrow$ *real*
assumes *decseq X*
and $\forall i.$ *B* $\leq$ *X i*
obtains *L* **where** *X* $\longrightarrow$ *L* $\forall i.$ *L* $\leq$ *X i*
⟨*proof*⟩

lemma *monoseq-convergent*:
fixes *X* :: *nat* $\Rightarrow$ *real*
assumes *X*: *monoseq X* **and** *B*: $\bigwedge i.$ |*X i*| $\leq$ *B*
obtains *L* **where** *X* $\longrightarrow$ *L*
⟨*proof*⟩

109.8 More about *filterlim* (thanks to Wenda Li)

lemma *filterlim-at-infinity-times*:
fixes *f* :: *'a* $\Rightarrow$ *'b*::*real-normed-field*
assumes *filterlim f at-infinity F filterlim g at-infinity F*
shows *filterlim* ($\lambda x.$ *f x* $*$ *g x*) *at-infinity F*
⟨*proof*⟩

lemma *filterlim-at-top-at-bot*[*elim*]:
fixes *f*::*'a* $\Rightarrow$ *'b*::*unbounded-dense-linorder* **and** *F*::*'a filter*
assumes *top*:*filterlim f at-top F* **and** *bot*: *filterlim f at-bot F* **and** *F*$\neq$*bot*
shows *False*
⟨*proof*⟩

lemma *filterlim-at-top-nhds*[*elim*]:
fixes *f*::*'a* $\Rightarrow$ *'b*::{*unbounded-dense-linorder*,*order-topology*} **and** *F*::*'a filter*
assumes *top*:*filterlim f at-top F* **and** *tendsto*: (*f* $\longrightarrow$ *c*) *F* **and** *F*$\neq$*bot*
shows *False*
⟨*proof*⟩

lemma *filterlim-at-bot-nhds*[*elim*]:
fixes *f*::*'a* $\Rightarrow$ *'b*::{*unbounded-dense-linorder*,*order-topology*} **and** *F*::*'a filter*
assumes *top*:*filterlim f at-bot F* **and** *tendsto*: (*f* $\longrightarrow$ *c*) *F* **and** *F*$\neq$*bot*

shows *False*
⟨*proof*⟩

lemma *eventually-times-inverse-1*:
fixes f::$'a \Rightarrow 'b$::{*field*,*t2-space*}
assumes $(f \longrightarrow c)\ F\ c \neq 0$
shows $\forall_F\ x\ in\ F.\ inverse\ (f\ x) * f\ x = 1$
⟨*proof*⟩

lemma *filterlim-at-infinity-divide-iff*:
fixes f::$'a \Rightarrow 'b$::*real-normed-field*
assumes $(f \longrightarrow c)\ F\ c \neq 0$
shows $(LIM\ x\ F.\ f\ x\ /\ g\ x :> at\text{-}infinity) \longleftrightarrow (LIM\ x\ F.\ g\ x :> at\ 0)$
⟨*proof*⟩

lemma *filterlim-tendsto-pos-mult-at-top-iff*:
fixes f::$'a \Rightarrow real$
assumes $(f \longrightarrow c)\ F$ **and** $0 < c$
shows $(LIM\ x\ F.\ (f\ x * g\ x) :> at\text{-}top) \longleftrightarrow (LIM\ x\ F.\ g\ x :> at\text{-}top)$
⟨*proof*⟩

lemma *filterlim-tendsto-pos-mult-at-bot-iff*:
fixes $c :: real$
assumes $(f \longrightarrow c)\ F\ 0 < c$
shows $(LIM\ x\ F.\ f\ x * g\ x :> at\text{-}bot) \longleftrightarrow filterlim\ g\ at\text{-}bot\ F$
⟨*proof*⟩

lemma *filterlim-tendsto-neg-mult-at-top-iff*:
fixes f::$'a \Rightarrow real$
assumes $(f \longrightarrow c)\ F$ **and** $c < 0$
shows $(LIM\ x\ F.\ (f\ x * g\ x) :> at\text{-}top) \longleftrightarrow (LIM\ x\ F.\ g\ x :> at\text{-}bot)$
⟨*proof*⟩

lemma *filterlim-tendsto-neg-mult-at-bot-iff*:
fixes $c :: real$
assumes $(f \longrightarrow c)\ F\ 0 > c$
shows $(LIM\ x\ F.\ f\ x * g\ x :> at\text{-}bot) \longleftrightarrow filterlim\ g\ at\text{-}top\ F$
⟨*proof*⟩

109.9 Power Sequences

lemma *Bseq-realpow*: $0 \leq x \Longrightarrow x \leq 1 \Longrightarrow Bseq\ (\lambda n.\ x\ \hat{}\ n)$
for $x :: real$
⟨*proof*⟩

lemma *monoseq-realpow*: $0 \leq x \Longrightarrow x \leq 1 \Longrightarrow monoseq\ (\lambda n.\ x\ \hat{}\ n)$
for $x :: real$
⟨*proof*⟩

lemma *convergent-realpow*: $0 \leq x \Longrightarrow x \leq 1 \Longrightarrow$ *convergent* $(\lambda n.\ x \mathbin{\hat{}} n)$
for x :: *real*
⟨*proof*⟩

lemma *LIMSEQ-inverse-realpow-zero*: $1 < x \Longrightarrow (\lambda n.\ \mathit{inverse}\ (x \mathbin{\hat{}} n)) \longrightarrow 0$
for x :: *real*
⟨*proof*⟩

lemma *LIMSEQ-realpow-zero*:
fixes x :: *real*
assumes $0 \leq x\ x < 1$
shows $(\lambda n.\ x \mathbin{\hat{}} n) \longrightarrow 0$
⟨*proof*⟩

lemma *LIMSEQ-power-zero* [*tendsto-intros*]: *norm* $x < 1 \Longrightarrow (\lambda n.\ x \mathbin{\hat{}} n) \longrightarrow 0$
for x :: $'a$::*real-normed-algebra-1*
⟨*proof*⟩

lemma *LIMSEQ-divide-realpow-zero*: $1 < x \Longrightarrow (\lambda n.\ a\ /\ (x \mathbin{\hat{}} n) :: \mathit{real}) \longrightarrow 0$
⟨*proof*⟩

lemma
tendsto-power-zero:
fixes x::$'a$::*real-normed-algebra-1*
assumes *filterlim f at-top F*
assumes *norm* $x < 1$
shows $((\lambda y.\ x \mathbin{\hat{}} (f\ y)) \longrightarrow 0)\ F$
⟨*proof*⟩

Limit of c^n for $|c| < (1::'a)$.

lemma *LIMSEQ-abs-realpow-zero*: $|c| < 1 \Longrightarrow (\lambda n.\ |c| \mathbin{\hat{}} n :: \mathit{real}) \longrightarrow 0$
⟨*proof*⟩

lemma *LIMSEQ-abs-realpow-zero2*: $|c| < 1 \Longrightarrow (\lambda n.\ c \mathbin{\hat{}} n :: \mathit{real}) \longrightarrow 0$
⟨*proof*⟩

109.10 Limits of Functions

lemma *LIM-eq*: $f\ -a\to\ L = (\forall r{>}0.\ \exists s{>}0.\ \forall x.\ x \neq a \wedge \mathit{norm}\ (x - a) < s \longrightarrow \mathit{norm}\ (f\ x - L) < r)$
for a :: $'a$::*real-normed-vector* **and** L :: $'b$::*real-normed-vector*
⟨*proof*⟩

lemma *LIM-I*:
$(\bigwedge r.\ 0 < r \Longrightarrow \exists s{>}0.\ \forall x.\ x \neq a \wedge \mathit{norm}\ (x - a) < s \longrightarrow \mathit{norm}\ (f\ x - L) < r) \Longrightarrow f\ -a\to\ L$
for a :: $'a$::*real-normed-vector* **and** L :: $'b$::*real-normed-vector*
⟨*proof*⟩

lemma *LIM-D*: $f \ -a\to \ L \Longrightarrow \ 0 \ < \ r \Longrightarrow \exists \, s{>}0.\forall \, x. \ x \neq \ a \wedge \ norm \ (x \ - \ a) \ < \ s \longrightarrow \ norm \ (f \, x \ - \ L) \ < \ r$
for a :: $'a$::*real-normed-vector* **and** L :: $'b$::*real-normed-vector*
⟨*proof*⟩

lemma *LIM-offset*: $f \ -a\to \ L \Longrightarrow \ (\lambda x. \ f \ (x \ + \ k)) \ -(a \ - \ k)\to \ L$
for a :: $'a$::*real-normed-vector*
⟨*proof*⟩

lemma *LIM-offset-zero*: $f \ -a\to \ L \Longrightarrow \ (\lambda h. \ f \ (a \ + \ h)) \ -0\to \ L$
for a :: $'a$::*real-normed-vector*
⟨*proof*⟩

lemma *LIM-offset-zero-cancel*: $(\lambda h. \ f \ (a \ + \ h)) \ -0\to \ L \Longrightarrow \ f \ -a\to \ L$
for a :: $'a$::*real-normed-vector*
⟨*proof*⟩

lemma *LIM-offset-zero-iff*: *NO-MATCH 0 a* $\Longrightarrow \ f \ -a\to \ L \longleftrightarrow \ (\lambda h. \ f \ (a \ + \ h)) \ -0\to \ L$
for f :: $'a$ :: *real-normed-vector* ⇒ -
⟨*proof*⟩

lemma *tendsto-offset-zero-iff*:
fixes f :: $'a$ :: *real-normed-vector* ⇒ -
assumes *NO-MATCH 0 a a* ∈ *S open S*
shows $(f \longrightarrow \ L) \ (at \ a \ within \ S) \longleftrightarrow \ ((\lambda h. \ f \ (a \ + \ h)) \longrightarrow \ L) \ (at \ 0)$
⟨*proof*⟩

lemma *LIM-zero*: $(f \longrightarrow \ l) \ F \Longrightarrow \ ((\lambda x. \ f \, x \ - \ l) \longrightarrow \ 0) \ F$
for f :: $'a \Rightarrow \ 'b$::*real-normed-vector*
⟨*proof*⟩

lemma *LIM-zero-cancel*:
fixes f :: $'a \Rightarrow \ 'b$::*real-normed-vector*
shows $((\lambda x. \ f \, x \ - \ l) \longrightarrow \ 0) \ F \Longrightarrow \ (f \longrightarrow \ l) \ F$
⟨*proof*⟩

lemma *LIM-zero-iff*: $((\lambda x. \ f \, x \ - \ l) \longrightarrow \ 0) \ F \ = \ (f \longrightarrow \ l) \ F$
for f :: $'a \Rightarrow \ 'b$::*real-normed-vector*
⟨*proof*⟩

lemma *LIM-imp-LIM*:
fixes f :: $'a$::*topological-space* ⇒ $'b$::*real-normed-vector*
fixes g :: $'a$::*topological-space* ⇒ $'c$::*real-normed-vector*
assumes *f*: $f \ -a\to \ l$
and *le*: $\bigwedge x. \ x \neq \ a \Longrightarrow \ norm \ (g \, x \ - \ m) \leq \ norm \ (f \, x \ - \ l)$
shows $g \ -a\to \ m$
⟨*proof*⟩

lemma *LIM-equal2*:
fixes $f\ g :: 'a{::}real\text{-}normed\text{-}vector \Rightarrow 'b{::}topological\text{-}space$
assumes $0 < R$
and $\bigwedge x.\ x \neq a \Longrightarrow norm\ (x - a) < R \Longrightarrow f\ x = g\ x$
shows $g \ -a\rightarrow\ l \Longrightarrow f \ -a\rightarrow\ l$
⟨*proof*⟩

lemma *LIM-compose2*:
fixes $a :: 'a{::}real\text{-}normed\text{-}vector$
assumes *f*: $f \ -a\rightarrow\ b$
and *g*: $g \ -b\rightarrow\ c$
and *inj*: $\exists\, d{>}0.\ \forall x.\ x \neq a \wedge norm\ (x - a) < d \longrightarrow f\ x \neq b$
shows $(\lambda x.\ g\ (f\ x)) \ -a\rightarrow\ c$
⟨*proof*⟩

lemma *real-LIM-sandwich-zero*:
fixes $f\ g :: 'a{::}topological\text{-}space \Rightarrow real$
assumes *f*: $f \ -a\rightarrow\ 0$
and *1*: $\bigwedge x.\ x \neq a \Longrightarrow 0 \leq g\ x$
and *2*: $\bigwedge x.\ x \neq a \Longrightarrow g\ x \leq f\ x$
shows $g \ -a\rightarrow\ 0$
⟨*proof*⟩

109.11 Continuity

lemma *LIM-isCont-iff*: $(f \ -a\rightarrow\ f\ a) = ((\lambda h.\ f\ (a + h)) \ -0\rightarrow\ f\ a)$
for $f :: 'a{::}real\text{-}normed\text{-}vector \Rightarrow 'b{::}topological\text{-}space$
⟨*proof*⟩

lemma *isCont-iff*: $isCont\ f\ x = (\lambda h.\ f\ (x + h)) \ -0\rightarrow\ f\ x$
for $f :: 'a{::}real\text{-}normed\text{-}vector \Rightarrow 'b{::}topological\text{-}space$
⟨*proof*⟩

lemma *isCont-LIM-compose2*:
fixes $a :: 'a{::}real\text{-}normed\text{-}vector$
assumes *f* [*unfolded isCont-def*]: $isCont\ f\ a$
and *g*: $g \ -f\ a\rightarrow\ l$
and *inj*: $\exists\, d{>}0.\ \forall x.\ x \neq a \wedge norm\ (x - a) < d \longrightarrow f\ x \neq f\ a$
shows $(\lambda x.\ g\ (f\ x)) \ -a\rightarrow\ l$
⟨*proof*⟩

lemma *isCont-norm* [*simp*]: $isCont\ f\ a \Longrightarrow isCont\ (\lambda x.\ norm\ (f\ x))\ a$
for $f :: 'a{::}t2\text{-}space \Rightarrow 'b{::}real\text{-}normed\text{-}vector$
⟨*proof*⟩

lemma *isCont-rabs* [*simp*]: $isCont\ f\ a \Longrightarrow isCont\ (\lambda x.\ |f\ x|)\ a$
for $f :: 'a{::}t2\text{-}space \Rightarrow real$
⟨*proof*⟩

lemma *isCont-add* [*simp*]: *isCont f a* $\Longrightarrow$ *isCont g a* $\Longrightarrow$ *isCont* ($\lambda x.\ f\ x + g\ x$) *a*
for *f* :: *'a::t2-space* $\Rightarrow$ *'b::topological-monoid-add*
⟨*proof*⟩

lemma *isCont-minus* [*simp*]: *isCont f a* $\Longrightarrow$ *isCont* ($\lambda x.\ -\ f\ x$) *a*
for *f* :: *'a::t2-space* $\Rightarrow$ *'b::real-normed-vector*
⟨*proof*⟩

lemma *isCont-diff* [*simp*]: *isCont f a* $\Longrightarrow$ *isCont g a* $\Longrightarrow$ *isCont* ($\lambda x.\ f\ x - g\ x$) *a*
for *f* :: *'a::t2-space* $\Rightarrow$ *'b::real-normed-vector*
⟨*proof*⟩

lemma *isCont-mult* [*simp*]: *isCont f a* $\Longrightarrow$ *isCont g a* $\Longrightarrow$ *isCont* ($\lambda x.\ f\ x * g\ x$) *a*
for *f g* :: *'a::t2-space* $\Rightarrow$ *'b::real-normed-algebra*
⟨*proof*⟩

lemma (**in** *bounded-linear*) *isCont*: *isCont g a* $\Longrightarrow$ *isCont* ($\lambda x.\ f\ (g\ x)$) *a*
⟨*proof*⟩

lemma (**in** *bounded-bilinear*) *isCont*: *isCont f a* $\Longrightarrow$ *isCont g a* $\Longrightarrow$ *isCont* ($\lambda x.\ f\ x$ ** $g\ x$) *a*
⟨*proof*⟩

lemmas *isCont-scaleR* [*simp*] =
bounded-bilinear.isCont [*OF bounded-bilinear-scaleR*]

lemmas *isCont-of-real* [*simp*] =
bounded-linear.isCont [*OF bounded-linear-of-real*]

lemma *isCont-power* [*simp*]: *isCont f a* $\Longrightarrow$ *isCont* ($\lambda x.\ f\ x\ \hat{}\ n$) *a*
for *f* :: *'a::t2-space* $\Rightarrow$ *'b::*{*power*,*real-normed-algebra*}
⟨*proof*⟩

lemma *isCont-sum* [*simp*]: $\forall\, i{\in}A.$ *isCont* (*f i*) *a* $\Longrightarrow$ *isCont* ($\lambda x.\ \sum i{\in}A.\ f\ i\ x$) *a*
for *f* :: *'a* $\Rightarrow$ *'b::t2-space* $\Rightarrow$ *'c::topological-comm-monoid-add*
⟨*proof*⟩

109.12 Uniform Continuity

lemma *uniformly-continuous-on-def*:
fixes *f* :: *'a::metric-space* $\Rightarrow$ *'b::metric-space*
shows *uniformly-continuous-on s f* $\longleftrightarrow$
($\forall\, e{>}0.\ \exists\, d{>}0.\ \forall\, x{\in}s.\ \forall\, x'{\in}s.$ *dist x' x* $<$ *d* $\longrightarrow$ *dist* (*f x'*) (*f x*) $<$ *e*)
⟨*proof*⟩

abbreviation *isUCont* :: [*'a::metric-space* $\Rightarrow$ *'b::metric-space*] $\Rightarrow$ *bool*
where *isUCont f* $\equiv$ *uniformly-continuous-on UNIV f*

lemma *isUCont-def*: *isUCont f* $\longleftrightarrow$ $(\forall r{>}0.\ \exists s{>}0.\ \forall x\ y.\ dist\ x\ y < s \longrightarrow dist\ (f\ x)\ (f\ y) < r)$
⟨*proof*⟩

lemma *isUCont-isCont*: *isUCont f* $\Longrightarrow$ *isCont f x*
⟨*proof*⟩

lemma *uniformly-continuous-on-Cauchy*:
fixes *f* :: *'a*::*metric-space* ⇒ *'b*::*metric-space*
assumes *uniformly-continuous-on S f Cauchy X* $\bigwedge n.\ X\ n \in S$
shows *Cauchy* $(\lambda n.\ f\ (X\ n))$
⟨*proof*⟩

lemma *isUCont-Cauchy*: *isUCont f* $\Longrightarrow$ *Cauchy X* $\Longrightarrow$ *Cauchy* $(\lambda n.\ f\ (X\ n))$
⟨*proof*⟩

lemma (**in** *bounded-linear*) *isUCont*: *isUCont f*
⟨*proof*⟩

lemma (**in** *bounded-linear*) *Cauchy*: *Cauchy X* $\Longrightarrow$ *Cauchy* $(\lambda n.\ f\ (X\ n))$
⟨*proof*⟩

lemma *LIM-less-bound*:
fixes *f* :: *real* ⇒ *real*
assumes *ev*: $b < x \ \forall\ x' \in \{\ b <..< x\}.\ 0 \le f\ x'$ **and** *isCont f x*
shows $0 \le f\ x$
⟨*proof*⟩

109.13 Nested Intervals and Bisection – Needed for Compactness

lemma *nested-sequence-unique*:
assumes $\forall n.\ f\ n \le f\ (Suc\ n)\ \forall n.\ g\ (Suc\ n) \le g\ n\ \forall n.\ f\ n \le g\ n\ (\lambda n.\ f\ n - g\ n) \longrightarrow 0$
shows $\exists l{::}real.\ ((\forall n.\ f\ n \le l) \wedge f \longrightarrow l) \wedge ((\forall n.\ l \le g\ n) \wedge g \longrightarrow l)$
⟨*proof*⟩

lemma *Bolzano*[*consumes 1*, *case-names trans local*]:
fixes *P* :: *real* ⇒ *real* ⇒ *bool*
assumes [*arith*]: $a \le b$
and *trans*: $\bigwedge a\ b\ c.\ P\ a\ b \Longrightarrow P\ b\ c \Longrightarrow a \le b \Longrightarrow b \le c \Longrightarrow P\ a\ c$
and *local*: $\bigwedge x.\ a \le x \Longrightarrow x \le b \Longrightarrow \exists d{>}0.\ \forall a\ b.\ a \le x \wedge x \le b \wedge b - a < d \longrightarrow P\ a\ b$
shows *P a b*
⟨*proof*⟩

lemma *compact-Icc*[*simp*, *intro*]: *compact* {*a* .. *b*::*real*}
⟨*proof*⟩

lemma *continuous-image-closed-interval*:
fixes a b **and** f :: *real* ⇒ *real*
defines $S \equiv \{a..b\}$
assumes $a \leq b$ **and** f: *continuous-on* S f
shows $\exists\, c\ d.\ f`S = \{c..d\} \wedge c \leq d$
⟨*proof*⟩

lemma *open-Collect-positive*:
fixes f :: $'a$::*topological-space* ⇒ *real*
assumes f: *continuous-on* s f
shows $\exists\, A.$ *open* $A \wedge A \cap s = \{x{\in}s.\ 0 < f\, x\}$
⟨*proof*⟩

lemma *open-Collect-less-Int*:
fixes f g :: $'a$::*topological-space* ⇒ *real*
assumes f: *continuous-on* s f
and g: *continuous-on* s g
shows $\exists\, A.$ *open* $A \wedge A \cap s = \{x{\in}s.\ f\, x < g\, x\}$
⟨*proof*⟩

109.14 Boundedness of continuous functions

By bisection, function continuous on closed interval is bounded above

lemma *isCont-eq-Ub*:
fixes f :: *real* ⇒ $'a$::*linorder-topology*
shows $a \leq b \Longrightarrow \forall\, x{::}real.\ a \leq x \wedge x \leq b \longrightarrow isCont\, f\, x \Longrightarrow$
$\exists\, M.\ (\forall\, x.\ a \leq x \wedge x \leq b \longrightarrow f\, x \leq M) \wedge (\exists\, x.\ a \leq x \wedge x \leq b \wedge f\, x = M)$
⟨*proof*⟩

lemma *isCont-eq-Lb*:
fixes f :: *real* ⇒ $'a$::*linorder-topology*
shows $a \leq b \Longrightarrow \forall\, x.\ a \leq x \wedge x \leq b \longrightarrow isCont\, f\, x \Longrightarrow$
$\exists\, M.\ (\forall\, x.\ a \leq x \wedge x \leq b \longrightarrow M \leq f\, x) \wedge (\exists\, x.\ a \leq x \wedge x \leq b \wedge f\, x = M)$
⟨*proof*⟩

lemma *isCont-bounded*:
fixes f :: *real* ⇒ $'a$::*linorder-topology*
shows $a \leq b \Longrightarrow \forall\, x.\ a \leq x \wedge x \leq b \longrightarrow isCont\, f\, x \Longrightarrow \exists\, M.\ \forall\, x.\ a \leq x \wedge x \leq b \longrightarrow f\, x \leq M$
⟨*proof*⟩

lemma *isCont-has-Ub*:
fixes f :: *real* ⇒ $'a$::*linorder-topology*
shows $a \leq b \Longrightarrow \forall\, x.\ a \leq x \wedge x \leq b \longrightarrow isCont\, f\, x \Longrightarrow$
$\exists\, M.\ (\forall\, x.\ a \leq x \wedge x \leq b \longrightarrow f\, x \leq M) \wedge (\forall\, N.\ N < M \longrightarrow (\exists\, x.\ a \leq x \wedge x \leq b \wedge N < f\, x))$
⟨*proof*⟩

lemma *isCont-Lb-Ub*:
fixes $f :: real \Rightarrow real$
assumes $a \leq b \; \forall x.\; a \leq x \wedge x \leq b \longrightarrow isCont\; f\; x$
shows $\exists L\; M.\; (\forall x.\; a \leq x \wedge x \leq b \longrightarrow L \leq f\; x \wedge f\; x \leq M) \wedge$
$(\forall y.\; L \leq y \wedge y \leq M \longrightarrow (\exists x.\; a \leq x \wedge x \leq b \wedge (f\; x = y)))$
⟨*proof*⟩

Continuity of inverse function.

lemma *isCont-inverse-function*:
fixes $f\; g :: real \Rightarrow real$
assumes d: $0 < d$
and *inj*: $\bigwedge z.\; |z-x| \leq d \Longrightarrow g\; (f\; z) = z$
and *cont*: $\bigwedge z.\; |z-x| \leq d \Longrightarrow isCont\; f\; z$
shows $isCont\; g\; (f\; x)$
⟨*proof*⟩

lemma *isCont-inverse-function2*:
fixes $f\; g :: real \Rightarrow real$
shows
$[\![a < x;\; x < b;$
$\bigwedge z.\; [\![a \leq z;\; z \leq b]\!] \Longrightarrow g\; (f\; z) = z;$
$\bigwedge z.\; [\![a \leq z;\; z \leq b]\!] \Longrightarrow isCont\; f\; z]\!] \Longrightarrow isCont\; g\; (f\; x)$
⟨*proof*⟩

Bartle/Sherbert: Introduction to Real Analysis, Theorem 4.2.9, p. 110.

lemma *LIM-fun-gt-zero*: $f\; -c\rightarrow\; l \Longrightarrow 0 < l \Longrightarrow \exists r.\; 0 < r \wedge (\forall x.\; x \neq c \wedge |c - x| < r \longrightarrow 0 < f\; x)$
for $f :: real \Rightarrow real$
⟨*proof*⟩

lemma *LIM-fun-less-zero*: $f\; -c\rightarrow\; l \Longrightarrow l < 0 \Longrightarrow \exists r.\; 0 < r \wedge (\forall x.\; x \neq c \wedge |c - x| < r \longrightarrow f\; x < 0)$
for $f :: real \Rightarrow real$
⟨*proof*⟩

lemma *LIM-fun-not-zero*: $f\; -c\rightarrow\; l \Longrightarrow l \neq 0 \Longrightarrow \exists r.\; 0 < r \wedge (\forall x.\; x \neq c \wedge |c - x| < r \longrightarrow f\; x \neq 0)$
for $f :: real \Rightarrow real$
⟨*proof*⟩

lemma *Lim-topological*:
$(f \longrightarrow l)\; net \longleftrightarrow$
trivial-limit net $\vee$ $(\forall S.\; open\; S \longrightarrow l \in S \longrightarrow eventually\; (\lambda x.\; f\; x \in S)\; net)$
⟨*proof*⟩

lemma *eventually-within-Un*:
$eventually\; P\; (at\; x\; within\; (s \cup t)) \longleftrightarrow$
$eventually\; P\; (at\; x\; within\; s) \wedge eventually\; P\; (at\; x\; within\; t)$
⟨*proof*⟩

lemma *Lim-within-Un*:
$(f \longrightarrow l)\ (at\ x\ within\ (s \cup t)) \longleftrightarrow$
$(f \longrightarrow l)\ (at\ x\ within\ s) \wedge (f \longrightarrow l)\ (at\ x\ within\ t)$
⟨*proof*⟩

end

theory *Inequalities*
imports *Real-Vector-Spaces*
begin

lemma *Chebyshev-sum-upper*:
fixes $a\ b$::*nat* ⇒ *'a*::*linordered-idom*
assumes $\bigwedge i\ j.\ i \leq j \Longrightarrow j < n \Longrightarrow a\ i \leq a\ j$
assumes $\bigwedge i\ j.\ i \leq j \Longrightarrow j < n \Longrightarrow b\ i \geq b\ j$
shows *of-nat* $n * (\sum k{=}0..{<}n.\ a\ k * b\ k) \leq (\sum k{=}0..{<}n.\ a\ k) * (\sum k{=}0..{<}n.\ b\ k)$
⟨*proof*⟩

lemma *Chebyshev-sum-upper-nat*:
fixes $a\ b$ :: *nat* ⇒ *nat*
shows $(\bigwedge i\ j.\ [\![\ i{\leq}j;\ j{<}n\ ]\!] \Longrightarrow a\ i \leq a\ j) \Longrightarrow$
$(\bigwedge i\ j.\ [\![\ i{\leq}j;\ j{<}n\ ]\!] \Longrightarrow b\ i \geq b\ j) \Longrightarrow$
$n * (\sum i{=}0..{<}n.\ a\ i * b\ i) \leq (\sum i{=}0..{<}n.\ a\ i) * (\sum i{=}0..{<}n.\ b\ i)$
⟨*proof*⟩

end

110 Infinite Series

theory *Series*
imports *Limits Inequalities*
begin

110.1 Definition of infinite summability

definition *sums* :: (*nat* ⇒ *'a*::{*topological-space*, *comm-monoid-add*}) ⇒ *'a* ⇒ *bool*
(**infixr** *sums 80*)
where $f\ sums\ s \longleftrightarrow (\lambda n.\ \sum i{<}n.\ f\ i) \longrightarrow s$

definition *summable* :: (*nat* ⇒ *'a*::{*topological-space*, *comm-monoid-add*}) ⇒ *bool*
where $summable\ f \longleftrightarrow (\exists s.\ f\ sums\ s)$

definition *suminf* :: (*nat* ⇒ *'a*::{*topological-space*, *comm-monoid-add*}) ⇒ *'a*
(**binder** $\sum$ *10*)
where $suminf\ f = (THE\ s.\ f\ sums\ s)$

Variants of the definition

lemma *sums-def'*: f *sums* $s \longleftrightarrow (\lambda n.\ \sum i = 0..n.\ f\ i) \longrightarrow s$
⟨*proof*⟩

lemma *sums-def-le*: f *sums* $s \longleftrightarrow (\lambda n.\ \sum i{\leq}n.\ f\ i) \longrightarrow s$
⟨*proof*⟩

lemma *bounded-imp-summable*:
fixes $a :: nat \Rightarrow 'a$::{*conditionally-complete-linorder*,*linorder-topology*,*linordered-comm-semiring-strict*}
assumes *0*: $\bigwedge n.\ a\ n \geq 0$ **and** *bounded*: $\bigwedge n.\ (\sum k{\leq}n.\ a\ k) \leq B$
shows *summable a*
⟨*proof*⟩

110.2 Infinite summability on topological monoids

lemma *sums-subst*[*trans*]: $f = g \Longrightarrow g$ *sums* $z \Longrightarrow f$ *sums* z
⟨*proof*⟩

lemma *sums-cong*: $(\bigwedge n.\ f\ n = g\ n) \Longrightarrow f$ *sums* $c \longleftrightarrow g$ *sums* c
⟨*proof*⟩

lemma *sums-summable*: f *sums* $l \Longrightarrow$ *summable* f
⟨*proof*⟩

lemma *summable-iff-convergent*: *summable* $f \longleftrightarrow$ *convergent* $(\lambda n.\ \sum i{<}n.\ f\ i)$
⟨*proof*⟩

lemma *summable-iff-convergent'*: *summable* $f \longleftrightarrow$ *convergent* $(\lambda n.\ sum\ f\ \{..n\})$
⟨*proof*⟩

lemma *suminf-eq-lim*: *suminf* f = *lim* $(\lambda n.\ \sum i{<}n.\ f\ i)$
⟨*proof*⟩

lemma *sums-zero*[*simp*, *intro*]: $(\lambda n.\ 0)$ *sums 0*
⟨*proof*⟩

lemma *summable-zero*[*simp*, *intro*]: *summable* $(\lambda n.\ 0)$
⟨*proof*⟩

lemma *sums-group*: f *sums* $s \Longrightarrow 0 < k \Longrightarrow (\lambda n.\ sum\ f\ \{n * k\ ..{<}\ n * k + k\})$
sums s
⟨*proof*⟩

lemma *suminf-cong*: $(\bigwedge n.\ f\ n = g\ n) \Longrightarrow$ *suminf* f = *suminf* g
⟨*proof*⟩

lemma *summable-cong*:
fixes $f\ g :: nat \Rightarrow 'a$::*real-normed-vector*
assumes *eventually* $(\lambda x.\ f\ x = g\ x)$ *sequentially*

shows *summable f = summable g*
⟨*proof*⟩

lemma *sums-finite*:
assumes [*simp*]: *finite N*
and *f*: $\bigwedge n.\ n \notin N \Longrightarrow f\ n = 0$
shows *f sums* $(\sum n{\in}N.\ f\ n)$
⟨*proof*⟩

corollary *sums-0*: $(\bigwedge n.\ f\ n = 0) \Longrightarrow (f\ sums\ 0)$
⟨*proof*⟩

lemma *summable-finite*: *finite N* $\Longrightarrow (\bigwedge n.\ n \notin N \Longrightarrow f\ n = 0) \Longrightarrow$ *summable f*
⟨*proof*⟩

lemma *sums-If-finite-set*: *finite A* $\Longrightarrow (\lambda r.\ if\ r \in A\ then\ f\ r\ else\ 0)\ sums\ (\sum r{\in}A.\ f\ r)$
⟨*proof*⟩

lemma *summable-If-finite-set*[*simp*, *intro*]: *finite A* $\Longrightarrow$ *summable* $(\lambda r.\ if\ r \in A\ then\ f\ r\ else\ 0)$
⟨*proof*⟩

lemma *sums-If-finite*: *finite* $\{r.\ P\ r\} \Longrightarrow (\lambda r.\ if\ P\ r\ then\ f\ r\ else\ 0)\ sums\ (\sum r \mid P\ r.\ f\ r)$
⟨*proof*⟩

lemma *summable-If-finite*[*simp*, *intro*]: *finite* $\{r.\ P\ r\} \Longrightarrow$ *summable* $(\lambda r.\ if\ P\ r\ then\ f\ r\ else\ 0)$
⟨*proof*⟩

lemma *sums-single*: $(\lambda r.\ if\ r = i\ then\ f\ r\ else\ 0)\ sums\ f\ i$
⟨*proof*⟩

lemma *summable-single*[*simp*, *intro*]: *summable* $(\lambda r.\ if\ r = i\ then\ f\ r\ else\ 0)$
⟨*proof*⟩

context
fixes *f* :: *nat* ⇒ *'a*::{*t2-space*,*comm-monoid-add*}
begin

lemma *summable-sums*[*intro*]: *summable f* $\Longrightarrow$ *f sums* (*suminf f*)
⟨*proof*⟩

lemma *summable-LIMSEQ*: *summable f* $\Longrightarrow (\lambda n.\ \sum i{<}n.\ f\ i) \longrightarrow$ *suminf f*
⟨*proof*⟩

lemma *summable-LIMSEQ'*: *summable f* $\Longrightarrow (\lambda n.\ \sum i{\leq}n.\ f\ i) \longrightarrow$ *suminf f*
⟨*proof*⟩

lemma *sums-unique*: f *sums* $s \Longrightarrow s = \mathit{suminf}\ f$
⟨*proof*⟩

lemma *sums-iff*: f *sums* $x \longleftrightarrow \mathit{summable}\ f \wedge \mathit{suminf}\ f = x$
⟨*proof*⟩

lemma *summable-sums-iff*: *summable* $f \longleftrightarrow f$ *sums* *suminf* f
⟨*proof*⟩

lemma *sums-unique2*: f *sums* $a \Longrightarrow f$ *sums* $b \Longrightarrow a = b$
for $a\ b :: 'a$
⟨*proof*⟩

lemma *sums-Uniq*: $\exists_{\leq 1} a.\ f$ *sums* a
for $a\ b :: 'a$
⟨*proof*⟩

lemma *suminf-finite*:
assumes N: *finite* N
and f: $\bigwedge n.\ n \notin N \Longrightarrow f\ n = 0$
shows $\mathit{suminf}\ f = (\sum n{\in}N.\ f\ n)$
⟨*proof*⟩

end

lemma *suminf-zero*[*simp*]: *suminf* $(\lambda n.\ 0{::}'a{::}\{$*t2-space*, *comm-monoid-add*$\}) = 0$
⟨*proof*⟩

110.3 Infinite summability on ordered, topological monoids

lemma *sums-le*: $(\bigwedge n.\ f\ n \leq g\ n) \Longrightarrow f$ *sums* $s \Longrightarrow g$ *sums* $t \Longrightarrow s \leq t$
for $f\ g :: \mathit{nat} \Rightarrow 'a{::}\{$*ordered-comm-monoid-add*,*linorder-topology*$\}$
⟨*proof*⟩

context
fixes $f :: \mathit{nat} \Rightarrow 'a{::}\{$*ordered-comm-monoid-add*,*linorder-topology*$\}$
begin

lemma *suminf-le*: $(\bigwedge n.\ f\ n \leq g\ n) \Longrightarrow$ *summable* $f \Longrightarrow$ *summable* $g \Longrightarrow \mathit{suminf}\ f \leq \mathit{suminf}\ g$
⟨*proof*⟩

lemma *sum-le-suminf*:
shows *summable* $f \Longrightarrow$ *finite* $I \Longrightarrow (\bigwedge n.\ n \in\!- I \Longrightarrow 0 \leq f\ n) \Longrightarrow \mathit{sum}\ f\ I \leq \mathit{suminf}\ f$
⟨*proof*⟩

lemma *suminf-nonneg*: *summable* $f \Longrightarrow (\bigwedge n.\ 0 \leq f\ n) \Longrightarrow 0 \leq \mathit{suminf}\ f$

⟨*proof*⟩

lemma *suminf-le-const*: *summable f* ⟹ (⋀*n*. *sum f* {..<*n*} ≤ *x*) ⟹ *suminf f* ≤ *x*
⟨*proof*⟩

lemma *suminf-eq-zero-iff*:
assumes *summable f* **and** *pos*: ⋀*n*. *0* ≤ *f n*
shows *suminf f* = *0* ⟷ (∀ *n*. *f n* = *0*)
⟨*proof*⟩

lemma *suminf-pos-iff*: *summable f* ⟹ (⋀*n*. *0* ≤ *f n*) ⟹ *0* < *suminf f* ⟷ (∃ *i*. *0* < *f i*)
⟨*proof*⟩

lemma *suminf-pos2*:
assumes *summable f* ⋀*n*. *0* ≤ *f n* *0* < *f i*
shows *0* < *suminf f*
⟨*proof*⟩

lemma *suminf-pos*: *summable f* ⟹ (⋀*n*. *0* < *f n*) ⟹ *0* < *suminf f*
⟨*proof*⟩

end

context
fixes *f* :: *nat* ⇒ *'a*::{*ordered-cancel-comm-monoid-add*,*linorder-topology*}
begin

lemma *sum-less-suminf2*:
summable f ⟹ (⋀*m*. *m*≥*n* ⟹ *0* ≤ *f m*) ⟹ *n* ≤ *i* ⟹ *0* < *f i* ⟹ *sum f* {..<*n*} < *suminf f*
⟨*proof*⟩

lemma *sum-less-suminf*: *summable f* ⟹ (⋀*m*. *m*≥*n* ⟹ *0* < *f m*) ⟹ *sum f* {..<*n*} < *suminf f*
⟨*proof*⟩

end

lemma *summableI-nonneg-bounded*:
fixes *f* :: *nat* ⇒ *'a*::{*ordered-comm-monoid-add*,*linorder-topology*,*conditionally-complete-linorder*}
assumes *pos*[*simp*]: ⋀*n*. *0* ≤ *f n*
and *le*: ⋀*n*. (∑ *i*<*n*. *f i*) ≤ *x*
shows *summable f*
⟨*proof*⟩

lemma *summableI*[*intro*, *simp*]: *summable f*
for *f* :: *nat* ⇒ *'a*::{*canonically-ordered-monoid-add*,*linorder-topology*,*complete-linorder*}

⟨*proof*⟩

lemma *suminf-eq-SUP-real*:
assumes *X*: *summable X* ⋀*i*. *0* ≤ *X i* **shows** *suminf X* = (*SUP i*. ∑*n*<*i*. *X n*::*real*)
⟨*proof*⟩

110.4 Infinite summability on topological monoids

context
fixes *f g* :: *nat* ⇒ *'a*::{*t2-space*,*topological-comm-monoid-add*}
begin

lemma *sums-Suc*:
assumes (λ*n*. *f* (*Suc n*)) *sums l*
shows *f sums* (*l* + *f* 0)
⟨*proof*⟩

lemma *sums-add*: *f sums a* ⟹ *g sums b* ⟹ (λ*n*. *f n* + *g n*) *sums* (*a* + *b*)
⟨*proof*⟩

lemma *summable-add*: *summable f* ⟹ *summable g* ⟹ *summable* (λ*n*. *f n* + *g n*)
⟨*proof*⟩

lemma *suminf-add*: *summable f* ⟹ *summable g* ⟹ *suminf f* + *suminf g* = (∑ *n*. *f n* + *g n*)
⟨*proof*⟩

end

context
fixes *f* :: *'i* ⇒ *nat* ⇒ *'a*::{*t2-space*,*topological-comm-monoid-add*}
and *I* :: *'i set*
begin

lemma *sums-sum*: (⋀*i*. *i* ∈ *I* ⟹ (*f i*) *sums* (*x i*)) ⟹ (λ*n*. ∑ *i*∈*I*. *f i n*) *sums* (∑ *i*∈*I*. *x i*)
⟨*proof*⟩

lemma *suminf-sum*: (⋀*i*. *i* ∈ *I* ⟹ *summable* (*f i*)) ⟹ (∑ *n*. ∑ *i*∈*I*. *f i n*) = (∑ *i*∈*I*. ∑ *n*. *f i n*)
⟨*proof*⟩

lemma *summable-sum*: (⋀*i*. *i* ∈ *I* ⟹ *summable* (*f i*)) ⟹ *summable* (λ*n*. ∑ *i*∈*I*. *f i n*)
⟨*proof*⟩

end

lemma *sums-If-finite-set'*:
fixes *f g* :: *nat* ⇒ *'a*::{*t2-space*,*topological-ab-group-add*}
assumes *g sums S* **and** *finite A* **and** $S' = S + (\sum n{\in}A.\ f\ n - g\ n)$
shows $(\lambda n.\ \textit{if}\ n \in A\ \textit{then}\ f\ n\ \textit{else}\ g\ n)\ \textit{sums}\ S'$
⟨*proof*⟩

110.5 Infinite summability on real normed vector spaces

context
fixes *f* :: *nat* ⇒ *'a*::*real-normed-vector*
begin

lemma *sums-Suc-iff*: $(\lambda n.\ f\ (Suc\ n))\ sums\ s \longleftrightarrow f\ sums\ (s + f\ 0)$
⟨*proof*⟩

lemma *summable-Suc-iff*: $summable\ (\lambda n.\ f\ (Suc\ n)) = summable\ f$
⟨*proof*⟩

lemma *sums-Suc-imp*: $f\ 0 = 0 \Longrightarrow (\lambda n.\ f\ (Suc\ n))\ sums\ s \Longrightarrow (\lambda n.\ f\ n)\ sums\ s$
⟨*proof*⟩

end

context
fixes *f* :: *nat* ⇒ *'a*::*real-normed-vector*
begin

lemma *sums-diff*: $f\ sums\ a \Longrightarrow g\ sums\ b \Longrightarrow (\lambda n.\ f\ n - g\ n)\ sums\ (a - b)$
⟨*proof*⟩

lemma *summable-diff*: $summable\ f \Longrightarrow summable\ g \Longrightarrow summable\ (\lambda n.\ f\ n - g\ n)$
⟨*proof*⟩

lemma *suminf-diff*: $summable\ f \Longrightarrow summable\ g \Longrightarrow suminf\ f - suminf\ g = (\sum n.\ f\ n - g\ n)$
⟨*proof*⟩

lemma *sums-minus*: $f\ sums\ a \Longrightarrow (\lambda n.\ -\ f\ n)\ sums\ (-\ a)$
⟨*proof*⟩

lemma *summable-minus*: $summable\ f \Longrightarrow summable\ (\lambda n.\ -\ f\ n)$
⟨*proof*⟩

lemma *suminf-minus*: $summable\ f \Longrightarrow (\sum n.\ -\ f\ n) = -\ (\sum n.\ f\ n)$
⟨*proof*⟩

lemma *sums-iff-shift*: $(\lambda i.\ f\ (i + n))\ sums\ s \longleftrightarrow f\ sums\ (s + (\sum i{<}n.\ f\ i))$

⟨*proof*⟩

corollary *sums-iff-shift'*: $(\lambda i.\ f\ (i + n))\ sums\ (s - (\sum i{<}n.\ f\ i)) \longleftrightarrow f\ sums\ s$
⟨*proof*⟩

lemma *sums-zero-iff-shift*:
assumes $\bigwedge i.\ i < n \Longrightarrow f\ i = 0$
shows $(\lambda i.\ f\ (i{+}n))\ sums\ s \longleftrightarrow (\lambda i.\ f\ i)\ sums\ s$
⟨*proof*⟩

lemma *summable-iff-shift* [*simp*]: $summable\ (\lambda n.\ f\ (n + k)) \longleftrightarrow summable\ f$
⟨*proof*⟩

lemma *sums-split-initial-segment*: $f\ sums\ s \Longrightarrow (\lambda i.\ f\ (i + n))\ sums\ (s - (\sum i{<}n.\ f\ i))$
⟨*proof*⟩

lemma *summable-ignore-initial-segment*: $summable\ f \Longrightarrow summable\ (\lambda n.\ f(n + k))$
⟨*proof*⟩

lemma *suminf-minus-initial-segment*: $summable\ f \Longrightarrow (\sum n.\ f\ (n + k)) = (\sum n.\ f\ n) - (\sum i{<}k.\ f\ i)$
⟨*proof*⟩

lemma *suminf-split-initial-segment*: $summable\ f \Longrightarrow suminf\ f = (\sum n.\ f(n + k)) + (\sum i{<}k.\ f\ i)$
⟨*proof*⟩

lemma *suminf-split-head*: $summable\ f \Longrightarrow (\sum n.\ f\ (Suc\ n)) = suminf\ f - f\ 0$
⟨*proof*⟩

lemma *suminf-exist-split*:
fixes $r :: real$
assumes $0 < r$ **and** $summable\ f$
shows $\exists N.\ \forall n{\geq}N.\ norm\ (\sum i.\ f\ (i + n)) < r$
⟨*proof*⟩

lemma *summable-LIMSEQ-zero*:
assumes $summable\ f$ **shows** $f \longrightarrow 0$
⟨*proof*⟩

lemma *summable-imp-convergent*: $summable\ f \Longrightarrow convergent\ f$
⟨*proof*⟩

lemma *summable-imp-Bseq*: $summable\ f \Longrightarrow Bseq\ f$
⟨*proof*⟩

end

lemma *summable-minus-iff*: *summable* ($\lambda n.\ -\ f\ n$) $\longleftrightarrow$ *summable f*
for *f* :: *nat* $\Rightarrow$ *'a*::*real-normed-vector*
⟨*proof*⟩

lemma (**in** *bounded-linear*) *sums*: ($\lambda n.\ X\ n$) *sums a* $\Longrightarrow$ ($\lambda n.\ f\ (X\ n)$) *sums* (*f a*)
⟨*proof*⟩

lemma (**in** *bounded-linear*) *summable*: *summable* ($\lambda n.\ X\ n$) $\Longrightarrow$ *summable* ($\lambda n.\ f$ (*X n*))
⟨*proof*⟩

lemma (**in** *bounded-linear*) *suminf*: *summable* ($\lambda n.\ X\ n$) $\Longrightarrow$ *f* ($\sum n.\ X\ n$) = ($\sum n.\ f\ (X\ n)$)
⟨*proof*⟩

lemmas *sums-of-real* = *bounded-linear.sums* [*OF bounded-linear-of-real*]
lemmas *summable-of-real* = *bounded-linear.summable* [*OF bounded-linear-of-real*]
lemmas *suminf-of-real* = *bounded-linear.suminf* [*OF bounded-linear-of-real*]

lemmas *sums-scaleR-left* = *bounded-linear.sums*[*OF bounded-linear-scaleR-left*]
lemmas *summable-scaleR-left* = *bounded-linear.summable*[*OF bounded-linear-scaleR-left*]
lemmas *suminf-scaleR-left* = *bounded-linear.suminf*[*OF bounded-linear-scaleR-left*]

lemmas *sums-scaleR-right* = *bounded-linear.sums*[*OF bounded-linear-scaleR-right*]
lemmas *summable-scaleR-right* = *bounded-linear.summable*[*OF bounded-linear-scaleR-right*]
lemmas *suminf-scaleR-right* = *bounded-linear.suminf*[*OF bounded-linear-scaleR-right*]

lemma *summable-const-iff*: *summable* (λ-. *c*) $\longleftrightarrow$ *c* = *0*
for *c* :: *'a*::*real-normed-vector*
⟨*proof*⟩

110.6 Infinite summability on real normed algebras

context
fixes *f* :: *nat* $\Rightarrow$ *'a*::*real-normed-algebra*
begin

lemma *sums-mult*: *f sums a* $\Longrightarrow$ ($\lambda n.\ c * f\ n$) *sums* (*c* * *a*)
⟨*proof*⟩

lemma *summable-mult*: *summable f* $\Longrightarrow$ *summable* ($\lambda n.\ c * f\ n$)
⟨*proof*⟩

lemma *suminf-mult*: *summable f* $\Longrightarrow$ *suminf* ($\lambda n.\ c * f\ n$) = *c* * *suminf f*
⟨*proof*⟩

lemma *sums-mult2*: *f sums a* $\Longrightarrow$ ($\lambda n.\ f\ n * c$) *sums* (*a* * *c*)
⟨*proof*⟩

lemma *summable-mult2*: *summable f* $\Longrightarrow$ *summable* ($\lambda n.$ *f n* $*$ *c*)
⟨*proof*⟩

lemma *suminf-mult2*: *summable f* $\Longrightarrow$ *suminf f* $*$ *c* = ($\sum n.$ *f n* $*$ *c*)
⟨*proof*⟩

end

lemma *sums-mult-iff*:
fixes *f* :: *nat* $\Rightarrow$ *'a*::{*real-normed-algebra*,*field*}
assumes *c* $\neq$ *0*
shows ($\lambda n.$ *c* $*$ *f n*) *sums* (*c* $*$ *d*) $\longleftrightarrow$ *f sums d*
⟨*proof*⟩

lemma *sums-mult2-iff*:
fixes *f* :: *nat* $\Rightarrow$ *'a*::{*real-normed-algebra*,*field*}
assumes *c* $\neq$ *0*
shows ($\lambda n.$ *f n* $*$ *c*) *sums* (*d* $*$ *c*) $\longleftrightarrow$ *f sums d*
⟨*proof*⟩

lemma *sums-of-real-iff*:
($\lambda n.$ *of-real* (*f n*) :: *'a*::*real-normed-div-algebra*) *sums of-real c* $\longleftrightarrow$ *f sums c*
⟨*proof*⟩

110.7 Infinite summability on real normed fields

context
fixes *c* :: *'a*::*real-normed-field*
begin

lemma *sums-divide*: *f sums a* $\Longrightarrow$ ($\lambda n.$ *f n* / *c*) *sums* (*a* / *c*)
⟨*proof*⟩

lemma *summable-divide*: *summable f* $\Longrightarrow$ *summable* ($\lambda n.$ *f n* / *c*)
⟨*proof*⟩

lemma *suminf-divide*: *summable f* $\Longrightarrow$ *suminf* ($\lambda n.$ *f n* / *c*) = *suminf f* / *c*
⟨*proof*⟩

lemma *summable-inverse-divide*: *summable* (*inverse* $\circ$ *f*) $\Longrightarrow$ *summable* ($\lambda n.$ *c* / *f n*)
⟨*proof*⟩

lemma *sums-mult-D*: ($\lambda n.$ *c* $*$ *f n*) *sums a* $\Longrightarrow$ *c* $\neq$ *0* $\Longrightarrow$ *f sums* (*a*/*c*)
⟨*proof*⟩

lemma *summable-mult-D*: *summable* ($\lambda n.$ *c* $*$ *f n*) $\Longrightarrow$ *c* $\neq$ *0* $\Longrightarrow$ *summable f*
⟨*proof*⟩

Sum of a geometric progression.

lemma *geometric-sums*:
assumes *norm c < 1*
shows *(λn. c^n) sums (1 / (1 − c))*
⟨*proof*⟩

lemma *summable-geometric*: *norm c < 1 ⟹ summable (λn. c^n)*
⟨*proof*⟩

lemma *suminf-geometric*: *norm c < 1 ⟹ suminf (λn. c^n) = 1 / (1 − c)*
⟨*proof*⟩

lemma *summable-geometric-iff* [*simp*]: *summable (λn. c ^ n) ⟷ norm c < 1*
⟨*proof*⟩

end

Biconditional versions for constant factors

context
fixes *c* :: *'a*::*real-normed-field*
begin

lemma *summable-cmult-iff* [*simp*]: *summable (λn. c * f n) ⟷ c=0 ∨ summable f*
⟨*proof*⟩

lemma *summable-divide-iff* [*simp*]: *summable (λn. f n / c) ⟷ c=0 ∨ summable f*
⟨*proof*⟩

end

lemma *power-half-series*: *(λn. (1/2::real)^Suc n) sums 1*
⟨*proof*⟩

110.8 Telescoping

lemma *telescope-sums*:
fixes *c* :: *'a*::*real-normed-vector*
assumes *f ⟶ c*
shows *(λn. f (Suc n) − f n) sums (c − f 0)*
⟨*proof*⟩

lemma *telescope-sums'*:
fixes *c* :: *'a*::*real-normed-vector*
assumes *f ⟶ c*
shows *(λn. f n − f (Suc n)) sums (f 0 − c)*
⟨*proof*⟩

lemma *telescope-summable*:
fixes $c :: 'a$::*real-normed-vector*
assumes $f \longrightarrow c$
shows *summable* $(\lambda n.\ f\ (Suc\ n) - f\ n)$
⟨*proof*⟩

lemma *telescope-summable'*:
fixes $c :: 'a$::*real-normed-vector*
assumes $f \longrightarrow c$
shows *summable* $(\lambda n.\ f\ n - f\ (Suc\ n))$
⟨*proof*⟩

110.9 Infinite summability on Banach spaces

Cauchy-type criterion for convergence of series (c.f. Harrison).

lemma *summable-Cauchy*: *summable* $f \longleftrightarrow (\forall\, e{>}0.\ \exists\, N.\ \forall\, m{\geq}N.\ \forall\, n.\ norm\ (sum\ f\ \{m..{<}n\}) < e)$ (**is** - = *?rhs*)
for $f :: nat \Rightarrow 'a$::*banach*
⟨*proof*⟩

lemma *summable-Cauchy'*:
fixes $f :: nat \Rightarrow 'a :: banach$
assumes *ev*: *eventually* $(\lambda m.\ \forall\, n{\geq}m.\ norm\ (sum\ f\ \{m..{<}n\}) \leq g\ m)$ *sequentially*
assumes *g0*: $g \longrightarrow 0$
shows *summable f*
⟨*proof*⟩

context
fixes $f :: nat \Rightarrow 'a$::*banach*
begin

Absolute convergence imples normal convergence.

lemma *summable-norm-cancel*: *summable* $(\lambda n.\ norm\ (f\ n)) \Longrightarrow$ *summable f*
⟨*proof*⟩

lemma *summable-norm*: *summable* $(\lambda n.\ norm\ (f\ n)) \Longrightarrow norm\ (suminf\ f) \leq (\sum n.\ norm\ (f\ n))$
⟨*proof*⟩

Comparison tests.

lemma *summable-comparison-test*:
assumes *fg*: $\exists\, N.\ \forall\, n{\geq}N.\ norm\ (f\ n) \leq g\ n$ **and** *g*: *summable g*
shows *summable f*
⟨*proof*⟩

lemma *summable-comparison-test-ev*:
eventually $(\lambda n.\ norm\ (f\ n) \leq g\ n)$ *sequentially* $\Longrightarrow$ *summable g* $\Longrightarrow$ *summable f*
⟨*proof*⟩

A better argument order.

lemma *summable-comparison-test'*: *summable g* $\Longrightarrow$ ($\bigwedge n.\ n \geq N \Longrightarrow$ *norm* (*f n*) $\leq$ *g n*) $\Longrightarrow$ *summable f*
⟨*proof*⟩

110.10 The Ratio Test

lemma *summable-ratio-test*:
assumes $c < 1$ $\bigwedge n.\ n \geq N \Longrightarrow$ *norm* (*f* (*Suc n*)) $\leq c *$ *norm* (*f n*)
shows *summable f*
⟨*proof*⟩

end

Application to convergence of the log function

lemma *norm-summable-ln-series*:
fixes *z* :: *'a* :: {*real-normed-field*, *banach*}
assumes *norm z* < *1*
shows *summable* (λ*n*. *norm* (*z* ^ *n* / *of-nat n*))
⟨*proof*⟩

Relations among convergence and absolute convergence for power series.

lemma *Abel-lemma*:
fixes *a* :: *nat* ⇒ *'a*::*real-normed-vector*
assumes *r*: *0* ≤ *r*
and *r0*: *r* < *r0*
and *M*: $\bigwedge n.$ *norm* (*a n*) * *r0*^*n* ≤ *M*
shows *summable* (λ*n*. *norm* (*a n*) * *r*^*n*)
⟨*proof*⟩

Summability of geometric series for real algebras.

lemma *complete-algebra-summable-geometric*:
fixes *x* :: *'a*::{*real-normed-algebra-1*,*banach*}
assumes *norm x* < *1*
shows *summable* (λ*n*. *x* ^ *n*)
⟨*proof*⟩

110.11 Cauchy Product Formula

Proof based on Analysis WebNotes: Chapter 07, Class 41 http://www.math.unl.edu/~webnotes/classes/class41/prp77.htm

lemma *Cauchy-product-sums*:
fixes *a b* :: *nat* ⇒ *'a*::{*real-normed-algebra*,*banach*}
assumes *a*: *summable* (λ*k*. *norm* (*a k*))
and *b*: *summable* (λ*k*. *norm* (*b k*))
shows (λ*k*. $\sum i{\leq}k.$ *a i* * *b* (*k* − *i*)) *sums* (($\sum k.$ *a k*) * ($\sum k.$ *b k*))
⟨*proof*⟩

lemma *Cauchy-product*:
fixes *a b* :: *nat* ⇒ *'a*::{*real-normed-algebra*,*banach*}
assumes *summable* (λ*k*. *norm* (*a k*))
and *summable* (λ*k*. *norm* (*b k*))
shows (∑ *k*. *a k*) ∗ (∑ *k*. *b k*) = (∑ *k*. ∑ *i*≤*k*. *a i* ∗ *b* (*k* − *i*))
⟨*proof*⟩

lemma *summable-Cauchy-product*:
fixes *a b* :: *nat* ⇒ *'a*::{*real-normed-algebra*,*banach*}
assumes *summable* (λ*k*. *norm* (*a k*))
and *summable* (λ*k*. *norm* (*b k*))
shows *summable* (λ*k*. ∑ *i*≤*k*. *a i* ∗ *b* (*k* − *i*))
⟨*proof*⟩

110.12 Series on *real*s

lemma *summable-norm-comparison-test*:
∃ *N*. ∀ *n*≥*N*. *norm* (*f n*) ≤ *g n* ⟹ *summable g* ⟹ *summable* (λ*n*. *norm* (*f n*))
⟨*proof*⟩

lemma *summable-rabs-comparison-test*: ∃ *N*. ∀ *n*≥*N*. |*f n*| ≤ *g n* ⟹ *summable g*
⟹ *summable* (λ*n*. |*f n*|)
for *f* :: *nat* ⇒ *real*
⟨*proof*⟩

lemma *summable-rabs-cancel*: *summable* (λ*n*. |*f n*|) ⟹ *summable f*
for *f* :: *nat* ⇒ *real*
⟨*proof*⟩

lemma *summable-rabs*: *summable* (λ*n*. |*f n*|) ⟹ |*suminf f*| ≤ (∑ *n*. |*f n*|)
for *f* :: *nat* ⇒ *real*
⟨*proof*⟩

lemma *norm-suminf-le*:
assumes ⋀*n*. *norm* (*f n* :: *'a* :: *banach*) ≤ *g n summable g*
shows *norm* (*suminf f*) ≤ *suminf g*
⟨*proof*⟩

lemma *summable-zero-power* [*simp*]: *summable* (λ*n*. *0* ^ *n* :: *'a*::{*comm-ring-1*,*topological-space*})
⟨*proof*⟩

lemma *summable-zero-power'* [*simp*]: *summable* (λ*n*. *f n* ∗ *0* ^ *n* :: *'a*::{*ring-1*,*topological-space*})
⟨*proof*⟩

lemma *summable-power-series*:
fixes *z* :: *real*
assumes *le-1*: ⋀*i*. *f i* ≤ *1*
and *nonneg*: ⋀*i*. *0* ≤ *f i*

and *z*: $0 \leq z\ z < 1$
shows *summable* $(\lambda i.\ f\ i * z\hat{}i)$
⟨*proof*⟩

lemma *summable-0-powser*: *summable* $(\lambda n.\ f\ n * 0\ \hat{}\ n :: 'a{::}$*real-normed-div-algebra*$)$
⟨*proof*⟩

lemma *summable-powser-split-head*:
summable $(\lambda n.\ f\ (Suc\ n) * z\ \hat{}\ n :: 'a{::}$*real-normed-div-algebra*$) =$ *summable* $(\lambda n.\ f\ n * z\ \hat{}\ n)$
⟨*proof*⟩

lemma *summable-powser-ignore-initial-segment*:
fixes $f :: nat \Rightarrow 'a ::$ *real-normed-div-algebra*
shows *summable* $(\lambda n.\ f\ (n + m) * z\ \hat{}\ n) \longleftrightarrow$ *summable* $(\lambda n.\ f\ n * z\ \hat{}\ n)$
⟨*proof*⟩

lemma *powser-split-head*:
fixes $f :: nat \Rightarrow 'a{::}\{$*real-normed-div-algebra*,*banach*$\}$
assumes *summable* $(\lambda n.\ f\ n * z\ \hat{}\ n)$
shows *suminf* $(\lambda n.\ f\ n * z\ \hat{}\ n) = f\ 0 +$ *suminf* $(\lambda n.\ f\ (Suc\ n) * z\ \hat{}\ n) * z$
and *suminf* $(\lambda n.\ f\ (Suc\ n) * z\ \hat{}\ n) * z =$ *suminf* $(\lambda n.\ f\ n * z\ \hat{}\ n) - f\ 0$
and *summable* $(\lambda n.\ f\ (Suc\ n) * z\ \hat{}\ n)$
⟨*proof*⟩

lemma *summable-partial-sum-bound*:
fixes $f :: nat \Rightarrow 'a ::$ *banach*
and $e ::$ *real*
assumes *summable*: *summable f*
and *e*: $e > 0$
obtains N **where** $\bigwedge m\ n.\ m \geq N \Longrightarrow$ *norm* $(\sum k{=}m..n.\ f\ k) < e$
⟨*proof*⟩

lemma *powser-sums-if*:
$(\lambda n.\ (if\ n = m\ then\ (1 :: 'a{::}\{$*ring-1*,*topological-space*$\})\ else\ 0) * z\hat{}n)$ *sums* $z\hat{}m$
⟨*proof*⟩

lemma
fixes $f :: nat \Rightarrow real$
assumes *summable f*
and *inj g*
and *pos*: $\bigwedge x.\ 0 \leq f\ x$
shows *summable-reindex*: *summable* $(f \circ g)$
and *suminf-reindex-mono*: *suminf* $(f \circ g) \leq$ *suminf f*
and *suminf-reindex*: $(\bigwedge x.\ x \notin range\ g \Longrightarrow f\ x = 0) \Longrightarrow$ *suminf* $(f \circ g) =$ *suminf f*
⟨*proof*⟩

lemma *sums-mono-reindex*:

assumes *subseq*: *strict-mono g*
and *zero*: $\bigwedge n.\ n \notin range\ g \Longrightarrow f\ n = 0$
shows $(\lambda n.\ f\ (g\ n))\ sums\ c \longleftrightarrow f\ sums\ c$
⟨*proof*⟩

lemma *summable-mono-reindex*:
assumes *subseq*: *strict-mono g*
and *zero*: $\bigwedge n.\ n \notin range\ g \Longrightarrow f\ n = 0$
shows $summable\ (\lambda n.\ f\ (g\ n)) \longleftrightarrow summable\ f$
⟨*proof*⟩

lemma *suminf-mono-reindex*:
fixes $f :: nat \Rightarrow 'a$::{*t2-space*,*comm-monoid-add*}
assumes *strict-mono g* $\bigwedge n.\ n \notin range\ g \Longrightarrow f\ n = 0$
shows $suminf\ (\lambda n.\ f\ (g\ n)) = suminf\ f$
⟨*proof*⟩

lemma *summable-bounded-partials*:
fixes $f :: nat \Rightarrow 'a ::$ {*real-normed-vector*,*complete-space*}
assumes *bound*: *eventually* $(\lambda x0.\ \forall a{\geq}x0.\ \forall b{>}a.\ norm\ (sum\ f\ \{a{<}..b\}) \leq g\ a)$ *sequentially*
assumes *g*: $g \longrightarrow 0$
shows $summable\ f$ ⟨*proof*⟩

end

111 Differentiation

theory *Deriv*
imports *Limits*
begin

111.1 Frechet derivative

definition *has-derivative* :: ('*a*::*real-normed-vector* ⇒ '*b*::*real-normed-vector*) ⇒
('*a* ⇒ '*b*) ⇒ '*a filter* ⇒ *bool* (**infix** (*has'-derivative*) *50*)
where $(f\ has\text{-}derivative\ f')\ F \longleftrightarrow$
bounded-linear $f' \wedge$
$((\lambda y.\ ((f\ y - f\ (Lim\ F\ (\lambda x.\ x))) - f'\ (y - Lim\ F\ (\lambda x.\ x)))\ /_R\ norm\ (y - Lim\ F\ (\lambda x.\ x))) \longrightarrow 0)\ F$

Usually the filter F is $at\ x\ within\ s$. $(f\ has\text{-}derivative\ D)\ (at\ x\ within\ s)$ means: D is the derivative of function f at point x within the set s. Where s is used to express left or right sided derivatives. In most cases s is either a variable or *UNIV*.

These are the only cases we'll care about, probably.

lemma *has-derivative-within*: $(f\ has\text{-}derivative\ f')\ (at\ x\ within\ s) \longleftrightarrow$

bounded-linear $f' \wedge ((\lambda y.\ (1\ /\ norm(y - x)) *_R (f\ y - (f\ x + f'\ (y - x)))) \longrightarrow 0)\ (at\ x\ within\ s)$
⟨*proof*⟩

lemma *has-derivative-eq-rhs*: $(f\ has\text{-}derivative\ f')\ F \Longrightarrow f' = g' \Longrightarrow (f\ has\text{-}derivative\ g')\ F$
⟨*proof*⟩

definition *has-field-derivative* :: $('a{::}real\text{-}normed\text{-}field \Rightarrow 'a) \Rightarrow 'a \Rightarrow 'a\ filter \Rightarrow bool$
(**infix** $(has'\text{-}field'\text{-}derivative)\ 50$)
where $(f\ has\text{-}field\text{-}derivative\ D)\ F \longleftrightarrow (f\ has\text{-}derivative\ (*)\ D)\ F$

lemma *DERIV-cong*: $(f\ has\text{-}field\text{-}derivative\ X)\ F \Longrightarrow X = Y \Longrightarrow (f\ has\text{-}field\text{-}derivative\ Y)\ F$
⟨*proof*⟩

definition *has-vector-derivative* :: $(real \Rightarrow 'b{::}real\text{-}normed\text{-}vector) \Rightarrow 'b \Rightarrow real\ filter \Rightarrow bool$
(**infix** $has'\text{-}vector'\text{-}derivative\ 50$)
where $(f\ has\text{-}vector\text{-}derivative\ f')\ net \longleftrightarrow (f\ has\text{-}derivative\ (\lambda x.\ x *_R f'))\ net$

lemma *has-vector-derivative-eq-rhs*:
$(f\ has\text{-}vector\text{-}derivative\ X)\ F \Longrightarrow X = Y \Longrightarrow (f\ has\text{-}vector\text{-}derivative\ Y)\ F$
⟨*proof*⟩

named-theorems *derivative-intros structural introduction rules for derivatives*
⟨*ML*⟩

The following syntax is only used as a legacy syntax.

abbreviation (*input*)
$FDERIV :: ('a{::}real\text{-}normed\text{-}vector \Rightarrow 'b{::}real\text{-}normed\text{-}vector) \Rightarrow 'a \Rightarrow ('a \Rightarrow 'b) \Rightarrow bool$
$((FDERIV\ (\text{-})/\ (\text{-})/\ :>\ (\text{-}))\ [1000,\ 1000,\ 60]\ 60)$
where $FDERIV\ f\ x :> f' \equiv (f\ has\text{-}derivative\ f')\ (at\ x)$

lemma *has-derivative-bounded-linear*: $(f\ has\text{-}derivative\ f')\ F \Longrightarrow bounded\text{-}linear\ f'$
⟨*proof*⟩

lemma *has-derivative-linear*: $(f\ has\text{-}derivative\ f')\ F \Longrightarrow linear\ f'$
⟨*proof*⟩

lemma *has-derivative-ident*[*derivative-intros*, *simp*]: $((\lambda x.\ x)\ has\text{-}derivative\ (\lambda x.\ x))\ F$
⟨*proof*⟩

lemma *has-derivative-id* [*derivative-intros*, *simp*]: $(id\ has\text{-}derivative\ id)\ F$
⟨*proof*⟩

lemma *shift-has-derivative-id*: $((+)\ d\ \textit{has-derivative}\ (\lambda x.\ x))\ F$
⟨*proof*⟩

lemma *has-derivative-const*[*derivative-intros*, *simp*]: $((\lambda x.\ c)\ \textit{has-derivative}\ (\lambda x.\ 0))\ F$
⟨*proof*⟩

lemma (**in** *bounded-linear*) *bounded-linear*: *bounded-linear f* ⟨*proof*⟩

lemma (**in** *bounded-linear*) *has-derivative*:
$(g\ \textit{has-derivative}\ g')\ F \Longrightarrow ((\lambda x.\ f\ (g\ x))\ \textit{has-derivative}\ (\lambda x.\ f\ (g'\ x)))\ F$
⟨*proof*⟩

lemmas *has-derivative-scaleR-right* [*derivative-intros*] =
bounded-linear.has-derivative [*OF bounded-linear-scaleR-right*]

lemmas *has-derivative-scaleR-left* [*derivative-intros*] =
bounded-linear.has-derivative [*OF bounded-linear-scaleR-left*]

lemmas *has-derivative-mult-right* [*derivative-intros*] =
bounded-linear.has-derivative [*OF bounded-linear-mult-right*]

lemmas *has-derivative-mult-left* [*derivative-intros*] =
bounded-linear.has-derivative [*OF bounded-linear-mult-left*]

lemmas *has-derivative-of-real*[*derivative-intros*, *simp*] =
bounded-linear.has-derivative[*OF bounded-linear-of-real*]

lemma *has-derivative-add*[*simp*, *derivative-intros*]:
assumes *f*: $(f\ \textit{has-derivative}\ f')\ F$
and *g*: $(g\ \textit{has-derivative}\ g')\ F$
shows $((\lambda x.\ f\ x + g\ x)\ \textit{has-derivative}\ (\lambda x.\ f'\ x + g'\ x))\ F$
⟨*proof*⟩

lemma *has-derivative-sum*[*simp*, *derivative-intros*]:
$(\bigwedge i.\ i \in I \Longrightarrow (f\ i\ \textit{has-derivative}\ f'\ i)\ F) \Longrightarrow$
$((\lambda x.\ \sum i{\in}I.\ f\ i\ x)\ \textit{has-derivative}\ (\lambda x.\ \sum i{\in}I.\ f'\ i\ x))\ F$
⟨*proof*⟩

lemma *has-derivative-minus*[*simp*, *derivative-intros*]:
$(f\ \textit{has-derivative}\ f')\ F \Longrightarrow ((\lambda x.\ -\ f\ x)\ \textit{has-derivative}\ (\lambda x.\ -\ f'\ x))\ F$
⟨*proof*⟩

lemma *has-derivative-diff*[*simp*, *derivative-intros*]:
$(f\ \textit{has-derivative}\ f')\ F \Longrightarrow (g\ \textit{has-derivative}\ g')\ F \Longrightarrow$
$((\lambda x.\ f\ x - g\ x)\ \textit{has-derivative}\ (\lambda x.\ f'\ x - g'\ x))\ F$
⟨*proof*⟩

lemma *has-derivative-at-within*:

(*f has-derivative f'*) (*at x within s*) ⟷
(*bounded-linear f'* ∧ ((λ*y*. ((*f y* − *f x*) − *f'* (*y* − *x*)) /$_R$ *norm* (*y* − *x*)) ⟶ *0*) (*at x within s*))
⟨*proof*⟩

lemma *has-derivative-iff-norm*:
(*f has-derivative f'*) (*at x within s*) ⟷
bounded-linear f' ∧ ((λ*y*. *norm* ((*f y* − *f x*) − *f'* (*y* − *x*)) / *norm* (*y* − *x*)) ⟶ *0*) (*at x within s*)
⟨*proof*⟩

lemma *has-derivative-at*:
(*f has-derivative D*) (*at x*) ⟷
(*bounded-linear D* ∧ (λ*h*. *norm* (*f* (*x* + *h*) − *f x* − *D h*) / *norm h*) −*0*→ *0*)
⟨*proof*⟩

lemma *field-has-derivative-at*:
fixes *x* :: *'a*::*real-normed-field*
shows (*f has-derivative* (∗) *D*) (*at x*) ⟷ (λ*h*. (*f* (*x* + *h*) − *f x*) / *h*) −*0*→ *D* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *has-derivative-iff-Ex*:
(*f has-derivative f'*) (*at x*) ⟷
bounded-linear f' ∧ (∃ *e*. (∀ *h*. *f* (*x*+*h*) = *f x* + *f' h* + *e h*) ∧ ((λ*h*. *norm* (*e h*) / *norm h*) ⟶ *0*) (*at 0*))
⟨*proof*⟩

lemma *has-derivative-at-within-iff-Ex*:
assumes *x* ∈ *S open S*
shows (*f has-derivative f'*) (*at x within S*) ⟷
bounded-linear f' ∧ (∃ *e*. (∀ *h*. *x*+*h* ∈ *S* ⟶ *f* (*x*+*h*) = *f x* + *f' h* + *e h*) ∧ ((λ*h*. *norm* (*e h*) / *norm h*) ⟶ *0*) (*at 0*))
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *has-derivativeI*:
bounded-linear f' ⟹
((λ*y*. ((*f y* − *f x*) − *f'* (*y* − *x*)) /$_R$ *norm* (*y* − *x*)) ⟶ *0*) (*at x within s*) ⟹
(*f has-derivative f'*) (*at x within s*)
⟨*proof*⟩

lemma *has-derivativeI-sandwich*:
assumes *e*: *0* < *e*
and *bounded*: *bounded-linear f'*
and *sandwich*: (⋀*y*. *y* ∈ *s* ⟹ *y* ≠ *x* ⟹ *dist y x* < *e* ⟹
norm ((*f y* − *f x*) − *f'* (*y* − *x*)) / *norm* (*y* − *x*) ≤ *H y*)
and (*H* ⟶ *0*) (*at x within s*)
shows (*f has-derivative f'*) (*at x within s*)

⟨*proof*⟩

lemma *has-derivative-subset*:
(*f has-derivative f'*) (*at x within s*) ⟹ *t* ⊆ *s* ⟹ (*f has-derivative f'*) (*at x within t*)
⟨*proof*⟩

lemma *has-derivative-within-singleton-iff*:
(*f has-derivative g*) (*at x within* {*x*}) ⟷ *bounded-linear g*
⟨*proof*⟩

111.1.1 Limit transformation for derivatives

lemma *has-derivative-transform-within*:
assumes (*f has-derivative f'*) (*at x within s*)
and *0* < *d*
and *x* ∈ *s*
and ⋀*x'*. ⟦*x'* ∈ *s*; *dist x' x* < *d*⟧ ⟹ *f x'* = *g x'*
shows (*g has-derivative f'*) (*at x within s*)
⟨*proof*⟩

lemma *has-derivative-transform-within-open*:
assumes (*f has-derivative f'*) (*at x within t*)
and *open s*
and *x* ∈ *s*
and ⋀*x*. *x*∈*s* ⟹ *f x* = *g x*
shows (*g has-derivative f'*) (*at x within t*)
⟨*proof*⟩

lemma *has-derivative-transform*:
assumes *x* ∈ *s* ⋀*x*. *x* ∈ *s* ⟹ *g x* = *f x*
assumes (*f has-derivative f'*) (*at x within s*)
shows (*g has-derivative f'*) (*at x within s*)
⟨*proof*⟩

lemma *has-derivative-transform-eventually*:
assumes (*f has-derivative f'*) (*at x within s*)
(∀$_F$ *x'* *in at x within s*. *f x'* = *g x'*)
assumes *f x* = *g x x* ∈ *s*
shows (*g has-derivative f'*) (*at x within s*)
⟨*proof*⟩

lemma *has-field-derivative-transform-within*:
assumes (*f has-field-derivative f'*) (*at a within S*)
and *0* < *d*
and *a* ∈ *S*
and ⋀*x*. ⟦*x* ∈ *S*; *dist x a* < *d*⟧ ⟹ *f x* = *g x*
shows (*g has-field-derivative f'*) (*at a within S*)
⟨*proof*⟩

lemma *has-field-derivative-transform-within-open*:
assumes (*f has-field-derivative f'*) (*at a*)
and *open S a* ∈ *S*
and ⋀*x*. *x* ∈ *S* ⟹ *f x* = *g x*
shows (*g has-field-derivative f'*) (*at a*)
⟨*proof*⟩

111.2 Continuity

lemma *has-derivative-continuous*:
assumes *f*: (*f has-derivative f'*) (*at x within s*)
shows *continuous* (*at x within s*) *f*
⟨*proof*⟩

111.3 Composition

lemma *tendsto-at-iff-tendsto-nhds-within*:
f x = *y* ⟹ (*f* ⟶ *y*) (*at x within s*) ⟷ (*f* ⟶ *y*) (*inf* (*nhds x*) (*principal s*))
⟨*proof*⟩

lemma *has-derivative-in-compose*:
assumes *f*: (*f has-derivative f'*) (*at x within s*)
and *g*: (*g has-derivative g'*) (*at* (*f x*) *within* (*f*‘*s*))
shows ((λ*x*. *g* (*f x*)) *has-derivative* (λ*x*. *g'* (*f'* *x*))) (*at x within s*)
⟨*proof*⟩

lemma *has-derivative-compose*:
(*f has-derivative f'*) (*at x within s*) ⟹ (*g has-derivative g'*) (*at* (*f x*)) ⟹
((λ*x*. *g* (*f x*)) *has-derivative* (λ*x*. *g'* (*f'* *x*))) (*at x within s*)
⟨*proof*⟩

lemma *has-derivative-in-compose2*:
assumes ⋀*x*. *x* ∈ *t* ⟹ (*g has-derivative g'* *x*) (*at x within t*)
assumes *f* ‘ *s* ⊆ *t x* ∈ *s*
assumes (*f has-derivative f'*) (*at x within s*)
shows ((λ*x*. *g* (*f x*)) *has-derivative* (λ*y*. *g'* (*f x*) (*f'* *y*))) (*at x within s*)
⟨*proof*⟩

lemma (**in** *bounded-bilinear*) *FDERIV*:
assumes *f*: (*f has-derivative f'*) (*at x within s*) **and** *g*: (*g has-derivative g'*) (*at x within s*)
shows ((λ*x*. *f x* ** *g x*) *has-derivative* (λ*h*. *f x* ** *g'* *h* + *f'* *h* ** *g x*)) (*at x within s*)
⟨*proof*⟩

lemmas *has-derivative-mult*[*simp*, *derivative-intros*] = *bounded-bilinear.FDERIV*[*OF bounded-bilinear-mult*]

lemmas *has-derivative-scaleR*[*simp*, *derivative-intros*] = *bounded-bilinear.FDERIV*[*OF bounded-bilinear-scaleR*]

lemma *has-derivative-prod*[*simp*, *derivative-intros*]:
fixes $f :: 'i \Rightarrow 'a$::*real-normed-vector* $\Rightarrow 'b$::*real-normed-field*
shows $(\bigwedge i.\ i \in I \Longrightarrow (f\ i\ \textit{has-derivative}\ f'\ i)\ (\textit{at}\ x\ \textit{within}\ S)) \Longrightarrow$
$((\lambda x.\ \prod i{\in}I.\ f\ i\ x)\ \textit{has-derivative}\ (\lambda y.\ \sum i{\in}I.\ f'\ i\ y * (\prod j{\in}I - \{i\}.\ f\ j\ x)))$
$(\textit{at}\ x\ \textit{within}\ S)$
⟨*proof*⟩

lemma *has-derivative-power*[*simp*, *derivative-intros*]:
fixes $f :: 'a :: \textit{real-normed-vector} \Rightarrow 'b :: \textit{real-normed-field}$
assumes *f*: $(f\ \textit{has-derivative}\ f')\ (\textit{at}\ x\ \textit{within}\ S)$
shows $((\lambda x.\ f\ x\hat{}n)\ \textit{has-derivative}\ (\lambda y.\ \textit{of-nat}\ n * f'\ y * f\ x\hat{}(n - 1)))\ (\textit{at}\ x\ \textit{within}\ S)$
⟨*proof*⟩

lemma *has-derivative-inverse'*:
fixes $x :: 'a$::*real-normed-div-algebra*
assumes *x*: $x \neq 0$
shows $(\textit{inverse}\ \textit{has-derivative}\ (\lambda h.\ -\ (\textit{inverse}\ x * h * \textit{inverse}\ x)))\ (\textit{at}\ x\ \textit{within}\ S)$
(**is** (- *has-derivative ?f*) -)
⟨*proof*⟩

lemma *has-derivative-inverse*[*simp*, *derivative-intros*]:
fixes $f :: - \Rightarrow 'a$::*real-normed-div-algebra*
assumes *x*: $f\ x \neq 0$
and *f*: $(f\ \textit{has-derivative}\ f')\ (\textit{at}\ x\ \textit{within}\ S)$
shows $((\lambda x.\ \textit{inverse}\ (f\ x))\ \textit{has-derivative}\ (\lambda h.\ -\ (\textit{inverse}\ (f\ x) * f'\ h * \textit{inverse}\ (f\ x))))$
$(\textit{at}\ x\ \textit{within}\ S)$
⟨*proof*⟩

lemma *has-derivative-divide*[*simp*, *derivative-intros*]:
fixes $f :: - \Rightarrow 'a$::*real-normed-div-algebra*
assumes *f*: $(f\ \textit{has-derivative}\ f')\ (\textit{at}\ x\ \textit{within}\ S)$
and *g*: $(g\ \textit{has-derivative}\ g')\ (\textit{at}\ x\ \textit{within}\ S)$
assumes *x*: $g\ x \neq 0$
shows $((\lambda x.\ f\ x\ /\ g\ x)\ \textit{has-derivative}$
$(\lambda h.\ -\ f\ x * (\textit{inverse}\ (g\ x) * g'\ h * \textit{inverse}\ (g\ x)) + f'\ h\ /\ g\ x))\ (\textit{at}\ x\ \textit{within}\ S)$
⟨*proof*⟩

lemma *has-derivative-power-int'*:
fixes $x :: 'a$::*real-normed-field*
assumes *x*: $x \neq 0$
shows $((\lambda x.\ \textit{power-int}\ x\ n)\ \textit{has-derivative}\ (\lambda y.\ y * (\textit{of-int}\ n * \textit{power-int}\ x\ (n - 1))))\ (\textit{at}\ x\ \textit{within}\ S)$

⟨*proof*⟩

lemma *has-derivative-power-int*[*simp*, *derivative-intros*]:
fixes f :: - ⇒ 'a::*real-normed-field*
assumes x: $f\ x \neq 0$
and f: (f *has-derivative* f') (*at* x *within* S)
shows ((λx. *power-int* ($f\ x$) n) *has-derivative* (λh. $f'\ h$ ∗ (*of-int* n ∗ *power-int* (f x) (n − *1*)))) (*at* x *within* S)
⟨*proof*⟩

Conventional form requires mult-AC laws. Types real and complex only.

lemma *has-derivative-divide'*[*derivative-intros*]:
fixes f :: - ⇒ 'a::*real-normed-field*
assumes f: (f *has-derivative* f') (*at* x *within* S)
and g: (g *has-derivative* g') (*at* x *within* S)
and x: $g\ x \neq 0$
shows ((λx. $f\ x$ / $g\ x$) *has-derivative* (λh. ($f'\ h$ ∗ $g\ x$ − $f\ x$ ∗ $g'\ h$) / ($g\ x$ ∗ $g\ x$))) (*at* x *within* S)
⟨*proof*⟩

111.4 Uniqueness

This can not generally shown for (*has-derivative*), as we need to approach the point from all directions. There is a proof in *Analysis* for *euclidean-space*.

lemma *has-derivative-at2*: (f *has-derivative* f') (*at* x) ⟷
bounded-linear f' ∧ ((λy. (*1* / (*norm*(y − x))) ∗$_R$ ($f\ y$ − ($f\ x$ + f' (y − x)))) ⟶ *0*) (*at* x)
⟨*proof*⟩

lemma *has-derivative-zero-unique*:
assumes ((λx. *0*) *has-derivative* F) (*at* x)
shows F = (λh. *0*)
⟨*proof*⟩

lemma *has-derivative-unique*:
assumes (f *has-derivative* F) (*at* x)
and (f *has-derivative* F') (*at* x)
shows $F = F'$
⟨*proof*⟩

lemma *has-derivative-Uniq*: ∃$_{\leq 1}$$F$. ($f$ *has-derivative* F) (*at* x)
⟨*proof*⟩

111.5 Differentiability predicate

definition *differentiable* :: ('a::*real-normed-vector* ⇒ 'b::*real-normed-vector*) ⇒ 'a *filter* ⇒ *bool*
(**infix** *differentiable 50*)

where *f differentiable F* $\longleftrightarrow$ $(\exists\, D.\ (f\ \textit{has-derivative}\ D)\ F)$

lemma *differentiable-subset*:
f differentiable (*at x within s*) $\Longrightarrow$ $t \subseteq s \Longrightarrow$ *f differentiable* (*at x within t*)
⟨*proof*⟩

lemmas *differentiable-within-subset* = *differentiable-subset*

lemma *differentiable-ident* [*simp*, *derivative-intros*]: $(\lambda x.\ x)$ *differentiable F*
⟨*proof*⟩

lemma *differentiable-const* [*simp*, *derivative-intros*]: $(\lambda z.\ a)$ *differentiable F*
⟨*proof*⟩

lemma *differentiable-in-compose*:
f differentiable (*at* (*g x*) *within* (*g's*)) $\Longrightarrow$ *g differentiable* (*at x within s*) $\Longrightarrow$
$(\lambda x.\ f\ (g\ x))$ *differentiable* (*at x within s*)
⟨*proof*⟩

lemma *differentiable-compose*:
f differentiable (*at* (*g x*)) $\Longrightarrow$ *g differentiable* (*at x within s*) $\Longrightarrow$
$(\lambda x.\ f\ (g\ x))$ *differentiable* (*at x within s*)
⟨*proof*⟩

lemma *differentiable-add* [*simp*, *derivative-intros*]:
f differentiable F $\Longrightarrow$ *g differentiable F* $\Longrightarrow$ $(\lambda x.\ f\ x + g\ x)$ *differentiable F*
⟨*proof*⟩

lemma *differentiable-sum*[*simp*, *derivative-intros*]:
assumes *finite s* $\forall\, a{\in}s.\ (f\ a)$ *differentiable net*
shows $(\lambda x.\ \textit{sum}\ (\lambda a.\ f\ a\ x)\ s)$ *differentiable net*
⟨*proof*⟩

lemma *differentiable-minus* [*simp*, *derivative-intros*]:
f differentiable F $\Longrightarrow$ $(\lambda x.\ -\ f\ x)$ *differentiable F*
⟨*proof*⟩

lemma *differentiable-diff* [*simp*, *derivative-intros*]:
f differentiable F $\Longrightarrow$ *g differentiable F* $\Longrightarrow$ $(\lambda x.\ f\ x - g\ x)$ *differentiable F*
⟨*proof*⟩

lemma *differentiable-mult* [*simp*, *derivative-intros*]:
fixes *f g* :: *'a*::*real-normed-vector* $\Rightarrow$ *'b*::*real-normed-algebra*
shows *f differentiable* (*at x within s*) $\Longrightarrow$ *g differentiable* (*at x within s*) $\Longrightarrow$
$(\lambda x.\ f\ x * g\ x)$ *differentiable* (*at x within s*)
⟨*proof*⟩

lemma *differentiable-cmult-left-iff* [*simp*]:
fixes *c*::*'a*::*real-normed-field*

shows $(\lambda t.\ c * q\ t)$ *differentiable at* $t \longleftrightarrow c = 0 \vee (\lambda t.\ q\ t)$ *differentiable at* t
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *differentiable-cmult-right-iff* [*simp*]:
fixes $c::'a::$*real-normed-field*
shows $(\lambda t.\ q\ t * c)$ *differentiable at* $t \longleftrightarrow c = 0 \vee (\lambda t.\ q\ t)$ *differentiable at* t
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *differentiable-inverse* [*simp*, *derivative-intros*]:
fixes $f :: 'a::$*real-normed-vector* $\Rightarrow 'b::$*real-normed-field*
shows f *differentiable* $(at\ x\ within\ s) \Longrightarrow f\ x \neq 0 \Longrightarrow$
$(\lambda x.\ inverse\ (f\ x))$ *differentiable* $(at\ x\ within\ s)$
⟨*proof*⟩

lemma *differentiable-divide* [*simp*, *derivative-intros*]:
fixes $f\ g :: 'a::$*real-normed-vector* $\Rightarrow 'b::$*real-normed-field*
shows f *differentiable* $(at\ x\ within\ s) \Longrightarrow g$ *differentiable* $(at\ x\ within\ s) \Longrightarrow$
$g\ x \neq 0 \Longrightarrow (\lambda x.\ f\ x\ /\ g\ x)$ *differentiable* $(at\ x\ within\ s)$
⟨*proof*⟩

lemma *differentiable-power* [*simp*, *derivative-intros*]:
fixes $f\ g :: 'a::$*real-normed-vector* $\Rightarrow 'b::$*real-normed-field*
shows f *differentiable* $(at\ x\ within\ s) \Longrightarrow (\lambda x.\ f\ x\ \hat{}\ n)$ *differentiable* $(at\ x\ within\ s)$
⟨*proof*⟩

lemma *differentiable-power-int* [*simp*, *derivative-intros*]:
fixes $f :: 'a::$*real-normed-vector* $\Rightarrow 'b::$*real-normed-field*
shows f *differentiable* $(at\ x\ within\ s) \Longrightarrow f\ x \neq 0 \Longrightarrow$
$(\lambda x.\ power\text{-}int\ (f\ x)\ n)$ *differentiable* $(at\ x\ within\ s)$
⟨*proof*⟩

lemma *differentiable-scaleR* [*simp*, *derivative-intros*]:
f *differentiable* $(at\ x\ within\ s) \Longrightarrow g$ *differentiable* $(at\ x\ within\ s) \Longrightarrow$
$(\lambda x.\ f\ x *_R g\ x)$ *differentiable* $(at\ x\ within\ s)$
⟨*proof*⟩

lemma *has-derivative-imp-has-field-derivative*:
$(f$ *has-derivative* $D)\ F \Longrightarrow (\bigwedge x.\ x * D' = D\ x) \Longrightarrow (f$ *has-field-derivative* $D')\ F$
⟨*proof*⟩

lemma *has-field-derivative-imp-has-derivative*:
$(f$ *has-field-derivative* $D)\ F \Longrightarrow (f$ *has-derivative* $(*)\ D)\ F$
⟨*proof*⟩

lemma *DERIV-subset*:
$(f$ *has-field-derivative* $f')\ (at\ x\ within\ s) \Longrightarrow t \subseteq s \Longrightarrow$

(*f has-field-derivative f'*) (*at x within t*)
⟨*proof*⟩

lemma *has-field-derivative-at-within*:
(*f has-field-derivative f'*) (*at x*) ⟹ (*f has-field-derivative f'*) (*at x within s*)
⟨*proof*⟩

abbreviation (*input*)
DERIV :: ('*a*::*real-normed-field* ⇒ '*a*) ⇒ '*a* ⇒ '*a* ⇒ *bool*
((*DERIV* (-)/ (-)/ :> (-)) [*1000*, *1000*, *60*] *60*)
where *DERIV f x :> D* ≡ (*f has-field-derivative D*) (*at x*)

abbreviation *has-real-derivative* :: (*real* ⇒ *real*) ⇒ *real* ⇒ *real filter* ⇒ *bool*
(**infix** (*has'-real'-derivative*) *50*)
where (*f has-real-derivative D*) *F* ≡ (*f has-field-derivative D*) *F*

lemma *real-differentiable-def*:
f differentiable at x within s ⟷ (∃ *D*. (*f has-real-derivative D*) (*at x within s*))
⟨*proof*⟩

lemma *real-differentiableE* [*elim?*]:
assumes *f*: *f differentiable* (*at x within s*)
obtains *df* **where** (*f has-real-derivative df*) (*at x within s*)
⟨*proof*⟩

lemma *has-field-derivative-iff*:
(*f has-field-derivative D*) (*at x within S*) ⟷
((λ*y*. (*f y* − *f x*) / (*y* − *x*)) ⟶ *D*) (*at x within S*)
⟨*proof*⟩

lemma *DERIV-def*: *DERIV f x :> D* ⟷ (λ*h*. (*f* (*x* + *h*) − *f x*) / *h*) −*0*→ *D*
⟨*proof*⟩

due to Christian Pardillo Laursen, replacing a proper epsilon-delta horror

lemma *field-derivative-lim-unique*:
assumes *f*: (*f has-field-derivative df*) (*at z*)
and *s*: *s* ⟶ *0* ⋀*n*. *s n* ≠ *0*
and *a*: (λ*n*. (*f* (*z* + *s n*) − *f z*) / *s n*) ⟶ *a*
shows *df* = *a*
⟨*proof*⟩

lemma *mult-commute-abs*: (λ*x*. *x* ∗ *c*) = (∗) *c*
for *c* :: '*a*::*ab-semigroup-mult*
⟨*proof*⟩

lemma *DERIV-compose-FDERIV*:
fixes *f*::*real*⇒*real*
assumes *DERIV f* (*g x*) :> *f'*
assumes (*g has-derivative g'*) (*at x within s*)

shows $((\lambda x.\ f\ (g\ x))\ \textit{has-derivative}\ (\lambda x.\ g'\ x * f'))\ (\textit{at}\ x\ \textit{within}\ s)$
⟨*proof*⟩

111.6 Vector derivative

It's for real derivatives only, and not obviously generalisable to field derivatives

lemma *has-real-derivative-iff-has-vector-derivative*:
$(f\ \textit{has-real-derivative}\ y)\ F \longleftrightarrow (f\ \textit{has-vector-derivative}\ y)\ F$
⟨*proof*⟩

lemma *has-field-derivative-subset*:
$(f\ \textit{has-field-derivative}\ y)\ (\textit{at}\ x\ \textit{within}\ s) \Longrightarrow t \subseteq s \Longrightarrow$
$(f\ \textit{has-field-derivative}\ y)\ (\textit{at}\ x\ \textit{within}\ t)$
⟨*proof*⟩

lemma *has-vector-derivative-const*[*simp*, *derivative-intros*]: $((\lambda x.\ c)\ \textit{has-vector-derivative}\ 0)\ \textit{net}$
⟨*proof*⟩

lemma *has-vector-derivative-id*[*simp*, *derivative-intros*]: $((\lambda x.\ x)\ \textit{has-vector-derivative}\ 1)\ \textit{net}$
⟨*proof*⟩

lemma *has-vector-derivative-minus*[*derivative-intros*]:
$(f\ \textit{has-vector-derivative}\ f')\ \textit{net} \Longrightarrow ((\lambda x.\ -\ f\ x)\ \textit{has-vector-derivative}\ (-\ f'))\ \textit{net}$
⟨*proof*⟩

lemma *has-vector-derivative-add*[*derivative-intros*]:
$(f\ \textit{has-vector-derivative}\ f')\ \textit{net} \Longrightarrow (g\ \textit{has-vector-derivative}\ g')\ \textit{net} \Longrightarrow$
$((\lambda x.\ f\ x + g\ x)\ \textit{has-vector-derivative}\ (f' + g'))\ \textit{net}$
⟨*proof*⟩

lemma *has-vector-derivative-sum*[*derivative-intros*]:
$(\bigwedge i.\ i \in I \Longrightarrow (f\ i\ \textit{has-vector-derivative}\ f'\ i)\ \textit{net}) \Longrightarrow$
$((\lambda x.\ \sum i{\in}I.\ f\ i\ x)\ \textit{has-vector-derivative}\ (\sum i{\in}I.\ f'\ i))\ \textit{net}$
⟨*proof*⟩

lemma *has-vector-derivative-diff*[*derivative-intros*]:
$(f\ \textit{has-vector-derivative}\ f')\ \textit{net} \Longrightarrow (g\ \textit{has-vector-derivative}\ g')\ \textit{net} \Longrightarrow$
$((\lambda x.\ f\ x - g\ x)\ \textit{has-vector-derivative}\ (f' - g'))\ \textit{net}$
⟨*proof*⟩

lemma *has-vector-derivative-add-const*:
$((\lambda t.\ g\ t + z)\ \textit{has-vector-derivative}\ f')\ \textit{net} = ((\lambda t.\ g\ t)\ \textit{has-vector-derivative}\ f')\ \textit{net}$
⟨*proof*⟩

lemma *has-vector-derivative-diff-const*:

((λt. g t − z) has-vector-derivative f′) net = ((λt. g t) has-vector-derivative f′) net
⟨proof⟩

lemma (**in** *bounded-linear*) *has-vector-derivative*:
assumes (g has-vector-derivative g′) F
shows ((λx. f (g x)) has-vector-derivative f g′) F
⟨proof⟩

lemma (**in** *bounded-bilinear*) *has-vector-derivative*:
assumes (f has-vector-derivative f′) (at x within s)
and (g has-vector-derivative g′) (at x within s)
shows ((λx. f x ** g x) has-vector-derivative (f x ** g′ + f′ ** g x)) (at x within s)
⟨proof⟩

lemma *has-vector-derivative-scaleR*[*derivative-intros*]:
(f has-field-derivative f′) (at x within s) ⟹ (g has-vector-derivative g′) (at x within s) ⟹
((λx. f x *R g x) has-vector-derivative (f x *R g′ + f′ *R g x)) (at x within s)
⟨proof⟩

lemma *has-vector-derivative-mult*[*derivative-intros*]:
(f has-vector-derivative f′) (at x within s) ⟹ (g has-vector-derivative g′) (at x within s) ⟹
((λx. f x * g x) has-vector-derivative (f x * g′ + f′ * g x)) (at x within s)
for f g :: *real* ⇒ 'a::*real-normed-algebra*
⟨proof⟩

lemma *has-vector-derivative-of-real*[*derivative-intros*]:
(f has-field-derivative D) F ⟹ ((λx. of-real (f x)) has-vector-derivative (of-real D)) F
⟨proof⟩

lemma *has-vector-derivative-real-field*:
(f has-field-derivative f′) (at (of-real a)) ⟹ ((λx. f (of-real x)) has-vector-derivative f′) (at a within s)
⟨proof⟩

lemma *has-vector-derivative-continuous*:
(f has-vector-derivative D) (at x within s) ⟹ continuous (at x within s) f
⟨proof⟩

lemma *continuous-on-vector-derivative*:
(⋀x. x ∈ S ⟹ (f has-vector-derivative f′ x) (at x within S)) ⟹ continuous-on S f
⟨proof⟩

lemma *has-vector-derivative-mult-right*[*derivative-intros*]:

fixes $a :: 'a$::*real-normed-algebra*
shows $(f$ *has-vector-derivative* $x)\ F \Longrightarrow ((\lambda x.\ a * f\ x)$ *has-vector-derivative* $(a * x))\ F$
⟨*proof*⟩

lemma *has-vector-derivative-mult-left*[*derivative-intros*]:
fixes $a :: 'a$::*real-normed-algebra*
shows $(f$ *has-vector-derivative* $x)\ F \Longrightarrow ((\lambda x.\ f\ x * a)$ *has-vector-derivative* $(x * a))\ F$
⟨*proof*⟩

lemma *has-vector-derivative-divide*[*derivative-intros*]:
fixes $a :: 'a$::*real-normed-field*
shows $(f$ *has-vector-derivative* $x)\ F \Longrightarrow ((\lambda x.\ f\ x\ /\ a)$ *has-vector-derivative* $(x / a))\ F$
⟨*proof*⟩

111.7 Derivatives

lemma *DERIV-D*: $DERIV\ f\ x :> D \Longrightarrow (\lambda h.\ (f\ (x + h) - f\ x)\ /\ h) -0\to D$
⟨*proof*⟩

lemma *has-field-derivativeD*:
$(f$ *has-field-derivative* $D)$ $(at\ x\ within\ S) \Longrightarrow$
$((\lambda y.\ (f\ y - f\ x)\ /\ (y - x)) \longrightarrow D)\ (at\ x\ within\ S)$
⟨*proof*⟩

lemma *DERIV-const* [*simp*, *derivative-intros*]: $((\lambda x.\ k)$ *has-field-derivative* $0)\ F$
⟨*proof*⟩

lemma *DERIV-ident* [*simp*, *derivative-intros*]: $((\lambda x.\ x)$ *has-field-derivative* $1)\ F$
⟨*proof*⟩

lemma *field-differentiable-add*[*derivative-intros*]:
$(f$ *has-field-derivative* $f')\ F \Longrightarrow (g$ *has-field-derivative* $g')\ F \Longrightarrow$
$((\lambda z.\ f\ z + g\ z)$ *has-field-derivative* $f' + g')\ F$
⟨*proof*⟩

corollary *DERIV-add*:
$(f$ *has-field-derivative* $D)$ $(at\ x\ within\ s) \Longrightarrow (g$ *has-field-derivative* $E)$ $(at\ x\ within\ s) \Longrightarrow$
$((\lambda x.\ f\ x + g\ x)$ *has-field-derivative* $D + E)$ $(at\ x\ within\ s)$
⟨*proof*⟩

lemma *field-differentiable-minus*[*derivative-intros*]:
$(f$ *has-field-derivative* $f')\ F \Longrightarrow ((\lambda z.\ -\ (f\ z))$ *has-field-derivative* $-f')\ F$
⟨*proof*⟩

corollary *DERIV-minus*:

(*f has-field-derivative D*) (*at x within s*) ⟹
((λ*x*. − *f x*) *has-field-derivative* −*D*) (*at x within s*)
⟨*proof*⟩

lemma *field-differentiable-diff*[*derivative-intros*]:
(*f has-field-derivative f'*) *F* ⟹
(*g has-field-derivative g'*) *F* ⟹ ((λ*z*. *f z* − *g z*) *has-field-derivative f'* − *g'*) *F*
⟨*proof*⟩

corollary *DERIV-diff*:
(*f has-field-derivative D*) (*at x within s*) ⟹
(*g has-field-derivative E*) (*at x within s*) ⟹
((λ*x*. *f x* − *g x*) *has-field-derivative D* − *E*) (*at x within s*)
⟨*proof*⟩

lemma *DERIV-continuous*: (*f has-field-derivative D*) (*at x within s*) ⟹ *continuous* (*at x within s*) *f*
⟨*proof*⟩

corollary *DERIV-isCont*: *DERIV f x :> D* ⟹ *isCont f x*
⟨*proof*⟩

lemma *DERIV-atLeastAtMost-imp-continuous-on*:
assumes ⋀*x*. ⟦*a* ≤ *x*; *x* ≤ *b*⟧ ⟹ ∃ *y*. *DERIV f x :> y*
shows *continuous-on* {*a*..*b*} *f*
⟨*proof*⟩

lemma *DERIV-continuous-on*:
(⋀*x*. *x* ∈ *s* ⟹ (*f has-field-derivative* (*D x*)) (*at x within s*)) ⟹ *continuous-on s f*
⟨*proof*⟩

lemma *DERIV-mult'*:
(*f has-field-derivative D*) (*at x within s*) ⟹ (*g has-field-derivative E*) (*at x within s*) ⟹
((λ*x*. *f x* ∗ *g x*) *has-field-derivative f x* ∗ *E* + *D* ∗ *g x*) (*at x within s*)
⟨*proof*⟩

lemma *DERIV-mult*[*derivative-intros*]:
(*f has-field-derivative Da*) (*at x within s*) ⟹ (*g has-field-derivative Db*) (*at x within s*) ⟹
((λ*x*. *f x* ∗ *g x*) *has-field-derivative Da* ∗ *g x* + *Db* ∗ *f x*) (*at x within s*)
⟨*proof*⟩

Derivative of linear multiplication

lemma *DERIV-cmult*:
(*f has-field-derivative D*) (*at x within s*) ⟹
((λ*x*. *c* ∗ *f x*) *has-field-derivative c* ∗ *D*) (*at x within s*)
⟨*proof*⟩

lemma *DERIV-cmult-right*:
(*f has-field-derivative D*) (*at x within s*) ⟹
((λ*x*. *f x* ∗ *c*) *has-field-derivative D* ∗ *c*) (*at x within s*)
⟨*proof*⟩

lemma *DERIV-cmult-Id* [*simp*]: ((∗) *c has-field-derivative c*) (*at x within s*)
⟨*proof*⟩

lemma *DERIV-cdivide*:
(*f has-field-derivative D*) (*at x within s*) ⟹
((λ*x*. *f x* / *c*) *has-field-derivative D* / *c*) (*at x within s*)
⟨*proof*⟩

lemma *DERIV-unique*: *DERIV f x* :> *D* ⟹ *DERIV f x* :> *E* ⟹ *D* = *E*
⟨*proof*⟩

lemma *DERIV-Uniq*: ∃$_{\leq 1}$*D*. *DERIV f x* :> *D*
⟨*proof*⟩

lemma *DERIV-sum*[*derivative-intros*]:
(⋀ *n*. *n* ∈ *S* ⟹ ((λ*x*. *f x n*) *has-field-derivative* (*f*′ *x n*)) *F*) ⟹
((λ*x*. *sum* (*f x*) *S*) *has-field-derivative sum* (*f*′ *x*) *S*) *F*
⟨*proof*⟩

lemma *DERIV-inverse*′[*derivative-intros*]:
assumes (*f has-field-derivative D*) (*at x within s*)
and *f x* ≠ *0*
shows ((λ*x*. *inverse* (*f x*)) *has-field-derivative* − (*inverse* (*f x*) ∗ *D* ∗ *inverse* (*f x*)))
(*at x within s*)
⟨*proof*⟩

Power of −*1*

lemma *DERIV-inverse*:
x ≠ *0* ⟹ ((λ*x*. *inverse*(*x*)) *has-field-derivative* − (*inverse x* ^ *Suc* (*Suc 0*))) (*at x within s*)
⟨*proof*⟩

Derivative of inverse

lemma *DERIV-inverse-fun*:
(*f has-field-derivative d*) (*at x within s*) ⟹ *f x* ≠ *0* ⟹
((λ*x*. *inverse* (*f x*)) *has-field-derivative* (− (*d* ∗ *inverse*(*f x* ^ *Suc* (*Suc 0*))))) (*at x within s*)
⟨*proof*⟩

Derivative of quotient

lemma *DERIV-divide*[*derivative-intros*]:
(*f has-field-derivative D*) (*at x within s*) ⟹

$(g$ *has-field-derivative* $E)$ $(at\ x\ within\ s) \Longrightarrow g\ x \neq 0 \Longrightarrow$
$((\lambda x.\ f\ x\ /\ g\ x)$ *has-field-derivative* $(D * g\ x - f\ x * E) / (g\ x * g\ x))$ $(at\ x\ within\ s)$
⟨*proof*⟩

lemma *DERIV-quotient*:
$(f$ *has-field-derivative* $d)$ $(at\ x\ within\ s) \Longrightarrow$
$(g$ *has-field-derivative* $e)$ $(at\ x\ within\ s) \Longrightarrow g\ x \neq 0 \Longrightarrow$
$((\lambda y.\ f\ y\ /\ g\ y)$ *has-field-derivative* $(d * g\ x - (e * f\ x)) / (g\ x\ \hat{}\ Suc\ (Suc\ 0)))$ $(at\ x\ within\ s)$
⟨*proof*⟩

lemma *DERIV-power-Suc*:
$(f$ *has-field-derivative* $D)$ $(at\ x\ within\ s) \Longrightarrow$
$((\lambda x.\ f\ x\ \hat{}\ Suc\ n)$ *has-field-derivative* $(1 +$ *of-nat* $n) * (D * f\ x\ \hat{}\ n))$ $(at\ x\ within\ s)$
⟨*proof*⟩

lemma *DERIV-power*[*derivative-intros*]:
$(f$ *has-field-derivative* $D)$ $(at\ x\ within\ s) \Longrightarrow$
$((\lambda x.\ f\ x\ \hat{}\ n)$ *has-field-derivative* *of-nat* $n * (D * f\ x\ \hat{}\ (n - Suc\ 0)))$ $(at\ x\ within\ s)$
⟨*proof*⟩

lemma *DERIV-pow*: $((\lambda x.\ x\ \hat{}\ n)$ *has-field-derivative* *real* $n * (x\ \hat{}\ (n - Suc\ 0)))$ $(at\ x\ within\ s)$
⟨*proof*⟩

lemma *DERIV-power-int* [*derivative-intros*]:
assumes [*derivative-intros*]: $(f$ *has-field-derivative* $d)$ $(at\ x\ within\ s)$ **and** [*simp*]: $f\ x \neq 0$
shows $((\lambda x.\ power\text{-}int\ (f\ x)\ n)$ *has-field-derivative*
$(of\text{-}int\ n * power\text{-}int\ (f\ x)\ (n - 1) * d))$ $(at\ x\ within\ s)$
⟨*proof*⟩

lemma *DERIV-chain'*: $(f$ *has-field-derivative* $D)$ $(at\ x\ within\ s) \Longrightarrow DERIV\ g\ (f\ x) :> E \Longrightarrow$
$((\lambda x.\ g\ (f\ x))$ *has-field-derivative* $E * D)$ $(at\ x\ within\ s)$
⟨*proof*⟩

corollary *DERIV-chain2*: $DERIV\ f\ (g\ x) :> Da \Longrightarrow (g$ *has-field-derivative* $Db)$ $(at\ x\ within\ s) \Longrightarrow$
$((\lambda x.\ f\ (g\ x))$ *has-field-derivative* $Da * Db)$ $(at\ x\ within\ s)$
⟨*proof*⟩

Standard version

lemma *DERIV-chain*:
$DERIV\ f\ (g\ x) :> Da \Longrightarrow (g$ *has-field-derivative* $Db)$ $(at\ x\ within\ s) \Longrightarrow$
$(f \circ g$ *has-field-derivative* $Da * Db)$ $(at\ x\ within\ s)$

⟨*proof*⟩

lemma *DERIV-image-chain*:
(*f has-field-derivative Da*) (*at* (*g x*) *within* (*g ' s*)) ⟹
(*g has-field-derivative Db*) (*at x within s*) ⟹
(*f* ∘ *g has-field-derivative Da* ∗ *Db*) (*at x within s*)
⟨*proof*⟩

lemma *DERIV-chain-s*:
assumes (⋀*x*. *x* ∈ *s* ⟹ *DERIV g x* :> *g'*(*x*))
and *DERIV f x* :> *f'*
and *f x* ∈ *s*
shows *DERIV* (λ*x*. *g*(*f x*)) *x* :> *f'* ∗ *g'*(*f x*)
⟨*proof*⟩

lemma *DERIV-chain3*:
assumes (⋀*x*. *DERIV g x* :> *g'*(*x*))
and *DERIV f x* :> *f'*
shows *DERIV* (λ*x*. *g*(*f x*)) *x* :> *f'* ∗ *g'*(*f x*)
⟨*proof*⟩

Alternative definition for differentiability

lemma *DERIV-LIM-iff*:
fixes *f* :: *'a*::{*real-normed-vector*,*inverse*} ⇒ *'a*
shows ((λ*h*. (*f* (*a* + *h*) − *f a*) / *h*) −*0*→ *D*) = ((λ*x*. (*f x* − *f a*) / (*x* − *a*)) −*a*→ *D*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *has-field-derivative-cong-ev*:
assumes *x* = *y*
and ∗: *eventually* (λ*x*. *x* ∈ *S* ⟶ *f x* = *g x*) (*nhds x*)
and *u* = *v S* = *t x* ∈ *S*
shows (*f has-field-derivative u*) (*at x within S*) = (*g has-field-derivative v*) (*at y within t*)
⟨*proof*⟩

lemma *has-field-derivative-cong-eventually*:
assumes *eventually* (λ*x*. *f x* = *g x*) (*at x within S*) *f x* = *g x*
shows (*f has-field-derivative u*) (*at x within S*) = (*g has-field-derivative u*) (*at x within S*)
⟨*proof*⟩

lemma *DERIV-cong-ev*:
x = *y* ⟹ *eventually* (λ*x*. *f x* = *g x*) (*nhds x*) ⟹ *u* = *v* ⟹
DERIV f x :> *u* ⟷ *DERIV g y* :> *v*
⟨*proof*⟩

lemma *DERIV-mirror*: (*DERIV f* (− *x*) :> *y*) ⟷ (*DERIV* (λ*x*. *f* (− *x*)) *x* :>

$-$ y)
for f :: *real* $\Rightarrow$ *real* **and** x y :: *real*
⟨*proof*⟩

lemma *DERIV-shift*:
(f *has-field-derivative* y) (*at* (x + z)) = ((λx. f (x + z)) *has-field-derivative* y) (*at* x)
⟨*proof*⟩

lemma *DERIV-at-within-shift-lemma*:
assumes (f *has-field-derivative* y) (*at* (z+x) *within* (+) z ' S)
shows (f ∘ (+)z *has-field-derivative* y) (*at* x *within* S)
⟨*proof*⟩

lemma *DERIV-at-within-shift*:
(f *has-field-derivative* y) (*at* (z+x) *within* (+) z ' S) ⟷
((λx. f (z+x)) *has-field-derivative* y) (*at* x *within* S) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *floor-has-real-derivative*:
fixes f :: *real* ⇒ 'a::{*floor-ceiling*,*order-topology*}
assumes *isCont* f x
and f x ∉ ℤ
shows ((λx. *floor* (f x)) *has-real-derivative* 0) (*at* x)
⟨*proof*⟩

lemmas *has-derivative-floor*[*derivative-intros*] =
floor-has-real-derivative[*THEN DERIV-compose-FDERIV*]

lemma *continuous-floor*:
fixes x::*real*
shows x ∉ ℤ ⟹ *continuous* (*at* x) (*real-of-int* ∘ *floor*)
⟨*proof*⟩

lemma *continuous-frac*:
fixes x::*real*
assumes x ∉ ℤ
shows *continuous* (*at* x) *frac*
⟨*proof*⟩

Caratheodory formulation of derivative at a point

lemma *CARAT-DERIV*:
(*DERIV* f x :> l) ⟷ (∃ g. (∀ z. f z − f x = g z ∗ (z − x)) ∧ *isCont* g x ∧ g x = l)
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

111.8 Local extrema

If $(0::'a) < f'\ x$ then x is Locally Strictly Increasing At The Right.

lemma *has-real-derivative-pos-inc-right*:
fixes $f :: real \Rightarrow real$
assumes *der*: $(f$ *has-real-derivative* $l)$ $(at\ x\ within\ S)$
and l: $0 < l$
shows $\exists\, d > 0.\ \forall\, h > 0.\ x + h \in S \longrightarrow h < d \longrightarrow f\ x < f\ (x + h)$
⟨*proof*⟩

lemma *DERIV-pos-inc-right*:
fixes $f :: real \Rightarrow real$
assumes *der*: $DERIV\ f\ x :> l$
and l: $0 < l$
shows $\exists\, d > 0.\ \forall\, h > 0.\ h < d \longrightarrow f\ x < f\ (x + h)$
⟨*proof*⟩

lemma *has-real-derivative-neg-dec-left*:
fixes $f :: real \Rightarrow real$
assumes *der*: $(f$ *has-real-derivative* $l)$ $(at\ x\ within\ S)$
and $l < 0$
shows $\exists\, d > 0.\ \forall\, h > 0.\ x - h \in S \longrightarrow h < d \longrightarrow f\ x < f\ (x - h)$
⟨*proof*⟩

lemma *DERIV-neg-dec-left*:
fixes $f :: real \Rightarrow real$
assumes *der*: $DERIV\ f\ x :> l$
and l: $l < 0$
shows $\exists\, d > 0.\ \forall\, h > 0.\ h < d \longrightarrow f\ x < f\ (x - h)$
⟨*proof*⟩

lemma *has-real-derivative-pos-inc-left*:
fixes $f :: real \Rightarrow real$
shows $(f$ *has-real-derivative* $l)$ $(at\ x\ within\ S) \Longrightarrow 0 < l \Longrightarrow$
$\exists\, d{>}0.\ \forall\, h{>}0.\ x - h \in S \longrightarrow h < d \longrightarrow f\ (x - h) < f\ x$
⟨*proof*⟩

lemma *DERIV-pos-inc-left*:
fixes $f :: real \Rightarrow real$
shows $DERIV\ f\ x :> l \Longrightarrow 0 < l \Longrightarrow \exists\, d > 0.\ \forall\, h > 0.\ h < d \longrightarrow f\ (x - h)$
$< f\ x$
⟨*proof*⟩

lemma *has-real-derivative-neg-dec-right*:
fixes $f :: real \Rightarrow real$
shows $(f$ *has-real-derivative* $l)$ $(at\ x\ within\ S) \Longrightarrow l < 0 \Longrightarrow$
$\exists\, d > 0.\ \forall\, h > 0.\ x + h \in S \longrightarrow h < d \longrightarrow f\ x > f\ (x + h)$
⟨*proof*⟩

lemma *DERIV-neg-dec-right*:
fixes $f :: real \Rightarrow real$
shows $DERIV\ f\ x :> l \Longrightarrow l < 0 \Longrightarrow \exists\, d > 0.\ \forall\, h > 0.\ h < d \longrightarrow f\ x > f\ (x + h)$
⟨*proof*⟩

lemma *DERIV-local-max*:
fixes $f :: real \Rightarrow real$
assumes *der*: $DERIV\ f\ x :> l$
and *d*: $0 < d$
and *le*: $\forall\, y.\ |x - y| < d \longrightarrow f\ y \leq f\ x$
shows $l = 0$
⟨*proof*⟩

Similar theorem for a local minimum

lemma *DERIV-local-min*:
fixes $f :: real \Rightarrow real$
shows $DERIV\ f\ x :> l \Longrightarrow 0 < d \Longrightarrow \forall\, y.\ |x - y| < d \longrightarrow f\ x \leq f\ y \Longrightarrow l = 0$
⟨*proof*⟩

In particular, if a function is locally flat

lemma *DERIV-local-const*:
fixes $f :: real \Rightarrow real$
shows $DERIV\ f\ x :> l \Longrightarrow 0 < d \Longrightarrow \forall\, y.\ |x - y| < d \longrightarrow f\ x = f\ y \Longrightarrow l = 0$
⟨*proof*⟩

111.9 Rolle's Theorem

Lemma about introducing open ball in open interval

lemma *lemma-interval-lt*:
fixes $a\ b\ x :: real$
assumes $a < x\ x < b$
shows $\exists\, d.\ 0 < d \wedge (\forall\, y.\ |x - y| < d \longrightarrow a < y \wedge y < b)$
⟨*proof*⟩

lemma *lemma-interval*: $a < x \Longrightarrow x < b \Longrightarrow \exists\, d.\ 0 < d \wedge (\forall\, y.\ |x - y| < d \longrightarrow a \leq y \wedge y \leq b)$
for $a\ b\ x :: real$
⟨*proof*⟩

Rolle's Theorem. If f is defined and continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), and $f\ a = f\ b$, then there exists $x0 \in (a,b)$ such that $f'\ x0 = (0::'a)$

theorem *Rolle-deriv*:
fixes $f :: real \Rightarrow real$
assumes $a < b$
and *fab*: $f\ a = f\ b$
and *contf*: *continuous-on* $\{a..b\}\ f$

and *derf*: $\bigwedge x.\ [\![a < x;\ x < b]\!] \Longrightarrow (f\ \textit{has-derivative}\ f'\ x)\ (\textit{at}\ x)$
shows $\exists z.\ a < z \wedge z < b \wedge f'\ z = (\lambda v.\ 0)$
⟨*proof*⟩

corollary *Rolle*:
fixes $a\ b :: \textit{real}$
assumes *ab*: $a < b\ f\ a = f\ b\ \textit{continuous-on}\ \{a..b\}\ f$
and *dif* [*rule-format*]: $\bigwedge x.\ [\![a < x;\ x < b]\!] \Longrightarrow f\ \textit{differentiable}\ (\textit{at}\ x)$
shows $\exists z.\ a < z \wedge z < b \wedge \textit{DERIV}\ f\ z :> 0$
⟨*proof*⟩

111.10 Mean Value Theorem

theorem *mvt*:
fixes $f :: \textit{real} \Rightarrow \textit{real}$
assumes $a < b$
and *contf*: $\textit{continuous-on}\ \{a..b\}\ f$
and *derf*: $\bigwedge x.\ [\![a < x;\ x < b]\!] \Longrightarrow (f\ \textit{has-derivative}\ f'\ x)\ (\textit{at}\ x)$
obtains ξ **where** $a < \xi\ \xi < b\ f\ b - f\ a = (f'\ \xi)\ (b - a)$
⟨*proof*⟩

theorem *MVT*:
fixes $a\ b :: \textit{real}$
assumes *lt*: $a < b$
and *contf*: $\textit{continuous-on}\ \{a..b\}\ f$
and *dif*: $\bigwedge x.\ [\![a < x;\ x < b]\!] \Longrightarrow f\ \textit{differentiable}\ (\textit{at}\ x)$
shows $\exists l\ z.\ a < z \wedge z < b \wedge \textit{DERIV}\ f\ z :> l \wedge f\ b - f\ a = (b - a) * l$
⟨*proof*⟩

corollary *MVT2*:
assumes $a < b$ **and** *der*: $\bigwedge x.\ [\![a \leq x;\ x \leq b]\!] \Longrightarrow \textit{DERIV}\ f\ x :> f'\ x$
shows $\exists z{::}\textit{real}.\ a < z \wedge z < b \wedge (f\ b - f\ a = (b - a) * f'\ z)$
⟨*proof*⟩

lemma *pos-deriv-imp-strict-mono*:
assumes $\bigwedge x.\ (f\ \textit{has-real-derivative}\ f'\ x)\ (\textit{at}\ x)$
assumes $\bigwedge x.\ f'\ x > 0$
shows *strict-mono f*
⟨*proof*⟩

proposition *deriv-nonneg-imp-mono*:
assumes *deriv*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow (g\ \textit{has-real-derivative}\ g'\ x)\ (\textit{at}\ x)$
assumes *nonneg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow g'\ x \geq 0$
assumes *ab*: $a \leq b$
shows $g\ a \leq g\ b$
⟨*proof*⟩

111.10.1 A function is constant if its derivative is 0 over an interval.

lemma *DERIV-isconst-end*:
fixes $f :: real \Rightarrow real$
assumes $a < b$ **and** *contf*: *continuous-on* $\{a..b\}$ f
and *0*: $\bigwedge x.\ [\![a < x; x < b]\!] \Longrightarrow DERIV\ f\ x :> 0$
shows $f\ b = f\ a$
⟨*proof*⟩

lemma *DERIV-isconst2*:
fixes $f :: real \Rightarrow real$
assumes $a < b$ **and** *contf*: *continuous-on* $\{a..b\}$ f **and** *derf*: $\bigwedge x.\ [\![a < x; x < b]\!] \Longrightarrow DERIV\ f\ x :> 0$
and $a \leq x\ x \leq b$
shows $f\ x = f\ a$
⟨*proof*⟩

lemma *DERIV-isconst3*:
fixes $a\ b\ x\ y :: real$
assumes $a < b$
and $x \in \{a <..< b\}$
and $y \in \{a <..< b\}$
and *derivable*: $\bigwedge x.\ x \in \{a <..< b\} \Longrightarrow DERIV\ f\ x :> 0$
shows $f\ x = f\ y$
⟨*proof*⟩

lemma *DERIV-isconst-all*:
fixes $f :: real \Rightarrow real$
shows $\forall x.\ DERIV\ f\ x :> 0 \Longrightarrow f\ x = f\ y$
⟨*proof*⟩

lemma *DERIV-const-ratio-const*:
fixes $f :: real \Rightarrow real$
assumes $a \neq b$ **and** *df*: $\bigwedge x.\ DERIV\ f\ x :> k$
shows $f\ b - f\ a = (b - a) * k$
⟨*proof*⟩

lemma *DERIV-const-ratio-const2*:
fixes $f :: real \Rightarrow real$
assumes $a \neq b$ **and** *df*: $\bigwedge x.\ DERIV\ f\ x :> k$
shows $(f\ b - f\ a)\ /\ (b - a) = k$
⟨*proof*⟩

lemma *real-average-minus-first* [*simp*]: $(a + b)\ /\ 2 - a = (b - a)\ /\ 2$
for $a\ b :: real$
⟨*proof*⟩

lemma *real-average-minus-second* [*simp*]: $(b + a)\ /\ 2 - a = (b - a)\ /\ 2$
for $a\ b :: real$

⟨*proof*⟩

Gallileo's "trick": average velocity = av. of end velocities.

lemma *DERIV-const-average*:
fixes $v :: real \Rightarrow real$
and $a\ b :: real$
assumes *neq*: $a \neq b$
and *der*: $\bigwedge x.\ DERIV\ v\ x :> k$
shows $v\ ((a + b)\ /\ 2) = (v\ a + v\ b)\ /\ 2$
⟨*proof*⟩

111.10.2 A function with positive derivative is increasing

A simple proof using the MVT, by Jeremy Avigad. And variants.

lemma *DERIV-pos-imp-increasing-open*:
fixes $a\ b :: real$
and $f :: real \Rightarrow real$
assumes $a < b$
and $\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow (\exists y.\ DERIV\ f\ x :> y \wedge y > 0)$
and *con*: *continuous-on* $\{a..b\}\ f$
shows $f\ a < f\ b$
⟨*proof*⟩

lemma *DERIV-pos-imp-increasing*:
fixes $a\ b :: real$ **and** $f :: real \Rightarrow real$
assumes $a < b$
and *der*: $\bigwedge x.\ [\![a \leq x; x \leq b]\!] \Longrightarrow \exists y.\ DERIV\ f\ x :> y \wedge y > 0$
shows $f\ a < f\ b$
⟨*proof*⟩

lemma *DERIV-nonneg-imp-nondecreasing*:
fixes $a\ b :: real$
and $f :: real \Rightarrow real$
assumes $a \leq b$
and $\bigwedge x.\ [\![a \leq x; x \leq b]\!] \Longrightarrow \exists y.\ DERIV\ f\ x :> y \wedge y \geq 0$
shows $f\ a \leq f\ b$
⟨*proof*⟩

lemma *DERIV-neg-imp-decreasing-open*:
fixes $a\ b :: real$
and $f :: real \Rightarrow real$
assumes $a < b$
and $\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow \exists y.\ DERIV\ f\ x :> y \wedge y < 0$
and *con*: *continuous-on* $\{a..b\}\ f$
shows $f\ a > f\ b$
⟨*proof*⟩

lemma *DERIV-neg-imp-decreasing*:
fixes $a\ b :: real$ **and** $f :: real \Rightarrow real$

assumes $a < b$
and *der*: $\bigwedge x.\ [\![a \le x;\ x \le b]\!] \Longrightarrow \exists\, y.\ DERIV\ f\ x :> y \wedge y < 0$
shows $f\ a > f\ b$
⟨*proof*⟩

lemma *DERIV-nonpos-imp-nonincreasing*:
fixes $a\ b :: real$
and $f :: real \Rightarrow real$
assumes $a \le b$
and $\bigwedge x.\ [\![a \le x;\ x \le b]\!] \Longrightarrow \exists\, y.\ DERIV\ f\ x :> y \wedge y \le 0$
shows $f\ a \ge f\ b$
⟨*proof*⟩

lemma *DERIV-pos-imp-increasing-at-bot*:
fixes $f :: real \Rightarrow real$
assumes $\bigwedge x.\ x \le b \Longrightarrow (\exists\, y.\ DERIV\ f\ x :> y \wedge y > 0)$
and *lim*: $(f \longrightarrow flim)\ at\text{-}bot$
shows $flim < f\ b$
⟨*proof*⟩

lemma *DERIV-neg-imp-decreasing-at-top*:
fixes $f :: real \Rightarrow real$
assumes *der*: $\bigwedge x.\ x \ge b \Longrightarrow \exists\, y.\ DERIV\ f\ x :> y \wedge y < 0$
and *lim*: $(f \longrightarrow flim)\ at\text{-}top$
shows $flim < f\ b$
⟨*proof*⟩

Derivative of inverse function

lemma *DERIV-inverse-function*:
fixes $f\ g :: real \Rightarrow real$
assumes *der*: $DERIV\ f\ (g\ x) :> D$
and *neq*: $D \neq 0$
and *x*: $a < x\ x < b$
and *inj*: $\bigwedge y.\ [\![a < y;\ y < b]\!] \Longrightarrow f\ (g\ y) = y$
and *cont*: $isCont\ g\ x$
shows $DERIV\ g\ x :> inverse\ D$
⟨*proof*⟩

111.11 Generalized Mean Value Theorem

theorem *GMVT*:
fixes $a\ b :: real$
assumes *alb*: $a < b$
and *fc*: $\forall\, x.\ a \le x \wedge x \le b \longrightarrow isCont\ f\ x$
and *fd*: $\forall\, x.\ a < x \wedge x < b \longrightarrow f\ differentiable\ (at\ x)$
and *gc*: $\forall\, x.\ a \le x \wedge x \le b \longrightarrow isCont\ g\ x$
and *gd*: $\forall\, x.\ a < x \wedge x < b \longrightarrow g\ differentiable\ (at\ x)$
shows $\exists\, g'c\ f'c\ c.$
$DERIV\ g\ c :> g'c \wedge DERIV\ f\ c :> f'c \wedge a < c \wedge c < b \wedge (f\ b - f\ a) * g'c =$

$(g\ b - g\ a) * f'c$
⟨*proof*⟩

lemma *GMVT'*:
fixes $f\ g :: real \Rightarrow real$
assumes $a < b$
and *isCont-f*: $\bigwedge z.\ a \leq z \Longrightarrow z \leq b \Longrightarrow isCont\ f\ z$
and *isCont-g*: $\bigwedge z.\ a \leq z \Longrightarrow z \leq b \Longrightarrow isCont\ g\ z$
and *DERIV-g*: $\bigwedge z.\ a < z \Longrightarrow z < b \Longrightarrow DERIV\ g\ z :> (g'\ z)$
and *DERIV-f*: $\bigwedge z.\ a < z \Longrightarrow z < b \Longrightarrow DERIV\ f\ z :> (f'\ z)$
shows $\exists c.\ a < c \wedge c < b \wedge (f\ b - f\ a) * g'\ c = (g\ b - g\ a) * f'\ c$
⟨*proof*⟩

111.12 L'Hopitals rule

lemma *isCont-If-ge*:
fixes $a :: 'a :: linorder\text{-}topology$
assumes *continuous* (*at-left a*) *g* **and** *f*: $(f \longrightarrow g\ a)$ (*at-right a*)
shows *isCont* $(\lambda x.\ if\ x \leq a\ then\ g\ x\ else\ f\ x)\ a$ (**is** *isCont ?gf a*)
⟨*proof*⟩

lemma *lhopital-right-0*:
fixes *f0 g0* :: *real* ⇒ *real*
assumes *f-0*: (*f0* ⟶ *0*) (*at-right 0*)
and *g-0*: (*g0* ⟶ *0*) (*at-right 0*)
and *ev*:
eventually $(\lambda x.\ g0\ x \neq 0)$ (*at-right 0*)
eventually $(\lambda x.\ g'\ x \neq 0)$ (*at-right 0*)
eventually $(\lambda x.\ DERIV\ f0\ x :> f'\ x)$ (*at-right 0*)
eventually $(\lambda x.\ DERIV\ g0\ x :> g'\ x)$ (*at-right 0*)
and *lim*: *filterlim* $(\lambda\ x.\ (f'\ x\ /\ g'\ x))\ F$ (*at-right 0*)
shows *filterlim* $(\lambda\ x.\ f0\ x\ /\ g0\ x)\ F$ (*at-right 0*)
⟨*proof*⟩

lemma *lhopital-right*:
$(f \longrightarrow 0)$ (*at-right x*) $\Longrightarrow$ $(g \longrightarrow 0)$ (*at-right x*) $\Longrightarrow$
eventually $(\lambda x.\ g\ x \neq 0)$ (*at-right x*) $\Longrightarrow$
eventually $(\lambda x.\ g'\ x \neq 0)$ (*at-right x*) $\Longrightarrow$
eventually $(\lambda x.\ DERIV\ f\ x :> f'\ x)$ (*at-right x*) $\Longrightarrow$
eventually $(\lambda x.\ DERIV\ g\ x :> g'\ x)$ (*at-right x*) $\Longrightarrow$
filterlim $(\lambda\ x.\ (f'\ x\ /\ g'\ x))\ F$ (*at-right x*) $\Longrightarrow$
filterlim $(\lambda\ x.\ f\ x\ /\ g\ x)\ F$ (*at-right x*)
for $x :: real$
⟨*proof*⟩

lemma *lhopital-left*:
$(f \longrightarrow 0)$ (*at-left x*) $\Longrightarrow$ $(g \longrightarrow 0)$ (*at-left x*) $\Longrightarrow$
eventually $(\lambda x.\ g\ x \neq 0)$ (*at-left x*) $\Longrightarrow$
eventually $(\lambda x.\ g'\ x \neq 0)$ (*at-left x*) $\Longrightarrow$

eventually (λx. *DERIV f x :> f' x*) (*at-left x*) ⟹
eventually (λx. *DERIV g x :> g' x*) (*at-left x*) ⟹
filterlim (λ *x*. (*f' x* / *g' x*)) *F* (*at-left x*) ⟹
filterlim (λ *x*. *f x* / *g x*) *F* (*at-left x*)
for *x* :: *real*
⟨*proof*⟩

lemma *lhopital*:
(*f* ⟶ *0*) (*at x*) ⟹ (*g* ⟶ *0*) (*at x*) ⟹
eventually (λx. *g x* ≠ *0*) (*at x*) ⟹
eventually (λx. *g' x* ≠ *0*) (*at x*) ⟹
eventually (λx. *DERIV f x :> f' x*) (*at x*) ⟹
eventually (λx. *DERIV g x :> g' x*) (*at x*) ⟹
filterlim (λ *x*. (*f' x* / *g' x*)) *F* (*at x*) ⟹
filterlim (λ *x*. *f x* / *g x*) *F* (*at x*)
for *x* :: *real*
⟨*proof*⟩

lemma *lhopital-right-0-at-top*:
fixes *f g* :: *real* ⇒ *real*
assumes *g-0*: *LIM x at-right 0*. *g x :> at-top*
and *ev*:
eventually (λx. *g' x* ≠ *0*) (*at-right 0*)
eventually (λx. *DERIV f x :> f' x*) (*at-right 0*)
eventually (λx. *DERIV g x :> g' x*) (*at-right 0*)
and *lim*: ((λ *x*. (*f' x* / *g' x*)) ⟶ *x*) (*at-right 0*)
shows ((λ *x*. *f x* / *g x*) ⟶ *x*) (*at-right 0*)
⟨*proof*⟩

lemma *lhopital-right-at-top*:
LIM x at-right x. (*g*::*real* ⇒ *real*) *x :> at-top* ⟹
eventually (λx. *g' x* ≠ *0*) (*at-right x*) ⟹
eventually (λx. *DERIV f x :> f' x*) (*at-right x*) ⟹
eventually (λx. *DERIV g x :> g' x*) (*at-right x*) ⟹
((λ *x*. (*f' x* / *g' x*)) ⟶ *y*) (*at-right x*) ⟹
((λ *x*. *f x* / *g x*) ⟶ *y*) (*at-right x*)
⟨*proof*⟩

lemma *lhopital-left-at-top*:
LIM x at-left x. *g x :> at-top* ⟹
eventually (λx. *g' x* ≠ *0*) (*at-left x*) ⟹
eventually (λx. *DERIV f x :> f' x*) (*at-left x*) ⟹
eventually (λx. *DERIV g x :> g' x*) (*at-left x*) ⟹
((λ *x*. (*f' x* / *g' x*)) ⟶ *y*) (*at-left x*) ⟹
((λ *x*. *f x* / *g x*) ⟶ *y*) (*at-left x*)
for *x* :: *real*
⟨*proof*⟩

lemma *lhopital-at-top*:
LIM x at x. (g::real ⇒ real) x :> at-top ⟹
eventually (λx. g' x ≠ 0) (at x) ⟹
eventually (λx. DERIV f x :> f' x) (at x) ⟹
eventually (λx. DERIV g x :> g' x) (at x) ⟹
((λ x. (f' x / g' x)) ⟶ y) (at x) ⟹
((λ x. f x / g x) ⟶ y) (at x)
⟨*proof*⟩

lemma *lhospital-at-top-at-top*:
fixes *f g* :: *real ⇒ real*
assumes *g-0*: *LIM x at-top. g x :> at-top*
and *g'*: *eventually (λx. g' x ≠ 0) at-top*
and *Df*: *eventually (λx. DERIV f x :> f' x) at-top*
and *Dg*: *eventually (λx. DERIV g x :> g' x) at-top*
and *lim*: *((λ x. (f' x / g' x)) ⟶ x) at-top*
shows *((λ x. f x / g x) ⟶ x) at-top*
⟨*proof*⟩

lemma *lhopital-right-at-top-at-top*:
fixes *f g* :: *real ⇒ real*
assumes *f-0*: *LIM x at-right a. f x :> at-top*
assumes *g-0*: *LIM x at-right a. g x :> at-top*
and *ev*:
eventually (λx. DERIV f x :> f' x) (at-right a)
eventually (λx. DERIV g x :> g' x) (at-right a)
and *lim*: *filterlim (λ x. (f' x / g' x)) at-top (at-right a)*
shows *filterlim (λ x. f x / g x) at-top (at-right a)*
⟨*proof*⟩

lemma *lhopital-right-at-top-at-bot*:
fixes *f g* :: *real ⇒ real*
assumes *f-0*: *LIM x at-right a. f x :> at-top*
assumes *g-0*: *LIM x at-right a. g x :> at-bot*
and *ev*:
eventually (λx. DERIV f x :> f' x) (at-right a)
eventually (λx. DERIV g x :> g' x) (at-right a)
and *lim*: *filterlim (λ x. (f' x / g' x)) at-bot (at-right a)*
shows *filterlim (λ x. f x / g x) at-bot (at-right a)*
⟨*proof*⟩

lemma *lhopital-left-at-top-at-top*:
fixes *f g* :: *real ⇒ real*
assumes *f-0*: *LIM x at-left a. f x :> at-top*
assumes *g-0*: *LIM x at-left a. g x :> at-top*
and *ev*:
eventually (λx. DERIV f x :> f' x) (at-left a)
eventually (λx. DERIV g x :> g' x) (at-left a)
and *lim*: *filterlim (λ x. (f' x / g' x)) at-top (at-left a)*

shows *filterlim* (λ *x*. *f x* / *g x*) *at-top* (*at-left a*)
⟨*proof*⟩

lemma *lhopital-left-at-top-at-bot*:
fixes *f g* :: *real* ⇒ *real*
assumes *f-0*: *LIM x at-left a*. *f x* :> *at-top*
assumes *g-0*: *LIM x at-left a*. *g x* :> *at-bot*
and *ev*:
eventually (λ*x*. *DERIV f x* :> *f′ x*) (*at-left a*)
eventually (λ*x*. *DERIV g x* :> *g′ x*) (*at-left a*)
and *lim*: *filterlim* (λ *x*. (*f′ x* / *g′ x*)) *at-bot* (*at-left a*)
shows *filterlim* (λ *x*. *f x* / *g x*) *at-bot* (*at-left a*)
⟨*proof*⟩

lemma *lhopital-at-top-at-top*:
fixes *f g* :: *real* ⇒ *real*
assumes *f-0*: *LIM x at a*. *f x* :> *at-top*
assumes *g-0*: *LIM x at a*. *g x* :> *at-top*
and *ev*:
eventually (λ*x*. *DERIV f x* :> *f′ x*) (*at a*)
eventually (λ*x*. *DERIV g x* :> *g′ x*) (*at a*)
and *lim*: *filterlim* (λ *x*. (*f′ x* / *g′ x*)) *at-top* (*at a*)
shows *filterlim* (λ *x*. *f x* / *g x*) *at-top* (*at a*)
⟨*proof*⟩

lemma *lhopital-at-top-at-bot*:
fixes *f g* :: *real* ⇒ *real*
assumes *f-0*: *LIM x at a*. *f x* :> *at-top*
assumes *g-0*: *LIM x at a*. *g x* :> *at-bot*
and *ev*:
eventually (λ*x*. *DERIV f x* :> *f′ x*) (*at a*)
eventually (λ*x*. *DERIV g x* :> *g′ x*) (*at a*)
and *lim*: *filterlim* (λ *x*. (*f′ x* / *g′ x*)) *at-bot* (*at a*)
shows *filterlim* (λ *x*. *f x* / *g x*) *at-bot* (*at a*)
⟨*proof*⟩

end

112 Nth Roots of Real Numbers

theory *NthRoot*
imports *Deriv*
begin

112.1 Existence of Nth Root

Existence follows from the Intermediate Value Theorem

lemma *realpow-pos-nth*:

fixes $a :: real$
assumes n: $0 < n$
and a: $0 < a$
shows $\exists r{>}0.\ r \mathbin{\hat{}} n = a$
⟨*proof*⟩

lemma *realpow-pos-nth2*: $(0::real) < a \Longrightarrow \exists r{>}0.\ r \mathbin{\hat{}} Suc\ n = a$
⟨*proof*⟩

Uniqueness of nth positive root.

lemma *realpow-pos-nth-unique*: $0 < n \Longrightarrow 0 < a \Longrightarrow \exists!r.\ 0 < r \wedge r \mathbin{\hat{}} n = a$ **for** $a :: real$
⟨*proof*⟩

112.2 Nth Root

We define roots of negative reals such that $root\ n\ (-x) = -\ root\ n\ x$. This allows us to omit side conditions from many theorems.

lemma *inj-sgn-power*:
assumes $0 < n$
shows *inj* $(\lambda y.\ sgn\ y * |y| \mathbin{\hat{}} n :: real)$
(**is** *inj ?f*)
⟨*proof*⟩

lemma *sgn-power-injE*:
$sgn\ a * |a| \mathbin{\hat{}} n = x \Longrightarrow x = sgn\ b * |b| \mathbin{\hat{}} n \Longrightarrow 0 < n \Longrightarrow a = b$
for $a\ b :: real$
⟨*proof*⟩

definition *root* :: $nat \Rightarrow real \Rightarrow real$
where $root\ n\ x = (if\ n = 0\ then\ 0\ else\ the\text{-}inv\ (\lambda y.\ sgn\ y * |y| \mathbin{\hat{}} n)\ x)$

lemma *root-0* [*simp*]: $root\ 0\ x = 0$
⟨*proof*⟩

lemma *root-sgn-power*: $0 < n \Longrightarrow root\ n\ (sgn\ y * |y| \mathbin{\hat{}} n) = y$
⟨*proof*⟩

lemma *sgn-power-root*:
assumes $0 < n$
shows $sgn\ (root\ n\ x) * |(root\ n\ x)| \mathbin{\hat{}} n = x$
(**is** *?f* $(root\ n\ x) = x$)
⟨*proof*⟩

lemma *split-root*: $P\ (root\ n\ x) \longleftrightarrow (n = 0 \longrightarrow P\ 0) \wedge (0 < n \longrightarrow (\forall y.\ sgn\ y * |y| \mathbin{\hat{}} n = x \longrightarrow P\ y))$
⟨*proof*⟩

lemma *real-root-zero* [*simp*]: *root n 0 = 0*
⟨*proof*⟩

lemma *real-root-minus*: *root n (− x) = − root n x*
⟨*proof*⟩

lemma *real-root-less-mono*: *0 < n ⟹ x < y ⟹ root n x < root n y*
⟨*proof*⟩

lemma *real-root-gt-zero*: *0 < n ⟹ 0 < x ⟹ 0 < root n x*
⟨*proof*⟩

lemma *real-root-ge-zero*: *0 ≤ x ⟹ 0 ≤ root n x*
⟨*proof*⟩

lemma *real-root-pow-pos*: *0 < n ⟹ 0 < x ⟹ root n x ^ n = x*
⟨*proof*⟩

lemma *real-root-pow-pos2* [*simp*]: *0 < n ⟹ 0 ≤ x ⟹ root n x ^ n = x*
⟨*proof*⟩

lemma *sgn-root*: *0 < n ⟹ sgn (root n x) = sgn x*
⟨*proof*⟩

lemma *odd-real-root-pow*: *odd n ⟹ root n x ^ n = x*
⟨*proof*⟩

lemma *real-root-power-cancel*: *0 < n ⟹ 0 ≤ x ⟹ root n (x ^ n) = x*
⟨*proof*⟩

lemma *odd-real-root-power-cancel*: *odd n ⟹ root n (x ^ n) = x*
⟨*proof*⟩

lemma *real-root-pos-unique*: *0 < n ⟹ 0 ≤ y ⟹ y ^ n = x ⟹ root n x = y*
⟨*proof*⟩

lemma *odd-real-root-unique*: *odd n ⟹ y ^ n = x ⟹ root n x = y*
⟨*proof*⟩

lemma *real-root-one* [*simp*]: *0 < n ⟹ root n 1 = 1*
⟨*proof*⟩

Root function is strictly monotonic, hence injective.

lemma *real-root-le-mono*: *0 < n ⟹ x ≤ y ⟹ root n x ≤ root n y*
⟨*proof*⟩

lemma *real-root-less-iff* [*simp*]: *0 < n ⟹ root n x < root n y ⟷ x < y*
⟨*proof*⟩

lemma *real-root-le-iff* [*simp*]: $0 < n \Longrightarrow root\ n\ x \leq root\ n\ y \longleftrightarrow x \leq y$
⟨*proof*⟩

lemma *real-root-eq-iff* [*simp*]: $0 < n \Longrightarrow root\ n\ x = root\ n\ y \longleftrightarrow x = y$
⟨*proof*⟩

lemmas *real-root-gt-0-iff* [*simp*] = *real-root-less-iff* [**where** *x=0*, *simplified*]
lemmas *real-root-lt-0-iff* [*simp*] = *real-root-less-iff* [**where** *y=0*, *simplified*]
lemmas *real-root-ge-0-iff* [*simp*] = *real-root-le-iff* [**where** *x=0*, *simplified*]
lemmas *real-root-le-0-iff* [*simp*] = *real-root-le-iff* [**where** *y=0*, *simplified*]
lemmas *real-root-eq-0-iff* [*simp*] = *real-root-eq-iff* [**where** *y=0*, *simplified*]

lemma *real-root-gt-1-iff* [*simp*]: $0 < n \Longrightarrow 1 < root\ n\ y \longleftrightarrow 1 < y$
⟨*proof*⟩

lemma *real-root-lt-1-iff* [*simp*]: $0 < n \Longrightarrow root\ n\ x < 1 \longleftrightarrow x < 1$
⟨*proof*⟩

lemma *real-root-ge-1-iff* [*simp*]: $0 < n \Longrightarrow 1 \leq root\ n\ y \longleftrightarrow 1 \leq y$
⟨*proof*⟩

lemma *real-root-le-1-iff* [*simp*]: $0 < n \Longrightarrow root\ n\ x \leq 1 \longleftrightarrow x \leq 1$
⟨*proof*⟩

lemma *real-root-eq-1-iff* [*simp*]: $0 < n \Longrightarrow root\ n\ x = 1 \longleftrightarrow x = 1$
⟨*proof*⟩

Roots of multiplication and division.

lemma *real-root-mult*: $root\ n\ (x * y) = root\ n\ x * root\ n\ y$
⟨*proof*⟩

lemma *real-root-inverse*: $root\ n\ (inverse\ x) = inverse\ (root\ n\ x)$
⟨*proof*⟩

lemma *real-root-divide*: $root\ n\ (x\ /\ y) = root\ n\ x\ /\ root\ n\ y$
⟨*proof*⟩

lemma *real-root-abs*: $0 < n \Longrightarrow root\ n\ |x| = |root\ n\ x|$
⟨*proof*⟩

lemma *root-abs-power*: $n > 0 \Longrightarrow abs\ (root\ n\ (y\ \hat{}\ n)) = abs\ y$
⟨*proof*⟩

lemma *real-root-power*: $0 < n \Longrightarrow root\ n\ (x\ \hat{}\ k) = root\ n\ x\ \hat{}\ k$
⟨*proof*⟩

Roots of roots.

lemma *real-root-Suc-0* [*simp*]: $root\ (Suc\ 0)\ x = x$

⟨*proof*⟩

lemma *real-root-mult-exp*: *root* (*m* ∗ *n*) *x* = *root m* (*root n x*)
⟨*proof*⟩

lemma *real-root-commute*: *root m* (*root n x*) = *root n* (*root m x*)
⟨*proof*⟩

Monotonicity in first argument.

lemma *real-root-strict-decreasing*:
assumes *0* < *n n* < *N 1* < *x*
shows *root N x* < *root n x*
⟨*proof*⟩

lemma *real-root-strict-increasing*:
assumes *0* < *n n* < *N 0* < *x x* < *1*
shows *root n x* < *root N x*
⟨*proof*⟩

lemma *real-root-decreasing*: *0* < *n* ⟹ *n* ≤ *N* ⟹ *1* ≤ *x* ⟹ *root N x* ≤ *root n x*
⟨*proof*⟩

lemma *real-root-increasing*: *0* < *n* ⟹ *n* ≤ *N* ⟹ *0* ≤ *x* ⟹ *x* ≤ *1* ⟹ *root n x* ≤ *root N x*
⟨*proof*⟩

Continuity and derivatives.

lemma *isCont-real-root*: *isCont* (*root n*) *x*
⟨*proof*⟩

lemma *tendsto-real-root* [*tendsto-intros*]:
(*f* ⟶ *x*) *F* ⟹ ((λ*x*. *root n* (*f x*)) ⟶ *root n x*) *F*
⟨*proof*⟩

lemma *continuous-real-root* [*continuous-intros*]:
continuous F f ⟹ *continuous F* (λ*x*. *root n* (*f x*))
⟨*proof*⟩

lemma *continuous-on-real-root* [*continuous-intros*]:
continuous-on s f ⟹ *continuous-on s* (λ*x*. *root n* (*f x*))
⟨*proof*⟩

lemma *DERIV-real-root*:
assumes *n*: *0* < *n*
and *x*: *0* < *x*
shows *DERIV* (*root n*) *x* :> *inverse* (*real n* ∗ *root n x* ^ (*n* − *Suc 0*))
⟨*proof*⟩

lemma *DERIV-odd-real-root*:

assumes n: *odd n*
and x: $x \neq 0$
shows *DERIV* (*root n*) x :> *inverse* (*real n* $*$ *root n x* ^ (*n* − *Suc 0*))
⟨*proof*⟩

lemma *DERIV-even-real-root*:
assumes n: $0 < n$
and *even n*
and x: $x < 0$
shows *DERIV* (*root n*) x :> *inverse* (− *real n* $*$ *root n x* ^ (*n* − *Suc 0*))
⟨*proof*⟩

lemma *DERIV-real-root-generic*:
assumes $0 < n$
and $x \neq 0$
and *even n* ⟹ $0 < x$ ⟹ D = *inverse* (*real n* $*$ *root n x* ^ (*n* − *Suc 0*))
and *even n* ⟹ $x < 0$ ⟹ $D = -$ *inverse* (*real n* $*$ *root n x* ^ (*n* − *Suc 0*))
and *odd n* ⟹ D = *inverse* (*real n* $*$ *root n x* ^ (*n* − *Suc 0*))
shows *DERIV* (*root n*) x :> D
⟨*proof*⟩

lemma *power-tendsto-0-iff* [*simp*]:
fixes f :: $'a \Rightarrow$ *real*
assumes $n > 0$
shows ((λx. $f\ x$ ^ n) ⟶ 0) *F* ⟷ (f ⟶ 0) *F*
⟨*proof*⟩

112.3 Square Root

definition *sqrt* :: *real* ⇒ *real*
where *sqrt* = *root 2*

lemma *pos2*: $0 <$ (2::*nat*)
⟨*proof*⟩

lemma *real-sqrt-unique*: $y^2 = x \Longrightarrow 0 \leq y \Longrightarrow$ *sqrt* $x = y$
⟨*proof*⟩

lemma *real-sqrt-abs* [*simp*]: *sqrt* $(x^2) = |x|$
⟨*proof*⟩

lemma *real-sqrt-pow2* [*simp*]: $0 \leq x \Longrightarrow (\mathit{sqrt}\ x)^2 = x$
⟨*proof*⟩

lemma *real-sqrt-pow2-iff* [*simp*]: $(\mathit{sqrt}\ x)^2 = x \longleftrightarrow 0 \leq x$
⟨*proof*⟩

lemma *real-sqrt-zero* [*simp*]: *sqrt* 0 = 0
⟨*proof*⟩

lemma *real-sqrt-one* [*simp*]: $sqrt\ 1 = 1$
⟨*proof*⟩

lemma *real-sqrt-four* [*simp*]: $sqrt\ 4 = 2$
⟨*proof*⟩

lemma *real-sqrt-minus*: $sqrt\ (-x) = -\ sqrt\ x$
⟨*proof*⟩

lemma *real-sqrt-mult*: $sqrt\ (x * y) = sqrt\ x * sqrt\ y$
⟨*proof*⟩

lemma *real-sqrt-mult-self*[*simp*]: $sqrt\ a * sqrt\ a = |a|$
⟨*proof*⟩

lemma *real-sqrt-inverse*: $sqrt\ (inverse\ x) = inverse\ (sqrt\ x)$
⟨*proof*⟩

lemma *real-sqrt-divide*: $sqrt\ (x\ /\ y) = sqrt\ x\ /\ sqrt\ y$
⟨*proof*⟩

lemma *real-sqrt-power*: $sqrt\ (x\ \hat{}\ k) = sqrt\ x\ \hat{}\ k$
⟨*proof*⟩

lemma *real-sqrt-gt-zero*: $0 < x \Longrightarrow 0 < sqrt\ x$
⟨*proof*⟩

lemma *real-sqrt-ge-zero*: $0 \leq x \Longrightarrow 0 \leq sqrt\ x$
⟨*proof*⟩

lemma *real-sqrt-less-mono*: $x < y \Longrightarrow sqrt\ x < sqrt\ y$
⟨*proof*⟩

lemma *real-sqrt-le-mono*: $x \leq y \Longrightarrow sqrt\ x \leq sqrt\ y$
⟨*proof*⟩

lemma *real-sqrt-less-iff* [*simp*]: $sqrt\ x < sqrt\ y \longleftrightarrow x < y$
⟨*proof*⟩

lemma *real-sqrt-le-iff* [*simp*]: $sqrt\ x \leq sqrt\ y \longleftrightarrow x \leq y$
⟨*proof*⟩

lemma *real-sqrt-eq-iff* [*simp*]: $sqrt\ x = sqrt\ y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *real-less-lsqrt*: $0 \leq y \Longrightarrow x < y^2 \Longrightarrow sqrt\ x < y$
⟨*proof*⟩

lemma *real-le-lsqrt*: $0 \leq y \Longrightarrow x \leq y^2 \Longrightarrow sqrt\ x \leq y$
⟨*proof*⟩

lemma *real-le-rsqrt*: $x^2 \leq y \Longrightarrow x \leq sqrt\ y$
⟨*proof*⟩

lemma *real-less-rsqrt*: $x^2 < y \Longrightarrow x < sqrt\ y$
⟨*proof*⟩

lemma *real-sqrt-power-even*:
assumes *even n* $x \geq 0$
shows $sqrt\ x \mathbin{\hat{}} n = x \mathbin{\hat{}} (n\ div\ 2)$
⟨*proof*⟩

lemma *sqrt-le-D*: $sqrt\ x \leq y \Longrightarrow x \leq y^2$
⟨*proof*⟩

lemma *sqrt-ge-absD*: $|x| \leq sqrt\ y \Longrightarrow x^2 \leq y$
⟨*proof*⟩

lemma *sqrt-even-pow2*:
assumes *n*: *even n*
shows $sqrt\ (2 \mathbin{\hat{}} n) = 2 \mathbin{\hat{}} (n\ div\ 2)$
⟨*proof*⟩

lemmas *real-sqrt-gt-0-iff* [*simp*] = *real-sqrt-less-iff* [**where** *x=0*, *unfolded real-sqrt-zero*]
lemmas *real-sqrt-lt-0-iff* [*simp*] = *real-sqrt-less-iff* [**where** *y=0*, *unfolded real-sqrt-zero*]
lemmas *real-sqrt-ge-0-iff* [*simp*] = *real-sqrt-le-iff* [**where** *x=0*, *unfolded real-sqrt-zero*]
lemmas *real-sqrt-le-0-iff* [*simp*] = *real-sqrt-le-iff* [**where** *y=0*, *unfolded real-sqrt-zero*]
lemmas *real-sqrt-eq-0-iff* [*simp*] = *real-sqrt-eq-iff* [**where** *y=0*, *unfolded real-sqrt-zero*]

lemmas *real-sqrt-gt-1-iff* [*simp*] = *real-sqrt-less-iff* [**where** *x=1*, *unfolded real-sqrt-one*]
lemmas *real-sqrt-lt-1-iff* [*simp*] = *real-sqrt-less-iff* [**where** *y=1*, *unfolded real-sqrt-one*]
lemmas *real-sqrt-ge-1-iff* [*simp*] = *real-sqrt-le-iff* [**where** *x=1*, *unfolded real-sqrt-one*]
lemmas *real-sqrt-le-1-iff* [*simp*] = *real-sqrt-le-iff* [**where** *y=1*, *unfolded real-sqrt-one*]
lemmas *real-sqrt-eq-1-iff* [*simp*] = *real-sqrt-eq-iff* [**where** *y=1*, *unfolded real-sqrt-one*]

lemma *sqrt-add-le-add-sqrt*:
assumes $0 \leq x\ 0 \leq y$
shows $sqrt\ (x + y) \leq sqrt\ x + sqrt\ y$
⟨*proof*⟩

lemma *isCont-real-sqrt*: *isCont sqrt x*
⟨*proof*⟩

lemma *tendsto-real-sqrt* [*tendsto-intros*]:
$(f \longrightarrow x)\ F \Longrightarrow ((\lambda x.\ sqrt\ (f\ x)) \longrightarrow sqrt\ x)\ F$
⟨*proof*⟩

lemma *continuous-real-sqrt* [*continuous-intros*]:
continuous F f $\Longrightarrow$ *continuous F* ($\lambda x.$ *sqrt* (*f x*))
⟨*proof*⟩

lemma *continuous-on-real-sqrt* [*continuous-intros*]:
continuous-on s f $\Longrightarrow$ *continuous-on s* ($\lambda x.$ *sqrt* (*f x*))
⟨*proof*⟩

lemma *DERIV-real-sqrt-generic*:
assumes $x \neq 0$
and $x > 0 \Longrightarrow D =$ *inverse* (*sqrt x*) / 2
and $x < 0 \Longrightarrow D = -$ *inverse* (*sqrt x*) / 2
shows *DERIV sqrt x* :> *D*
⟨*proof*⟩

lemma *DERIV-real-sqrt*: $0 < x \Longrightarrow$ *DERIV sqrt x* :> *inverse* (*sqrt x*) / 2
⟨*proof*⟩

declare
DERIV-real-sqrt-generic[*THEN DERIV-chain2*, *derivative-intros*]
DERIV-real-root-generic[*THEN DERIV-chain2*, *derivative-intros*]

lemmas *has-derivative-real-sqrt*[*derivative-intros*] = *DERIV-real-sqrt*[*THEN DERIV-compose-FDERIV*]

lemma *not-real-square-gt-zero* [*simp*]: $\neg\ 0 < x * x \longleftrightarrow x = 0$
for *x* :: *real*
⟨*proof*⟩

lemma *real-sqrt-abs2* [*simp*]: *sqrt* ($x * x$) = $|x|$
⟨*proof*⟩

lemma *real-inv-sqrt-pow2*: $0 < x \Longrightarrow$ (*inverse* (*sqrt x*))^2 = *inverse x*
⟨*proof*⟩

lemma *real-sqrt-eq-zero-cancel*: $0 \leq x \Longrightarrow$ *sqrt x* = $0 \Longrightarrow x = 0$
⟨*proof*⟩

lemma *real-sqrt-ge-one*: $1 \leq x \Longrightarrow 1 \leq$ *sqrt x*
⟨*proof*⟩

lemma *sqrt-divide-self-eq*:
assumes *nneg*: $0 \leq x$
shows *sqrt x* / *x* = *inverse* (*sqrt x*)
⟨*proof*⟩

lemma *real-div-sqrt*: $0 \leq x \Longrightarrow$ *x* / *sqrt x* = *sqrt x*
⟨*proof*⟩

lemma *real-divide-square-eq* [*simp*]: $(r * a) / (r * r) = a / r$
for $a\ r$ :: *real*
⟨*proof*⟩

lemma *lemma-real-divide-sqrt-less*: $0 < u \Longrightarrow u / sqrt\ 2 < u$
⟨*proof*⟩

lemma *four-x-squared*: $4 * x^2 = (2 * x)^2$
for x :: *real*
⟨*proof*⟩

lemma *sqrt-at-top*: *LIM x at-top. sqrt x* :: *real* :> *at-top*
⟨*proof*⟩

112.4 Square Root of Sum of Squares

lemma *sum-squares-bound*: $2 * x * y \leq x^2 + y^2$
for $x\ y$ :: *'a*::*linordered-field*
⟨*proof*⟩

lemma *arith-geo-mean*:
fixes u :: *'a*::*linordered-field*
assumes $u^2 = x * y\ x \geq 0\ y \geq 0$
shows $u \leq (x + y)/2$
⟨*proof*⟩

lemma *arith-geo-mean-sqrt*:
fixes x :: *real*
assumes $x \geq 0\ y \geq 0$
shows $sqrt\ (x * y) \leq (x + y)/2$
⟨*proof*⟩

lemma *real-sqrt-sum-squares-mult-ge-zero* [*simp*]: $0 \leq sqrt\ ((x^2 + y^2) * (xa^2 + ya^2))$
⟨*proof*⟩

lemma *real-sqrt-sum-squares-mult-squared-eq* [*simp*]:
$(sqrt\ ((x^2 + y^2) * (xa^2 + ya^2)))^2 = (x^2 + y^2) * (xa^2 + ya^2)$
⟨*proof*⟩

lemma *real-sqrt-sum-squares-eq-cancel*: $sqrt\ (x^2 + y^2) = x \Longrightarrow y = 0$
⟨*proof*⟩

lemma *real-sqrt-sum-squares-eq-cancel2*: $sqrt\ (x^2 + y^2) = y \Longrightarrow x = 0$
⟨*proof*⟩

lemma *real-sqrt-sum-squares-ge1* [*simp*]: $x \leq sqrt\ (x^2 + y^2)$
⟨*proof*⟩

lemma *real-sqrt-sum-squares-ge2* [*simp*]: $y \leq sqrt\ (x^2 + y^2)$
⟨*proof*⟩

lemma *real-sqrt-ge-abs1* [*simp*]: $|x| \leq sqrt\ (x^2 + y^2)$
⟨*proof*⟩

lemma *real-sqrt-ge-abs2* [*simp*]: $|y| \leq sqrt\ (x^2 + y^2)$
⟨*proof*⟩

lemma *le-real-sqrt-sumsq* [*simp*]: $x \leq sqrt\ (x * x + y * y)$
⟨*proof*⟩

lemma *sqrt-sum-squares-le-sum*:
$[\![0 \leq x;\ 0 \leq y]\!] \Longrightarrow sqrt\ (x^2 + y^2) \leq x + y$
⟨*proof*⟩

lemma *L2-set-mult-ineq-lemma*:
fixes *a b c d* :: *real*
shows $2 * (a * c) * (b * d) \leq a^2 * d^2 + b^2 * c^2$
⟨*proof*⟩

lemma *sqrt-sum-squares-le-sum-abs*: $sqrt\ (x^2 + y^2) \leq |x| + |y|$
⟨*proof*⟩

lemma *real-sqrt-sum-squares-triangle-ineq*:
$sqrt\ ((a + c)^2 + (b + d)^2) \leq sqrt\ (a^2 + b^2) + sqrt\ (c^2 + d^2)$
⟨*proof*⟩

lemma *real-sqrt-sum-squares-less*: $|x| < u\ /\ sqrt\ 2 \Longrightarrow |y| < u\ /\ sqrt\ 2 \Longrightarrow sqrt\ (x^2 + y^2) < u$
⟨*proof*⟩

lemma *sqrt2-less-2*: $sqrt\ 2 < (2::real)$
⟨*proof*⟩

lemma *sqrt-sum-squares-half-less*:
$x < u/2 \Longrightarrow y < u/2 \Longrightarrow 0 \leq x \Longrightarrow 0 \leq y \Longrightarrow sqrt\ (x^2 + y^2) < u$
⟨*proof*⟩

lemma *LIMSEQ-root*: $(\lambda n.\ root\ n\ n) \longrightarrow 1$
⟨*proof*⟩

lemma *LIMSEQ-root-const*:
assumes $0 < c$
shows $(\lambda n.\ root\ n\ c) \longrightarrow 1$
⟨*proof*⟩

Legacy theorem names:

lemmas *real-root-pos2* = *real-root-power-cancel*

lemmas *real-root-pos-pos* = *real-root-gt-zero* [*THEN order-less-imp-le*]
lemmas *real-root-pos-pos-le* = *real-root-ge-zero*
lemmas *real-sqrt-eq-zero-cancel-iff* = *real-sqrt-eq-0-iff*

end

113 Power Series, Transcendental Functions etc.

theory *Transcendental*
imports *Series Deriv NthRoot*
begin

A theorem about the factcorial function on the reals.

lemma *square-fact-le-2-fact*: *fact n* ∗ *fact n* ≤ (*fact* (*2* ∗ *n*) :: *real*)
⟨*proof*⟩

lemma *fact-in-Reals*: *fact n* ∈ ℝ
⟨*proof*⟩

lemma *of-real-fact* [*simp*]: *of-real* (*fact n*) = *fact n*
⟨*proof*⟩

lemma *pochhammer-of-real*: *pochhammer* (*of-real x*) *n* = *of-real* (*pochhammer x n*)
⟨*proof*⟩

lemma *norm-fact* [*simp*]: *norm* (*fact n* :: *'a*::*real-normed-algebra-1*) = *fact n*
⟨*proof*⟩

lemma *root-test-convergence*:
fixes *f* :: *nat* ⇒ *'a*::*banach*
assumes *f*: (λ*n*. *root n* (*norm* (*f n*))) ⟶ *x* — could be weakened to lim sup
and *x* < *1*
shows *summable f*
⟨*proof*⟩

113.1 More facts about binomial coefficients

These facts could have been proven before, but having real numbers makes the proofs a lot easier.

lemma *central-binomial-odd*:
odd n ⟹ *n choose* (*Suc* (*n div 2*)) = *n choose* (*n div 2*)
⟨*proof*⟩

lemma *binomial-less-binomial-Suc*:
assumes *k*: *k* < *n div 2*
shows *n choose k* < *n choose* (*Suc k*)
⟨*proof*⟩

lemma *binomial-strict-mono*:
assumes $k < k'$ $2*k' \leq n$
shows n *choose* $k < n$ *choose* k'
⟨*proof*⟩

lemma *binomial-mono*:
assumes $k \leq k'$ $2*k' \leq n$
shows n *choose* $k \leq n$ *choose* k'
⟨*proof*⟩

lemma *binomial-strict-antimono*:
assumes $k < k'$ $2 * k \geq n$ $k' \leq n$
shows n *choose* $k > n$ *choose* k'
⟨*proof*⟩

lemma *binomial-antimono*:
assumes $k \leq k'$ $k \geq n$ *div* 2 $k' \leq n$
shows n *choose* $k \geq n$ *choose* k'
⟨*proof*⟩

lemma *binomial-maximum*: n *choose* $k \leq n$ *choose* $(n$ *div* $2)$
⟨*proof*⟩

lemma *binomial-maximum'*: $(2*n)$ *choose* $k \leq (2*n)$ *choose* n
⟨*proof*⟩

lemma *central-binomial-lower-bound*:
assumes $n > 0$
shows $4\hat{}n$ / $(2*$*real* $n) \leq$ *real* $((2*n)$ *choose* $n)$
⟨*proof*⟩

113.2 Properties of Power Series

lemma *powser-zero* [*simp*]: $(\sum n.\ f\ n * 0\ \hat{}\ n) = f\ 0$
for f :: $nat \Rightarrow$ *'a*::*real-normed-algebra-1*
⟨*proof*⟩

lemma *powser-sums-zero*: $(\lambda n.\ a\ n * 0\hat{}n)$ *sums* $a\ 0$
for a :: $nat \Rightarrow$ *'a*::*real-normed-div-algebra*
⟨*proof*⟩

lemma *powser-sums-zero-iff* [*simp*]: $(\lambda n.\ a\ n * 0\hat{}n)$ *sums* $x \longleftrightarrow a\ 0 = x$
for a :: $nat \Rightarrow$ *'a*::*real-normed-div-algebra*
⟨*proof*⟩

Power series has a circle or radius of convergence: if it sums for x, then it sums absolutely for z with $|z| < |x|$.

lemma *powser-insidea*:

fixes *x z* :: *'a*::*real-normed-div-algebra*
assumes *1*: *summable* (λ*n*. *f n* ∗ *x*^*n*)
and *2*: *norm z* < *norm x*
shows *summable* (λ*n*. *norm* (*f n* ∗ *z* ^ *n*))
⟨*proof*⟩

lemma *powser-inside*:
fixes *f* :: *nat* ⇒ *'a*::{*real-normed-div-algebra*,*banach*}
shows
summable (λ*n*. *f n* ∗ (*x*^*n*)) ⟹ *norm z* < *norm x* ⟹
summable (λ*n*. *f n* ∗ (*z* ^ *n*))
⟨*proof*⟩

lemma *powser-times-n-limit-0*:
fixes *x* :: *'a*::{*real-normed-div-algebra*,*banach*}
assumes *norm x* < *1*
shows (λ*n*. *of-nat n* ∗ *x* ^ *n*) ⟶ *0*
⟨*proof*⟩

corollary *lim-n-over-pown*:
fixes *x* :: *'a*::{*real-normed-field*,*banach*}
shows *1* < *norm x* ⟹ ((λ*n*. *of-nat n* / *x*^*n*) ⟶ *0*) *sequentially*
⟨*proof*⟩

lemma *sum-split-even-odd*:
fixes *f* :: *nat* ⇒ *real*
shows (∑ *i*<*2* ∗ *n*. *if even i then f i else g i*) = (∑ *i*<*n*. *f* (*2* ∗ *i*)) + (∑ *i*<*n*. *g* (*2* ∗ *i* + *1*))
⟨*proof*⟩

lemma *sums-if'*:
fixes *g* :: *nat* ⇒ *real*
assumes *g sums x*
shows (λ *n*. *if even n then 0 else g* ((*n* − *1*) *div 2*)) *sums x*
⟨*proof*⟩

lemma *sums-if*:
fixes *g* :: *nat* ⇒ *real*
assumes *g sums x* **and** *f sums y*
shows (λ *n*. *if even n then f* (*n div 2*) *else g* ((*n* − *1*) *div 2*)) *sums* (*x* + *y*)
⟨*proof*⟩

113.3 Alternating series test / Leibniz formula

lemma *sums-alternating-upper-lower*:
fixes *a* :: *nat* ⇒ *real*
assumes *mono*: ⋀*n*. *a* (*Suc n*) ≤ *a n*
and *a-pos*: ⋀*n*. *0* ≤ *a n*
and *a* ⟶ *0*

shows $\exists l.\ ((\forall n.\ (\sum i{<}2{*}n.\ (-\ 1)\hat{}i{*}a\ i) \leq l) \wedge (\lambda\ n.\ \sum i{<}2{*}n.\ (-\ 1)\hat{}i{*}a\ i) \longrightarrow l) \wedge$
$((\forall n.\ l \leq (\sum i{<}2{*}n\ +\ 1.\ (-\ 1)\hat{}i{*}a\ i)) \wedge (\lambda\ n.\ \sum i{<}2{*}n\ +\ 1.\ (-\ 1)\hat{}i{*}a\ i) \longrightarrow l)$
(**is** $\exists l.\ ((\forall n.\ ?f\ n \leq l) \wedge$ -) $\wedge\ ((\forall n.\ l \leq\ ?g\ n) \wedge$ -))
⟨*proof*⟩

lemma *summable-Leibniz'*:
fixes a :: *nat* ⇒ *real*
assumes *a-zero*: $a \longrightarrow 0$
and *a-pos*: $\bigwedge n.\ 0 \leq a\ n$
and *a-monotone*: $\bigwedge n.\ a\ (Suc\ n) \leq a\ n$
shows *summable*: *summable* $(\lambda\ n.\ (-1)\hat{}n * a\ n)$
and $\bigwedge n.\ (\sum i{<}2{*}n.\ (-1)\hat{}i{*}a\ i) \leq (\sum i.\ (-1)\hat{}i{*}a\ i)$
and $(\lambda n.\ \sum i{<}2{*}n.\ (-1)\hat{}i{*}a\ i) \longrightarrow (\sum i.\ (-1)\hat{}i{*}a\ i)$
and $\bigwedge n.\ (\sum i.\ (-1)\hat{}i{*}a\ i) \leq (\sum i{<}2{*}n{+}1.\ (-1)\hat{}i{*}a\ i)$
and $(\lambda n.\ \sum i{<}2{*}n{+}1.\ (-1)\hat{}i{*}a\ i) \longrightarrow (\sum i.\ (-1)\hat{}i{*}a\ i)$
⟨*proof*⟩

theorem *summable-Leibniz*:
fixes a :: *nat* ⇒ *real*
assumes *a-zero*: $a \longrightarrow 0$
and *monoseq a*
shows *summable* $(\lambda\ n.\ (-1)\hat{}n * a\ n)$ (**is** *?summable*)
and $0 < a\ 0 \longrightarrow$
$(\forall n.\ (\sum i.\ (-\ 1)\hat{}i{*}a\ i) \in \{\ \sum i{<}2{*}n.\ (-\ 1)\hat{}i * a\ i\ ..\ \sum i{<}2{*}n{+}1.\ (-\ 1)\hat{}i * a\ i\})$ (**is** *?pos*)
and $a\ 0 < 0 \longrightarrow$
$(\forall n.\ (\sum i.\ (-\ 1)\hat{}i{*}a\ i) \in \{\ \sum i{<}2{*}n{+}1.\ (-\ 1)\hat{}i * a\ i\ ..\ \sum i{<}2{*}n.\ (-\ 1)\hat{}i * a\ i\})$ (**is** *?neg*)
and $(\lambda n.\ \sum i{<}2{*}n.\ (-\ 1)\hat{}i{*}a\ i) \longrightarrow (\sum i.\ (-\ 1)\hat{}i{*}a\ i)$ (**is** *?f*)
and $(\lambda n.\ \sum i{<}2{*}n{+}1.\ (-\ 1)\hat{}i{*}a\ i) \longrightarrow (\sum i.\ (-\ 1)\hat{}i{*}a\ i)$ (**is** *?g*)
⟨*proof*⟩

113.4 Term-by-Term Differentiability of Power Series

definition *diffs* :: $(nat \Rightarrow 'a{::}ring\text{-}1) \Rightarrow nat \Rightarrow 'a$
where *diffs* $c = (\lambda n.\ of\text{-}nat\ (Suc\ n) * c\ (Suc\ n))$

Lemma about distributing negation over it.

lemma *diffs-minus*: *diffs* $(\lambda n.\ -\ c\ n) = (\lambda n.\ -\ diffs\ c\ n)$
⟨*proof*⟩

lemma *diffs-equiv*:
fixes x :: $'a{::}\{real\text{-}normed\text{-}vector, ring\text{-}1\}$
shows *summable* $(\lambda n.\ diffs\ c\ n * x\hat{}n) \Longrightarrow$
$(\lambda n.\ of\text{-}nat\ n * c\ n * x\hat{}(n - Suc\ 0))\ sums\ (\sum n.\ diffs\ c\ n * x\hat{}n)$
⟨*proof*⟩

lemma *lemma-termdiff1*:
fixes $z :: 'a :: \{monoid\text{-}mult, comm\text{-}ring\}$
shows $(\sum p<m.\ (((z + h) \hat{} (m - p)) * (z \hat{} p)) - (z \hat{} m)) =$
$(\sum p<m.\ (z \hat{} p) * (((z + h) \hat{} (m - p)) - (z \hat{} (m - p))))$
⟨*proof*⟩

lemma *sumr-diff-mult-const2*: $sum\ f\ \{..<n\} - of\text{-}nat\ n * r = (\sum i<n.\ f\ i - r)$
for $r :: 'a::ring\text{-}1$
⟨*proof*⟩

lemma *lemma-termdiff2*:
fixes $h :: 'a::field$
assumes *h*: $h \neq 0$
shows $((z + h) \hat{} n - z \hat{} n) / h - of\text{-}nat\ n * z \hat{} (n - Suc\ 0) =$
$h * (\sum p< n - Suc\ 0.\ \sum q< n - Suc\ 0 - p.\ (z + h) \hat{} q * z \hat{} (n - 2 - q))$
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *real-sum-nat-ivl-bounded2*:
fixes $K :: 'a::linordered\text{-}semidom$
assumes *f*: $\bigwedge p::nat.\ p < n \Longrightarrow f\ p \leq K$ **and** *K*: $0 \leq K$
shows $sum\ f\ \{..<n-k\} \leq of\text{-}nat\ n * K$
⟨*proof*⟩

lemma *lemma-termdiff3*:
fixes $h\ z :: 'a::real\text{-}normed\text{-}field$
assumes *1*: $h \neq 0$
and *2*: $norm\ z \leq K$
and *3*: $norm\ (z + h) \leq K$
shows $norm\ (((z + h) \hat{} n - z \hat{} n) / h - of\text{-}nat\ n * z \hat{} (n - Suc\ 0)) \leq$
$of\text{-}nat\ n * of\text{-}nat\ (n - Suc\ 0) * K \hat{} (n - 2) * norm\ h$
⟨*proof*⟩

lemma *lemma-termdiff4*:
fixes $f :: 'a::real\text{-}normed\text{-}vector \Rightarrow 'b::real\text{-}normed\text{-}vector$
and $k :: real$
assumes *k*: $0 < k$
and *le*: $\bigwedge h.\ h \neq 0 \Longrightarrow norm\ h < k \Longrightarrow norm\ (f\ h) \leq K * norm\ h$
shows $f\ -0\rightarrow\ 0$
⟨*proof*⟩

lemma *lemma-termdiff5*:
fixes $g :: 'a::real\text{-}normed\text{-}vector \Rightarrow nat \Rightarrow 'b::banach$
and $k :: real$
assumes *k*: $0 < k$
and *f*: *summable f*
and *le*: $\bigwedge h\ n.\ h \neq 0 \Longrightarrow norm\ h < k \Longrightarrow norm\ (g\ h\ n) \leq f\ n * norm\ h$

shows $(\lambda h.\ suminf\ (g\ h)) \ -0\rightarrow\ 0$
⟨*proof*⟩

lemma *termdiffs-aux*:
fixes $x :: 'a::\{real\text{-}normed\text{-}field,banach\}$
assumes *1*: $summable\ (\lambda n.\ diffs\ (diffs\ c)\ n * K\ \hat{}\ n)$
and *2*: $norm\ x < norm\ K$
shows $(\lambda h.\ \sum n.\ c\ n * (((x + h)\ \hat{}\ n - x\hat{}n)\ /\ h - of\text{-}nat\ n * x\ \hat{}\ (n - Suc\ 0)))$
$-0\rightarrow\ 0$
⟨*proof*⟩

lemma *termdiffs*:
fixes $K\ x :: 'a::\{real\text{-}normed\text{-}field,banach\}$
assumes *1*: $summable\ (\lambda n.\ c\ n * K\ \hat{}\ n)$
and *2*: $summable\ (\lambda n.\ (diffs\ c)\ n * K\ \hat{}\ n)$
and *3*: $summable\ (\lambda n.\ (diffs\ (diffs\ c))\ n * K\ \hat{}\ n)$
and *4*: $norm\ x < norm\ K$
shows $DERIV\ (\lambda x.\ \sum n.\ c\ n * x\hat{}n)\ x :>\ (\sum n.\ (diffs\ c)\ n * x\hat{}n)$
⟨*proof*⟩

113.5 The Derivative of a Power Series Has the Same Radius of Convergence

lemma *termdiff-converges*:
fixes $x :: 'a::\{real\text{-}normed\text{-}field,banach\}$
assumes *K*: $norm\ x < K$
and *sm*: $\bigwedge x.\ norm\ x < K \Longrightarrow summable(\lambda n.\ c\ n * x\ \hat{}\ n)$
shows $summable\ (\lambda n.\ diffs\ c\ n * x\ \hat{}\ n)$
⟨*proof*⟩

lemma *termdiff-converges-all*:
fixes $x :: 'a::\{real\text{-}normed\text{-}field,banach\}$
assumes $\bigwedge x.\ summable\ (\lambda n.\ c\ n * x\hat{}n)$
shows $summable\ (\lambda n.\ diffs\ c\ n * x\hat{}n)$
⟨*proof*⟩

lemma *termdiffs-strong*:
fixes $K\ x :: 'a::\{real\text{-}normed\text{-}field,banach\}$
assumes *sm*: $summable\ (\lambda n.\ c\ n * K\ \hat{}\ n)$
and *K*: $norm\ x < norm\ K$
shows $DERIV\ (\lambda x.\ \sum n.\ c\ n * x\hat{}n)\ x :>\ (\sum n.\ diffs\ c\ n * x\hat{}n)$
⟨*proof*⟩

lemma *termdiffs-strong-converges-everywhere*:
fixes $K\ x :: 'a::\{real\text{-}normed\text{-}field,banach\}$
assumes $\bigwedge y.\ summable\ (\lambda n.\ c\ n * y\ \hat{}\ n)$

shows $((\lambda x.\ \sum n.\ c\ n * x\hat{}n)$ *has-field-derivative* $(\sum n.\ diffs\ c\ n * x\hat{}n))\ (at\ x)$
⟨*proof*⟩

lemma *termdiffs-strong'*:
fixes $z :: 'a :: \{real\text{-}normed\text{-}field, banach\}$
assumes $\bigwedge z.\ norm\ z < K \Longrightarrow summable\ (\lambda n.\ c\ n * z\ \hat{}\ n)$
assumes $norm\ z < K$
shows $((\lambda z.\ \sum n.\ c\ n * z\hat{}n)$ *has-field-derivative* $(\sum n.\ diffs\ c\ n * z\hat{}n))\ (at\ z)$
⟨*proof*⟩

lemma *termdiffs-sums-strong*:
fixes $z :: 'a :: \{banach, real\text{-}normed\text{-}field\}$
assumes *sums*: $\bigwedge z.\ norm\ z < K \Longrightarrow (\lambda n.\ c\ n * z\ \hat{}\ n)\ sums\ f\ z$
assumes *deriv*: $(f$ *has-field-derivative* $f')\ (at\ z)$
assumes *norm*: $norm\ z < K$
shows $(\lambda n.\ diffs\ c\ n * z\ \hat{}\ n)\ sums\ f'$
⟨*proof*⟩

lemma *isCont-powser*:
fixes $K\ x :: 'a::\{real\text{-}normed\text{-}field, banach\}$
assumes $summable\ (\lambda n.\ c\ n * K\ \hat{}\ n)$
assumes $norm\ x < norm\ K$
shows $isCont\ (\lambda x.\ \sum n.\ c\ n * x\hat{}n)\ x$
⟨*proof*⟩

lemmas *isCont-powser'* = *isCont-o2*[*OF - isCont-powser*]

lemma *isCont-powser-converges-everywhere*:
fixes $K\ x :: 'a::\{real\text{-}normed\text{-}field, banach\}$
assumes $\bigwedge y.\ summable\ (\lambda n.\ c\ n * y\ \hat{}\ n)$
shows $isCont\ (\lambda x.\ \sum n.\ c\ n * x\hat{}n)\ x$
⟨*proof*⟩

lemma *powser-limit-0*:
fixes $a :: nat \Rightarrow 'a::\{real\text{-}normed\text{-}field, banach\}$
assumes s: $0 < s$
and *sm*: $\bigwedge x.\ norm\ x < s \Longrightarrow (\lambda n.\ a\ n * x\ \hat{}\ n)\ sums\ (f\ x)$
shows $(f \longrightarrow a\ 0)\ (at\ 0)$
⟨*proof*⟩

lemma *powser-limit-0-strong*:
fixes $a :: nat \Rightarrow 'a::\{real\text{-}normed\text{-}field, banach\}$
assumes s: $0 < s$
and *sm*: $\bigwedge x.\ x \neq 0 \Longrightarrow norm\ x < s \Longrightarrow (\lambda n.\ a\ n * x\ \hat{}\ n)\ sums\ (f\ x)$
shows $(f \longrightarrow a\ 0)\ (at\ 0)$
⟨*proof*⟩

113.6 Derivability of power series

lemma *DERIV-series'*:
fixes $f :: real \Rightarrow nat \Rightarrow real$
assumes *DERIV-f*: $\bigwedge n.\ DERIV\ (\lambda x.\ f\ x\ n)\ x0\ :>\ (f'\ x0\ n)$
and *allf-summable*: $\bigwedge x.\ x \in \{a <..< b\} \Longrightarrow summable\ (f\ x)$
and *x0-in-I*: $x0 \in \{a <..< b\}$
and $summable\ (f'\ x0)$
and $summable\ L$
and *L-def*: $\bigwedge n\ x\ y.\ x \in \{a <..< b\} \Longrightarrow y \in \{a <..< b\} \Longrightarrow |f\ x\ n - f\ y\ n| \leq L\ n * |x - y|$
shows $DERIV\ (\lambda x.\ suminf\ (f\ x))\ x0\ :>\ (suminf\ (f'\ x0))$
⟨*proof*⟩

lemma *DERIV-power-series'*:
fixes $f :: nat \Rightarrow real$
assumes *converges*: $\bigwedge x.\ x \in \{-R <..< R\} \Longrightarrow summable\ (\lambda n.\ f\ n * real\ (Suc\ n) * x\hat{}n)$
and *x0-in-I*: $x0 \in \{-R <..< R\}$
and $0 < R$
shows $DERIV\ (\lambda x.\ (\sum n.\ f\ n * x\hat{}(Suc\ n)))\ x0\ :>\ (\sum n.\ f\ n * real\ (Suc\ n) * x0\hat{}n)$
(**is** $DERIV\ (\lambda x.\ suminf\ (?f\ x))\ x0\ :>\ suminf\ (?f'\ x0)$)
⟨*proof*⟩

lemma *geometric-deriv-sums*:
fixes $z :: 'a :: \{real\text{-}normed\text{-}field, banach\}$
assumes $norm\ z < 1$
shows $(\lambda n.\ of\text{-}nat\ (Suc\ n) * z\ \hat{}\ n)\ sums\ (1\ /\ (1 - z)\hat{}2)$
⟨*proof*⟩

lemma *isCont-pochhammer* [*continuous-intros*]: $isCont\ (\lambda z.\ pochhammer\ z\ n)\ z$
for $z :: 'a :: real\text{-}normed\text{-}field$
⟨*proof*⟩

lemma *continuous-on-pochhammer* [*continuous-intros*]: $continuous\text{-}on\ A\ (\lambda z.\ pochhammer\ z\ n)$
for $A :: 'a :: real\text{-}normed\text{-}field\ set$
⟨*proof*⟩

lemmas *continuous-on-pochhammer'* [*continuous-intros*] =
continuous-on-compose2[*OF continuous-on-pochhammer - subset-UNIV*]

113.7 Exponential Function

definition $exp :: 'a \Rightarrow 'a :: \{real\text{-}normed\text{-}algebra\text{-}1, banach\}$
where $exp = (\lambda x.\ \sum n.\ x\hat{}n\ /_R\ fact\ n)$

lemma *summable-exp-generic*:
fixes $x :: 'a :: \{real\text{-}normed\text{-}algebra\text{-}1, banach\}$

defines *S-def*: $S \equiv \lambda n.\ x\hat{}n\ /_R\ fact\ n$
shows *summable S*
⟨*proof*⟩

lemma *summable-norm-exp*: *summable* $(\lambda n.\ norm\ (x\hat{}n\ /_R\ fact\ n))$
for $x :: 'a::\{real\text{-}normed\text{-}algebra\text{-}1, banach\}$
⟨*proof*⟩

lemma *summable-exp*: *summable* $(\lambda n.\ inverse\ (fact\ n) * x\hat{}n)$
for $x :: 'a::\{real\text{-}normed\text{-}field, banach\}$
⟨*proof*⟩

lemma *exp-converges*: $(\lambda n.\ x\hat{}n\ /_R\ fact\ n)$ *sums exp x*
⟨*proof*⟩

lemma *exp-fdiffs*:
diffs $(\lambda n.\ inverse\ (fact\ n)) = (\lambda n.\ inverse\ (fact\ n :: 'a::\{real\text{-}normed\text{-}field, banach\}))$
⟨*proof*⟩

lemma *diffs-of-real*: *diffs* $(\lambda n.\ of\text{-}real\ (f\ n)) = (\lambda n.\ of\text{-}real\ (diffs\ f\ n))$
⟨*proof*⟩

lemma *DERIV-exp* [*simp*]: *DERIV exp x :> exp x*
⟨*proof*⟩

declare *DERIV-exp*[*THEN DERIV-chain2*, *derivative-intros*]
and *DERIV-exp*[*THEN DERIV-chain2*, *unfolded has-field-derivative-def*, *derivative-intros*]

lemmas *has-derivative-exp*[*derivative-intros*] = *DERIV-exp*[*THEN DERIV-compose-FDERIV*]

lemma *norm-exp*: *norm* $(exp\ x) \leq exp\ (norm\ x)$
⟨*proof*⟩

lemma *isCont-exp*: *isCont exp x*
for $x :: 'a::\{real\text{-}normed\text{-}field, banach\}$
⟨*proof*⟩

lemma *isCont-exp'* [*simp*]: *isCont f a* $\Longrightarrow$ *isCont* $(\lambda x.\ exp\ (f\ x))\ a$
for $f :: - \Rightarrow 'a::\{real\text{-}normed\text{-}field, banach\}$
⟨*proof*⟩

lemma *tendsto-exp* [*tendsto-intros*]: $(f \longrightarrow a)\ F \Longrightarrow ((\lambda x.\ exp\ (f\ x)) \longrightarrow exp\ a)\ F$
for $f :: - \Rightarrow 'a::\{real\text{-}normed\text{-}field, banach\}$
⟨*proof*⟩

lemma *continuous-exp* [*continuous-intros*]: *continuous F f* $\Longrightarrow$ *continuous F* $(\lambda x.\ exp\ (f\ x))$

for f :: - $\Rightarrow$'a::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *continuous-on-exp* [*continuous-intros*]: *continuous-on s f* $\Longrightarrow$ *continuous-on s* ($\lambda x.$ *exp* (*f x*))
for f :: - $\Rightarrow$'a::{*real-normed-field*,*banach*}
⟨*proof*⟩

113.7.1 Properties of the Exponential Function

lemma *exp-zero* [*simp*]: *exp 0* = *1*
⟨*proof*⟩

lemma *exp-series-add-commuting*:
fixes *x y* :: '*a*::{*real-normed-algebra-1*,*banach*}
defines *S-def*: $S \equiv \lambda x\ n.\ x\hat{}n\ /_R\ fact\ n$
assumes *comm*: *x* * *y* = *y* * *x*
shows *S* (*x* + *y*) *n* = ($\sum i{\le}n.$ *S x i* * *S y* (*n* − *i*))
⟨*proof*⟩

lemma *exp-add-commuting*: *x* * *y* = *y* * *x* $\Longrightarrow$ *exp* (*x* + *y*) = *exp x* * *exp y*
⟨*proof*⟩

lemma *exp-times-arg-commute*: *exp A* * *A* = *A* * *exp A*
⟨*proof*⟩

lemma *exp-add*: *exp* (*x* + *y*) = *exp x* * *exp y*
for *x y* :: '*a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *exp-double*: *exp*(*2* * *z*) = *exp z* ^ *2*
⟨*proof*⟩

lemmas *mult-exp-exp* = *exp-add* [*symmetric*]

lemma *exp-of-real*: *exp* (*of-real x*) = *of-real* (*exp x*)
⟨*proof*⟩

lemmas *of-real-exp* = *exp-of-real*[*symmetric*]

corollary *exp-in-Reals* [*simp*]: *z* ∈ ℝ $\Longrightarrow$ *exp z* ∈ ℝ
⟨*proof*⟩

lemma *exp-not-eq-zero* [*simp*]: *exp x* ≠ *0*
⟨*proof*⟩

lemma *exp-minus-inverse*: *exp x* * *exp* (− *x*) = *1*
⟨*proof*⟩

lemma *exp-minus*: *exp* (− *x*) = *inverse* (*exp x*)
  for *x* :: *'a*::{*real-normed-field*,*banach*}
  ⟨*proof*⟩

lemma *exp-diff*: *exp* (*x* − *y*) = *exp x* / *exp y*
  for *x* :: *'a*::{*real-normed-field*,*banach*}
  ⟨*proof*⟩

lemma *exp-of-nat-mult*: *exp* (*of-nat n* ∗ *x*) = *exp x* ^ *n*
  for *x* :: *'a*::{*real-normed-field*,*banach*}
  ⟨*proof*⟩

corollary *exp-of-nat2-mult*: *exp* (*x* ∗ *of-nat n*) = *exp x* ^ *n*
  for *x* :: *'a*::{*real-normed-field*,*banach*}
  ⟨*proof*⟩

lemma *exp-sum*: *finite I* ⟹ *exp* (*sum f I*) = *prod* (λ*x*. *exp* (*f x*)) *I*
  ⟨*proof*⟩

lemma *exp-divide-power-eq*:
  fixes *x* :: *'a*::{*real-normed-field*,*banach*}
  assumes *n* > *0*
  shows *exp* (*x* / *of-nat n*) ^ *n* = *exp x*
  ⟨*proof*⟩

lemma *exp-power-int*:
  fixes *x* :: *'a*::{*real-normed-field*,*banach*}
  shows *exp x powi n* = *exp* (*of-int n* ∗ *x*)
⟨*proof*⟩

113.7.2 Properties of the Exponential Function on Reals

Comparisons of *exp x* with zero.

Proof: because every exponential can be seen as a square.

lemma *exp-ge-zero* [*simp*]: *0* ≤ *exp x*
  for *x* :: *real*
⟨*proof*⟩

lemma *exp-gt-zero* [*simp*]: *0* < *exp x*
  for *x* :: *real*
  ⟨*proof*⟩

lemma *not-exp-less-zero* [*simp*]: ¬ *exp x* < *0*
  for *x* :: *real*
  ⟨*proof*⟩

lemma *not-exp-le-zero* [*simp*]: ¬ *exp x* ≤ *0*
  for *x* :: *real*

⟨*proof*⟩

lemma *abs-exp-cancel* [*simp*]: |*exp x*| = *exp x*
for *x* :: *real*
⟨*proof*⟩

Strict monotonicity of exponential.

lemma *exp-ge-add-one-self-aux*:
fixes *x* :: *real*
assumes *0* ≤ *x*
shows *1* + *x* ≤ *exp x*
⟨*proof*⟩

lemma *exp-gt-one*: *0* < *x* ⟹ *1* < *exp x*
for *x* :: *real*
⟨*proof*⟩

lemma *exp-less-mono*:
fixes *x y* :: *real*
assumes *x* < *y*
shows *exp x* < *exp y*
⟨*proof*⟩

lemma *exp-less-cancel*: *exp x* < *exp y* ⟹ *x* < *y*
for *x y* :: *real*
⟨*proof*⟩

lemma *exp-less-cancel-iff* [*iff*]: *exp x* < *exp y* ⟷ *x* < *y*
for *x y* :: *real*
⟨*proof*⟩

lemma *exp-le-cancel-iff* [*iff*]: *exp x* ≤ *exp y* ⟷ *x* ≤ *y*
for *x y* :: *real*
⟨*proof*⟩

lemma *exp-inj-iff* [*iff*]: *exp x* = *exp y* ⟷ *x* = *y*
for *x y* :: *real*
⟨*proof*⟩

Comparisons of *exp x* with one.

lemma *one-less-exp-iff* [*simp*]: *1* < *exp x* ⟷ *0* < *x*
for *x* :: *real*
⟨*proof*⟩

lemma *exp-less-one-iff* [*simp*]: *exp x* < *1* ⟷ *x* < *0*
for *x* :: *real*
⟨*proof*⟩

lemma *one-le-exp-iff* [*simp*]: *1* ≤ *exp x* ⟷ *0* ≤ *x*

for x :: *real*
⟨*proof*⟩

lemma *exp-le-one-iff* [*simp*]: $exp\ x \leq 1 \longleftrightarrow x \leq 0$
for x :: *real*
⟨*proof*⟩

lemma *exp-eq-one-iff* [*simp*]: $exp\ x = 1 \longleftrightarrow x = 0$
for x :: *real*
⟨*proof*⟩

lemma *lemma-exp-total*: $1 \leq y \Longrightarrow \exists x.\ 0 \leq x \wedge x \leq y - 1 \wedge exp\ x = y$
for y :: *real*
⟨*proof*⟩

lemma *exp-total*: $0 < y \Longrightarrow \exists x.\ exp\ x = y$
for y :: *real*
⟨*proof*⟩

113.8 Natural Logarithm

class *ln* = *real-normed-algebra-1* + *banach* +
fixes *ln* :: $'a \Rightarrow 'a$
assumes *ln-one* [*simp*]: $ln\ 1 = 0$

definition *powr* :: $'a \Rightarrow 'a \Rightarrow 'a$::*ln* (**infixr** *powr* 80)
— exponentation via ln and exp
where $x\ powr\ a \equiv if\ x = 0\ then\ 0\ else\ exp\ (a * ln\ x)$

lemma *powr-0* [*simp*]: $0\ powr\ z = 0$
⟨*proof*⟩

instantiation *real* :: *ln*
begin

definition *ln-real* :: *real* ⇒ *real*
where $ln\text{-}real\ x = (THE\ u.\ exp\ u = x)$

instance
⟨*proof*⟩

end

lemma *powr-eq-0-iff* [*simp*]: $w\ powr\ z = 0 \longleftrightarrow w = 0$
⟨*proof*⟩

lemma *ln-exp* [*simp*]: $ln\ (exp\ x) = x$
for x :: *real*

⟨*proof*⟩

lemma *exp-ln* [*simp*]: $0 < x \Longrightarrow \exp(\ln x) = x$
for x :: *real*
⟨*proof*⟩

lemma *exp-ln-iff* [*simp*]: $\exp(\ln x) = x \longleftrightarrow 0 < x$
for x :: *real*
⟨*proof*⟩

lemma *ln-unique*: $\exp y = x \Longrightarrow \ln x = y$
for x :: *real*
⟨*proof*⟩

lemma *ln-mult*: $0 < x \Longrightarrow 0 < y \Longrightarrow \ln(x * y) = \ln x + \ln y$
for x :: *real*
⟨*proof*⟩

lemma *ln-prod*: *finite* $I \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow f\ i > 0) \Longrightarrow \ln(\mathit{prod}\ f\ I) = \mathit{sum}\ (\lambda x.\ \ln(f\ x))\ I$
for f :: $'a \Rightarrow$ *real*
⟨*proof*⟩

lemma *ln-inverse*: $0 < x \Longrightarrow \ln(\mathit{inverse}\ x) = -\ln x$
for x :: *real*
⟨*proof*⟩

lemma *ln-div*: $0 < x \Longrightarrow 0 < y \Longrightarrow \ln(x\ /\ y) = \ln x - \ln y$
for x :: *real*
⟨*proof*⟩

lemma *ln-realpow*: $0 < x \Longrightarrow \ln(x\hat{}n) = \mathit{real}\ n * \ln x$
⟨*proof*⟩

lemma *ln-less-cancel-iff* [*simp*]: $0 < x \Longrightarrow 0 < y \Longrightarrow \ln x < \ln y \longleftrightarrow x < y$
for x :: *real*
⟨*proof*⟩

lemma *ln-le-cancel-iff* [*simp*]: $0 < x \Longrightarrow 0 < y \Longrightarrow \ln x \leq \ln y \longleftrightarrow x \leq y$
for x :: *real*
⟨*proof*⟩

lemma *ln-inj-iff* [*simp*]: $0 < x \Longrightarrow 0 < y \Longrightarrow \ln x = \ln y \longleftrightarrow x = y$
for x :: *real*
⟨*proof*⟩

lemma *ln-add-one-self-le-self*: $0 \leq x \Longrightarrow \ln(1 + x) \leq x$
for x :: *real*
⟨*proof*⟩

lemma *ln-less-self* [*simp*]: $0 < x \Longrightarrow \ln x < x$
for x :: *real*
⟨*proof*⟩

lemma *ln-ge-iff*: $\bigwedge x$::*real*. $0 < x \Longrightarrow y \leq \ln x \longleftrightarrow \exp y \leq x$
⟨*proof*⟩

lemma *ln-ge-zero* [*simp*]: $1 \leq x \Longrightarrow 0 \leq \ln x$
for x :: *real*
⟨*proof*⟩

lemma *ln-ge-zero-imp-ge-one*: $0 \leq \ln x \Longrightarrow 0 < x \Longrightarrow 1 \leq x$
for x :: *real*
⟨*proof*⟩

lemma *ln-ge-zero-iff* [*simp*]: $0 < x \Longrightarrow 0 \leq \ln x \longleftrightarrow 1 \leq x$
for x :: *real*
⟨*proof*⟩

lemma *ln-less-zero-iff* [*simp*]: $0 < x \Longrightarrow \ln x < 0 \longleftrightarrow x < 1$
for x :: *real*
⟨*proof*⟩

lemma *ln-le-zero-iff* [*simp*]: $0 < x \Longrightarrow \ln x \leq 0 \longleftrightarrow x \leq 1$
for x :: *real*
⟨*proof*⟩

lemma *ln-gt-zero*: $1 < x \Longrightarrow 0 < \ln x$
for x :: *real*
⟨*proof*⟩

lemma *ln-gt-zero-imp-gt-one*: $0 < \ln x \Longrightarrow 0 < x \Longrightarrow 1 < x$
for x :: *real*
⟨*proof*⟩

lemma *ln-gt-zero-iff* [*simp*]: $0 < x \Longrightarrow 0 < \ln x \longleftrightarrow 1 < x$
for x :: *real*
⟨*proof*⟩

lemma *ln-eq-zero-iff* [*simp*]: $0 < x \Longrightarrow \ln x = 0 \longleftrightarrow x = 1$
for x :: *real*
⟨*proof*⟩

lemma *ln-less-zero*: $0 < x \Longrightarrow x < 1 \Longrightarrow \ln x < 0$
for x :: *real*
⟨*proof*⟩

lemma *ln-neg-is-const*: $x \leq 0 \Longrightarrow \ln x = (\mathit{THE}\ x.\ \mathit{False})$

for x :: *real*
⟨*proof*⟩

lemma *powr-eq-one-iff* [*simp*]:
a *powr* $x = 1 \longleftrightarrow x = 0$ **if** $a > 1$ **for** $a\ x$ :: *real*
⟨*proof*⟩

lemma *isCont-ln*:
fixes x :: *real*
assumes $x \neq 0$
shows *isCont ln x*
⟨*proof*⟩

lemma *tendsto-ln* [*tendsto-intros*]: $(f \longrightarrow a)\ F \Longrightarrow a \neq 0 \Longrightarrow ((\lambda x.\ ln\ (f\ x)) \longrightarrow ln\ a)\ F$
for a :: *real*
⟨*proof*⟩

lemma *continuous-ln*:
continuous $F\ f \Longrightarrow f\ (Lim\ F\ (\lambda x.\ x)) \neq 0 \Longrightarrow$ *continuous* $F\ (\lambda x.\ ln\ (f\ x ::\ real))$
⟨*proof*⟩

lemma *isCont-ln'* [*continuous-intros*]:
continuous $(at\ x)\ f \Longrightarrow f\ x \neq 0 \Longrightarrow$ *continuous* $(at\ x)\ (\lambda x.\ ln\ (f\ x ::\ real))$
⟨*proof*⟩

lemma *continuous-within-ln* [*continuous-intros*]:
continuous $(at\ x\ within\ s)\ f \Longrightarrow f\ x \neq 0 \Longrightarrow$ *continuous* $(at\ x\ within\ s)\ (\lambda x.\ ln\ (f\ x ::\ real))$
⟨*proof*⟩

lemma *continuous-on-ln* [*continuous-intros*]:
continuous-on $s\ f \Longrightarrow (\forall x{\in}s.\ f\ x \neq 0) \Longrightarrow$ *continuous-on* $s\ (\lambda x.\ ln\ (f\ x ::\ real))$
⟨*proof*⟩

lemma *DERIV-ln*: $0 < x \Longrightarrow DERIV\ ln\ x :> inverse\ x$
for x :: *real*
⟨*proof*⟩

lemma *DERIV-ln-divide*: $0 < x \Longrightarrow DERIV\ ln\ x :> 1\ /\ x$
for x :: *real*
⟨*proof*⟩

declare *DERIV-ln-divide*[*THEN DERIV-chain2*, *derivative-intros*]
and *DERIV-ln-divide*[*THEN DERIV-chain2*, *unfolded has-field-derivative-def*, *derivative-intros*]

lemmas *has-derivative-ln*[*derivative-intros*] = *DERIV-ln*[*THEN DERIV-compose-FDERIV*]

lemma *ln-series*:
assumes $0 < x$ **and** $x < 2$
shows $ln\ x = (\sum\ n.\ (-1)\hat{}n * (1\ /\ real\ (n + 1)) * (x - 1)\hat{}(Suc\ n))$
(**is** $ln\ x = suminf\ (?f\ (x - 1))$)
⟨*proof*⟩

lemma *exp-first-terms*:
fixes $x :: 'a::\{real\text{-}normed\text{-}algebra\text{-}1, banach\}$
shows $exp\ x = (\sum n<k.\ inverse(fact\ n) *_R (x\ \hat{}\ n)) + (\sum n.\ inverse(fact\ (n + k)) *_R (x\ \hat{}\ (n + k)))$
⟨*proof*⟩

lemma *exp-first-term*: $exp\ x = 1 + (\sum n.\ inverse\ (fact\ (Suc\ n)) *_R (x\ \hat{}\ Suc\ n))$
for $x :: 'a::\{real\text{-}normed\text{-}algebra\text{-}1, banach\}$
⟨*proof*⟩

lemma *exp-first-two-terms*: $exp\ x = 1 + x + (\sum n.\ inverse\ (fact\ (n + 2)) *_R (x\ \hat{}\ (n + 2)))$
for $x :: 'a::\{real\text{-}normed\text{-}algebra\text{-}1, banach\}$
⟨*proof*⟩

lemma *exp-bound*:
fixes $x :: real$
assumes *a*: $0 \leq x$
and *b*: $x \leq 1$
shows $exp\ x \leq 1 + x + x^2$
⟨*proof*⟩

corollary *exp-half-le2*: $exp(1/2) \leq (2::real)$
⟨*proof*⟩

corollary *exp-le*: $exp\ 1 \leq (3::real)$
⟨*proof*⟩

lemma *exp-bound-half*: $norm\ z \leq 1/2 \Longrightarrow norm\ (exp\ z) \leq 2$
⟨*proof*⟩

lemma *exp-bound-lemma*:
assumes $norm\ z \leq 1/2$
shows $norm\ (exp\ z) \leq 1 + 2 * norm\ z$
⟨*proof*⟩

lemma *real-exp-bound-lemma*: $0 \leq x \Longrightarrow x \leq 1/2 \Longrightarrow exp\ x \leq 1 + 2 * x$
for $x :: real$
⟨*proof*⟩

lemma *ln-one-minus-pos-upper-bound*:
fixes $x :: real$
assumes *a*: $0 \leq x$ **and** *b*: $x < 1$

shows $ln\ (1 - x) \leq - x$
⟨*proof*⟩

lemma *exp-ge-add-one-self* [*simp*]: $1 + x \leq exp\ x$
for x :: *real*
⟨*proof*⟩

lemma *ln-one-plus-pos-lower-bound*:
fixes x :: *real*
assumes *a*: $0 \leq x$ **and** *b*: $x \leq 1$
shows $x - x^2 \leq ln\ (1 + x)$
⟨*proof*⟩

lemma *ln-one-minus-pos-lower-bound*:
fixes x :: *real*
assumes *a*: $0 \leq x$ **and** *b*: $x \leq 1/2$
shows $- x - 2 * x^2 \leq ln\ (1 - x)$
⟨*proof*⟩

lemma *ln-add-one-self-le-self2*:
fixes x :: *real*
shows $-1 < x \Longrightarrow ln\ (1 + x) \leq x$
⟨*proof*⟩

lemma *abs-ln-one-plus-x-minus-x-bound-nonneg*:
fixes x :: *real*
assumes *x*: $0 \leq x$ **and** *x1*: $x \leq 1$
shows $|ln\ (1 + x) - x| \leq x^2$
⟨*proof*⟩

lemma *abs-ln-one-plus-x-minus-x-bound-nonpos*:
fixes x :: *real*
assumes *a*: $-(1/2) \leq x$ **and** *b*: $x \leq 0$
shows $|ln\ (1 + x) - x| \leq 2 * x^2$
⟨*proof*⟩

lemma *abs-ln-one-plus-x-minus-x-bound*:
fixes x :: *real*
assumes $|x| \leq 1/2$
shows $|ln\ (1 + x) - x| \leq 2 * x^2$
⟨*proof*⟩

lemma *ln-x-over-x-mono*:
fixes x :: *real*
assumes *x*: $exp\ 1 \leq x\ x \leq y$
shows $ln\ y\ /\ y \leq ln\ x\ /\ x$
⟨*proof*⟩

lemma *ln-le-minus-one*: $0 < x \Longrightarrow ln\ x \leq x - 1$

for x :: *real*
⟨*proof*⟩

corollary *ln-diff-le*: $0 < x \Longrightarrow 0 < y \Longrightarrow ln\ x - ln\ y \leq (x - y)\ /\ y$
for x :: *real*
⟨*proof*⟩

lemma *ln-eq-minus-one*:
fixes x :: *real*
assumes $0 < x\ ln\ x = x - 1$
shows $x = 1$
⟨*proof*⟩

lemma *ln-x-over-x-tendsto-0*: $((\lambda x{::}real.\ ln\ x\ /\ x) \longrightarrow 0)\ at\text{-}top$
⟨*proof*⟩

lemma *exp-ge-one-plus-x-over-n-power-n*:
assumes $x \geq -\ real\ n\ n > 0$
shows $(1 + x\ /\ of\text{-}nat\ n)\ \hat{}\ n \leq exp\ x$
⟨*proof*⟩

lemma *exp-ge-one-minus-x-over-n-power-n*:
assumes $x \leq real\ n\ n > 0$
shows $(1 - x\ /\ of\text{-}nat\ n)\ \hat{}\ n \leq exp\ (-x)$
⟨*proof*⟩

lemma *exp-at-bot*: $(exp \longrightarrow (0{::}real))\ at\text{-}bot$
⟨*proof*⟩

lemma *exp-at-top*: *LIM x at-top. exp x* :: *real* :> *at-top*
⟨*proof*⟩

lemma *lim-exp-minus-1*: $((\lambda z{::}'a.\ (exp(z) - 1)\ /\ z) \longrightarrow 1)\ (at\ 0)$
for x :: $'a{::}\{real\text{-}normed\text{-}field, banach\}$
⟨*proof*⟩

lemma *ln-at-0*: *LIM x at-right 0. ln* $(x{::}real)$:> *at-bot*
⟨*proof*⟩

lemma *ln-at-top*: *LIM x at-top. ln* $(x{::}real)$:> *at-top*
⟨*proof*⟩

lemma *filtermap-ln-at-top*: *filtermap* $(ln{::}real \Rightarrow real)\ at\text{-}top = at\text{-}top$
⟨*proof*⟩

lemma *filtermap-exp-at-top*: *filtermap* $(exp{::}real \Rightarrow real)\ at\text{-}top = at\text{-}top$
⟨*proof*⟩

lemma *filtermap-ln-at-right*: *filtermap ln* $(at\text{-}right\ (0{::}real)) = at\text{-}bot$

⟨*proof*⟩

lemma *tendsto-power-div-exp-0*: $((\lambda x.\ x\ \hat{}\ k\ /\ exp\ x) \longrightarrow (0::real))\ at\text{-}top$
⟨*proof*⟩

113.8.1 A couple of simple bounds

lemma *exp-plus-inverse-exp*:
fixes $x::real$
shows $2 \leq exp\ x + inverse\ (exp\ x)$
⟨*proof*⟩

lemma *real-le-x-sinh*:
fixes $x::real$
assumes $0 \leq x$
shows $x \leq (exp\ x - inverse(exp\ x))\ /\ 2$
⟨*proof*⟩

lemma *real-le-abs-sinh*:
fixes $x::real$
shows $abs\ x \leq abs((exp\ x - inverse(exp\ x))\ /\ 2)$
⟨*proof*⟩

113.9 The general logarithm

definition $log :: real \Rightarrow real \Rightarrow real$
— logarithm of x to base a
where $log\ a\ x = ln\ x\ /\ ln\ a$

lemma *tendsto-log* [*tendsto-intros*]:
$(f \longrightarrow a)\ F \Longrightarrow (g \longrightarrow b)\ F \Longrightarrow 0 < a \Longrightarrow a \neq 1 \Longrightarrow 0 < b \Longrightarrow$
$((\lambda x.\ log\ (f\ x)\ (g\ x)) \longrightarrow log\ a\ b)\ F$
⟨*proof*⟩

lemma *continuous-log*:
assumes *continuous F f*
and *continuous F g*
and $0 < f\ (Lim\ F\ (\lambda x.\ x))$
and $f\ (Lim\ F\ (\lambda x.\ x)) \neq 1$
and $0 < g\ (Lim\ F\ (\lambda x.\ x))$
shows *continuous F* $(\lambda x.\ log\ (f\ x)\ (g\ x))$
⟨*proof*⟩

lemma *continuous-at-within-log*[*continuous-intros*]:
assumes *continuous* $(at\ a\ within\ s)\ f$
and *continuous* $(at\ a\ within\ s)\ g$
and $0 < f\ a$
and $f\ a \neq 1$
and $0 < g\ a$
shows *continuous* $(at\ a\ within\ s)\ (\lambda x.\ log\ (f\ x)\ (g\ x))$

⟨*proof*⟩

lemma *isCont-log*[*continuous-intros*, *simp*]:
assumes *isCont f a isCont g a* 0 < *f a f a* ≠ 1 0 < *g a*
shows *isCont* (λ*x*. *log* (*f x*) (*g x*)) *a*
⟨*proof*⟩

lemma *continuous-on-log*[*continuous-intros*]:
assumes *continuous-on s f continuous-on s g*
and ∀ *x*∈*s*. 0 < *f x* ∀ *x*∈*s*. *f x* ≠ 1 ∀ *x*∈*s*. 0 < *g x*
shows *continuous-on s* (λ*x*. *log* (*f x*) (*g x*))
⟨*proof*⟩

lemma *powr-one-eq-one* [*simp*]: *1 powr a* = *1*
⟨*proof*⟩

lemma *powr-zero-eq-one* [*simp*]: *x powr 0* = (*if x* = *0 then 0 else 1*)
⟨*proof*⟩

lemma *powr-one-gt-zero-iff* [*simp*]: *x powr 1* = *x* ⟷ *0* ≤ *x*
for *x* :: *real*
⟨*proof*⟩
declare *powr-one-gt-zero-iff* [*THEN iffD2*, *simp*]

lemma *powr-diff*:
fixes *w*:: *'a*::{*ln*,*real-normed-field*} **shows** *w powr* (*z1* − *z2*) = *w powr z1* / *w powr z2*
⟨*proof*⟩

lemma *powr-mult*: *0* ≤ *x* ⟹ *0* ≤ *y* ⟹ (*x* ∗ *y*) *powr a* = (*x powr a*) ∗ (*y powr a*)
for *a x y* :: *real*
⟨*proof*⟩

lemma *powr-ge-pzero* [*simp*]: *0* ≤ *x powr y*
for *x y* :: *real*
⟨*proof*⟩

lemma *powr-non-neg*[*simp*]: ¬*a powr x* < *0* **for** *a x*::*real*
⟨*proof*⟩

lemma *inverse-powr*: ⋀*y*::*real*. *0* ≤ *y* ⟹ *inverse y powr a* = *inverse* (*y powr a*)
⟨*proof*⟩

lemma *powr-divide*: ⟦*0* ≤ *x*; *0* ≤ *y*⟧ ⟹ (*x* / *y*) *powr a* = (*x powr a*) / (*y powr a*)
for *a b x* :: *real*
⟨*proof*⟩

lemma *powr-add*: *x powr* (*a* + *b*) = (*x powr a*) ∗ (*x powr b*)

for *a b x* :: *'a*::{*ln*,*real-normed-field*}
⟨*proof*⟩

lemma *powr-mult-base*: $0 \leq x \Longrightarrow x * x \text{ powr } y = x \text{ powr } (1 + y)$
for *x* :: *real*
⟨*proof*⟩

lemma *powr-powr*: $(x \text{ powr } a) \text{ powr } b = x \text{ powr } (a * b)$
for *a b x* :: *real*
⟨*proof*⟩

lemma *powr-powr-swap*: $(x \text{ powr } a) \text{ powr } b = (x \text{ powr } b) \text{ powr } a$
for *a b x* :: *real*
⟨*proof*⟩

lemma *powr-minus*: $x \text{ powr } (- a) = \text{inverse } (x \text{ powr } a)$
for *a x* :: *'a*::{*ln*,*real-normed-field*}
⟨*proof*⟩

lemma *powr-minus-divide*: $x \text{ powr } (- a) = 1/(x \text{ powr } a)$
for *a x* :: *'a*::{*ln*,*real-normed-field*}
⟨*proof*⟩

lemma *powr-sum*: $x \neq 0 \Longrightarrow \text{finite } A \Longrightarrow x \text{ powr sum } f A = (\prod y \in A.\ x \text{ powr } f\ y)$
⟨*proof*⟩

lemma *divide-powr-uminus*: $a\ /\ b \text{ powr } c = a * b \text{ powr } (- c)$
for *a b c* :: *real*
⟨*proof*⟩

lemma *powr-less-mono*: $a < b \Longrightarrow 1 < x \Longrightarrow x \text{ powr } a < x \text{ powr } b$
for *a b x* :: *real*
⟨*proof*⟩

lemma *powr-less-cancel*: $x \text{ powr } a < x \text{ powr } b \Longrightarrow 1 < x \Longrightarrow a < b$
for *a b x* :: *real*
⟨*proof*⟩

lemma *powr-less-cancel-iff* [*simp*]: $1 < x \Longrightarrow x \text{ powr } a < x \text{ powr } b \longleftrightarrow a < b$
for *a b x* :: *real*
⟨*proof*⟩

lemma *powr-le-cancel-iff* [*simp*]: $1 < x \Longrightarrow x \text{ powr } a \leq x \text{ powr } b \longleftrightarrow a \leq b$
for *a b x* :: *real*
⟨*proof*⟩

lemma *powr-realpow*: $0 < x \Longrightarrow x \text{ powr } (\text{real } n) = x\hat{}n$
⟨*proof*⟩

lemma *powr-realpow'*: $(z :: real) \geq 0 \Longrightarrow n \neq 0 \Longrightarrow z\ powr\ of\text{-}nat\ n = z\ \hat{}\ n$
⟨*proof*⟩

lemma *powr-real-of-int'*:
assumes $x \geq 0\ x \neq 0 \vee n > 0$
shows $x\ powr\ real\text{-}of\text{-}int\ n = power\text{-}int\ x\ n$
⟨*proof*⟩

lemma *log-ln*: $ln\ x = log\ (exp(1))\ x$
⟨*proof*⟩

lemma *DERIV-log*:
assumes $x > 0$
shows $DERIV\ (\lambda y.\ log\ b\ y)\ x :> 1\ /\ (ln\ b * x)$
⟨*proof*⟩

lemmas *DERIV-log*[*THEN DERIV-chain2*, *derivative-intros*]
and *DERIV-log*[*THEN DERIV-chain2*, *unfolded has-field-derivative-def*, *derivative-intros*]

lemma *powr-log-cancel* [*simp*]: $0 < a \Longrightarrow a \neq 1 \Longrightarrow 0 < x \Longrightarrow a\ powr\ (log\ a\ x) = x$
⟨*proof*⟩

lemma *log-powr-cancel* [*simp*]: $0 < a \Longrightarrow a \neq 1 \Longrightarrow log\ a\ (a\ powr\ y) = y$
⟨*proof*⟩

lemma *log-mult*:
$0 < a \Longrightarrow a \neq 1 \Longrightarrow 0 < x \Longrightarrow 0 < y \Longrightarrow$
$log\ a\ (x * y) = log\ a\ x + log\ a\ y$
⟨*proof*⟩

lemma *log-eq-div-ln-mult-log*:
$0 < a \Longrightarrow a \neq 1 \Longrightarrow 0 < b \Longrightarrow b \neq 1 \Longrightarrow 0 < x \Longrightarrow$
$log\ a\ x = (ln\ b / ln\ a) * log\ b\ x$
⟨*proof*⟩

Base 10 logarithms

lemma *log-base-10-eq1*: $0 < x \Longrightarrow log\ 10\ x = (ln\ (exp\ 1)\ /\ ln\ 10) * ln\ x$
⟨*proof*⟩

lemma *log-base-10-eq2*: $0 < x \Longrightarrow log\ 10\ x = (log\ 10\ (exp\ 1)) * ln\ x$
⟨*proof*⟩

lemma *log-one* [*simp*]: $log\ a\ 1 = 0$
⟨*proof*⟩

lemma *log-eq-one* [*simp*]: $0 < a \Longrightarrow a \neq 1 \Longrightarrow log\ a\ a = 1$
⟨*proof*⟩

lemma *log-inverse*: $0 < a \Longrightarrow a \neq 1 \Longrightarrow 0 < x \Longrightarrow log\ a\ (inverse\ x) = -\ log\ a\ x$
⟨*proof*⟩

lemma *log-divide*: $0 < a \Longrightarrow a \neq 1 \Longrightarrow 0 < x \Longrightarrow 0 < y \Longrightarrow log\ a\ (x/y) = log\ a\ x - log\ a\ y$
⟨*proof*⟩

lemma *powr-gt-zero* [*simp*]: $0 < x\ powr\ a \longleftrightarrow x \neq 0$
for $a\ x :: real$
⟨*proof*⟩

lemma *powr-nonneg-iff*[*simp*]: $a\ powr\ x \leq 0 \longleftrightarrow a = 0$
for $a\ x$::*real*
⟨*proof*⟩

lemma *log-add-eq-powr*: $0 < b \Longrightarrow b \neq 1 \Longrightarrow 0 < x \Longrightarrow log\ b\ x + y = log\ b\ (x * b\ powr\ y)$
and *add-log-eq-powr*: $0 < b \Longrightarrow b \neq 1 \Longrightarrow 0 < x \Longrightarrow y + log\ b\ x = log\ b\ (b\ powr\ y * x)$
and *log-minus-eq-powr*: $0 < b \Longrightarrow b \neq 1 \Longrightarrow 0 < x \Longrightarrow log\ b\ x - y = log\ b\ (x * b\ powr\ -y)$
and *minus-log-eq-powr*: $0 < b \Longrightarrow b \neq 1 \Longrightarrow 0 < x \Longrightarrow y - log\ b\ x = log\ b\ (b\ powr\ y\ /\ x)$
⟨*proof*⟩

lemma *log-less-cancel-iff* [*simp*]: $1 < a \Longrightarrow 0 < x \Longrightarrow 0 < y \Longrightarrow log\ a\ x < log\ a\ y \longleftrightarrow x < y$
⟨*proof*⟩

lemma *log-inj*:
assumes $1 < b$
shows *inj-on* $(log\ b)\ \{0 <..\}$
⟨*proof*⟩

lemma *log-le-cancel-iff* [*simp*]: $1 < a \Longrightarrow 0 < x \Longrightarrow 0 < y \Longrightarrow log\ a\ x \leq log\ a\ y \longleftrightarrow x \leq y$
⟨*proof*⟩

lemma *zero-less-log-cancel-iff*[*simp*]: $1 < a \Longrightarrow 0 < x \Longrightarrow 0 < log\ a\ x \longleftrightarrow 1 < x$
⟨*proof*⟩

lemma *zero-le-log-cancel-iff*[*simp*]: $1 < a \Longrightarrow 0 < x \Longrightarrow 0 \leq log\ a\ x \longleftrightarrow 1 \leq x$
⟨*proof*⟩

lemma *log-less-zero-cancel-iff*[*simp*]: $1 < a \Longrightarrow 0 < x \Longrightarrow log\ a\ x < 0 \longleftrightarrow x < 1$
⟨*proof*⟩

lemma *log-le-zero-cancel-iff*[*simp*]: $1 < a \Longrightarrow 0 < x \Longrightarrow log\ a\ x \leq 0 \longleftrightarrow x \leq 1$
⟨*proof*⟩

lemma *one-less-log-cancel-iff*[*simp*]: $1 < a \Longrightarrow 0 < x \Longrightarrow 1 < log\ a\ x \longleftrightarrow a < x$
⟨*proof*⟩

lemma *one-le-log-cancel-iff*[*simp*]: $1 < a \Longrightarrow 0 < x \Longrightarrow 1 \leq log\ a\ x \longleftrightarrow a \leq x$
⟨*proof*⟩

lemma *log-less-one-cancel-iff*[*simp*]: $1 < a \Longrightarrow 0 < x \Longrightarrow log\ a\ x < 1 \longleftrightarrow x < a$
⟨*proof*⟩

lemma *log-le-one-cancel-iff*[*simp*]: $1 < a \Longrightarrow 0 < x \Longrightarrow log\ a\ x \leq 1 \longleftrightarrow x \leq a$
⟨*proof*⟩

lemma *le-log-iff*:
fixes $b\ x\ y :: real$
assumes $1 < b\ x > 0$
shows $y \leq log\ b\ x \longleftrightarrow b\ powr\ y \leq x$
⟨*proof*⟩

lemma *less-log-iff*:
assumes $1 < b\ x > 0$
shows $y < log\ b\ x \longleftrightarrow b\ powr\ y < x$
⟨*proof*⟩

lemma
assumes $1 < b\ x > 0$
shows *log-less-iff*: $log\ b\ x < y \longleftrightarrow x < b\ powr\ y$
and *log-le-iff*: $log\ b\ x \leq y \longleftrightarrow x \leq b\ powr\ y$
⟨*proof*⟩

lemmas *powr-le-iff* = *le-log-iff*[*symmetric*]
and *powr-less-iff* = *less-log-iff*[*symmetric*]
and *less-powr-iff* = *log-less-iff*[*symmetric*]
and *le-powr-iff* = *log-le-iff*[*symmetric*]

lemma *le-log-of-power*:
assumes $b \mathbin{\hat{}} n \leq m\ 1 < b$
shows $n \leq log\ b\ m$
⟨*proof*⟩

lemma *le-log2-of-power*: $2 \mathbin{\hat{}} n \leq m \Longrightarrow n \leq log\ 2\ m$ **for** $m\ n :: nat$
⟨*proof*⟩

lemma *log-of-power-le*: $[\![\ m \leq b \mathbin{\hat{}} n;\ b > 1;\ m > 0\ ]\!] \Longrightarrow log\ b\ (real\ m) \leq n$
⟨*proof*⟩

lemma *log2-of-power-le*: $[\![m \leq 2 \mathbin{\hat{}} n; m > 0]\!] \Longrightarrow log\ 2\ m \leq n$ **for** $m\ n :: nat$
⟨*proof*⟩

lemma *log-of-power-less*: $[\![m < b \mathbin{\hat{}} n; b > 1; m > 0]\!] \Longrightarrow log\ b\ (real\ m) < n$
⟨*proof*⟩

lemma *log2-of-power-less*: $[\![m < 2 \mathbin{\hat{}} n; m > 0]\!] \Longrightarrow log\ 2\ m < n$ **for** $m\ n :: nat$
⟨*proof*⟩

lemma *less-log-of-power*:
assumes $b \mathbin{\hat{}} n < m\ 1 < b$
shows $n < log\ b\ m$
⟨*proof*⟩

lemma *less-log2-of-power*: $2 \mathbin{\hat{}} n < m \Longrightarrow n < log\ 2\ m$ **for** $m\ n :: nat$
⟨*proof*⟩

lemma *gr-one-powr*[*simp*]:
fixes $x\ y :: real$ **shows** $[\![x > 1; y > 0]\!] \Longrightarrow 1 < x\ powr\ y$
⟨*proof*⟩

lemma *log-pow-cancel* [*simp*]:
$a > 0 \Longrightarrow a \neq 1 \Longrightarrow log\ a\ (a \mathbin{\hat{}} b) = b$
⟨*proof*⟩

lemma *floor-log-eq-powr-iff*: $x > 0 \Longrightarrow b > 1 \Longrightarrow \lfloor log\ b\ x \rfloor = k \longleftrightarrow b\ powr\ k \leq x \wedge x < b\ powr\ (k + 1)$
⟨*proof*⟩

lemma *floor-log-nat-eq-powr-iff*:
fixes $b\ n\ k :: nat$
shows $[\![b \geq 2; k > 0]\!] \Longrightarrow floor\ (log\ b\ (real\ k)) = n \longleftrightarrow b\hat{}n \leq k \wedge k < b\hat{}(n+1)$
⟨*proof*⟩

lemma *floor-log-nat-eq-if*:
fixes $b\ n\ k :: nat$
assumes $b\hat{}n \leq k\ k < b\hat{}(n+1)\ b \geq 2$
shows $floor\ (log\ b\ (real\ k)) = n$
⟨*proof*⟩

lemma *ceiling-log-eq-powr-iff*:
$[\![x > 0; b > 1]\!] \Longrightarrow \lceil log\ b\ x \rceil = int\ k + 1 \longleftrightarrow b\ powr\ k < x \wedge x \leq b\ powr\ (k + 1)$
⟨*proof*⟩

lemma *ceiling-log-nat-eq-powr-iff*:
fixes $b\ n\ k :: nat$
shows $[\![b \geq 2; k > 0]\!] \Longrightarrow ceiling\ (log\ b\ (real\ k)) = int\ n + 1 \longleftrightarrow (b\hat{}n < k$

$\wedge\ k \leq b\hat{}(n+1))$
⟨*proof*⟩

lemma *ceiling-log-nat-eq-if*:
fixes *b n k* :: *nat*
assumes $b\hat{}n < k\ k \leq b\hat{}(n+1)\ b \geq 2$
shows $\lceil log\ (real\ b)\ (real\ k) \rceil = int\ n + 1$
⟨*proof*⟩

lemma *floor-log2-div2*:
fixes *n* :: *nat*
assumes $n \geq 2$
shows $\lfloor log\ 2\ (real\ n) \rfloor = \lfloor log\ 2\ (n\ div\ 2) \rfloor + 1$
⟨*proof*⟩

lemma *ceiling-log2-div2*:
assumes $n \geq 2$
shows $ceiling(log\ 2\ (real\ n)) = ceiling(log\ 2\ ((n-1)\ div\ 2 + 1)) + 1$
⟨*proof*⟩

lemma *powr-real-of-int*:
$x > 0 \Longrightarrow x\ powr\ real\text{-}of\text{-}int\ n = (if\ n \geq 0\ then\ x\ \hat{}\ nat\ n\ else\ inverse\ (x\ \hat{}\ nat\ (-\ n)))$
⟨*proof*⟩

lemma *powr-numeral* [*simp*]: $0 \leq x \Longrightarrow x\ powr\ (numeral\ n :: real) = x\ \hat{}(numeral\ n)$
⟨*proof*⟩

lemma *powr-int*:
assumes $x > 0$
shows $x\ powr\ i = (if\ i \geq 0\ then\ x\ \hat{}\ nat\ i\ else\ 1\ /\ x\ \hat{}\ nat\ (-i))$
⟨*proof*⟩

lemma *power-of-nat-log-ge*: $b > 1 \Longrightarrow b\ \hat{}\ nat\ \lceil log\ b\ x \rceil \geq x$
⟨*proof*⟩

lemma *power-of-nat-log-le*:
assumes $b > 1\ x \geq 1$
shows $b\ \hat{}\ nat\ \lfloor log\ b\ x \rfloor \leq x$
⟨*proof*⟩

definition *powr-real* :: *real* ⇒ *real* ⇒ *real*
where [*code-abbrev*, *simp*]: *powr-real* = *Transcendental.powr*

lemma *compute-powr-real* [*code*]:
powr-real b i =
(*if b* ≤ *0 then Code.abort* (*STR ''powr-real with nonpositive base''*) (λ-. *powr-real b i*)

else if $\lfloor i \rfloor = i$ *then* (*if* $0 \leq i$ *then* $b \mathbin{\hat{}} nat \lfloor i \rfloor$ *else* $1 \mathbin{/} b \mathbin{\hat{}} nat \lfloor - i \rfloor$)
else Code.abort (*STR ''powr-real with non−integer exponent''*) (λ-. *powr-real b i*))
for *b i* :: *real*
⟨*proof*⟩

lemma *powr-one*: $0 \leq x \Longrightarrow x \; powr \; 1 = x$
for *x* :: *real*
⟨*proof*⟩

lemma *powr-neg-one*: $0 < x \Longrightarrow x \; powr - 1 = 1 \mathbin{/} x$
for *x* :: *real*
⟨*proof*⟩

lemma *powr-neg-numeral*: $0 < x \Longrightarrow x \; powr - numeral \; n = 1 \mathbin{/} x \mathbin{\hat{}} numeral \; n$
for *x* :: *real*
⟨*proof*⟩

lemma *root-powr-inverse*: $0 < n \Longrightarrow 0 < x \Longrightarrow root \; n \; x = x \; powr \; (1/n)$
⟨*proof*⟩

lemma *ln-powr*: $x \neq 0 \Longrightarrow ln \; (x \; powr \; y) = y * ln \; x$
for *x* :: *real*
⟨*proof*⟩

lemma *ln-root*: $n > 0 \Longrightarrow b > 0 \Longrightarrow ln \; (root \; n \; b) = \; ln \; b \mathbin{/} n$
⟨*proof*⟩

lemma *ln-sqrt*: $0 < x \Longrightarrow ln \; (sqrt \; x) = ln \; x \mathbin{/} 2$
⟨*proof*⟩

lemma *log-root*: $n > 0 \Longrightarrow a > 0 \Longrightarrow log \; b \; (root \; n \; a) = \; log \; b \; a \mathbin{/} n$
⟨*proof*⟩

lemma *log-powr*: $x \neq 0 \Longrightarrow log \; b \; (x \; powr \; y) = y * log \; b \; x$
⟨*proof*⟩

lemma *log-nat-power*: $0 < x \Longrightarrow log \; b \; (x \mathbin{\hat{}} n) = real \; n * log \; b \; x$
⟨*proof*⟩

lemma *log-of-power-eq*:
assumes $m = b \mathbin{\hat{}} n \; b > 1$
shows $n = log \; b \; (real \; m)$
⟨*proof*⟩

lemma *log2-of-power-eq*: $m = 2 \mathbin{\hat{}} n \Longrightarrow n = log \; 2 \; m$ **for** *m n* :: *nat*
⟨*proof*⟩

lemma *log-base-change*: $0 < a \Longrightarrow a \neq 1 \Longrightarrow log\ b\ x = log\ a\ x\ /\ log\ a\ b$
⟨*proof*⟩

lemma *log-base-pow*: $0 < a \Longrightarrow log\ (a\ \hat{}\ n)\ x = log\ a\ x\ /\ n$
⟨*proof*⟩

lemma *log-base-powr*: $a \neq 0 \Longrightarrow log\ (a\ powr\ b)\ x = log\ a\ x\ /\ b$
⟨*proof*⟩

lemma *log-base-root*: $n > 0 \Longrightarrow b > 0 \Longrightarrow log\ (root\ n\ b)\ x = n * (log\ b\ x)$
⟨*proof*⟩

lemma *ln-bound*: $0 < x \Longrightarrow ln\ x \leq x$ **for** $x :: real$
⟨*proof*⟩

lemma *powr-mono*:
fixes $x :: real$
assumes $a \leq b$ **and** $1 \leq x$ **shows** $x\ powr\ a \leq x\ powr\ b$
⟨*proof*⟩

lemma *ge-one-powr-ge-zero*: $1 \leq x \Longrightarrow 0 \leq a \Longrightarrow 1 \leq x\ powr\ a$
for $x :: real$
⟨*proof*⟩

lemma *powr-less-mono2*: $0 < a \Longrightarrow 0 \leq x \Longrightarrow x < y \Longrightarrow x\ powr\ a < y\ powr\ a$
for $x :: real$
⟨*proof*⟩

lemma *powr-less-mono2-neg*: $a < 0 \Longrightarrow 0 < x \Longrightarrow x < y \Longrightarrow y\ powr\ a < x\ powr\ a$
for $x :: real$
⟨*proof*⟩

lemma *powr-mono2*: $x\ powr\ a \leq y\ powr\ a$ **if** $0 \leq a\ 0 \leq x\ x \leq y$
for $x :: real$
⟨*proof*⟩

lemma *powr-le1*: $0 \leq a \Longrightarrow 0 \leq x \Longrightarrow x \leq 1 \Longrightarrow x\ powr\ a \leq 1$
for $x :: real$
⟨*proof*⟩

lemma *powr-mono2'*:
fixes $a\ x\ y :: real$
assumes $a \leq 0\ x > 0\ x \leq y$
shows $x\ powr\ a \geq y\ powr\ a$
⟨*proof*⟩

lemma *powr-mono-both*:
fixes $x :: real$

assumes $0 \leq a$ $a \leq b$ $1 \leq x$ $x \leq y$
shows $x\ powr\ a \leq y\ powr\ b$
⟨*proof*⟩

lemma *powr-inj*: $0 < a \Longrightarrow a \neq 1 \Longrightarrow a\ powr\ x = a\ powr\ y \longleftrightarrow x = y$
for $x :: real$
⟨*proof*⟩

lemma *powr-half-sqrt*: $0 \leq x \Longrightarrow x\ powr\ (1/2) = sqrt\ x$
⟨*proof*⟩

lemma *square-powr-half* [*simp*]:
fixes x::*real* **shows** $x^2\ powr\ (1/2) = |x|$
⟨*proof*⟩

lemma *ln-powr-bound*: $1 \leq x \Longrightarrow 0 < a \Longrightarrow ln\ x \leq (x\ powr\ a)\ /\ a$
for $x :: real$
⟨*proof*⟩

lemma *ln-powr-bound2*:
fixes $x :: real$
assumes $1 < x$ **and** $0 < a$
shows $(ln\ x)\ powr\ a \leq (a\ powr\ a) * x$
⟨*proof*⟩

lemma *tendsto-powr*:
fixes $a\ b :: real$
assumes f: $(f \longrightarrow a)\ F$
and g: $(g \longrightarrow b)\ F$
and a: $a \neq 0$
shows $((\lambda x.\ f\ x\ powr\ g\ x) \longrightarrow a\ powr\ b)\ F$
⟨*proof*⟩

lemma *tendsto-powr'*[*tendsto-intros*]:
fixes $a :: real$
assumes f: $(f \longrightarrow a)\ F$
and g: $(g \longrightarrow b)\ F$
and a: $a \neq 0 \vee (b > 0 \wedge eventually\ (\lambda x.\ f\ x \geq 0)\ F)$
shows $((\lambda x.\ f\ x\ powr\ g\ x) \longrightarrow a\ powr\ b)\ F$
⟨*proof*⟩

lemma *continuous-powr*:
assumes *continuous F f*
and *continuous F g*
and $f\ (Lim\ F\ (\lambda x.\ x)) \neq 0$
shows *continuous F* $(\lambda x.\ (f\ x)\ powr\ (g\ x :: real))$
⟨*proof*⟩

lemma *continuous-at-within-powr*[*continuous-intros*]:

fixes $f\ g :: - \Rightarrow real$
assumes *continuous* $(at\ a\ within\ s)\ f$
and *continuous* $(at\ a\ within\ s)\ g$
and $f\ a \neq 0$
shows *continuous* $(at\ a\ within\ s)\ (\lambda x.\ (f\ x)\ powr\ (g\ x))$
⟨*proof*⟩

lemma *isCont-powr*[*continuous-intros*, *simp*]:
fixes $f\ g :: - \Rightarrow real$
assumes *isCont* $f\ a$ *isCont* $g\ a$ $f\ a \neq 0$
shows *isCont* $(\lambda x.\ (f\ x)\ powr\ g\ x)\ a$
⟨*proof*⟩

lemma *continuous-on-powr*[*continuous-intros*]:
fixes $f\ g :: - \Rightarrow real$
assumes *continuous-on* $s\ f$ *continuous-on* $s\ g$ **and** $\forall x{\in}s.\ f\ x \neq 0$
shows *continuous-on* $s\ (\lambda x.\ (f\ x)\ powr\ (g\ x))$
⟨*proof*⟩

lemma *tendsto-powr2*:
fixes $a :: real$
assumes f: $(f \longrightarrow a)\ F$
and g: $(g \longrightarrow b)\ F$
and $\forall_F\ x\ in\ F.\ 0 \leq f\ x$
and b: $0 < b$
shows $((\lambda x.\ f\ x\ powr\ g\ x) \longrightarrow a\ powr\ b)\ F$
⟨*proof*⟩

lemma *has-derivative-powr*[*derivative-intros*]:
assumes g[*derivative-intros*]: $(g\ has\text{-}derivative\ g')\ (at\ x\ within\ X)$
and f[*derivative-intros*]:$(f\ has\text{-}derivative\ f')\ (at\ x\ within\ X)$
assumes *pos*: $0 < g\ x$ **and** $x \in X$
shows $((\lambda x.\ g\ x\ powr\ f\ x{::}real)\ has\text{-}derivative\ (\lambda h.\ (g\ x\ powr\ f\ x) * (f'\ h * ln\ (g\ x) + g'\ h * f\ x\ /\ g\ x)))\ (at\ x\ within\ X)$
⟨*proof*⟩

lemma *DERIV-powr*:
fixes $r :: real$
assumes g: $DERIV\ g\ x :> m$
and *pos*: $g\ x > 0$
and f: $DERIV\ f\ x :> r$
shows $DERIV\ (\lambda x.\ g\ x\ powr\ f\ x)\ x :> (g\ x\ powr\ f\ x) * (r * ln\ (g\ x) + m * f\ x\ /\ g\ x)$
⟨*proof*⟩

lemma *DERIV-fun-powr*:
fixes $r :: real$
assumes g: $DERIV\ g\ x :> m$
and *pos*: $g\ x > 0$

shows *DERIV* $(\lambda x.\ (g\ x)\ powr\ r)\ x :> r * (g\ x)\ powr\ (r - of\text{-}nat\ 1) * m$
⟨*proof*⟩

lemma *has-real-derivative-powr*:
assumes $z > 0$
shows $((\lambda z.\ z\ powr\ r)\ has\text{-}real\text{-}derivative\ r * z\ powr\ (r - 1))\ (at\ z)$
⟨*proof*⟩

declare *has-real-derivative-powr*[*THEN DERIV-chain2*, *derivative-intros*]

lemma *tendsto-zero-powrI*:
assumes $(f \longrightarrow (0::real))\ F\ (g \longrightarrow b)\ F\ \forall_F\ x\ in\ F.\ 0 \le f\ x\ 0 < b$
shows $((\lambda x.\ f\ x\ powr\ g\ x) \longrightarrow 0)\ F$
⟨*proof*⟩

lemma *continuous-on-powr'*:
fixes $f\ g :: - \Rightarrow real$
assumes *continuous-on s f continuous-on s g*
and $\forall x{\in}s.\ f\ x \ge 0 \wedge (f\ x = 0 \longrightarrow g\ x > 0)$
shows *continuous-on s* $(\lambda x.\ (f\ x)\ powr\ (g\ x))$
⟨*proof*⟩

lemma *tendsto-neg-powr*:
assumes $s < 0$
and *f*: *LIM x F. f x :> at-top*
shows $((\lambda x.\ f\ x\ powr\ s) \longrightarrow (0::real))\ F$
⟨*proof*⟩

lemma *tendsto-exp-limit-at-right*: $((\lambda y.\ (1 + x * y)\ powr\ (1\ /\ y)) \longrightarrow exp\ x)\ (at\text{-}right\ 0)$
for $x :: real$
⟨*proof*⟩

lemma *tendsto-exp-limit-at-top*: $((\lambda y.\ (1 + x\ /\ y)\ powr\ y) \longrightarrow exp\ x)\ at\text{-}top$
for $x :: real$
⟨*proof*⟩

lemma *tendsto-exp-limit-sequentially*: $(\lambda n.\ (1 + x\ /\ n)\ \hat{}\ n) \longrightarrow exp\ x$
for $x :: real$
⟨*proof*⟩

113.10 Sine and Cosine

definition *sin-coeff* :: $nat \Rightarrow real$
where $sin\text{-}coeff = (\lambda n.\ if\ even\ n\ then\ 0\ else\ (-1)\ \hat{}\ ((n - Suc\ 0)\ div\ 2)\ /\ (fact\ n))$

definition *cos-coeff* :: $nat \Rightarrow real$
where $cos\text{-}coeff = (\lambda n.\ if\ even\ n\ then\ ((-1)\ \hat{}\ (n\ div\ 2))\ /\ (fact\ n)\ else\ 0)$

definition *sin* :: *'a* ⇒ *'a*::{*real-normed-algebra-1*,*banach*}
  where *sin* = (λ*x*. ∑ *n*. *sin-coeff n* ∗$_R$ *x*^*n*)

definition *cos* :: *'a* ⇒ *'a*::{*real-normed-algebra-1*,*banach*}
  where *cos* = (λ*x*. ∑ *n*. *cos-coeff n* ∗$_R$ *x*^*n*)

lemma *sin-coeff-0* [*simp*]: *sin-coeff 0* = *0*
  ⟨*proof*⟩

lemma *cos-coeff-0* [*simp*]: *cos-coeff 0* = *1*
  ⟨*proof*⟩

lemma *sin-coeff-Suc*: *sin-coeff* (*Suc n*) = *cos-coeff n* / *real* (*Suc n*)
  ⟨*proof*⟩

lemma *cos-coeff-Suc*: *cos-coeff* (*Suc n*) = − *sin-coeff n* / *real* (*Suc n*)
  ⟨*proof*⟩

lemma *summable-norm-sin*: *summable* (λ*n*. *norm* (*sin-coeff n* ∗$_R$ *x*^*n*))
  for *x* :: *'a*::{*real-normed-algebra-1*,*banach*}
⟨*proof*⟩

lemma *summable-norm-cos*: *summable* (λ*n*. *norm* (*cos-coeff n* ∗$_R$ *x*^*n*))
  for *x* :: *'a*::{*real-normed-algebra-1*,*banach*}
⟨*proof*⟩

lemma *sin-converges*: (λ*n*. *sin-coeff n* ∗$_R$ *x*^*n*) *sums sin x*
  ⟨*proof*⟩

lemma *cos-converges*: (λ*n*. *cos-coeff n* ∗$_R$ *x*^*n*) *sums cos x*
  ⟨*proof*⟩

lemma *sin-of-real*: *sin* (*of-real x*) = *of-real* (*sin x*)
  for *x* :: *real*
⟨*proof*⟩

corollary *sin-in-Reals* [*simp*]: *z* ∈ ℝ ⟹ *sin z* ∈ ℝ
  ⟨*proof*⟩

lemma *cos-of-real*: *cos* (*of-real x*) = *of-real* (*cos x*)
  for *x* :: *real*
⟨*proof*⟩

corollary *cos-in-Reals* [*simp*]: *z* ∈ ℝ ⟹ *cos z* ∈ ℝ
  ⟨*proof*⟩

lemma *diffs-sin-coeff*: *diffs sin-coeff* = *cos-coeff*

⟨*proof*⟩

lemma *diffs-cos-coeff*: *diffs cos-coeff* = (λ*n*. − *sin-coeff n*)
⟨*proof*⟩

lemma *sin-int-times-real*: *sin* (*of-int m* ∗ *of-real x*) = *of-real* (*sin* (*of-int m* ∗ *x*))
⟨*proof*⟩

lemma *cos-int-times-real*: *cos* (*of-int m* ∗ *of-real x*) = *of-real* (*cos* (*of-int m* ∗ *x*))
⟨*proof*⟩

Now at last we can get the derivatives of exp, sin and cos.

lemma *DERIV-sin* [*simp*]: *DERIV sin x* :> *cos x*
for *x* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

declare *DERIV-sin*[*THEN DERIV-chain2*, *derivative-intros*]
and *DERIV-sin*[*THEN DERIV-chain2*, *unfolded has-field-derivative-def*, *derivative-intros*]

lemmas *has-derivative-sin*[*derivative-intros*] = *DERIV-sin*[*THEN DERIV-compose-FDERIV*]

lemma *DERIV-cos* [*simp*]: *DERIV cos x* :> − *sin x*
for *x* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

declare *DERIV-cos*[*THEN DERIV-chain2*, *derivative-intros*]
and *DERIV-cos*[*THEN DERIV-chain2*, *unfolded has-field-derivative-def*, *derivative-intros*]

lemmas *has-derivative-cos*[*derivative-intros*] = *DERIV-cos*[*THEN DERIV-compose-FDERIV*]

lemma *isCont-sin*: *isCont sin x*
for *x* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *continuous-on-sin-real*: *continuous-on* {*a*..*b*} *sin* **for** *a*::*real*
⟨*proof*⟩

lemma *isCont-cos*: *isCont cos x*
for *x* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *continuous-on-cos-real*: *continuous-on* {*a*..*b*} *cos* **for** *a*::*real*
⟨*proof*⟩

context
fixes *f* :: *'a*::*t2-space* ⇒ *'b*::{*real-normed-field*,*banach*}

begin

lemma *isCont-sin'* [*simp*]: *isCont f a* $\Longrightarrow$ *isCont* ($\lambda x.$ *sin* (*f x*)) *a*
⟨*proof*⟩

lemma *isCont-cos'* [*simp*]: *isCont f a* $\Longrightarrow$ *isCont* ($\lambda x.$ *cos* (*f x*)) *a*
⟨*proof*⟩

lemma *tendsto-sin* [*tendsto-intros*]: (*f* $\longrightarrow$ *a*) *F* $\Longrightarrow$ (($\lambda x.$ *sin* (*f x*)) $\longrightarrow$ *sin a*) *F*
⟨*proof*⟩

lemma *tendsto-cos* [*tendsto-intros*]: (*f* $\longrightarrow$ *a*) *F* $\Longrightarrow$ (($\lambda x.$ *cos* (*f x*)) $\longrightarrow$ *cos a*) *F*
⟨*proof*⟩

lemma *continuous-sin* [*continuous-intros*]: *continuous F f* $\Longrightarrow$ *continuous F* ($\lambda x.$ *sin* (*f x*))
⟨*proof*⟩

lemma *continuous-on-sin* [*continuous-intros*]: *continuous-on s f* $\Longrightarrow$ *continuous-on s* ($\lambda x.$ *sin* (*f x*))
⟨*proof*⟩

lemma *continuous-cos* [*continuous-intros*]: *continuous F f* $\Longrightarrow$ *continuous F* ($\lambda x.$ *cos* (*f x*))
⟨*proof*⟩

lemma *continuous-on-cos* [*continuous-intros*]: *continuous-on s f* $\Longrightarrow$ *continuous-on s* ($\lambda x.$ *cos* (*f x*))
⟨*proof*⟩

end

lemma *continuous-within-sin*: *continuous* (*at z within s*) *sin*
for *z* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *continuous-within-cos*: *continuous* (*at z within s*) *cos*
for *z* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

113.11 Properties of Sine and Cosine

lemma *sin-zero* [*simp*]: *sin 0 = 0*
⟨*proof*⟩

lemma *cos-zero* [*simp*]: *cos 0 = 1*
⟨*proof*⟩

lemma *DERIV-fun-sin*: *DERIV g x :> m ⟹ DERIV* (λ*x*. *sin* (*g x*)) *x :> cos* (*g x*) ∗ *m*
⟨*proof*⟩

lemma *DERIV-fun-cos*: *DERIV g x :> m ⟹ DERIV* (λ*x*. *cos*(*g x*)) *x :> − sin* (*g x*) ∗ *m*
⟨*proof*⟩

113.12 Deriving the Addition Formulas

The product of two cosine series.

lemma *cos-x-cos-y*:
fixes *x* :: *'a*::{*real-normed-field*,*banach*}
shows
(λ*p*. ∑ *n*≤*p*.
if even p ∧ *even n*
then ((−*1*) ^ (*p div 2*) ∗ (*p choose n*) / (*fact p*)) ∗$_R$ (*x*^*n*) ∗ *y*^(*p*−*n*) *else 0*)
sums (*cos x* ∗ *cos y*)
⟨*proof*⟩

The product of two sine series.

lemma *sin-x-sin-y*:
fixes *x* :: *'a*::{*real-normed-field*,*banach*}
shows
(λ*p*. ∑ *n*≤*p*.
if even p ∧ *odd n*
then − ((−*1*) ^ (*p div 2*) ∗ (*p choose n*) / (*fact p*)) ∗$_R$ (*x*^*n*) ∗ *y*^(*p*−*n*)
else 0)
sums (*sin x* ∗ *sin y*)
⟨*proof*⟩

lemma *sums-cos-x-plus-y*:
fixes *x* :: *'a*::{*real-normed-field*,*banach*}
shows
(λ*p*. ∑ *n*≤*p*.
if even p
then ((−*1*) ^ (*p div 2*) ∗ (*p choose n*) / (*fact p*)) ∗$_R$ (*x*^*n*) ∗ *y*^(*p*−*n*)
else 0)
sums cos (*x* + *y*)
⟨*proof*⟩

theorem *cos-add*:
fixes *x* :: *'a*::{*real-normed-field*,*banach*}
shows *cos* (*x* + *y*) = *cos x* ∗ *cos y* − *sin x* ∗ *sin y*
⟨*proof*⟩

lemma *sin-minus-converges*: (λ*n*. − (*sin-coeff n* ∗$_R$ (−*x*)^*n*)) *sums sin x*

⟨*proof*⟩

lemma *sin-minus* [*simp*]: *sin* (− *x*) = − *sin x*
for *x* :: *'a*::{*real-normed-algebra-1*,*banach*}
⟨*proof*⟩

lemma *cos-minus-converges*: (λ*n*. (*cos-coeff n* ∗$_R$ (−*x*)^*n*)) *sums cos x*
⟨*proof*⟩

lemma *cos-minus* [*simp*]: *cos* (−*x*) = *cos x*
for *x* :: *'a*::{*real-normed-algebra-1*,*banach*}
⟨*proof*⟩

lemma *cos-abs-real* [*simp*]: *cos* |*x* :: *real*| = *cos x*
⟨*proof*⟩

lemma *sin-cos-squared-add* [*simp*]: (*sin x*)^2 + (*cos x*)^2 = *1*
for *x* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *sin-cos-squared-add2* [*simp*]: (*cos x*)^2 + (*sin x*)^2 = *1*
for *x* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *sin-cos-squared-add3* [*simp*]: *cos x* ∗ *cos x* + *sin x* ∗ *sin x* = *1*
for *x* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *sin-squared-eq*: (*sin x*)^2 = *1* − (*cos x*)^2
for *x* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *cos-squared-eq*: (*cos x*)^2 = *1* − (*sin x*)^2
for *x* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *abs-sin-le-one* [*simp*]: |*sin x*| ≤ *1*
for *x* :: *real*
⟨*proof*⟩

lemma *sin-ge-minus-one* [*simp*]: − *1* ≤ *sin x*
for *x* :: *real*
⟨*proof*⟩

lemma *sin-le-one* [*simp*]: *sin x* ≤ *1*
for *x* :: *real*
⟨*proof*⟩

lemma *abs-cos-le-one* [*simp*]: |*cos x*| ≤ *1*

for x :: *real*
⟨*proof*⟩

lemma *cos-ge-minus-one* [*simp*]: $-1 \leq \cos x$
for x :: *real*
⟨*proof*⟩

lemma *cos-le-one* [*simp*]: $\cos x \leq 1$
for x :: *real*
⟨*proof*⟩

lemma *cos-diff*: $\cos(x - y) = \cos x * \cos y + \sin x * \sin y$
for x :: $'a$::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *cos-double*: $\cos(2*x) = (\cos x)^2 - (\sin x)^2$
for x :: $'a$::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *sin-cos-le1*: $|\sin x * \sin y + \cos x * \cos y| \leq 1$
for x :: *real*
⟨*proof*⟩

lemma *DERIV-fun-pow*: *DERIV* $g\ x :> m \Longrightarrow$ *DERIV* $(\lambda x.\ (g\ x)\ \hat{}\ n)\ x :>$ *real* $n * (g\ x)\ \hat{}\ (n - 1) * m$
⟨*proof*⟩

lemma *DERIV-fun-exp*: *DERIV* $g\ x :> m \Longrightarrow$ *DERIV* $(\lambda x.\ \exp\ (g\ x))\ x :> \exp\ (g\ x) * m$
⟨*proof*⟩

113.13 The Constant Pi

definition *pi* :: *real*
where $\pi = 2 * (\text{THE } x.\ 0 \leq x \wedge x \leq 2 \wedge \cos x = 0)$

Show that there's a least positive x with $\cos x = (0::'a)$; hence define pi.

lemma *sin-paired*: $(\lambda n.\ (-1)\ \hat{}\ n\ /\ (\text{fact}\ (2 * n + 1)) * x\ \hat{}\ (2 * n + 1))$ *sums* $\sin x$
for x :: *real*
⟨*proof*⟩

lemma *sin-gt-zero-02*:
fixes x :: *real*
assumes $0 < x$ **and** $x < 2$
shows $0 < \sin x$
⟨*proof*⟩

lemma *cos-double-less-one*: $0 < x \Longrightarrow x < 2 \Longrightarrow \cos\ (2 * x) < 1$

for x :: *real*
⟨*proof*⟩

lemma *cos-paired*: $(\lambda n.\ (-\ 1)\ \hat{}\ n\ /\ (fact\ (2 * n)) * x\ \hat{}\ (2 * n))$ *sums cos x*
for x :: *real*
⟨*proof*⟩

lemma *sum-pos-lt-pair*:
fixes f :: $nat \Rightarrow real$
assumes f: *summable f* **and** *fplus*: $\bigwedge d.\ 0 < f\ (k + (Suc(Suc\ 0) * d)) + f\ (k + ((Suc\ (Suc\ 0) * d) + 1))$
shows *sum f* $\{..<k\} <$ *suminf f*
⟨*proof*⟩

lemma *cos-two-less-zero* [*simp*]: *cos 2* $<$ $(0::real)$
⟨*proof*⟩

lemmas *cos-two-neq-zero* [*simp*] = *cos-two-less-zero* [*THEN less-imp-neq*]
lemmas *cos-two-le-zero* [*simp*] = *cos-two-less-zero* [*THEN order-less-imp-le*]

lemma *cos-is-zero*: $\exists!x::real.\ 0 \le x \wedge x \le 2 \wedge cos\ x = 0$
⟨*proof*⟩

lemma *pi-half*: $pi/2 = (THE\ x.\ 0 \le x \wedge x \le 2 \wedge cos\ x = 0)$
⟨*proof*⟩

lemma *cos-pi-half* [*simp*]: *cos* $(pi/2) = 0$
⟨*proof*⟩

lemma *cos-of-real-pi-half* [*simp*]: *cos* $((of\text{-}real\ pi/2) :: 'a) = 0$
if *SORT-CONSTRAINT*('a::{*real-field*,*banach*,*real-normed-algebra-1*})
⟨*proof*⟩

lemma *pi-half-gt-zero* [*simp*]: $0 < pi/2$
⟨*proof*⟩

lemmas *pi-half-neq-zero* [*simp*] = *pi-half-gt-zero* [*THEN less-imp-neq*, *symmetric*]
lemmas *pi-half-ge-zero* [*simp*] = *pi-half-gt-zero* [*THEN order-less-imp-le*]

lemma *pi-half-less-two* [*simp*]: $pi/2 < 2$
⟨*proof*⟩

lemmas *pi-half-neq-two* [*simp*] = *pi-half-less-two* [*THEN less-imp-neq*]
lemmas *pi-half-le-two* [*simp*] = *pi-half-less-two* [*THEN order-less-imp-le*]

lemma *pi-gt-zero* [*simp*]: $0 < pi$
⟨*proof*⟩

lemma *pi-ge-zero* [*simp*]: $0 \le pi$

⟨*proof*⟩

lemma *pi-neq-zero* [*simp*]: *pi* ≠ *0*
⟨*proof*⟩

lemma *pi-not-less-zero* [*simp*]: ¬ *pi* < *0*
⟨*proof*⟩

lemma *minus-pi-half-less-zero*: −(*pi*/*2*) < *0*
⟨*proof*⟩

lemma *m2pi-less-pi*: − (*2*∗*pi*) < *pi*
⟨*proof*⟩

lemma *sin-pi-half* [*simp*]: *sin*(*pi*/*2*) = *1*
⟨*proof*⟩

lemma *sin-of-real-pi-half* [*simp*]: *sin* ((*of-real pi*/*2*) :: *'a*) = *1*
if *SORT-CONSTRAINT*(*'a*::{*real-field*,*banach*,*real-normed-algebra-1*})
⟨*proof*⟩

lemma *sin-cos-eq*: *sin x* = *cos* (*of-real pi*/*2* − *x*)
for *x* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *minus-sin-cos-eq*: − *sin x* = *cos* (*x* + *of-real pi*/*2*)
for *x* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *cos-sin-eq*: *cos x* = *sin* (*of-real pi*/*2* − *x*)
for *x* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *sin-add*: *sin* (*x* + *y*) = *sin x* ∗ *cos y* + *cos x* ∗ *sin y*
for *x* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *sin-diff*: *sin* (*x* − *y*) = *sin x* ∗ *cos y* − *cos x* ∗ *sin y*
for *x* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *sin-double*: *sin*(*2* ∗ *x*) = *2* ∗ *sin x* ∗ *cos x*
for *x* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *cos-of-real-pi* [*simp*]: *cos* (*of-real pi*) = −*1*
⟨*proof*⟩

lemma *sin-of-real-pi* [*simp*]: *sin* (*of-real pi*) = *0*

⟨*proof*⟩

lemma *cos-pi* [*simp*]: *cos pi* = *−1*
⟨*proof*⟩

lemma *sin-pi* [*simp*]: *sin pi* = *0*
⟨*proof*⟩

lemma *sin-periodic-pi* [*simp*]: *sin* (*x* + *pi*) = *− sin x*
⟨*proof*⟩

lemma *sin-periodic-pi2* [*simp*]: *sin* (*pi* + *x*) = *− sin x*
⟨*proof*⟩

lemma *cos-periodic-pi* [*simp*]: *cos* (*x* + *pi*) = *− cos x*
⟨*proof*⟩

lemma *cos-periodic-pi2* [*simp*]: *cos* (*pi* + *x*) = *− cos x*
⟨*proof*⟩

lemma *sin-periodic* [*simp*]: *sin* (*x* + *2* ∗ *pi*) = *sin x*
⟨*proof*⟩

lemma *cos-periodic* [*simp*]: *cos* (*x* + *2* ∗ *pi*) = *cos x*
⟨*proof*⟩

lemma *cos-npi* [*simp*]: *cos* (*real n* ∗ *pi*) = (*− 1*) ^ *n*
⟨*proof*⟩

lemma *cos-npi2* [*simp*]: *cos* (*pi* ∗ *real n*) = (*− 1*) ^ *n*
⟨*proof*⟩

lemma *sin-npi* [*simp*]: *sin* (*real n* ∗ *pi*) = *0*
for *n* :: *nat*
⟨*proof*⟩

lemma *sin-npi2* [*simp*]: *sin* (*pi* ∗ *real n*) = *0*
for *n* :: *nat*
⟨*proof*⟩

lemma *cos-two-pi* [*simp*]: *cos* (*2* ∗ *pi*) = *1*
⟨*proof*⟩

lemma *sin-two-pi* [*simp*]: *sin* (*2* ∗ *pi*) = *0*
⟨*proof*⟩

context
fixes *w* :: *'a*::{*real-normed-field*,*banach*}

begin

lemma *sin-times-sin*: $sin\ w * sin\ z = (cos\ (w - z) - cos\ (w + z)) / 2$
⟨*proof*⟩

lemma *sin-times-cos*: $sin\ w * cos\ z = (sin\ (w + z) + sin\ (w - z)) / 2$
⟨*proof*⟩

lemma *cos-times-sin*: $cos\ w * sin\ z = (sin\ (w + z) - sin\ (w - z)) / 2$
⟨*proof*⟩

lemma *cos-times-cos*: $cos\ w * cos\ z = (cos\ (w - z) + cos\ (w + z)) / 2$
⟨*proof*⟩

lemma *cos-double-cos*: $cos\ (2 * w) = 2 * cos\ w\ \hat{}\ 2 - 1$
⟨*proof*⟩

lemma *cos-double-sin*: $cos\ (2 * w) = 1 - 2 * sin\ w\ \hat{}\ 2$
⟨*proof*⟩

end

lemma *sin-plus-sin*: $sin\ w + sin\ z = 2 * sin\ ((w + z) / 2) * cos\ ((w - z) / 2)$
for $w :: 'a::\{real\text{-}normed\text{-}field, banach\}$
⟨*proof*⟩

lemma *sin-diff-sin*: $sin\ w - sin\ z = 2 * sin\ ((w - z) / 2) * cos\ ((w + z) / 2)$
for $w :: 'a::\{real\text{-}normed\text{-}field, banach\}$
⟨*proof*⟩

lemma *cos-plus-cos*: $cos\ w + cos\ z = 2 * cos\ ((w + z) / 2) * cos\ ((w - z) / 2)$
for $w :: 'a::\{real\text{-}normed\text{-}field, banach, field\}$
⟨*proof*⟩

lemma *cos-diff-cos*: $cos\ w - cos\ z = 2 * sin\ ((w + z) / 2) * sin\ ((z - w) / 2)$
for $w :: 'a::\{real\text{-}normed\text{-}field, banach, field\}$
⟨*proof*⟩

lemma *sin-pi-minus* [*simp*]: $sin\ (pi - x) = sin\ x$
⟨*proof*⟩

lemma *cos-pi-minus* [*simp*]: $cos\ (pi - x) = -\ (cos\ x)$
⟨*proof*⟩

lemma *sin-minus-pi* [*simp*]: $sin\ (x - pi) = -\ (sin\ x)$
⟨*proof*⟩

lemma *cos-minus-pi* [*simp*]: $cos\ (x - pi) = -\ (cos\ x)$
⟨*proof*⟩

lemma *sin-2pi-minus* [*simp*]: *sin* (*2* ∗ *pi* − *x*) = − (*sin x*)
⟨*proof*⟩

lemma *cos-2pi-minus* [*simp*]: *cos* (*2* ∗ *pi* − *x*) = *cos x*
⟨*proof*⟩

lemma *sin-gt-zero2*: *0* < *x* ⟹ *x* < *pi*/*2* ⟹ *0* < *sin x*
⟨*proof*⟩

lemma *sin-less-zero*:
assumes − *pi*/*2* < *x* **and** *x* < *0*
shows *sin x* < *0*
⟨*proof*⟩

lemma *pi-less-4*: *pi* < *4*
⟨*proof*⟩

lemma *cos-gt-zero*: *0* < *x* ⟹ *x* < *pi*/*2* ⟹ *0* < *cos x*
⟨*proof*⟩

lemma *cos-gt-zero-pi*: −(*pi*/*2*) < *x* ⟹ *x* < *pi*/*2* ⟹ *0* < *cos x*
⟨*proof*⟩

lemma *cos-ge-zero*: −(*pi*/*2*) ≤ *x* ⟹ *x* ≤ *pi*/*2* ⟹ *0* ≤ *cos x*
⟨*proof*⟩

lemma *sin-gt-zero*: *0* < *x* ⟹ *x* < *pi* ⟹ *0* < *sin x*
⟨*proof*⟩

lemma *sin-lt-zero*: *pi* < *x* ⟹ *x* < *2* ∗ *pi* ⟹ *sin x* < *0*
⟨*proof*⟩

lemma *pi-ge-two*: *2* ≤ *pi*
⟨*proof*⟩

lemma *sin-ge-zero*: *0* ≤ *x* ⟹ *x* ≤ *pi* ⟹ *0* ≤ *sin x*
⟨*proof*⟩

lemma *sin-le-zero*: *pi* ≤ *x* ⟹ *x* < *2* ∗ *pi* ⟹ *sin x* ≤ *0*
⟨*proof*⟩

lemma *sin-pi-divide-n-ge-0* [*simp*]:
assumes *n* ≠ *0*
shows *0* ≤ *sin* (*pi*/*real n*)
⟨*proof*⟩

lemma *sin-pi-divide-n-gt-0*:
assumes *2* ≤ *n*

shows $0 < sin\ (pi/real\ n)$
⟨*proof*⟩

Proof resembles that of *cos-is-zero* but with *pi* for the upper bound

lemma *cos-total*:
assumes *y*: $-1 \leq y\ y \leq 1$
shows $\exists!x.\ 0 \leq x \wedge x \leq pi \wedge cos\ x = y$
⟨*proof*⟩

lemma *sin-total*:
assumes *y*: $-1 \leq y\ y \leq 1$
shows $\exists!x.\ -(pi/2) \leq x \wedge x \leq pi/2 \wedge sin\ x = y$
⟨*proof*⟩

lemma *cos-zero-lemma*:
assumes $0 \leq x\ cos\ x = 0$
shows $\exists\, n.\ odd\ n \wedge x = of\text{-}nat\ n * (pi/2)$
⟨*proof*⟩

lemma *sin-zero-lemma*:
assumes $0 \leq x\ sin\ x = 0$
shows $\exists\, n{::}nat.\ even\ n \wedge x = real\ n * (pi/2)$
⟨*proof*⟩

lemma *cos-zero-iff*:
$cos\ x = 0 \longleftrightarrow ((\exists\, n.\ odd\ n \wedge x = real\ n * (pi/2)) \vee (\exists\, n.\ odd\ n \wedge x = -(real\ n * (pi/2))))$
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *sin-zero-iff*:
$sin\ x = 0 \longleftrightarrow ((\exists\, n.\ even\ n \wedge x = real\ n * (pi/2)) \vee (\exists\, n.\ even\ n \wedge x = -(real\ n * (pi/2))))$
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *sin-zero-pi-iff*:
fixes *x*::*real*
assumes $|x| < pi$
shows $sin\ x = 0 \longleftrightarrow x = 0$
⟨*proof*⟩

lemma *cos-zero-iff-int*: $cos\ x = 0 \longleftrightarrow (\exists\, i.\ odd\ i \wedge x = of\text{-}int\ i * (pi/2))$
⟨*proof*⟩

lemma *sin-zero-iff-int*: $sin\ x = 0 \longleftrightarrow (\exists\, i.\ even\ i \wedge x = of\text{-}int\ i * (pi/2))$ (**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *sin-zero-iff-int2*: $sin\ x = 0 \longleftrightarrow (\exists\, i{::}int.\ x = of\text{-}int\ i * pi)$
⟨*proof*⟩

lemma *cos-zero-iff-int2*:
fixes $x{::}real$
shows $cos\ x = 0 \longleftrightarrow (\exists\, n{::}int.\ x = n * pi + \ pi/2)$
⟨*proof*⟩

lemma *sin-npi-int* [*simp*]: $sin\ (pi * of\text{-}int\ n) = 0$
⟨*proof*⟩

lemma *cos-monotone-0-pi*:
assumes $0 \leq y$ **and** $y < x$ **and** $x \leq pi$
shows $cos\ x < cos\ y$
⟨*proof*⟩

lemma *cos-monotone-0-pi-le*:
assumes $0 \leq y$ **and** $y \leq x$ **and** $x \leq pi$
shows $cos\ x \leq cos\ y$
⟨*proof*⟩

lemma *cos-monotone-minus-pi-0*:
assumes $-\ pi \leq y$ **and** $y < x$ **and** $x \leq 0$
shows $cos\ y < cos\ x$
⟨*proof*⟩

lemma *cos-monotone-minus-pi-0'*:
assumes $-\ pi \leq y$ **and** $y \leq x$ **and** $x \leq 0$
shows $cos\ y \leq cos\ x$
⟨*proof*⟩

lemma *sin-monotone-2pi*:
assumes $-\ (pi/2) \leq y$ **and** $y < x$ **and** $x \leq pi/2$
shows $sin\ y < sin\ x$
⟨*proof*⟩

lemma *sin-monotone-2pi-le*:
assumes $-\ (pi/2) \leq y$ **and** $y \leq x$ **and** $x \leq pi/2$
shows $sin\ y \leq sin\ x$
⟨*proof*⟩

lemma *sin-x-le-x*:
fixes $x :: real$
assumes $x \geq 0$
shows $sin\ x \leq x$
⟨*proof*⟩

lemma *sin-x-ge-neg-x*:
fixes $x :: real$

assumes *x*: $x \geq 0$
shows $sin\ x \geq -\ x$
⟨*proof*⟩

lemma *abs-sin-x-le-abs-x*: $|sin\ x| \leq |x|$
for *x* :: *real*
⟨*proof*⟩

113.14 More Corollaries about Sine and Cosine

lemma *sin-cos-npi* [*simp*]: $sin\ (real\ (Suc\ (2 * n)) * pi/2) = (-1)\ \hat{}\ n$
⟨*proof*⟩

lemma *cos-2npi* [*simp*]: $cos\ (2 * real\ n * pi) = 1$
for *n* :: *nat*
⟨*proof*⟩

lemma *cos-3over2-pi* [*simp*]: $cos\ (3/2*pi) = 0$
⟨*proof*⟩

lemma *sin-2npi* [*simp*]: $sin\ (2 * real\ n * pi) = 0$
for *n* :: *nat*
⟨*proof*⟩

lemma *sin-3over2-pi* [*simp*]: $sin\ (3/2*pi) = -\ 1$
⟨*proof*⟩

lemma *cos-pi-eq-zero* [*simp*]: $cos\ (pi * real\ (Suc\ (2 * m))\ /\ 2) = 0$
⟨*proof*⟩

lemma *DERIV-cos-add* [*simp*]: $DERIV\ (\lambda x.\ cos\ (x + k))\ xa :> -\ sin\ (xa + k)$
⟨*proof*⟩

lemma *sin-zero-norm-cos-one*:
fixes *x* :: *'a*::{*real-normed-field*,*banach*}
assumes $sin\ x = 0$
shows $norm\ (cos\ x) = 1$
⟨*proof*⟩

lemma *sin-zero-abs-cos-one*: $sin\ x = 0 \Longrightarrow |cos\ x| = (1::real)$
⟨*proof*⟩

lemma *cos-one-sin-zero*:
fixes *x* :: *'a*::{*real-normed-field*,*banach*}
assumes $cos\ x = 1$
shows $sin\ x = 0$
⟨*proof*⟩

lemma *sin-times-pi-eq-0*: $sin\ (x * pi) = 0 \longleftrightarrow x \in \mathbb{Z}$

⟨*proof*⟩

lemma *cos-one-2pi*: *cos x = 1 ⟷ (∃ n::nat. x = n * 2 * pi) ∨ (∃ n::nat. x = − (n * 2 * pi))*
(is *?lhs = ?rhs*)
⟨*proof*⟩

lemma *cos-one-2pi-int*: *cos x = 1 ⟷ (∃ n::int. x = n * 2 * pi)* **(is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *cos-npi-int* [*simp*]:
fixes *n::int* **shows** *cos (pi * of-int n) = (if even n then 1 else −1)*
⟨*proof*⟩

lemma *sin-cos-sqrt*: *0 ≤ sin x ⟹ sin x = sqrt (1 − (cos(x) ^ 2))*
⟨*proof*⟩

lemma *sin-eq-0-pi*: *− pi < x ⟹ x < pi ⟹ sin x = 0 ⟹ x = 0*
⟨*proof*⟩

lemma *cos-treble-cos*: *cos (3 * x) = 4 * cos x ^ 3 − 3 * cos x*
for *x :: 'a::{real-normed-field,banach}*
⟨*proof*⟩

lemma *cos-45*: *cos (pi/4) = sqrt 2 / 2*
⟨*proof*⟩

lemma *cos-30*: *cos (pi/6) = sqrt 3/2*
⟨*proof*⟩

lemma *sin-45*: *sin (pi/4) = sqrt 2 / 2*
⟨*proof*⟩

lemma *sin-60*: *sin (pi/3) = sqrt 3/2*
⟨*proof*⟩

lemma *cos-60*: *cos (pi/3) = 1/2*
⟨*proof*⟩

lemma *sin-30*: *sin (pi/6) = 1/2*
⟨*proof*⟩

lemma *cos-120*: *cos (2 * pi/3) = −1/2*
and *sin-120*: *sin (2 * pi/3) = sqrt 3 / 2*
⟨*proof*⟩

lemma *cos-120'*: *cos (pi * 2 / 3) = −1/2*
⟨*proof*⟩

lemma *sin-120'*: *sin* (*pi* ∗ *2* / *3*) = *sqrt 3* / *2*
⟨*proof*⟩

lemma *cos-integer-2pi*: *n* ∈ ℤ ⟹ *cos*(*2* ∗ *pi* ∗ *n*) = *1*
⟨*proof*⟩

lemma *sin-integer-2pi*: *n* ∈ ℤ ⟹ *sin*(*2* ∗ *pi* ∗ *n*) = *0*
⟨*proof*⟩

lemma *cos-int-2pin* [*simp*]: *cos* ((*2* ∗ *pi*) ∗ *of-int n*) = *1*
⟨*proof*⟩

lemma *sin-int-2pin* [*simp*]: *sin* ((*2* ∗ *pi*) ∗ *of-int n*) = *0*
⟨*proof*⟩

lemma *sincos-principal-value*: ∃ *y*. (− *pi* < *y* ∧ *y* ≤ *pi*) ∧ (*sin y* = *sin x* ∧ *cos y* = *cos x*)
⟨*proof*⟩

113.15 Tangent

definition *tan* :: *'a* ⇒ *'a*::{*real-normed-field*,*banach*}
where *tan* = (λ*x*. *sin x* / *cos x*)

lemma *tan-of-real*: *of-real* (*tan x*) = (*tan* (*of-real x*) :: *'a*::{*real-normed-field*,*banach*})
⟨*proof*⟩

lemma *tan-in-Reals* [*simp*]: *z* ∈ ℝ ⟹ *tan z* ∈ ℝ
for *z* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *tan-zero* [*simp*]: *tan 0* = *0*
⟨*proof*⟩

lemma *tan-pi* [*simp*]: *tan pi* = *0*
⟨*proof*⟩

lemma *tan-npi* [*simp*]: *tan* (*real n* ∗ *pi*) = *0*
for *n* :: *nat*
⟨*proof*⟩

lemma *tan-pi-half* [*simp*]: *tan* (*pi* / *2*) = *0*
⟨*proof*⟩

lemma *tan-minus* [*simp*]: *tan* (− *x*) = − *tan x*
⟨*proof*⟩

lemma *tan-periodic* [*simp*]: *tan* (*x* + *2* ∗ *pi*) = *tan x*
⟨*proof*⟩

lemma *lemma-tan-add1*: $cos\ x \neq 0 \Longrightarrow cos\ y \neq 0 \Longrightarrow 1 - tan\ x * tan\ y = cos\ (x + y)/(cos\ x * cos\ y)$
⟨*proof*⟩

lemma *add-tan-eq*: $cos\ x \neq 0 \Longrightarrow cos\ y \neq 0 \Longrightarrow tan\ x + tan\ y = sin(x + y)/(cos\ x * cos\ y)$
for $x :: 'a::\{real\text{-}normed\text{-}field,banach\}$
⟨*proof*⟩

lemma *tan-eq-0-cos-sin*: $tan\ x = 0 \longleftrightarrow cos\ x = 0 \vee sin\ x = 0$
⟨*proof*⟩

Note: half of these zeros would normally be regarded as undefined cases.

lemma *tan-eq-0-Ex*:
assumes $tan\ x = 0$
obtains $k::int$ **where** $x = (k/2) * pi$
⟨*proof*⟩

lemma *tan-add*:
$cos\ x \neq 0 \Longrightarrow cos\ y \neq 0 \Longrightarrow cos\ (x + y) \neq 0 \Longrightarrow tan\ (x + y) = (tan\ x + tan\ y)/(1 - tan\ x * tan\ y)$
for $x :: 'a::\{real\text{-}normed\text{-}field,banach\}$
⟨*proof*⟩

lemma *tan-double*: $cos\ x \neq 0 \Longrightarrow cos\ (2 * x) \neq 0 \Longrightarrow tan\ (2 * x) = (2 * tan\ x) / (1 - (tan\ x)^2)$
for $x :: 'a::\{real\text{-}normed\text{-}field,banach\}$
⟨*proof*⟩

lemma *tan-gt-zero*: $0 < x \Longrightarrow x < pi/2 \Longrightarrow 0 < tan\ x$
⟨*proof*⟩

lemma *tan-less-zero*:
assumes $-\ pi/2 < x$ **and** $x < 0$
shows $tan\ x < 0$
⟨*proof*⟩

lemma *tan-half*: $tan\ x = sin\ (2 * x) / (cos\ (2 * x) + 1)$
for $x :: 'a::\{real\text{-}normed\text{-}field,banach,field\}$
⟨*proof*⟩

lemma *tan-30*: $tan\ (pi/6) = 1 / sqrt\ 3$
⟨*proof*⟩

lemma *tan-45*: $tan\ (pi/4) = 1$
⟨*proof*⟩

lemma *tan-60*: $tan\ (pi/3) = sqrt\ 3$

⟨*proof*⟩

lemma *DERIV-tan* [*simp*]: *cos x* ≠ *0* ⟹ *DERIV tan x* :> *inverse* ((*cos x*)²)
for *x* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

declare *DERIV-tan*[*THEN DERIV-chain2*, *derivative-intros*]
and *DERIV-tan*[*THEN DERIV-chain2*, *unfolded has-field-derivative-def*, *derivative-intros*]

lemmas *has-derivative-tan*[*derivative-intros*] = *DERIV-tan*[*THEN DERIV-compose-FDERIV*]

lemma *isCont-tan*: *cos x* ≠ *0* ⟹ *isCont tan x*
for *x* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *isCont-tan'* [*simp*,*continuous-intros*]:
fixes *a* :: *'a*::{*real-normed-field*,*banach*} **and** *f* :: *'a* ⇒ *'a*
shows *isCont f a* ⟹ *cos* (*f a*) ≠ *0* ⟹ *isCont* (λ*x*. *tan* (*f x*)) *a*
⟨*proof*⟩

lemma *tendsto-tan* [*tendsto-intros*]:
fixes *f* :: *'a* ⇒ *'a*::{*real-normed-field*,*banach*}
shows (*f* ⟶ *a*) *F* ⟹ *cos a* ≠ *0* ⟹ ((λ*x*. *tan* (*f x*)) ⟶ *tan a*) *F*
⟨*proof*⟩

lemma *continuous-tan*:
fixes *f* :: *'a* ⇒ *'a*::{*real-normed-field*,*banach*}
shows *continuous F f* ⟹ *cos* (*f* (*Lim F* (λ*x*. *x*))) ≠ *0* ⟹ *continuous F* (λ*x*. *tan* (*f x*))
⟨*proof*⟩

lemma *continuous-on-tan* [*continuous-intros*]:
fixes *f* :: *'a* ⇒ *'a*::{*real-normed-field*,*banach*}
shows *continuous-on s f* ⟹ (∀*x*∈*s*. *cos* (*f x*) ≠ *0*) ⟹ *continuous-on s* (λ*x*. *tan* (*f x*))
⟨*proof*⟩

lemma *continuous-within-tan* [*continuous-intros*]:
fixes *f* :: *'a* ⇒ *'a*::{*real-normed-field*,*banach*}
shows *continuous* (*at x within s*) *f* ⟹
cos (*f x*) ≠ *0* ⟹ *continuous* (*at x within s*) (λ*x*. *tan* (*f x*))
⟨*proof*⟩

lemma *LIM-cos-div-sin*: (λ*x*. *cos*(*x*)/*sin*(*x*)) −*pi*/*2*→ *0*
⟨*proof*⟩

lemma *lemma-tan-total*:
assumes *0* < *y* **shows** ∃*x*. *0* < *x* ∧ *x* < *pi*/*2* ∧ *y* < *tan x*

⟨*proof*⟩

lemma *tan-total-pos*:
assumes $0 \leq y$ **shows** $\exists x.\ 0 \leq x \wedge x < pi/2 \wedge tan\ x = y$
⟨*proof*⟩

lemma *lemma-tan-total1*: $\exists x.\ -(pi/2) < x \wedge x < (pi/2) \wedge tan\ x = y$
⟨*proof*⟩

proposition *tan-total*: $\exists!\ x.\ -(pi/2) < x \wedge x < (pi/2) \wedge tan\ x = y$
⟨*proof*⟩

lemma *tan-monotone*:
assumes $-\ (pi/2) < y$ **and** $y < x$ **and** $x < pi/2$
shows $tan\ y < tan\ x$
⟨*proof*⟩

lemma *tan-monotone'*:
assumes $-\ (pi/2) < y$
and $y < pi/2$
and $-\ (pi/2) < x$
and $x < pi/2$
shows $y < x \longleftrightarrow tan\ y < tan\ x$
⟨*proof*⟩

lemma *tan-inverse*: $1\ /\ (tan\ y) = tan\ (pi/2 - y)$
⟨*proof*⟩

lemma *tan-periodic-pi*[*simp*]: $tan\ (x + pi) = tan\ x$
⟨*proof*⟩

lemma *tan-periodic-nat*[*simp*]: $tan\ (x + real\ n * pi) = tan\ x$
⟨*proof*⟩

lemma *tan-periodic-int*[*simp*]: $tan\ (x + of\text{-}int\ i * pi) = tan\ x$
⟨*proof*⟩

lemma *tan-periodic-n*[*simp*]: $tan\ (x + numeral\ n * pi) = tan\ x$
⟨*proof*⟩

lemma *tan-minus-45* [*simp*]: $tan\ (-(pi/4)) = -1$
⟨*proof*⟩

lemma *tan-diff*:
$cos\ x \neq 0 \Longrightarrow cos\ y \neq 0 \Longrightarrow cos\ (x - y) \neq 0 \Longrightarrow tan\ (x - y) = (tan\ x - tan\ y)/(1 + tan\ x * tan\ y)$
for $x :: 'a$::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *tan-pos-pi2-le*: $0 \leq x \Longrightarrow x < pi/2 \Longrightarrow 0 \leq tan\ x$
⟨*proof*⟩

lemma *cos-tan*: $|x| < pi/2 \Longrightarrow cos\ x = 1\ /\ sqrt\ (1 + tan\ x\ \hat{}\ 2)$
⟨*proof*⟩

lemma *cos-tan-half*: $cos\ x \neq 0 \Longrightarrow\ cos\ (2{*}x) = (1 - (tan\ x)\hat{}2)\ /\ (1 + (tan\ x)\hat{}2)$
⟨*proof*⟩

lemma *sin-tan*: $|x| < pi/2 \Longrightarrow sin\ x = tan\ x\ /\ sqrt\ (1 + tan\ x\ \hat{}\ 2)$
⟨*proof*⟩

lemma *sin-tan-half*: $sin\ (2{*}x) = 2 * tan\ x\ /\ (1 + (tan\ x)\hat{}2)$
⟨*proof*⟩

lemma *tan-mono-le*: $-(pi/2) < x \Longrightarrow x \leq y \Longrightarrow y < pi/2 \Longrightarrow tan\ x \leq tan\ y$
⟨*proof*⟩

lemma *tan-mono-lt-eq*:
$-(pi/2) < x \Longrightarrow x < pi/2 \Longrightarrow -(pi/2) < y \Longrightarrow y < pi/2 \Longrightarrow tan\ x < tan\ y \longleftrightarrow x < y$
⟨*proof*⟩

lemma *tan-mono-le-eq*:
$-(pi/2) < x \Longrightarrow x < pi/2 \Longrightarrow -(pi/2) < y \Longrightarrow y < pi/2 \Longrightarrow tan\ x \leq tan\ y \longleftrightarrow x \leq y$
⟨*proof*⟩

lemma *tan-bound-pi2*: $|x| < pi/4 \Longrightarrow |tan\ x| < 1$
⟨*proof*⟩

lemma *tan-cot*: $tan(pi/2 - x) = inverse(tan\ x)$
⟨*proof*⟩

113.16 Cotangent

definition *cot* :: *'a* ⇒ *'a*::{*real-normed-field*,*banach*}
where $cot = (\lambda x.\ cos\ x\ /\ sin\ x)$

lemma *cot-of-real*: $of\text{-}real\ (cot\ x) = (cot\ (of\text{-}real\ x) :: 'a::\{real\text{-}normed\text{-}field,banach\})$
⟨*proof*⟩

lemma *cot-in-Reals* [*simp*]: $z \in \mathbb{R} \Longrightarrow cot\ z \in \mathbb{R}$
for *z* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *cot-zero* [*simp*]: $cot\ 0 = 0$
⟨*proof*⟩

lemma *cot-pi* [*simp*]: *cot pi = 0*
⟨*proof*⟩

lemma *cot-npi* [*simp*]: *cot (real n * pi) = 0*
for *n* :: *nat*
⟨*proof*⟩

lemma *cot-minus* [*simp*]: *cot (- x) = - cot x*
⟨*proof*⟩

lemma *cot-periodic* [*simp*]: *cot (x + 2 * pi) = cot x*
⟨*proof*⟩

lemma *cot-altdef*: *cot x = inverse (tan x)*
⟨*proof*⟩

lemma *tan-altdef*: *tan x = inverse (cot x)*
⟨*proof*⟩

lemma *tan-cot'*: *tan (pi/2 - x) = cot x*
⟨*proof*⟩

lemma *cot-gt-zero*: *0 < x ⟹ x < pi/2 ⟹ 0 < cot x*
⟨*proof*⟩

lemma *cot-less-zero*:
assumes *lb*: *- pi/2 < x* **and** *x < 0*
shows *cot x < 0*
⟨*proof*⟩

lemma *DERIV-cot* [*simp*]: *sin x ≠ 0 ⟹ DERIV cot x :> -inverse ((sin x)²)*
for *x* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *isCont-cot*: *sin x ≠ 0 ⟹ isCont cot x*
for *x* :: *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *isCont-cot'* [*simp*,*continuous-intros*]:
isCont f a ⟹ sin (f a) ≠ 0 ⟹ isCont (λx. cot (f x)) a
for *a* :: *'a*::{*real-normed-field*,*banach*} **and** *f* :: *'a ⇒ 'a*
⟨*proof*⟩

lemma *tendsto-cot* [*tendsto-intros*]: *(f ⟶ a) F ⟹ sin a ≠ 0 ⟹ ((λx. cot (f x)) ⟶ cot a) F*
for *f* :: *'a ⇒ 'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *continuous-cot*:
continuous F f $\Longrightarrow$ *sin* (*f* (*Lim F* (λ*x*. *x*))) $\neq$ *0* $\Longrightarrow$ *continuous F* (λ*x*. *cot* (*f x*))
for *f* :: *'a* $\Rightarrow$ *'a*::{*real-normed-field*,*banach*}
⟨*proof*⟩

lemma *continuous-on-cot* [*continuous-intros*]:
fixes *f* :: *'a* $\Rightarrow$ *'a*::{*real-normed-field*,*banach*}
shows *continuous-on s f* $\Longrightarrow$ (∀ *x*∈*s*. *sin* (*f x*) $\neq$ *0*) $\Longrightarrow$ *continuous-on s* (λ*x*. *cot* (*f x*))
⟨*proof*⟩

lemma *continuous-within-cot* [*continuous-intros*]:
fixes *f* :: *'a* $\Rightarrow$ *'a*::{*real-normed-field*,*banach*}
shows *continuous* (*at x within s*) *f* $\Longrightarrow$ *sin* (*f x*) $\neq$ *0* $\Longrightarrow$ *continuous* (*at x within s*) (λ*x*. *cot* (*f x*))
⟨*proof*⟩

113.17 Inverse Trigonometric Functions

definition *arcsin* :: *real* $\Rightarrow$ *real*
where *arcsin y* = (*THE x*. −(*pi*/*2*) $\leq$ *x* ∧ *x* $\leq$ *pi*/*2* ∧ *sin x* = *y*)

definition *arccos* :: *real* $\Rightarrow$ *real*
where *arccos y* = (*THE x*. *0* $\leq$ *x* ∧ *x* $\leq$ *pi* ∧ *cos x* = *y*)

definition *arctan* :: *real* $\Rightarrow$ *real*
where *arctan y* = (*THE x*. −(*pi*/*2*) < *x* ∧ *x* < *pi*/*2* ∧ *tan x* = *y*)

lemma *arcsin*: − *1* $\leq$ *y* $\Longrightarrow$ *y* $\leq$ *1* $\Longrightarrow$ − (*pi*/*2*) $\leq$ *arcsin y* ∧ *arcsin y* $\leq$ *pi*/*2* ∧ *sin* (*arcsin y*) = *y*
⟨*proof*⟩

lemma *arcsin-pi*: − *1* $\leq$ *y* $\Longrightarrow$ *y* $\leq$ *1* $\Longrightarrow$ − (*pi*/*2*) $\leq$ *arcsin y* ∧ *arcsin y* $\leq$ *pi* ∧ *sin* (*arcsin y*) = *y*
⟨*proof*⟩

lemma *sin-arcsin* [*simp*]: − *1* $\leq$ *y* $\Longrightarrow$ *y* $\leq$ *1* $\Longrightarrow$ *sin* (*arcsin y*) = *y*
⟨*proof*⟩

lemma *arcsin-bounded*: − *1* $\leq$ *y* $\Longrightarrow$ *y* $\leq$ *1* $\Longrightarrow$ − (*pi*/*2*) $\leq$ *arcsin y* ∧ *arcsin y* $\leq$ *pi*/*2*
⟨*proof*⟩

lemma *arcsin-lbound*: − *1* $\leq$ *y* $\Longrightarrow$ *y* $\leq$ *1* $\Longrightarrow$ − (*pi*/*2*) $\leq$ *arcsin y*
⟨*proof*⟩

lemma *arcsin-ubound*: − *1* $\leq$ *y* $\Longrightarrow$ *y* $\leq$ *1* $\Longrightarrow$ *arcsin y* $\leq$ *pi*/*2*
⟨*proof*⟩

lemma *arcsin-lt-bounded*:
assumes $-1 < y$ $y < 1$
shows $-(pi/2) < arcsin\ y \wedge arcsin\ y < pi/2$
⟨*proof*⟩

lemma *arcsin-sin*: $-(pi/2) \leq x \Longrightarrow x \leq pi/2 \Longrightarrow arcsin\ (sin\ x) = x$
⟨*proof*⟩

lemma *arcsin-unique*:
assumes $-pi/2 \leq x$ **and** $x \leq pi/2$ **and** $sin\ x = y$ **shows** $arcsin\ y = x$
⟨*proof*⟩

lemma *arcsin-0* [*simp*]: $arcsin\ 0 = 0$
⟨*proof*⟩

lemma *arcsin-1* [*simp*]: $arcsin\ 1 = pi/2$
⟨*proof*⟩

lemma *arcsin-minus-1* [*simp*]: $arcsin\ (-1) = -(pi/2)$
⟨*proof*⟩

lemma *arcsin-minus*: $-1 \leq x \Longrightarrow x \leq 1 \Longrightarrow arcsin\ (-x) = -\ arcsin\ x$
⟨*proof*⟩

lemma *arcsin-one-half* [*simp*]: $arcsin\ (1/2) = pi\ /\ 6$
and *arcsin-minus-one-half* [*simp*]: $arcsin\ (-(1/2)) = -pi\ /\ 6$
⟨*proof*⟩

lemma *arcsin-one-over-sqrt-2*: $arcsin\ (1\ /\ sqrt\ 2) = pi\ /\ 4$
⟨*proof*⟩

lemma *arcsin-eq-iff*: $|x| \leq 1 \Longrightarrow |y| \leq 1 \Longrightarrow arcsin\ x = arcsin\ y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *cos-arcsin-nonzero*: $-1 < x \Longrightarrow x < 1 \Longrightarrow cos\ (arcsin\ x) \neq 0$
⟨*proof*⟩

lemma *arccos*: $-1 \leq y \Longrightarrow y \leq 1 \Longrightarrow 0 \leq arccos\ y \wedge arccos\ y \leq pi \wedge cos\ (arccos\ y) = y$
⟨*proof*⟩

lemma *cos-arccos* [*simp*]: $-1 \leq y \Longrightarrow y \leq 1 \Longrightarrow cos\ (arccos\ y) = y$
⟨*proof*⟩

lemma *arccos-bounded*: $-1 \leq y \Longrightarrow y \leq 1 \Longrightarrow 0 \leq arccos\ y \wedge arccos\ y \leq pi$
⟨*proof*⟩

lemma *arccos-lbound*: $-1 \leq y \Longrightarrow y \leq 1 \Longrightarrow 0 \leq arccos\ y$
⟨*proof*⟩

lemma *arccos-ubound*: $-1 \leq y \Longrightarrow y \leq 1 \Longrightarrow \arccos\ y \leq pi$
⟨*proof*⟩

lemma *arccos-lt-bounded*:
assumes $-1 < y\ y < 1$
shows $0 < \arccos\ y \wedge \arccos\ y < pi$
⟨*proof*⟩

lemma *arccos-cos*: $0 \leq x \Longrightarrow x \leq pi \Longrightarrow \arccos\ (\cos\ x) = x$
⟨*proof*⟩

lemma *arccos-cos2*: $x \leq 0 \Longrightarrow -pi \leq x \Longrightarrow \arccos\ (\cos\ x) = -x$
⟨*proof*⟩

lemma *arccos-unique*:
assumes $0 \leq x$ **and** $x \leq pi$ **and** $\cos\ x = y$ **shows** $\arccos\ y = x$
⟨*proof*⟩

lemma *cos-arcsin*:
assumes $-1 \leq x\ x \leq 1$
shows $\cos\ (\arcsin\ x) = sqrt\ (1 - x^2)$
⟨*proof*⟩

lemma *sin-arccos*:
assumes $-1 \leq x\ x \leq 1$
shows $\sin\ (\arccos\ x) = sqrt\ (1 - x^2)$
⟨*proof*⟩

lemma *arccos-0* [*simp*]: $\arccos\ 0 = pi/2$
⟨*proof*⟩

lemma *arccos-1* [*simp*]: $\arccos\ 1 = 0$
⟨*proof*⟩

lemma *arccos-minus-1* [*simp*]: $\arccos\ (-1) = pi$
⟨*proof*⟩

lemma *arccos-minus*: $-1 \leq x \Longrightarrow x \leq 1 \Longrightarrow \arccos\ (-x) = pi - \arccos\ x$
⟨*proof*⟩

lemma *arccos-one-half* [*simp*]: $\arccos\ (1/2) = pi\ /\ 3$
and *arccos-minus-one-half* [*simp*]: $\arccos\ (-(1/2)) = 2 * pi\ /\ 3$
⟨*proof*⟩

lemma *arccos-one-over-sqrt-2*: $\arccos\ (1\ /\ sqrt\ 2) = pi\ /\ 4$
⟨*proof*⟩

corollary *arccos-minus-abs*:

assumes $|x| \leq 1$
shows $arccos\ (-\ x) = pi - arccos\ x$
⟨*proof*⟩

lemma *sin-arccos-nonzero*: $-\ 1 < x \Longrightarrow x < 1 \Longrightarrow sin\ (arccos\ x) \neq 0$
⟨*proof*⟩

lemma *arctan*: $-\ (pi/2) < arctan\ y \wedge arctan\ y < pi/2 \wedge tan\ (arctan\ y) = y$
⟨*proof*⟩

lemma *tan-arctan*: $tan\ (arctan\ y) = y$
⟨*proof*⟩

lemma *arctan-bounded*: $-\ (pi/2) < arctan\ y \wedge arctan\ y < pi/2$
⟨*proof*⟩

lemma *arctan-lbound*: $-\ (pi/2) < arctan\ y$
⟨*proof*⟩

lemma *arctan-ubound*: $arctan\ y < pi/2$
⟨*proof*⟩

lemma *arctan-unique*:
assumes $-(pi/2) < x$
and $x < pi/2$
and $tan\ x = y$
shows $arctan\ y = x$
⟨*proof*⟩

lemma *arctan-tan*: $-(pi/2) < x \Longrightarrow x < pi/2 \Longrightarrow arctan\ (tan\ x) = x$
⟨*proof*⟩

lemma *arctan-zero-zero* [*simp*]: $arctan\ 0 = 0$
⟨*proof*⟩

lemma *arctan-minus*: $arctan\ (-\ x) = -\ arctan\ x$
⟨*proof*⟩

lemma *cos-arctan-not-zero* [*simp*]: $cos\ (arctan\ x) \neq 0$
⟨*proof*⟩

lemma *tan-eq-arctan-Ex*:
shows $tan\ x = y \longleftrightarrow (\exists\, k{::}int.\ x = arctan\ y + k{*}pi \vee (x = pi/2 + k{*}pi \wedge y{=}0))$
⟨*proof*⟩

lemma *arctan-tan-eq-abs-pi*:
assumes $cos\ \vartheta \neq 0$
obtains k **where** $arctan\ (tan\ \vartheta) = \vartheta - of\text{-}int\ k * pi$

⟨*proof*⟩

lemma *tan-eq*:
assumes *tan x* = *tan y tan x* ≠ *0*
obtains *k*::*int* **where** *x* = *y* + *k* ∗ *pi*
⟨*proof*⟩

lemma *cos-arctan*: *cos* (*arctan x*) = *1* / *sqrt* (*1* + x^2)
⟨*proof*⟩

lemma *sin-arctan*: *sin* (*arctan x*) = *x* / *sqrt* (*1* + x^2)
⟨*proof*⟩

lemma *tan-sec*: *cos x* ≠ *0* ⟹ *1* + $(tan\ x)^2$ = $(inverse\ (cos\ x))^2$
for *x* :: *'a*::{*real-normed-field*,*banach*,*field*}
⟨*proof*⟩

lemma *arctan-less-iff*: *arctan x* < *arctan y* ⟷ *x* < *y*
⟨*proof*⟩

lemma *arctan-le-iff*: *arctan x* ≤ *arctan y* ⟷ *x* ≤ *y*
⟨*proof*⟩

lemma *arctan-eq-iff*: *arctan x* = *arctan y* ⟷ *x* = *y*
⟨*proof*⟩

lemma *zero-less-arctan-iff* [*simp*]: *0* < *arctan x* ⟷ *0* < *x*
⟨*proof*⟩

lemma *arctan-less-zero-iff* [*simp*]: *arctan x* < *0* ⟷ *x* < *0*
⟨*proof*⟩

lemma *zero-le-arctan-iff* [*simp*]: *0* ≤ *arctan x* ⟷ *0* ≤ *x*
⟨*proof*⟩

lemma *arctan-le-zero-iff* [*simp*]: *arctan x* ≤ *0* ⟷ *x* ≤ *0*
⟨*proof*⟩

lemma *arctan-eq-zero-iff* [*simp*]: *arctan x* = *0* ⟷ *x* = *0*
⟨*proof*⟩

lemma *continuous-on-arcsin'*: *continuous-on* {−*1* .. *1*} *arcsin*
⟨*proof*⟩

lemma *continuous-on-arcsin* [*continuous-intros*]:
continuous-on s f ⟹ (∀ *x*∈*s*. −*1* ≤ *f x* ∧ *f x* ≤ *1*) ⟹ *continuous-on s* (λ*x*. *arcsin* (*f x*))
⟨*proof*⟩

lemma *isCont-arcsin*: $-1 < x \Longrightarrow x < 1 \Longrightarrow$ *isCont arcsin x*
⟨*proof*⟩

lemma *continuous-on-arccos'*: *continuous-on* $\{-1 \mathinner{..} 1\}$ *arccos*
⟨*proof*⟩

lemma *continuous-on-arccos* [*continuous-intros*]:
continuous-on s f $\Longrightarrow$ $(\forall x{\in}s.\ -1 \leq f\ x \wedge f\ x \leq 1) \Longrightarrow$ *continuous-on s* $(\lambda x.$ *arccos* $(f\ x))$
⟨*proof*⟩

lemma *isCont-arccos*: $-1 < x \Longrightarrow x < 1 \Longrightarrow$ *isCont arccos x*
⟨*proof*⟩

lemma *isCont-arctan*: *isCont arctan x*
⟨*proof*⟩

lemma *tendsto-arctan* [*tendsto-intros*]: $(f \longrightarrow x)\ F \Longrightarrow ((\lambda x.$ *arctan* $(f\ x)) \longrightarrow$ *arctan x*$)\ F$
⟨*proof*⟩

lemma *continuous-arctan* [*continuous-intros*]: *continuous F f* $\Longrightarrow$ *continuous F* $(\lambda x.$ *arctan* $(f\ x))$
⟨*proof*⟩

lemma *continuous-on-arctan* [*continuous-intros*]:
continuous-on s f $\Longrightarrow$ *continuous-on s* $(\lambda x.$ *arctan* $(f\ x))$
⟨*proof*⟩

lemma *DERIV-arcsin*:
assumes $-1 < x\ x < 1$
shows *DERIV arcsin x :> inverse* $(sqrt\ (1 - x^2))$
⟨*proof*⟩

lemma *DERIV-arccos*:
assumes $-1 < x\ x < 1$
shows *DERIV arccos x :> inverse* $(-\ sqrt\ (1 - x^2))$
⟨*proof*⟩

lemma *DERIV-arctan*: *DERIV arctan x :> inverse* $(1 + x^2)$
⟨*proof*⟩

declare
DERIV-arcsin[*THEN DERIV-chain2*, *derivative-intros*]
DERIV-arcsin[*THEN DERIV-chain2*, *unfolded has-field-derivative-def*, *derivative-intros*]
DERIV-arccos[*THEN DERIV-chain2*, *derivative-intros*]
DERIV-arccos[*THEN DERIV-chain2*, *unfolded has-field-derivative-def*, *derivative-intros*]

DERIV-arctan[*THEN DERIV-chain2*, *derivative-intros*]
DERIV-arctan[*THEN DERIV-chain2*, *unfolded has-field-derivative-def*, *derivative-intros*]

lemmas *has-derivative-arctan*[*derivative-intros*] = *DERIV-arctan*[*THEN DERIV-compose-FDERIV*]
and *has-derivative-arccos*[*derivative-intros*] = *DERIV-arccos*[*THEN DERIV-compose-FDERIV*]
and *has-derivative-arcsin*[*derivative-intros*] = *DERIV-arcsin*[*THEN DERIV-compose-FDERIV*]

lemma *filterlim-tan-at-right*: *filterlim tan at-bot* (*at-right* (− (*pi*/2)))
⟨*proof*⟩

lemma *filterlim-tan-at-left*: *filterlim tan at-top* (*at-left* (*pi*/2))
⟨*proof*⟩

lemma *tendsto-arctan-at-top*: (*arctan* ⟶ (*pi*/2)) *at-top*
⟨*proof*⟩

lemma *tendsto-arctan-at-bot*: (*arctan* ⟶ − (*pi*/2)) *at-bot*
⟨*proof*⟩

113.18 Prove Totality of the Trigonometric Functions

lemma *cos-arccos-abs*: |*y*| ≤ 1 ⟹ *cos* (*arccos y*) = *y*
⟨*proof*⟩

lemma *sin-arccos-abs*: |*y*| ≤ 1 ⟹ *sin* (*arccos y*) = *sqrt* (1 − y^2)
⟨*proof*⟩

lemma *sin-mono-less-eq*:
− (*pi*/2) ≤ *x* ⟹ *x* ≤ *pi*/2 ⟹ − (*pi*/2) ≤ *y* ⟹ *y* ≤ *pi*/2 ⟹ *sin x* < *sin y* ⟷ *x* < *y*
⟨*proof*⟩

lemma *sin-mono-le-eq*:
− (*pi*/2) ≤ *x* ⟹ *x* ≤ *pi*/2 ⟹ − (*pi*/2) ≤ *y* ⟹ *y* ≤ *pi*/2 ⟹ *sin x* ≤ *sin y* ⟷ *x* ≤ *y*
⟨*proof*⟩

lemma *sin-inj-pi*:
− (*pi*/2) ≤ *x* ⟹ *x* ≤ *pi*/2 ⟹ − (*pi*/2) ≤ *y* ⟹ *y* ≤ *pi*/2 ⟹ *sin x* = *sin y* ⟹ *x* = *y*
⟨*proof*⟩

lemma *arcsin-le-iff*:
assumes *x* ≥ −1 *x* ≤ 1 *y* ≥ −*pi*/2 *y* ≤ *pi*/2
shows *arcsin x* ≤ *y* ⟷ *x* ≤ *sin y*
⟨*proof*⟩

lemma *le-arcsin-iff*:

assumes $x \geq -1$ $x \leq 1$ $y \geq -pi/2$ $y \leq pi/2$
shows $arcsin\ x \geq y \longleftrightarrow x \geq sin\ y$
⟨*proof*⟩

lemma *cos-mono-less-eq*: $0 \leq x \Longrightarrow x \leq pi \Longrightarrow 0 \leq y \Longrightarrow y \leq pi \Longrightarrow cos\ x < cos\ y \longleftrightarrow y < x$
⟨*proof*⟩

lemma *cos-mono-le-eq*: $0 \leq x \Longrightarrow x \leq pi \Longrightarrow 0 \leq y \Longrightarrow y \leq pi \Longrightarrow cos\ x \leq cos\ y \longleftrightarrow y \leq x$
⟨*proof*⟩

lemma *cos-inj-pi*: $0 \leq x \Longrightarrow x \leq pi \Longrightarrow 0 \leq y \Longrightarrow y \leq pi \Longrightarrow cos\ x = cos\ y \Longrightarrow x = y$
⟨*proof*⟩

lemma *arccos-le-pi2*: $[\![0 \leq y; y \leq 1]\!] \Longrightarrow arccos\ y \leq pi/2$
⟨*proof*⟩

lemma *sincos-total-pi-half*:
assumes $0 \leq x$ $0 \leq y$ $x^2 + y^2 = 1$
shows $\exists t.\ 0 \leq t \wedge t \leq pi/2 \wedge x = cos\ t \wedge y = sin\ t$
⟨*proof*⟩

lemma *sincos-total-pi*:
assumes $0 \leq y$ $x^2 + y^2 = 1$
shows $\exists t.\ 0 \leq t \wedge t \leq pi \wedge x = cos\ t \wedge y = sin\ t$
⟨*proof*⟩

lemma *sincos-total-2pi-le*:
assumes $x^2 + y^2 = 1$
shows $\exists t.\ 0 \leq t \wedge t \leq 2 * pi \wedge x = cos\ t \wedge y = sin\ t$
⟨*proof*⟩

lemma *sincos-total-2pi*:
assumes $x^2 + y^2 = 1$
obtains t **where** $0 \leq t$ $t < 2*pi$ $x = cos\ t$ $y = sin\ t$
⟨*proof*⟩

lemma *arcsin-less-mono*: $|x| \leq 1 \Longrightarrow |y| \leq 1 \Longrightarrow arcsin\ x < arcsin\ y \longleftrightarrow x < y$
⟨*proof*⟩

lemma *arcsin-le-mono*: $|x| \leq 1 \Longrightarrow |y| \leq 1 \Longrightarrow arcsin\ x \leq arcsin\ y \longleftrightarrow x \leq y$
⟨*proof*⟩

lemma *arcsin-less-arcsin*: $-1 \leq x \Longrightarrow x < y \Longrightarrow y \leq 1 \Longrightarrow arcsin\ x < arcsin\ y$
⟨*proof*⟩

lemma *arcsin-le-arcsin*: $-1 \leq x \Longrightarrow x \leq y \Longrightarrow y \leq 1 \Longrightarrow arcsin\ x \leq arcsin\ y$

⟨*proof*⟩

lemma *arcsin-nonneg*: $x \in \{0..1\} \Longrightarrow \arcsin\ x \geq 0$
⟨*proof*⟩

lemma *arccos-less-mono*: $|x| \leq 1 \Longrightarrow |y| \leq 1 \Longrightarrow \arccos\ x < \arccos\ y \longleftrightarrow y < x$
⟨*proof*⟩

lemma *arccos-le-mono*: $|x| \leq 1 \Longrightarrow |y| \leq 1 \Longrightarrow \arccos\ x \leq \arccos\ y \longleftrightarrow y \leq x$
⟨*proof*⟩

lemma *arccos-less-arccos*: $-1 \leq x \Longrightarrow x < y \Longrightarrow y \leq 1 \Longrightarrow \arccos\ y < \arccos\ x$
⟨*proof*⟩

lemma *arccos-le-arccos*: $-1 \leq x \Longrightarrow x \leq y \Longrightarrow y \leq 1 \Longrightarrow \arccos\ y \leq \arccos\ x$
⟨*proof*⟩

lemma *arccos-eq-iff*: $|x| \leq 1 \wedge |y| \leq 1 \Longrightarrow \arccos\ x = \arccos\ y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *arccos-cos-eq-abs*:
assumes $|\vartheta| \leq pi$
shows $\arccos\ (\cos\ \vartheta) = |\vartheta|$
⟨*proof*⟩

lemma *arccos-cos-eq-abs-2pi*:
obtains k **where** $\arccos\ (\cos\ \vartheta) = |\vartheta - \text{of-int}\ k * (2 * pi)|$
⟨*proof*⟩

lemma *arccos-arctan*:
assumes $-1 < x\ x < 1$
shows $\arccos\ x = pi/2 - \arctan(x\ /\ \text{sqrt}(1 - x^2))$
⟨*proof*⟩

lemma *arcsin-plus-arccos*:
assumes $-1 \leq x\ x \leq 1$
shows $\arcsin\ x + \arccos\ x = pi/2$
⟨*proof*⟩

lemma *arcsin-arccos-eq*: $-1 \leq x \Longrightarrow x \leq 1 \Longrightarrow \arcsin\ x = pi/2 - \arccos\ x$
⟨*proof*⟩

lemma *arccos-arcsin-eq*: $-1 \leq x \Longrightarrow x \leq 1 \Longrightarrow \arccos\ x = pi/2 - \arcsin\ x$
⟨*proof*⟩

lemma *arcsin-arctan*: $-1 < x \Longrightarrow x < 1 \Longrightarrow \arcsin\ x = \arctan(x\ /\ \text{sqrt}(1 - x^2))$
⟨*proof*⟩

lemma *arcsin-arccos-sqrt-pos*: $0 \leq x \Longrightarrow x \leq 1 \Longrightarrow \arcsin x = \arccos(\sqrt{1 - x^2})$
⟨*proof*⟩

lemma *arcsin-arccos-sqrt-neg*: $-1 \leq x \Longrightarrow x \leq 0 \Longrightarrow \arcsin x = -\arccos(\sqrt{1 - x^2})$
⟨*proof*⟩

lemma *arccos-arcsin-sqrt-pos*: $0 \leq x \Longrightarrow x \leq 1 \Longrightarrow \arccos x = \arcsin(\sqrt{1 - x^2})$
⟨*proof*⟩

lemma *arccos-arcsin-sqrt-neg*: $-1 \leq x \Longrightarrow x \leq 0 \Longrightarrow \arccos x = pi - \arcsin(\sqrt{1 - x^2})$
⟨*proof*⟩

lemma *cos-limit-1*:
assumes $(\lambda j.\ \cos(\vartheta\ j)) \longrightarrow 1$
shows $\exists k.\ (\lambda j.\ \vartheta\ j - \text{of-int}\ (k\ j) * (2 * pi)) \longrightarrow 0$
⟨*proof*⟩

lemma *cos-diff-limit-1*:
assumes $(\lambda j.\ \cos(\vartheta\ j - \Theta)) \longrightarrow 1$
obtains k **where** $(\lambda j.\ \vartheta\ j - \text{of-int}\ (k\ j) * (2 * pi)) \longrightarrow \Theta$
⟨*proof*⟩

113.19 Machin's formula

lemma *arctan-one*: $\arctan 1 = pi/4$
⟨*proof*⟩

lemma *tan-total-pi4*:
assumes $|x| < 1$
shows $\exists z.\ -(pi/4) < z \wedge z < pi/4 \wedge \tan z = x$
⟨*proof*⟩

lemma *arctan-add*:
assumes $|x| \leq 1$ $|y| < 1$
shows $\arctan x + \arctan y = \arctan((x + y) / (1 - x * y))$
⟨*proof*⟩

lemma *arctan-double*: $|x| < 1 \Longrightarrow 2 * \arctan x = \arctan((2 * x) / (1 - x^2))$
⟨*proof*⟩

theorem *machin*: $pi/4 = 4 * \arctan(1 / 5) - \arctan(1/239)$
⟨*proof*⟩

lemma *machin-Euler*: $5 * \arctan(1 / 7) + 2 * \arctan(3 / 79) = pi/4$
⟨*proof*⟩

113.20 Introducing the inverse tangent power series

lemma *monoseq-arctan-series*:
fixes x :: *real*
assumes $|x| \leq 1$
shows *monoseq* $(\lambda n.\ 1\ /\ real\ (n * 2 + 1) * x\hat{}(n * 2 + 1))$
(**is** *monoseq ?a*)
⟨*proof*⟩

lemma *zeroseq-arctan-series*:
fixes x :: *real*
assumes $|x| \leq 1$
shows $(\lambda n.\ 1\ /\ real\ (n * 2 + 1) * x\hat{}(n * 2 + 1)) \longrightarrow 0$
(**is** $?a \longrightarrow 0$)
⟨*proof*⟩

lemma *summable-arctan-series*:
fixes n :: *nat*
assumes $|x| \leq 1$
shows *summable* $(\lambda\ k.\ (-1)\hat{}k * (1\ /\ real\ (k*2+1) * x\ \hat{}\ (k*2+1)))$
(**is** *summable* (*?c x*))
⟨*proof*⟩

lemma *DERIV-arctan-series*:
assumes $|x| < 1$
shows *DERIV* $(\lambda x'.\ \sum k.\ (-1)\hat{}k * (1\ /\ real\ (k * 2 + 1) * x'\ \hat{}\ (k * 2 + 1)))$
x :>
$(\sum k.\ (-1)\hat{}k * x\hat{}(k * 2))$
(**is** *DERIV ?arctan - :> ?Int*)
⟨*proof*⟩

lemma *arctan-series*:
assumes $|x| \leq 1$
shows *arctan* $x = (\sum k.\ (-1)\hat{}k * (1\ /\ real\ (k * 2 + 1) * x\ \hat{}\ (k * 2 + 1)))$
(**is** - = *suminf* $(\lambda\ n.\ ?c\ x\ n)$)
⟨*proof*⟩

lemma *arctan-half*: *arctan* $x = 2 *$ *arctan* $(x\ /\ (1 + sqrt(1 + x^2)))$
for x :: *real*
⟨*proof*⟩

lemma *arctan-monotone*: $x < y \Longrightarrow$ *arctan* $x <$ *arctan* y
⟨*proof*⟩

lemma *arctan-monotone'*: $x \leq y \Longrightarrow$ *arctan* $x \leq$ *arctan* y
⟨*proof*⟩

lemma *arctan-inverse*:
assumes $x \neq 0$
shows *arctan* $(1\ /\ x) =$ *sgn* $x *$ *pi*$/2 -$ *arctan* x

⟨*proof*⟩

theorem *pi-series*: $pi/4 = (\sum k.\ (-1)\hat{}k * 1\ /\ real\ (k * 2 + 1))$
(**is** - = *?SUM*)
⟨*proof*⟩

113.21 Existence of Polar Coordinates

lemma *cos-x-y-le-one*: $|x\ /\ sqrt\ (x^2 + y^2)| \leq 1$
⟨*proof*⟩

lemma *polar-Ex*: $\exists\, r{::}real.\ \exists\, a.\ x = r * cos\ a \wedge y = r * sin\ a$
⟨*proof*⟩

113.22 Basics about polynomial functions: products, extremal behaviour and root counts

lemma *pairs-le-eq-Sigma*: $\{(i,\ j).\ i + j \leq m\} = Sigma\ (atMost\ m)\ (\lambda r.\ atMost\ (m - r))$
for m :: *nat*
⟨*proof*⟩

lemma *sum-up-index-split*: $(\sum k{\leq}m + n.\ f\ k) = (\sum k{\leq}m.\ f\ k) + (\sum k = Suc\ m..m + n.\ f\ k)$
⟨*proof*⟩

lemma *Sigma-interval-disjoint*: $(SIGMA\ i{:}A.\ \{..v\ i\}) \cap (SIGMA\ i{:}A.\{v\ i{<}..w\}) = \{\}$
for w :: *'a*::*order*
⟨*proof*⟩

lemma *product-atMost-eq-Un*: $A \times \{..m\} = (SIGMA\ i{:}A.\{..m - i\}) \cup (SIGMA\ i{:}A.\{m - i{<}..m\})$
for m :: *nat*
⟨*proof*⟩

lemma *polynomial-product*:
fixes x :: *'a*::*idom*
assumes m: $\bigwedge i.\ i > m \Longrightarrow a\ i = 0$
and n: $\bigwedge j.\ j > n \Longrightarrow b\ j = 0$
shows $(\sum i{\leq}m.\ (a\ i) * x\ \hat{}\ i) * (\sum j{\leq}n.\ (b\ j) * x\ \hat{}\ j) =$
$(\sum r{\leq}m + n.\ (\sum k{\leq}r.\ (a\ k) * (b\ (r - k))) * x\ \hat{}\ r)$
⟨*proof*⟩

lemma *polynomial-product-nat*:
fixes x :: *nat*
assumes m: $\bigwedge i.\ i > m \Longrightarrow a\ i = 0$
and n: $\bigwedge j.\ j > n \Longrightarrow b\ j = 0$
shows $(\sum i{\leq}m.\ (a\ i) * x\ \hat{}\ i) * (\sum j{\leq}n.\ (b\ j) * x\ \hat{}\ j) =$

$(\sum r{\leq}m + n.\ (\sum k{\leq}r.\ (a\ k) * (b\ (r - k))) * x\ \widehat{}\ r)$
⟨*proof*⟩

lemma *polyfun-diff*:
fixes $x :: 'a{::}idom$
assumes $1 \leq n$
shows $(\sum i{\leq}n.\ a\ i * x\widehat{}i) - (\sum i{\leq}n.\ a\ i * y\widehat{}i) =$
$(x - y) * (\sum j{<}n.\ (\sum i{=}Suc\ j..n.\ a\ i * y\widehat{}(i - j - 1)) * x\widehat{}j)$
⟨*proof*⟩

lemma *polyfun-diff-alt*:
fixes $x :: 'a{::}idom$
assumes $1 \leq n$
shows $(\sum i{\leq}n.\ a\ i * x\widehat{}i) - (\sum i{\leq}n.\ a\ i * y\widehat{}i) =$
$(x - y) * ((\sum j{<}n.\ \sum k{<}n{-}j.\ a(j + k + 1) * y\widehat{}k * x\widehat{}j))$
⟨*proof*⟩

lemma *polyfun-linear-factor*:
fixes $a :: 'a{::}idom$
shows $\exists b.\ \forall z.\ (\sum i{\leq}n.\ c(i) * z\widehat{}i) = (z - a) * (\sum i{<}n.\ b(i) * z\widehat{}i) + (\sum i{\leq}n.\ c(i) * a\widehat{}i)$
⟨*proof*⟩

lemma *polyfun-linear-factor-root*:
fixes $a :: 'a{::}idom$
assumes $(\sum i{\leq}n.\ c(i) * a\widehat{}i) = 0$
obtains b **where** $\bigwedge z.\ (\sum i{\leq}n.\ c\ i * z\widehat{}i) = (z - a) * (\sum i{<}n.\ b\ i * z\widehat{}i)$
⟨*proof*⟩

lemma *isCont-polynom*: *isCont* $(\lambda w.\ \sum i{\leq}n.\ c\ i * w\widehat{}i)\ a$
for $c :: nat \Rightarrow 'a{::}real\text{-}normed\text{-}div\text{-}algebra$
⟨*proof*⟩

lemma *zero-polynom-imp-zero-coeffs*:
fixes $c :: nat \Rightarrow 'a{::}\{ab\text{-}semigroup\text{-}mult,real\text{-}normed\text{-}div\text{-}algebra\}$
assumes $\bigwedge w.\ (\sum i{\leq}n.\ c\ i * w\widehat{}i) = 0\ \ k \leq n$
shows $c\ k = 0$
⟨*proof*⟩

lemma *polyfun-rootbound*:
fixes $c :: nat \Rightarrow 'a{::}\{idom,real\text{-}normed\text{-}div\text{-}algebra\}$
assumes $c\ k \neq 0\ k{\leq}n$
shows *finite* $\{z.\ (\sum i{\leq}n.\ c(i) * z\widehat{}i) = 0\} \wedge card\ \{z.\ (\sum i{\leq}n.\ c(i) * z\widehat{}i) = 0\} \leq n$
⟨*proof*⟩

lemma

fixes $c :: nat \Rightarrow 'a::\{idom,real\text{-}normed\text{-}div\text{-}algebra\}$
assumes $c\ k \neq 0\ k \leq n$
shows *polyfun-roots-finite*: *finite* $\{z.\ (\sum i \leq n.\ c(i) * z\hat{}i) = 0\}$
and *polyfun-roots-card*: *card* $\{z.\ (\sum i \leq n.\ c(i) * z\hat{}i) = 0\} \leq n$
⟨*proof*⟩

lemma *polyfun-finite-roots*:
fixes $c :: nat \Rightarrow 'a::\{idom,real\text{-}normed\text{-}div\text{-}algebra\}$
shows *finite* $\{x.\ (\sum i \leq n.\ c\ i * x\hat{}i) = 0\} \longleftrightarrow (\exists\, i \leq n.\ c\ i \neq 0)$
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *polyfun-eq-0*: $(\forall\, x.\ (\sum i \leq n.\ c\ i * x\hat{}i) = 0) \longleftrightarrow (\forall\, i \leq n.\ c\ i = 0)$
for $c :: nat \Rightarrow 'a::\{idom,real\text{-}normed\text{-}div\text{-}algebra\}$

⟨*proof*⟩

lemma *polyfun-eq-coeffs*: $(\forall\, x.\ (\sum i \leq n.\ c\ i * x\hat{}i) = (\sum i \leq n.\ d\ i * x\hat{}i)) \longleftrightarrow (\forall\, i \leq n.\ c\ i = d\ i)$
for $c :: nat \Rightarrow 'a::\{idom,real\text{-}normed\text{-}div\text{-}algebra\}$
⟨*proof*⟩

lemma *polyfun-eq-const*:
fixes $c :: nat \Rightarrow 'a::\{idom,real\text{-}normed\text{-}div\text{-}algebra\}$
shows $(\forall\, x.\ (\sum i \leq n.\ c\ i * x\hat{}i) = k) \longleftrightarrow c\ 0 = k \wedge (\forall\, i \in \{1..n\}.\ c\ i = 0)$
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *root-polyfun*:
fixes $z :: 'a::idom$
assumes $1 \leq n$
shows $z\hat{}n = a \longleftrightarrow (\sum i \leq n.\ (\textit{if } i = 0 \textit{ then } -a \textit{ else if } i{=}n \textit{ then } 1 \textit{ else } 0) * z\hat{}i) = 0$
⟨*proof*⟩

lemma
assumes *SORT-CONSTRAINT*$('a::\{idom,real\text{-}normed\text{-}div\text{-}algebra\})$
and $1 \leq n$
shows *finite-roots-unity*: *finite* $\{z::'a.\ z\hat{}n = 1\}$
and *card-roots-unity*: *card* $\{z::'a.\ z\hat{}n = 1\} \leq n$
⟨*proof*⟩

113.23 Hyperbolic functions

definition *sinh* $:: 'a :: \{banach, real\text{-}normed\text{-}algebra\text{-}1\} \Rightarrow 'a$ **where**
$sinh\ x = (exp\ x - exp\ (-x))\ /_R\ 2$

definition *cosh* $:: 'a :: \{banach, real\text{-}normed\text{-}algebra\text{-}1\} \Rightarrow 'a$ **where**
$cosh\ x = (exp\ x + exp\ (-x))\ /_R\ 2$

definition *tanh* :: *'a* :: {*banach*, *real-normed-field*} ⇒ *'a* **where**
tanh x = sinh x / cosh x

definition *arsinh* :: *'a* :: {*banach*, *real-normed-algebra-1*, *ln*} ⇒ *'a* **where**
arsinh x = ln (x + (x^2 + 1) powr of-real (1/2))

definition *arcosh* :: *'a* :: {*banach*, *real-normed-algebra-1*, *ln*} ⇒ *'a* **where**
arcosh x = ln (x + (x^2 − 1) powr of-real (1/2))

definition *artanh* :: *'a* :: {*banach*, *real-normed-field*, *ln*} ⇒ *'a* **where**
artanh x = ln ((1 + x) / (1 − x)) / 2

lemma *arsinh-0* [*simp*]: *arsinh 0 = 0*
⟨*proof*⟩

lemma *arcosh-1* [*simp*]: *arcosh 1 = 0*
⟨*proof*⟩

lemma *artanh-0* [*simp*]: *artanh 0 = 0*
⟨*proof*⟩

lemma *tanh-altdef*:
tanh x = (exp x − exp (−x)) / (exp x + exp (−x))
⟨*proof*⟩

lemma *tanh-real-altdef*: *tanh (x::real) = (1 − exp (− 2 ∗ x)) / (1 + exp (− 2 ∗ x))*
⟨*proof*⟩

lemma *sinh-converges*: *(λn. if even n then 0 else x ^ n /R fact n) sums sinh x*
⟨*proof*⟩

lemma *cosh-converges*: *(λn. if even n then x ^ n /R fact n else 0) sums cosh x*
⟨*proof*⟩

lemma *sinh-0* [*simp*]: *sinh 0 = 0*
⟨*proof*⟩

lemma *cosh-0* [*simp*]: *cosh 0 = 1*
⟨*proof*⟩

lemma *tanh-0* [*simp*]: *tanh 0 = 0*
⟨*proof*⟩

lemma *sinh-minus* [*simp*]: *sinh (− x) = −sinh x*
⟨*proof*⟩

lemma *cosh-minus* [*simp*]: *cosh* (− *x*) = *cosh x*
⟨*proof*⟩

lemma *tanh-minus* [*simp*]: *tanh* (−*x*) = −*tanh x*
⟨*proof*⟩

lemma *sinh-ln-real*: *x* > *0* ⟹ *sinh* (*ln x* :: *real*) = (*x* − *inverse x*) / *2*
⟨*proof*⟩

lemma *cosh-ln-real*: *x* > *0* ⟹ *cosh* (*ln x* :: *real*) = (*x* + *inverse x*) / *2*
⟨*proof*⟩

lemma *tanh-ln-real*:
tanh (*ln x* :: *real*) = (*x* ^ *2* − *1*) / (*x* ^ *2* + *1*) **if** *x* > *0*
⟨*proof*⟩

lemma *has-field-derivative-scaleR-right* [*derivative-intros*]:
(*f has-field-derivative D*) *F* ⟹ ((λ*x*. *c* *$_R$ *f x*) *has-field-derivative* (*c* *$_R$ *D*)) *F*
⟨*proof*⟩

lemma *has-field-derivative-sinh* [*THEN DERIV-chain2*, *derivative-intros*]:
(*sinh has-field-derivative cosh x*) (*at* (*x* :: *'a* :: {*banach*, *real-normed-field*}))
⟨*proof*⟩

lemma *has-field-derivative-cosh* [*THEN DERIV-chain2*, *derivative-intros*]:
(*cosh has-field-derivative sinh x*) (*at* (*x* :: *'a* :: {*banach*, *real-normed-field*}))
⟨*proof*⟩

lemma *has-field-derivative-tanh* [*THEN DERIV-chain2*, *derivative-intros*]:
cosh x ≠ *0* ⟹ (*tanh has-field-derivative 1* − *tanh x* ^ *2*)
(*at* (*x* :: *'a* :: {*banach*, *real-normed-field*}))
⟨*proof*⟩

lemma *has-derivative-sinh* [*derivative-intros*]:
fixes *g* :: *'a* ⇒ (*'a* :: {*banach*, *real-normed-field*})
assumes (*g has-derivative* (λ*x*. *Db* ∗ *x*)) (*at x within s*)
shows ((λ*x*. *sinh* (*g x*)) *has-derivative* (λ*y*. (*cosh* (*g x*) ∗ *Db*) ∗ *y*)) (*at x within s*)
⟨*proof*⟩

lemma *has-derivative-cosh* [*derivative-intros*]:
fixes *g* :: *'a* ⇒ (*'a* :: {*banach*, *real-normed-field*})
assumes (*g has-derivative* (λ*y*. *Db* ∗ *y*)) (*at x within s*)
shows ((λ*x*. *cosh* (*g x*)) *has-derivative* (λ*y*. (*sinh* (*g x*) ∗ *Db*) ∗ *y*)) (*at x within s*)
⟨*proof*⟩

lemma *sinh-plus-cosh*: *sinh x* + *cosh x* = *exp x*
⟨*proof*⟩

lemma *cosh-plus-sinh*: $cosh\ x + sinh\ x = exp\ x$
⟨*proof*⟩

lemma *cosh-minus-sinh*: $cosh\ x - sinh\ x = exp\ (-x)$
⟨*proof*⟩

lemma *sinh-minus-cosh*: $sinh\ x - cosh\ x = -exp\ (-x)$
⟨*proof*⟩

context
fixes $x :: 'a :: \{real\text{-}normed\text{-}field, banach\}$
begin

lemma *sinh-zero-iff*: $sinh\ x = 0 \longleftrightarrow exp\ x \in \{1, -1\}$
⟨*proof*⟩

lemma *cosh-zero-iff*: $cosh\ x = 0 \longleftrightarrow exp\ x\ \hat{}\ 2 = -1$
⟨*proof*⟩

lemma *cosh-square-eq*: $cosh\ x\ \hat{}\ 2 = sinh\ x\ \hat{}\ 2 + 1$
⟨*proof*⟩

lemma *sinh-square-eq*: $sinh\ x\ \hat{}\ 2 = cosh\ x\ \hat{}\ 2 - 1$
⟨*proof*⟩

lemma *hyperbolic-pythagoras*: $cosh\ x\ \hat{}\ 2 - sinh\ x\ \hat{}\ 2 = 1$
⟨*proof*⟩

lemma *sinh-add*: $sinh\ (x + y) = sinh\ x * cosh\ y + cosh\ x * sinh\ y$
⟨*proof*⟩

lemma *sinh-diff*: $sinh\ (x - y) = sinh\ x * cosh\ y - cosh\ x * sinh\ y$
⟨*proof*⟩

lemma *cosh-add*: $cosh\ (x + y) = cosh\ x * cosh\ y + sinh\ x * sinh\ y$
⟨*proof*⟩

lemma *cosh-diff*: $cosh\ (x - y) = cosh\ x * cosh\ y - sinh\ x * sinh\ y$
⟨*proof*⟩

lemma *tanh-add*:
$tanh\ (x + y) = (tanh\ x + tanh\ y)\ /\ (1 + tanh\ x * tanh\ y)$
if $cosh\ x \neq 0\ cosh\ y \neq 0$
⟨*proof*⟩

lemma *sinh-double*: $sinh\ (2 * x) = 2 * sinh\ x * cosh\ x$
⟨*proof*⟩

lemma *cosh-double*: $cosh\ (2 * x) = cosh\ x\ \hat{}\ 2 + sinh\ x\ \hat{}\ 2$
⟨*proof*⟩

end

lemma *sinh-field-def*: $sinh\ z = (exp\ z - exp\ (-z))\ /\ (2 :: 'a :: \{banach, real\text{-}normed\text{-}field\})$
⟨*proof*⟩

lemma *cosh-field-def*: $cosh\ z = (exp\ z + exp\ (-z))\ /\ (2 :: 'a :: \{banach, real\text{-}normed\text{-}field\})$
⟨*proof*⟩

113.23.1 More specific properties of the real functions

lemma *plus-inverse-ge-2*:
fixes $x :: real$
assumes $x > 0$
shows $x + inverse\ x \geq 2$
⟨*proof*⟩

lemma *sinh-real-nonneg-iff* [*simp*]: $sinh\ (x :: real) \geq 0 \longleftrightarrow x \geq 0$
⟨*proof*⟩

lemma *sinh-real-pos-iff* [*simp*]: $sinh\ (x :: real) > 0 \longleftrightarrow x > 0$
⟨*proof*⟩

lemma *sinh-real-nonpos-iff* [*simp*]: $sinh\ (x :: real) \leq 0 \longleftrightarrow x \leq 0$
⟨*proof*⟩

lemma *sinh-real-neg-iff* [*simp*]: $sinh\ (x :: real) < 0 \longleftrightarrow x < 0$
⟨*proof*⟩

lemma *cosh-real-ge-1*: $cosh\ (x :: real) \geq 1$
⟨*proof*⟩

lemma *cosh-real-pos* [*simp*]: $cosh\ (x :: real) > 0$
⟨*proof*⟩

lemma *cosh-real-nonneg*[*simp*]: $cosh\ (x :: real) \geq 0$
⟨*proof*⟩

lemma *cosh-real-nonzero* [*simp*]: $cosh\ (x :: real) \neq 0$
⟨*proof*⟩

lemma *arsinh-real-def*: $arsinh\ (x::real) = ln\ (x + sqrt\ (x\hat{}2 + 1))$
⟨*proof*⟩

lemma *arcosh-real-def*: $x \geq 1 \Longrightarrow arcosh\ (x::real) = ln\ (x + sqrt\ (x\hat{}2 - 1))$
⟨*proof*⟩

lemma *arsinh-real-aux*: $0 < x + sqrt\ (x\ \hat{}\ 2 + 1 :: real)$
⟨*proof*⟩

lemma *arsinh-minus-real* [*simp*]: $arsinh\ (-x::real) = -arsinh\ x$
⟨*proof*⟩

lemma *artanh-minus-real* [*simp*]:
assumes $abs\ x < 1$
shows $artanh\ (-x::real) = -artanh\ x$
⟨*proof*⟩

lemma *sinh-less-cosh-real*: $sinh\ (x :: real) < cosh\ x$
⟨*proof*⟩

lemma *sinh-le-cosh-real*: $sinh\ (x :: real) \leq cosh\ x$
⟨*proof*⟩

lemma *tanh-real-lt-1*: $tanh\ (x :: real) < 1$
⟨*proof*⟩

lemma *tanh-real-gt-neg1*: $tanh\ (x :: real) > -1$
⟨*proof*⟩

lemma *tanh-real-bounds*: $tanh\ (x :: real) \in \{-1<..<1\}$
⟨*proof*⟩

context
fixes $x :: real$
begin

lemma *arsinh-sinh-real*: $arsinh\ (sinh\ x) = x$
⟨*proof*⟩

lemma *arcosh-cosh-real*: $x \geq 0 \Longrightarrow arcosh\ (cosh\ x) = x$
⟨*proof*⟩

lemma *artanh-tanh-real*: $artanh\ (tanh\ x) = x$
⟨*proof*⟩

lemma *sinh-real-zero-iff* [*simp*]: $sinh\ x = 0 \longleftrightarrow x = 0$
⟨*proof*⟩

lemma *cosh-real-one-iff* [*simp*]: $cosh\ x = 1 \longleftrightarrow x = 0$
⟨*proof*⟩

lemma *tanh-real-nonneg-iff* [*simp*]: $tanh\ x \geq 0 \longleftrightarrow x \geq 0$
⟨*proof*⟩

lemma *tanh-real-pos-iff* [*simp*]: $tanh\ x > 0 \longleftrightarrow x > 0$
⟨*proof*⟩

lemma *tanh-real-nonpos-iff* [*simp*]: $tanh\ x \leq 0 \longleftrightarrow x \leq 0$
⟨*proof*⟩

lemma *tanh-real-neg-iff* [*simp*]: $tanh\ x < 0 \longleftrightarrow x < 0$
⟨*proof*⟩

lemma *tanh-real-zero-iff* [*simp*]: $tanh\ x = 0 \longleftrightarrow x = 0$
⟨*proof*⟩

end

lemma *sinh-real-strict-mono*: *strict-mono* $(sinh :: real \Rightarrow real)$
⟨*proof*⟩

lemma *cosh-real-strict-mono*:
assumes $0 \leq x$ **and** $x < (y::real)$
shows $cosh\ x < cosh\ y$
⟨*proof*⟩

lemma *tanh-real-strict-mono*: *strict-mono* $(tanh :: real \Rightarrow real)$
⟨*proof*⟩

lemma *sinh-real-abs* [*simp*]: $sinh\ (abs\ x :: real) = abs\ (sinh\ x)$
⟨*proof*⟩

lemma *cosh-real-abs* [*simp*]: $cosh\ (abs\ x :: real) = cosh\ x$
⟨*proof*⟩

lemma *tanh-real-abs* [*simp*]: $tanh\ (abs\ x :: real) = abs\ (tanh\ x)$
⟨*proof*⟩

lemma *sinh-real-eq-iff* [*simp*]: $sinh\ x = sinh\ y \longleftrightarrow x = (y :: real)$
⟨*proof*⟩

lemma *tanh-real-eq-iff* [*simp*]: $tanh\ x = tanh\ y \longleftrightarrow x = (y :: real)$
⟨*proof*⟩

lemma *cosh-real-eq-iff* [*simp*]: $cosh\ x = cosh\ y \longleftrightarrow abs\ x = abs\ (y :: real)$
⟨*proof*⟩

lemma *sinh-real-le-iff* [*simp*]: $sinh\ x \leq sinh\ y \longleftrightarrow x \leq (y::real)$
⟨*proof*⟩

lemma *cosh-real-nonneg-le-iff*: $x \geq 0 \Longrightarrow y \geq 0 \Longrightarrow cosh\ x \leq cosh\ y \longleftrightarrow x \leq (y::real)$
⟨*proof*⟩

lemma *cosh-real-nonpos-le-iff*: $x \leq 0 \Longrightarrow y \leq 0 \Longrightarrow \cosh x \leq \cosh y \longleftrightarrow x \geq (y::real)$
⟨*proof*⟩

lemma *tanh-real-le-iff* [*simp*]: $\tanh x \leq \tanh y \longleftrightarrow x \leq (y::real)$
⟨*proof*⟩

lemma *sinh-real-less-iff* [*simp*]: $\sinh x < \sinh y \longleftrightarrow x < (y::real)$
⟨*proof*⟩

lemma *cosh-real-nonneg-less-iff*: $x \geq 0 \Longrightarrow y \geq 0 \Longrightarrow \cosh x < \cosh y \longleftrightarrow x < (y::real)$
⟨*proof*⟩

lemma *cosh-real-nonpos-less-iff*: $x \leq 0 \Longrightarrow y \leq 0 \Longrightarrow \cosh x < \cosh y \longleftrightarrow x > (y::real)$
⟨*proof*⟩

lemma *tanh-real-less-iff* [*simp*]: $\tanh x < \tanh y \longleftrightarrow x < (y::real)$
⟨*proof*⟩

113.23.2 Limits

lemma *sinh-real-at-top*: *filterlim* $(\sinh :: real \Rightarrow real)$ *at-top at-top*
⟨*proof*⟩

lemma *sinh-real-at-bot*: *filterlim* $(\sinh :: real \Rightarrow real)$ *at-bot at-bot*
⟨*proof*⟩

lemma *cosh-real-at-top*: *filterlim* $(\cosh :: real \Rightarrow real)$ *at-top at-top*
⟨*proof*⟩

lemma *cosh-real-at-bot*: *filterlim* $(\cosh :: real \Rightarrow real)$ *at-top at-bot*
⟨*proof*⟩

lemma *tanh-real-at-top*: $(\tanh \longrightarrow (1::real))$ *at-top*
⟨*proof*⟩

lemma *tanh-real-at-bot*: $(\tanh \longrightarrow (-1::real))$ *at-bot*
⟨*proof*⟩

113.23.3 Properties of the inverse hyperbolic functions

lemma *isCont-sinh*: *isCont sinh* $(x :: 'a :: \{real\text{-}normed\text{-}field, banach\})$
⟨*proof*⟩

lemma *isCont-cosh*: *isCont cosh* $(x :: 'a :: \{real\text{-}normed\text{-}field, banach\})$
⟨*proof*⟩

lemma *isCont-tanh*: *cosh x ≠ 0 ⟹ isCont tanh (x :: 'a :: {real-normed-field, banach})*
⟨*proof*⟩

lemma *continuous-on-sinh* [*continuous-intros*]:
fixes *f* :: - ⇒'a::{*real-normed-field*,*banach*}
assumes *continuous-on A f*
shows *continuous-on A (λx. sinh (f x))*
⟨*proof*⟩

lemma *continuous-on-cosh* [*continuous-intros*]:
fixes *f* :: - ⇒'a::{*real-normed-field*,*banach*}
assumes *continuous-on A f*
shows *continuous-on A (λx. cosh (f x))*
⟨*proof*⟩

lemma *continuous-sinh* [*continuous-intros*]:
fixes *f* :: - ⇒'a::{*real-normed-field*,*banach*}
assumes *continuous F f*
shows *continuous F (λx. sinh (f x))*
⟨*proof*⟩

lemma *continuous-cosh* [*continuous-intros*]:
fixes *f* :: - ⇒'a::{*real-normed-field*,*banach*}
assumes *continuous F f*
shows *continuous F (λx. cosh (f x))*
⟨*proof*⟩

lemma *continuous-on-tanh* [*continuous-intros*]:
fixes *f* :: - ⇒'a::{*real-normed-field*,*banach*}
assumes *continuous-on A f* ⋀*x. x ∈ A ⟹ cosh (f x) ≠ 0*
shows *continuous-on A (λx. tanh (f x))*
⟨*proof*⟩

lemma *continuous-at-within-tanh* [*continuous-intros*]:
fixes *f* :: - ⇒'a::{*real-normed-field*,*banach*}
assumes *continuous (at x within A) f cosh (f x) ≠ 0*
shows *continuous (at x within A) (λx. tanh (f x))*
⟨*proof*⟩

lemma *continuous-tanh* [*continuous-intros*]:
fixes *f* :: - ⇒'a::{*real-normed-field*,*banach*}
assumes *continuous F f cosh (f (Lim F (λx. x))) ≠ 0*
shows *continuous F (λx. tanh (f x))*
⟨*proof*⟩

lemma *tendsto-sinh* [*tendsto-intros*]:
fixes *f* :: - ⇒'a::{*real-normed-field*,*banach*}

shows $(f \longrightarrow a)\ F \Longrightarrow ((\lambda x.\ sinh\ (f\ x)) \longrightarrow sinh\ a)\ F$
⟨*proof*⟩

lemma *tendsto-cosh* [*tendsto-intros*]:
fixes $f :: - \Rightarrow 'a::\{real\text{-}normed\text{-}field, banach\}$
shows $(f \longrightarrow a)\ F \Longrightarrow ((\lambda x.\ cosh\ (f\ x)) \longrightarrow cosh\ a)\ F$
⟨*proof*⟩

lemma *tendsto-tanh* [*tendsto-intros*]:
fixes $f :: - \Rightarrow 'a::\{real\text{-}normed\text{-}field, banach\}$
shows $(f \longrightarrow a)\ F \Longrightarrow cosh\ a \neq 0 \Longrightarrow ((\lambda x.\ tanh\ (f\ x)) \longrightarrow tanh\ a)\ F$
⟨*proof*⟩

lemma *arsinh-real-has-field-derivative* [*derivative-intros*]:
fixes $x :: real$
shows $(arsinh\ has\text{-}field\text{-}derivative\ (1\ /\ (sqrt\ (x\ \hat{}\ 2\ +\ 1))))\ (at\ x\ within\ A)$
⟨*proof*⟩

lemma *arcosh-real-has-field-derivative* [*derivative-intros*]:
fixes $x :: real$
assumes $x > 1$
shows $(arcosh\ has\text{-}field\text{-}derivative\ (1\ /\ (sqrt\ (x\ \hat{}\ 2\ -\ 1))))\ (at\ x\ within\ A)$
⟨*proof*⟩

lemma *artanh-real-has-field-derivative* [*derivative-intros*]:
$(artanh\ has\text{-}field\text{-}derivative\ (1\ /\ (1\ -\ x\ \hat{}\ 2)))\ (at\ x\ within\ A)$ **if**
$|x| < 1$ **for** $x :: real$
⟨*proof*⟩

lemma *continuous-on-arsinh* [*continuous-intros*]: $continuous\text{-}on\ A\ (arsinh :: real \Rightarrow real)$
⟨*proof*⟩

lemma *continuous-on-arcosh* [*continuous-intros*]:
assumes $A \subseteq \{1..\}$
shows $continuous\text{-}on\ A\ (arcosh :: real \Rightarrow real)$
⟨*proof*⟩

lemma *continuous-on-artanh* [*continuous-intros*]:
assumes $A \subseteq \{-1<..<1\}$
shows $continuous\text{-}on\ A\ (artanh :: real \Rightarrow real)$
⟨*proof*⟩

lemma *continuous-on-arsinh'* [*continuous-intros*]:
fixes $f :: real \Rightarrow real$
assumes $continuous\text{-}on\ A\ f$
shows $continuous\text{-}on\ A\ (\lambda x.\ arsinh\ (f\ x))$
⟨*proof*⟩

lemma *continuous-on-arcosh'* [*continuous-intros*]:
fixes $f :: real \Rightarrow real$
assumes *continuous-on* A f $\bigwedge x.\ x \in A \Longrightarrow f\ x \geq 1$
shows *continuous-on* A $(\lambda x.\ arcosh\ (f\ x))$
⟨*proof*⟩

lemma *continuous-on-artanh'* [*continuous-intros*]:
fixes $f :: real \Rightarrow real$
assumes *continuous-on* A f $\bigwedge x.\ x \in A \Longrightarrow f\ x \in \{-1<..<1\}$
shows *continuous-on* A $(\lambda x.\ artanh\ (f\ x))$
⟨*proof*⟩

lemma *isCont-arsinh* [*continuous-intros*]: *isCont arsinh* $(x :: real)$
⟨*proof*⟩

lemma *isCont-arcosh* [*continuous-intros*]:
assumes $x > 1$
shows *isCont arcosh* $(x :: real)$
⟨*proof*⟩

lemma *isCont-artanh* [*continuous-intros*]:
assumes $x > -1$ $x < 1$
shows *isCont artanh* $(x :: real)$
⟨*proof*⟩

lemma *tendsto-arsinh* [*tendsto-intros*]: $(f \longrightarrow a)\ F \Longrightarrow ((\lambda x.\ arsinh\ (f\ x)) \longrightarrow arsinh\ a)\ F$
for $f :: - \Rightarrow real$
⟨*proof*⟩

lemma *tendsto-arcosh-strong* [*tendsto-intros*]:
fixes $f :: - \Rightarrow real$
assumes $(f \longrightarrow a)\ F$ $a \geq 1$ *eventually* $(\lambda x.\ f\ x \geq 1)\ F$
shows $((\lambda x.\ arcosh\ (f\ x)) \longrightarrow arcosh\ a)\ F$
⟨*proof*⟩

lemma *tendsto-arcosh*:
fixes $f :: - \Rightarrow real$
assumes $(f \longrightarrow a)\ F$ $a > 1$
shows $((\lambda x.\ arcosh\ (f\ x)) \longrightarrow arcosh\ a)\ F$
⟨*proof*⟩

lemma *tendsto-arcosh-at-left-1*: $(arcosh \longrightarrow 0)\ (at\text{-}right\ (1::real))$
⟨*proof*⟩

lemma *tendsto-artanh* [*tendsto-intros*]:
fixes $f :: 'a \Rightarrow real$
assumes $(f \longrightarrow a)\ F$ $a > -1$ $a < 1$

shows $((\lambda x.\ artanh\ (f\ x)) \longrightarrow artanh\ a)\ F$
⟨*proof*⟩

lemma *continuous-arsinh* [*continuous-intros*]:
$continuous\ F\ f \Longrightarrow continuous\ F\ (\lambda x.\ arsinh\ (f\ x :: real))$
⟨*proof*⟩

lemma *continuous-arcosh-strong* [*continuous-intros*]:
assumes *continuous F f eventually* $(\lambda x.\ f\ x \geq 1)\ F$
shows $continuous\ F\ (\lambda x.\ arcosh\ (f\ x :: real))$
⟨*proof*⟩

lemma *continuous-arcosh*:
$continuous\ F\ f \Longrightarrow f\ (Lim\ F\ (\lambda x.\ x)) > 1 \Longrightarrow continuous\ F\ (\lambda x.\ arcosh\ (f\ x :: real))$
⟨*proof*⟩

lemma *continuous-artanh* [*continuous-intros*]:
$continuous\ F\ f \Longrightarrow f\ (Lim\ F\ (\lambda x.\ x)) \in \{-1<..<1\} \Longrightarrow continuous\ F\ (\lambda x.\ artanh\ (f\ x :: real))$
⟨*proof*⟩

lemma *arsinh-real-at-top*:
$filterlim\ (arsinh :: real \Rightarrow real)\ at\text{-}top\ at\text{-}top$
⟨*proof*⟩

lemma *arsinh-real-at-bot*:
$filterlim\ (arsinh :: real \Rightarrow real)\ at\text{-}bot\ at\text{-}bot$
⟨*proof*⟩

lemma *arcosh-real-at-top*:
$filterlim\ (arcosh :: real \Rightarrow real)\ at\text{-}top\ at\text{-}top$
⟨*proof*⟩

lemma *artanh-real-at-left-1*:
$filterlim\ (artanh :: real \Rightarrow real)\ at\text{-}top\ (at\text{-}left\ 1)$
⟨*proof*⟩

lemma *artanh-real-at-right-1*:
$filterlim\ (artanh :: real \Rightarrow real)\ at\text{-}bot\ (at\text{-}right\ (-1))$
⟨*proof*⟩

113.24 Simprocs for root and power literals

lemma *numeral-powr-numeral-real* [*simp*]:
$numeral\ m\ powr\ numeral\ n = (numeral\ m\ \hat{}\ numeral\ n :: real)$
⟨*proof*⟩

context
begin

private lemma *sqrt-numeral-simproc-aux*:
assumes $m * m \equiv n$
shows *sqrt* (*numeral n* :: *real*) ≡ *numeral m*
⟨*proof*⟩ **lemma** *root-numeral-simproc-aux*:
assumes *Num.pow m n* ≡ *x*
shows *root* (*numeral n*) (*numeral x* :: *real*) ≡ *numeral m*
⟨*proof*⟩ **lemma** *powr-numeral-simproc-aux*:
assumes *Num.pow y n* = *x*
shows *numeral x powr* (*m* / *numeral n* :: *real*) ≡ *numeral y powr m*
⟨*proof*⟩ **lemma** *numeral-powr-inverse-eq*:
numeral x powr (*inverse* (*numeral n*)) = *numeral x powr* (*1* / *numeral n* :: *real*)
⟨*proof*⟩

⟨*ML*⟩

end

⟨*ML*⟩

lemma *root 100 1267650600228229401496703205376* = *2*
⟨*proof*⟩

lemma *sqrt 196* = *14*
⟨*proof*⟩

lemma *256 powr* (*7* / *4* :: *real*) = *16384*
⟨*proof*⟩

lemma *27 powr* (*inverse 3*) = (*3*::*real*)
⟨*proof*⟩

end

114 Complex Numbers: Rectangular and Polar Representations

theory *Complex*
imports *Transcendental Real-Vector-Spaces*
begin

We use the **codatatype** command to define the type of complex numbers. This allows us to use **primcorec** to define complex functions by defining their real and imaginary result separately.

codatatype *complex* = *Complex* (*Re*: *real*) (*Im*: *real*)

lemma *complex-surj*: *Complex* (*Re z*) (*Im z*) = *z*
⟨*proof*⟩

lemma *complex-eqI* [*intro?*]: *Re x* = *Re y* ⟹ *Im x* = *Im y* ⟹ *x* = *y*
⟨*proof*⟩

lemma *complex-eq-iff*: *x* = *y* ⟷ *Re x* = *Re y* ∧ *Im x* = *Im y*
⟨*proof*⟩

114.1 Addition and Subtraction

instantiation *complex* :: *ab-group-add*
begin

primcorec *zero-complex*
where
Re 0 = *0*
| *Im 0* = *0*

primcorec *plus-complex*
where
Re (*x* + *y*) = *Re x* + *Re y*
| *Im* (*x* + *y*) = *Im x* + *Im y*

primcorec *uminus-complex*
where
Re (− *x*) = − *Re x*
| *Im* (− *x*) = − *Im x*

primcorec *minus-complex*
where
Re (*x* − *y*) = *Re x* − *Re y*
| *Im* (*x* − *y*) = *Im x* − *Im y*

instance
⟨*proof*⟩

end

114.2 Multiplication and Division

instantiation *complex* :: *field*
begin

primcorec *one-complex*
where
Re 1 = *1*
| *Im 1* = *0*

primcorec *times-complex*
where
Re $(x * y) = Re\ x * Re\ y - Im\ x * Im\ y$
| *Im* $(x * y) = Re\ x * Im\ y + Im\ x * Re\ y$

primcorec *inverse-complex*
where
Re $(inverse\ x) = Re\ x\ /\ ((Re\ x)^2 + (Im\ x)^2)$
| *Im* $(inverse\ x) = -\ Im\ x\ /\ ((Re\ x)^2 + (Im\ x)^2)$

definition $x\ div\ y = x * inverse\ y$ **for** $x\ y$ :: *complex*

instance
⟨*proof*⟩

end

lemma *Re-divide*: $Re\ (x\ /\ y) = (Re\ x * Re\ y + Im\ x * Im\ y)\ /\ ((Re\ y)^2 + (Im\ y)^2)$
⟨*proof*⟩

lemma *Im-divide*: $Im\ (x\ /\ y) = (Im\ x * Re\ y - Re\ x * Im\ y)\ /\ ((Re\ y)^2 + (Im\ y)^2)$
⟨*proof*⟩

lemma *Complex-divide*:
$(x\ /\ y) = Complex\ ((Re\ x * Re\ y + Im\ x * Im\ y)\ /\ ((Re\ y)^2 + (Im\ y)^2))$
$((Im\ x * Re\ y - Re\ x * Im\ y)\ /\ ((Re\ y)^2 + (Im\ y)^2))$
⟨*proof*⟩

lemma *Re-power2*: $Re\ (x\ \hat{}\ 2) = (Re\ x)\hat{}2 - (Im\ x)\hat{}2$
⟨*proof*⟩

lemma *Im-power2*: $Im\ (x\ \hat{}\ 2) = 2 * Re\ x * Im\ x$
⟨*proof*⟩

lemma *Re-power-real* [*simp*]: $Im\ x = 0 \Longrightarrow Re\ (x\ \hat{}\ n) = Re\ x\ \hat{}\ n$
⟨*proof*⟩

lemma *Im-power-real* [*simp*]: $Im\ x = 0 \Longrightarrow Im\ (x\ \hat{}\ n) = 0$
⟨*proof*⟩

114.3 Scalar Multiplication

instantiation *complex* :: *real-field*
begin

primcorec *scaleR-complex*

where
Re (*scaleR r x*) = *r* ∗ *Re x*
| *Im* (*scaleR r x*) = *r* ∗ *Im x*

instance
⟨*proof*⟩

end

114.4 Numerals, Arithmetic, and Embedding from R

declare [[*coercion of-real* :: *real* ⇒ *complex*]]
declare [[*coercion of-rat* :: *rat* ⇒ *complex*]]
declare [[*coercion of-int* :: *int* ⇒ *complex*]]
declare [[*coercion of-nat* :: *nat* ⇒ *complex*]]

abbreviation *complex-of-nat*::*nat* ⇒ *complex*
where *complex-of-nat* ≡ *of-nat*

abbreviation *complex-of-int*::*int* ⇒ *complex*
where *complex-of-int* ≡ *of-int*

abbreviation *complex-of-rat*::*rat* ⇒ *complex*
where *complex-of-rat* ≡ *of-rat*

abbreviation *complex-of-real* :: *real* ⇒ *complex*
where *complex-of-real* ≡ *of-real*

lemma *complex-Re-of-nat* [*simp*]: *Re* (*of-nat n*) = *of-nat n*
⟨*proof*⟩

lemma *complex-Im-of-nat* [*simp*]: *Im* (*of-nat n*) = *0*
⟨*proof*⟩

lemma *complex-Re-of-int* [*simp*]: *Re* (*of-int z*) = *of-int z*
⟨*proof*⟩

lemma *complex-Im-of-int* [*simp*]: *Im* (*of-int z*) = *0*
⟨*proof*⟩

lemma *complex-Re-numeral* [*simp*]: *Re* (*numeral v*) = *numeral v*
⟨*proof*⟩

lemma *complex-Im-numeral* [*simp*]: *Im* (*numeral v*) = *0*
⟨*proof*⟩

lemma *Re-complex-of-real* [*simp*]: *Re* (*complex-of-real z*) = *z*
⟨*proof*⟩

lemma *Im-complex-of-real* [*simp*]: *Im* (*complex-of-real z*) = *0*
⟨*proof*⟩

lemma *Re-divide-numeral* [*simp*]: *Re* (*z* / *numeral w*) = *Re z* / *numeral w*
⟨*proof*⟩

lemma *Im-divide-numeral* [*simp*]: *Im* (*z* / *numeral w*) = *Im z* / *numeral w*
⟨*proof*⟩

lemma *Re-divide-of-nat* [*simp*]: *Re* (*z* / *of-nat n*) = *Re z* / *of-nat n*
⟨*proof*⟩

lemma *Im-divide-of-nat* [*simp*]: *Im* (*z* / *of-nat n*) = *Im z* / *of-nat n*
⟨*proof*⟩

lemma *Re-inverse* [*simp*]: *r* ∈ ℝ ⟹ *Re* (*inverse r*) = *inverse* (*Re r*)
⟨*proof*⟩

lemma *Im-inverse* [*simp*]: *r* ∈ ℝ ⟹ *Im* (*inverse r*) = *0*
⟨*proof*⟩

lemma *of-real-Re* [*simp*]: *z* ∈ ℝ ⟹ *of-real* (*Re z*) = *z*
⟨*proof*⟩

lemma *complex-Re-fact* [*simp*]: *Re* (*fact n*) = *fact n*
⟨*proof*⟩

lemma *surj-Re*: *surj Re*
⟨*proof*⟩

lemma *surj-Im*: *surj Im*
⟨*proof*⟩

lemma *complex-Im-fact* [*simp*]: *Im* (*fact n*) = *0*
⟨*proof*⟩

lemma *Re-prod-Reals*: (⋀*x*. *x* ∈ *A* ⟹ *f x* ∈ ℝ) ⟹ *Re* (*prod f A*) = *prod* (λ*x*. *Re* (*f x*)) *A*
⟨*proof*⟩

114.5 The Complex Number *i*

primcorec *imaginary-unit* :: *complex* (i)
where
Re i = *0*
| *Im* i = *1*

lemma *Complex-eq*: *Complex a b* = *a* + i ∗ *b*
⟨*proof*⟩

lemma *complex-eq*: $a = Re\ a + \mathrm{i} * Im\ a$
⟨*proof*⟩

lemma *fun-complex-eq*: $f = (\lambda x.\ Re\ (f\ x) + \mathrm{i} * Im\ (f\ x))$
⟨*proof*⟩

lemma *i-squared* [*simp*]: $\mathrm{i} * \mathrm{i} = -1$
⟨*proof*⟩

lemma *power2-i* [*simp*]: $\mathrm{i}^2 = -1$
⟨*proof*⟩

lemma *inverse-i* [*simp*]: *inverse* $\mathrm{i} = -\mathrm{i}$
⟨*proof*⟩

lemma *divide-i* [*simp*]: $x\ /\ \mathrm{i} = -\mathrm{i} * x$
⟨*proof*⟩

lemma *complex-i-mult-minus* [*simp*]: $\mathrm{i} * (\mathrm{i} * x) = -x$
⟨*proof*⟩

lemma *complex-i-not-zero* [*simp*]: $\mathrm{i} \neq 0$
⟨*proof*⟩

lemma *complex-i-not-one* [*simp*]: $\mathrm{i} \neq 1$
⟨*proof*⟩

lemma *complex-i-not-numeral* [*simp*]: $\mathrm{i} \neq numeral\ w$
⟨*proof*⟩

lemma *complex-i-not-neg-numeral* [*simp*]: $\mathrm{i} \neq -\ numeral\ w$
⟨*proof*⟩

lemma *complex-split-polar*: $\exists r\ a.\ z = complex\text{-}of\text{-}real\ r * (cos\ a + \mathrm{i} * sin\ a)$
⟨*proof*⟩

lemma *i-even-power* [*simp*]: $\mathrm{i}\ \hat{}\ (n * 2) = (-1)\ \hat{}\ n$
⟨*proof*⟩

lemma *i-even-power'* [*simp*]: $even\ n \Longrightarrow \mathrm{i}\ \hat{}\ n = (-1)\ \hat{}\ (n\ div\ 2)$
⟨*proof*⟩

lemma *Re-i-times* [*simp*]: $Re\ (\mathrm{i} * z) = -\ Im\ z$
⟨*proof*⟩

lemma *Im-i-times* [*simp*]: $Im\ (\mathrm{i} * z) = Re\ z$
⟨*proof*⟩

lemma *i-times-eq-iff*: i $*$ $w = z \longleftrightarrow w = -$ (i $*$ z)
⟨*proof*⟩

lemma *divide-numeral-i* [*simp*]: z / (*numeral* n $*$ i) $= -$ (i $*$ z) / *numeral* n
⟨*proof*⟩

lemma *imaginary-eq-real-iff* [*simp*]:
assumes $y \in$ *Reals* $x \in$ *Reals*
shows i $*$ $y = x \longleftrightarrow x{=}0 \wedge y{=}0$
⟨*proof*⟩

lemma *real-eq-imaginary-iff* [*simp*]:
assumes $y \in$ *Reals* $x \in$ *Reals*
shows $x =$ i $*$ $y \longleftrightarrow x{=}0 \wedge y{=}0$
⟨*proof*⟩

114.6 Vector Norm

instantiation *complex* :: *real-normed-field*
begin

definition *norm* z = *sqrt* $((Re\ z)^2 + (Im\ z)^2)$

abbreviation *cmod* :: *complex* ⇒ *real*
where *cmod* ≡ *norm*

definition *complex-sgn-def*: *sgn* $x = x\ /_R$ *cmod* x

definition *dist-complex-def*: *dist* $x\ y$ = *cmod* $(x - y)$

definition *uniformity-complex-def* [*code del*]:
(*uniformity* :: (*complex* × *complex*) *filter*) = (*INF* e∈{0 <..}. *principal* {(x, y). *dist* $x\ y < e$})

definition *open-complex-def* [*code del*]:
open (U :: *complex set*) ⟷ (∀ x∈U. *eventually* (λ(x', y). $x' = x \longrightarrow y \in U$) *uniformity*)

instance
⟨*proof*⟩

end

declare *uniformity-Abort*[**where** *'a* = *complex*, *code*]

lemma *norm-ii* [*simp*]: *norm* i $=$ *1*
⟨*proof*⟩

lemma *cmod-unit-one*: *cmod* (*cos* a + i $*$ *sin* a) = *1*

⟨*proof*⟩

lemma *cmod-complex-polar*: *cmod* $(r * (cos\ a + \mathrm{i} * sin\ a)) = |r|$
⟨*proof*⟩

lemma *complex-Re-le-cmod*: $Re\ x \leq cmod\ x$
⟨*proof*⟩

lemma *complex-mod-minus-le-complex-mod*: $-\ cmod\ x \leq cmod\ x$
⟨*proof*⟩

lemma *complex-mod-triangle-ineq2*: $cmod\ (b + a) - cmod\ b \leq cmod\ a$
⟨*proof*⟩

lemma *abs-Re-le-cmod*: $|Re\ x| \leq cmod\ x$
⟨*proof*⟩

lemma *abs-Im-le-cmod*: $|Im\ x| \leq cmod\ x$
⟨*proof*⟩

lemma *cmod-le*: $cmod\ z \leq |Re\ z| + |Im\ z|$
⟨*proof*⟩

lemma *cmod-eq-Re*: $Im\ z = 0 \Longrightarrow cmod\ z = |Re\ z|$
⟨*proof*⟩

lemma *cmod-eq-Im*: $Re\ z = 0 \Longrightarrow cmod\ z = |Im\ z|$
⟨*proof*⟩

lemma *cmod-power2*: $(cmod\ z)^2 = (Re\ z)^2 + (Im\ z)^2$
⟨*proof*⟩

lemma *cmod-plus-Re-le-0-iff*: $cmod\ z + Re\ z \leq 0 \longleftrightarrow Re\ z = -\ cmod\ z$
⟨*proof*⟩

lemma *cmod-Re-le-iff*: $Im\ x = Im\ y \Longrightarrow cmod\ x \leq cmod\ y \longleftrightarrow |Re\ x| \leq |Re\ y|$
⟨*proof*⟩

lemma *cmod-Im-le-iff*: $Re\ x = Re\ y \Longrightarrow cmod\ x \leq cmod\ y \longleftrightarrow |Im\ x| \leq |Im\ y|$
⟨*proof*⟩

lemma *Im-eq-0*: $|Re\ z| = cmod\ z \Longrightarrow Im\ z = 0$
⟨*proof*⟩

lemma *abs-sqrt-wlog*: $(\bigwedge x.\ x \geq 0 \Longrightarrow P\ x\ (x^2)) \Longrightarrow P\ |x|\ (x^2)$
for x::$'a$::*linordered-idom*
⟨*proof*⟩

lemma *complex-abs-le-norm*: $|Re\ z| + |Im\ z| \leq sqrt\ 2 * norm\ z$

⟨*proof*⟩

lemma *complex-unit-circle*: $z \neq 0 \Longrightarrow (Re\ z\ /\ cmod\ z)^2 + (Im\ z\ /\ cmod\ z)^2 = 1$
⟨*proof*⟩

Properties of complex signum.

lemma *sgn-eq*: $sgn\ z = z\ /\ complex\text{-}of\text{-}real\ (cmod\ z)$
⟨*proof*⟩

lemma *Re-sgn* [*simp*]: $Re(sgn\ z) = Re(z)/cmod\ z$
⟨*proof*⟩

lemma *Im-sgn* [*simp*]: $Im(sgn\ z) = Im(z)/cmod\ z$
⟨*proof*⟩

114.7 Absolute value

instantiation *complex* :: *field-abs-sgn*
begin

definition *abs-complex* :: *complex* ⇒ *complex*
where *abs-complex* = *of-real* ∘ *norm*

instance
⟨*proof*⟩
end

114.8 Completeness of the Complexes

lemma *bounded-linear-Re*: *bounded-linear Re*
⟨*proof*⟩

lemma *bounded-linear-Im*: *bounded-linear Im*
⟨*proof*⟩

lemmas *Cauchy-Re* = *bounded-linear.Cauchy* [*OF bounded-linear-Re*]
lemmas *Cauchy-Im* = *bounded-linear.Cauchy* [*OF bounded-linear-Im*]
lemmas *tendsto-Re* [*tendsto-intros*] = *bounded-linear.tendsto* [*OF bounded-linear-Re*]
lemmas *tendsto-Im* [*tendsto-intros*] = *bounded-linear.tendsto* [*OF bounded-linear-Im*]
lemmas *isCont-Re* [*simp*] = *bounded-linear.isCont* [*OF bounded-linear-Re*]
lemmas *isCont-Im* [*simp*] = *bounded-linear.isCont* [*OF bounded-linear-Im*]
lemmas *continuous-Re* [*simp*] = *bounded-linear.continuous* [*OF bounded-linear-Re*]
lemmas *continuous-Im* [*simp*] = *bounded-linear.continuous* [*OF bounded-linear-Im*]
lemmas *continuous-on-Re* [*continuous-intros*] = *bounded-linear.continuous-on*[*OF bounded-linear-Re*]
lemmas *continuous-on-Im* [*continuous-intros*] = *bounded-linear.continuous-on*[*OF bounded-linear-Im*]
lemmas *has-derivative-Re* [*derivative-intros*] = *bounded-linear.has-derivative*[*OF bounded-linear-Re*]

lemmas *has-derivative-Im* [*derivative-intros*] = *bounded-linear.has-derivative*[*OF bounded-linear-Im*]
lemmas *sums-Re* = *bounded-linear.sums* [*OF bounded-linear-Re*]
lemmas *sums-Im* = *bounded-linear.sums* [*OF bounded-linear-Im*]
lemmas *Re-suminf* = *bounded-linear.suminf*[*OF bounded-linear-Re*]
lemmas *Im-suminf* = *bounded-linear.suminf*[*OF bounded-linear-Im*]

lemma *tendsto-Complex* [*tendsto-intros*]:
(*f* ⟶ *a*) *F* ⟹ (*g* ⟶ *b*) *F* ⟹ ((λ*x*. *Complex* (*f x*) (*g x*)) ⟶ *Complex a b*) *F*
⟨*proof*⟩

lemma *tendsto-complex-iff*:
(*f* ⟶ *x*) *F* ⟷ (((λ*x*. *Re* (*f x*)) ⟶ *Re x*) *F* ∧ ((λ*x*. *Im* (*f x*)) ⟶ *Im x*) *F*)
⟨*proof*⟩

lemma *continuous-complex-iff*:
continuous F f ⟷ *continuous F* (λ*x*. *Re* (*f x*)) ∧ *continuous F* (λ*x*. *Im* (*f x*))
⟨*proof*⟩

lemma *continuous-on-of-real-o-iff* [*simp*]:
continuous-on S (λ*x*. *complex-of-real* (*g x*)) = *continuous-on S g*
⟨*proof*⟩

lemma *continuous-on-of-real-id* [*simp*]:
continuous-on S (*of-real* :: *real* ⇒ *'a*::*real-normed-algebra-1*)
⟨*proof*⟩

lemma *has-vector-derivative-complex-iff*: (*f has-vector-derivative x*) *F* ⟷
((λ*x*. *Re* (*f x*)) *has-field-derivative* (*Re x*)) *F* ∧
((λ*x*. *Im* (*f x*)) *has-field-derivative* (*Im x*)) *F*
⟨*proof*⟩

lemma *has-field-derivative-Re*[*derivative-intros*]:
(*f has-vector-derivative D*) *F* ⟹ ((λ*x*. *Re* (*f x*)) *has-field-derivative* (*Re D*)) *F*
⟨*proof*⟩

lemma *has-field-derivative-Im*[*derivative-intros*]:
(*f has-vector-derivative D*) *F* ⟹ ((λ*x*. *Im* (*f x*)) *has-field-derivative* (*Im D*)) *F*
⟨*proof*⟩

instance *complex* :: *banach*
⟨*proof*⟩

declare *DERIV-power*[**where** *'a*=*complex*, *unfolded of-nat-def*[*symmetric*], *derivative-intros*]

114.9 Complex Conjugation

primcorec *cnj* :: *complex* ⇒ *complex*
where
Re (*cnj z*) = *Re z*
| *Im* (*cnj z*) = − *Im z*

lemma *complex-cnj-cancel-iff* [*simp*]: *cnj x* = *cnj y* ⟷ *x* = *y*
⟨*proof*⟩

lemma *complex-cnj-cnj* [*simp*]: *cnj* (*cnj z*) = *z*
⟨*proof*⟩

lemma *in-image-cnj-iff*: *z* ∈ *cnj* ‘ *A* ⟷ *cnj z* ∈ *A*
⟨*proof*⟩

lemma *image-cnj-conv-vimage-cnj*: *cnj* ‘ *A* = *cnj* −‘ *A*
⟨*proof*⟩

lemma *complex-cnj-zero* [*simp*]: *cnj* 0 = 0
⟨*proof*⟩

lemma *complex-cnj-zero-iff* [*iff*]: *cnj z* = 0 ⟷ *z* = 0
⟨*proof*⟩

lemma *complex-cnj-one-iff* [*simp*]: *cnj z* = 1 ⟷ *z* = 1
⟨*proof*⟩

lemma *complex-cnj-add* [*simp*]: *cnj* (*x* + *y*) = *cnj x* + *cnj y*
⟨*proof*⟩

lemma *cnj-sum* [*simp*]: *cnj* (*sum f s*) = (∑ *x*∈*s*. *cnj* (*f x*))
⟨*proof*⟩

lemma *complex-cnj-diff* [*simp*]: *cnj* (*x* − *y*) = *cnj x* − *cnj y*
⟨*proof*⟩

lemma *complex-cnj-minus* [*simp*]: *cnj* (− *x*) = − *cnj x*
⟨*proof*⟩

lemma *complex-cnj-one* [*simp*]: *cnj* 1 = 1
⟨*proof*⟩

lemma *complex-cnj-mult* [*simp*]: *cnj* (*x* ∗ *y*) = *cnj x* ∗ *cnj y*
⟨*proof*⟩

lemma *cnj-prod* [*simp*]: *cnj* (*prod f s*) = (∏ *x*∈*s*. *cnj* (*f x*))
⟨*proof*⟩

lemma *complex-cnj-inverse* [*simp*]: *cnj* (*inverse x*) = *inverse* (*cnj x*)

⟨*proof*⟩

lemma *complex-cnj-divide* [*simp*]: *cnj* (*x* / *y*) = *cnj x* / *cnj y*
⟨*proof*⟩

lemma *complex-cnj-power* [*simp*]: *cnj* (*x* ^ *n*) = *cnj x* ^ *n*
⟨*proof*⟩

lemma *complex-cnj-of-nat* [*simp*]: *cnj* (*of-nat n*) = *of-nat n*
⟨*proof*⟩

lemma *complex-cnj-of-int* [*simp*]: *cnj* (*of-int z*) = *of-int z*
⟨*proof*⟩

lemma *complex-cnj-numeral* [*simp*]: *cnj* (*numeral w*) = *numeral w*
⟨*proof*⟩

lemma *complex-cnj-neg-numeral* [*simp*]: *cnj* (− *numeral w*) = − *numeral w*
⟨*proof*⟩

lemma *complex-cnj-scaleR* [*simp*]: *cnj* (*scaleR r x*) = *scaleR r* (*cnj x*)
⟨*proof*⟩

lemma *complex-mod-cnj* [*simp*]: *cmod* (*cnj z*) = *cmod z*
⟨*proof*⟩

lemma *complex-cnj-complex-of-real* [*simp*]: *cnj* (*of-real x*) = *of-real x*
⟨*proof*⟩

lemma *complex-cnj-i* [*simp*]: *cnj* i = − i
⟨*proof*⟩

lemma *complex-add-cnj*: *z* + *cnj z* = *complex-of-real* (2 ∗ *Re z*)
⟨*proof*⟩

lemma *complex-diff-cnj*: *z* − *cnj z* = *complex-of-real* (2 ∗ *Im z*) ∗ i
⟨*proof*⟩

lemma *Ints-cnj* [*intro*]: *x* ∈ ℤ ⟹ *cnj x* ∈ ℤ
⟨*proof*⟩

lemma *cnj-in-Ints-iff* [*simp*]: *cnj x* ∈ ℤ ⟷ *x* ∈ ℤ
⟨*proof*⟩

lemma *complex-mult-cnj*: *z* ∗ *cnj z* = *complex-of-real* ((*Re z*)^2 + (*Im z*)^2)
⟨*proof*⟩

lemma *cnj-add-mult-eq-Re*: *z* ∗ *cnj w* + *cnj z* ∗ *w* = 2 ∗ *Re* (*z* ∗ *cnj w*)
⟨*proof*⟩

lemma *complex-mod-mult-cnj*: $cmod\ (z * cnj\ z) = (cmod\ z)^2$
⟨*proof*⟩

lemma *complex-mod-sqrt-Re-mult-cnj*: $cmod\ z = sqrt\ (Re\ (z * cnj\ z))$
⟨*proof*⟩

lemma *complex-In-mult-cnj-zero* [*simp*]: $Im\ (z * cnj\ z) = 0$
⟨*proof*⟩

lemma *complex-cnj-fact* [*simp*]: $cnj\ (fact\ n) = fact\ n$
⟨*proof*⟩

lemma *complex-cnj-pochhammer* [*simp*]: $cnj\ (pochhammer\ z\ n) = pochhammer\ (cnj\ z)\ n$
⟨*proof*⟩

lemma *bounded-linear-cnj*: *bounded-linear cnj*
⟨*proof*⟩

lemma *linear-cnj*: *linear cnj*
⟨*proof*⟩

lemmas *tendsto-cnj* [*tendsto-intros*] = *bounded-linear.tendsto* [*OF bounded-linear-cnj*]
and *isCont-cnj* [*simp*] = *bounded-linear.isCont* [*OF bounded-linear-cnj*]
and *continuous-cnj* [*simp*, *continuous-intros*] = *bounded-linear.continuous* [*OF bounded-linear-cnj*]
and *continuous-on-cnj* [*simp*, *continuous-intros*] = *bounded-linear.continuous-on* [*OF bounded-linear-cnj*]
and *has-derivative-cnj* [*simp*, *derivative-intros*] = *bounded-linear.has-derivative* [*OF bounded-linear-cnj*]

lemma *lim-cnj*: $((\lambda x.\ cnj(f\ x)) \longrightarrow cnj\ l)\ F \longleftrightarrow (f \longrightarrow l)\ F$
⟨*proof*⟩

lemma *sums-cnj*: $((\lambda x.\ cnj(f\ x))\ sums\ cnj\ l) \longleftrightarrow (f\ sums\ l)$
⟨*proof*⟩

lemma *differentiable-cnj-iff*:
$(\lambda z.\ cnj\ (f\ z))$ *differentiable at x within A* $\longleftrightarrow$ *f differentiable at x within A*
⟨*proof*⟩

lemma *has-vector-derivative-cnj* [*derivative-intros*]:
assumes $(f\ has\text{-}vector\text{-}derivative\ f')\ (at\ z\ within\ A)$
shows $((\lambda z.\ cnj\ (f\ z))\ has\text{-}vector\text{-}derivative\ cnj\ f')\ (at\ z\ within\ A)$
⟨*proof*⟩

lemma *has-field-derivative-cnj-cnj*:
assumes $(f\ has\text{-}field\text{-}derivative\ F)\ (at\ (cnj\ z))$

shows $((cnj \circ f \circ cnj)\ has\text{-}field\text{-}derivative\ cnj\ F)\ (at\ z)$
⟨*proof*⟩

114.10 Basic Lemmas

lemma *complex-of-real-code*[*code-unfold*]: $of\text{-}real = (\lambda x.\ Complex\ x\ 0)$
⟨*proof*⟩

lemma *complex-eq-0*: $z{=}0 \longleftrightarrow (Re\ z)^2 + (Im\ z)^2 = 0$
⟨*proof*⟩

lemma *complex-neq-0*: $z{\neq}0 \longleftrightarrow (Re\ z)^2 + (Im\ z)^2 > 0$
⟨*proof*⟩

lemma *complex-norm-square*: $of\text{-}real\ ((norm\ z)^2) = z * cnj\ z$
⟨*proof*⟩

lemma *complex-div-cnj*: $a\ /\ b = (a * cnj\ b)\ /\ (norm\ b)^2$
⟨*proof*⟩

lemma *Re-complex-div-eq-0*: $Re\ (a\ /\ b) = 0 \longleftrightarrow Re\ (a * cnj\ b) = 0$
⟨*proof*⟩

lemma *Im-complex-div-eq-0*: $Im\ (a\ /\ b) = 0 \longleftrightarrow Im\ (a * cnj\ b) = 0$
⟨*proof*⟩

lemma *complex-div-gt-0*: $(Re\ (a\ /\ b) > 0 \longleftrightarrow Re\ (a * cnj\ b) > 0) \wedge (Im\ (a\ /\ b) > 0 \longleftrightarrow Im\ (a * cnj\ b) > 0)$
⟨*proof*⟩

lemma *Re-complex-div-gt-0*: $Re\ (a\ /\ b) > 0 \longleftrightarrow Re\ (a * cnj\ b) > 0$
and *Im-complex-div-gt-0*: $Im\ (a\ /\ b) > 0 \longleftrightarrow Im\ (a * cnj\ b) > 0$
⟨*proof*⟩

lemma *Re-complex-div-ge-0*: $Re\ (a\ /\ b) \geq 0 \longleftrightarrow Re\ (a * cnj\ b) \geq 0$
⟨*proof*⟩

lemma *Im-complex-div-ge-0*: $Im\ (a\ /\ b) \geq 0 \longleftrightarrow Im\ (a * cnj\ b) \geq 0$
⟨*proof*⟩

lemma *Re-complex-div-lt-0*: $Re\ (a\ /\ b) < 0 \longleftrightarrow Re\ (a * cnj\ b) < 0$
⟨*proof*⟩

lemma *Im-complex-div-lt-0*: $Im\ (a\ /\ b) < 0 \longleftrightarrow Im\ (a * cnj\ b) < 0$
⟨*proof*⟩

lemma *Re-complex-div-le-0*: $Re\ (a\ /\ b) \leq 0 \longleftrightarrow Re\ (a * cnj\ b) \leq 0$
⟨*proof*⟩

lemma *Im-complex-div-le-0*: $Im\ (a\ /\ b) \leq 0 \longleftrightarrow Im\ (a * cnj\ b) \leq 0$
⟨*proof*⟩

lemma *Re-divide-of-real* [*simp*]: $Re\ (z\ /\ of\text{-}real\ r) = Re\ z\ /\ r$
⟨*proof*⟩

lemma *Im-divide-of-real* [*simp*]: $Im\ (z\ /\ of\text{-}real\ r) = Im\ z\ /\ r$
⟨*proof*⟩

lemma *Re-divide-Reals* [*simp*]: $r \in \mathbb{R} \Longrightarrow Re\ (z\ /\ r) = Re\ z\ /\ Re\ r$
⟨*proof*⟩

lemma *Im-divide-Reals* [*simp*]: $r \in \mathbb{R} \Longrightarrow Im\ (z\ /\ r) = Im\ z\ /\ Re\ r$
⟨*proof*⟩

lemma *Re-sum*[*simp*]: $Re\ (sum\ f\ s) = (\sum x{\in}s.\ Re\ (f\ x))$
⟨*proof*⟩

lemma *Im-sum*[*simp*]: $Im\ (sum\ f\ s) = (\sum x{\in}s.\ Im(f\ x))$
⟨*proof*⟩

lemma *sum-Re-le-cmod*: $(\sum i{\in}I.\ Re\ (z\ i)) \leq cmod\ (\sum i{\in}I.\ z\ i)$
⟨*proof*⟩

lemma *sum-Im-le-cmod*: $(\sum i{\in}I.\ Im\ (z\ i)) \leq cmod\ (\sum i{\in}I.\ z\ i)$
⟨*proof*⟩

lemma *sums-complex-iff*: $f\ sums\ x \longleftrightarrow ((\lambda x.\ Re\ (f\ x))\ sums\ Re\ x) \wedge ((\lambda x.\ Im\ (f\ x))\ sums\ Im\ x)$
⟨*proof*⟩

lemma *summable-complex-iff*: $summable\ f \longleftrightarrow summable\ (\lambda x.\ Re\ (f\ x)) \wedge summable\ (\lambda x.\ Im\ (f\ x))$
⟨*proof*⟩

lemma *summable-complex-of-real* [*simp*]: $summable\ (\lambda n.\ complex\text{-}of\text{-}real\ (f\ n)) \longleftrightarrow summable\ f$
⟨*proof*⟩

lemma *summable-Re*: $summable\ f \Longrightarrow summable\ (\lambda x.\ Re\ (f\ x))$
⟨*proof*⟩

lemma *summable-Im*: $summable\ f \Longrightarrow summable\ (\lambda x.\ Im\ (f\ x))$
⟨*proof*⟩

lemma *complex-is-Nat-iff*: $z \in \mathbb{N} \longleftrightarrow Im\ z = 0 \wedge (\exists i.\ Re\ z = of\text{-}nat\ i)$
⟨*proof*⟩

lemma *complex-is-Int-iff*: $z \in \mathbb{Z} \longleftrightarrow Im\ z = 0 \wedge (\exists i.\ Re\ z = of\text{-}int\ i)$

⟨*proof*⟩

lemma *complex-is-Real-iff*: $z \in \mathbb{R} \longleftrightarrow Im\ z = 0$
⟨*proof*⟩

lemma *Reals-cnj-iff*: $z \in \mathbb{R} \longleftrightarrow cnj\ z = z$
⟨*proof*⟩

lemma *in-Reals-norm*: $z \in \mathbb{R} \Longrightarrow norm\ z = |Re\ z|$
⟨*proof*⟩

lemma *Re-Reals-divide*: $r \in \mathbb{R} \Longrightarrow Re\ (r\ /\ z) = Re\ r * Re\ z\ /\ (norm\ z)^2$
⟨*proof*⟩

lemma *Im-Reals-divide*: $r \in \mathbb{R} \Longrightarrow Im\ (r\ /\ z) = -Re\ r * Im\ z\ /\ (norm\ z)^2$
⟨*proof*⟩

lemma *series-comparison-complex*:
fixes f:: *nat* ⇒ $'a$::*banach*
assumes *sg*: *summable g*
and $\bigwedge n.\ g\ n \in \mathbb{R}\ \bigwedge n.\ Re\ (g\ n) \geq 0$
and *fg*: $\bigwedge n.\ n \geq N \Longrightarrow norm(f\ n) \leq norm(g\ n)$
shows *summable f*
⟨*proof*⟩

114.11 Polar Form for Complex Numbers

lemma *complex-unimodular-polar*:
assumes *norm z = 1*
obtains *t* **where** $0 \leq t\ t < 2 * pi\ z = Complex\ (cos\ t)\ (sin\ t)$
⟨*proof*⟩

114.11.1 $\cos\theta + i\sin\theta$

primcorec *cis* :: *real* ⇒ *complex*
where
Re (*cis a*) = *cos a*
| *Im* (*cis a*) = *sin a*

lemma *cis-zero* [*simp*]: *cis 0 = 1*
⟨*proof*⟩

lemma *norm-cis* [*simp*]: *norm* (*cis a*) = *1*
⟨*proof*⟩

lemma *sgn-cis* [*simp*]: *sgn* (*cis a*) = *cis a*
⟨*proof*⟩

lemma *cis-2pi* [*simp*]: *cis* (*2* ∗ *pi*) = *1*
⟨*proof*⟩

lemma *cis-neq-zero* [*simp*]: $cis\ a \neq 0$
⟨*proof*⟩

lemma *cis-cnj*: $cnj\ (cis\ t) = cis\ (-t)$
⟨*proof*⟩

lemma *cis-mult*: $cis\ a * cis\ b = cis\ (a + b)$
⟨*proof*⟩

lemma *DeMoivre*: $(cis\ a)\ \hat{}\ n = cis\ (real\ n * a)$
⟨*proof*⟩

lemma *cis-inverse* [*simp*]: $inverse\ (cis\ a) = cis\ (-\ a)$
⟨*proof*⟩

lemma *cis-divide*: $cis\ a\ /\ cis\ b = cis\ (a - b)$
⟨*proof*⟩

lemma *divide-conv-cnj*: $norm\ z = 1 \Longrightarrow x\ /\ z = x * cnj\ z$
⟨*proof*⟩

lemma *i-not-in-Reals* [*simp*, *intro*]: $\mathrm{i} \notin \mathbb{R}$
⟨*proof*⟩

lemma *powr-power-complex*: $z \neq 0 \vee n \neq 0 \Longrightarrow (z\ powr\ u :: complex)\ \hat{}\ n = z\ powr\ (of\text{-}nat\ n * u)$
⟨*proof*⟩

lemma *cos-n-Re-cis-pow-n*: $cos\ (real\ n * a) = Re\ (cis\ a\ \hat{}\ n)$
⟨*proof*⟩

lemma *sin-n-Im-cis-pow-n*: $sin\ (real\ n * a) = Im\ (cis\ a\ \hat{}\ n)$
⟨*proof*⟩

lemma *cis-pi* [*simp*]: $cis\ pi = -1$
⟨*proof*⟩

lemma *cis-pi-half*[*simp*]: $cis\ (pi\ /\ 2) = \mathrm{i}$
⟨*proof*⟩

lemma *cis-minus-pi-half*[*simp*]: $cis\ (-(pi\ /\ 2)) = -\mathrm{i}$
⟨*proof*⟩

lemma *cis-multiple-2pi*[*simp*]: $n \in \mathbb{Z} \Longrightarrow cis\ (2 * pi * n) = 1$
⟨*proof*⟩

lemma *minus-cis*: $-cis\ x = cis\ (x + pi)$
⟨*proof*⟩

lemma *minus-cis'*: $-cis\ x = cis\ (x - pi)$
⟨*proof*⟩

114.11.2 $r(\cos\theta + i\sin\theta)$

definition *rcis* :: *real* ⇒ *real* ⇒ *complex*
where *rcis r a* = *complex-of-real r* ∗ *cis a*

lemma *Re-rcis* [*simp*]: $Re(rcis\ r\ a) = r * cos\ a$
⟨*proof*⟩

lemma *Im-rcis* [*simp*]: $Im(rcis\ r\ a) = r * sin\ a$
⟨*proof*⟩

lemma *rcis-Ex*: ∃ *r a*. *z* = *rcis r a*
⟨*proof*⟩

lemma *complex-mod-rcis* [*simp*]: $cmod\ (rcis\ r\ a) = |r|$
⟨*proof*⟩

lemma *cis-rcis-eq*: *cis a* = *rcis 1 a*
⟨*proof*⟩

lemma *rcis-mult*: *rcis r1 a* ∗ *rcis r2 b* = *rcis* (*r1* ∗ *r2*) (*a* + *b*)
⟨*proof*⟩

lemma *rcis-zero-mod* [*simp*]: *rcis 0 a* = *0*
⟨*proof*⟩

lemma *rcis-zero-arg* [*simp*]: *rcis r 0* = *complex-of-real r*
⟨*proof*⟩

lemma *rcis-eq-zero-iff* [*simp*]: *rcis r a* = *0* ⟷ *r* = *0*
⟨*proof*⟩

lemma *DeMoivre2*: (*rcis r a*) ^ *n* = *rcis* (*r* ^ *n*) (*real n* ∗ *a*)
⟨*proof*⟩

lemma *rcis-inverse*: *inverse*(*rcis r a*) = *rcis* (*1* / *r*) (− *a*)
⟨*proof*⟩

lemma *rcis-divide*: *rcis r1 a* / *rcis r2 b* = *rcis* (*r1* / *r2*) (*a* − *b*)
⟨*proof*⟩

114.11.3 Complex exponential

lemma *exp-Reals-eq*:
assumes *z* ∈ ℝ
shows *exp z* = *of-real* (*exp* (*Re z*))

⟨*proof*⟩

lemma *cis-conv-exp*: *cis b* = *exp* (i ∗ *b*)
⟨*proof*⟩

lemma *exp-eq-polar*: *exp z* = *exp* (*Re z*) ∗ *cis* (*Im z*)
⟨*proof*⟩

lemma *Re-exp*: *Re* (*exp z*) = *exp* (*Re z*) ∗ *cos* (*Im z*)
⟨*proof*⟩

lemma *Im-exp*: *Im* (*exp z*) = *exp* (*Re z*) ∗ *sin* (*Im z*)
⟨*proof*⟩

lemma *norm-cos-sin* [*simp*]: *norm* (*Complex* (*cos t*) (*sin t*)) = *1*
⟨*proof*⟩

lemma *norm-exp-eq-Re* [*simp*]: *norm* (*exp z*) = *exp* (*Re z*)
⟨*proof*⟩

lemma *complex-exp-exists*: ∃ *a r*. *z* = *complex-of-real r* ∗ *exp a*
⟨*proof*⟩

lemma *exp-pi-i* [*simp*]: *exp* (*of-real pi* ∗ i) = −*1*
⟨*proof*⟩

lemma *exp-pi-i′* [*simp*]: *exp* (i ∗ *of-real pi*) = −*1*
⟨*proof*⟩

lemma *exp-two-pi-i* [*simp*]: *exp* (*2* ∗ *of-real pi* ∗ i) = *1*
⟨*proof*⟩

lemma *exp-two-pi-i′* [*simp*]: *exp* (i ∗ (*of-real pi* ∗ *2*)) = *1*
⟨*proof*⟩

lemma *continuous-on-cis* [*continuous-intros*]:
continuous-on A f ⟹ *continuous-on A* (λ*x*. *cis* (*f x*))
⟨*proof*⟩

lemma *tendsto-exp-0-Re-at-bot*: (*exp* ⟶ *0*) (*filtercomap Re at-bot*)
⟨*proof*⟩

lemma *filterlim-exp-at-infinity-Re-at-top*: *filterlim exp at-infinity* (*filtercomap Re at-top*)
⟨*proof*⟩

114.11.4 Complex argument

definition *Arg* :: *complex* ⇒ *real*

where $Arg\ z = (if\ z = 0\ then\ 0\ else\ (SOME\ a.\ sgn\ z = cis\ a \wedge -\ pi < a \wedge a \leq pi))$

lemma *Arg-zero*: $Arg\ 0 = 0$
⟨*proof*⟩

lemma *cis-Arg-unique*:
assumes $sgn\ z = cis\ x$ **and** $-pi < x$ **and** $x \leq pi$
shows $Arg\ z = x$
⟨*proof*⟩

lemma *Arg-correct*:
assumes $z \neq 0$
shows $sgn\ z = cis\ (Arg\ z) \wedge -pi < Arg\ z \wedge Arg\ z \leq pi$
⟨*proof*⟩

lemma *Arg-bounded*: $-\ pi < Arg\ z \wedge Arg\ z \leq pi$
⟨*proof*⟩

lemma *cis-Arg*: $z \neq 0 \Longrightarrow cis\ (Arg\ z) = sgn\ z$
⟨*proof*⟩

lemma *rcis-cmod-Arg*: $rcis\ (cmod\ z)\ (Arg\ z) = z$
⟨*proof*⟩

lemma *rcis-cnj*:
shows $cnj\ a = rcis\ (cmod\ a)\ (-\ Arg\ a)$
⟨*proof*⟩

lemma *cos-Arg-i-mult-zero* [*simp*]: $y \neq 0 \Longrightarrow Re\ y = 0 \Longrightarrow cos\ (Arg\ y) = 0$
⟨*proof*⟩

lemma *Arg-ii* [*simp*]: $Arg\ \mathrm{i} = pi/2$
⟨*proof*⟩

lemma *Arg-minus-ii* [*simp*]: $Arg\ (-\mathrm{i}) = -pi/2$
⟨*proof*⟩

lemma *cos-Arg*: $z \neq 0 \Longrightarrow cos\ (Arg\ z) = Re\ z\ /\ norm\ z$
⟨*proof*⟩

lemma *sin-Arg*: $z \neq 0 \Longrightarrow sin\ (Arg\ z) = Im\ z\ /\ norm\ z$
⟨*proof*⟩

114.12 Complex n-th roots

lemma *bij-betw-roots-unity*:
assumes $n > 0$
shows *bij-betw* $(\lambda k.\ cis\ (2 * pi * real\ k\ /\ real\ n))\ \{..<n\}\ \{z.\ z\ \hat{}\ n = 1\}$

(**is** *bij-betw ?f - -*)
⟨*proof*⟩

lemma *card-roots-unity-eq*:
assumes $n > 0$
shows *card* {*z*::*complex*. $z \hat{} n = 1$} = n
⟨*proof*⟩

lemma *bij-betw-nth-root-unity*:
fixes c :: *complex* **and** n :: *nat*
assumes *c*: $c \neq 0$ **and** *n*: $n > 0$
defines $c' \equiv$ *root n* (*norm c*) ∗ *cis* (*Arg c* / *n*)
shows *bij-betw* (λ*z*. $c' * z$) {*z*. $z \hat{} n = 1$} {*z*. $z \hat{} n = c$}
⟨*proof*⟩

lemma *finite-nth-roots* [*intro*]:
assumes $n > 0$
shows *finite* {*z*::*complex*. $z \hat{} n = c$}
⟨*proof*⟩

lemma *card-nth-roots*:
assumes $c \neq 0$ $n > 0$
shows *card* {*z*::*complex*. $z \hat{} n = c$} = n
⟨*proof*⟩

lemma *sum-roots-unity*:
assumes $n > 1$
shows $\sum$ {*z*::*complex*. $z \hat{} n = 1$} = 0
⟨*proof*⟩

lemma *sum-nth-roots*:
assumes $n > 1$
shows $\sum$ {*z*::*complex*. $z \hat{} n = c$} = 0
⟨*proof*⟩

114.13 Square root of complex numbers

primcorec *csqrt* :: *complex* ⇒ *complex*
where
Re (*csqrt z*) = *sqrt* ((*cmod z* + *Re z*) / 2)
| *Im* (*csqrt z*) = (*if Im z* = 0 *then* 1 *else sgn* (*Im z*)) ∗ *sqrt* ((*cmod z* − *Re z*) / 2)

lemma *csqrt-of-real-nonneg* [*simp*]: *Im x* = 0 ⟹ *Re x* ≥ 0 ⟹ *csqrt x* = *sqrt* (*Re x*)
⟨*proof*⟩

lemma *csqrt-of-real-nonpos* [*simp*]: *Im x* = 0 ⟹ *Re x* ≤ 0 ⟹ *csqrt x* = i ∗ *sqrt* |*Re x*|

⟨*proof*⟩

lemma *of-real-sqrt*: $x \geq 0 \Longrightarrow$ *of-real* (*sqrt* x) = *csqrt* (*of-real* x)
⟨*proof*⟩

lemma *csqrt-0* [*simp*]: *csqrt* 0 = 0
⟨*proof*⟩

lemma *csqrt-1* [*simp*]: *csqrt* 1 = 1
⟨*proof*⟩

lemma *csqrt-ii* [*simp*]: *csqrt* i = (1 + i) / *sqrt* 2
⟨*proof*⟩

lemma *power2-csqrt*[*simp*,*algebra*]: (*csqrt* z)^2 = z
⟨*proof*⟩

lemma *csqrt-power-even*:
assumes *even* n
shows *csqrt* $z \mathbin{\hat{}} n = z \mathbin{\hat{}} (n$ *div* 2)
⟨*proof*⟩

lemma *norm-csqrt* [*simp*]: *norm* (*csqrt* z) = *sqrt* (*norm* z)
⟨*proof*⟩

lemma *csqrt-eq-0* [*simp*]: *csqrt* z = 0 $\longleftrightarrow$ z = 0
⟨*proof*⟩

lemma *csqrt-eq-1* [*simp*]: *csqrt* z = 1 $\longleftrightarrow$ z = 1
⟨*proof*⟩

lemma *csqrt-principal*: 0 < *Re* (*csqrt* z) ∨ *Re* (*csqrt* z) = 0 ∧ 0 ≤ *Im* (*csqrt* z)
⟨*proof*⟩

lemma *Re-csqrt*: 0 ≤ *Re* (*csqrt* z)
⟨*proof*⟩

lemma *csqrt-square*:
assumes 0 < *Re* b ∨ (*Re* b = 0 ∧ 0 ≤ *Im* b)
shows *csqrt* ($b\hat{}2$) = b
⟨*proof*⟩

lemma *csqrt-unique*: $w^2 = z \Longrightarrow$ 0 < *Re* w ∨ *Re* w = 0 ∧ 0 ≤ *Im* $w \Longrightarrow$ *csqrt* z = w
⟨*proof*⟩

lemma *csqrt-minus* [*simp*]:
assumes *Im* x < 0 ∨ (*Im* x = 0 ∧ 0 ≤ *Re* x)
shows *csqrt* (− x) = i ∗ *csqrt* x

⟨*proof*⟩

Legacy theorem names

lemmas *cmod-def = norm-complex-def*

lemma *legacy-Complex-simps*:
shows *Complex-eq-0*: *Complex a b = 0 ⟷ a = 0 ∧ b = 0*
and *complex-add*: *Complex a b + Complex c d = Complex (a + c) (b + d)*
and *complex-minus*: *− (Complex a b) = Complex (− a) (− b)*
and *complex-diff*: *Complex a b − Complex c d = Complex (a − c) (b − d)*
and *Complex-eq-1*: *Complex a b = 1 ⟷ a = 1 ∧ b = 0*
and *Complex-eq-neg-1*: *Complex a b = − 1 ⟷ a = − 1 ∧ b = 0*
and *complex-mult*: *Complex a b ∗ Complex c d = Complex (a ∗ c − b ∗ d) (a ∗ d + b ∗ c)*
and *complex-inverse*: *inverse (Complex a b) = Complex (a / (a² + b²)) (− b / (a² + b²))*
and *Complex-eq-numeral*: *Complex a b = numeral w ⟷ a = numeral w ∧ b = 0*
and *Complex-eq-neg-numeral*: *Complex a b = − numeral w ⟷ a = − numeral w ∧ b = 0*
and *complex-scaleR*: *scaleR r (Complex a b) = Complex (r ∗ a) (r ∗ b)*
and *Complex-eq-i*: *Complex x y = i ⟷ x = 0 ∧ y = 1*
and *i-mult-Complex*: *i ∗ Complex a b = Complex (− b) a*
and *Complex-mult-i*: *Complex a b ∗ i = Complex (− b) a*
and *i-complex-of-real*: *i ∗ complex-of-real r = Complex 0 r*
and *complex-of-real-i*: *complex-of-real r ∗ i = Complex 0 r*
and *Complex-add-complex-of-real*: *Complex x y + complex-of-real r = Complex (x+r) y*
and *complex-of-real-add-Complex*: *complex-of-real r + Complex x y = Complex (r+x) y*
and *Complex-mult-complex-of-real*: *Complex x y ∗ complex-of-real r = Complex (x∗r) (y∗r)*
and *complex-of-real-mult-Complex*: *complex-of-real r ∗ Complex x y = Complex (r∗x) (r∗y)*
and *complex-eq-cancel-iff2*: *(Complex x y = complex-of-real xa) = (x = xa ∧ y = 0)*
and *complex-cnj*: *cnj (Complex a b) = Complex a (− b)*
and *Complex-sum′*: *sum (λx. Complex (f x) 0) s = Complex (sum f s) 0*
and *Complex-sum*: *Complex (sum f s) 0 = sum (λx. Complex (f x) 0) s*
and *complex-of-real-def*: *complex-of-real r = Complex r 0*
and *complex-norm*: *cmod (Complex x y) = sqrt (x² + y²)*
⟨*proof*⟩

lemma *Complex-in-Reals*: *Complex x 0 ∈ ℝ*
⟨*proof*⟩

Express a complex number as a linear combination of two others, not collinear with the origin

lemma *complex-axes*:

assumes *Im* $(y/x) \neq 0$
obtains a b **where** $z = \textit{of-real}\ a * x + \textit{of-real}\ b * y$
⟨*proof*⟩

end

115 MacLaurin and Taylor Series

theory *MacLaurin*
imports *Transcendental*
begin

115.1 Maclaurin's Theorem with Lagrange Form of Remainder

This is a very long, messy proof even now that it's been broken down into lemmas.

lemma *Maclaurin-lemma*:
$0 < h \Longrightarrow$
$\exists B{::}\textit{real}.\ f\ h = (\sum m{<}n.\ (j\ m\ /\ (\textit{fact}\ m)) * (h\hat{}m)) + (B * ((h\hat{}n)\ /(\textit{fact}\ n)))$
⟨*proof*⟩

lemma *eq-diff-eq'*: $x = y - z \longleftrightarrow y = x + z$
for $x\ y\ z :: \textit{real}$
⟨*proof*⟩

lemma *fact-diff-Suc*: $n < \textit{Suc}\ m \Longrightarrow \textit{fact}\ (\textit{Suc}\ m - n) = (\textit{Suc}\ m - n) * \textit{fact}\ (m - n)$
⟨*proof*⟩

lemma *Maclaurin-lemma2*:
fixes B
assumes *DERIV*: $\forall m\ t.\ m < n \wedge 0{\leq}t \wedge t{\leq}h \longrightarrow \textit{DERIV}\ (\textit{diff}\ m)\ t :> \textit{diff}\ (\textit{Suc}\ m)\ t$
and *INIT*: $n = \textit{Suc}\ k$
defines *difg* $\equiv$
$(\lambda m\ t{::}\textit{real}.\ \textit{diff}\ m\ t -$
$((\sum p{<}n - m.\ \textit{diff}\ (m + p)\ 0\ /\ \textit{fact}\ p * t\ \hat{}\ p) + B * (t\ \hat{}\ (n - m)\ /\ \textit{fact}\ (n - m))))$
(**is** *difg* $\equiv (\lambda m\ t.\ \textit{diff}\ m\ t - ?\textit{difg}\ m\ t)$)
shows $\forall m\ t.\ m < n \wedge 0 \leq t \wedge t \leq h \longrightarrow \textit{DERIV}\ (\textit{difg}\ m)\ t :> \textit{difg}\ (\textit{Suc}\ m)\ t$
⟨*proof*⟩

lemma *Maclaurin*:
assumes *h*: $0 < h$
and *n*: $0 < n$
and *diff-0*: $\textit{diff}\ 0 = f$

and *diff-Suc*: $\forall\, m\ t.\ m < n \wedge 0 \leq t \wedge t \leq h \longrightarrow DERIV\ (diff\ m)\ t :> diff\ (Suc\ m)\ t$
shows
$\exists\, t{::}real.\ 0 < t \wedge t < h \wedge$
$f\ h = sum\ (\lambda m.\ (diff\ m\ 0\ /\ fact\ m) * h\ \hat{}\ m)\ \{..<n\} + (diff\ n\ t\ /\ fact\ n) * h\ \hat{}\ n$
⟨*proof*⟩

lemma *Maclaurin2*:
fixes $n :: nat$
and $h :: real$
assumes *INIT1*: $0 < h$
and *INIT2*: $diff\ 0 = f$
and *DERIV*: $\forall\, m\ t.\ m < n \wedge 0 \leq t \wedge t \leq h \longrightarrow DERIV\ (diff\ m)\ t :> diff\ (Suc\ m)\ t$
shows $\exists\, t.\ 0 < t \wedge t \leq h \wedge f\ h = (\sum m{<}n.\ diff\ m\ 0\ /\ (fact\ m) * h\ \hat{}\ m) + diff\ n\ t\ /\ fact\ n * h\ \hat{}\ n$
⟨*proof*⟩

lemma *Maclaurin-minus*:
fixes $n :: nat$ **and** $h :: real$
assumes $h < 0\ 0 < n\ diff\ 0 = f$
and *DERIV*: $\forall\, m\ t.\ m < n \wedge h \leq t \wedge t \leq 0 \longrightarrow DERIV\ (diff\ m)\ t :> diff\ (Suc\ m)\ t$
shows $\exists\, t.\ h < t \wedge t < 0 \wedge f\ h = (\sum m{<}n.\ diff\ m\ 0\ /\ fact\ m * h\ \hat{}\ m) + diff\ n\ t\ /\ fact\ n * h\ \hat{}\ n$
⟨*proof*⟩

115.2 More Convenient "Bidirectional" Version.

lemma *Maclaurin-bi-le*:
fixes $n :: nat$ **and** $x :: real$
assumes $diff\ 0 = f$
and *DERIV* : $\forall\, m\ t.\ m < n \wedge |t| \leq |x| \longrightarrow DERIV\ (diff\ m)\ t :> diff\ (Suc\ m)\ t$
shows $\exists\, t.\ |t| \leq |x| \wedge f\ x = (\sum m{<}n.\ diff\ m\ 0\ /\ (fact\ m) * x\ \hat{}\ m) + diff\ n\ t\ /\ (fact\ n) * x\ \hat{}\ n$
(**is** $\exists\, t.\ \text{-} \wedge f\ x = ?f\ x\ t$)
⟨*proof*⟩

lemma *Maclaurin-all-lt*:
fixes $x :: real$
assumes *INIT1*: $diff\ 0 = f$
and *INIT2*: $0 < n$
and *INIT3*: $x \neq 0$
and *DERIV*: $\forall\, m\ x.\ DERIV\ (diff\ m)\ x :> diff(Suc\ m)\ x$
shows $\exists\, t.\ 0 < |t| \wedge |t| < |x| \wedge f\ x =$
$(\sum m{<}n.\ (diff\ m\ 0\ /\ fact\ m) * x\ \hat{}\ m) + (diff\ n\ t\ /\ fact\ n) * x\ \hat{}\ n$
(**is** $\exists\, t.\ \text{-} \wedge \text{-} \wedge f\ x = ?f\ x\ t$)

⟨*proof*⟩

lemma *Maclaurin-zero*: $x = 0 \Longrightarrow n \neq 0 \Longrightarrow (\sum m{<}n.\ (diff\ m\ 0\ /\ fact\ m) * x\ \widehat{}\ m) = diff\ 0\ 0$
for x :: *real* **and** n :: *nat*
⟨*proof*⟩

lemma *Maclaurin-all-le*:
fixes x :: *real* **and** n :: *nat*
assumes *INIT*: $diff\ 0 = f$
and *DERIV*: $\forall m\ x.\ DERIV\ (diff\ m)\ x :> diff\ (Suc\ m)\ x$
shows $\exists t.\ |t| \leq |x| \wedge f\ x = (\sum m{<}n.\ (diff\ m\ 0\ /\ fact\ m) * x\ \widehat{}\ m) + (diff\ n\ t\ /\ fact\ n) * x\ \widehat{}\ n$
(**is** $\exists t.\ - \wedge f\ x = ?f\ x\ t$)
⟨*proof*⟩

lemma *Maclaurin-all-le-objl*:
$diff\ 0 = f \wedge (\forall m\ x.\ DERIV\ (diff\ m)\ x :> diff\ (Suc\ m)\ x) \longrightarrow$
$(\exists t{::}real.\ |t| \leq |x| \wedge f\ x = (\sum m{<}n.\ (diff\ m\ 0\ /\ fact\ m) * x\ \widehat{}\ m) + (diff\ n\ t\ /\ fact\ n) * x\ \widehat{}\ n)$
for x :: *real* **and** n :: *nat*
⟨*proof*⟩

115.3 Version for Exponential Function

lemma *Maclaurin-exp-lt*:
fixes x :: *real* **and** n :: *nat*
shows
$x \neq 0 \Longrightarrow n > 0 \Longrightarrow$
$(\exists t.\ 0 < |t| \wedge |t| < |x| \wedge exp\ x = (\sum m{<}n.\ (x\ \widehat{}\ m)\ /\ fact\ m) + (exp\ t\ /\ fact\ n) * x\ \widehat{}\ n)$
⟨*proof*⟩

lemma *Maclaurin-exp-le*:
fixes x :: *real* **and** n :: *nat*
shows $\exists t.\ |t| \leq |x| \wedge exp\ x = (\sum m{<}n.\ (x\ \widehat{}\ m)\ /\ fact\ m) + (exp\ t\ /\ fact\ n) * x\ \widehat{}\ n$
⟨*proof*⟩

corollary *exp-lower-Taylor-quadratic*: $0 \leq x \Longrightarrow 1 + x + x^2\ /\ 2 \leq exp\ x$
for x :: *real*
⟨*proof*⟩

corollary *ln-2-less-1*: $ln\ 2 < (1{::}real)$
⟨*proof*⟩

115.4 Version for Sine Function

lemma *mod-exhaust-less-4*: $m \bmod 4 = 0 \vee m \bmod 4 = 1 \vee m \bmod 4 = 2 \vee m \bmod 4 = 3$
for $m :: nat$
⟨*proof*⟩

It is unclear why so many variant results are needed.

lemma *sin-expansion-lemma*: $sin\ (x + real\ (Suc\ m) * pi\ /\ 2) = cos\ (x + real\ m * pi\ /\ 2)$
⟨*proof*⟩

lemma *Maclaurin-sin-expansion2*:
$\exists t.\ |t| \leq |x| \wedge$
$sin\ x = (\sum m{<}n.\ sin\text{-}coeff\ m * x\ \hat{}\ m) + (sin\ (t + 1/2 * real\ n * pi)\ /\ fact\ n) * x\ \hat{}\ n$
⟨*proof*⟩

lemma *Maclaurin-sin-expansion*:
$\exists t.\ sin\ x = (\sum m{<}n.\ sin\text{-}coeff\ m * x\ \hat{}\ m) + (sin\ (t + 1/2 * real\ n * pi)\ /\ fact\ n) * x\ \hat{}\ n$
⟨*proof*⟩

lemma *Maclaurin-sin-expansion3*:
assumes $n > 0\ x > 0$
shows $\exists t.\ 0 < t \wedge t < x \wedge$
$sin\ x = (\sum m{<}n.\ sin\text{-}coeff\ m * x\ \hat{}\ m) + (sin\ (t + 1/2 * real\ n * pi)\ /\ fact\ n) * x\ \hat{}\ n$
⟨*proof*⟩

lemma *Maclaurin-sin-expansion4*:
assumes $0 < x$
shows $\exists t.\ 0 < t \wedge t \leq x \wedge sin\ x = (\sum m{<}n.\ sin\text{-}coeff\ m * x\ \hat{}\ m) + (sin\ (t + 1/2 * real\ n * pi)\ /\ fact\ n) * x\ \hat{}\ n$
⟨*proof*⟩

115.5 Maclaurin Expansion for Cosine Function

lemma *sumr-cos-zero-one* [*simp*]: $(\sum m{<}Suc\ n.\ cos\text{-}coeff\ m * 0\ \hat{}\ m) = 1$
⟨*proof*⟩

lemma *cos-expansion-lemma*: $cos\ (x + real\ (Suc\ m) * pi\ /\ 2) = -\ sin\ (x + real\ m * pi\ /\ 2)$
⟨*proof*⟩

lemma *Maclaurin-cos-expansion*:
$\exists t{::}real.\ |t| \leq |x| \wedge$
$cos\ x = (\sum m{<}n.\ cos\text{-}coeff\ m * x\ \hat{}\ m) + (cos(t + 1/2 * real\ n * pi)\ /\ fact\ n) * x\ \hat{}\ n$
⟨*proof*⟩

lemma *Maclaurin-cos-expansion2*:
assumes $x > 0$ $n > 0$
shows $\exists t.\ 0 < t \wedge t < x \wedge$
$cos\ x = (\sum m{<}n.\ cos\text{-}coeff\ m * x\ \hat{}\ m) + (cos\ (t + 1/2 * real\ n * pi)\ /\ fact\ n) * x\ \hat{}\ n$
⟨*proof*⟩

lemma *Maclaurin-minus-cos-expansion*:
assumes $n > 0$ $x < 0$
shows $\exists t.\ x < t \wedge t < 0 \wedge$
$cos\ x = (\sum m{<}n.\ cos\text{-}coeff\ m * x\ \hat{}\ m) + ((cos\ (t + 1/2 * real\ n * pi)\ /\ fact\ n) * x\ \hat{}\ n)$
⟨*proof*⟩

lemma *sin-bound-lemma*: $x = y \Longrightarrow |u| \leq v \Longrightarrow |(x + u) - y| \leq v$
for $x\ y\ u\ v$ :: *real*
⟨*proof*⟩

lemma *Maclaurin-sin-bound*: $|sin\ x - (\sum m{<}n.\ sin\text{-}coeff\ m * x\ \hat{}\ m)| \leq inverse\ (fact\ n) * |x|\ \hat{}\ n$
⟨*proof*⟩

116 Taylor series

We use MacLaurin and the translation of the expansion point c to 0 to prove Taylor's theorem.

lemma *Taylor-up*:
assumes *INIT*: $n > 0$ $diff\ 0 = f$
and *DERIV*: $\forall m\ t.\ m < n \wedge a \leq t \wedge t \leq b \longrightarrow DERIV\ (diff\ m)\ t :> (diff\ (Suc\ m)\ t)$
and *INTERV*: $a \leq c$ $c < b$
shows $\exists t{::}real.\ c < t \wedge t < b \wedge$
$f\ b = (\sum m{<}n.\ (diff\ m\ c\ /\ fact\ m) * (b - c)\hat{}m) + (diff\ n\ t\ /\ fact\ n) * (b - c)\hat{}n$
⟨*proof*⟩

lemma *Taylor-down*:
fixes a :: *real* **and** n :: *nat*
assumes *INIT*: $n > 0$ $diff\ 0 = f$
and *DERIV*: $(\forall m\ t.\ m < n \wedge a \leq t \wedge t \leq b \longrightarrow DERIV\ (diff\ m)\ t :> diff\ (Suc\ m)\ t)$
and *INTERV*: $a < c$ $c \leq b$
shows $\exists t.\ a < t \wedge t < c \wedge$

$f\ a = (\sum m{<}n.\ (diff\ m\ c\ /\ fact\ m) * (a - c)\hat{}m) + (diff\ n\ t\ /\ fact\ n) * (a - c)\hat{}n$
⟨*proof*⟩

theorem *Taylor*:
fixes a :: *real* **and** n :: *nat*
assumes *INIT*: $n > 0$ $diff\ 0 = f$
and *DERIV*: $\forall m\ t.\ m < n \wedge a \leq t \wedge t \leq b \longrightarrow DERIV\ (diff\ m)\ t :> diff\ (Suc\ m)\ t$
and *INTERV*: $a \leq c\ \ c \leq b\ a \leq x\ x \leq b\ x \neq c$
shows $\exists t.$
$(if\ x < c\ then\ x < t \wedge t < c\ else\ c < t \wedge t < x) \wedge$
$f\ x = (\sum m{<}n.\ (diff\ m\ c\ /\ fact\ m) * (x - c)\hat{}m) + (diff\ n\ t\ /\ fact\ n) * (x - c)\hat{}n$
⟨*proof*⟩

end

117 Comprehensive Complex Theory

theory *Complex-Main*
imports
Complex
MacLaurin
begin

end

References

[1] H. Davenport. *The Higher Arithmetic.* Cambridge University Press, 1992.

[2] M. Gordon. HOL: A machine oriented formulation of higher-order logic. Technical Report 68, University of Cambridge, Computer Laboratory, 1985.

[3] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics: A Foundation for Computer Science.* Addison–Wesley, Boston, MA, USA, 2nd edition, 1994.

[4] M. Holz, K. Steffens, and E. Weitz. *Introduction to Cardinal Arithmetic.* Birkhäuser, 1999.

[5] D. Leijen. Division and modulus for computer scientists. 2001.

[6] C. Paulin-Mohring. Inductive definitions in the system Coq: Rules and properties. In M. Bezem and J. Groote, editors, *Typed Lambda Calculi and Applications*, LNCS 664, pages 328–345. Springer, 1993.