Measure and Probability Theory

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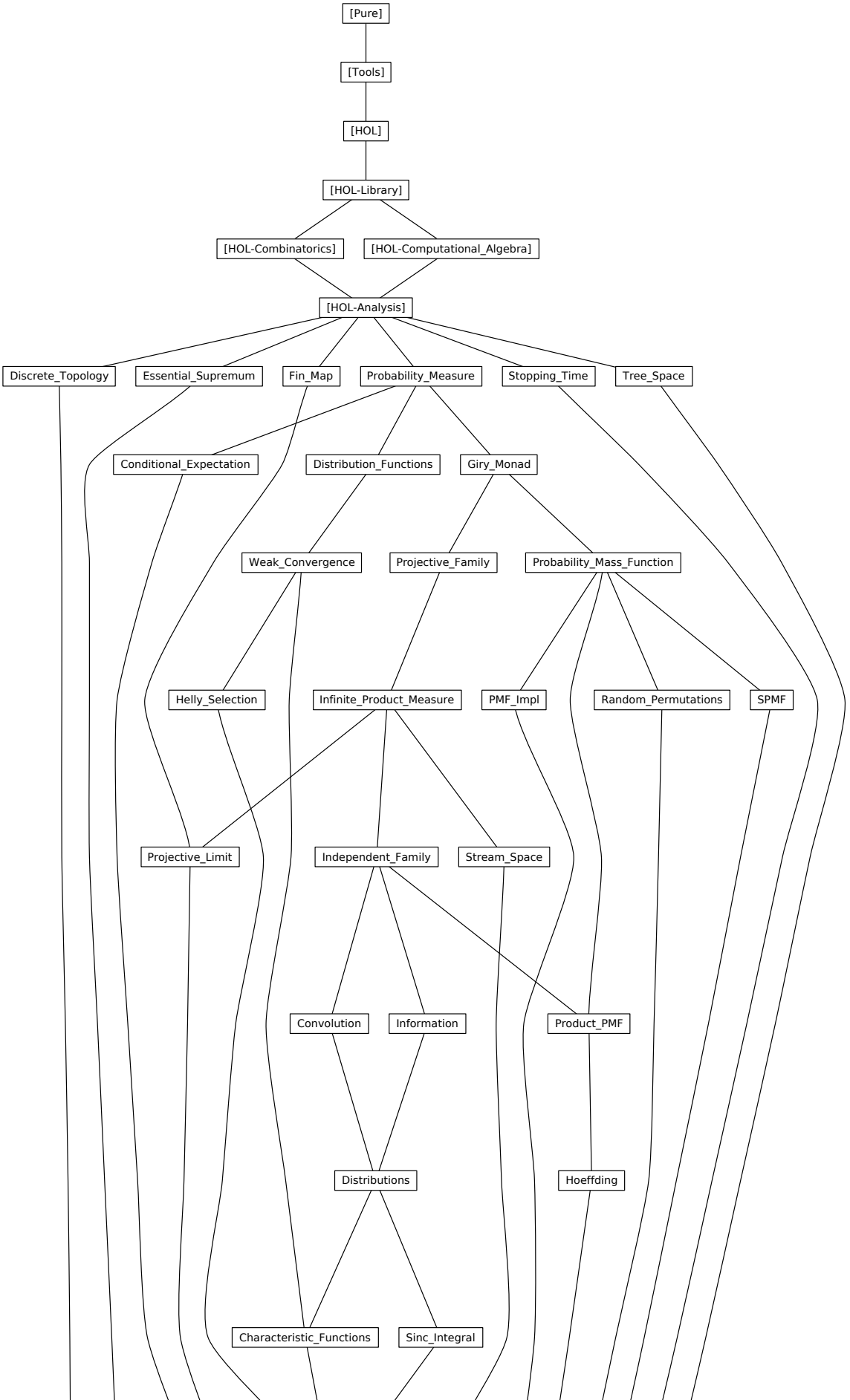
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1 Probability measure

theory *Probability-Measure*
imports *HOL−Analysis.Analysis*
begin

locale *prob-space* = *finite-measure* +
assumes *emeasure-space-1*: *emeasure M* (*space M*) = *1*

lemma *prob-spaceI*[*Pure.intro*!]:
assumes ∗: *emeasure M* (*space M*) = *1*
shows *prob-space M*
⟨*proof*⟩

lemma *prob-space-imp-sigma-finite*: *prob-space M* ⟹ *sigma-finite-measure M*
⟨*proof*⟩

abbreviation (**in** *prob-space*) *events* ≡ *sets M*
abbreviation (**in** *prob-space*) *prob* ≡ *measure M*
abbreviation (**in** *prob-space*) *random-variable M′ X* ≡ *X* ∈ *measurable M M′*
abbreviation (**in** *prob-space*) *expectation* ≡ *integral*^{L} *M*
abbreviation (**in** *prob-space*) *variance X* ≡ *integral*^{L} *M* (λ*x*. (*X x* − *expectation X*)^2)

lemma (**in** *prob-space*) *finite-measure* [*simp*]: *finite-measure M*
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-space-distr*:
assumes *f*: *f* ∈ *measurable M M′* **shows** *prob-space* (*distr M M′ f*)
⟨*proof*⟩

lemma *prob-space-distrD*:
assumes *f*: *f* ∈ *measurable M N* **and** *M*: *prob-space* (*distr M N f*) **shows** *prob-space M*
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-space*: *prob* (*space M*) = *1*
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-le-1*[*simp*, *intro*]: *prob A* ≤ *1*
⟨*proof*⟩

lemma (**in** *prob-space*) *not-empty*: *space M* ≠ {}
⟨*proof*⟩

lemma (**in** *prob-space*) *emeasure-eq-1-AE*:
S ∈ *sets M* ⟹ *AE x in M*. *x* ∈ *S* ⟹ *emeasure M S* = *1*
⟨*proof*⟩

lemma (**in** *prob-space*) *emeasure-le-1*: $emeasure\ M\ S \leq 1$
⟨*proof*⟩

lemma (**in** *prob-space*) *emeasure-ge-1-iff*: $emeasure\ M\ A \geq 1 \longleftrightarrow emeasure\ M\ A = 1$
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-iff-emeasure-eq-1*:
assumes [*measurable*]: *Measurable.pred M P*
shows $(AE\ x\ in\ M.\ P\ x) \longleftrightarrow emeasure\ M\ \{x \in space\ M.\ P\ x\} = 1$
⟨*proof*⟩

lemma (**in** *prob-space*) *measure-le-1*: $emeasure\ M\ X \leq 1$
⟨*proof*⟩

lemma (**in** *prob-space*) *measure-ge-1-iff*: $measure\ M\ A \geq 1 \longleftrightarrow measure\ M\ A = 1$
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-I-eq-1*:
assumes $emeasure\ M\ \{x \in space\ M.\ P\ x\} = 1$ $\{x \in space\ M.\ P\ x\} \in sets\ M$
shows $AE\ x\ in\ M.\ P\ x$
⟨*proof*⟩

lemma *prob-space-restrict-space*:
$S \in sets\ M \Longrightarrow emeasure\ M\ S = 1 \Longrightarrow prob\text{-}space\ (restrict\text{-}space\ M\ S)$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-compl*:
assumes *A*: $A \in events$
shows $prob\ (space\ M - A) = 1 - prob\ A$
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-in-set-eq-1*:
assumes *A*[*measurable*]: $A \in events$ **shows** $(AE\ x\ in\ M.\ x \in A) \longleftrightarrow prob\ A = 1$
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-False*: $(AE\ x\ in\ M.\ False) \longleftrightarrow False$
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-prob-1*:
assumes $prob\ A = 1$ **shows** $AE\ x\ in\ M.\ x \in A$
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-const*[*simp*]: $(AE\ x\ in\ M.\ P) \longleftrightarrow P$
⟨*proof*⟩

lemma (**in** *prob-space*) *ae-filter-bot*: $ae\text{-}filter\ M \neq bot$

⟨*proof*⟩

lemma (**in** *prob-space*) *AE-contr*:
assumes *ae*: *AE* ω *in* *M*. *P* ω *AE* ω *in* *M*. ¬ *P* ω
shows *False*
⟨*proof*⟩

lemma (**in** *prob-space*) *integral-ge-const*:
fixes *c* :: *real*
shows *integrable* *M* *f* ⟹ (*AE* *x* *in* *M*. *c* ≤ *f* *x*) ⟹ *c* ≤ (∫ *x*. *f* *x* ∂*M*)
⟨*proof*⟩

lemma (**in** *prob-space*) *integral-le-const*:
fixes *c* :: *real*
shows *integrable* *M* *f* ⟹ (*AE* *x* *in* *M*. *f* *x* ≤ *c*) ⟹ (∫ *x*. *f* *x* ∂*M*) ≤ *c*
⟨*proof*⟩

lemma (**in** *prob-space*) *nn-integral-ge-const*:
(*AE* *x* *in* *M*. *c* ≤ *f* *x*) ⟹ *c* ≤ (∫⁺ *x*. *f* *x* ∂*M*)
⟨*proof*⟩

lemma (**in** *prob-space*) *expectation-less*:
fixes *X* :: - ⇒ *real*
assumes [*simp*]: *integrable* *M* *X*
assumes *gt*: *AE* *x* *in* *M*. *X* *x* < *b*
shows *expectation* *X* < *b*
⟨*proof*⟩

lemma (**in** *prob-space*) *expectation-greater*:
fixes *X* :: - ⇒ *real*
assumes [*simp*]: *integrable* *M* *X*
assumes *gt*: *AE* *x* *in* *M*. *a* < *X* *x*
shows *a* < *expectation* *X*
⟨*proof*⟩

lemma (**in** *prob-space*) *jensens-inequality*:
fixes *q* :: *real* ⇒ *real*
assumes *X*: *integrable* *M* *X* *AE* *x* *in* *M*. *X* *x* ∈ *I*
assumes *I*: *I* = {*a* <..< *b*} ∨ *I* = {*a* <..} ∨ *I* = {..< *b*} ∨ *I* = *UNIV*
assumes *q*: *integrable* *M* (λ*x*. *q* (*X* *x*)) *convex-on* *I* *q*
shows *q* (*expectation* *X*) ≤ *expectation* (λ*x*. *q* (*X* *x*))
⟨*proof*⟩

1.1 Introduce binder for probability

syntax
-*prob* :: *pttrn* ⇒ *logic* ⇒ *logic* ⇒ *logic* (‹('𝒫'((/- *in* -./ -)'))›)

translations

$\mathcal{P}(x\ in\ M.\ P) => CONST\ measure\ M\ \{x \in CONST\ space\ M.\ P\}$

⟨*ML*⟩

definition
$cond\text{-}prob\ M\ P\ Q = \mathcal{P}(\omega\ in\ M.\ P\ \omega \wedge Q\ \omega)\ /\ \mathcal{P}(\omega\ in\ M.\ Q\ \omega)$

syntax
$\text{-}conditional\text{-}prob :: pttrn \Rightarrow logic \Rightarrow logic \Rightarrow logic \Rightarrow logic\ (\langle(\mathcal{P}(\text{- in -. - |/ -}))\rangle)$

translations
$\mathcal{P}(x\ in\ M.\ P \mid Q) => CONST\ cond\text{-}prob\ M\ (\lambda x.\ P)\ (\lambda x.\ Q)$

lemma (**in** *prob-space*) *AE-E-prob*:
assumes *ae*: $AE\ x\ in\ M.\ P\ x$
obtains S **where** $S \subseteq \{x \in space\ M.\ P\ x\}\ S \in events\ prob\ S = 1$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-neg*: $\{x{\in}space\ M.\ P\ x\} \in events \Longrightarrow \mathcal{P}(x\ in\ M.\ \neg\ P\ x) = 1 - \mathcal{P}(x\ in\ M.\ P\ x)$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-eq-AE*:
$(AE\ x\ in\ M.\ P\ x \longleftrightarrow Q\ x) \Longrightarrow \{x{\in}space\ M.\ P\ x\} \in events \Longrightarrow \{x{\in}space\ M.\ Q\ x\} \in events \Longrightarrow \mathcal{P}(x\ in\ M.\ P\ x) = \mathcal{P}(x\ in\ M.\ Q\ x)$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-eq-0-AE*:
assumes *not*: $AE\ x\ in\ M.\ \neg\ P\ x$ **shows** $\mathcal{P}(x\ in\ M.\ P\ x) = 0$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-Collect-eq-0*:
$\{x \in space\ M.\ P\ x\} \in sets\ M \Longrightarrow \mathcal{P}(x\ in\ M.\ P\ x) = 0 \longleftrightarrow (AE\ x\ in\ M.\ \neg\ P\ x)$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-Collect-eq-1*:
$\{x \in space\ M.\ P\ x\} \in sets\ M \Longrightarrow \mathcal{P}(x\ in\ M.\ P\ x) = 1 \longleftrightarrow (AE\ x\ in\ M.\ P\ x)$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-eq-0*:
$A \in sets\ M \Longrightarrow prob\ A = 0 \longleftrightarrow (AE\ x\ in\ M.\ x \notin A)$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-eq-1*:
$A \in sets\ M \Longrightarrow prob\ A = 1 \longleftrightarrow (AE\ x\ in\ M.\ x \in A)$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-sums*:
assumes P: $\bigwedge n.\ \{x{\in}space\ M.\ P\ n\ x\} \in events$

assumes *Q*: $\{x \in space\ M.\ Q\ x\} \in events$
assumes *ae*: $AE\ x\ in\ M.\ (\forall n.\ P\ n\ x \longrightarrow Q\ x) \wedge (Q\ x \longrightarrow (\exists! n.\ P\ n\ x))$
shows $(\lambda n.\ \mathcal{P}(x\ in\ M.\ P\ n\ x))\ sums\ \mathcal{P}(x\ in\ M.\ Q\ x)$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-sum*:
assumes [*simp*, *intro*]: *finite I*
assumes *P*: $\bigwedge n.\ n \in I \Longrightarrow \{x \in space\ M.\ P\ n\ x\} \in events$
assumes *Q*: $\{x \in space\ M.\ Q\ x\} \in events$
assumes *ae*: $AE\ x\ in\ M.\ (\forall n \in I.\ P\ n\ x \longrightarrow Q\ x) \wedge (Q\ x \longrightarrow (\exists! n \in I.\ P\ n\ x))$
shows $\mathcal{P}(x\ in\ M.\ Q\ x) = (\sum n \in I.\ \mathcal{P}(x\ in\ M.\ P\ n\ x))$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-EX-countable*:
assumes *sets*: $\bigwedge i.\ i \in I \Longrightarrow \{x \in space\ M.\ P\ i\ x\} \in sets\ M$ **and** *I*: *countable I*
assumes *disj*: $AE\ x\ in\ M.\ \forall i \in I.\ \forall j \in I.\ P\ i\ x \longrightarrow P\ j\ x \longrightarrow i = j$
shows $\mathcal{P}(x\ in\ M.\ \exists i \in I.\ P\ i\ x) = (\int^+ i.\ \mathcal{P}(x\ in\ M.\ P\ i\ x)\ \partial count\text{-}space\ I)$
⟨*proof*⟩

lemma (**in** *prob-space*) *cond-prob-eq-AE*:
assumes *P*: $AE\ x\ in\ M.\ Q\ x \longrightarrow P\ x \longleftrightarrow P'\ x$ $\{x \in space\ M.\ P\ x\} \in events$ $\{x \in space\ M.\ P'\ x\} \in events$
assumes *Q*: $AE\ x\ in\ M.\ Q\ x \longleftrightarrow Q'\ x$ $\{x \in space\ M.\ Q\ x\} \in events$ $\{x \in space\ M.\ Q'\ x\} \in events$
shows *cond-prob M P Q* = *cond-prob M P' Q'*
⟨*proof*⟩

lemma (**in** *prob-space*) *joint-distribution-Times-le-fst*:
random-variable MX X $\Longrightarrow$ *random-variable MY Y* $\Longrightarrow$ $A \in sets\ MX \Longrightarrow B \in sets\ MY$
$\Longrightarrow$ *emeasure* (*distr M* (*MX* $\bigotimes_M$ *MY*) ($\lambda x.$ (*X x*, *Y x*))) ($A \times B$) $\leq$ *emeasure* (*distr M MX X*) *A*
⟨*proof*⟩

lemma (**in** *prob-space*) *joint-distribution-Times-le-snd*:
random-variable MX X $\Longrightarrow$ *random-variable MY Y* $\Longrightarrow$ $A \in sets\ MX \Longrightarrow B \in sets\ MY$
$\Longrightarrow$ *emeasure* (*distr M* (*MX* $\bigotimes_M$ *MY*) ($\lambda x.$ (*X x*, *Y x*))) ($A \times B$) $\leq$ *emeasure* (*distr M MY Y*) *B*
⟨*proof*⟩

lemma (**in** *prob-space*) *variance-eq*:
fixes *X* :: $'a \Rightarrow real$
assumes [*simp*]: *integrable M X*
assumes [*simp*]: *integrable M* ($\lambda x.$ $(X\ x)^2$)
shows *variance X* = *expectation* ($\lambda x.$ $(X\ x)^2$) − $(expectation\ X)^2$
⟨*proof*⟩

lemma (**in** *prob-space*) *variance-positive*: $0 \leq$ *variance* (X::$'a \Rightarrow$ *real*)
⟨*proof*⟩

lemma (**in** *prob-space*) *variance-mean-zero*:
expectation $X = 0 \Longrightarrow$ *variance* $X =$ *expectation* ($\lambda x.$ (X x)^2)
⟨*proof*⟩

theorem (**in** *prob-space*) *Chebyshev-inequality*:
assumes [*measurable*]: *random-variable borel f*
assumes *integrable M* ($\lambda x.$ *f x* ^ 2)
defines $\mu \equiv$ *expectation f*
assumes $a > 0$
shows *prob* $\{x \in$ *space M*. $|f\ x - \mu| \geq a\} \leq$ *variance f* / a^2
unfolding μ-*def*
proof (*rule second-moment-method*)
have *integrable*: *integrable M f*
using *assms* **by** (*blast dest*: *square-integrable-imp-integrable*)
show *integrable M* ($\lambda x.$ ($f\ x -$ *expectation f*)^2)
using *assms integrable* **unfolding** *power2-eq-square ring-distribs*
by (*intro Bochner-Integration.integrable-diff*) *auto*
qed (*use assms* **in** *auto*)

locale *pair-prob-space* = *pair-sigma-finite M1 M2* + *M1*: *prob-space M1* + *M2*: *prob-space M2* **for** *M1 M2*

sublocale *pair-prob-space* ⊆ *P?*: *prob-space M1* $\bigotimes_M$ *M2*
⟨*proof*⟩

locale *product-prob-space* = *product-sigma-finite M* **for** *M* :: $'i \Rightarrow$ $'a$ *measure* +
fixes *I* :: $'i$ *set*
assumes *prob-space*: $\bigwedge i.$ *prob-space* (*M i*)

sublocale *product-prob-space* ⊆ *M?*: *prob-space M i* **for** *i*
⟨*proof*⟩

locale *finite-product-prob-space* = *finite-product-sigma-finite M I* + *product-prob-space M I* **for** *M I*

sublocale *finite-product-prob-space* ⊆ *prob-space* Π_M $i \in I.$ *M i*
⟨*proof*⟩

lemma (**in** *finite-product-prob-space*) *prob-times*:
assumes *X*: $\bigwedge i.$ $i \in I \Longrightarrow$ *X i* ∈ *sets* (*M i*)
shows *prob* (Π_E $i \in I.$ *X i*) = ($\prod i \in I.$ *M.prob i* (*X i*))
⟨*proof*⟩

lemma *product-prob-spaceI*:
assumes $\bigwedge i.$ *prob-space* (*M i*)
shows *product-prob-space M*

⟨*proof*⟩

1.2 Distributions

definition *distributed* :: *'a measure* ⇒ *'b measure* ⇒ (*'a* ⇒ *'b*) ⇒ (*'b* ⇒ *ennreal*) ⇒ *bool*
where
distributed M N X f ⟷
distr M N X = *density N f* ∧ *f* ∈ *borel-measurable N* ∧ *X* ∈ *measurable M N*

lemma
assumes *distributed M N X f*
shows *distributed-distr-eq-density*: *distr M N X* = *density N f*
and *distributed-measurable*: *X* ∈ *measurable M N*
and *distributed-borel-measurable*: *f* ∈ *borel-measurable N*
⟨*proof*⟩

lemma
assumes *D*: *distributed M N X f*
shows *distributed-measurable'*[*measurable-dest*]:
g ∈ *measurable L M* ⟹ (λ*x*. *X* (*g x*)) ∈ *measurable L N*
and *distributed-borel-measurable'*[*measurable-dest*]:
h ∈ *measurable L N* ⟹ (λ*x*. *f* (*h x*)) ∈ *borel-measurable L*
⟨*proof*⟩

lemma *distributed-real-measurable*:
(⋀*x*. *x* ∈ *space N* ⟹ 0 ≤ *f x*) ⟹ *distributed M N X* (λ*x*. *ennreal* (*f x*)) ⟹ *f* ∈ *borel-measurable N*
⟨*proof*⟩

lemma *distributed-real-measurable'*:
(⋀*x*. *x* ∈ *space N* ⟹ 0 ≤ *f x*) ⟹ *distributed M N X* (λ*x*. *ennreal* (*f x*)) ⟹
h ∈ *measurable L N* ⟹ (λ*x*. *f* (*h x*)) ∈ *borel-measurable L*
⟨*proof*⟩

lemma *joint-distributed-measurable1*:
distributed M (*S* ⨂$_M$ *T*) (λ*x*. (*X x*, *Y x*)) *f* ⟹ *h1* ∈ *measurable N M* ⟹ (λ*x*. *X* (*h1 x*)) ∈ *measurable N S*
⟨*proof*⟩

lemma *joint-distributed-measurable2*:
distributed M (*S* ⨂$_M$ *T*) (λ*x*. (*X x*, *Y x*)) *f* ⟹ *h2* ∈ *measurable N M* ⟹ (λ*x*. *Y* (*h2 x*)) ∈ *measurable N T*
⟨*proof*⟩

lemma *distributed-count-space*:
assumes *X*: *distributed M* (*count-space A*) *X P* **and** *a*: *a* ∈ *A* **and** *A*: *finite A*
shows *P a* = *emeasure M* (*X* −‘ {*a*} ∩ *space M*)
⟨*proof*⟩

lemma *distributed-cong-density*:
(*AE x in N. f x = g x*) ⟹ *g* ∈ *borel-measurable N* ⟹ *f* ∈ *borel-measurable N* ⟹
distributed M N X f ⟷ *distributed M N X g*
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-imp-emeasure-nonzero*:
assumes *X*: *distributed M MX X Px*
shows *emeasure MX* {*x* ∈ *space MX*. *Px x* ≠ *0*} ≠ *0*
⟨*proof*⟩

lemma *subdensity*:
assumes *T*: *T* ∈ *measurable P Q*
assumes *f*: *distributed M P X f*
assumes *g*: *distributed M Q Y g*
assumes *Y*: *Y* = *T* ∘ *X*
shows *AE x in P. g* (*T x*) = *0* ⟶ *f x* = *0*
⟨*proof*⟩

lemma *subdensity-real*:
fixes *g* :: *'a* ⇒ *real* **and** *f* :: *'b* ⇒ *real*
assumes *T*: *T* ∈ *measurable P Q*
assumes *f*: *distributed M P X f*
assumes *g*: *distributed M Q Y g*
assumes *Y*: *Y* = *T* ∘ *X*
shows (*AE x in P. 0* ≤ *g* (*T x*)) ⟹ (*AE x in P. 0* ≤ *f x*) ⟹ *AE x in P. g* (*T x*) = *0* ⟶ *f x* = *0*
⟨*proof*⟩

lemma *distributed-emeasure*:
distributed M N X f ⟹ *A* ∈ *sets N* ⟹ *emeasure M* (*X* −‘ *A* ∩ *space M*) = (∫⁺*x*. *f x* ∗ *indicator A x* ∂*N*)
⟨*proof*⟩

lemma *distributed-nn-integral*:
distributed M N X f ⟹ *g* ∈ *borel-measurable N* ⟹ (∫⁺*x*. *f x* ∗ *g x* ∂*N*) = (∫⁺*x*. *g* (*X x*) ∂*M*)
⟨*proof*⟩

lemma *distributed-integral*:
distributed M N X f ⟹ *g* ∈ *borel-measurable N* ⟹ (⋀*x*. *x* ∈ *space N* ⟹ *0* ≤ *f x*) ⟹
(∫ *x*. *f x* ∗ *g x* ∂*N*) = (∫ *x*. *g* (*X x*) ∂*M*)
⟨*proof*⟩

lemma *distributed-transform-integral*:
assumes *Px*: *distributed M N X Px* ⋀*x*. *x* ∈ *space N* ⟹ *0* ≤ *Px x*
assumes *distributed M P Y Py* ⋀*x*. *x* ∈ *space P* ⟹ *0* ≤ *Py x*

assumes Y: $Y = T \circ X$ **and** T: $T \in$ *measurable* $N\ P$ **and** f: $f \in$ *borel-measurable* P
shows $(\int x.\ Py\ x * f\ x\ \partial P) = (\int x.\ Px\ x * f\ (T\ x)\ \partial N)$
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-unique*:
assumes Px: *distributed* $M\ S\ X\ Px$
assumes Py: *distributed* $M\ S\ X\ Py$
shows $AE\ x\ in\ S.\ Px\ x = Py\ x$
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-jointI*:
assumes *sigma-finite-measure* S *sigma-finite-measure* T
assumes X[*measurable*]: $X \in$ *measurable* $M\ S$ **and** Y[*measurable*]: $Y \in$ *measurable* $M\ T$
assumes [*measurable*]: $f \in$ *borel-measurable* $(S \bigotimes_M T)$ **and** f: $AE\ x\ in\ S \bigotimes_M T.\ 0 \le f\ x$
assumes *eq*: $\bigwedge A\ B.\ A \in sets\ S \Longrightarrow B \in sets\ T \Longrightarrow$
emeasure $M\ \{x \in space\ M.\ X\ x \in A \wedge Y\ x \in B\} = (\int^+ x.\ (\int^+ y.\ f\ (x,\ y) * indicator\ B\ y\ \partial T) * indicator\ A\ x\ \partial S)$
shows *distributed* $M\ (S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))\ f$
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-swap*:
assumes *sigma-finite-measure* S *sigma-finite-measure* T
assumes Pxy: *distributed* $M\ (S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))\ Pxy$
shows *distributed* $M\ (T \bigotimes_M S)\ (\lambda x.\ (Y\ x,\ X\ x))\ (\lambda(x,\ y).\ Pxy\ (y,\ x))$
⟨*proof*⟩

lemma (**in** *prob-space*) *distr-marginal1*:
assumes *sigma-finite-measure* S *sigma-finite-measure* T
assumes Pxy: *distributed* $M\ (S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))\ Pxy$
defines $Px \equiv \lambda x.\ (\int^+ z.\ Pxy\ (x,\ z)\ \partial T)$
shows *distributed* $M\ S\ X\ Px$
⟨*proof*⟩

lemma (**in** *prob-space*) *distr-marginal2*:
assumes S: *sigma-finite-measure* S **and** T: *sigma-finite-measure* T
assumes Pxy: *distributed* $M\ (S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))\ Pxy$
shows *distributed* $M\ T\ Y\ (\lambda y.\ (\int^+ x.\ Pxy\ (x,\ y)\ \partial S))$
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-marginal-eq-joint1*:
assumes T: *sigma-finite-measure* T
assumes S: *sigma-finite-measure* S
assumes Px: *distributed* $M\ S\ X\ Px$
assumes Pxy: *distributed* $M\ (S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))\ Pxy$
shows $AE\ x\ in\ S.\ Px\ x = (\int^+ y.\ Pxy\ (x,\ y)\ \partial T)$
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-marginal-eq-joint2*:
assumes T: *sigma-finite-measure T*
assumes S: *sigma-finite-measure S*
assumes *Py*: *distributed M T Y Py*
assumes *Pxy*: *distributed M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
shows *AE y in T. Py y* $= (\int^+ x.\ Pxy\ (x,\ y)\ \partial S)$
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-joint-indep'*:
assumes S: *sigma-finite-measure S* **and** T: *sigma-finite-measure T*
assumes X[*measurable*]: *distributed M S X Px* **and** Y[*measurable*]: *distributed M T Y Py*
assumes *indep*: *distr M S X* $\bigotimes_M$ *distr M T Y* $=$ *distr M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$
shows *distributed M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ $(\lambda(x,\ y).\ Px\ x * Py\ y)$
⟨*proof*⟩

lemma *distributed-integrable*:
distributed M N X f $\Longrightarrow$ *g* $\in$ *borel-measurable N* $\Longrightarrow$ $(\bigwedge x.\ x \in space\ N \Longrightarrow 0 \le f\ x) \Longrightarrow$
integrable N $(\lambda x.\ f\ x * g\ x) \longleftrightarrow$ *integrable M* $(\lambda x.\ g\ (X\ x))$
⟨*proof*⟩

lemma *distributed-transform-integrable*:
assumes *Px*: *distributed M N X Px* $\bigwedge x.\ x \in space\ N \Longrightarrow 0 \le Px\ x$
assumes *distributed M P Y Py* $\bigwedge x.\ x \in space\ P \Longrightarrow 0 \le Py\ x$
assumes Y: $Y = (\lambda x.\ T\ (X\ x))$ **and** T: $T \in measurable\ N\ P$ **and** f: $f \in borel\text{-}measurable\ P$
shows *integrable P* $(\lambda x.\ Py\ x * f\ x) \longleftrightarrow$ *integrable N* $(\lambda x.\ Px\ x * f\ (T\ x))$
⟨*proof*⟩

lemma *distributed-integrable-var*:
fixes $X :: 'a \Rightarrow real$
shows *distributed M lborel X* $(\lambda x.\ ennreal\ (f\ x)) \Longrightarrow (\bigwedge x.\ 0 \le f\ x) \Longrightarrow$
integrable lborel $(\lambda x.\ f\ x * x) \Longrightarrow$ *integrable M X*
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-variance*:
fixes f::*real* $\Rightarrow$ *real*
assumes D: *distributed M lborel X f* **and** [*simp*]: $\bigwedge x.\ 0 \le f\ x$
shows *variance X* $= (\int x.\ x^2 * f\ (x + expectation\ X)\ \partial lborel)$
⟨*proof*⟩

lemma (**in** *prob-space*) *variance-affine*:
fixes f::*real* $\Rightarrow$ *real*
assumes [*arith*]: $b \neq 0$
assumes D[*intro*]: *distributed M lborel X f*
assumes [*simp*]: *prob-space* (*density lborel f*)

assumes *I*[*simp*]: *integrable M X*
assumes *I2*[*simp*]: *integrable M* $(\lambda x.\ (X\ x)^2)$
shows *variance* $(\lambda x.\ a + b * X\ x) = b^2 * variance\ X$
⟨*proof*⟩

definition
simple-distributed M X f $\longleftrightarrow$
$(\forall x.\ 0 \leq f\ x) \wedge$
distributed M (*count-space* (*X*'*space M*)) *X* $(\lambda x.\ ennreal\ (f\ x)) \wedge$
finite (*X*'*space M*)

lemma *simple-distributed-nonneg*[*dest*]: *simple-distributed M X f* $\Longrightarrow 0 \leq f\ x$
⟨*proof*⟩

lemma *simple-distributed*:
simple-distributed M X Px $\Longrightarrow$ *distributed M* (*count-space* (*X*'*space M*)) *X Px*
⟨*proof*⟩

lemma *simple-distributed-finite*[*dest*]: *simple-distributed M X P* $\Longrightarrow$ *finite* (*X*'*space M*)
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-simple-function-superset*:
assumes *X*: *simple-function M X* $\bigwedge x.\ x \in X$ ' *space M* $\Longrightarrow$ *P x* = *measure M* $(X -$ ' $\{x\} \cap$ *space M*)
assumes *A*: *X*'*space M* $\subseteq$ *A finite A*
defines *S* $\equiv$ *count-space A* **and** $P' \equiv (\lambda x.$ *if x* $\in$ *X*'*space M then P x else 0*)
shows *distributed M S X P*′
⟨*proof*⟩

lemma (**in** *prob-space*) *simple-distributedI*:
assumes *X*: *simple-function M X*
$\bigwedge x.\ 0 \leq P\ x$
$\bigwedge x.\ x \in X$ ' *space M* $\Longrightarrow$ *P x* = *measure M* $(X -$ ' $\{x\} \cap$ *space M*)
shows *simple-distributed M X P*
⟨*proof*⟩

lemma *simple-distributed-joint-finite*:
assumes *X*: *simple-distributed M* $(\lambda x.\ (X\ x,\ Y\ x))$ *Px*
shows *finite* (*X* ' *space M*) *finite* (*Y* ' *space M*)
⟨*proof*⟩

lemma *simple-distributed-joint2-finite*:
assumes *X*: *simple-distributed M* $(\lambda x.\ (X\ x,\ Y\ x,\ Z\ x))$ *Px*
shows *finite* (*X* ' *space M*) *finite* (*Y* ' *space M*) *finite* (*Z* ' *space M*)
⟨*proof*⟩

lemma *simple-distributed-simple-function*:
simple-distributed M X Px $\Longrightarrow$ *simple-function M X*

⟨*proof*⟩

lemma *simple-distributed-measure*:
simple-distributed M X P ⟹ *a* ∈ *X*‘*space M* ⟹ *P a* = *measure M* (*X* −‘ {*a*} ∩ *space M*)
⟨*proof*⟩

lemma (**in** *prob-space*) *simple-distributed-joint*:
assumes *X*: *simple-distributed M* (λ*x*. (*X x*, *Y x*)) *Px*
defines *S* ≡ *count-space* (*X*‘*space M*) ⨂$_M$ *count-space* (*Y*‘*space M*)
defines *P* ≡ (λ*x*. *if x* ∈ (λ*x*. (*X x*, *Y x*))‘*space M then Px x else 0*)
shows *distributed M S* (λ*x*. (*X x*, *Y x*)) *P*
⟨*proof*⟩

lemma (**in** *prob-space*) *simple-distributed-joint2*:
assumes *X*: *simple-distributed M* (λ*x*. (*X x*, *Y x*, *Z x*)) *Px*
defines *S* ≡ *count-space* (*X*‘*space M*) ⨂$_M$ *count-space* (*Y*‘*space M*) ⨂$_M$ *count-space* (*Z*‘*space M*)
defines *P* ≡ (λ*x*. *if x* ∈ (λ*x*. (*X x*, *Y x*, *Z x*))‘*space M then Px x else 0*)
shows *distributed M S* (λ*x*. (*X x*, *Y x*, *Z x*)) *P*
⟨*proof*⟩

lemma (**in** *prob-space*) *simple-distributed-sum-space*:
assumes *X*: *simple-distributed M X f*
shows *sum f* (*X*‘*space M*) = *1*
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-marginal-eq-joint-simple*:
assumes *Px*: *simple-function M X*
assumes *Py*: *simple-distributed M Y Py*
assumes *Pxy*: *simple-distributed M* (λ*x*. (*X x*, *Y x*)) *Pxy*
assumes *y*: *y* ∈ *Y*‘*space M*
shows *Py y* = (∑ *x*∈*X*‘*space M*. *if* (*x*, *y*) ∈ (λ*x*. (*X x*, *Y x*)) ‘ *space M then Pxy* (*x*, *y*) *else 0*)
⟨*proof*⟩

lemma *distributedI-real*:
fixes *f* :: ′*a* ⇒ *real*
assumes *gen*: *sets M1* = *sigma-sets* (*space M1*) *E* **and** *Int-stable E*
and *A*: *range A* ⊆ *E* (⋃ *i*::*nat*. *A i*) = *space M1* ⋀*i*. *emeasure* (*distr M M1 X*) (*A i*) ≠ ∞
and *X*: *X* ∈ *measurable M M1*
and *f*: *f* ∈ *borel-measurable M1 AE x in M1*. *0* ≤ *f x*
and *eq*: ⋀*A*. *A* ∈ *E* ⟹ *emeasure M* (*X* −‘ *A* ∩ *space M*) = (∫$^+$ *x*. *f x* ∗ *indicator A x* ∂*M1*)
shows *distributed M M1 X f*
⟨*proof*⟩

lemma *distributedI-borel-atMost*:

fixes $f :: real \Rightarrow real$
assumes [*measurable*]: $X \in$ *borel-measurable M*
and [*measurable*]: $f \in$ *borel-measurable borel* **and** f[*simp*]: *AE x in lborel.* $0 \leq f\ x$
and *g-eq*: $\bigwedge a.\ (\int^+ x.\ f\ x * indicator\ \{..a\}\ x\ \partial lborel) = ennreal\ (g\ a)$
and *M-eq*: $\bigwedge a.\ emeasure\ M\ \{x \in space\ M.\ X\ x \leq a\} = ennreal\ (g\ a)$
shows *distributed M lborel X f*
⟨*proof*⟩

lemma (**in** *prob-space*) *uniform-distributed-params*:
assumes *X*: *distributed M MX X* ($\lambda x.$ *indicator A x / measure MX A*)
shows $A \in sets\ MX$ *measure MX A* $\neq 0$
⟨*proof*⟩

lemma *prob-space-uniform-measure*:
assumes *A*: *emeasure M A* $\neq 0$ *emeasure M A* $\neq \infty$
shows *prob-space* (*uniform-measure M A*)
⟨*proof*⟩

lemma *prob-space-uniform-count-measure*: *finite* $A \Longrightarrow A \neq \{\} \Longrightarrow$ *prob-space* (*uniform-count-measure A*)
⟨*proof*⟩

lemma (**in** *prob-space*) *measure-uniform-measure-eq-cond-prob*:
assumes [*measurable*]: *Measurable.pred M P Measurable.pred M Q*
shows $\mathcal{P}(x\ in\ uniform\text{-}measure\ M\ \{x \in space\ M.\ Q\ x\}.\ P\ x) = \mathcal{P}(x\ in\ M.\ P\ x \mid Q\ x)$
⟨*proof*⟩

lemma *prob-space-point-measure*:
finite $S \Longrightarrow (\bigwedge s.\ s \in S \Longrightarrow 0 \leq p\ s) \Longrightarrow (\sum s \in S.\ p\ s) = 1 \Longrightarrow$ *prob-space* (*point-measure S p*)
⟨*proof*⟩

lemma (**in** *prob-space*) *distr-pair-fst*: $distr\ (N \bigotimes_M M)\ N\ fst = N$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *distr-reorder*:
assumes *inj-on t J t* $\in J \to K$ *finite K*
shows $distr\ (PiM\ K\ M)\ (Pi_M\ J\ (\lambda x.\ M\ (t\ x)))\ (\lambda \omega.\ \lambda n \in J.\ \omega\ (t\ n)) = PiM\ J\ (\lambda x.\ M\ (t\ x))$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *distr-restrict*:
$J \subseteq K \Longrightarrow$ *finite* $K \Longrightarrow (\Pi_M\ i \in J.\ M\ i) = distr\ (\Pi_M\ i \in K.\ M\ i)\ (\Pi_M\ i \in J.\ M\ i)\ (\lambda f.\ restrict\ f\ J)$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *emeasure-prod-emb*[*simp*]:

assumes L: $J \subseteq L$ *finite* L **and** X: $X \in$ *sets* $(Pi_M\ J\ M)$
shows *emeasure* $(Pi_M\ L\ M)$ (*prod-emb* $L\ M\ J\ X$) = *emeasure* $(Pi_M\ J\ M)\ X$
⟨*proof*⟩

lemma *emeasure-distr-restrict*:
assumes $I \subseteq K$ **and** Q[*measurable-cong*]: *sets* Q = *sets* $(PiM\ K\ M)$ **and** A[*measurable*]: $A \in$ *sets* $(PiM\ I\ M)$
shows *emeasure* (*distr* Q $(PiM\ I\ M)$ $(\lambda\omega.\ restrict\ \omega\ I))\ A$ = *emeasure* Q (*prod-emb* $K\ M\ I\ A$)
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-space-completion*: *prob-space* (*completion* M)
⟨*proof*⟩

lemma *distr-PiM-finite-prob-space*:
assumes *fin*: *finite* I
assumes *product-prob-space* M
assumes *product-prob-space* M'
assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow f \in$ *measurable* $(M\ i)\ (M'\ i)$
shows *distr* $(PiM\ I\ M)\ (PiM\ I\ M')$ (*compose* $I\ f$) = $PiM\ I\ (\lambda i.\ distr\ (M\ i)\ (M'\ i)\ f)$
⟨*proof*⟩

end

2 Distribution Functions

Shows that the cumulative distribution function (cdf) of a distribution (a measure on the reals) is nondecreasing and right continuous, which tends to 0 and 1 in either direction.

Conversely, every such function is the cdf of a unique distribution. This direction defines the measure in the obvious way on half-open intervals, and then applies the Caratheodory extension theorem.

theory *Distribution-Functions*
imports *Probability-Measure*
begin

lemma *UN-Ioc-eq-UNIV*: $(\bigcup n.\ \{-real\ n <..\ real\ n\}) = UNIV$
⟨*proof*⟩

2.1 Properties of cdf's

definition
cdf :: *real measure* ⇒ *real* ⇒ *real*
where
cdf $M \equiv \lambda x.$ *measure* M $\{..x\}$

lemma *cdf-def2*: *cdf M x = measure M {..x}*
⟨*proof*⟩

locale *finite-borel-measure = finite-measure M* **for** *M :: real measure* +
assumes *M-is-borel*: *sets M = sets borel*
begin

lemma *sets-M*[*intro*]: *a ∈ sets borel ⟹ a ∈ sets M*
⟨*proof*⟩

lemma *cdf-diff-eq*:
assumes *x < y*
shows *cdf M y − cdf M x = measure M {x<..y}*
⟨*proof*⟩

lemma *cdf-nondecreasing*: *x ≤ y ⟹ cdf M x ≤ cdf M y*
⟨*proof*⟩

lemma *borel-UNIV*: *space M = UNIV*
⟨*proof*⟩

lemma *cdf-nonneg*: *cdf M x ≥ 0*
⟨*proof*⟩

lemma *cdf-bounded*: *cdf M x ≤ measure M (space M)*
⟨*proof*⟩

lemma *cdf-lim-infty*:
((λi. cdf M (real i)) ⟶ measure M (space M))
⟨*proof*⟩

lemma *cdf-lim-at-top*: *(cdf M ⟶ measure M (space M)) at-top*
⟨*proof*⟩

lemma *cdf-lim-neg-infty*: *((λi. cdf M (− real i)) ⟶ 0)*
⟨*proof*⟩

lemma *cdf-lim-at-bot*: *(cdf M ⟶ 0) at-bot*
⟨*proof*⟩

lemma *cdf-is-right-cont*: *continuous (at-right a) (cdf M)*
⟨*proof*⟩

lemma *cdf-at-left*: *(cdf M ⟶ measure M {..<a}) (at-left a)*
⟨*proof*⟩

lemma *isCont-cdf*: *isCont (cdf M) x ⟷ measure M {x} = 0*
⟨*proof*⟩

lemma *countable-atoms*: *countable* $\{x.\ measure\ M\ \{x\} > 0\}$
⟨*proof*⟩

end

locale *real-distribution* = *prob-space M* **for** *M* :: *real measure* +
assumes *events-eq-borel* [*simp*, *measurable-cong*]: *sets M* = *sets borel*
begin

lemma *finite-borel-measure-M*: *finite-borel-measure M*
⟨*proof*⟩

sublocale *finite-borel-measure M*
⟨*proof*⟩

lemma *space-eq-univ* [*simp*]: *space M* = *UNIV*
⟨*proof*⟩

lemma *cdf-bounded-prob*: $\bigwedge x.$ *cdf M x* $\leq$ *1*
⟨*proof*⟩

lemma *cdf-lim-infty-prob*: $(\lambda i.$ *cdf M* (*real i*)) $\longrightarrow$ *1*
⟨*proof*⟩

lemma *cdf-lim-at-top-prob*: (*cdf M* $\longrightarrow$ *1*) *at-top*
⟨*proof*⟩

lemma *measurable-finite-borel* [*simp*]:
f $\in$ *borel-measurable borel* $\Longrightarrow$ *f* $\in$ *borel-measurable M*
⟨*proof*⟩

end

lemma (**in** *prob-space*) *real-distribution-distr* [*intro*, *simp*]:
random-variable borel X $\Longrightarrow$ *real-distribution* (*distr M borel X*)
⟨*proof*⟩

2.2 Uniqueness

lemma (**in** *finite-borel-measure*) *emeasure-Ioc*:
assumes *a* $\leq$ *b* **shows** *emeasure M* $\{a <..\ b\}$ = *cdf M b* − *cdf M a*
⟨*proof*⟩

lemma *cdf-unique'*:
fixes *M1 M2*
assumes *finite-borel-measure M1* **and** *finite-borel-measure M2*
assumes *cdf M1* = *cdf M2*
shows *M1* = *M2*
⟨*proof*⟩

lemma *cdf-unique*:
real-distribution M1 $\Longrightarrow$ *real-distribution M2* $\Longrightarrow$ *cdf M1* $=$ *cdf M2* $\Longrightarrow$ *M1* $=$ *M2*
⟨*proof*⟩

lemma
fixes *F* :: *real* $\Rightarrow$ *real*
assumes *nondecF* : $\bigwedge$ *x y*. *x* $\leq$ *y* $\Longrightarrow$ *F x* $\leq$ *F y*
and *right-cont-F* : $\bigwedge$*a*. *continuous* (*at-right a*) *F*
and *lim-F-at-bot* : (*F* $\longrightarrow$ *0*) *at-bot*
and *lim-F-at-top* : (*F* $\longrightarrow$ *m*) *at-top*
and *m*: *0* $\leq$ *m*
shows *interval-measure-UNIV*: *emeasure* (*interval-measure F*) *UNIV* $=$ *m*
and *finite-borel-measure-interval-measure*: *finite-borel-measure* (*interval-measure F*)
⟨*proof*⟩

lemma *real-distribution-interval-measure*:
fixes *F* :: *real* $\Rightarrow$ *real*
assumes *nondecF* : $\bigwedge$ *x y*. *x* $\leq$ *y* $\Longrightarrow$ *F x* $\leq$ *F y* **and**
right-cont-F : $\bigwedge$*a*. *continuous* (*at-right a*) *F* **and**
lim-F-at-bot : (*F* $\longrightarrow$ *0*) *at-bot* **and**
lim-F-at-top : (*F* $\longrightarrow$ *1*) *at-top*
shows *real-distribution* (*interval-measure F*)
⟨*proof*⟩

lemma
fixes *F* :: *real* $\Rightarrow$ *real*
assumes *nondecF* : $\bigwedge$ *x y*. *x* $\leq$ *y* $\Longrightarrow$ *F x* $\leq$ *F y* **and**
right-cont-F : $\bigwedge$*a*. *continuous* (*at-right a*) *F* **and**
lim-F-at-bot : (*F* $\longrightarrow$ *0*) *at-bot*
shows *emeasure-interval-measure-Iic*: *emeasure* (*interval-measure F*) {.. *x*} $=$ *F x*
and *measure-interval-measure-Iic*: *measure* (*interval-measure F*) {.. *x*} $=$ *F x*
⟨*proof*⟩

lemma *cdf-interval-measure*:
($\bigwedge$ *x y*. *x* $\leq$ *y* $\Longrightarrow$ *F x* $\leq$ *F y*) $\Longrightarrow$ ($\bigwedge$*a*. *continuous* (*at-right a*) *F*) $\Longrightarrow$ (*F* $\longrightarrow$ *0*) *at-bot* $\Longrightarrow$ *cdf* (*interval-measure F*) $=$ *F*
⟨*proof*⟩

end

3 Weak Convergence of Functions and Distributions

Properties of weak convergence of functions and measures, including the portmanteau theorem.

theory *Weak-Convergence*
imports *Distribution-Functions*
begin

4 Weak Convergence of Functions

definition
weak-conv :: $(nat \Rightarrow (real \Rightarrow real)) \Rightarrow (real \Rightarrow real) \Rightarrow bool$
where
$weak\text{-}conv\ F\text{-}seq\ F \equiv \forall x.\ isCont\ F\ x \longrightarrow (\lambda n.\ F\text{-}seq\ n\ x) \longrightarrow F\ x$

5 Weak Convergence of Distributions

definition
weak-conv-m :: $(nat \Rightarrow real\ measure) \Rightarrow real\ measure \Rightarrow bool$
where
$weak\text{-}conv\text{-}m\ M\text{-}seq\ M \equiv weak\text{-}conv\ (\lambda n.\ cdf\ (M\text{-}seq\ n))\ (cdf\ M)$

6 Skorohod's theorem

locale *right-continuous-mono* =
fixes $f :: real \Rightarrow real$ **and** $a\ b :: real$
assumes *cont*: $\bigwedge x.\ continuous\ (at\text{-}right\ x)\ f$
assumes *mono*: $mono\ f$
assumes *bot*: $(f \longrightarrow a)\ at\text{-}bot$
assumes *top*: $(f \longrightarrow b)\ at\text{-}top$
begin

abbreviation $I :: real \Rightarrow real$ **where**
$I\ \omega \equiv Inf\ \{x.\ \omega \leq f\ x\}$

lemma *pseudoinverse*: **assumes** $a < \omega\ \omega < b$ **shows** $\omega \leq f\ x \longleftrightarrow I\ \omega \leq x$
⟨*proof*⟩

lemma *pseudoinverse'*: $\forall \omega \in \{a<..<b\}.\ \forall x.\ \omega \leq f\ x \longleftrightarrow I\ \omega \leq x$
⟨*proof*⟩

lemma *mono-I*: $mono\text{-}on\ \{a <..< b\}\ I$
⟨*proof*⟩

end

locale *cdf-distribution = real-distribution*
begin

abbreviation $C \equiv cdf\ M$

sublocale *right-continuous-mono C 0 1*
⟨*proof*⟩

lemma *measurable-C*[*measurable*]: $C \in borel\text{-}measurable\ borel$
⟨*proof*⟩

lemma *measurable-CI*[*measurable*]: $I \in borel\text{-}measurable\ (restrict\text{-}space\ borel\ \{0<..<1\})$
⟨*proof*⟩

lemma *emeasure-distr-I*: $emeasure\ (distr\ (restrict\text{-}space\ lborel\ \{0<..<1::real\})\ borel\ I)\ UNIV = 1$
⟨*proof*⟩

lemma *distr-I-eq-M*: $distr\ (restrict\text{-}space\ lborel\ \{0<..<1::real\})\ borel\ I = M$ (**is** *?I = -*)
⟨*proof*⟩

end

context
fixes $\mu :: nat \Rightarrow real\ measure$
and $M :: real\ measure$
assumes μ: $\bigwedge n.\ real\text{-}distribution\ (\mu\ n)$
assumes M: $real\text{-}distribution\ M$
assumes $\mu\text{-}to\text{-}M$: $weak\text{-}conv\text{-}m\ \mu\ M$
begin

theorem *Skorohod*:
$\exists\ (\Omega :: real\ measure)\ (Y\text{-}seq :: nat \Rightarrow real \Rightarrow real)\ (Y :: real \Rightarrow real).$
$prob\text{-}space\ \Omega \wedge$
$(\forall n.\ Y\text{-}seq\ n \in measurable\ \Omega\ borel) \wedge$
$(\forall n.\ distr\ \Omega\ borel\ (Y\text{-}seq\ n) = \mu\ n) \wedge$
$Y \in measurable\ \Omega\ lborel \wedge$
$distr\ \Omega\ borel\ Y = M \wedge$
$(\forall x \in space\ \Omega.\ (\lambda n.\ Y\text{-}seq\ n\ x) \longrightarrow Y\ x)$
⟨*proof*⟩

The Portmanteau theorem, that is, the equivalence of various definitions of weak convergence.

theorem *weak-conv-imp-bdd-ae-continuous-conv*:
fixes
$f :: real \Rightarrow 'a::\{banach, second\text{-}countable\text{-}topology\}$

assumes
discont-null: $M\ (\{x.\ \neg\ isCont\ f\ x\}) = 0$ **and**
f-bdd: $\bigwedge x.\ norm\ (f\ x) \leq B$ **and**
[*measurable*]: $f \in borel\text{-}measurable\ borel$
shows
$(\lambda\ n.\ integral^L\ (\mu\ n)\ f) \longrightarrow integral^L\ M\ f$
⟨*proof*⟩

theorem *weak-conv-imp-integral-bdd-continuous-conv*:
fixes f :: *real* ⇒ $'a$::{*banach*, *second-countable-topology*}
assumes
$\bigwedge x.\ isCont\ f\ x$ **and**
$\bigwedge x.\ norm\ (f\ x) \leq B$
shows
$(\lambda\ n.\ integral^L\ (\mu\ n)\ f) \longrightarrow integral^L\ M\ f$
⟨*proof*⟩

theorem *weak-conv-imp-continuity-set-conv*:
fixes f :: *real* ⇒ *real*
assumes [*measurable*]: $A \in sets\ borel$ **and** $M\ (frontier\ A) = 0$
shows $(\lambda n.\ measure\ (\mu\ n)\ A) \longrightarrow measure\ M\ A$
⟨*proof*⟩

end

definition
cts-step :: *real* ⇒ *real* ⇒ *real* ⇒ *real*
where
$cts\text{-}step\ a\ b\ x \equiv if\ x \leq a\ then\ 1\ else\ if\ x \geq b\ then\ 0\ else\ (b - x)\ /\ (b - a)$

lemma *cts-step-uniformly-continuous*:
assumes [*arith*]: $a < b$
shows *uniformly-continuous-on UNIV* (*cts-step a b*)
⟨*proof*⟩

lemma (**in** *real-distribution*) *integrable-cts-step*: $a < b \Longrightarrow integrable\ M\ (cts\text{-}step\ a\ b)$
⟨*proof*⟩

lemma (**in** *real-distribution*) *cdf-cts-step*:
assumes [*arith*]: $x < y$
shows $cdf\ M\ x \leq integral^L\ M\ (cts\text{-}step\ x\ y)$ **and** $integral^L\ M\ (cts\text{-}step\ x\ y) \leq cdf\ M\ y$
⟨*proof*⟩

context
fixes *M-seq* :: *nat* ⇒ *real measure*
and *M* :: *real measure*
assumes *distr-M-seq* [*simp*]: $\bigwedge n.\ real\text{-}distribution\ (M\text{-}seq\ n)$

assumes *distr-M* [*simp*]: *real-distribution M*
begin

theorem *continuity-set-conv-imp-weak-conv*:
fixes *f* :: *real* ⇒ *real*
assumes ∗: ⋀*A*. *A* ∈ *sets borel* ⟹ *M* (*frontier A*) = *0* ⟹ (λ *n*. (*measure* (*M-seq n*) *A*)) ⟶ *measure M A*
shows *weak-conv-m M-seq M*
⟨*proof*⟩

theorem *integral-cts-step-conv-imp-weak-conv*:
assumes *integral-conv*: ⋀*x y*. *x* < *y* ⟹ (λ*n*. *integral*^{L} (*M-seq n*) (*cts-step x y*)) ⟶ *integral*^{L} *M* (*cts-step x y*)
shows *weak-conv-m M-seq M*
⟨*proof*⟩

theorem *integral-bdd-continuous-conv-imp-weak-conv*:
assumes
⋀*f*. (⋀*x*. *isCont f x*) ⟹ (⋀*x*. *abs* (*f x*) ≤ *1*) ⟹ (λ*n*. *integral*^{L} (*M-seq n*) *f*::*real*) ⟶ *integral*^{L} *M f*
shows
weak-conv-m M-seq M
⟨*proof*⟩

end

end

7 The Giry monad

theory *Giry-Monad*
imports *Probability-Measure HOL−Library.Monad-Syntax*
begin

7.1 Sub-probability spaces

locale *subprob-space* = *finite-measure* +
assumes *emeasure-space-le-1*: *emeasure M* (*space M*) ≤ *1*
assumes *subprob-not-empty*: *space M* ≠ {}

lemma *subprob-spaceI*[*Pure.intro*!]:
assumes ∗: *emeasure M* (*space M*) ≤ *1*
assumes *space M* ≠ {}
shows *subprob-space M*
⟨*proof*⟩

lemma (**in** *subprob-space*) *emeasure-subprob-space-less-top*: *emeasure M A* ≠ *top*
⟨*proof*⟩

lemma *prob-space-imp-subprob-space*:
prob-space M $\Longrightarrow$ *subprob-space M*
⟨*proof*⟩

lemma *subprob-space-imp-sigma-finite*: *subprob-space M* $\Longrightarrow$ *sigma-finite-measure M*
⟨*proof*⟩

sublocale *prob-space* $\subseteq$ *subprob-space*
⟨*proof*⟩

lemma *subprob-space-sigma* [*simp*]: $\Omega \neq \{\}$ $\Longrightarrow$ *subprob-space* (*sigma* Ω *X*)
⟨*proof*⟩

lemma *subprob-space-null-measure*: *space M* $\neq \{\}$ $\Longrightarrow$ *subprob-space* (*null-measure M*)
⟨*proof*⟩

lemma (**in** *subprob-space*) *subprob-space-distr*:
assumes *f*: *f* $\in$ *measurable M M'* **and** *space M'* $\neq \{\}$ **shows** *subprob-space* (*distr M M' f*)
⟨*proof*⟩

lemma (**in** *subprob-space*) *subprob-emeasure-le-1*: *emeasure M X* $\leq$ *1*
⟨*proof*⟩

lemma (**in** *subprob-space*) *subprob-measure-le-1*: *measure M X* $\leq$ *1*
⟨*proof*⟩

lemma (**in** *subprob-space*) *nn-integral-le-const*:
assumes *0* $\leq$ *c AE x in M*. *f x* $\leq$ *c*
shows $(\int^+ x.\ f\ x\ \partial M) \leq c$
⟨*proof*⟩

lemma *emeasure-density-distr-interval*:
fixes *h* :: *real* $\Rightarrow$ *real* **and** *g* :: *real* $\Rightarrow$ *real* **and** *g'* :: *real* $\Rightarrow$ *real*
assumes [*simp*]: *a* $\leq$ *b*
assumes *Mf*[*measurable*]: *f* $\in$ *borel-measurable borel*
assumes *Mg*[*measurable*]: *g* $\in$ *borel-measurable borel*
assumes *Mg'*[*measurable*]: *g'* $\in$ *borel-measurable borel*
assumes *Mh*[*measurable*]: *h* $\in$ *borel-measurable borel*
assumes *prob*: *subprob-space* (*density lborel f*)
assumes *nonnegf*: $\bigwedge x.\ f\ x \geq 0$
assumes *derivg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow$ (*g has-real-derivative g' x*) (*at x*)
assumes *contg'*: *continuous-on* $\{a..b\}$ *g'*
assumes *mono*: *strict-mono-on* $\{a..b\}$ *g* **and** *inv*: $\bigwedge x.\ h\ x \in \{a..b\} \Longrightarrow g\ (h\ x) = x$
assumes *range*: $\{a..b\} \subseteq$ *range h*
shows *emeasure* (*distr* (*density lborel f*) *lborel h*) $\{a..b\}$ =

emeasure (*density lborel* (λ*x*. *f* (*g x*) ∗ *g*′ *x*)) {*a*..*b*}
⟨*proof*⟩

locale *pair-subprob-space* =
pair-sigma-finite M1 M2 + *M1*: *subprob-space M1* + *M2*: *subprob-space M2* **for** *M1 M2*

sublocale *pair-subprob-space* ⊆ *P?*: *subprob-space M1* ⨂$_M$ *M2*
⟨*proof*⟩

lemma *subprob-space-null-measure-iff*:
subprob-space (*null-measure M*) ⟷ *space M* ≠ {}
⟨*proof*⟩

lemma *subprob-space-restrict-space*:
assumes *M*: *subprob-space M*
and *A*: *A* ∩ *space M* ∈ *sets M A* ∩ *space M* ≠ {}
shows *subprob-space* (*restrict-space M A*)
⟨*proof*⟩

definition *subprob-algebra* :: *'a measure* ⇒ *'a measure measure* **where**
subprob-algebra K =
(*SUP A* ∈ *sets K*. *vimage-algebra* {*M*. *subprob-space M* ∧ *sets M* = *sets K*}
(λ*M*. *emeasure M A*) *borel*)

lemma *space-subprob-algebra*: *space* (*subprob-algebra A*) = {*M*. *subprob-space M* ∧ *sets M* = *sets A*}
⟨*proof*⟩

lemma *subprob-algebra-cong*: *sets M* = *sets N* ⟹ *subprob-algebra M* = *subprob-algebra N*
⟨*proof*⟩

lemma *measurable-emeasure-subprob-algebra*[*measurable*]:
a ∈ *sets A* ⟹ (λ*M*. *emeasure M a*) ∈ *borel-measurable* (*subprob-algebra A*)
⟨*proof*⟩

lemma *measurable-measure-subprob-algebra*[*measurable*]:
a ∈ *sets A* ⟹ (λ*M*. *measure M a*) ∈ *borel-measurable* (*subprob-algebra A*)
⟨*proof*⟩

lemma *subprob-measurableD*:
assumes *N*: *N* ∈ *measurable M* (*subprob-algebra S*) **and** *x*: *x* ∈ *space M*
shows *space* (*N x*) = *space S*
and *sets* (*N x*) = *sets S*
and *measurable* (*N x*) *K* = *measurable S K*
and *measurable K* (*N x*) = *measurable K S*
⟨*proof*⟩

⟨*ML*⟩

context
fixes *K M N* **assumes** *K*: $K \in$ *measurable M* (*subprob-algebra N*)
begin

lemma *subprob-space-kernel*: $a \in$ *space M* $\Longrightarrow$ *subprob-space* (*K a*)
⟨*proof*⟩

lemma *sets-kernel*: $a \in$ *space M* $\Longrightarrow$ *sets* (*K a*) = *sets N*
⟨*proof*⟩

lemma *measurable-emeasure-kernel*[*measurable*]:
$A \in$ *sets N* $\Longrightarrow$ (λa. *emeasure* (*K a*) *A*) $\in$ *borel-measurable M*
⟨*proof*⟩

end

lemma *measurable-subprob-algebra*:
($\bigwedge a$. $a \in$ *space M* $\Longrightarrow$ *subprob-space* (*K a*)) $\Longrightarrow$
($\bigwedge a$. $a \in$ *space M* $\Longrightarrow$ *sets* (*K a*) = *sets N*) $\Longrightarrow$
($\bigwedge A$. $A \in$ *sets N* $\Longrightarrow$ (λa. *emeasure* (*K a*) *A*) $\in$ *borel-measurable M*) $\Longrightarrow$
K $\in$ *measurable M* (*subprob-algebra N*)
⟨*proof*⟩

lemma *measurable-submarkov*:
K $\in$ *measurable M* (*subprob-algebra M*) $\longleftrightarrow$
($\forall x \in$*space M*. *subprob-space* (*K x*) $\wedge$ *sets* (*K x*) = *sets M*) $\wedge$
($\forall A \in$*sets M*. (λx. *emeasure* (*K x*) *A*) $\in$ *measurable M borel*)
⟨*proof*⟩

lemma *measurable-subprob-algebra-generated*:
assumes *eq*: *sets N* = *sigma-sets* Ω *G* **and** *Int-stable G G* $\subseteq$ *Pow* Ω
assumes *subsp*: $\bigwedge a$. $a \in$ *space M* $\Longrightarrow$ *subprob-space* (*K a*)
assumes *sets*: $\bigwedge a$. $a \in$ *space M* $\Longrightarrow$ *sets* (*K a*) = *sets N*
assumes $\bigwedge A$. $A \in G \Longrightarrow$ (λa. *emeasure* (*K a*) *A*) $\in$ *borel-measurable M*
assumes Ω: (λa. *emeasure* (*K a*) Ω) $\in$ *borel-measurable M*
shows *K* $\in$ *measurable M* (*subprob-algebra N*)
⟨*proof*⟩

lemma *space-subprob-algebra-empty-iff*:
space (*subprob-algebra N*) = {} $\longleftrightarrow$ *space N* = {}
⟨*proof*⟩

lemma *nn-integral-measurable-subprob-algebra*[*measurable*]:
assumes *f*: *f* $\in$ *borel-measurable N*
shows (λM. *integral*^{N} *M f*) $\in$ *borel-measurable* (*subprob-algebra N*) (**is** - $\in$ *?B*)
⟨*proof*⟩

lemma *measurable-distr*:
assumes [*measurable*]: $f \in$ *measurable* M N
shows $(\lambda M'.$ *distr* M' N $f) \in$ *measurable* (*subprob-algebra* M) (*subprob-algebra* N)
⟨*proof*⟩

lemma *emeasure-space-subprob-algebra*[*measurable*]:
$(\lambda a.$ *emeasure* a (*space* a)) $\in$ *borel-measurable* (*subprob-algebra* N)
⟨*proof*⟩

lemma *integrable-measurable-subprob-algebra*[*measurable*]:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes [*measurable*]: $f \in$ *borel-measurable* N
shows *Measurable.pred* (*subprob-algebra* N) $(\lambda M.$ *integrable* M $f)$
⟨*proof*⟩

lemma *integral-measurable-subprob-algebra*[*measurable*]:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes f [*measurable*]: $f \in$ *borel-measurable* N
shows $(\lambda M.$ *integral*^{L} M $f) \in$ *subprob-algebra* $N \rightarrow_M$ *borel*
⟨*proof*⟩

lemma *measurable-pair-measure*:
assumes f: $f \in$ *measurable* M (*subprob-algebra* N)
assumes g: $g \in$ *measurable* M (*subprob-algebra* L)
shows $(\lambda x.$ f $x \bigotimes_M$ g $x) \in$ *measurable* M (*subprob-algebra* $(N \bigotimes_M L)$)
⟨*proof*⟩

lemma *restrict-space-measurable*:
assumes X: $X \neq \{\}$ $X \in$ *sets* K
assumes N: $N \in$ *measurable* M (*subprob-algebra* K)
shows $(\lambda x.$ *restrict-space* $(N\,x)$ $X) \in$ *measurable* M (*subprob-algebra* (*restrict-space* K X))
⟨*proof*⟩

7.2 Properties of "return"

definition *return* :: $'a$ *measure* $\Rightarrow$ $'a \Rightarrow$ $'a$ *measure* **where**
return R x = *measure-of* (*space* R) (*sets* R) $(\lambda A.$ *indicator* A $x)$

lemma *space-return*[*simp*]: *space* (*return* M x) = *space* M
⟨*proof*⟩

lemma *sets-return*[*simp*]: *sets* (*return* M x) = *sets* M
⟨*proof*⟩

lemma *measurable-return1*[*simp*]: *measurable* (*return* N x) L = *measurable* N L
⟨*proof*⟩

lemma *measurable-return2*[*simp*]: *measurable L* (*return N x*) = *measurable L N*
⟨*proof*⟩

lemma *return-sets-cong*: *sets M* = *sets N* ⟹ *return M* = *return N*
⟨*proof*⟩

lemma *return-cong*: *sets A* = *sets B* ⟹ *return A x* = *return B x*
⟨*proof*⟩

lemma *emeasure-return*[*simp*]:
assumes *A* ∈ *sets M*
shows *emeasure* (*return M x*) *A* = *indicator A x*
⟨*proof*⟩

lemma *prob-space-return*: *x* ∈ *space M* ⟹ *prob-space* (*return M x*)
⟨*proof*⟩

lemma *subprob-space-return*: *x* ∈ *space M* ⟹ *subprob-space* (*return M x*)
⟨*proof*⟩

lemma *subprob-space-return-ne*:
assumes *space M* ≠ {} **shows** *subprob-space* (*return M x*)
⟨*proof*⟩

lemma *measure-return*: **assumes** *X*: *X* ∈ *sets M* **shows** *measure* (*return M x*) *X* = *indicator X x*
⟨*proof*⟩

lemma *AE-return*:
assumes [*simp*]: *x* ∈ *space M* **and** [*measurable*]: *Measurable.pred M P*
shows (*AE y in return M x*. *P y*) ⟷ *P x*
⟨*proof*⟩

lemma *nn-integral-return*:
assumes *x* ∈ *space M g* ∈ *borel-measurable M*
shows (∫⁺ *a*. *g a* ∂*return M x*) = *g x*
⟨*proof*⟩

lemma *integral-return*:
fixes *g* :: - ⇒ *'a* :: {*banach*, *second-countable-topology*}
assumes *x* ∈ *space M g* ∈ *borel-measurable M*
shows (∫ *a*. *g a* ∂*return M x*) = *g x*
⟨*proof*⟩

lemma *return-measurable*[*measurable*]: *return N* ∈ *measurable N* (*subprob-algebra N*)
⟨*proof*⟩

lemma *distr-return*:
assumes $f \in$ *measurable* M N **and** $x \in$ *space* M
shows *distr* (*return* M x) N f = *return* N (f x)
⟨*proof*⟩

lemma *return-restrict-space*:
$\Omega \in$ *sets* $M \Longrightarrow$ *return* (*restrict-space* M Ω) x = *restrict-space* (*return* M x) Ω
⟨*proof*⟩

lemma *measurable-distr2*:
assumes f[*measurable*]: *case-prod* $f \in$ *measurable* ($L \bigotimes_M M$) N
assumes g[*measurable*]: $g \in$ *measurable* L (*subprob-algebra* M)
shows (λx. *distr* (g x) N (f x)) $\in$ *measurable* L (*subprob-algebra* N)
⟨*proof*⟩

lemma *nn-integral-measurable-subprob-algebra2*:
assumes f[*measurable*]: ($\lambda(x, y)$. f x y) $\in$ *borel-measurable* ($M \bigotimes_M N$)
assumes N[*measurable*]: $L \in$ *measurable* M (*subprob-algebra* N)
shows (λx. *integral*^{N} (L x) (f x)) $\in$ *borel-measurable* M
⟨*proof*⟩

lemma *emeasure-measurable-subprob-algebra2*:
assumes A[*measurable*]: (*SIGMA* x:*space* M. A x) $\in$ *sets* ($M \bigotimes_M N$)
assumes L[*measurable*]: $L \in$ *measurable* M (*subprob-algebra* N)
shows (λx. *emeasure* (L x) (A x)) $\in$ *borel-measurable* M
⟨*proof*⟩

lemma *measure-measurable-subprob-algebra2*:
assumes A[*measurable*]: (*SIGMA* x:*space* M. A x) $\in$ *sets* ($M \bigotimes_M N$)
assumes L[*measurable*]: $L \in$ *measurable* M (*subprob-algebra* N)
shows (λx. *measure* (L x) (A x)) $\in$ *borel-measurable* M
⟨*proof*⟩

definition *select-sets* M = (*SOME* N. *sets* M = *sets* (*subprob-algebra* N))

lemma *select-sets1*:
sets M = *sets* (*subprob-algebra* N) $\Longrightarrow$ *sets* M = *sets* (*subprob-algebra* (*select-sets* M))
⟨*proof*⟩

lemma *sets-select-sets*[*simp*]:
assumes *sets*: *sets* M = *sets* (*subprob-algebra* N)
shows *sets* (*select-sets* M) = *sets* N
⟨*proof*⟩

lemma *space-select-sets*[*simp*]:
sets M = *sets* (*subprob-algebra* N) $\Longrightarrow$ *space* (*select-sets* M) = *space* N
⟨*proof*⟩

7.3 Join

definition *join* :: *'a measure measure* ⇒ *'a measure* **where**
join M = *measure-of* (*space* (*select-sets M*)) (*sets* (*select-sets M*)) (λ*B*. ∫⁺ *M'*. *emeasure M' B* ∂*M*)

lemma
shows *space-join*[*simp*]: *space* (*join M*) = *space* (*select-sets M*)
and *sets-join*[*simp*]: *sets* (*join M*) = *sets* (*select-sets M*)
⟨*proof*⟩

lemma *emeasure-join*:
assumes *M*[*simp*, *measurable-cong*]: *sets M* = *sets* (*subprob-algebra N*) **and** *A*: *A* ∈ *sets N*
shows *emeasure* (*join M*) *A* = (∫⁺ *M'*. *emeasure M' A* ∂*M*)
⟨*proof*⟩

lemma *measurable-join*:
join ∈ *measurable* (*subprob-algebra* (*subprob-algebra N*)) (*subprob-algebra N*)
⟨*proof*⟩

lemma *nn-integral-join*:
assumes *f*: *f* ∈ *borel-measurable N*
and *M*[*measurable-cong*]: *sets M* = *sets* (*subprob-algebra N*)
shows (∫⁺*x*. *f x* ∂*join M*) = (∫⁺*M'*. ∫⁺*x*. *f x* ∂*M'* ∂*M*)
⟨*proof*⟩

lemma *measurable-join1*:
⟦ *f* ∈ *measurable N K*; *sets M* = *sets* (*subprob-algebra N*) ⟧
⟹ *f* ∈ *measurable* (*join M*) *K*
⟨*proof*⟩

lemma
fixes *f* :: *-* ⇒ *real*
assumes *f-measurable* [*measurable*]: *f* ∈ *borel-measurable N*
and *f-bounded*: ⋀*x*. *x* ∈ *space N* ⟹ |*f x*| ≤ *B*
and *M* [*measurable-cong*]: *sets M* = *sets* (*subprob-algebra N*)
and *fin*: *finite-measure M*
and *M-bounded*: *AE M' in M*. *emeasure M'* (*space M'*) ≤ *ennreal B'*
shows *integrable-join*: *integrable* (*join M*) *f* (**is** *?integrable*)
and *integral-join*: *integral*^{L} (*join M*) *f* = ∫ *M'*. *integral*^{L} *M' f* ∂*M* (**is** *?integral*)
⟨*proof*⟩

lemma *join-assoc*:
assumes *M*[*measurable-cong*]: *sets M* = *sets* (*subprob-algebra* (*subprob-algebra N*))
shows *join* (*distr M* (*subprob-algebra N*) *join*) = *join* (*join M*)
⟨*proof*⟩

lemma *join-return*:

assumes *sets M* = *sets N* **and** *subprob-space M*
shows *join* (*return* (*subprob-algebra N*) *M*) = *M*
⟨*proof*⟩

lemma *join-return'*:
assumes *sets N* = *sets M*
shows *join* (*distr M* (*subprob-algebra N*) (*return N*)) = *M*
⟨*proof*⟩

lemma *join-distr-distr*:
fixes *f* :: *'a* ⇒ *'b* **and** *M* :: *'a measure measure* **and** *N* :: *'b measure*
assumes *sets M* = *sets* (*subprob-algebra R*) **and** *f* ∈ *measurable R N*
shows *join* (*distr M* (*subprob-algebra N*) (λ*M*. *distr M N f*)) = *distr* (*join M*) *N f* (**is** *?r* = *?l*)
⟨*proof*⟩

definition *bind* :: *'a measure* ⇒ (*'a* ⇒ *'b measure*) ⇒ *'b measure* **where**
bind M f = (*if space M* = {} *then count-space* {} *else*
join (*distr M* (*subprob-algebra* (*f* (*SOME x*. *x* ∈ *space M*))) *f*))

adhoc-overloading *Monad-Syntax.bind bind*

lemma *bind-empty*:
space M = {} ⟹ *bind M f* = *count-space* {}
⟨*proof*⟩

lemma *bind-nonempty*:
space M ≠ {} ⟹ *bind M f* = *join* (*distr M* (*subprob-algebra* (*f* (*SOME x*. *x* ∈ *space M*))) *f*)
⟨*proof*⟩

lemma *sets-bind-empty*: *sets M* = {} ⟹ *sets* (*bind M f*) = {{}}
⟨*proof*⟩

lemma *space-bind-empty*: *space M* = {} ⟹ *space* (*bind M f*) = {}
⟨*proof*⟩

lemma *sets-bind*[*simp*, *measurable-cong*]:
assumes *f*: ⋀*x*. *x* ∈ *space M* ⟹ *sets* (*f x*) = *sets N* **and** *M*: *space M* ≠ {}
shows *sets* (*bind M f*) = *sets N*
⟨*proof*⟩

lemma *space-bind*[*simp*]:
assumes ⋀*x*. *x* ∈ *space M* ⟹ *sets* (*f x*) = *sets N* **and** *space M* ≠ {}
shows *space* (*bind M f*) = *space N*
⟨*proof*⟩

lemma *bind-cong-All*:
assumes ∀ *x* ∈ *space M*. *f x* = *g x*

shows *bind M f = bind M g*
⟨*proof*⟩

lemma *bind-cong*:
M = N ⟹ (⋀x. x ∈ space M ⟹ f x = g x) ⟹ bind M f = bind N g
⟨*proof*⟩

lemma *bind-nonempty′*:
assumes *f ∈ measurable M (subprob-algebra N) x ∈ space M*
shows *bind M f = join (distr M (subprob-algebra N) f)*
⟨*proof*⟩

lemma *bind-nonempty″*:
assumes *f ∈ measurable M (subprob-algebra N) space M ≠ {}*
shows *bind M f = join (distr M (subprob-algebra N) f)*
⟨*proof*⟩

lemma *emeasure-bind*:
⟦space M ≠ {}; f ∈ measurable M (subprob-algebra N);X ∈ sets N⟧
⟹ emeasure (M ⤜ f) X = ∫⁺x. emeasure (f x) X ∂M
⟨*proof*⟩

lemma *nn-integral-bind*:
assumes *f*: *f ∈ borel-measurable B*
assumes *N*: *N ∈ measurable M (subprob-algebra B)*
shows *(∫⁺x. f x ∂(M ⤜ N)) = (∫⁺x. ∫⁺y. f y ∂N x ∂M)*
⟨*proof*⟩

lemma *AE-bind*:
assumes *N*[*measurable*]: *N ∈ measurable M (subprob-algebra B)*
assumes *P*[*measurable*]: *Measurable.pred B P*
shows *(AE x in M ⤜ N. P x) ⟷ (AE x in M. AE y in N x. P y)*
⟨*proof*⟩

lemma *measurable-bind′*:
assumes *M1*: *f ∈ measurable M (subprob-algebra N)* **and**
M2: *case-prod g ∈ measurable (M ⨂M N) (subprob-algebra R)*
shows *(λx. bind (f x) (g x)) ∈ measurable M (subprob-algebra R)*
⟨*proof*⟩

lemma *measurable-bind*[*measurable (raw)*]:
assumes *M1*: *f ∈ measurable M (subprob-algebra N)* **and**
M2: *(λx. g (fst x) (snd x)) ∈ measurable (M ⨂M N) (subprob-algebra R)*
shows *(λx. bind (f x) (g x)) ∈ measurable M (subprob-algebra R)*
⟨*proof*⟩

lemma *measurable-bind2*:
assumes *f ∈ measurable M (subprob-algebra N)* **and** *g ∈ measurable N (subprob-algebra R)*

shows $(\lambda x.\ bind\ (f\ x)\ g) \in measurable\ M\ (subprob\text{-}algebra\ R)$
⟨*proof*⟩

lemma *subprob-space-bind*:
assumes $subprob\text{-}space\ M\ f \in measurable\ M\ (subprob\text{-}algebra\ N)$
shows $subprob\text{-}space\ (M \gg\!= f)$
⟨*proof*⟩

lemma
fixes $f :: - \Rightarrow real$
assumes *f-measurable* [*measurable*]: $f \in borel\text{-}measurable\ K$
and *f-bounded*: $\bigwedge x.\ x \in space\ K \Longrightarrow |f\ x| \leq B$
and N [*measurable*]: $N \in measurable\ M\ (subprob\text{-}algebra\ K)$
and *fin*: *finite-measure M*
and *M-bounded*: $AE\ x\ in\ M.\ emeasure\ (N\ x)\ (space\ (N\ x)) \leq ennreal\ B'$
shows *integrable-bind*: $integrable\ (bind\ M\ N)\ f$ (**is** *?integrable*)
and *integral-bind*: $integral^L\ (bind\ M\ N)\ f = \int x.\ integral^L\ (N\ x)\ f\ \partial M$ (**is** *?integral*)
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-space-bind*:
assumes *ae*: $AE\ x\ in\ M.\ prob\text{-}space\ (N\ x)$
and N[*measurable*]: $N \in measurable\ M\ (subprob\text{-}algebra\ S)$
shows $prob\text{-}space\ (M \gg\!= N)$
⟨*proof*⟩

lemma (**in** *subprob-space*) *bind-in-space*:
$A \in measurable\ M\ (subprob\text{-}algebra\ N) \Longrightarrow (M \gg\!= A) \in space\ (subprob\text{-}algebra\ N)$
⟨*proof*⟩

lemma (**in** *subprob-space*) *measure-bind*:
assumes *f*: $f \in measurable\ M\ (subprob\text{-}algebra\ N)$ **and** *X*: $X \in sets\ N$
shows $measure\ (M \gg\!= f)\ X = \int x.\ measure\ (f\ x)\ X\ \partial M$
⟨*proof*⟩

lemma *emeasure-bind-const*:
$space\ M \neq \{\} \Longrightarrow X \in sets\ N \Longrightarrow subprob\text{-}space\ N \Longrightarrow$
$emeasure\ (M \gg\!= (\lambda x.\ N))\ X = emeasure\ N\ X * emeasure\ M\ (space\ M)$
⟨*proof*⟩

lemma *emeasure-bind-const'*:
assumes *subprob-space M subprob-space N*
shows $emeasure\ (M \gg\!= (\lambda x.\ N))\ X = emeasure\ N\ X * emeasure\ M\ (space\ M)$
⟨*proof*⟩

lemma *emeasure-bind-const-prob-space*:
assumes *prob-space M subprob-space N*
shows $emeasure\ (M \gg\!= (\lambda x.\ N))\ X = emeasure\ N\ X$

⟨*proof*⟩

lemma *bind-return*:
assumes $f \in$ *measurable* M (*subprob-algebra* N) **and** $x \in$ *space* M
shows *bind* (*return* M x) $f = f\ x$
⟨*proof*⟩

lemma *bind-return'*:
shows *bind* M (*return* M) $= M$
⟨*proof*⟩

lemma *distr-bind*:
assumes *N*: $N \in$ *measurable* M (*subprob-algebra* K) *space* $M \neq \{\}$
assumes *f*: $f \in$ *measurable* K R
shows *distr* ($M \gg\!= N$) R f $= (M \gg\!= (\lambda x.\ distr\ (N\ x)\ R\ f))$
⟨*proof*⟩

lemma *bind-distr*:
assumes *f*[*measurable*]: $f \in$ *measurable* M X
assumes *N*[*measurable*]: $N \in$ *measurable* X (*subprob-algebra* K) **and** *space* $M \neq \{\}$
shows (*distr* M X $f \gg\!= N$) $= (M \gg\!= (\lambda x.\ N\ (f\ x)))$
⟨*proof*⟩

lemma *bind-count-space-singleton*:
assumes *subprob-space* ($f\ x$)
shows *count-space* $\{x\} \gg\!= f = f\ x$
⟨*proof*⟩

lemma *restrict-space-bind*:
assumes *N*: $N \in$ *measurable* M (*subprob-algebra* K)
assumes *space* $M \neq \{\}$
assumes *X*[*simp*]: $X \in$ *sets* K $X \neq \{\}$
shows *restrict-space* (*bind* M N) X $=$ *bind* M ($\lambda x.$ *restrict-space* ($N\ x$) X)
⟨*proof*⟩

lemma *bind-restrict-space*:
assumes *A*: $A \cap$ *space* $M \neq \{\}$ $A \cap$ *space* $M \in$ *sets* M
and *f*: $f \in$ *measurable* (*restrict-space* M A) (*subprob-algebra* N)
shows *restrict-space* M $A \gg\!= f = M \gg\!= (\lambda x.$ *if* $x \in A$ *then* $f\ x$ *else null-measure* (f (*SOME* $x.\ x \in A \wedge x \in$ *space* M)))
(**is** *?lhs* $=$ *?rhs* **is** - $= M \gg\!=$ *?f*)
⟨*proof*⟩

lemma *bind-const'*: ⟦*prob-space* M; *subprob-space* N⟧ $\Longrightarrow$ $M \gg\!= (\lambda x.\ N) = N$
⟨*proof*⟩

lemma *bind-return-distr*:
space $M \neq \{\} \Longrightarrow f \in$ *measurable* M $N \Longrightarrow$ *bind* M (*return* $N \circ f$) $=$ *distr* M

N f
⟨*proof*⟩

lemma *bind-return-distr'*:
space M ≠ {} ⟹ *f* ∈ *measurable M N* ⟹ *bind M* (λ*x*. *return N* (*f x*)) = *distr M N f*
⟨*proof*⟩

lemma *bind-assoc*:
fixes *f* :: *'a* ⇒ *'b measure* **and** *g* :: *'b* ⇒ *'c measure*
assumes *M1*: *f* ∈ *measurable M* (*subprob-algebra N*) **and** *M2*: *g* ∈ *measurable N* (*subprob-algebra R*)
shows *bind* (*bind M f*) *g* = *bind M* (λ*x*. *bind* (*f x*) *g*)
⟨*proof*⟩

lemma *double-bind-assoc*:
assumes *Mg*: *g* ∈ *measurable N* (*subprob-algebra N'*)
assumes *Mf*: *f* ∈ *measurable M* (*subprob-algebra M'*)
assumes *Mh*: *case-prod h* ∈ *measurable* (*M* ⨂$_M$ *M'*) *N*
shows *do* {*x* ← *M*; *y* ← *f x*; *g* (*h x y*)} = *do* {*x* ← *M*; *y* ← *f x*; *return N* (*h x y*)} ⤜ *g*
⟨*proof*⟩

lemma (**in** *prob-space*) *M-in-subprob*[*measurable* (*raw*)]: *M* ∈ *space* (*subprob-algebra M*)
⟨*proof*⟩

lemma (**in** *pair-prob-space*) *pair-measure-eq-bind*:
(*M1* ⨂$_M$ *M2*) = (*M1* ⤜ (λ*x*. *M2* ⤜ (λ*y*. *return* (*M1* ⨂$_M$ *M2*) (*x*, *y*))))
⟨*proof*⟩

lemma (**in** *pair-prob-space*) *bind-rotate*:
assumes *C*[*measurable*]: (λ(*x*, *y*). *C x y*) ∈ *measurable* (*M1* ⨂$_M$ *M2*) (*subprob-algebra N*)
shows (*M1* ⤜ (λ*x*. *M2* ⤜ (λ*y*. *C x y*))) = (*M2* ⤜ (λ*y*. *M1* ⤜ (λ*x*. *C x y*)))
⟨*proof*⟩

lemma *bind-return''*: *sets M* = *sets N* ⟹ *M* ⤜ *return N* = *M*
⟨*proof*⟩

lemma (**in** *prob-space*) *distr-const*[*simp*]:
c ∈ *space N* ⟹ *distr M N* (λ*x*. *c*) = *return N c*
⟨*proof*⟩

lemma *return-count-space-eq-density*:
return (*count-space M*) *x* = *density* (*count-space M*) (*indicator* {*x*})
⟨*proof*⟩

lemma *null-measure-in-space-subprob-algebra* [*simp*]:
null-measure M ∈ *space* (*subprob-algebra M*) ⟷ *space M* ≠ {}
⟨*proof*⟩

7.4 Giry monad on probability spaces

definition *prob-algebra* :: *'a measure* ⇒ *'a measure measure* **where**
prob-algebra K = *restrict-space* (*subprob-algebra K*) {*M*. *prob-space M*}

lemma *space-prob-algebra*: *space* (*prob-algebra M*) = {*N*. *sets N* = *sets M* ∧ *prob-space N*}
⟨*proof*⟩

lemma *measurable-measure-prob-algebra*[*measurable*]:
a ∈ *sets A* ⟹ (λ*M*. *Sigma-Algebra.measure M a*) ∈ *prob-algebra A* →$_M$ *borel*
⟨*proof*⟩

lemma *measurable-prob-algebraD*:
f ∈ *N* →$_M$ *prob-algebra M* ⟹ *f* ∈ *N* →$_M$ *subprob-algebra M*
⟨*proof*⟩

lemma *measure-measurable-prob-algebra2*:
Sigma (*space M*) *A* ∈ *sets* (*M* ⨂$_M$ *N*) ⟹ *L* ∈ *M* →$_M$ *prob-algebra N* ⟹
(λ*x*. *Sigma-Algebra.measure* (*L x*) (*A x*)) ∈ *borel-measurable M*
⟨*proof*⟩

lemma *measurable-prob-algebraI*:
(⋀*x*. *x* ∈ *space N* ⟹ *prob-space* (*f x*)) ⟹ *f* ∈ *N* →$_M$ *subprob-algebra M* ⟹
f ∈ *N* →$_M$ *prob-algebra M*
⟨*proof*⟩

lemma *measurable-distr-prob-space*:
assumes *f*: *f* ∈ *M* →$_M$ *N*
shows (λ*M'*. *distr M' N f*) ∈ *prob-algebra M* →$_M$ *prob-algebra N*
⟨*proof*⟩

lemma *measurable-return-prob-space*[*measurable*]: *return N* ∈ *N* →$_M$ *prob-algebra N*
⟨*proof*⟩

lemma *measurable-distr-prob-space2*[*measurable* (*raw*)]:
assumes *f*: *g* ∈ *L* →$_M$ *prob-algebra M* (λ(*x*, *y*). *f x y*) ∈ *L* ⨂$_M$ *M* →$_M$ *N*
shows (λ*x*. *distr* (*g x*) *N* (*f x*)) ∈ *L* →$_M$ *prob-algebra N*
⟨*proof*⟩

lemma *measurable-bind-prob-space*:
assumes *f*: *f* ∈ *M* →$_M$ *prob-algebra N* **and** *g*: *g* ∈ *N* →$_M$ *prob-algebra R*
shows (λ*x*. *bind* (*f x*) *g*) ∈ *M* →$_M$ *prob-algebra R*
⟨*proof*⟩

lemma *measurable-bind-prob-space2*[*measurable* (*raw*)]:
assumes *f*: $f \in M \to_M$ *prob-algebra* N **and** *g*: $(\lambda(x, y).\ g\ x\ y) \in (M \bigotimes_M N) \to_M$ *prob-algebra* R
shows $(\lambda x.\ bind\ (f\ x)\ (g\ x)) \in M \to_M$ *prob-algebra* R
⟨*proof*⟩

lemma *measurable-prob-algebra-generated*:
assumes *eq*: *sets* N = *sigma-sets* $\Omega\ G$ **and** *Int-stable* $G\ G \subseteq Pow\ \Omega$
assumes *subsp*: $\bigwedge a.\ a \in space\ M \Longrightarrow prob\text{-}space\ (K\ a)$
assumes *sets*: $\bigwedge a.\ a \in space\ M \Longrightarrow sets\ (K\ a) = sets\ N$
assumes $\bigwedge A.\ A \in G \Longrightarrow (\lambda a.\ emeasure\ (K\ a)\ A) \in borel\text{-}measurable\ M$
shows $K \in measurable\ M\ (prob\text{-}algebra\ N)$
⟨*proof*⟩

lemma *in-space-prob-algebra*:
$x \in space\ (prob\text{-}algebra\ M) \Longrightarrow emeasure\ x\ (space\ M) = 1$
⟨*proof*⟩

lemma *prob-space-pair*:
assumes *prob-space M prob-space N* **shows** *prob-space* $(M \bigotimes_M N)$
⟨*proof*⟩

lemma *measurable-pair-prob*[*measurable*]:
$f \in M \to_M$ *prob-algebra* $N \Longrightarrow g \in M \to_M$ *prob-algebra* $L \Longrightarrow (\lambda x.\ f\ x \bigotimes_M g\ x) \in M \to_M$ *prob-algebra* $(N \bigotimes_M L)$
⟨*proof*⟩

lemma *emeasure-bind-prob-algebra*:
assumes *A*: $A \in space\ (prob\text{-}algebra\ N)$
assumes *B*: $B \in N \to_M$ *prob-algebra* L
assumes *X*: $X \in sets\ L$
shows $emeasure\ (bind\ A\ B)\ X = (\int^+ x.\ emeasure\ (B\ x)\ X\ \partial A)$
⟨*proof*⟩

lemma *prob-space-bind′*:
assumes *A*: $A \in space\ (prob\text{-}algebra\ M)$ **and** *B*: $B \in M \to_M$ *prob-algebra* N
shows *prob-space* $(A \gg\!= B)$
⟨*proof*⟩

lemma *sets-bind′*:
assumes *A*: $A \in space\ (prob\text{-}algebra\ M)$ **and** *B*: $B \in M \to_M$ *prob-algebra* N
shows $sets\ (A \gg\!= B) = sets\ N$
⟨*proof*⟩

lemma *bind-cong-AE′*:
assumes *M*: $M \in space\ (prob\text{-}algebra\ L)$
and *f*: $f \in L \to_M$ *prob-algebra* N **and** *g*: $g \in L \to_M$ *prob-algebra* N

and *ae*: $AE\ x\ in\ M.\ f\ x = g\ x$
shows $bind\ M\ f = bind\ M\ g$
⟨*proof*⟩

lemma *density-discrete*:
$countable\ A \Longrightarrow sets\ N = Set.Pow\ A \Longrightarrow (\bigwedge x.\ f\ x \geq 0) \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow f\ x = emeasure\ N\ \{x\}) \Longrightarrow$
$density\ (count\text{-}space\ A)\ f = N$
⟨*proof*⟩

lemma *distr-density-discrete*:
fixes f'
assumes $countable\ A$
assumes $f' \in borel\text{-}measurable\ M$
assumes $g \in measurable\ M\ (count\text{-}space\ A)$
defines $f \equiv \lambda x.\ \int^+ t.\ (if\ g\ t = x\ then\ 1\ else\ 0) * f'\ t\ \partial M$
assumes $\bigwedge x.\ \ x \in space\ M \Longrightarrow g\ x \in A$
shows $density\ (count\text{-}space\ A)\ (\lambda x.\ f\ x) = distr\ (density\ M\ f')\ (count\text{-}space\ A)\ g$
⟨*proof*⟩

lemma *bind-cong-AE*:
assumes $M = N$
assumes *f*: $f \in measurable\ N\ (subprob\text{-}algebra\ B)$
assumes *g*: $g \in measurable\ N\ (subprob\text{-}algebra\ B)$
assumes *ae*: $AE\ x\ in\ N.\ f\ x = g\ x$
shows $bind\ M\ f = bind\ N\ g$
⟨*proof*⟩

lemma *bind-cong-simp*: $M = N \Longrightarrow (\bigwedge x.\ x{\in}space\ M\ =simp{=}>\ f\ x = g\ x) \Longrightarrow bind\ M\ f = bind\ N\ g$
⟨*proof*⟩

lemma *sets-bind-measurable*:
assumes *f*: $f \in measurable\ M\ (subprob\text{-}algebra\ B)$
assumes *M*: $space\ M \neq \{\}$
shows $sets\ (M \gg\!= f) = sets\ B$
⟨*proof*⟩

lemma *space-bind-measurable*:
assumes *f*: $f \in measurable\ M\ (subprob\text{-}algebra\ B)$
assumes *M*: $space\ M \neq \{\}$
shows $space\ (M \gg\!= f) = space\ B$
⟨*proof*⟩

lemma *bind-distr-return*:
$f \in M \rightarrow_M N \Longrightarrow g \in N \rightarrow_M L \Longrightarrow space\ M \neq \{\} \Longrightarrow$
$distr\ M\ N\ f \gg\!= (\lambda x.\ return\ L\ (g\ x)) = distr\ M\ L\ (\lambda x.\ g\ (f\ x))$
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-eq-constD*:
assumes *AE x in M. x = y*
shows *M = return M y y ∈ space M*
⟨*proof*⟩

end

8 Projective Family

theory *Projective-Family*
imports *Giry-Monad*
begin

lemma *vimage-restrict-preseve-mono*:
assumes *J*: *J ⊆ I*
and *sets*: *A ⊆ (Π$_E$ i∈J. S i) B ⊆ (Π$_E$ i∈J. S i)* **and** *ne*: *(Π$_E$ i∈I. S i) ≠ {}*
and *eq*: *(λx. restrict x J) −‘ A ∩ (Π$_E$ i∈I. S i) ⊆ (λx. restrict x J) −‘ B ∩ (Π$_E$ i∈I. S i)*
shows *A ⊆ B*
⟨*proof*⟩

locale *projective-family* =
fixes *I* :: *'i set* **and** *P* :: *'i set ⇒ ('i ⇒ 'a) measure* **and** *M* :: *'i ⇒ 'a measure*
assumes *P*: *⋀J H. J ⊆ H ⟹ finite H ⟹ H ⊆ I ⟹ P J = distr (P H) (PiM J M) (λf. restrict f J)*
assumes *prob-space-P*: *⋀J. finite J ⟹ J ⊆ I ⟹ prob-space (P J)*
begin

lemma *sets-P*: *finite J ⟹ J ⊆ I ⟹ sets (P J) = sets (PiM J M)*
⟨*proof*⟩

lemma *space-P*: *finite J ⟹ J ⊆ I ⟹ space (P J) = space (PiM J M)*
⟨*proof*⟩

lemma *not-empty-M*: *i ∈ I ⟹ space (M i) ≠ {}*
⟨*proof*⟩

lemma *not-empty*: *space (PiM I M) ≠ {}*
⟨*proof*⟩

abbreviation
emb L K ≡ prod-emb L M K

lemma *emb-preserve-mono*:
assumes *J ⊆ L L ⊆ I* **and** *sets*: *X ∈ sets (Pi$_M$ J M) Y ∈ sets (Pi$_M$ J M)*
assumes *emb L J X ⊆ emb L J Y*
shows *X ⊆ Y*
⟨*proof*⟩

lemma *emb-injective*:
assumes *L*: $J \subseteq L$ $L \subseteq I$ **and** *X*: $X \in sets\ (Pi_M\ J\ M)$ **and** *Y*: $Y \in sets\ (Pi_M\ J\ M)$
shows $emb\ L\ J\ X = emb\ L\ J\ Y \Longrightarrow X = Y$
⟨*proof*⟩

lemma *emeasure-P*: $J \subseteq K \Longrightarrow finite\ K \Longrightarrow K \subseteq I \Longrightarrow X \in sets\ (PiM\ J\ M) \Longrightarrow P\ K\ (emb\ K\ J\ X) = P\ J\ X$
⟨*proof*⟩

inductive-set *generator* :: $('i \Rightarrow 'a)\ set\ set$ **where**
$finite\ J \Longrightarrow J \subseteq I \Longrightarrow X \in sets\ (Pi_M\ J\ M) \Longrightarrow emb\ I\ J\ X \in generator$

lemma *algebra-generator*: $algebra\ (space\ (PiM\ I\ M))\ generator$
⟨*proof*⟩

interpretation *generator*: *algebra space* $(PiM\ I\ M)$ *generator*
⟨*proof*⟩

lemma *sets-PiM-generator*: $sets\ (PiM\ I\ M) = sigma\text{-}sets\ (space\ (PiM\ I\ M))\ generator$
⟨*proof*⟩

definition *mu-G* (μG) **where**
$\mu G\ A = (THE\ x.\ \forall J{\subseteq}I.\ finite\ J \longrightarrow (\forall X{\in}sets\ (Pi_M\ J\ M).\ A = emb\ I\ J\ X \longrightarrow x = emeasure\ (P\ J)\ X))$

definition *lim* :: $('i \Rightarrow 'a)\ measure$ **where**
$lim = extend\text{-}measure\ (space\ (PiM\ I\ M))\ generator\ (\lambda x.\ x)\ \mu G$

lemma *space-lim*[*simp*]: $space\ lim = space\ (PiM\ I\ M)$
⟨*proof*⟩

lemma *sets-lim*[*simp*, *measurable*]: $sets\ lim = sets\ (PiM\ I\ M)$
⟨*proof*⟩

lemma *mu-G-spec*:
assumes *J*: $finite\ J$ $J \subseteq I$ $X \in sets\ (Pi_M\ J\ M)$
shows $\mu G\ (emb\ I\ J\ X) = emeasure\ (P\ J)\ X$
⟨*proof*⟩

lemma *positive-mu-G*: $positive\ generator\ \mu G$
⟨*proof*⟩

lemma *additive-mu-G*: $additive\ generator\ \mu G$
⟨*proof*⟩

lemma *emeasure-lim*:
assumes *JX*: $finite\ J$ $J \subseteq I$ $X \in sets\ (PiM\ J\ M)$

assumes *cont*: $\bigwedge J\ X.\ (\bigwedge i.\ J\ i \subseteq I) \Longrightarrow incseq\ J \Longrightarrow (\bigwedge i.\ finite\ (J\ i)) \Longrightarrow (\bigwedge i.\ X\ i \in sets\ (PiM\ (J\ i)\ M)) \Longrightarrow$
$decseq\ (\lambda i.\ emb\ I\ (J\ i)\ (X\ i)) \Longrightarrow 0 < (INF\ i.\ P\ (J\ i)\ (X\ i)) \Longrightarrow (\bigcap i.\ emb\ I\ (J\ i)\ (X\ i)) \neq \{\}$
shows *emeasure lim* $(emb\ I\ J\ X) = P\ J\ X$
⟨*proof*⟩

end

sublocale *product-prob-space* ⊆ *projective-family I* $\lambda J.\ PiM\ J\ M\ M$
⟨*proof*⟩

Proof due to Ionescu Tulcea.

locale *Ionescu-Tulcea* =
fixes $P :: nat \Rightarrow (nat \Rightarrow 'a) \Rightarrow 'a\ measure$ **and** $M :: nat \Rightarrow 'a\ measure$
assumes *P*[*measurable*]: $\bigwedge i.\ P\ i \in measurable\ (PiM\ \{0..<i\}\ M)\ (subprob\text{-}algebra\ (M\ i))$
assumes *prob-space-P*: $\bigwedge i\ x.\ x \in space\ (PiM\ \{0..<i\}\ M) \Longrightarrow prob\text{-}space\ (P\ i\ x)$
begin

lemma *non-empty*[*simp*]: $space\ (M\ i) \neq \{\}$
⟨*proof*⟩

lemma *space-PiM-not-empty*[*simp*]: $space\ (PiM\ UNIV\ M) \neq \{\}$
⟨*proof*⟩

lemma *space-P*: $x \in space\ (PiM\ \{0..<n\}\ M) \Longrightarrow space\ (P\ n\ x) = space\ (M\ n)$
⟨*proof*⟩

lemma *sets-P*[*measurable-cong*]: $x \in space\ (PiM\ \{0..<n\}\ M) \Longrightarrow sets\ (P\ n\ x) = sets\ (M\ n)$
⟨*proof*⟩

definition $eP :: nat \Rightarrow (nat \Rightarrow 'a) \Rightarrow (nat \Rightarrow 'a)\ measure$ **where**
$eP\ n\ \omega = distr\ (P\ n\ \omega)\ (PiM\ \{0..<Suc\ n\}\ M)\ (fun\text{-}upd\ \omega\ n)$

lemma *measurable-eP*[*measurable*]:
$eP\ n \in measurable\ (PiM\ \{0..<n\}\ M)\ (subprob\text{-}algebra\ (PiM\ \{0..<Suc\ n\}\ M))$
⟨*proof*⟩

lemma *space-eP*:
$x \in space\ (PiM\ \{0..<n\}\ M) \Longrightarrow space\ (eP\ n\ x) = space\ (PiM\ \{0..<Suc\ n\}\ M)$
⟨*proof*⟩

lemma *sets-eP*[*measurable*]:
$x \in space\ (PiM\ \{0..<n\}\ M) \Longrightarrow sets\ (eP\ n\ x) = sets\ (PiM\ \{0..<Suc\ n\}\ M)$
⟨*proof*⟩

lemma *prob-space-eP*: $x \in space\ (PiM\ \{0..<n\}\ M) \Longrightarrow prob\text{-}space\ (eP\ n\ x)$
⟨*proof*⟩

lemma *nn-integral-eP*:
$\omega \in space\ (PiM\ \{0..<n\}\ M) \Longrightarrow f \in borel\text{-}measurable\ (PiM\ \{0..<Suc\ n\}\ M) \Longrightarrow$
$(\int^+ x.\ f\ x\ \partial eP\ n\ \omega) = (\int^+ x.\ f\ (\omega(n := x))\ \partial P\ n\ \omega)$
⟨*proof*⟩

lemma *emeasure-eP*:
assumes ω[*simp*]: $\omega \in space\ (PiM\ \{0..<n\}\ M)$ **and** A[*measurable*]: $A \in sets\ (PiM\ \{0..<Suc\ n\}\ M)$
shows $eP\ n\ \omega\ A = P\ n\ \omega\ ((\lambda x.\ \omega(n := x)) -` A \cap space\ (M\ n))$
⟨*proof*⟩

primrec $C :: nat \Rightarrow nat \Rightarrow (nat \Rightarrow 'a) \Rightarrow (nat \Rightarrow 'a)\ measure$ **where**
$C\ n\ 0\ \omega = return\ (PiM\ \{0..<n\}\ M)\ \omega$
| $C\ n\ (Suc\ m)\ \omega = C\ n\ m\ \omega \gg= eP\ (n + m)$

lemma *measurable-C*[*measurable*]:
$C\ n\ m \in measurable\ (PiM\ \{0..<n\}\ M)\ (subprob\text{-}algebra\ (PiM\ \{0..<n + m\}\ M))$
⟨*proof*⟩

lemma *space-C*:
$x \in space\ (PiM\ \{0..<n\}\ M) \Longrightarrow space\ (C\ n\ m\ x) = space\ (PiM\ \{0..<n + m\}\ M)$
⟨*proof*⟩

lemma *sets-C*[*measurable-cong*]:
$x \in space\ (PiM\ \{0..<n\}\ M) \Longrightarrow sets\ (C\ n\ m\ x) = sets\ (PiM\ \{0..<n + m\}\ M)$
⟨*proof*⟩

lemma *prob-space-C*: $x \in space\ (PiM\ \{0..<n\}\ M) \Longrightarrow prob\text{-}space\ (C\ n\ m\ x)$
⟨*proof*⟩

lemma *split-C*:
assumes ω: $\omega \in space\ (PiM\ \{0..<n\}\ M)$ **shows** $(C\ n\ m\ \omega \gg= C\ (n + m)\ l) = C\ n\ (m + l)\ \omega$
⟨*proof*⟩

lemma *nn-integral-C*:
assumes $m \leq m'$ **and** f[*measurable*]: $f \in borel\text{-}measurable\ (PiM\ \{0..<n+m\}\ M)$
and *nonneg*: $\bigwedge x.\ x \in space\ (PiM\ \{0..<n+m\}\ M) \Longrightarrow 0 \leq f\ x$
and x: $x \in space\ (PiM\ \{0..<n\}\ M)$
shows $(\int^+ x.\ f\ x\ \partial C\ n\ m\ x) = (\int^+ x.\ f\ (restrict\ x\ \{0..<n+m\})\ \partial C\ n\ m'\ x)$
⟨*proof*⟩

lemma *emeasure-C*:
assumes $m \leq m'$ **and** A[*measurable*]: $A \in sets\ (PiM\ \{0..<n+m\}\ M)$ **and** [*simp*]: $x \in space\ (PiM\ \{0..<n\}\ M)$
shows $emeasure\ (C\ n\ m'\ x)\ (prod\text{-}emb\ \{0..<n+m'\}\ M\ \{0..<n+m\}\ A) = emeasure\ (C\ n\ m\ x)\ A$
⟨*proof*⟩

lemma *distr-C*:
assumes $m \leq m'$ **and** [*simp*]: $x \in space\ (PiM\ \{0..<n\}\ M)$
shows $C\ n\ m\ x = distr\ (C\ n\ m'\ x)\ (PiM\ \{0..<n+m\}\ M)\ (\lambda x.\ restrict\ x\ \{0..<n+m\})$
⟨*proof*⟩

definition *up-to* :: *nat set* ⇒ *nat* **where**
$up\text{-}to\ J = (LEAST\ n.\ \forall i{\geq}n.\ i \notin J)$

lemma *up-to-less*: *finite* $J \Longrightarrow i \in J \Longrightarrow i < up\text{-}to\ J$
⟨*proof*⟩

lemma *up-to-iff*: *finite* $J \Longrightarrow up\text{-}to\ J \leq n \longleftrightarrow (\forall i{\in}J.\ i < n)$
⟨*proof*⟩

lemma *up-to-iff-Ico*: *finite* $J \Longrightarrow up\text{-}to\ J \leq n \longleftrightarrow J \subseteq \{0..<n\}$
⟨*proof*⟩

lemma *up-to*: *finite* $J \Longrightarrow J \subseteq \{0..<up\text{-}to\ J\}$
⟨*proof*⟩

lemma *up-to-mono*: $J \subseteq H \Longrightarrow$ *finite* $H \Longrightarrow up\text{-}to\ J \leq up\text{-}to\ H$
⟨*proof*⟩

definition *CI* :: *nat set* ⇒ (*nat* ⇒ *'a*) *measure* **where**
$CI\ J = distr\ (C\ 0\ (up\text{-}to\ J)\ (\lambda x.\ undefined))\ (PiM\ J\ M)\ (\lambda f.\ restrict\ f\ J)$

sublocale *PF*: *projective-family UNIV CI*
⟨*proof*⟩

lemma *emeasure-CI'*:
finite $J \Longrightarrow X \in sets\ (PiM\ J\ M) \Longrightarrow CI\ J\ X = C\ 0\ (up\text{-}to\ J)\ (\lambda\text{-}.\ undefined)\ (PF.emb\ \{0..<up\text{-}to\ J\}\ J\ X)$
⟨*proof*⟩

lemma *emeasure-CI*:
$J \subseteq \{0..<n\} \Longrightarrow X \in sets\ (PiM\ J\ M) \Longrightarrow CI\ J\ X = C\ 0\ n\ (\lambda\text{-}.\ undefined)\ (PF.emb\ \{0..<n\}\ J\ X)$
⟨*proof*⟩

lemma *lim*:
assumes J: *finite* J **and** X: $X \in sets\ (PiM\ J\ M)$

shows *emeasure PF.lim* (*PF.emb UNIV J X*) = *emeasure* (*CI J*) *X*
⟨*proof*⟩

lemma *distr-lim*: **assumes** *J*[*simp*]: *finite J* **shows** *distr PF.lim* (*PiM J M*) (λ*x*. *restrict x J*) = *CI J*
⟨*proof*⟩

end

lemma (**in** *product-prob-space*) *emeasure-lim-emb*:
assumes ∗: *finite J J* ⊆ *I X* ∈ *sets* (*PiM J M*)
shows *emeasure lim* (*emb I J X*) = *emeasure* (*Pi*$_M$ *J M*) *X*
⟨*proof*⟩

end

9 Infinite Product Measure

theory *Infinite-Product-Measure*
imports *Probability-Measure Projective-Family*
begin

lemma (**in** *product-prob-space*) *distr-PiM-restrict-finite*:
assumes *finite J J* ⊆ *I*
shows *distr* (*PiM I M*) (*PiM J M*) (λ*x*. *restrict x J*) = *PiM J M*
⟨*proof*⟩

lemma (**in** *product-prob-space*) *emeasure-PiM-emb′*:
J ⊆ *I* ⟹ *finite J* ⟹ *X* ∈ *sets* (*PiM J M*) ⟹ *emeasure* (*Pi*$_M$ *I M*) (*emb I J X*) = *PiM J M X*
⟨*proof*⟩

lemma (**in** *product-prob-space*) *emeasure-PiM-emb*:
J ⊆ *I* ⟹ *finite J* ⟹ (⋀*i*. *i* ∈ *J* ⟹ *X i* ∈ *sets* (*M i*)) ⟹
emeasure (*Pi*$_M$ *I M*) (*emb I J* (*Pi*$_E$ *J X*)) = (∏ *i*∈*J*. *emeasure* (*M i*) (*X i*))
⟨*proof*⟩

sublocale *product-prob-space* ⊆ *P?*: *prob-space Pi*$_M$ *I M*
⟨*proof*⟩

lemma *prob-space-PiM*:
assumes *M*: ⋀*i*. *i* ∈ *I* ⟹ *prob-space* (*M i*) **shows** *prob-space* (*PiM I M*)
⟨*proof*⟩

lemma (**in** *product-prob-space*) *emeasure-PiM-Collect*:
assumes *X*: *J* ⊆ *I finite J* ⋀*i*. *i* ∈ *J* ⟹ *X i* ∈ *sets* (*M i*)
shows *emeasure* (*Pi*$_M$ *I M*) {*x*∈*space* (*Pi*$_M$ *I M*). ∀ *i*∈*J*. *x i* ∈ *X i*} = (∏ *i*∈*J*. *emeasure* (*M i*) (*X i*))
⟨*proof*⟩

lemma (**in** *product-prob-space*) *emeasure-PiM-Collect-single*:
assumes *X*: $i \in I$ $A \in sets\ (M\ i)$
shows $emeasure\ (Pi_M\ I\ M)\ \{x \in space\ (Pi_M\ I\ M).\ x\ i \in A\} = emeasure\ (M\ i)\ A$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *measure-PiM-emb*:
assumes $J \subseteq I$ *finite* J $\bigwedge i.\ i \in J \Longrightarrow X\ i \in sets\ (M\ i)$
shows $measure\ (PiM\ I\ M)\ (emb\ I\ J\ (Pi_E\ J\ X)) = (\prod\ i \in J.\ measure\ (M\ i)\ (X\ i))$
⟨*proof*⟩

lemma *sets-Collect-single'*:
$i \in I \Longrightarrow \{x \in space\ (M\ i).\ P\ x\} \in sets\ (M\ i) \Longrightarrow \{x \in space\ (PiM\ I\ M).\ P\ (x\ i)\} \in sets\ (PiM\ I\ M)$
⟨*proof*⟩

lemma (**in** *finite-product-prob-space*) *finite-measure-PiM-emb*:
$(\bigwedge i.\ i \in I \Longrightarrow A\ i \in sets\ (M\ i)) \Longrightarrow measure\ (PiM\ I\ M)\ (Pi_E\ I\ A) = (\prod i \in I.\ measure\ (M\ i)\ (A\ i))$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *PiM-component*:
assumes $i \in I$
shows $distr\ (PiM\ I\ M)\ (M\ i)\ (\lambda\omega.\ \omega\ i) = M\ i$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *PiM-eq*:
assumes *M'*: $sets\ M' = sets\ (PiM\ I\ M)$
assumes *eq*: $\bigwedge J\ F.\ finite\ J \Longrightarrow J \subseteq I \Longrightarrow (\bigwedge j.\ j \in J \Longrightarrow F\ j \in sets\ (M\ j)) \Longrightarrow emeasure\ M'\ (prod\text{-}emb\ I\ M\ J\ (\Pi_E\ j \in J.\ F\ j)) = (\prod j \in J.\ emeasure\ (M\ j)\ (F\ j))$
shows $M' = (PiM\ I\ M)$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *AE-component*: $i \in I \Longrightarrow AE\ x\ in\ M\ i.\ P\ x \Longrightarrow AE\ x\ in\ PiM\ I\ M.\ P\ (x\ i)$
⟨*proof*⟩

lemma *emeasure-PiM-emb*:
assumes *M*: $\bigwedge i.\ i \in I \Longrightarrow prob\text{-}space\ (M\ i)$
assumes *J*: $J \subseteq I$ *finite* J **and** *A*: $\bigwedge i.\ i \in J \Longrightarrow A\ i \in sets\ (M\ i)$
shows $emeasure\ (Pi_M\ I\ M)\ (prod\text{-}emb\ I\ M\ J\ (Pi_E\ J\ A)) = (\prod i \in J.\ emeasure\ (M\ i)\ (A\ i))$
⟨*proof*⟩

lemma *distr-pair-PiM-eq-PiM*:
fixes i' :: *'i* **and** I :: *'i set* **and** M :: $'i \Rightarrow 'a\ measure$
assumes *M*: $\bigwedge i.\ i \in I \Longrightarrow prob\text{-}space\ (M\ i)\ prob\text{-}space\ (M\ i')$

shows *distr* $(M\ i' \bigotimes_M (\Pi_M\ i{\in}I.\ M\ i))\ (\Pi_M\ i{\in}insert\ i'\ I.\ M\ i)\ (\lambda(x,\ X).\ X(i' := x)) =$
$(\Pi_M\ i{\in}insert\ i'\ I.\ M\ i)$ (**is** *?L* = -)
⟨*proof*⟩

lemma *distr-PiM-reindex*:
assumes *M*: $\bigwedge i.\ i \in K \Longrightarrow$ *prob-space* $(M\ i)$
assumes *f*: *inj-on f I f* $\in I \to K$
shows *distr* $(Pi_M\ K\ M)\ (\Pi_M\ i{\in}I.\ M\ (f\ i))\ (\lambda\omega.\ \lambda n{\in}I.\ \omega\ (f\ n)) = (\Pi_M\ i{\in}I.\ M\ (f\ i))$
(**is** *distr ?K ?I ?t* = *?I*)
⟨*proof*⟩

lemma *distr-PiM-component*:
assumes *M*: $\bigwedge i.\ i \in I \Longrightarrow$ *prob-space* $(M\ i)$
assumes $i \in I$
shows *distr* $(Pi_M\ I\ M)\ (M\ i)\ (\lambda\omega.\ \omega\ i) = M\ i$
⟨*proof*⟩

lemma *AE-PiM-component*:
$(\bigwedge i.\ i \in I \Longrightarrow$ *prob-space* $(M\ i)) \Longrightarrow i \in I \Longrightarrow$ *AE x in M i. P x* $\Longrightarrow$ *AE x in PiM I M. P* $(x\ i)$
⟨*proof*⟩

lemma *decseq-emb-PiE*:
incseq J $\Longrightarrow$ *decseq* $(\lambda i.$ *prod-emb I M* $(J\ i)\ (\Pi_E\ j{\in}J\ i.\ X\ j))$
⟨*proof*⟩

9.1 Sequence space

definition *comb-seq* :: *nat* ⇒ $(nat \Rightarrow 'a) \Rightarrow (nat \Rightarrow 'a) \Rightarrow (nat \Rightarrow 'a)$ **where**
comb-seq i ω ω' j = (*if j* < *i then ω j else ω'* $(j - i)$)

lemma *split-comb-seq*: *P* $(comb\text{-}seq\ i\ \omega\ \omega'\ j) \longleftrightarrow (j < i \longrightarrow P\ (\omega\ j)) \wedge (\forall k.\ j = i + k \longrightarrow P\ (\omega'\ k))$
⟨*proof*⟩

lemma *split-comb-seq-asm*: *P* $(comb\text{-}seq\ i\ \omega\ \omega'\ j) \longleftrightarrow \neg\ ((j < i \wedge \neg\ P\ (\omega\ j)) \vee (\exists k.\ j = i + k \wedge \neg\ P\ (\omega'\ k)))$
⟨*proof*⟩

lemma *measurable-comb-seq*:
$(\lambda(\omega, \omega').$ *comb-seq i ω ω'*$) \in$ *measurable* $((\Pi_M\ i{\in}UNIV.\ M) \bigotimes_M (\Pi_M\ i{\in}UNIV.\ M))\ (\Pi_M\ i{\in}UNIV.\ M)$
⟨*proof*⟩

lemma *measurable-comb-seq'*[*measurable* (*raw*)]:
assumes *f*: $f \in$ *measurable N* $(\Pi_M\ i{\in}UNIV.\ M)$ **and** *g*: $g \in$ *measurable N* $(\Pi_M\ i{\in}UNIV.\ M)$

shows $(\lambda x.\ comb\text{-}seq\ i\ (f\ x)\ (g\ x)) \in measurable\ N\ (\Pi_M\ i{\in}UNIV.\ M)$
⟨*proof*⟩

lemma *comb-seq-0*: $comb\text{-}seq\ 0\ \omega\ \omega' = \omega'$
⟨*proof*⟩

lemma *comb-seq-Suc*: $comb\text{-}seq\ (Suc\ n)\ \omega\ \omega' = comb\text{-}seq\ n\ \omega\ (case\text{-}nat\ (\omega\ n)\ \omega')$
⟨*proof*⟩

lemma *comb-seq-Suc-0*[*simp*]: $comb\text{-}seq\ (Suc\ 0)\ \omega = case\text{-}nat\ (\omega\ 0)$
⟨*proof*⟩

lemma *comb-seq-less*: $i < n \Longrightarrow comb\text{-}seq\ n\ \omega\ \omega'\ i = \omega\ i$
⟨*proof*⟩

lemma *comb-seq-add*: $comb\text{-}seq\ n\ \omega\ \omega'\ (i + n) = \omega'\ i$
⟨*proof*⟩

lemma *case-nat-comb-seq*: $case\text{-}nat\ s'\ (comb\text{-}seq\ n\ \omega\ \omega')\ (i + n) = case\text{-}nat\ (case\text{-}nat\ s'\ \omega\ n)\ \omega'\ i$
⟨*proof*⟩

lemma *case-nat-comb-seq'*:
$case\text{-}nat\ s\ (comb\text{-}seq\ i\ \omega\ \omega') = comb\text{-}seq\ (Suc\ i)\ (case\text{-}nat\ s\ \omega)\ \omega'$
⟨*proof*⟩

locale *sequence-space* = *product-prob-space* $\lambda i.\ M\ UNIV$ :: *nat set* **for** *M*
begin

abbreviation $S \equiv \Pi_M\ i{\in}UNIV{::}nat\ set.\ M$

lemma *infprod-in-sets*[*intro*]:
fixes $E :: nat \Rightarrow 'a\ set$ **assumes** E: $\bigwedge i.\ E\ i \in sets\ M$
shows $Pi\ UNIV\ E \in sets\ S$
⟨*proof*⟩

lemma *measure-PiM-countable*:
fixes $E :: nat \Rightarrow 'a\ set$ **assumes** E: $\bigwedge i.\ E\ i \in sets\ M$
shows $(\lambda n.\ \prod i{\le}n.\ measure\ M\ (E\ i)) \longrightarrow measure\ S\ (Pi\ UNIV\ E)$
⟨*proof*⟩

lemma *nat-eq-diff-eq*:
fixes $a\ b\ c :: nat$
shows $c \le b \Longrightarrow a = b - c \longleftrightarrow a + c = b$
⟨*proof*⟩

lemma *PiM-comb-seq*:
$distr\ (S \bigotimes_M S)\ S\ (\lambda(\omega, \omega').\ comb\text{-}seq\ i\ \omega\ \omega') = S$ (**is** *?D* = -)
⟨*proof*⟩

lemma *PiM-iter*:
distr $(M \bigotimes_M S)$ S $(\lambda(s, \omega).$ *case-nat* s $\omega) = S$ (**is** *?D* = -)
⟨*proof*⟩

end

lemma *PiM-return*:
assumes *finite I*
assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow \{a\ i\} \in$ *sets* $(M\ i)$
shows *PiM I* $(\lambda i.$ *return* $(M\ i)$ $(a\ i)) =$ *return* $(PiM\ I\ M)$ (*restrict a I*)
⟨*proof*⟩

lemma *distr-PiM-finite-prob-space'*:
assumes *fin*: *finite I*
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *prob-space* $(M\ i)$
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *prob-space* $(M'\ i)$
assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow f \in$ *measurable* $(M\ i)$ $(M'\ i)$
shows *distr* $(PiM\ I\ M)$ $(PiM\ I\ M')$ $(compose\ I\ f) = PiM\ I$ $(\lambda i.$ *distr* $(M\ i)$ $(M'\ i)$ $f)$
⟨*proof*⟩

end

10 Independent families of events, event sets, and random variables

theory *Independent-Family*
imports *Infinite-Product-Measure*
begin

definition (**in** *prob-space*)
indep-sets F I $\longleftrightarrow$ $(\forall i{\in}I.\ F\ i \subseteq events) \wedge$
$(\forall J{\subseteq}I.\ J \neq \{\} \longrightarrow$ *finite* $J \longrightarrow (\forall A{\in}Pi\ J\ F.$ *prob* $(\bigcap j{\in}J.\ A\ j) = (\prod j{\in}J.$ *prob* $(A\ j))))$

definition (**in** *prob-space*)
indep-set A B $\longleftrightarrow$ *indep-sets* (*case-bool A B*) *UNIV*

definition (**in** *prob-space*)
indep-events-def-alt: *indep-events A I* $\longleftrightarrow$ *indep-sets* $(\lambda i.\ \{A\ i\})\ I$

lemma (**in** *prob-space*) *indep-events-def*:
indep-events A I $\longleftrightarrow$ $(A`I \subseteq events) \wedge$
$(\forall J{\subseteq}I.\ J \neq \{\} \longrightarrow$ *finite* $J \longrightarrow$ *prob* $(\bigcap j{\in}J.\ A\ j) = (\prod j{\in}J.$ *prob* $(A\ j)))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-eventsI*:

$(\bigwedge i.\ i \in I \Longrightarrow F\ i \in sets\ M) \Longrightarrow (\bigwedge J.\ J \subseteq I \Longrightarrow finite\ J \Longrightarrow J \neq \{\} \Longrightarrow prob\ (\bigcap i{\in}J.\ F\ i) = (\prod i{\in}J.\ prob\ (F\ i))) \Longrightarrow indep\text{-}events\ F\ I$
⟨*proof*⟩

definition (**in** *prob-space*)
$indep\text{-}event\ A\ B \longleftrightarrow indep\text{-}events\ (case\text{-}bool\ A\ B)\ UNIV$

lemma (**in** *prob-space*) *indep-sets-cong*:
$I = J \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow F\ i = G\ i) \Longrightarrow indep\text{-}sets\ F\ I \longleftrightarrow indep\text{-}sets\ G\ J$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-events-finite-index-events*:
$indep\text{-}events\ F\ I \longleftrightarrow (\forall J{\subseteq}I.\ J \neq \{\} \longrightarrow finite\ J \longrightarrow indep\text{-}events\ F\ J)$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-finite-index-sets*:
$indep\text{-}sets\ F\ I \longleftrightarrow (\forall J{\subseteq}I.\ J \neq \{\} \longrightarrow finite\ J \longrightarrow indep\text{-}sets\ F\ J)$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-mono-index*:
$J \subseteq I \Longrightarrow indep\text{-}sets\ F\ I \Longrightarrow indep\text{-}sets\ F\ J$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-mono-sets*:
assumes *indep*: $indep\text{-}sets\ F\ I$
assumes *mono*: $\bigwedge i.\ i{\in}I \Longrightarrow G\ i \subseteq F\ i$
shows $indep\text{-}sets\ G\ I$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-mono*:
assumes *indep*: $indep\text{-}sets\ F\ I$
assumes *mono*: $J \subseteq I\ \bigwedge i.\ i{\in}J \Longrightarrow G\ i \subseteq F\ i$
shows $indep\text{-}sets\ G\ J$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-setsI*:
assumes $\bigwedge i.\ i \in I \Longrightarrow F\ i \subseteq events$
and $\bigwedge A\ J.\ J \neq \{\} \Longrightarrow J \subseteq I \Longrightarrow finite\ J \Longrightarrow (\forall j{\in}J.\ A\ j \in F\ j) \Longrightarrow prob\ (\bigcap j{\in}J.\ A\ j) = (\prod j{\in}J.\ prob\ (A\ j))$
shows $indep\text{-}sets\ F\ I$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-setsD*:
assumes $indep\text{-}sets\ F\ I$ **and** $J \subseteq I\ J \neq \{\}\ finite\ J\ \forall j{\in}J.\ A\ j \in F\ j$
shows $prob\ (\bigcap j{\in}J.\ A\ j) = (\prod j{\in}J.\ prob\ (A\ j))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-setI*:
assumes *ev*: $A \subseteq events\ B \subseteq events$

and *indep*: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in B \Longrightarrow prob\ (a \cap b) = prob\ a * prob\ b$
shows *indep-set A B*
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-setD*:
assumes *indep*: *indep-set A B* **and** *ev*: $a \in A\ b \in B$
shows *prob* $(a \cap b) =$ *prob a* $*$ *prob b*
⟨*proof*⟩

lemma (**in** *prob-space*)
assumes *indep*: *indep-set A B*
shows *indep-setD-ev1*: $A \subseteq events$
and *indep-setD-ev2*: $B \subseteq events$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-Dynkin*:
assumes *indep*: *indep-sets F I*
shows *indep-sets* $(\lambda i.\ Dynkin\ (space\ M)\ (F\ i))\ I$
(**is** *indep-sets ?F I*)
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-sigma*:
assumes *indep*: *indep-sets F I*
assumes *stable*: $\bigwedge i.\ i \in I \Longrightarrow Int\text{-}stable\ (F\ i)$
shows *indep-sets* $(\lambda i.\ sigma\text{-}sets\ (space\ M)\ (F\ i))\ I$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-sigma-sets-iff*:
assumes $\bigwedge i.\ i \in I \Longrightarrow Int\text{-}stable\ (F\ i)$
shows *indep-sets* $(\lambda i.\ sigma\text{-}sets\ (space\ M)\ (F\ i))\ I \longleftrightarrow$ *indep-sets F I*
⟨*proof*⟩

definition (**in** *prob-space*)
indep-vars-def2: *indep-vars M' X I* $\longleftrightarrow$
$(\forall i \in I.\ random\text{-}variable\ (M'\ i)\ (X\ i)) \wedge$
indep-sets $(\lambda i.\ \{\ X\ i\ -`\ A \cap space\ M \mid A.\ A \in sets\ (M'\ i)\})\ I$

definition (**in** *prob-space*)
indep-var Ma A Mb B $\longleftrightarrow$ *indep-vars* (*case-bool Ma Mb*) (*case-bool A B*) *UNIV*

lemma (**in** *prob-space*) *indep-vars-def*:
indep-vars M' X I $\longleftrightarrow$
$(\forall i \in I.\ random\text{-}variable\ (M'\ i)\ (X\ i)) \wedge$
indep-sets $(\lambda i.\ sigma\text{-}sets\ (space\ M)\ \{\ X\ i\ -`\ A \cap space\ M \mid A.\ A \in sets\ (M'\ i)\})\ I$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-var-eq*:
indep-var S X T Y $\longleftrightarrow$

(*random-variable S X* ∧ *random-variable T Y*) ∧
indep-set
(*sigma-sets* (*space M*) { *X* −‘ *A* ∩ *space M* | *A*. *A* ∈ *sets S*})
(*sigma-sets* (*space M*) { *Y* −‘ *A* ∩ *space M* | *A*. *A* ∈ *sets T*})
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets2-eq*:
indep-set A B ⟷ *A* ⊆ *events* ∧ *B* ⊆ *events* ∧ (∀ *a*∈*A*. ∀ *b*∈*B*. *prob* (*a* ∩ *b*) = *prob a* ∗ *prob b*)
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-set-sigma-sets*:
assumes *indep-set A B*
assumes *A*: *Int-stable A* **and** *B*: *Int-stable B*
shows *indep-set* (*sigma-sets* (*space M*) *A*) (*sigma-sets* (*space M*) *B*)
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-eventsI-indep-vars*:
assumes *indep*: *indep-vars N X I*
assumes *P*: ⋀*i*. *i* ∈ *I* ⟹ {*x*∈*space* (*N i*). *P i x*} ∈ *sets* (*N i*)
shows *indep-events* (λ*i*. {*x*∈*space M*. *P i* (*X i x*)}) *I*
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-collect-sigma*:
fixes *I* :: *'j* ⇒ *'i set* **and** *J* :: *'j set* **and** *E* :: *'i* ⇒ *'a set set*
assumes *indep*: *indep-sets E* (⋃*j*∈*J*. *I j*)
assumes *Int-stable*: ⋀*i j*. *j* ∈ *J* ⟹ *i* ∈ *I j* ⟹ *Int-stable* (*E i*)
assumes *disjoint*: *disjoint-family-on I J*
shows *indep-sets* (λ*j*. *sigma-sets* (*space M*) (⋃*i*∈*I j*. *E i*)) *J*
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-restrict*:
assumes *ind*: *indep-vars M' X I* **and** *K*: ⋀*j*. *j* ∈ *L* ⟹ *K j* ⊆ *I* **and** *J*: *disjoint-family-on K L*
shows *indep-vars* (λ*j*. *PiM* (*K j*) *M'*) (λ*j ω*. *restrict* (λ*i*. *X i ω*) (*K j*)) *L*
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-var-restrict*:
assumes *ind*: *indep-vars M' X I* **and** *AB*: *A* ∩ *B* = {} *A* ⊆ *I B* ⊆ *I*
shows *indep-var* (*PiM A M'*) (λ*ω*. *restrict* (λ*i*. *X i ω*) *A*) (*PiM B M'*) (λ*ω*. *restrict* (λ*i*. *X i ω*) *B*)
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-subset*:
assumes *indep-vars M' X I J* ⊆ *I*
shows *indep-vars M' X J*
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-cong*:

$I = J \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow X\ i = Y\ i) \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow M'\ i = N'\ i) \Longrightarrow$ *indep-vars* $M'\ X\ I \longleftrightarrow$ *indep-vars* $N'\ Y\ J$
⟨*proof*⟩

definition (**in** *prob-space*) *tail-events* **where**
tail-events $A = (\bigcap n.$ *sigma-sets* (*space* M) $(\bigcup\ (A\ `\ \{n..\})))$

lemma (**in** *prob-space*) *tail-events-sets*:
assumes A: $\bigwedge i$::*nat*. $A\ i \subseteq$ *events*
shows *tail-events* $A \subseteq$ *events*
⟨*proof*⟩

lemma (**in** *prob-space*) *sigma-algebra-tail-events*:
assumes $\bigwedge i$::*nat*. *sigma-algebra* (*space* M) $(A\ i)$
shows *sigma-algebra* (*space* M) (*tail-events* A)
⟨*proof*⟩

lemma (**in** *prob-space*) *kolmogorov-0-1-law*:
fixes A :: *nat* ⇒ *'a set set*
assumes $\bigwedge i$::*nat*. *sigma-algebra* (*space* M) $(A\ i)$
assumes *indep*: *indep-sets* A *UNIV*
and X: $X \in$ *tail-events* A
shows *prob* $X = 0 \vee$ *prob* $X = 1$
⟨*proof*⟩

lemma (**in** *prob-space*) *borel-0-1-law*:
fixes F :: *nat* ⇒ *'a set*
assumes *F2*: *indep-events* F *UNIV*
shows *prob* $(\bigcap n.\ \bigcup m{\in}\{n..\}.\ F\ m) = 0 \vee$ *prob* $(\bigcap n.\ \bigcup m{\in}\{n..\}.\ F\ m) = 1$
⟨*proof*⟩

lemma (**in** *prob-space*) *borel-0-1-law-AE*:
fixes P :: *nat* ⇒ *'a* ⇒ *bool*
assumes *indep-events* $(\lambda m.\ \{x{\in}$*space* $M.\ P\ m\ x\})$ *UNIV* (**is** *indep-events ?P* -)
shows (*AE* x *in* M. *infinite* $\{m.\ P\ m\ x\}) \vee$ (*AE* x *in* M. *finite* $\{m.\ P\ m\ x\}$)
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-finite*:
assumes I: $I \neq \{\}$ *finite* I
and F: $\bigwedge i.\ i \in I \Longrightarrow F\ i \subseteq$ *events* $\bigwedge i.\ i \in I \Longrightarrow$ *space* $M \in F\ i$
shows *indep-sets* $F\ I \longleftrightarrow (\forall A{\in}Pi\ I\ F.$ *prob* $(\bigcap j{\in}I.\ A\ j) = (\prod j{\in}I.$ *prob* $(A\ j)))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-finite*:
fixes I :: *'i set*
assumes I: $I \neq \{\}$ *finite* I
and M': $\bigwedge i.\ i \in I \Longrightarrow$ *sets* $(M'\ i) =$ *sigma-sets* (*space* $(M'\ i)$) $(E\ i)$
and *rv*: $\bigwedge i.\ i \in I \Longrightarrow$ *random-variable* $(M'\ i)\ (X\ i)$

and *Int-stable*: $\bigwedge i.\ i \in I \Longrightarrow$ *Int-stable* $(E\ i)$
and *space*: $\bigwedge i.\ i \in I \Longrightarrow$ *space* $(M'\ i) \in E\ i$ **and** *closed*: $\bigwedge i.\ i \in I \Longrightarrow E\ i \subseteq$ *Pow* (*space* $(M'\ i)$)
shows *indep-vars* $M'\ X\ I \longleftrightarrow$
$(\forall A \in (\Pi\ i \in I.\ E\ i).$ *prob* $(\bigcap j \in I.\ X\ j\ -`\ A\ j \cap$ *space* $M) = (\prod j \in I.$ *prob* $(X\ j\ -`\ A\ j \cap$ *space* $M)))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-compose*:
assumes *indep-vars* $M'\ X\ I$
assumes *rv*: $\bigwedge i.\ i \in I \Longrightarrow Y\ i \in$ *measurable* $(M'\ i)\ (N\ i)$
shows *indep-vars* $N\ (\lambda i.\ Y\ i \circ X\ i)\ I$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-compose2*:
assumes *indep-vars* $M'\ X\ I$
assumes *rv*: $\bigwedge i.\ i \in I \Longrightarrow Y\ i \in$ *measurable* $(M'\ i)\ (N\ i)$
shows *indep-vars* $N\ (\lambda i\ x.\ Y\ i\ (X\ i\ x))\ I$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-var-compose*:
assumes *indep-var M1 X1 M2 X2 Y1* $\in$ *measurable M1 N1 Y2* $\in$ *measurable M2 N2*
shows *indep-var N1* $(Y1 \circ X1)$ *N2* $(Y2 \circ X2)$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-Min*:
fixes $X :: 'i \Rightarrow 'a \Rightarrow$ *real*
assumes I: *finite* I $i \notin I$ **and** *indep*: *indep-vars* $(\lambda\text{-}.$ *borel*$)$ X $(insert\ i\ I)$
shows *indep-var borel* $(X\ i)$ *borel* $(\lambda\omega.$ *Min* $((\lambda i.\ X\ i\ \omega)\text{`}I))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-sum*:
fixes $X :: 'i \Rightarrow 'a \Rightarrow$ *real*
assumes I: *finite* I $i \notin I$ **and** *indep*: *indep-vars* $(\lambda\text{-}.$ *borel*$)$ X $(insert\ i\ I)$
shows *indep-var borel* $(X\ i)$ *borel* $(\lambda\omega.\ \sum i \in I.\ X\ i\ \omega)$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-prod*:
fixes $X :: 'i \Rightarrow 'a \Rightarrow$ *real*
assumes I: *finite* I $i \notin I$ **and** *indep*: *indep-vars* $(\lambda\text{-}.$ *borel*$)$ X $(insert\ i\ I)$
shows *indep-var borel* $(X\ i)$ *borel* $(\lambda\omega.\ \prod i \in I.\ X\ i\ \omega)$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-varsD-finite*:
assumes X: *indep-vars* $M'\ X\ I$
assumes I: $I \neq \{\}$ *finite* I $\bigwedge i.\ i \in I \Longrightarrow A\ i \in$ *sets* $(M'\ i)$
shows *prob* $(\bigcap i \in I.\ X\ i\ -`\ A\ i \cap$ *space* $M) = (\prod i \in I.$ *prob* $(X\ i\ -`\ A\ i \cap$ *space* $M))$

⟨*proof*⟩

lemma (**in** *prob-space*) *indep-varsD*:
assumes *X*: *indep-vars M' X I*
assumes *I*: $J \neq \{\}$ *finite J* $J \subseteq I$ $\bigwedge i.\ i \in J \Longrightarrow A\ i \in sets\ (M'\ i)$
shows *prob* $(\bigcap i{\in}J.\ X\ i -` A\ i \cap space\ M) = (\prod i{\in}J.\ prob\ (X\ i -` A\ i \cap space\ M))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-iff-distr-eq-PiM*:
fixes *I* :: *'i set* **and** *X* :: *'i* ⇒ *'a* ⇒ *'b*
assumes $I \neq \{\}$
assumes *rv*: $\bigwedge i.$ *random-variable* $(M'\ i)\ (X\ i)$
shows *indep-vars M' X I* ⟷
distr M $(\Pi_M\ i{\in}I.\ M'\ i)\ (\lambda x.\ \lambda i{\in}I.\ X\ i\ x) = (\Pi_M\ i{\in}I.\ distr\ M\ (M'\ i)\ (X\ i))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-iff-distr-eq-PiM'*:
fixes *I* :: *'i set* **and** *X* :: *'i* ⇒ *'a* ⇒ *'b*
assumes $I \neq \{\}$
assumes *rv*: $\bigwedge i.\ i \in I \Longrightarrow$ *random-variable* $(M'\ i)\ (X\ i)$
shows *indep-vars M' X I* ⟷
distr M $(\Pi_M\ i{\in}I.\ M'\ i)\ (\lambda x.\ \lambda i{\in}I.\ X\ i\ x) = (\Pi_M\ i{\in}I.\ distr\ M\ (M'\ i)\ (X\ i))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-varD*:
assumes *indep*: *indep-var Ma A Mb B*
assumes *sets*: *Xa* ∈ *sets Ma Xb* ∈ *sets Mb*
shows *prob* $((\lambda x.\ (A\ x,\ B\ x)) -` (Xa \times Xb) \cap space\ M) =$
prob $(A -` Xa \cap space\ M) * prob\ (B -` Xb \cap space\ M)$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-indep-random-variable*:
assumes *ind*[*simp*]: *indep-var N X N Y*
assumes [*simp*]: *A* ∈ *sets N B* ∈ *sets N*
shows $\mathcal{P}(x\ in\ M.\ X\ x \in A \wedge Y\ x \in B) = \mathcal{P}(x\ in\ M.\ X\ x \in A) * \mathcal{P}(x\ in\ M.\ Y\ x \in B)$
⟨*proof*⟩

lemma (**in** *prob-space*)
assumes *indep-var S X T Y*
shows *indep-var-rv1*: *random-variable S X*
and *indep-var-rv2*: *random-variable T Y*
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-var-distribution-eq*:
indep-var S X T Y ⟷ *random-variable S X* ∧ *random-variable T Y* ∧
distr M S X $\bigotimes_M$ *distr M T Y* = *distr M* $(S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))$ (**is** -

$\longleftrightarrow$ - $\wedge$ - $\wedge$ *?S* $\bigotimes_M$ *?T* = *?J*)
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-joint-indep*:
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*
assumes *X*: *distributed M S X Px* **and** *Y*: *distributed M T Y Py*
assumes *indep*: *indep-var S X T Y*
shows *distributed M* (*S* $\bigotimes_M$ *T*) (λ*x*. (*X x*, *Y x*)) (λ(*x*, *y*). *Px x* ∗ *Py y*)
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-nn-integral*:
assumes *I*: *finite I indep-vars* (λ-. *borel*) *X I* $\bigwedge i$ *ω*. *i* ∈ *I* ⟹ 0 ≤ *X i ω*
shows (∫ $^+\omega$. (∏ *i*∈*I*. *X i ω*) ∂*M*) = (∏ *i*∈*I*. ∫ $^+\omega$. *X i ω* ∂*M*)
⟨*proof*⟩

lemma (**in** *prob-space*)
fixes *X* :: *'i* ⇒ *'a* ⇒ *'b*::{*real-normed-field*, *banach*, *second-countable-topology*}
assumes *I*: *finite I indep-vars* (λ-. *borel*) *X I* $\bigwedge i$. *i* ∈ *I* ⟹ *integrable M* (*X i*)
shows *indep-vars-lebesgue-integral*: (∫ *ω*. (∏ *i*∈*I*. *X i ω*) ∂*M*) = (∏ *i*∈*I*. ∫ *ω*. *X i ω* ∂*M*) (**is** *?eq*)
and *indep-vars-integrable*: *integrable M* (λ*ω*. (∏ *i*∈*I*. *X i ω*)) (**is** *?int*)
⟨*proof*⟩

lemma (**in** *prob-space*)
fixes *X1 X2* :: *'a* ⇒ *'b*::{*real-normed-field*, *banach*, *second-countable-topology*}
assumes *indep-var borel X1 borel X2 integrable M X1 integrable M X2*
shows *indep-var-lebesgue-integral*: (∫ *ω*. *X1 ω* ∗ *X2 ω* ∂*M*) = (∫ *ω*. *X1 ω* ∂*M*) ∗ (∫ *ω*. *X2 ω* ∂*M*) (**is** *?eq*)
and *indep-var-integrable*: *integrable M* (λ*ω*. *X1 ω* ∗ *X2 ω*) (**is** *?int*)
⟨*proof*⟩

end

11 Convolution Measure

theory *Convolution*
imports *Independent-Family*
begin

lemma (**in** *finite-measure*) *sigma-finite-measure*: *sigma-finite-measure M*
⟨*proof*⟩

definition *convolution* :: (*'a* :: *ordered-euclidean-space*) *measure* ⇒ *'a measure* ⇒ *'a measure* (**infix** ⋆ *50*) **where**
convolution M N = *distr* (*M* $\bigotimes_M$ *N*) *borel* (λ(*x*, *y*). *x* + *y*)

lemma
shows *space-convolution*[*simp*]: *space* (*convolution M N*) = *space borel*
and *sets-convolution*[*simp*]: *sets* (*convolution M N*) = *sets borel*

and *measurable-convolution1*[*simp*]: *measurable A* (*convolution M N*) = *measurable A borel*
and *measurable-convolution2*[*simp*]: *measurable* (*convolution M N*) *B* = *measurable borel B*
⟨*proof*⟩

lemma *nn-integral-convolution*:
assumes *finite-measure M finite-measure N*
assumes [*measurable-cong*]: *sets N* = *sets borel sets M* = *sets borel*
assumes [*measurable*]: *f* ∈ *borel-measurable borel*
shows (∫⁺*x*. *f x* ∂*convolution M N*) = (∫⁺*x*. ∫⁺*y*. *f* (*x* + *y*) ∂*N* ∂*M*)
⟨*proof*⟩

lemma *convolution-emeasure*:
assumes *A* ∈ *sets borel finite-measure M finite-measure N*
assumes [*simp*]: *sets N* = *sets borel sets M* = *sets borel*
assumes [*simp*]: *space M* = *space N space N* = *space borel*
shows *emeasure* (*M* ⋆ *N*) *A* = ∫⁺*x*. (*emeasure N* {*a*. *a* + *x* ∈ *A*}) ∂*M*
⟨*proof*⟩

lemma *convolution-emeasure'*:
assumes [*simp*]:*A* ∈ *sets borel*
assumes [*simp*]: *finite-measure M finite-measure N*
assumes [*simp*]: *sets N* = *sets borel sets M* = *sets borel*
shows *emeasure* (*M* ⋆ *N*) *A* = ∫⁺*x*. ∫⁺*y*. (*indicator A* (*x* + *y*)) ∂*N* ∂*M*
⟨*proof*⟩

lemma *convolution-finite*:
assumes [*simp*]: *finite-measure M finite-measure N*
assumes [*measurable-cong*]: *sets N* = *sets borel sets M* = *sets borel*
shows *finite-measure* (*M* ⋆ *N*)
⟨*proof*⟩

lemma *convolution-emeasure-3*:
assumes [*simp*, *measurable*]: *A* ∈ *sets borel*
assumes [*simp*]: *finite-measure M finite-measure N finite-measure L*
assumes [*simp*]: *sets N* = *sets borel sets M* = *sets borel sets L* = *sets borel*
shows *emeasure* (*L* ⋆ (*M* ⋆ *N*)) *A* = ∫⁺*x*. ∫⁺*y*. ∫⁺*z*. *indicator A* (*x* + *y* + *z*) ∂*N* ∂*M* ∂*L*
⟨*proof*⟩

lemma *convolution-emeasure-3'*:
assumes [*simp*, *measurable*]:*A* ∈ *sets borel*
assumes [*simp*]: *finite-measure M finite-measure N finite-measure L*
assumes [*measurable-cong*, *simp*]: *sets N* = *sets borel sets M* = *sets borel sets L* = *sets borel*
shows *emeasure* ((*L* ⋆ *M*) ⋆ *N*) *A* = ∫⁺*x*. ∫⁺*y*. ∫⁺*z*. *indicator A* (*x* + *y* + *z*) ∂*N* ∂*M* ∂*L*
⟨*proof*⟩

lemma *convolution-commutative*:
assumes [*simp*]: *finite-measure M finite-measure N*
assumes [*measurable-cong*, *simp*]: *sets N = sets borel sets M = sets borel*
shows $(M \star N) = (N \star M)$
⟨*proof*⟩

lemma *convolution-associative*:
assumes [*simp*]: *finite-measure M finite-measure N finite-measure L*
assumes [*simp*]: *sets N = sets borel sets M = sets borel sets L = sets borel*
shows $(L \star (M \star N)) = ((L \star M) \star N)$
⟨*proof*⟩

lemma (**in** *prob-space*) *sum-indep-random-variable*:
assumes *ind*: *indep-var borel X borel Y*
assumes [*simp*, *measurable*]: *random-variable borel X*
assumes [*simp*, *measurable*]: *random-variable borel Y*
shows *distr M borel* $(\lambda x.\ X\ x + Y\ x)$ = *convolution* (*distr M borel X*) (*distr M borel Y*)
⟨*proof*⟩

lemma (**in** *prob-space*) *sum-indep-random-variable-lborel*:
assumes *ind*: *indep-var borel X borel Y*
assumes [*simp*, *measurable*]: *random-variable lborel X*
assumes [*simp*, *measurable*]:*random-variable lborel Y*
shows *distr M lborel* $(\lambda x.\ X\ x + Y\ x)$ = *convolution* (*distr M lborel X*) (*distr M lborel Y*)
⟨*proof*⟩

lemma *convolution-density*:
fixes *f g* :: *real* ⇒ *ennreal*
assumes [*measurable*]: *f* ∈ *borel-measurable borel g* ∈ *borel-measurable borel*
assumes [*simp*]:*finite-measure* (*density lborel f*) *finite-measure* (*density lborel g*)
shows *density lborel f* ⋆ *density lborel g* = *density lborel* $(\lambda x.\ \int^+ y.\ f\ (x - y) * g\ y\ \partial lborel)$
(**is** *?l* = *?r*)
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-finite-measure-density*:
distributed M N X f ⟹ *finite-measure* (*density N f*)
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-convolution*:
fixes *f* :: *real* ⇒ -
fixes *g* :: *real* ⇒ -
assumes *indep*: *indep-var borel X borel Y*
assumes *X*: *distributed M lborel X f*
assumes *Y*: *distributed M lborel Y g*

shows *distributed M lborel* $(\lambda x.\ X\ x + Y\ x)$ $(\lambda x.\ \int^+ y.\ f\ (x - y) * g\ y\ \partial lborel)$
⟨*proof*⟩

lemma *prob-space-convolution-density*:
fixes f:: *real* ⇒ -
fixes g:: *real* ⇒ -
assumes [*measurable*]: $f \in$ *borel-measurable borel*
assumes [*measurable*]: $g \in$ *borel-measurable borel*
assumes *gt-0*[*simp*]: $\bigwedge x.\ 0 \le f\ x\ \bigwedge x.\ 0 \le g\ x$
assumes *prob-space* (*density lborel f*) (**is** *prob-space ?F*)
assumes *prob-space* (*density lborel g*) (**is** *prob-space ?G*)
shows *prob-space* (*density lborel* $(\lambda x. \int^+ y.\ f\ (x - y) * g\ y\ \partial lborel)$) (**is** *prob-space ?D*)
⟨*proof*⟩

end

12 Information theory

theory *Information*
imports
Independent-Family
begin

lemma *log-le*: $1 < a \Longrightarrow 0 < x \Longrightarrow x \le y \Longrightarrow log\ a\ x \le log\ a\ y$
⟨*proof*⟩

lemma *log-less*: $1 < a \Longrightarrow 0 < x \Longrightarrow x < y \Longrightarrow log\ a\ x < log\ a\ y$
⟨*proof*⟩

lemma *sum-cartesian-product'*:
$(\sum x \in A \times B.\ f\ x) = (\sum x \in A.\ sum\ (\lambda y.\ f\ (x, y))\ B)$
⟨*proof*⟩

lemma *split-pairs*:
$((A, B) = X) \longleftrightarrow (fst\ X = A \wedge snd\ X = B)$ **and**
$(X = (A, B)) \longleftrightarrow (fst\ X = A \wedge snd\ X = B)$ ⟨*proof*⟩

12.1 Information theory

locale *information-space* = *prob-space* +
fixes b :: *real* **assumes** *b-gt-1*: $1 < b$

context *information-space*
begin

Introduce some simplification rules for logarithm of base b.

lemma *log-neg-const*:
assumes $x \le 0$

shows *log b x = log b 0*
⟨*proof*⟩

lemma *log-mult-eq*:
*log b (A * B) = (if 0 < A * B then log b |A| + log b |B| else log b 0)*
⟨*proof*⟩

lemma *log-inverse-eq*:
log b (inverse B) = (if 0 < B then − log b B else log b 0)
⟨*proof*⟩

lemma *log-divide-eq*:
*log b (A / B) = (if 0 < A * B then log b |A| − log b |B| else log b 0)*
⟨*proof*⟩

lemmas *log-simps = log-mult-eq log-inverse-eq log-divide-eq*

end

12.2 Kullback−Leibler divergence

The Kullback−Leibler divergence is also known as relative entropy or Kullback−Leibler distance.

definition
entropy-density b M N = log b ∘ enn2real ∘ RN-deriv M N

definition
KL-divergence b M N = integral^L N (entropy-density b M N)

lemma *measurable-entropy-density*[*measurable*]: *entropy-density b M N ∈ borel-measurable M*
⟨*proof*⟩

lemma (**in** *sigma-finite-measure*) *KL-density*:
fixes *f :: 'a ⇒ real*
assumes *1 < b*
assumes *f*[*measurable*]: *f ∈ borel-measurable M* **and** *nn*: *AE x in M. 0 ≤ f x*
shows *KL-divergence b M (density M f) = (∫ x. f x * log b (f x) ∂M)*
⟨*proof*⟩

lemma (**in** *sigma-finite-measure*) *KL-density-density*:
fixes *f g :: 'a ⇒ real*
assumes *1 < b*
assumes *f*: *f ∈ borel-measurable M AE x in M. 0 ≤ f x*
assumes *g*: *g ∈ borel-measurable M AE x in M. 0 ≤ g x*
assumes *ac*: *AE x in M. f x = 0 ⟶ g x = 0*
shows *KL-divergence b (density M f) (density M g) = (∫ x. g x * log b (g x / f x) ∂M)*
⟨*proof*⟩

lemma (**in** *information-space*) *KL-gt-0*:
fixes $D :: 'a \Rightarrow real$
assumes *prob-space* (*density M D*)
assumes *D*: $D \in$ *borel-measurable M AE x in M*. $0 \leq D\ x$
assumes *int*: *integrable M* (λx. $D\ x * log\ b\ (D\ x)$)
assumes *A*: *density M D* $\neq M$
shows $0 <$ *KL-divergence b M* (*density M D*)
⟨*proof*⟩

lemma (**in** *sigma-finite-measure*) *KL-same-eq-0*: *KL-divergence b M M* $= 0$
⟨*proof*⟩

lemma (**in** *information-space*) *KL-eq-0-iff-eq*:
fixes $D :: 'a \Rightarrow real$
assumes *prob-space* (*density M D*)
assumes *D*: $D \in$ *borel-measurable M AE x in M*. $0 \leq D\ x$
assumes *int*: *integrable M* (λx. $D\ x * log\ b\ (D\ x)$)
shows *KL-divergence b M* (*density M D*) $= 0 \longleftrightarrow$ *density M D* $= M$
⟨*proof*⟩

lemma (**in** *information-space*) *KL-eq-0-iff-eq-ac*:
fixes $D :: 'a \Rightarrow real$
assumes *prob-space N*
assumes *ac*: *absolutely-continuous M N sets N* $=$ *sets M*
assumes *int*: *integrable N* (*entropy-density b M N*)
shows *KL-divergence b M N* $= 0 \longleftrightarrow N = M$
⟨*proof*⟩

lemma (**in** *information-space*) *KL-nonneg*:
assumes *prob-space* (*density M D*)
assumes *D*: $D \in$ *borel-measurable M AE x in M*. $0 \leq D\ x$
assumes *int*: *integrable M* (λx. $D\ x * log\ b\ (D\ x)$)
shows $0 \leq$ *KL-divergence b M* (*density M D*)
⟨*proof*⟩

lemma (**in** *sigma-finite-measure*) *KL-density-density-nonneg*:
fixes $f\ g :: 'a \Rightarrow real$
assumes $1 < b$
assumes *f*: $f \in$ *borel-measurable M AE x in M*. $0 \leq f\ x$ *prob-space* (*density M f*)
assumes *g*: $g \in$ *borel-measurable M AE x in M*. $0 \leq g\ x$ *prob-space* (*density M g*)
assumes *ac*: *AE x in M*. $f\ x = 0 \longrightarrow g\ x = 0$
assumes *int*: *integrable M* (λx. $g\ x * log\ b\ (g\ x\ /\ f\ x)$)
shows $0 \leq$ *KL-divergence b* (*density M f*) (*density M g*)
⟨*proof*⟩

12.3 Finite Entropy

definition (**in** *information-space*) *finite-entropy* :: *'b measure* ⇒ (*'a* ⇒ *'b*) ⇒ (*'b* ⇒ *real*) ⇒ *bool*
where
finite-entropy S X f ⟷
distributed M S X f ∧
integrable S (λ*x*. *f x* ∗ *log b* (*f x*)) ∧
(∀ *x*∈*space S*. *0* ≤ *f x*)

lemma (**in** *information-space*) *finite-entropy-simple-function*:
assumes *X*: *simple-function M X*
shows *finite-entropy* (*count-space* (*X*'*space M*)) *X* (λ*a*. *measure M* {*x* ∈ *space M*. *X x* = *a*})
⟨*proof*⟩

lemma *ac-fst*:
assumes *sigma-finite-measure T*
shows *absolutely-continuous S* (*distr* (*S* ⨂$_M$ *T*) *S fst*)
⟨*proof*⟩

lemma *ac-snd*:
assumes *sigma-finite-measure T*
shows *absolutely-continuous T* (*distr* (*S* ⨂$_M$ *T*) *T snd*)
⟨*proof*⟩

lemma (**in** *information-space*) *finite-entropy-integrable*:
finite-entropy S X Px ⟹ *integrable S* (λ*x*. *Px x* ∗ *log b* (*Px x*))
⟨*proof*⟩

lemma (**in** *information-space*) *finite-entropy-distributed*:
finite-entropy S X Px ⟹ *distributed M S X Px*
⟨*proof*⟩

lemma (**in** *information-space*) *finite-entropy-nn*:
finite-entropy S X Px ⟹ *x* ∈ *space S* ⟹ *0* ≤ *Px x*
⟨*proof*⟩

lemma (**in** *information-space*) *finite-entropy-measurable*:
finite-entropy S X Px ⟹ *Px* ∈ *S* →$_M$ *borel*
⟨*proof*⟩

lemma (**in** *information-space*) *subdensity-finite-entropy*:
fixes *g* :: *'b* ⇒ *real* **and** *f* :: *'c* ⇒ *real*
assumes *T*: *T* ∈ *measurable P Q*
assumes *f*: *finite-entropy P X f*
assumes *g*: *finite-entropy Q Y g*
assumes *Y*: *Y* = *T* ∘ *X*
shows *AE x in P*. *g* (*T x*) = *0* ⟶ *f x* = *0*
⟨*proof*⟩

lemma (**in** *information-space*) *finite-entropy-integrable-transform*:
finite-entropy S X Px $\Longrightarrow$ *distributed M T Y Py* $\Longrightarrow$ ($\bigwedge x.\ x \in$ *space T* $\Longrightarrow$ $0 \leq$ *Py x*) $\Longrightarrow$
$X = (\lambda x.\ f\ (Y\ x)) \Longrightarrow f \in$ *measurable T S* $\Longrightarrow$ *integrable T* ($\lambda x.$ *Py x* $*$ *log b* (*Px* (*f x*)))
⟨*proof*⟩

12.4 Mutual Information

definition (**in** *prob-space*)
mutual-information b S T X Y =
KL-divergence b (*distr M S X* $\bigotimes_M$ *distr M T Y*) (*distr M* (*S* $\bigotimes_M$ *T*) ($\lambda x.$ (*X x*, *Y x*)))

lemma (**in** *information-space*) *mutual-information-indep-vars*:
fixes *S T X Y*
defines $P \equiv$ *distr M S X* $\bigotimes_M$ *distr M T Y*
defines $Q \equiv$ *distr M* (*S* $\bigotimes_M$ *T*) ($\lambda x.$ (*X x*, *Y x*))
shows *indep-var S X T Y* $\longleftrightarrow$
(*random-variable S X* $\wedge$ *random-variable T Y* $\wedge$
absolutely-continuous P Q $\wedge$ *integrable Q* (*entropy-density b P Q*) $\wedge$
mutual-information b S T X Y = 0)
⟨*proof*⟩

abbreviation (**in** *information-space*)
mutual-information-Pow ($\mathcal{I}'$(- ; -')) **where**
$\mathcal{I}(X\ ;\ Y) \equiv$ *mutual-information b* (*count-space* (*X*‘*space M*)) (*count-space* (*Y*‘*space M*)) *X Y*

lemma (**in** *information-space*)
fixes *Pxy* :: $'b \times 'c \Rightarrow$ *real* **and** *Px* :: $'b \Rightarrow$ *real* **and** *Py* :: $'c \Rightarrow$ *real*
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*
assumes *Fx*: *finite-entropy S X Px* **and** *Fy*: *finite-entropy T Y Py*
assumes *Fxy*: *finite-entropy* (*S* $\bigotimes_M$ *T*) ($\lambda x.$ (*X x*, *Y x*)) *Pxy*
defines $f \equiv \lambda x.$ *Pxy x* $*$ *log b* (*Pxy x* / (*Px* (*fst x*) $*$ *Py* (*snd x*)))
shows *mutual-information-distr'*: *mutual-information b S T X Y* = *integral*^{L} (*S* $\bigotimes_M$ *T*) *f* (**is** *?M* = *?R*)
and *mutual-information-nonneg'*: $0 \leq$ *mutual-information b S T X Y*
⟨*proof*⟩

lemma (**in** *information-space*)
fixes *Pxy* :: $'b \times 'c \Rightarrow$ *real* **and** *Px* :: $'b \Rightarrow$ *real* **and** *Py* :: $'c \Rightarrow$ *real*
assumes *sigma-finite-measure S sigma-finite-measure T*
assumes *Px*: *distributed M S X Px* **and** *Px-nn*: $\bigwedge x.\ x \in$ *space S* $\Longrightarrow$ $0 \leq$ *Px x*
and *Py*: *distributed M T Y Py* **and** *Py-nn*: $\bigwedge y.\ y \in$ *space T* $\Longrightarrow$ $0 \leq$ *Py y*
and *Pxy*: *distributed M* (*S* $\bigotimes_M$ *T*) ($\lambda x.$ (*X x*, *Y x*)) *Pxy*
and *Pxy-nn*: $\bigwedge x\ y.\ x \in$ *space S* $\Longrightarrow$ $y \in$ *space T* $\Longrightarrow$ $0 \leq$ *Pxy* (*x*, *y*)
defines $f \equiv \lambda x.$ *Pxy x* $*$ *log b* (*Pxy x* / (*Px* (*fst x*) $*$ *Py* (*snd x*)))

shows *mutual-information-distr*: *mutual-information b S T X Y* = *integral*^{L} (*S* $\bigotimes_M$ *T*) *f* (**is** *?M* = *?R*)
and *mutual-information-nonneg*: *integrable* (*S* $\bigotimes_M$ *T*) *f* $\Longrightarrow$ *0* $\leq$ *mutual-information b S T X Y*
⟨*proof*⟩

lemma (**in** *information-space*)
fixes *Pxy* :: *'b* × *'c* ⇒ *real* **and** *Px* :: *'b* ⇒ *real* **and** *Py* :: *'c* ⇒ *real*
assumes *sigma-finite-measure S sigma-finite-measure T*
assumes *Px*[*measurable*]: *distributed M S X Px* **and** *Px-nn*: $\bigwedge x.\ x \in$ *space S* $\Longrightarrow$ *0* $\leq$ *Px x*
and *Py*[*measurable*]: *distributed M T Y Py* **and** *Py-nn*: $\bigwedge x.\ x \in$ *space T* $\Longrightarrow$ *0* $\leq$ *Py x*
and *Pxy*[*measurable*]: *distributed M* (*S* $\bigotimes_M$ *T*) (λ*x*. (*X x*, *Y x*)) *Pxy*
and *Pxy-nn*: $\bigwedge x.\ x \in$ *space* (*S* $\bigotimes_M$ *T*) $\Longrightarrow$ *0* $\leq$ *Pxy x*
assumes *ae*: *AE x in S*. *AE y in T*. *Pxy* (*x*, *y*) = *Px x* ∗ *Py y*
shows *mutual-information-eq-0*: *mutual-information b S T X Y* = *0*
⟨*proof*⟩

lemma (**in** *information-space*) *mutual-information-simple-distributed*:
assumes *X*: *simple-distributed M X Px* **and** *Y*: *simple-distributed M Y Py*
assumes *XY*: *simple-distributed M* (λ*x*. (*X x*, *Y x*)) *Pxy*
shows $\mathcal{I}(X\ ;\ Y)$ = (∑ (*x*, *y*)∈(λ*x*. (*X x*, *Y x*))‘*space M*. *Pxy* (*x*, *y*) ∗ *log b* (*Pxy* (*x*, *y*) / (*Px x* ∗ *Py y*)))
⟨*proof*⟩

lemma (**in** *information-space*)
fixes *Pxy* :: *'b* × *'c* ⇒ *real* **and** *Px* :: *'b* ⇒ *real* **and** *Py* :: *'c* ⇒ *real*
assumes *Px*: *simple-distributed M X Px* **and** *Py*: *simple-distributed M Y Py*
assumes *Pxy*: *simple-distributed M* (λ*x*. (*X x*, *Y x*)) *Pxy*
assumes *ae*: ∀ *x*∈*space M*. *Pxy* (*X x*, *Y x*) = *Px* (*X x*) ∗ *Py* (*Y x*)
shows *mutual-information-eq-0-simple*: $\mathcal{I}(X\ ;\ Y) = 0$
⟨*proof*⟩

12.5 Entropy

definition (**in** *prob-space*) *entropy* :: *real* ⇒ *'b measure* ⇒ (*'a* ⇒ *'b*) ⇒ *real* **where**
entropy b S X = − *KL-divergence b S* (*distr M S X*)

abbreviation (**in** *information-space*)
entropy-Pow ($\mathcal{H}'$(-')) **where**
$\mathcal{H}(X)$ ≡ *entropy b* (*count-space* (*X*‘*space M*)) *X*

lemma (**in** *prob-space*) *distributed-RN-deriv*:
assumes *X*: *distributed M S X Px*
shows *AE x in S*. *RN-deriv S* (*density S Px*) *x* = *Px x*
⟨*proof*⟩

lemma (**in** *information-space*)

fixes $X :: 'a \Rightarrow 'b$
assumes X[*measurable*]: *distributed M MX X f* **and** *nn*: $\bigwedge x.\ x \in space\ MX \Longrightarrow 0 \leq f\ x$
shows *entropy-distr*: *entropy b MX X* $= -\ (\int x.\ f\ x * log\ b\ (f\ x)\ \partial MX)$ (**is** *?eq*)
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-le*:
fixes $Px :: 'b \Rightarrow real$ **and** $MX :: 'b\ measure$
assumes X[*measurable*]: *distributed M MX X Px* **and** *Px-nn*[*simp*]: $\bigwedge x.\ x \in space\ MX \Longrightarrow 0 \leq Px\ x$
and *fin*: *emeasure MX* $\{x \in space\ MX.\ Px\ x \neq 0\} \neq top$
and *int*: *integrable MX* $(\lambda x.\ -\ Px\ x * log\ b\ (Px\ x))$
shows *entropy b MX X* $\leq log\ b\ (measure\ MX\ \{x \in space\ MX.\ Px\ x \neq 0\})$
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-le-space*:
fixes $Px :: 'b \Rightarrow real$ **and** $MX :: 'b\ measure$
assumes *X*: *distributed M MX X Px* **and** *Px-nn*[*simp*]: $\bigwedge x.\ x \in space\ MX \Longrightarrow 0 \leq Px\ x$
and *fin*: *finite-measure MX*
and *int*: *integrable MX* $(\lambda x.\ -\ Px\ x * log\ b\ (Px\ x))$
shows *entropy b MX X* $\leq log\ b\ (measure\ MX\ (space\ MX))$
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-uniform*:
assumes *X*: *distributed M MX X* $(\lambda x.\ indicator\ A\ x\ /\ measure\ MX\ A)$ (**is** *distributed - - - ?f*)
shows *entropy b MX X* $= log\ b\ (measure\ MX\ A)$
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-simple-distributed*:
simple-distributed M X f $\Longrightarrow \mathcal{H}(X) = -\ (\sum x \in X`space\ M.\ f\ x * log\ b\ (f\ x))$
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-le-card-not-0*:
assumes *X*: *simple-distributed M X f*
shows $\mathcal{H}(X) \leq log\ b\ (card\ (X\ `\ space\ M \cap \{x.\ f\ x \neq 0\}))$
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-le-card*:
assumes *X*: *simple-distributed M X f*
shows $\mathcal{H}(X) \leq log\ b\ (real\ (card\ (X\ `\ space\ M)))$
⟨*proof*⟩

12.6 Conditional Mutual Information

definition (**in** *prob-space*)
conditional-mutual-information b MX MY MZ X Y Z ≡
mutual-information b MX $(MY \bigotimes_M MZ)$ *X* $(\lambda x.\ (Y\ x,\ Z\ x)) -$

mutual-information b MX MZ X Z

abbreviation (**in** *information-space*)
conditional-mutual-information-Pow ($\mathcal{I}'(\text{-};\text{-}\mid\text{-}')$) **where**
$\mathcal{I}(X\ ;\ Y\mid Z) \equiv$ *conditional-mutual-information b*
(*count-space* (*X ' space M*)) (*count-space* (*Y ' space M*)) (*count-space* (*Z ' space M*)) *X Y Z*

lemma (**in** *information-space*)
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T* **and** *P*: *sigma-finite-measure P*
assumes *Px*[*measurable*]: *distributed M S X Px*
and *Px-nn*[*simp*]: $\bigwedge x.\ x \in$ *space S* $\Longrightarrow$ *0* $\leq$ *Px x*
assumes *Pz*[*measurable*]: *distributed M P Z Pz*
and *Pz-nn*[*simp*]: $\bigwedge z.\ z \in$ *space P* $\Longrightarrow$ *0* $\leq$ *Pz z*
assumes *Pyz*[*measurable*]: *distributed M* $(T \bigotimes_M P)$ $(\lambda x.\ (Y\ x,\ Z\ x))$ *Pyz*
and *Pyz-nn*[*simp*]: $\bigwedge y\ z.\ y \in$ *space T* $\Longrightarrow$ $z \in$ *space P* $\Longrightarrow$ *0* $\leq$ *Pyz* $(y,\ z)$
assumes *Pxz*[*measurable*]: *distributed M* $(S \bigotimes_M P)$ $(\lambda x.\ (X\ x,\ Z\ x))$ *Pxz*
and *Pxz-nn*[*simp*]: $\bigwedge x\ z.\ x \in$ *space S* $\Longrightarrow$ $z \in$ *space P* $\Longrightarrow$ *0* $\leq$ *Pxz* $(x,\ z)$
assumes *Pxyz*[*measurable*]: *distributed M* $(S \bigotimes_M T \bigotimes_M P)$ $(\lambda x.\ (X\ x,\ Y\ x,\ Z\ x))$ *Pxyz*
and *Pxyz-nn*[*simp*]: $\bigwedge x\ y\ z.\ x \in$ *space S* $\Longrightarrow$ $y \in$ *space T* $\Longrightarrow$ $z \in$ *space P* $\Longrightarrow$ *0* $\leq$ *Pxyz* $(x,\ y,\ z)$
assumes *I1*: *integrable* $(S \bigotimes_M T \bigotimes_M P)$ $(\lambda(x,\ y,\ z).\ Pxyz\ (x,\ y,\ z) * log\ b\ (Pxyz\ (x,\ y,\ z)\ /\ (Px\ x * Pyz\ (y,\ z))))$
assumes *I2*: *integrable* $(S \bigotimes_M T \bigotimes_M P)$ $(\lambda(x,\ y,\ z).\ Pxyz\ (x,\ y,\ z) * log\ b\ (Pxz\ (x,\ z)\ /\ (Px\ x * Pz\ z)))$
shows *conditional-mutual-information-generic-eq*: *conditional-mutual-information b S T P X Y Z*
$= (\int (x,\ y,\ z).\ Pxyz\ (x,\ y,\ z) * log\ b\ (Pxyz\ (x,\ y,\ z)\ /\ (Pxz\ (x,\ z) * (Pyz\ (y,z)\ /\ Pz\ z)))\ \partial(S \bigotimes_M T \bigotimes_M P))$ (**is** *?eq*)
and *conditional-mutual-information-generic-nonneg*: *0* $\leq$ *conditional-mutual-information b S T P X Y Z* (**is** *?nonneg*)
⟨*proof*⟩

lemma (**in** *information-space*)
fixes *Px* :: *- ⇒ real*
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T* **and** *P*: *sigma-finite-measure P*
assumes *Fx*: *finite-entropy S X Px*
assumes *Fz*: *finite-entropy P Z Pz*
assumes *Fyz*: *finite-entropy* $(T \bigotimes_M P)$ $(\lambda x.\ (Y\ x,\ Z\ x))$ *Pyz*
assumes *Fxz*: *finite-entropy* $(S \bigotimes_M P)$ $(\lambda x.\ (X\ x,\ Z\ x))$ *Pxz*
assumes *Fxyz*: *finite-entropy* $(S \bigotimes_M T \bigotimes_M P)$ $(\lambda x.\ (X\ x,\ Y\ x,\ Z\ x))$ *Pxyz*
shows *conditional-mutual-information-generic-eq'*: *conditional-mutual-information b S T P X Y Z*
$= (\int (x,\ y,\ z).\ Pxyz\ (x,\ y,\ z) * log\ b\ (Pxyz\ (x,\ y,\ z)\ /\ (Pxz\ (x,\ z) * (Pyz\ (y,z)\ /\ Pz\ z)))\ \partial(S \bigotimes_M T \bigotimes_M P))$ (**is** *?eq*)
and *conditional-mutual-information-generic-nonneg'*: *0* $\leq$ *conditional-mutual-information*

b S T P X Y Z (**is** *?nonneg*)
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-mutual-information-eq*:
assumes *Pz*: *simple-distributed M Z Pz*
assumes *Pyz*: *simple-distributed M* (λ*x*. (*Y x*, *Z x*)) *Pyz*
assumes *Pxz*: *simple-distributed M* (λ*x*. (*X x*, *Z x*)) *Pxz*
assumes *Pxyz*: *simple-distributed M* (λ*x*. (*X x*, *Y x*, *Z x*)) *Pxyz*
shows $\mathcal{I}(X ; Y \mid Z) =$
(∑ (*x*, *y*, *z*)∈(λ*x*. (*X x*, *Y x*, *Z x*))‘*space M*. *Pxyz* (*x*, *y*, *z*) ∗ *log b* (*Pxyz* (*x*, *y*, *z*) / (*Pxz* (*x*, *z*) ∗ (*Pyz* (*y*,*z*) / *Pz z*))))
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-mutual-information-nonneg*:
assumes *X*: *simple-function M X* **and** *Y*: *simple-function M Y* **and** *Z*: *simple-function M Z*
shows $0 \leq \mathcal{I}(X ; Y \mid Z)$
⟨*proof*⟩

12.7 Conditional Entropy

definition (**in** *prob-space*)
conditional-entropy b S T X Y = − (∫ (*x*, *y*). *log b* (*enn2real* (*RN-deriv* (*S* ⨂$_M$ *T*) (*distr M* (*S* ⨂$_M$ *T*) (λ*x*. (*X x*, *Y x*))) (*x*, *y*)) /
enn2real (*RN-deriv T* (*distr M T Y*) *y*)) ∂*distr M* (*S* ⨂$_M$ *T*) (λ*x*. (*X x*, *Y x*)))

abbreviation (**in** *information-space*)
conditional-entropy-Pow ($\mathcal{H}'$(- | -')) **where**
$\mathcal{H}(X \mid Y)$ ≡ *conditional-entropy b* (*count-space* (*X*‘*space M*)) (*count-space* (*Y*‘*space M*)) *X Y*

lemma (**in** *information-space*) *conditional-entropy-generic-eq*:
fixes *Pxy* :: - ⇒ *real* **and** *Py* :: *'c* ⇒ *real*
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*
assumes *Py*[*measurable*]: *distributed M T Y Py* **and** *Py-nn*[*simp*]: ⋀*x*. *x* ∈ *space T* ⟹ 0 ≤ *Py x*
assumes *Pxy*[*measurable*]: *distributed M* (*S* ⨂$_M$ *T*) (λ*x*. (*X x*, *Y x*)) *Pxy*
and *Pxy-nn*[*simp*]: ⋀*x y*. *x* ∈ *space S* ⟹ *y* ∈ *space T* ⟹ 0 ≤ *Pxy* (*x*, *y*)
shows *conditional-entropy b S T X Y* = − (∫ (*x*, *y*). *Pxy* (*x*, *y*) ∗ *log b* (*Pxy* (*x*, *y*) / *Py y*) ∂(*S* ⨂$_M$ *T*))
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-entropy-eq-entropy*:
fixes *Px* :: *'b* ⇒ *real* **and** *Py* :: *'c* ⇒ *real*
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*
assumes *Py*[*measurable*]: *distributed M T Y Py*
and *Py-nn*[*simp*]: ⋀*x*. *x* ∈ *space T* ⟹ 0 ≤ *Py x*
assumes *Pxy*[*measurable*]: *distributed M* (*S* ⨂$_M$ *T*) (λ*x*. (*X x*, *Y x*)) *Pxy*

and *Pxy-nn*[*simp*]: $\bigwedge x\ y.\ x \in space\ S \Longrightarrow y \in space\ T \Longrightarrow 0 \leq Pxy\ (x,\ y)$
assumes *I1*: *integrable* $(S \bigotimes_M T)\ (\lambda x.\ Pxy\ x * log\ b\ (Pxy\ x))$
assumes *I2*: *integrable* $(S \bigotimes_M T)\ (\lambda x.\ Pxy\ x * log\ b\ (Py\ (snd\ x)))$
shows *conditional-entropy b S T X Y* = *entropy b* $(S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))$ − *entropy b T Y*
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-entropy-eq-entropy-simple*:
assumes *X*: *simple-function M X* **and** *Y*: *simple-function M Y*
shows $\mathcal{H}(X \mid Y)$ = *entropy b* (*count-space* (*X*'*space M*) $\bigotimes_M$ *count-space* (*Y*'*space M*)) $(\lambda x.\ (X\ x,\ Y\ x))$ − $\mathcal{H}(Y)$
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-entropy-eq*:
assumes *Y*: *simple-distributed M Y Py*
assumes *XY*: *simple-distributed M* $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
shows $\mathcal{H}(X \mid Y) = -\ (\sum (x,\ y) \in (\lambda x.\ (X\ x,\ Y\ x))\ `\ space\ M.\ Pxy\ (x,\ y) * log\ b\ (Pxy\ (x,\ y)\ /\ Py\ y))$
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-mutual-information-eq-conditional-entropy*:
assumes *X*: *simple-function M X* **and** *Y*: *simple-function M Y*
shows $\mathcal{I}(X\ ;\ X \mid Y) = \mathcal{H}(X \mid Y)$
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-entropy-nonneg*:
assumes *X*: *simple-function M X* **and** *Y*: *simple-function M Y* **shows** $0 \leq \mathcal{H}(X \mid Y)$
⟨*proof*⟩

12.8 Equalities

lemma (**in** *information-space*) *mutual-information-eq-entropy-conditional-entropy-distr*:
fixes *Px* :: $'b \Rightarrow real$ **and** *Py* :: $'c \Rightarrow real$ **and** *Pxy* :: $('b \times 'c) \Rightarrow real$
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*
assumes *Px*[*measurable*]: *distributed M S X Px*
and *Px-nn*[*simp*]: $\bigwedge x.\ x \in space\ S \Longrightarrow 0 \leq Px\ x$
and *Py*[*measurable*]: *distributed M T Y Py*
and *Py-nn*[*simp*]: $\bigwedge x.\ x \in space\ T \Longrightarrow 0 \leq Py\ x$
and *Pxy*[*measurable*]: *distributed M* $(S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
and *Pxy-nn*[*simp*]: $\bigwedge x\ y.\ x \in space\ S \Longrightarrow y \in space\ T \Longrightarrow 0 \leq Pxy\ (x,\ y)$
assumes *Ix*: *integrable*$(S \bigotimes_M T)\ (\lambda x.\ Pxy\ x * log\ b\ (Px\ (fst\ x)))$
assumes *Iy*: *integrable*$(S \bigotimes_M T)\ (\lambda x.\ Pxy\ x * log\ b\ (Py\ (snd\ x)))$
assumes *Ixy*: *integrable*$(S \bigotimes_M T)\ (\lambda x.\ Pxy\ x * log\ b\ (Pxy\ x))$
shows *mutual-information b S T X Y* = *entropy b S X* + *entropy b T Y* − *entropy b* $(S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))$
⟨*proof*⟩

lemma (**in** *information-space*) *mutual-information-eq-entropy-conditional-entropy'*:

fixes $Px :: 'b \Rightarrow real$ **and** $Py :: 'c \Rightarrow real$ **and** $Pxy :: ('b \times 'c) \Rightarrow real$
assumes S: *sigma-finite-measure* S **and** T: *sigma-finite-measure* T
assumes Px: *distributed* $M\ S\ X\ Px$ $\bigwedge x.\ x \in space\ S \Longrightarrow 0 \leq Px\ x$
and Py: *distributed* $M\ T\ Y\ Py$ $\bigwedge x.\ x \in space\ T \Longrightarrow 0 \leq Py\ x$
assumes Pxy: *distributed* $M\ (S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))\ Pxy$
$\bigwedge x.\ x \in space\ (S \bigotimes_M T) \Longrightarrow 0 \leq Pxy\ x$
assumes Ix: *integrable*$(S \bigotimes_M T)\ (\lambda x.\ Pxy\ x * log\ b\ (Px\ (fst\ x)))$
assumes Iy: *integrable*$(S \bigotimes_M T)\ (\lambda x.\ Pxy\ x * log\ b\ (Py\ (snd\ x)))$
assumes Ixy: *integrable*$(S \bigotimes_M T)\ (\lambda x.\ Pxy\ x * log\ b\ (Pxy\ x))$
shows *mutual-information* $b\ S\ T\ X\ Y = entropy\ b\ S\ X - conditional\text{-}entropy\ b\ S\ T\ X\ Y$
⟨*proof*⟩

lemma (**in** *information-space*) *mutual-information-eq-entropy-conditional-entropy*:
assumes *sf-X*: *simple-function* $M\ X$ **and** *sf-Y*: *simple-function* $M\ Y$
shows $\mathcal{I}(X\ ;\ Y) = \mathcal{H}(X) - \mathcal{H}(X \mid Y)$
⟨*proof*⟩

lemma (**in** *information-space*) *mutual-information-nonneg-simple*:
assumes *sf-X*: *simple-function* $M\ X$ **and** *sf-Y*: *simple-function* $M\ Y$
shows $0 \leq \mathcal{I}(X\ ;\ Y)$
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-entropy-less-eq-entropy*:
assumes X: *simple-function* $M\ X$ **and** Z: *simple-function* $M\ Z$
shows $\mathcal{H}(X \mid Z) \leq \mathcal{H}(X)$
⟨*proof*⟩

lemma (**in** *information-space*)
fixes $Px :: 'b \Rightarrow real$ **and** $Py :: 'c \Rightarrow real$ **and** $Pxy :: ('b \times 'c) \Rightarrow real$
assumes S: *sigma-finite-measure* S **and** T: *sigma-finite-measure* T
assumes Px: *finite-entropy* $S\ X\ Px$ **and** Py: *finite-entropy* $T\ Y\ Py$
assumes Pxy: *finite-entropy* $(S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))\ Pxy$
shows *conditional-entropy* $b\ S\ T\ X\ Y \leq entropy\ b\ S\ X$
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-chain-rule*:
assumes X: *simple-function* $M\ X$ **and** Y: *simple-function* $M\ Y$
shows $\mathcal{H}(\lambda x.\ (X\ x,\ Y\ x)) = \mathcal{H}(X) + \mathcal{H}(Y|X)$
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-partition*:
assumes X: *simple-function* $M\ X$
shows $\mathcal{H}(X) = \mathcal{H}(f \circ X) + \mathcal{H}(X|f \circ X)$
⟨*proof*⟩

corollary (**in** *information-space*) *entropy-data-processing*:
assumes X: *simple-function* $M\ X$ **shows** $\mathcal{H}(f \circ X) \leq \mathcal{H}(X)$
⟨*proof*⟩

corollary (**in** *information-space*) *entropy-of-inj*:
assumes *X*: *simple-function M X* **and** *inj*: *inj-on f* (*X'space M*)
shows $\mathcal{H}(f \circ X) = \mathcal{H}(X)$
⟨*proof*⟩

end

13 Properties of Various Distributions

theory *Distributions*
imports *Convolution Information*
begin

lemma (**in** *prob-space*) *distributed-affine*:
fixes *f* :: *real* ⇒ *ennreal*
assumes *f*: *distributed M lborel X f*
assumes *c*: $c \neq 0$
shows *distributed M lborel* (λ*x*. *t* + *c* ∗ *X x*) (λ*x*. *f* ((*x* − *t*) / *c*) / |*c*|)
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-affineI*:
fixes *f* :: *real* ⇒ *ennreal* **and** *c* :: *real*
assumes *f*: *distributed M lborel* (λ*x*. (*X x* − *t*) / *c*) (λ*x*. |*c*| ∗ *f* (*x* ∗ *c* + *t*))
assumes *c*: $c \neq 0$
shows *distributed M lborel X f*
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-AE2*:
assumes [*measurable*]: *distributed M N X f Measurable.pred N P*
shows (*AE x in M*. *P* (*X x*)) ⟷ (*AE x in N*. 0 < *f x* ⟶ *P x*)
⟨*proof*⟩

13.1 Erlang

lemma *nn-intergal-power-times-exp-Icc*:
assumes [*arith*]: $0 \leq a$
shows ($\int^+ x$. *ennreal* (*x*^*k* ∗ *exp* (−*x*)) ∗ *indicator* {0 .. *a*} *x* ∂*lborel*) =
(1 − (∑ *n*≤*k*. (*a*^*n* ∗ *exp* (−*a*)) / *fact n*)) ∗ *fact k* (**is** *?I* = -)
⟨*proof*⟩

lemma *nn-intergal-power-times-exp-Ici*:
shows ($\int^+ x$. *ennreal* (*x*^*k* ∗ *exp* (−*x*)) ∗ *indicator* {0 ..} *x* ∂*lborel*) = *real-of-nat* (*fact k*)
⟨*proof*⟩

definition *erlang-density* :: *nat* ⇒ *real* ⇒ *real* ⇒ *real* **where**
erlang-density k l x = (*if x* < 0 *then* 0 *else* (*l*^(*Suc k*) ∗ *x*^*k* ∗ *exp* (− *l* ∗ *x*)) / *fact k*)

definition *erlang-CDF* :: *nat* ⇒ *real* ⇒ *real* ⇒ *real* **where**
erlang-CDF k l x = (*if x* < *0 then 0 else 1* − (∑ *n*≤*k*. ((*l* ∗ *x*) ^ *n* ∗ *exp* (− *l* ∗ *x*) / *fact n*)))

lemma *erlang-density-nonneg*[*simp*]: *0* ≤ *l* ⟹ *0* ≤ *erlang-density k l x*
⟨*proof*⟩

lemma *borel-measurable-erlang-density*[*measurable*]: *erlang-density k l* ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *erlang-CDF-transform*: *0* < *l* ⟹ *erlang-CDF k l a* = *erlang-CDF k 1* (*l* ∗ *a*)
⟨*proof*⟩

lemma *erlang-CDF-nonneg*[*simp*]: **assumes** *0* < *l* **shows** *0* ≤ *erlang-CDF k l x*
⟨*proof*⟩

lemma *nn-integral-erlang-density*:
assumes [*arith*]: *0* < *l*
shows (∫⁺ *x*. *ennreal* (*erlang-density k l x*) ∗ *indicator* {.. *a*} *x* ∂*lborel*) = *erlang-CDF k l a*
⟨*proof*⟩

lemma *emeasure-erlang-density*:
0 < *l* ⟹ *emeasure* (*density lborel* (*erlang-density k l*)) {.. *a*} = *erlang-CDF k l a*
⟨*proof*⟩

lemma *nn-integral-erlang-ith-moment*:
fixes *k i* :: *nat* **and** *l* :: *real*
assumes [*arith*]: *0* < *l*
shows (∫⁺ *x*. *ennreal* (*erlang-density k l x* ∗ *x* ^ *i*) ∂*lborel*) = *fact* (*k* + *i*) / (*fact k* ∗ *l* ^ *i*)
⟨*proof*⟩

lemma *prob-space-erlang-density*:
assumes *l*[*arith*]: *0* < *l*
shows *prob-space* (*density lborel* (*erlang-density k l*)) (**is** *prob-space ?D*)
⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-distributed-le*:
assumes *D*: *distributed M lborel X* (*erlang-density k l*)
assumes [*simp*, *arith*]: *0* < *l 0* ≤ *a*
shows 𝒫(*x in M*. *X x* ≤ *a*) = *erlang-CDF k l a*
⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-distributed-gt*:

assumes *D*[*simp*]: *distributed M lborel X* (*erlang-density k l*)
assumes [*arith*]: $0 < l\ 0 \leq a$
shows $\mathcal{P}(x \text{ in } M.\ a < X\ x\ ) = 1 - (\text{erlang-CDF } k\ l\ a)$
⟨*proof*⟩

lemma *erlang-CDF-at0*: *erlang-CDF k l 0* = *0*
⟨*proof*⟩

lemma *erlang-distributedI*:
assumes *X*[*measurable*]: $X \in$ *borel-measurable M* **and** [*arith*]: $0 < l$
and *X-distr*: $\bigwedge a.\ 0 \leq a \Longrightarrow$ *emeasure M* {*x*∈*space M*. *X x* ≤ *a*} = *erlang-CDF k l a*
shows *distributed M lborel X* (*erlang-density k l*)
⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-distributed-iff*:
assumes [*arith*]: *0*<*l*
shows *distributed M lborel X* (*erlang-density k l*) ⟷
($X \in$ *borel-measurable M* $\wedge$ $0 < l \wedge$ ($\forall a \geq 0.\ \mathcal{P}(x \text{ in } M.\ X\ x \leq a) =$ *erlang-CDF k l a*))
⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-distributed-mult-const*:
assumes *erlX*: *distributed M lborel X* (*erlang-density k l*)
assumes *a-pos*[*arith*]: $0 < \alpha$ $0 < l$
shows *distributed M lborel* ($\lambda x.\ \alpha * X\ x$) (*erlang-density k* (*l* / α))
⟨*proof*⟩

lemma (**in** *prob-space*) *has-bochner-integral-erlang-ith-moment*:
fixes *k i* :: *nat* **and** *l* :: *real*
assumes [*arith*]: $0 < l$ **and** *D*: *distributed M lborel X* (*erlang-density k l*)
shows *has-bochner-integral M* ($\lambda x.\ X\ x\ \hat{}\ i$) (*fact* (*k* + *i*) / (*fact k* ∗ *l* $\hat{}$ *i*))
⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-ith-moment-integrable*:
$0 < l \Longrightarrow$ *distributed M lborel X* (*erlang-density k l*) $\Longrightarrow$ *integrable M* ($\lambda x.\ X\ x\ \hat{}\ i$)
⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-ith-moment*:
$0 < l \Longrightarrow$ *distributed M lborel X* (*erlang-density k l*) $\Longrightarrow$
expectation ($\lambda x.\ X\ x\ \hat{}\ i$) = *fact* (*k* + *i*) / (*fact k* ∗ *l* $\hat{}$ *i*)
⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-distributed-variance*:
assumes [*arith*]: $0 < l$ **and** *distributed M lborel X* (*erlang-density k l*)
shows *variance X* = (*k* + *1*) / l^2
⟨*proof*⟩

13.2 Exponential distribution

abbreviation *exponential-density* :: *real* ⇒ *real* ⇒ *real* **where**
exponential-density ≡ *erlang-density 0*

lemma *exponential-density-def*:
exponential-density l x = (*if x* < *0 then 0 else l* ∗ *exp* (− *x* ∗ *l*))
⟨*proof*⟩

lemma *erlang-CDF-0*: *erlang-CDF 0 l a* = (*if 0* ≤ *a then 1* − *exp* (− *l* ∗ *a*) *else 0*)
⟨*proof*⟩

lemma *prob-space-exponential-density*: *0* < *l* ⟹ *prob-space* (*density lborel* (*exponential-density l*))
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributedD-le*:
assumes *D*: *distributed M lborel X* (*exponential-density l*) **and** *a*: *0* ≤ *a* **and** *l*: *0* < *l*
shows $\mathcal{P}$(*x in M*. *X x* ≤ *a*) = *1* − *exp* (− *a* ∗ *l*)
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributedD-gt*:
assumes *D*: *distributed M lborel X* (*exponential-density l*) **and** *a*: *0* ≤ *a* **and** *l*: *0* < *l*
shows $\mathcal{P}$(*x in M*. *a* < *X x*) = *exp* (− *a* ∗ *l*)
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributed-memoryless*:
assumes *D*: *distributed M lborel X* (*exponential-density l*) **and** *a*: *0* ≤ *a* **and** *l*: *0* < *l* **and** *t*: *0* ≤ *t*
shows $\mathcal{P}$(*x in M*. *a* + *t* < *X x* | *a* < *X x*) = $\mathcal{P}$(*x in M*. *t* < *X x*)
⟨*proof*⟩

lemma *exponential-distributedI*:
assumes *X*[*measurable*]: *X* ∈ *borel-measurable M* **and** [*arith*]: *0* < *l*
and *X-distr*: ⋀*a*. *0* ≤ *a* ⟹ *emeasure M* {*x*∈*space M*. *X x* ≤ *a*} = *1* − *exp* (− *a* ∗ *l*)
shows *distributed M lborel X* (*exponential-density l*)
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributed-iff*:
assumes *0* < *l*
shows *distributed M lborel X* (*exponential-density l*) ⟷
(*X* ∈ *borel-measurable M* ∧ (∀ *a*≥*0*. $\mathcal{P}$(*x in M*. *X x* ≤ *a*) = *1* − *exp* (− *a* ∗ *l*)))
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributed-expectation*:

0 < l ⟹ distributed M lborel X (exponential-density l) ⟹ expectation X = 1 / l
⟨*proof*⟩

lemma *exponential-density-nonneg*: *0 < l ⟹ 0 ≤ exponential-density l x*
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributed-min*:
assumes *0 < l 0 < u*
assumes *expX*: *distributed M lborel X (exponential-density l)*
assumes *expY*: *distributed M lborel Y (exponential-density u)*
assumes *ind*: *indep-var borel X borel Y*
shows *distributed M lborel (λx. min (X x) (Y x)) (exponential-density (l + u))*
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributed-Min*:
assumes *finI*: *finite I*
assumes *A*: *I ≠ {}*
assumes *l*: *⋀i. i ∈ I ⟹ 0 < l i*
assumes *expX*: *⋀i. i ∈ I ⟹ distributed M lborel (X i) (exponential-density (l i))*
assumes *ind*: *indep-vars (λi. borel) X I*
shows *distributed M lborel (λx. Min ((λi. X i x)`I)) (exponential-density (∑ i∈I. l i))*
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributed-variance*:
0 < l ⟹ distributed M lborel X (exponential-density l) ⟹ variance X = 1 / l²
⟨*proof*⟩

lemma *nn-integral-zero'*: *AE x in M. f x = 0 ⟹ (∫⁺x. f x ∂M) = 0*
⟨*proof*⟩

lemma *convolution-erlang-density*:
fixes *k₁ k₂* :: *nat*
assumes [*simp*, *arith*]: *0 < l*
shows *(λx. ∫⁺y. ennreal (erlang-density k₁ l (x − y)) ∗ ennreal (erlang-density k₂ l y) ∂lborel) =*
(erlang-density (Suc k₁ + Suc k₂ − 1) l)
*(***is** *?LHS = ?RHS)*
⟨*proof*⟩

lemma (**in** *prob-space*) *sum-indep-erlang*:
assumes *indep*: *indep-var borel X borel Y*
assumes [*simp*, *arith*]: *0 < l*
assumes *erlX*: *distributed M lborel X (erlang-density k₁ l)*
assumes *erlY*: *distributed M lborel Y (erlang-density k₂ l)*
shows *distributed M lborel (λx. X x + Y x) (erlang-density (Suc k₁ + Suc k₂ − 1) l)*

⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-distributed-sum*:
assumes *finI* : *finite I*
assumes *A*: $I \neq \{\}$
assumes [*simp*, *arith*]: $0 < l$
assumes *expX*: $\bigwedge i.\ i \in I \Longrightarrow$ *distributed M lborel* (*X i*) (*erlang-density* (*k i*) *l*)
assumes *ind*: *indep-vars* ($\lambda i.$ *borel*) *X I*
shows *distributed M lborel* ($\lambda x.\ \sum i{\in}I.\ X\ i\ x$) (*erlang-density* ((∑ *i*∈*I*. *Suc* (*k i*)) − *1*) *l*)
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributed-sum*:
assumes *finI*: *finite I*
assumes *A*: $I \neq \{\}$
assumes *l*: $0 < l$
assumes *expX*: $\bigwedge i.\ i \in I \Longrightarrow$ *distributed M lborel* (*X i*) (*exponential-density l*)
assumes *ind*: *indep-vars* ($\lambda i.$ *borel*) *X I*
shows *distributed M lborel* ($\lambda x.\ \sum i{\in}I.\ X\ i\ x$) (*erlang-density* ((*card I*) − *1*) *l*)
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-exponential*:
assumes *l*[*simp*, *arith*]: $0 < l$
assumes *D*: *distributed M lborel X* (*exponential-density l*)
shows *entropy b lborel X* = *log b* (*exp 1* / *l*)
⟨*proof*⟩

13.3 Uniform distribution

lemma *uniform-distrI*:
assumes *X*: $X \in$ *measurable M M'*
and *A*: $A \in$ *sets M'* *emeasure M' A* $\neq \infty$ *emeasure M' A* $\neq 0$
assumes *distr*: $\bigwedge B.\ B \in$ *sets M'* $\Longrightarrow$ *emeasure M* (*X* −‘ *B* ∩ *space M*) = *emeasure M'* (*A* ∩ *B*) / *emeasure M' A*
shows *distr M M' X* = *uniform-measure M' A*
⟨*proof*⟩

lemma *uniform-distrI-borel*:
fixes *A* :: *real set*
assumes *X*[*measurable*]: $X \in$ *borel-measurable M* **and** *A*: *emeasure lborel A* = *ennreal r* $0 < r$
and [*measurable*]: $A \in$ *sets borel*
assumes *distr*: $\bigwedge a.$ *emeasure M* {*x*∈*space M*. *X x* ≤ *a*} = *emeasure lborel* (*A* ∩ {.. *a*}) / *r*
shows *distributed M lborel X* ($\lambda x.$ *indicator A x* / *measure lborel A*)
⟨*proof*⟩

lemma (**in** *prob-space*) *uniform-distrI-borel-atLeastAtMost*:
fixes *a b* :: *real*

assumes *X*: $X \in$ *borel-measurable M* **and** $a < b$
assumes *distr*: $\bigwedge t.\ a \leq t \Longrightarrow t \leq b \Longrightarrow \mathcal{P}(x\ in\ M.\ X\ x \leq t) = (t - a)\ /\ (b - a)$
shows *distributed M lborel X* ($\lambda x.$ *indicator* $\{a..b\}$ *x / measure lborel* $\{a..b\}$)
⟨*proof*⟩

lemma (**in** *prob-space*) *uniform-distributed-measure*:
fixes *a b* :: *real*
assumes *D*: *distributed M lborel X* ($\lambda x.$ *indicator* $\{a\ ..\ b\}$ *x / measure lborel* $\{a\ ..\ b\}$)
assumes *t*: $a \leq t\ t \leq b$
shows $\mathcal{P}(x\ in\ M.\ X\ x \leq t) = (t - a)\ /\ (b - a)$
⟨*proof*⟩

lemma (**in** *prob-space*) *uniform-distributed-bounds*:
fixes *a b* :: *real*
assumes *D*: *distributed M lborel X* ($\lambda x.$ *indicator* $\{a\ ..\ b\}$ *x / measure lborel* $\{a\ ..\ b\}$)
shows $a < b$
⟨*proof*⟩

lemma (**in** *prob-space*) *uniform-distributed-iff*:
fixes *a b* :: *real*
shows *distributed M lborel X* ($\lambda x.$ *indicator* $\{a..b\}$ *x / measure lborel* $\{a..b\}$) $\longleftrightarrow$
$(X \in$ *borel-measurable M* $\wedge\ a < b \wedge (\forall\, t{\in}\{a\ ..\ b\}.\ \mathcal{P}(x\ in\ M.\ X\ x \leq t) = (t - a)\ /\ (b - a)))$
⟨*proof*⟩

lemma (**in** *prob-space*) *uniform-distributed-expectation*:
fixes *a b* :: *real*
assumes *D*: *distributed M lborel X* ($\lambda x.$ *indicator* $\{a\ ..\ b\}$ *x / measure lborel* $\{a\ ..\ b\}$)
shows *expectation X* $= (a + b)\ /\ 2$
⟨*proof*⟩

lemma (**in** *prob-space*) *uniform-distributed-variance*:
fixes *a b* :: *real*
assumes *D*: *distributed M lborel X* ($\lambda x.$ *indicator* $\{a\ ..\ b\}$ *x / measure lborel* $\{a\ ..\ b\}$)
shows *variance X* $= (b - a)^2\ /\ 12$
⟨*proof*⟩

13.4 Normal distribution

definition *normal-density* :: *real* ⇒ *real* ⇒ *real* ⇒ *real* **where**
normal-density $\mu\ \sigma\ x = 1\ /\ sqrt\ (2 * pi * \sigma^2) * exp\ (-(x - \mu)^2 / (2 * \sigma^2))$

abbreviation *std-normal-density* :: *real* ⇒ *real* **where**
std-normal-density ≡ *normal-density 0 1*

lemma *std-normal-density-def*: *std-normal-density x = (1 / sqrt (2 * pi)) * exp (− x² / 2)*
⟨*proof*⟩

lemma *normal-density-nonneg*[*simp*]: *0 ≤ normal-density μ σ x*
⟨*proof*⟩

lemma *normal-density-pos*: *0 < σ ⟹ 0 < normal-density μ σ x*
⟨*proof*⟩

lemma *borel-measurable-normal-density*[*measurable*]: *normal-density μ σ ∈ borel-measurable borel*
⟨*proof*⟩

lemma *gaussian-moment-0*:
*has-bochner-integral lborel (λx. indicator {0..} x *ᵣ exp (− x²)) (sqrt pi / 2)*
⟨*proof*⟩

lemma *gaussian-moment-1*:
*has-bochner-integral lborel (λx::real. indicator {0..} x *ᵣ (exp (− x²) * x)) (1 / 2)*
⟨*proof*⟩

lemma
fixes *k* :: *nat*
shows *gaussian-moment-even-pos*:
*has-bochner-integral lborel (λx::real. indicator {0..} x *ᵣ (exp (−x²)*x^(2 * k)))*
*((sqrt pi / 2) * (fact (2 * k) / (2 ^ (2 * k) * fact k)))*
(**is** *?even*)
and *gaussian-moment-odd-pos*:
*has-bochner-integral lborel (λx::real. indicator {0..} x *ᵣ (exp (−x²)*x^(2 * k + 1))) (fact k / 2)*
(**is** *?odd*)
⟨*proof*⟩

context
fixes *k* :: *nat* **and** *μ σ* :: *real* **assumes** [*arith*]: *0 < σ*
begin

lemma *normal-moment-even*:
*has-bochner-integral lborel (λx. normal-density μ σ x * (x − μ) ^ (2 * k)) (fact (2 * k) / ((2 / σ²)^k * fact k))*
⟨*proof*⟩

lemma *normal-moment-abs-odd*:
*has-bochner-integral lborel (λx. normal-density μ σ x * |x − μ|^(2 * k + 1)) (2^k * σ^(2 * k + 1) * fact k * sqrt (2 / pi))*

⟨*proof*⟩

lemma *normal-moment-odd*:
has-bochner-integral lborel (λ*x*. *normal-density μ σ x* ∗ (*x* − *μ*)^(*2* ∗ *k* + *1*)) *0*
⟨*proof*⟩

lemma *integral-normal-moment-even*:
integral^L *lborel* (λ*x*. *normal-density μ σ x* ∗ (*x* − *μ*)^(*2* ∗ *k*)) = *fact* (*2* ∗ *k*) / ((*2* / *σ*^{2})^*k* ∗ *fact k*)
⟨*proof*⟩

lemma *integral-normal-moment-abs-odd*:
integral^L *lborel* (λ*x*. *normal-density μ σ x* ∗ |*x* − *μ*|^(*2* ∗ *k* + *1*)) = *2* ^ *k* ∗ *σ* ^ (*2* ∗ *k* + *1*) ∗ *fact k* ∗ *sqrt* (*2* / *pi*)
⟨*proof*⟩

lemma *integral-normal-moment-odd*:
integral^L *lborel* (λ*x*. *normal-density μ σ x* ∗ (*x* − *μ*)^(*2* ∗ *k* + *1*)) = *0*
⟨*proof*⟩

end

context
fixes *σ* :: *real*
assumes *σ-pos*[*arith*]: *0* < *σ*
begin

lemma *normal-moment-nz-1*: *has-bochner-integral lborel* (λ*x*. *normal-density μ σ x* ∗ *x*) *μ*
⟨*proof*⟩

lemma *integral-normal-moment-nz-1*:
integral^L *lborel* (λ*x*. *normal-density μ σ x* ∗ *x*) = *μ*
⟨*proof*⟩

lemma *integrable-normal-moment-nz-1*: *integrable lborel* (λ*x*. *normal-density μ σ x* ∗ *x*)
⟨*proof*⟩

lemma *integrable-normal-moment*: *integrable lborel* (λ*x*. *normal-density μ σ x* ∗ (*x* − *μ*)^*k*)
⟨*proof*⟩

lemma *integrable-normal-moment-abs*: *integrable lborel* (λ*x*. *normal-density μ σ x* ∗ |*x* − *μ*|^*k*)
⟨*proof*⟩

lemma *integrable-normal-density*[*simp*, *intro*]: *integrable lborel* (*normal-density μ*

σ)
⟨*proof*⟩

lemma *integral-normal-density*[*simp*]: ($\int$ *x*. *normal-density* μ σ *x* ∂*lborel*) = *1*
⟨*proof*⟩

lemma *prob-space-normal-density*:
prob-space (*density lborel* (*normal-density* μ σ))
⟨*proof*⟩

end

context
fixes *k* :: *nat*
begin

lemma *std-normal-moment-even*:
has-bochner-integral lborel (λ*x*. *std-normal-density x* * *x* ^ (*2* * *k*)) (*fact* (*2* * *k*) / (*2*^*k* * *fact k*))
⟨*proof*⟩

lemma *std-normal-moment-abs-odd*:
has-bochner-integral lborel (λ*x*. *std-normal-density x* * |*x*|^(*2* * *k* + *1*)) (*sqrt* (*2*/*pi*) * *2*^*k* * *fact k*)
⟨*proof*⟩

lemma *std-normal-moment-odd*:
has-bochner-integral lborel (λ*x*. *std-normal-density x* * *x*^(*2* * *k* + *1*)) *0*
⟨*proof*⟩

lemma *integral-std-normal-moment-even*:
integral^L *lborel* (λ*x*. *std-normal-density x* * *x*^(*2**k)) = *fact* (*2* * *k*) / (*2*^*k* * *fact k*)
⟨*proof*⟩

lemma *integral-std-normal-moment-abs-odd*:
integral^L *lborel* (λ*x*. *std-normal-density x* * |*x*|^(*2* * *k* + *1*)) = *sqrt* (*2* / *pi*) * *2*^*k* * *fact k*
⟨*proof*⟩

lemma *integral-std-normal-moment-odd*:
integral^L *lborel* (λ*x*. *std-normal-density x* * *x*^(*2* * *k* + *1*)) = *0*
⟨*proof*⟩

lemma *integrable-std-normal-moment-abs*: *integrable lborel* (λ*x*. *std-normal-density x* * |*x*|^*k*)
⟨*proof*⟩

lemma *integrable-std-normal-moment*: *integrable lborel* (*λx. std-normal-density x * x^k*)
⟨*proof*⟩

end

lemma (**in** *prob-space*) *normal-density-affine*:
assumes *X*: *distributed M lborel X* (*normal-density μ σ*)
assumes [*simp*, *arith*]: *0 < σ α ≠ 0*
shows *distributed M lborel* (*λx. β + α * X x*) (*normal-density* (*β + α * μ*) (|*α*| * *σ*))
⟨*proof*⟩

lemma (**in** *prob-space*) *normal-standard-normal-convert*:
assumes *pos-var*[*simp*, *arith*]: *0 < σ*
shows *distributed M lborel X* (*normal-density μ σ*) = *distributed M lborel* (*λx.* (*X x − μ*) / *σ*) *std-normal-density*
⟨*proof*⟩

lemma *conv-normal-density-zero-mean*:
assumes [*simp*, *arith*]: *0 < σ 0 < τ*
shows (*λx.* ∫⁺*y. ennreal* (*normal-density 0 σ* (*x − y*) * *normal-density 0 τ y*) *∂lborel*) =
normal-density 0 (*sqrt* (*σ*² + *τ*²)) (**is** *?LHS = ?RHS*)
⟨*proof*⟩

lemma *conv-std-normal-density*:
(*λx.* ∫⁺*y. ennreal* (*std-normal-density* (*x − y*) * *std-normal-density y*) *∂lborel*) =
(*normal-density 0* (*sqrt 2*))
⟨*proof*⟩

lemma (**in** *prob-space*) *add-indep-normal*:
assumes *indep*: *indep-var borel X borel Y*
assumes *pos-var*[*arith*]: *0 < σ 0 < τ*
assumes *normalX*[*simp*]: *distributed M lborel X* (*normal-density μ σ*)
assumes *normalY*[*simp*]: *distributed M lborel Y* (*normal-density ν τ*)
shows *distributed M lborel* (*λx. X x + Y x*) (*normal-density* (*μ + ν*) (*sqrt* (*σ*² + *τ*²)))
⟨*proof*⟩

lemma (**in** *prob-space*) *diff-indep-normal*:
assumes *indep*[*simp*]: *indep-var borel X borel Y*
assumes [*simp*, *arith*]: *0 < σ 0 < τ*
assumes *normalX*[*simp*]: *distributed M lborel X* (*normal-density μ σ*)
assumes *normalY*[*simp*]: *distributed M lborel Y* (*normal-density ν τ*)
shows *distributed M lborel* (*λx. X x − Y x*) (*normal-density* (*μ − ν*) (*sqrt* (*σ*² + *τ*²)))

⟨*proof*⟩

lemma (**in** *prob-space*) *sum-indep-normal*:
assumes *finite I* $I \neq \{\}$ *indep-vars* ($\lambda i.$ *borel*) *X I*
assumes $\bigwedge i.\ i \in I \Longrightarrow 0 < \sigma\ i$
assumes *normal*: $\bigwedge i.\ i \in I \Longrightarrow$ *distributed M lborel* (*X i*) (*normal-density* (μ *i*) (σ *i*))
shows *distributed M lborel* ($\lambda x.\ \sum i{\in}I.\ X\ i\ x$) (*normal-density* ($\sum i{\in}I.\ \mu\ i$) (*sqrt* ($\sum i{\in}I.\ (\sigma\ i)^2$)))
⟨*proof*⟩

lemma (**in** *prob-space*) *standard-normal-distributed-expectation*:
assumes *D*: *distributed M lborel X std-normal-density*
shows *expectation X* = *0*
⟨*proof*⟩

lemma (**in** *prob-space*) *normal-distributed-expectation*:
assumes σ[*arith*]: $0 < \sigma$
assumes *D*: *distributed M lborel X* (*normal-density* μ σ)
shows *expectation X* = μ
⟨*proof*⟩

lemma (**in** *prob-space*) *normal-distributed-variance*:
fixes *a b* :: *real*
assumes [*simp*, *arith*]: $0 < \sigma$
assumes *D*: *distributed M lborel X* (*normal-density* μ σ)
shows *variance X* = σ^2
⟨*proof*⟩

lemma (**in** *prob-space*) *standard-normal-distributed-variance*:
distributed M lborel X std-normal-density $\Longrightarrow$ *variance X* = *1*
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-normal-density*:
assumes [*arith*]: $0 < \sigma$
assumes *D*: *distributed M lborel X* (*normal-density* μ σ)
shows *entropy b lborel X* = *log b* (*2* $*$ *pi* $*$ *exp 1* $*$ σ^2) / *2*
⟨*proof*⟩

end

14 Characteristic Functions

theory *Characteristic-Functions*
imports *Weak-Convergence Independent-Family Distributions*
begin

lemma *mult-min-right*: $a \geq 0 \Longrightarrow (a :: real) * min\ b\ c = min\ (a * b)\ (a * c)$
⟨*proof*⟩

lemma *sequentially-even-odd*:
assumes *E*: *eventually* (λ*n*. *P* (*2* ∗ *n*)) *sequentially* **and** *O*: *eventually* (λ*n*. *P* (*2* ∗ *n* + *1*)) *sequentially*
shows *eventually P sequentially*
⟨*proof*⟩

lemma *limseq-even-odd*:
assumes (λ*n*. *f* (*2* ∗ *n*)) ⟶ (*l* :: *'a* :: *topological-space*)
and (λ*n*. *f* (*2* ∗ *n* + *1*)) ⟶ *l*
shows *f* ⟶ *l*
⟨*proof*⟩

14.1 Application of the FTC: integrating $e^i x$

abbreviation *iexp* :: *real* ⇒ *complex* **where**
iexp ≡ (λ*x*. *exp* (i ∗ *complex-of-real x*))

lemma *isCont-iexp* [*simp*]: *isCont iexp x*
⟨*proof*⟩

lemma *has-vector-derivative-iexp*[*derivative-intros*]:
(*iexp has-vector-derivative* i ∗ *iexp x*) (*at x within s*)
⟨*proof*⟩

lemma *interval-integral-iexp*:
fixes *a b* :: *real*
shows (*CLBINT x*=*a*..*b*. *iexp x*) = i ∗ *iexp a* − i ∗ *iexp b*
⟨*proof*⟩

14.2 The Characteristic Function of a Real Measure.

definition
char :: *real measure* ⇒ *real* ⇒ *complex*
where
char M t = *CLINT x*|*M*. *iexp* (*t* ∗ *x*)

lemma (**in** *real-distribution*) *char-zero*: *char M 0* = *1*
⟨*proof*⟩

lemma (**in** *prob-space*) *integrable-iexp*:
assumes *f*: *f* ∈ *borel-measurable M* ⋀*x*. *Im* (*f x*) = *0*
shows *integrable M* (λ*x*. *exp* (i ∗ (*f x*)))
⟨*proof*⟩

lemma (**in** *real-distribution*) *cmod-char-le-1*: *norm* (*char M t*) ≤ *1*
⟨*proof*⟩

lemma (**in** *real-distribution*) *isCont-char*: *isCont* (*char M*) *t*
⟨*proof*⟩

lemma (**in** *real-distribution*) *char-measurable* [*measurable*]: *char M* ∈ *borel-measurable borel*
⟨*proof*⟩

14.3 Independence

lemma (**in** *prob-space*) *char-distr-add*:
fixes *X1 X2* :: *'a* ⇒ *real* **and** *t* :: *real*
assumes *indep-var borel X1 borel X2*
shows *char* (*distr M borel* (λω. *X1* ω + *X2* ω)) *t* =
char (*distr M borel X1*) *t* ∗ *char* (*distr M borel X2*) *t*
⟨*proof*⟩

lemma (**in** *prob-space*) *char-distr-sum*:
indep-vars (λ*i*. *borel*) *X A* ⟹
char (*distr M borel* (λω. ∑ *i*∈*A*. *X i* ω)) *t* = (∏ *i*∈*A*. *char* (*distr M borel* (*X i*)) *t*)
⟨*proof*⟩

14.4 Approximations to e^{ix}

Proofs from Billingsley, page 343.

lemma *CLBINT-I0c-power-mirror-iexp*:
fixes *x* :: *real* **and** *n* :: *nat*
defines *f s m* ≡ *complex-of-real* ((*x* − *s*) ^ *m*)
shows (*CLBINT s=0..x. f s n* ∗ *iexp s*) =
x^*Suc n* / *Suc n* + (i / *Suc n*) ∗ (*CLBINT s=0..x. f s* (*Suc n*) ∗ *iexp s*)
⟨*proof*⟩

lemma *iexp-eq1*:
fixes *x* :: *real*
defines *f s m* ≡ *complex-of-real* ((*x* − *s*) ^ *m*)
shows *iexp x* =
(∑ *k* ≤ *n*. (i ∗ *x*)^*k* / (*fact k*)) + ((i ^ (*Suc n*)) / (*fact n*)) ∗ (*CLBINT s=0..x.* (*f s n*) ∗ (*iexp s*)) (**is** *?P n*)
⟨*proof*⟩

lemma *iexp-eq2*:
fixes *x* :: *real*
defines *f s m* ≡ *complex-of-real* ((*x* − *s*) ^ *m*)
shows *iexp x* = (∑ *k*≤*Suc n*. (i∗*x*)^*k*/*fact k*) + i^*Suc n*/*fact n* ∗ (*CLBINT s=0..x. f s n*∗(*iexp s* − *1*))
⟨*proof*⟩

lemma *abs-LBINT-I0c-abs-power-diff*:
|*LBINT s=0..x.* |(*x* − *s*)^*n*|| = |*x* ^ (*Suc n*) / (*Suc n*)|
⟨*proof*⟩

lemma *iexp-approx1*: *cmod* (*iexp x* − (∑ *k* ≤ *n*. (i ∗ *x*)^*k* / *fact k*)) ≤ |*x*|^(*Suc n*) / *fact* (*Suc n*)
⟨*proof*⟩

lemma *iexp-approx2*: *cmod* (*iexp x* − (∑ *k* ≤ *n*. (i ∗ *x*)^*k* / *fact k*)) ≤ *2* ∗ |*x*|^*n* / *fact n*
⟨*proof*⟩

lemma (**in** *real-distribution*) *char-approx1*:
assumes *integrable-moments*: ⋀*k*. *k* ≤ *n* ⟹ *integrable M* (λ*x*. *x*^*k*)
shows *cmod* (*char M t* − (∑ *k* ≤ *n*. ((i ∗ *t*)^*k* / *fact k*) ∗ *expectation* (λ*x*. *x*^*k*)))
≤
(*2*∗|*t*|^*n* / *fact n*) ∗ *expectation* (λ*x*. |*x*|^*n*) (**is** *cmod* (*char M t* − *?t1*) ≤ -)
⟨*proof*⟩

lemma (**in** *real-distribution*) *char-approx2*:
assumes *integrable-moments*: ⋀*k*. *k* ≤ *n* ⟹ *integrable M* (λ*x*. *x* ^ *k*)
shows *cmod* (*char M t* − (∑ *k* ≤ *n*. ((i ∗ *t*)^*k* / *fact k*) ∗ *expectation* (λ*x*. *x*^*k*)))
≤
(|*t*|^*n* / *fact* (*Suc n*)) ∗ *expectation* (λ*x*. *min* (*2* ∗ |*x*|^*n* ∗ *Suc n*) (|*t*| ∗ |*x*|^*Suc n*))
(**is** *cmod* (*char M t* − *?t1*) ≤ -)
⟨*proof*⟩

lemma (**in** *real-distribution*) *char-approx3*:
fixes *t*
assumes
integrable-1: *integrable M* (λ*x*. *x*) **and**
integral-1: *expectation* (λ*x*. *x*) = *0* **and**
integrable-2: *integrable M* (λ*x*. *x*^*2*) **and**
integral-2: *variance* (λ*x*. *x*) = *σ2*
shows *cmod* (*char M t* − (*1* − *t*^*2* ∗ *σ2* / *2*)) ≤
(*t*^*2* / *6*) ∗ *expectation* (λ*x*. *min* (*6* ∗ *x*^*2*) (*abs t* ∗ (*abs x*)^*3*))
⟨*proof*⟩

This is a more familiar textbook formulation in terms of random variables, but we will use the previous version for the CLT.

lemma (**in** *prob-space*) *char-approx3′*:
fixes *μ* :: *real measure* **and** *X*
assumes *rv-X* [*simp*]: *random-variable borel X*
and [*simp*]: *integrable M X integrable M* (λ*x*. (*X x*)^*2*) *expectation X* = *0*
and *var-X*: *variance X* = *σ2*
and *μ-def*: *μ* = *distr M borel X*
shows *cmod* (*char μ t* − (*1* − *t*^*2* ∗ *σ2* / *2*)) ≤
(*t*^*2* / *6*) ∗ *expectation* (λ*x*. *min* (*6* ∗ (*X x*)^*2*) (|*t*| ∗ |*X x*|^*3*))
⟨*proof*⟩

this is the formulation in the book – in terms of a random variable *with* the distribution, rather the distribution itself. I don't know which is more

useful, though in principal we can go back and forth between them.

lemma (**in** *prob-space*) *char-approx1'*:
fixes μ :: *real measure* **and** X
assumes *integrable-moments* : $\bigwedge k.\ k \leq n \Longrightarrow$ *integrable* M $(\lambda x.\ X\ x\ \widehat{}\ k)$
and *rv-X*[*measurable*]: *random-variable borel* X
and μ*-distr* : *distr* M *borel* $X = \mu$
shows *cmod* (*char* μ t $-$ $(\sum k \leq n.\ ((\mathrm{i} * t)\widehat{}k\ /\ fact\ k) * expectation\ (\lambda x.\ (X\ x)\widehat{}k))) \leq$
$(2 * |t|\widehat{}n\ /\ fact\ n) * expectation\ (\lambda x.\ |X\ x|\widehat{}n)$
⟨*proof*⟩

14.5 Calculation of the Characteristic Function of the Standard Distribution

abbreviation
std-normal-distribution ≡ *density lborel std-normal-density*

lemma *real-dist-normal-dist*: *real-distribution std-normal-distribution*
⟨*proof*⟩

lemma *std-normal-distribution-even-moments*:
fixes k :: *nat*
shows (*LINT* x|*std-normal-distribution*. $x\widehat{}(2 * k)) = fact\ (2 * k)\ /\ (2\widehat{}k * fact\ k)$
and *integrable std-normal-distribution* $(\lambda x.\ x\widehat{}(2 * k))$
⟨*proof*⟩

lemma *integrable-std-normal-distribution-moment*: *integrable std-normal-distribution* $(\lambda x.\ x\widehat{}k)$
⟨*proof*⟩

lemma *integral-std-normal-distribution-moment-odd*:
odd $k \Longrightarrow$ *integral*^{L} *std-normal-distribution* $(\lambda x.\ x\widehat{}k) = 0$
⟨*proof*⟩

lemma *std-normal-distribution-even-moments-abs*:
fixes k :: *nat*
shows (*LINT* x|*std-normal-distribution*. $|x|\widehat{}(2 * k)) = fact\ (2 * k)\ /\ (2\widehat{}k * fact\ k)$
⟨*proof*⟩

lemma *std-normal-distribution-odd-moments-abs*:
fixes k :: *nat*
shows (*LINT* x|*std-normal-distribution*. $|x|\widehat{}(2 * k + 1)) = sqrt\ (2\ /\ pi) * 2\ \widehat{}\ k * fact\ k$
⟨*proof*⟩

theorem *char-std-normal-distribution*:

char std-normal-distribution = (λ*t*. *complex-of-real* (*exp* (− (*t*^2) / 2)))
⟨*proof*⟩

end

15 Helly's selection theorem

The set of bounded, monotone, right continuous functions is sequentially compact

theory *Helly-Selection*
imports *HOL−Library.Diagonal-Subsequence Weak-Convergence*
begin

lemma *minus-one-less*: *x* − 1 < (*x*::*real*)
⟨*proof*⟩

theorem *Helly-selection*:
fixes *f* :: *nat* ⇒ *real* ⇒ *real*
assumes *rcont*: ⋀*n x*. *continuous* (*at-right x*) (*f n*)
assumes *mono*: ⋀*n*. *mono* (*f n*)
assumes *bdd*: ⋀*n x*. |*f n x*| ≤ *M*
shows ∃ *s*. *strict-mono* (*s*::*nat* ⇒ *nat*) ∧ (∃ *F*. (∀ *x*. *continuous* (*at-right x*) *F*) ∧ *mono F* ∧ (∀ *x*. |*F x*| ≤ *M*) ∧
(∀ *x*. *continuous* (*at x*) *F* ⟶ (λ*n*. *f* (*s n*) *x*) ⟶⟶ *F x*))
⟨*proof*⟩

definition
tight :: (*nat* ⇒ *real measure*) ⇒ *bool*
where
tight μ ≡ (∀ *n*. *real-distribution* (μ *n*)) ∧ (∀ (ε::*real*)>*0*. ∃ *a b*::*real*. *a* < *b* ∧ (∀ *n*. *measure* (μ *n*) {*a*<..*b*} > 1 − ε))

theorem *tight-imp-convergent-subsubsequence*:
assumes μ: *tight* μ *strict-mono s*
shows ∃ *r M*. *strict-mono* (*r* :: *nat* ⇒ *nat*) ∧ *real-distribution M* ∧ *weak-conv-m* (μ ∘ *s* ∘ *r*) *M*
⟨*proof*⟩

corollary *tight-subseq-weak-converge*:
fixes μ :: *nat* ⇒ *real measure* **and** *M* :: *real measure*
assumes ⋀*n*. *real-distribution* (μ *n*) *real-distribution M* **and** *tight*: *tight* μ **and**
subseq: ⋀*s* ν. *strict-mono s* ⟹ *real-distribution* ν ⟹ *weak-conv-m* (μ ∘ *s*) ν ⟹ *weak-conv-m* (μ ∘ *s*) *M*
shows *weak-conv-m* μ *M*
⟨*proof*⟩

end

16 Integral of sinc

theory *Sinc-Integral*
imports *Distributions*
begin

16.1 Various preparatory integrals

Naming convention The theorem name consists of the following parts:

- Kind of integral: *has-bochner-integral* / *integrable* / *LBINT*
- Interval: Interval (0 / infinity / open / closed) (infinity / open / closed)
- Name of the occurring constants: power, exp, m (for minus), scale, sin, ...

lemma *has-bochner-integral-I0i-power-exp-m'*:
has-bochner-integral lborel ($\lambda x.\ x\hat{}k * exp\ (-x) * indicator\ \{0\ ..\}\ x$::*real*) (*fact k*)
⟨*proof*⟩

lemma *has-bochner-integral-I0i-power-exp-m*:
has-bochner-integral lborel ($\lambda x.\ x\hat{}k * exp\ (-x) * indicator\ \{0\ <..\}\ x$::*real*) (*fact k*)
⟨*proof*⟩

lemma *integrable-I0i-exp-mscale*: $0 < (u$::*real*$) \Longrightarrow$ *set-integrable lborel* $\{0\ <..\}$ ($\lambda x.\ exp\ (-(x * u))$)
⟨*proof*⟩

lemma *LBINT-I0i-exp-mscale*: $0 < (u$::*real*$) \Longrightarrow$ *LBINT* $x{=}0..\infty.\ exp\ (-(x * u)) = 1 / u$
⟨*proof*⟩

lemma *LBINT-I0c-exp-mscale-sin*:
LBINT $x{=}0..t.\ exp\ (-(u * x)) * sin\ x =$
$(1 / (1 + u\hat{}2)) * (1 - exp\ (-(u * t)) * (u * sin\ t + cos\ t))$ (**is** - = *?F t*)
⟨*proof*⟩

lemma *LBINT-I0i-exp-mscale-sin*:
assumes $0 < x$
shows *LBINT* $u{=}0..\infty.\ |exp\ (-u * x) * sin\ x| = |sin\ x| / x$
⟨*proof*⟩

lemma

shows *integrable-inverse-1-plus-square*:
set-integrable lborel (*einterval* $(-\infty)$ ∞) $(\lambda x.\ inverse\ (1 + x\hat{}2))$
and *LBINT-inverse-1-plus-square*:
LBINT $x{=}-\infty..\infty.\ inverse\ (1 + x\hat{}2) = pi$
⟨*proof*⟩

lemma
shows *integrable-I0i-1-div-plus-square*:
interval-lebesgue-integrable lborel $0\ \infty\ (\lambda x.\ 1\ /\ (1 + x\hat{}2))$
and *LBINT-I0i-1-div-plus-square*:
LBINT $x{=}0..\infty.\ 1\ /\ (1 + x\hat{}2) = pi\ /\ 2$
⟨*proof*⟩

17 The sinc function, and the sine integral (Si)

abbreviation *sinc* :: *real* ⇒ *real* **where**
sinc ≡ $(\lambda x.\ if\ x = 0\ then\ 1\ else\ sin\ x\ /\ x)$

lemma *sinc-at-0*: $((\lambda x.\ sin\ x\ /\ x{::}real) \longrightarrow 1)\ (at\ 0)$
⟨*proof*⟩

lemma *isCont-sinc*: *isCont sinc x*
⟨*proof*⟩

lemma *continuous-on-sinc*[*continuous-intros*]:
continuous-on S f ⟹ *continuous-on S* $(\lambda x.\ sinc\ (f\ x))$
⟨*proof*⟩

lemma *borel-measurable-sinc*[*measurable*]: *sinc* ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *sinc-AE*: *AE x in lborel. sin x / x = sinc x*
⟨*proof*⟩

definition *Si* :: *real* ⇒ *real* **where** *Si t* ≡ *LBINT x=0..t. sin x / x*

lemma *sinc-neg* [*simp*]: *sinc* $(-\ x) = sinc\ x$
⟨*proof*⟩

lemma *Si-alt-def* : *Si t = LBINT x=0..t. sinc x*
⟨*proof*⟩

lemma *Si-neg*:
assumes $T \geq 0$ **shows** *Si* $(-\ T) = -\ Si\ T$
⟨*proof*⟩

lemma *integrable-sinc′*:
interval-lebesgue-integrable lborel (*ereal 0*) (*ereal T*) $(\lambda t.\ sin\ (t * \vartheta)\ /\ t)$

⟨*proof*⟩

lemma *DERIV-Si*: (*Si has-real-derivative sinc x*) (*at x*)
⟨*proof*⟩

lemma *isCont-Si*: *isCont Si x*
⟨*proof*⟩

lemma *borel-measurable-Si*[*measurable*]: *Si* ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *Si-at-top-LBINT*:
((λ*t*. (*LBINT x=0..∞*. *exp* (−(*x* ∗ *t*)) ∗ (*x* ∗ *sin t* + *cos t*) / (*1* + *x*^*2*))) ⟶ *0*) *at-top*
⟨*proof*⟩

lemma *Si-at-top-integrable*:
assumes *t* ≥ *0*
shows *interval-lebesgue-integrable lborel 0* ∞ (λ*x*. *exp* (− (*x* ∗ *t*)) ∗ (*x* ∗ *sin t* + *cos t*) / (*1* + x^2))
⟨*proof*⟩

lemma *Si-at-top*: (*Si* ⟶ *pi* / *2*) *at-top*
⟨*proof*⟩

17.1 The final theorems: boundedness and scalability

lemma *bounded-Si*: ∃ *B*. ∀ *T*. |*Si T*| ≤ *B*
⟨*proof*⟩

lemma *LBINT-I0c-sin-scale-divide*:
assumes *T* ≥ *0*
shows *LBINT t=0..T*. *sin* (*t* ∗ *ϑ*) / *t* = *sgn ϑ* ∗ *Si* (*T* ∗ |*ϑ*|)
⟨*proof*⟩

end

18 The Levy inversion theorem, and the Levy continuity theorem.

theory *Levy*
imports *Characteristic-Functions Helly-Selection Sinc-Integral*
begin

18.1 The Levy inversion theorem

lemma *Levy-Inversion-aux1*:
fixes *a b* :: *real*

assumes $a \leq b$
shows $((\lambda t.\ (iexp\ (-(t * a)) - iexp\ (-(t * b))) / (\mathrm{i} * t)) \longrightarrow b - a)\ (at\ 0)$
(**is** $(?F \longrightarrow \text{-})\ (at\ \text{-})$)
⟨*proof*⟩

lemma *Levy-Inversion-aux2*:
fixes $a\ b\ t$ :: *real*
assumes $a \leq b$ **and** $t \neq 0$
shows $cmod\ ((iexp\ (t * b) - iexp\ (t * a)) / (\mathrm{i} * t)) \leq b - a$ (**is** $?F \leq \text{-}$)
⟨*proof*⟩

theorem (**in** *real-distribution*) *Levy-Inversion*:
fixes $a\ b$ :: *real*
assumes $a \leq b$
defines $\mu \equiv measure\ M$ **and** $\varphi \equiv char\ M$
assumes $\mu\ \{a\} = 0$ **and** $\mu\ \{b\} = 0$
shows $(\lambda T.\ 1\ /\ (2 * pi) * (CLBINT\ t{=}{-}T..T.\ (iexp\ (-(t * a)) - iexp\ (-(t * b))) / (\mathrm{i} * t) * \varphi\ t))$
$\longrightarrow \mu\ \{a{<}..b\}$
(**is** $(\lambda T.\ 1\ /\ (2 * pi) * (CLBINT\ t{=}{-}T..T.\ ?F\ t * \varphi\ t)) \longrightarrow of\text{-}real\ (\mu\ \{a{<}..b\})$)
⟨*proof*⟩

theorem *Levy-uniqueness*:
fixes *M1 M2* :: *real measure*
assumes *real-distribution M1 real-distribution M2* **and**
char M1 = *char M2*
shows *M1* = *M2*
⟨*proof*⟩

18.2 The Levy continuity theorem

theorem *levy-continuity1*:
fixes M :: $nat \Rightarrow real\ measure$ **and** M' :: *real measure*
assumes $\bigwedge n.\ real\text{-}distribution\ (M\ n)\ real\text{-}distribution\ M'\ weak\text{-}conv\text{-}m\ M\ M'$
shows $(\lambda n.\ char\ (M\ n)\ t) \longrightarrow char\ M'\ t$
⟨*proof*⟩

theorem *levy-continuity*:
fixes M :: $nat \Rightarrow real\ measure$ **and** M' :: *real measure*
assumes *real-distr-M* : $\bigwedge n.\ real\text{-}distribution\ (M\ n)$
and *real-distr-M'*: *real-distribution M'*
and *char-conv*: $\bigwedge t.\ (\lambda n.\ char\ (M\ n)\ t) \longrightarrow char\ M'\ t$
shows *weak-conv-m M M'*
⟨*proof*⟩

end

19 The Central Limit Theorem

theory *Central-Limit-Theorem*
imports *Levy*
begin

theorem (**in** *prob-space*) *central-limit-theorem-zero-mean*:
fixes $X :: nat \Rightarrow 'a \Rightarrow real$
and $\mu :: real\ measure$
and $\sigma :: real$
and $S :: nat \Rightarrow 'a \Rightarrow real$
assumes *X-indep*: *indep-vars* $(\lambda i.\ borel)\ X\ UNIV$
and *X-mean-0*: $\bigwedge n.\ expectation\ (X\ n) = 0$
and σ*-pos*: $\sigma > 0$
and *X-square-integrable*: $\bigwedge n.\ integrable\ M\ (\lambda x.\ (X\ n\ x)^2)$
and *X-variance*: $\bigwedge n.\ variance\ (X\ n) = \sigma^2$
and *X-distrib*: $\bigwedge n.\ distr\ M\ borel\ (X\ n) = \mu$
defines $S\ n \equiv \lambda x.\ \sum i{<}n.\ X\ i\ x$
shows *weak-conv-m* $(\lambda n.\ distr\ M\ borel\ (\lambda x.\ S\ n\ x\ /\ sqrt\ (n * \sigma^2)))$ *std-normal-distribution*
⟨*proof*⟩

theorem (**in** *prob-space*) *central-limit-theorem*:
fixes $X :: nat \Rightarrow 'a \Rightarrow real$
and $\mu :: real\ measure$
and $\sigma :: real$
and $S :: nat \Rightarrow 'a \Rightarrow real$
assumes *X-indep*: *indep-vars* $(\lambda i.\ borel)\ X\ UNIV$
and *X-mean*: $\bigwedge n.\ expectation\ (X\ n) = m$
and σ*-pos*: $\sigma > 0$
and *X-square-integrable*: $\bigwedge n.\ integrable\ M\ (\lambda x.\ (X\ n\ x)^2)$
and *X-variance*: $\bigwedge n.\ variance\ (X\ n) = \sigma^2$
and *X-distrib*: $\bigwedge n.\ distr\ M\ borel\ (X\ n) = \mu$
defines $X'\ i\ x \equiv X\ i\ x - m$
shows *weak-conv-m* $(\lambda n.\ distr\ M\ borel\ (\lambda x.\ (\sum i{<}n.\ X'\ i\ x)\ /\ sqrt\ (n{*}\sigma^2)))$
std-normal-distribution
⟨*proof*⟩

end

theory *Discrete-Topology*
imports *HOL−Analysis.Analysis*
begin

Copy of discrete types with discrete topology. This space is polish.

typedef *'a discrete* = *UNIV*::*'a set*
morphisms *of-discrete discrete*
⟨*proof*⟩

instantiation *discrete* :: (*type*) *metric-space*
begin

definition *dist-discrete* :: *'a discrete* ⇒ *'a discrete* ⇒ *real*
  where *dist-discrete n m* = (*if n* = *m then 0 else 1*)

definition *uniformity-discrete* :: (*'a discrete* × *'a discrete*) *filter* **where**
  (*uniformity*::(*'a discrete* × *'a discrete*) *filter*) = (*INF e*∈{*0* <..}. *principal* {(*x*, *y*). *dist x y* < *e*})

definition *open-discrete* :: *'a discrete set* ⇒ *bool* **where**
  (*open*::*'a discrete set* ⇒ *bool*) *U* ⟷ (∀ *x*∈*U*. *eventually* (λ(*x'*, *y*). *x'* = *x* ⟶ *y* ∈ *U*) *uniformity*)

instance ⟨*proof*⟩
end

lemma *open-discrete*: *open* (*S* :: *'a discrete set*)
  ⟨*proof*⟩

instance *discrete* :: (*type*) *complete-space*
⟨*proof*⟩

instance *discrete* :: (*countable*) *countable*
⟨*proof*⟩

instance *discrete* :: (*countable*) *second-countable-topology*
⟨*proof*⟩

instance *discrete* :: (*countable*) *polish-space* ⟨*proof*⟩

end

20 Probability mass function

theory *Probability-Mass-Function*
imports
  Giry-Monad
  HOL−Library.Multiset
begin

Conflicting notation from *HOL−Analysis.Infinite-Sum*

no-notation *Infinite-Sum.abs-summable-on* (**infixr** *abs'-summable'-on 46*)

lemma *AE-emeasure-singleton*:
  assumes *x*: *emeasure M* {*x*} ≠ *0* **and** *ae*: *AE x in M*. *P x* **shows** *P x*
⟨*proof*⟩

lemma *AE-measure-singleton*: *measure M* {*x*} ≠ *0* ⟹ *AE x in M*. *P x* ⟹ *P x*

⟨*proof*⟩

lemma (**in** *finite-measure*) *AE-support-countable*:
assumes [*simp*]: *sets M* = *UNIV*
shows (*AE x in M. measure M* {*x*} ≠ *0*) ⟷ (∃ *S. countable S* ∧ (*AE x in M.* *x* ∈ *S*))
⟨*proof*⟩

20.1 PMF as measure

typedef *'a pmf* = {*M* :: *'a measure. prob-space M* ∧ *sets M* = *UNIV* ∧ (*AE x in M. measure M* {*x*} ≠ *0*)}
morphisms *measure-pmf Abs-pmf*
⟨*proof*⟩

declare [[*coercion measure-pmf*]]

lemma *prob-space-measure-pmf*: *prob-space* (*measure-pmf p*)
⟨*proof*⟩

interpretation *measure-pmf*: *prob-space measure-pmf M* **for** *M*
⟨*proof*⟩

interpretation *measure-pmf*: *subprob-space measure-pmf M* **for** *M*
⟨*proof*⟩

lemma *subprob-space-measure-pmf*: *subprob-space* (*measure-pmf x*)
⟨*proof*⟩

locale *pmf-as-measure*
begin

setup-lifting *type-definition-pmf*

end

context
begin

interpretation *pmf-as-measure* ⟨*proof*⟩

lemma *sets-measure-pmf*[*simp*]: *sets* (*measure-pmf p*) = *UNIV*
⟨*proof*⟩

lemma *sets-measure-pmf-count-space*[*measurable-cong*]:
sets (*measure-pmf M*) = *sets* (*count-space UNIV*)
⟨*proof*⟩

lemma *space-measure-pmf*[*simp*]: *space* (*measure-pmf p*) = *UNIV*

⟨*proof*⟩

lemma *measure-pmf-UNIV* [*simp*]: *measure* (*measure-pmf p*) *UNIV* = *1*
⟨*proof*⟩

lemma *measure-pmf-in-subprob-algebra*[*measurable* (*raw*)]: *measure-pmf x* ∈ *space* (*subprob-algebra* (*count-space UNIV*))
⟨*proof*⟩

lemma *measurable-pmf-measure1*[*simp*]: *measurable* (*M* :: *'a pmf*) *N* = *UNIV* → *space N*
⟨*proof*⟩

lemma *measurable-pmf-measure2*[*simp*]: *measurable N* (*M* :: *'a pmf*) = *measurable N* (*count-space UNIV*)
⟨*proof*⟩

lemma *measurable-pair-restrict-pmf2*:
assumes *countable A*
assumes [*measurable*]: ⋀*y*. *y* ∈ *A* ⟹ (λ*x*. *f* (*x*, *y*)) ∈ *measurable M L*
shows *f* ∈ *measurable* (*M* ⨂$_M$ *restrict-space* (*measure-pmf N*) *A*) *L* (**is** *f* ∈ *measurable ?M* -)
⟨*proof*⟩

lemma *measurable-pair-restrict-pmf1*:
assumes *countable A*
assumes [*measurable*]: ⋀*x*. *x* ∈ *A* ⟹ (λ*y*. *f* (*x*, *y*)) ∈ *measurable N L*
shows *f* ∈ *measurable* (*restrict-space* (*measure-pmf M*) *A* ⨂$_M$ *N*) *L*
⟨*proof*⟩

lift-definition *pmf* :: *'a pmf* ⇒ *'a* ⇒ *real* **is** λ*M x*. *measure M* {*x*} ⟨*proof*⟩

lift-definition *set-pmf* :: *'a pmf* ⇒ *'a set* **is** λ*M*. {*x*. *measure M* {*x*} ≠ *0*} ⟨*proof*⟩
declare [[*coercion set-pmf*]]

lemma *AE-measure-pmf*: *AE x in* (*M*::*'a pmf*). *x* ∈ *M*
⟨*proof*⟩

lemma *emeasure-pmf-single-eq-zero-iff*:
fixes *M* :: *'a pmf*
shows *emeasure M* {*y*} = *0* ⟷ *y* ∉ *M*
⟨*proof*⟩

lemma *AE-measure-pmf-iff*: (*AE x in measure-pmf M*. *P x*) ⟷ (∀ *y*∈*M*. *P y*)
⟨*proof*⟩

lemma *AE-pmfI*: (⋀*y*. *y* ∈ *set-pmf M* ⟹ *P y*) ⟹ *almost-everywhere* (*measure-pmf M*) *P*
⟨*proof*⟩

lemma *countable-set-pmf* [*simp*]: *countable* (*set-pmf p*)
⟨*proof*⟩

lemma *pmf-positive*: $x \in$ *set-pmf p* $\Longrightarrow$ *0* $<$ *pmf p x*
⟨*proof*⟩

lemma *pmf-nonneg*[*simp*]: *0* $\leq$ *pmf p x*
⟨*proof*⟩

lemma *pmf-not-neg* [*simp*]: $\neg$*pmf p x* $<$ *0*
⟨*proof*⟩

lemma *pmf-pos* [*simp*]: *pmf p x* $\neq$ *0* $\Longrightarrow$ *pmf p x* $>$ *0*
⟨*proof*⟩

lemma *pmf-le-1*: *pmf p x* $\leq$ *1*
⟨*proof*⟩

lemma *set-pmf-not-empty*: *set-pmf M* $\neq$ {}
⟨*proof*⟩

lemma *set-pmf-iff*: $x \in$ *set-pmf M* $\longleftrightarrow$ *pmf M x* $\neq$ *0*
⟨*proof*⟩

lemma *pmf-positive-iff*: *0* $<$ *pmf p x* $\longleftrightarrow$ $x \in$ *set-pmf p*
⟨*proof*⟩

lemma *set-pmf-eq*: *set-pmf M* $=$ {*x*. *pmf M x* $\neq$ *0*}
⟨*proof*⟩

lemma *set-pmf-eq'*: *set-pmf p* $=$ {*x*. *pmf p x* $>$ *0*}
⟨*proof*⟩

lemma *emeasure-pmf-single*:
fixes *M* :: *'a pmf*
shows *emeasure M* {*x*} $=$ *pmf M x*
⟨*proof*⟩

lemma *measure-pmf-single*: *measure* (*measure-pmf M*) {*x*} $=$ *pmf M x*
⟨*proof*⟩

lemma *emeasure-measure-pmf-finite*: *finite S* $\Longrightarrow$ *emeasure* (*measure-pmf M*) *S* $=$ ($\sum$ *s*$\in$*S*. *pmf M s*)
⟨*proof*⟩

lemma *measure-measure-pmf-finite*: *finite S* $\Longrightarrow$ *measure* (*measure-pmf M*) *S* $=$ *sum* (*pmf M*) *S*
⟨*proof*⟩

lemma *sum-pmf-eq-1*:
assumes *finite A set-pmf p* $\subseteq$ *A*
shows $(\sum x{\in}A.\ pmf\ p\ x) = 1$
⟨*proof*⟩

lemma *nn-integral-measure-pmf-support*:
fixes *f* :: *'a* ⇒ *ennreal*
assumes *f*: *finite A* **and** *nn*: $\bigwedge x.\ x \in A \Longrightarrow 0 \le f\ x\ \bigwedge x.\ x \in set\text{-}pmf\ M \Longrightarrow x \notin A \Longrightarrow f\ x = 0$
shows $(\int^+ x.\ f\ x\ \partial measure\text{-}pmf\ M) = (\sum x{\in}A.\ f\ x * pmf\ M\ x)$
⟨*proof*⟩

lemma *nn-integral-measure-pmf-finite*:
fixes *f* :: *'a* ⇒ *ennreal*
assumes *f*: *finite* (*set-pmf M*) **and** *nn*: $\bigwedge x.\ x \in set\text{-}pmf\ M \Longrightarrow 0 \le f\ x$
shows $(\int^+ x.\ f\ x\ \partial measure\text{-}pmf\ M) = (\sum x{\in}set\text{-}pmf\ M.\ f\ x * pmf\ M\ x)$
⟨*proof*⟩

lemma *integrable-measure-pmf-finite*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second-countable-topology*}
shows *finite* (*set-pmf M*) ⟹ *integrable M f*
⟨*proof*⟩

lemma *integral-measure-pmf-real*:
assumes [*simp*]: *finite A* **and** $\bigwedge a.\ a \in set\text{-}pmf\ M \Longrightarrow f\ a \neq 0 \Longrightarrow a \in A$
shows $(\int x.\ f\ x\ \partial measure\text{-}pmf\ M) = (\sum a{\in}A.\ f\ a * pmf\ M\ a)$
⟨*proof*⟩

lemma *integrable-pmf*: *integrable* (*count-space X*) (*pmf M*)
⟨*proof*⟩

lemma *integral-pmf*: $(\int x.\ pmf\ M\ x\ \partial count\text{-}space\ X) = measure\ M\ X$
⟨*proof*⟩

lemma *integral-pmf-restrict*:
(*f*::*'a* ⇒ *'b*::{*banach*, *second-countable-topology*}) ∈ *borel-measurable* (*count-space UNIV*) ⟹
$(\int x.\ f\ x\ \partial measure\text{-}pmf\ M) = (\int x.\ f\ x\ \partial restrict\text{-}space\ M\ M)$
⟨*proof*⟩

lemma *emeasure-pmf*: *emeasure* (*M*::*'a pmf*) *M* = *1*
⟨*proof*⟩

lemma *emeasure-pmf-UNIV* [*simp*]: *emeasure* (*measure-pmf M*) *UNIV* = *1*
⟨*proof*⟩

lemma *in-null-sets-measure-pmfI*:
A ∩ *set-pmf p* = {} ⟹ *A* ∈ *null-sets* (*measure-pmf p*)

⟨*proof*⟩

lemma *measure-subprob*: *measure-pmf M* ∈ *space* (*subprob-algebra* (*count-space UNIV*))
⟨*proof*⟩

20.2 Monad Interpretation

lemma *measurable-measure-pmf*[*measurable*]:
(λ*x*. *measure-pmf* (*M x*)) ∈ *measurable* (*count-space UNIV*) (*subprob-algebra* (*count-space UNIV*))
⟨*proof*⟩

lemma *bind-measure-pmf-cong*:
assumes ⋀*x*. *A x* ∈ *space* (*subprob-algebra N*) ⋀*x*. *B x* ∈ *space* (*subprob-algebra N*)
assumes ⋀*i*. *i* ∈ *set-pmf x* ⟹ *A i* = *B i*
shows *bind* (*measure-pmf x*) *A* = *bind* (*measure-pmf x*) *B*
⟨*proof*⟩

lift-definition *bind-pmf* :: *'a pmf* ⇒ (*'a* ⇒ *'b pmf*) ⇒ *'b pmf* **is** *bind*
⟨*proof*⟩

adhoc-overloading *Monad-Syntax.bind bind-pmf*

lemma *ennreal-pmf-bind*: *pmf* (*bind-pmf N f*) *i* = (∫⁺*x*. *pmf* (*f x*) *i* ∂*measure-pmf N*)
⟨*proof*⟩

lemma *pmf-bind*: *pmf* (*bind-pmf N f*) *i* = (∫ *x*. *pmf* (*f x*) *i* ∂*measure-pmf N*)
⟨*proof*⟩

lemma *bind-pmf-const*[*simp*]: *bind-pmf M* (λ*x*. *c*) = *c*
⟨*proof*⟩

lemma *set-bind-pmf*[*simp*]: *set-pmf* (*bind-pmf M N*) = (⋃*M*∈*set-pmf M*. *set-pmf* (*N M*))
⟨*proof*⟩

lemma *bind-pmf-cong* [*fundef-cong*]:
assumes *p* = *q*
shows (⋀*x*. *x* ∈ *set-pmf q* ⟹ *f x* = *g x*) ⟹ *bind-pmf p f* = *bind-pmf q g*
⟨*proof*⟩

lemma *bind-pmf-cong-simp*:
p = *q* ⟹ (⋀*x*. *x* ∈ *set-pmf q* =*simp*=> *f x* = *g x*) ⟹ *bind-pmf p f* = *bind-pmf q g*
⟨*proof*⟩

lemma *measure-pmf-bind*: *measure-pmf* (*bind-pmf M f*) = (*measure-pmf M* ⤜ (λ*x*. *measure-pmf* (*f x*)))
⟨*proof*⟩

lemma *nn-integral-bind-pmf*[*simp*]: (∫⁺*x*. *f x* ∂*bind-pmf M N*) = (∫⁺*x*. ∫⁺*y*. *f y* ∂*N x* ∂*M*)
⟨*proof*⟩

lemma *emeasure-bind-pmf*[*simp*]: *emeasure* (*bind-pmf M N*) *X* = (∫⁺*x*. *emeasure* (*N x*) *X* ∂*M*)
⟨*proof*⟩

lift-definition *return-pmf* :: *'a* ⇒ *'a pmf* **is** *return* (*count-space UNIV*)
⟨*proof*⟩

lemma *bind-return-pmf*: *bind-pmf* (*return-pmf x*) *f* = *f x*
⟨*proof*⟩

lemma *set-return-pmf*[*simp*]: *set-pmf* (*return-pmf x*) = {*x*}
⟨*proof*⟩

lemma *bind-return-pmf'*: *bind-pmf N return-pmf* = *N*
⟨*proof*⟩

lemma *bind-assoc-pmf*: *bind-pmf* (*bind-pmf A B*) *C* = *bind-pmf A* (λ*x*. *bind-pmf* (*B x*) *C*)
⟨*proof*⟩

definition *map-pmf f M* = *bind-pmf M* (λ*x*. *return-pmf* (*f x*))

lemma *map-bind-pmf*: *map-pmf f* (*bind-pmf M g*) = *bind-pmf M* (λ*x*. *map-pmf f* (*g x*))
⟨*proof*⟩

lemma *bind-map-pmf*: *bind-pmf* (*map-pmf f M*) *g* = *bind-pmf M* (λ*x*. *g* (*f x*))
⟨*proof*⟩

lemma *map-pmf-transfer*[*transfer-rule*]:
rel-fun (=) (*rel-fun cr-pmf cr-pmf*) (λ*f M*. *distr M* (*count-space UNIV*) *f*) *map-pmf*
⟨*proof*⟩

lemma *map-pmf-rep-eq*:
measure-pmf (*map-pmf f M*) = *distr* (*measure-pmf M*) (*count-space UNIV*) *f*
⟨*proof*⟩

lemma *map-pmf-id*[*simp*]: *map-pmf id* = *id*
⟨*proof*⟩

lemma *map-pmf-ident*[*simp*]: *map-pmf* $(\lambda x.\ x) = (\lambda x.\ x)$
⟨*proof*⟩

lemma *map-pmf-compose*: *map-pmf* $(f \circ g) =$ *map-pmf f* $\circ$ *map-pmf g*
⟨*proof*⟩

lemma *map-pmf-comp*: *map-pmf f* (*map-pmf g M*) = *map-pmf* $(\lambda x.\ f\ (g\ x))$ *M*
⟨*proof*⟩

lemma *map-pmf-cong*: $p = q \Longrightarrow (\bigwedge x.\ x \in$ *set-pmf q* $\Longrightarrow f\ x = g\ x) \Longrightarrow$ *map-pmf f p* = *map-pmf g q*
⟨*proof*⟩

lemma *pmf-set-map*: *set-pmf* $\circ$ *map-pmf f* = (‘) *f* $\circ$ *set-pmf*
⟨*proof*⟩

lemma *set-map-pmf*[*simp*]: *set-pmf* (*map-pmf f M*) = *f*‘*set-pmf M*
⟨*proof*⟩

lemma *emeasure-map-pmf*[*simp*]: *emeasure* (*map-pmf f M*) *X* = *emeasure M* (*f* −‘ *X*)
⟨*proof*⟩

lemma *measure-map-pmf*[*simp*]: *measure* (*map-pmf f M*) *X* = *measure M* (*f* −‘ *X*)
⟨*proof*⟩

lemma *nn-integral-map-pmf*[*simp*]: $(\int^+ x.\ f\ x\ \partial$*map-pmf g M*$) = (\int^+ x.\ f\ (g\ x)\ \partial M)$
⟨*proof*⟩

lemma *ennreal-pmf-map*: *pmf* (*map-pmf f p*) *x* = $(\int^+ y.$ *indicator* (*f* −‘ {*x*}) *y* ∂*measure-pmf p*)
⟨*proof*⟩

lemma *pmf-map*: *pmf* (*map-pmf f p*) *x* = *measure p* (*f* −‘ {*x*})
⟨*proof*⟩

lemma *nn-integral-pmf*: $(\int^+ x.$ *pmf p x* ∂*count-space A*$)$ = *emeasure* (*measure-pmf p*) *A*
⟨*proof*⟩

lemma *integral-map-pmf*[*simp*]:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second-countable-topology*}
shows *integral*^{L} (*map-pmf g p*) *f* = *integral*^{L} *p* $(\lambda x.\ f\ (g\ x))$
⟨*proof*⟩

lemma *integrable-map-pmf-eq* [*simp*]:
fixes *g* :: *'a* ⇒ *'b*::{*banach*, *second-countable-topology*}

shows *integrable* (*map-pmf f p*) *g* ⟷ *integrable* (*measure-pmf p*) (λ*x*. *g* (*f x*))
⟨*proof*⟩

lemma *integrable-map-pmf* [*intro*]:
fixes *g* :: *'a* ⇒ *'b*::{*banach*, *second-countable-topology*}
shows *integrable* (*measure-pmf p*) (λ*x*. *g* (*f x*)) ⟹ *integrable* (*map-pmf f p*) *g*
⟨*proof*⟩

lemma *pmf-abs-summable* [*intro*]: *pmf p abs-summable-on A*
⟨*proof*⟩

lemma *measure-pmf-conv-infsetsum*: *measure* (*measure-pmf p*) *A* = *infsetsum* (*pmf p*) *A*
⟨*proof*⟩

lemma *infsetsum-pmf-eq-1*:
assumes *set-pmf p* ⊆ *A*
shows *infsetsum* (*pmf p*) *A* = *1*
⟨*proof*⟩

lemma *map-return-pmf* [*simp*]: *map-pmf f* (*return-pmf x*) = *return-pmf* (*f x*)
⟨*proof*⟩

lemma *map-pmf-const*[*simp*]: *map-pmf* (λ-. *c*) *M* = *return-pmf c*
⟨*proof*⟩

lemma *pmf-return* [*simp*]: *pmf* (*return-pmf x*) *y* = *indicator* {*y*} *x*
⟨*proof*⟩

lemma *nn-integral-return-pmf*[*simp*]: *0* ≤ *f x* ⟹ (∫⁺*x*. *f x* ∂*return-pmf x*) = *f x*
⟨*proof*⟩

lemma *emeasure-return-pmf*[*simp*]: *emeasure* (*return-pmf x*) *X* = *indicator X x*
⟨*proof*⟩

lemma *measure-return-pmf* [*simp*]: *measure-pmf.prob* (*return-pmf x*) *A* = *indicator A x*
⟨*proof*⟩

lemma *return-pmf-inj*[*simp*]: *return-pmf x* = *return-pmf y* ⟷ *x* = *y*
⟨*proof*⟩

lemma *map-pmf-eq-return-pmf-iff*:
map-pmf f p = *return-pmf x* ⟷ (∀ *y* ∈ *set-pmf p*. *f y* = *x*)
⟨*proof*⟩

definition *pair-pmf A B* = *bind-pmf A* (λ*x*. *bind-pmf B* (λ*y*. *return-pmf* (*x*, *y*)))

lemma *pmf-pair*: *pmf* (*pair-pmf M N*) (*a*, *b*) = *pmf M a* ∗ *pmf N b*

⟨*proof*⟩

lemma *set-pair-pmf*[*simp*]: *set-pmf* (*pair-pmf A B*) = *set-pmf A* × *set-pmf B*
⟨*proof*⟩

lemma *measure-pmf-in-subprob-space*[*measurable* (*raw*)]:
measure-pmf M ∈ *space* (*subprob-algebra* (*count-space UNIV*))
⟨*proof*⟩

lemma *nn-integral-pair-pmf'*: (∫⁺*x*. *f x* ∂*pair-pmf A B*) = (∫⁺*a*. ∫⁺*b*. *f* (*a*, *b*) ∂*B* ∂*A*)
⟨*proof*⟩

lemma *bind-pair-pmf*:
assumes *M*[*measurable*]: *M* ∈ *measurable* (*count-space UNIV* ⨂*M* *count-space UNIV*) (*subprob-algebra N*)
shows *measure-pmf* (*pair-pmf A B*) ⤜ *M* = (*measure-pmf A* ⤜ (λ*x*. *measure-pmf B* ⤜ (λ*y*. *M* (*x*, *y*))))
(**is** *?L* = *?R*)
⟨*proof*⟩

lemma *map-fst-pair-pmf*: *map-pmf fst* (*pair-pmf A B*) = *A*
⟨*proof*⟩

lemma *map-snd-pair-pmf*: *map-pmf snd* (*pair-pmf A B*) = *B*
⟨*proof*⟩

lemma *nn-integral-pmf'*:
inj-on f A ⟹ (∫⁺*x*. *pmf p* (*f x*) ∂*count-space A*) = *emeasure p* (*f* ‘ *A*)
⟨*proof*⟩

lemma *pmf-le-0-iff*[*simp*]: *pmf M p* ≤ *0* ⟷ *pmf M p* = *0*
⟨*proof*⟩

lemma *min-pmf-0*[*simp*]: *min* (*pmf M p*) *0* = *0 min 0* (*pmf M p*) = *0*
⟨*proof*⟩

lemma *pmf-eq-0-set-pmf*: *pmf M p* = *0* ⟷ *p* ∉ *set-pmf M*
⟨*proof*⟩

lemma *pmf-map-inj*: *inj-on f* (*set-pmf M*) ⟹ *x* ∈ *set-pmf M* ⟹ *pmf* (*map-pmf f M*) (*f x*) = *pmf M x*
⟨*proof*⟩

lemma *pair-return-pmf* [*simp*]: *pair-pmf* (*return-pmf x*) (*return-pmf y*) = *return-pmf* (*x*, *y*)
⟨*proof*⟩

lemma *pmf-map-inj'*: *inj f* ⟹ *pmf* (*map-pmf f M*) (*f x*) = *pmf M x*

⟨*proof*⟩

lemma *expectation-pair-pmf-fst* [*simp*]:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
shows *measure-pmf.expectation* (*pair-pmf p q*) (λ*x*. *f* (*fst x*)) = *measure-pmf.expectation p f*
⟨*proof*⟩

lemma *expectation-pair-pmf-snd* [*simp*]:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
shows *measure-pmf.expectation* (*pair-pmf p q*) (λ*x*. *f* (*snd x*)) = *measure-pmf.expectation q f*
⟨*proof*⟩

lemma *pmf-map-outside*: $x \notin f$ ‘ *set-pmf M* ⟹ *pmf* (*map-pmf f M*) *x* = *0*
⟨*proof*⟩

lemma *measurable-set-pmf*[*measurable*]: *Measurable.pred* (*count-space UNIV*) (λ*x*. *x* ∈ *set-pmf M*)
⟨*proof*⟩

20.3 PMFs as function

context
fixes $f :: 'a \Rightarrow$ *real*
assumes *nonneg*: ⋀*x*. *0* ≤ *f x*
assumes *prob*: $(\int^+ x.\ f\ x\ \partial$*count-space UNIV*$) = 1$
begin

lift-definition *embed-pmf* :: $'a$ *pmf* **is** *density* (*count-space UNIV*) (*ennreal* ∘ *f*)
⟨*proof*⟩

lemma *pmf-embed-pmf*: *pmf embed-pmf x* = *f x*
⟨*proof*⟩

lemma *set-embed-pmf*: *set-pmf embed-pmf* = {*x*. *f x* ≠ *0*}
⟨*proof*⟩

end

lemma *embed-pmf-transfer*:
rel-fun (*eq-onp* (λ*f*. (∀ *x*. *0* ≤ *f x*) ∧ $(\int^+ x.$ *ennreal* (*f x*) ∂*count-space UNIV*) = *1*)) *pmf-as-measure.cr-pmf* (λ*f*. *density* (*count-space UNIV*) (*ennreal* ∘ *f*)) *embed-pmf*
⟨*proof*⟩

lemma *measure-pmf-eq-density*: *measure-pmf p* = *density* (*count-space UNIV*) (*pmf p*)
⟨*proof*⟩

lemma *td-pmf-embed-pmf*:
type-definition pmf embed-pmf $\{f::'a \Rightarrow real.\ (\forall x.\ 0 \leq f\ x) \wedge (\int^+ x.\ ennreal\ (f\ x)\ \partial count\text{-}space\ UNIV) = 1\}$
⟨*proof*⟩

end

lemma *nn-integral-measure-pmf*: $(\int^+ x.\ f\ x\ \partial measure\text{-}pmf\ p) = \int^+ x.\ ennreal\ (pmf\ p\ x) * f\ x\ \partial count\text{-}space\ UNIV$
⟨*proof*⟩

lemma *integral-measure-pmf*:
fixes $f :: 'a \Rightarrow 'b::\{banach, second\text{-}countable\text{-}topology\}$
assumes A: *finite A*
shows $(\bigwedge a.\ a \in set\text{-}pmf\ M \Longrightarrow f\ a \neq 0 \Longrightarrow a \in A) \Longrightarrow (LINT\ x|M.\ f\ x) = (\sum a{\in}A.\ pmf\ M\ a *_R f\ a)$
⟨*proof*⟩

lemma *expectation-return-pmf* [*simp*]:
fixes $f :: 'a \Rightarrow 'b::\{banach, second\text{-}countable\text{-}topology\}$
shows *measure-pmf.expectation* (*return-pmf x*) $f = f\ x$
⟨*proof*⟩

lemma *pmf-expectation-bind*:
fixes $p :: 'a\ pmf$ **and** $f :: 'a \Rightarrow 'b\ pmf$
and $h :: 'b \Rightarrow 'c::\{banach, second\text{-}countable\text{-}topology\}$
assumes *finite A* $\bigwedge x.\ x \in A \Longrightarrow finite\ (set\text{-}pmf\ (f\ x))$ *set-pmf p* $\subseteq A$
shows *measure-pmf.expectation* $(p \gg\!= f)\ h =$
$(\sum a{\in}A.\ pmf\ p\ a *_R measure\text{-}pmf.expectation\ (f\ a)\ h)$
⟨*proof*⟩

lemma *continuous-on-LINT-pmf*: — This is dominated convergence!?
fixes $f :: 'i \Rightarrow 'a::topological\text{-}space \Rightarrow 'b::\{banach, second\text{-}countable\text{-}topology\}$
assumes f: $\bigwedge i.\ i \in set\text{-}pmf\ M \Longrightarrow continuous\text{-}on\ A\ (f\ i)$
and *bnd*: $\bigwedge a\ i.\ a \in A \Longrightarrow i \in set\text{-}pmf\ M \Longrightarrow norm\ (f\ i\ a) \leq B$
shows *continuous-on A* $(\lambda a.\ LINT\ i|M.\ f\ i\ a)$
⟨*proof*⟩

lemma *continuous-on-LBINT*:
fixes $f :: real \Rightarrow real$
assumes f: $\bigwedge b.\ a \leq b \Longrightarrow set\text{-}integrable\ lborel\ \{a..b\}\ f$
shows *continuous-on UNIV* $(\lambda b.\ LBINT\ x{:}\{a..b\}.\ f\ x)$
⟨*proof*⟩

locale *pmf-as-function*
begin

setup-lifting *td-pmf-embed-pmf*

lemma *set-pmf-transfer*[*transfer-rule*]:
assumes *bi-total A*
shows *rel-fun* (*pcr-pmf A*) (*rel-set A*) (λ*f*. {*x*. *f x* ≠ *0*}) *set-pmf*
⟨*proof*⟩

end

context
begin

interpretation *pmf-as-function* ⟨*proof*⟩

lemma *pmf-eqI*: (⋀*i*. *pmf M i* = *pmf N i*) ⟹ *M* = *N*
⟨*proof*⟩

lemma *pmf-eq-iff*: *M* = *N* ⟷ (∀ *i*. *pmf M i* = *pmf N i*)
⟨*proof*⟩

lemma *pmf-neq-exists-less*:
assumes *M* ≠ *N*
shows ∃ *x*. *pmf M x* < *pmf N x*
⟨*proof*⟩

lemma *bind-commute-pmf*: *bind-pmf A* (λ*x*. *bind-pmf B* (*C x*)) = *bind-pmf B* (λ*y*. *bind-pmf A* (λ*x*. *C x y*))
⟨*proof*⟩

lemma *pair-map-pmf1*: *pair-pmf* (*map-pmf f A*) *B* = *map-pmf* (*apfst f*) (*pair-pmf A B*)
⟨*proof*⟩

lemma *pair-map-pmf2*: *pair-pmf A* (*map-pmf f B*) = *map-pmf* (*apsnd f*) (*pair-pmf A B*)
⟨*proof*⟩

lemma *map-pair*: *map-pmf* (λ(*a*, *b*). (*f a*, *g b*)) (*pair-pmf A B*) = *pair-pmf* (*map-pmf f A*) (*map-pmf g B*)
⟨*proof*⟩

end

lemma *pair-return-pmf1*: *pair-pmf* (*return-pmf x*) *y* = *map-pmf* (*Pair x*) *y*
⟨*proof*⟩

lemma *pair-return-pmf2*: *pair-pmf x* (*return-pmf y*) = *map-pmf* (λ*x*. (*x*, *y*)) *x*
⟨*proof*⟩

lemma *pair-pair-pmf*: *pair-pmf* (*pair-pmf u v*) *w* = *map-pmf* (λ(*x*, (*y*, *z*)). ((*x*,

$y)$, $z))$ $(pair\text{-}pmf\ u\ (pair\text{-}pmf\ v\ w))$
⟨*proof*⟩

lemma *pair-commute-pmf*: $pair\text{-}pmf\ x\ y = map\text{-}pmf\ (\lambda(x,\ y).\ (y,\ x))\ (pair\text{-}pmf\ y\ x)$
⟨*proof*⟩

lemma *set-pmf-subset-singleton*: $set\text{-}pmf\ p \subseteq \{x\} \longleftrightarrow p = return\text{-}pmf\ x$
⟨*proof*⟩

lemma *bind-eq-return-pmf*:
$bind\text{-}pmf\ p\ f = return\text{-}pmf\ x \longleftrightarrow (\forall\, y{\in}set\text{-}pmf\ p.\ f\ y = return\text{-}pmf\ x)$
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *pmf-False-conv-True*: $pmf\ p\ False = 1 - pmf\ p\ True$
⟨*proof*⟩

lemma *pmf-True-conv-False*: $pmf\ p\ True = 1 - pmf\ p\ False$
⟨*proof*⟩

20.4 Conditional Probabilities

lemma *measure-pmf-zero-iff*: $measure\ (measure\text{-}pmf\ p)\ s = 0 \longleftrightarrow set\text{-}pmf\ p \cap s = \{\}$
⟨*proof*⟩

context
fixes $p :: 'a\ pmf$ **and** $s :: 'a\ set$
assumes *not-empty*: $set\text{-}pmf\ p \cap s \neq \{\}$
begin

interpretation *pmf-as-measure* ⟨*proof*⟩

lemma *emeasure-measure-pmf-not-zero*: $emeasure\ (measure\text{-}pmf\ p)\ s \neq 0$
⟨*proof*⟩

lemma *measure-measure-pmf-not-zero*: $measure\ (measure\text{-}pmf\ p)\ s \neq 0$
⟨*proof*⟩

lift-definition *cond-pmf* :: $'a\ pmf$ **is**
$uniform\text{-}measure\ (measure\text{-}pmf\ p)\ s$
⟨*proof*⟩

lemma *pmf-cond*: $pmf\ cond\text{-}pmf\ x = (if\ x \in s\ then\ pmf\ p\ x\ /\ measure\ p\ s\ else\ 0)$
⟨*proof*⟩

lemma *set-cond-pmf*[*simp*]: $set\text{-}pmf\ cond\text{-}pmf = set\text{-}pmf\ p \cap s$
⟨*proof*⟩

end

lemma *measure-pmf-posI*: $x \in$ *set-pmf* $p \Longrightarrow x \in A \Longrightarrow$ *measure-pmf.prob* $p\ A > 0$
⟨*proof*⟩

lemma *cond-map-pmf*:
assumes *set-pmf* $p \cap f\ -`\ s \neq \{\}$
shows *cond-pmf* (*map-pmf* $f\ p$) $s =$ *map-pmf* f (*cond-pmf* p ($f\ -`\ s$))
⟨*proof*⟩

lemma *bind-cond-pmf-cancel*:
assumes [*simp*]: $\bigwedge x.\ x \in$ *set-pmf* $p \Longrightarrow$ *set-pmf* $q \cap \{y.\ R\ x\ y\} \neq \{\}$
assumes [*simp*]: $\bigwedge y.\ y \in$ *set-pmf* $q \Longrightarrow$ *set-pmf* $p \cap \{x.\ R\ x\ y\} \neq \{\}$
assumes [*simp*]: $\bigwedge x\ y.\ x \in$ *set-pmf* $p \Longrightarrow y \in$ *set-pmf* $q \Longrightarrow R\ x\ y \Longrightarrow$ *measure* $q\ \{y.\ R\ x\ y\} =$ *measure* $p\ \{x.\ R\ x\ y\}$
shows *bind-pmf* p ($\lambda x.$ *cond-pmf* $q\ \{y.\ R\ x\ y\}$) $= q$
⟨*proof*⟩

20.5 Relator

inductive *rel-pmf* :: ('a ⇒ 'b ⇒ *bool*) ⇒ 'a *pmf* ⇒ 'b *pmf* ⇒ *bool*
for $R\ p\ q$
where
⟦ $\bigwedge x\ y.\ (x, y) \in$ *set-pmf* $pq \Longrightarrow R\ x\ y$;
map-pmf fst $pq = p$; *map-pmf snd* $pq = q$ ⟧
$\Longrightarrow$ *rel-pmf* $R\ p\ q$

lemma *rel-pmfI*:
assumes R: *rel-set* R (*set-pmf* p) (*set-pmf* q)
assumes *eq*: $\bigwedge x\ y.\ x \in$ *set-pmf* $p \Longrightarrow y \in$ *set-pmf* $q \Longrightarrow R\ x\ y \Longrightarrow$ *measure* $p\ \{x.\ R\ x\ y\} =$ *measure* $q\ \{y.\ R\ x\ y\}$
shows *rel-pmf* $R\ p\ q$
⟨*proof*⟩

lemma *rel-pmf-imp-rel-set*: *rel-pmf* $R\ p\ q \Longrightarrow$ *rel-set* R (*set-pmf* p) (*set-pmf* q)
⟨*proof*⟩

lemma *rel-pmfD-measure*:
assumes *rel-R*: *rel-pmf* $R\ p\ q$ **and** R: $\bigwedge a\ b.\ R\ a\ b \Longrightarrow R\ a\ y \longleftrightarrow R\ x\ b$
assumes $x \in$ *set-pmf* p $y \in$ *set-pmf* q
shows *measure* $p\ \{x.\ R\ x\ y\} =$ *measure* $q\ \{y.\ R\ x\ y\}$
⟨*proof*⟩

lemma *rel-pmf-measureD*:
assumes *rel-pmf* $R\ p\ q$
shows *measure* (*measure-pmf* p) $A \leq$ *measure* (*measure-pmf* q) $\{y.\ \exists x{\in}A.\ R\ x\ y\}$ (**is** *?lhs* $\leq$ *?rhs*)

⟨*proof*⟩

lemma *rel-pmf-iff-measure*:
assumes *symp R transp R*
shows *rel-pmf R p q* ⟷
rel-set R (*set-pmf p*) (*set-pmf q*) ∧
(∀ *x*∈*set-pmf p*. ∀ *y*∈*set-pmf q*. *R x y* ⟶ *measure p* {*x*. *R x y*} = *measure q* {*y*. *R x y*})
⟨*proof*⟩

lemma *quotient-rel-set-disjoint*:
equivp R ⟹ *C* ∈ *UNIV* // {(*x*, *y*). *R x y*} ⟹ *rel-set R A B* ⟹ *A* ∩ *C* = {} ⟷ *B* ∩ *C* = {}
⟨*proof*⟩

lemma *quotientD*: *equiv X R* ⟹ *A* ∈ *X* // *R* ⟹ *x* ∈ *A* ⟹ *A* = *R* `` {*x*}
⟨*proof*⟩

lemma *rel-pmf-iff-equivp*:
assumes *equivp R*
shows *rel-pmf R p q* ⟷ (∀ *C*∈*UNIV* // {(*x*, *y*). *R x y*}. *measure p C* = *measure q C*)
(**is** - ⟷ (∀ *C*∈-// *?R*. -))
⟨*proof*⟩

bnf *pmf*: *'a pmf map*: *map-pmf sets*: *set-pmf bd* : *card-suc natLeq rel*: *rel-pmf*
⟨*proof*⟩

lemma *map-pmf-idI*: (⋀*x*. *x* ∈ *set-pmf p* ⟹ *f x* = *x*) ⟹ *map-pmf f p* = *p*
⟨*proof*⟩

lemma *rel-pmf-conj*[*simp*]:
rel-pmf (λ*x y*. *P* ∧ *Q x y*) *x y* ⟷ *P* ∧ *rel-pmf Q x y*
rel-pmf (λ*x y*. *Q x y* ∧ *P*) *x y* ⟷ *P* ∧ *rel-pmf Q x y*
⟨*proof*⟩

lemma *rel-pmf-top*[*simp*]: *rel-pmf top* = *top*
⟨*proof*⟩

lemma *rel-pmf-return-pmf1*: *rel-pmf R* (*return-pmf x*) *M* ⟷ (∀ *a*∈*M*. *R x a*)
⟨*proof*⟩

lemma *rel-pmf-return-pmf2*: *rel-pmf R M* (*return-pmf x*) ⟷ (∀ *a*∈*M*. *R a x*)
⟨*proof*⟩

lemma *rel-return-pmf*[*simp*]: *rel-pmf R* (*return-pmf x1*) (*return-pmf x2*) = *R x1 x2*
⟨*proof*⟩

lemma *rel-pmf-False*[*simp*]: *rel-pmf* (λ*x y*. *False*) *x y* = *False*
⟨*proof*⟩

lemma *rel-pmf-rel-prod*:
rel-pmf (*rel-prod R S*) (*pair-pmf A A'*) (*pair-pmf B B'*) ⟷ *rel-pmf R A B* ∧ *rel-pmf S A' B'*
⟨*proof*⟩

lemma *rel-pmf-reflI*:
assumes ⋀*x*. *x* ∈ *set-pmf p* ⟹ *P x x*
shows *rel-pmf P p p*
⟨*proof*⟩

lemma *rel-pmf-bij-betw*:
assumes *f*: *bij-betw f* (*set-pmf p*) (*set-pmf q*)
and *eq*: ⋀*x*. *x* ∈ *set-pmf p* ⟹ *pmf p x* = *pmf q* (*f x*)
shows *rel-pmf* (λ*x y*. *f x* = *y*) *p q*
⟨*proof*⟩

context
begin

interpretation *pmf-as-measure* ⟨*proof*⟩

definition *join-pmf M* = *bind-pmf M* (λ*x*. *x*)

lemma *bind-eq-join-pmf*: *bind-pmf M f* = *join-pmf* (*map-pmf f M*)
⟨*proof*⟩

lemma *join-eq-bind-pmf*: *join-pmf M* = *bind-pmf M id*
⟨*proof*⟩

lemma *pmf-join*: *pmf* (*join-pmf N*) *i* = (∫ *M*. *pmf M i* ∂*measure-pmf N*)
⟨*proof*⟩

lemma *ennreal-pmf-join*: *ennreal* (*pmf* (*join-pmf N*) *i*) = (∫⁺*M*. *pmf M i* ∂*measure-pmf N*)
⟨*proof*⟩

lemma *set-pmf-join-pmf*[*simp*]: *set-pmf* (*join-pmf f*) = (⋃*p*∈*set-pmf f*. *set-pmf p*)
⟨*proof*⟩

lemma *join-return-pmf*: *join-pmf* (*return-pmf M*) = *M*
⟨*proof*⟩

lemma *map-join-pmf*: *map-pmf f* (*join-pmf AA*) = *join-pmf* (*map-pmf* (*map-pmf f*) *AA*)
⟨*proof*⟩

lemma *join-map-return-pmf*: *join-pmf* (*map-pmf return-pmf A*) = *A*
⟨*proof*⟩

end

lemma *rel-pmf-joinI*:
assumes *rel-pmf* (*rel-pmf P*) *p q*
shows *rel-pmf P* (*join-pmf p*) (*join-pmf q*)
⟨*proof*⟩

lemma *rel-pmf-bindI*:
assumes *pq*: *rel-pmf R p q*
and *fg*: ⋀*x y*. *R x y* ⟹ *rel-pmf P* (*f x*) (*g y*)
shows *rel-pmf P* (*bind-pmf p f*) (*bind-pmf q g*)
⟨*proof*⟩

Proof that *rel-pmf* preserves orders. Antisymmetry proof follows Thm. 1 in N. Saheb-Djahromi, Cpo's of measures for nondeterminism, Theoretical Computer Science 12(1):19–37, 1980, https://doi.org/10.1016/0304-3975(80)90003-1

lemma
assumes ∗: *rel-pmf R p q*
and *refl*: *reflp R* **and** *trans*: *transp R*
shows *measure-Ici*: *measure p* {*y*. *R x y*} ≤ *measure q* {*y*. *R x y*} (**is** *?thesis1*)
and *measure-Ioi*: *measure p* {*y*. *R x y* ∧ ¬ *R y x*} ≤ *measure q* {*y*. *R x y* ∧ ¬ *R y x*} (**is** *?thesis2*)
⟨*proof*⟩

lemma *rel-pmf-inf*:
fixes *p q* :: *'a pmf*
assumes *1*: *rel-pmf R p q*
assumes *2*: *rel-pmf R q p*
and *refl*: *reflp R* **and** *trans*: *transp R*
shows *rel-pmf* (*inf R* R^{-1-1}) *p q*
⟨*proof*⟩

lemma *rel-pmf-antisym*:
fixes *p q* :: *'a pmf*
assumes *1*: *rel-pmf R p q*
assumes *2*: *rel-pmf R q p*
and *refl*: *reflp R* **and** *trans*: *transp R* **and** *antisym*: *antisymp R*
shows *p* = *q*
⟨*proof*⟩

lemma *reflp-rel-pmf*: *reflp R* ⟹ *reflp* (*rel-pmf R*)
⟨*proof*⟩

lemma *antisymp-rel-pmf*:

⟦ *reflp R*; *transp R*; *antisymp R* ⟧
⟹ *antisymp* (*rel-pmf R*)
⟨*proof*⟩

lemma *transp-rel-pmf*:
assumes *transp R*
shows *transp* (*rel-pmf R*)
⟨*proof*⟩

20.6 Distributions

context
begin

interpretation *pmf-as-function* ⟨*proof*⟩

20.6.1 Bernoulli Distribution

lift-definition *bernoulli-pmf* :: *real* ⇒ *bool pmf* **is**
λ*p b*. ((λ*p*. *if b then p else 1* − *p*) ∘ *min 1* ∘ *max 0*) *p*
⟨*proof*⟩

lemma *pmf-bernoulli-True*[*simp*]: *0* ≤ *p* ⟹ *p* ≤ *1* ⟹ *pmf* (*bernoulli-pmf p*) *True* = *p*
⟨*proof*⟩

lemma *pmf-bernoulli-False*[*simp*]: *0* ≤ *p* ⟹ *p* ≤ *1* ⟹ *pmf* (*bernoulli-pmf p*) *False* = *1* − *p*
⟨*proof*⟩

lemma *set-pmf-bernoulli*[*simp*]: *0* < *p* ⟹ *p* < *1* ⟹ *set-pmf* (*bernoulli-pmf p*) = *UNIV*
⟨*proof*⟩

lemma *nn-integral-bernoulli-pmf*[*simp*]:
assumes [*simp*]: *0* ≤ *p p* ≤ *1* ⋀*x*. *0* ≤ *f x*
shows (∫⁺*x*. *f x* ∂*bernoulli-pmf p*) = *f True* ∗ *p* + *f False* ∗ (*1* − *p*)
⟨*proof*⟩

lemma *integral-bernoulli-pmf*[*simp*]:
assumes [*simp*]: *0* ≤ *p p* ≤ *1*
shows (∫ *x*. *f x* ∂*bernoulli-pmf p*) = *f True* ∗ *p* + *f False* ∗ (*1* − *p*)
⟨*proof*⟩

lemma *pmf-bernoulli-half* [*simp*]: *pmf* (*bernoulli-pmf* (*1* / *2*)) *x* = *1* / *2*
⟨*proof*⟩

lemma *measure-pmf-bernoulli-half*: *measure-pmf* (*bernoulli-pmf* (*1* / *2*)) = *uniform-count-measure UNIV*
⟨*proof*⟩

20.6.2 Geometric Distribution

context
fixes $p :: real$ **assumes** $p[arith]: 0 < p\ p \leq 1$
begin

lift-definition $geometric\text{-}pmf :: nat\ pmf$ **is** $\lambda n.\ (1 - p)\hat{}n * p$
⟨*proof*⟩

lemma $pmf\text{-}geometric[simp]: pmf\ geometric\text{-}pmf\ n = (1 - p)\hat{}n * p$
⟨*proof*⟩

end

lemma $geometric\text{-}pmf\text{-}1\ [simp]: geometric\text{-}pmf\ 1 = return\text{-}pmf\ 0$
⟨*proof*⟩

lemma $set\text{-}pmf\text{-}geometric: 0 < p \Longrightarrow p < 1 \Longrightarrow set\text{-}pmf\ (geometric\text{-}pmf\ p) = UNIV$
⟨*proof*⟩

lemma *geometric-sums-times-n*:
fixes $c::'a::\{banach, real\text{-}normed\text{-}field\}$
assumes $norm\ c < 1$
shows $(\lambda n.\ c\hat{}n * of\text{-}nat\ n)\ sums\ (c\ /\ (1 - c)^2)$
⟨*proof*⟩

lemma *geometric-sums-times-norm*:
fixes $c::'a::\{banach, real\text{-}normed\text{-}field\}$
assumes $norm\ c < 1$
shows $(\lambda n.\ norm\ (c\hat{}n * of\text{-}nat\ n))\ sums\ (norm\ c\ /\ (1 - norm\ c)^2)$
⟨*proof*⟩

lemma *integrable-real-geometric-pmf*:
assumes $p \in \{0<..1\}$
shows $integrable\ (geometric\text{-}pmf\ p)\ real$
⟨*proof*⟩

lemma *expectation-geometric-pmf*:
assumes $p \in \{0<..1\}$
shows $measure\text{-}pmf.expectation\ (geometric\text{-}pmf\ p)\ real = (1 - p)\ /\ p$
⟨*proof*⟩

lemma *geometric-bind-pmf-unfold*:
assumes $p \in \{0<..1\}$
shows $geometric\text{-}pmf\ p =$
$do\ \{b \leftarrow bernoulli\text{-}pmf\ p;$
$if\ b\ then\ return\text{-}pmf\ 0\ else\ map\text{-}pmf\ Suc\ (geometric\text{-}pmf\ p)\}$
⟨*proof*⟩

20.6.3 Uniform Multiset Distribution

context
fixes $M :: 'a\ multiset$ **assumes** *M-not-empty*: $M \neq \{\#\}$
begin

lift-definition *pmf-of-multiset* :: $'a\ pmf$ **is** $\lambda x.\ count\ M\ x\ /\ size\ M$
⟨*proof*⟩

lemma *pmf-of-multiset*[*simp*]: *pmf pmf-of-multiset x* = *count M x / size M*
⟨*proof*⟩

lemma *set-pmf-of-multiset*[*simp*]: *set-pmf pmf-of-multiset* = *set-mset M*
⟨*proof*⟩

end

20.6.4 Uniform Distribution

context
fixes $S :: 'a\ set$ **assumes** *S-not-empty*: $S \neq \{\}$ **and** *S-finite*: *finite S*
begin

lift-definition *pmf-of-set* :: $'a\ pmf$ **is** $\lambda x.\ indicator\ S\ x\ /\ card\ S$
⟨*proof*⟩

lemma *pmf-of-set*[*simp*]: *pmf pmf-of-set x* = *indicator S x / card S*
⟨*proof*⟩

lemma *set-pmf-of-set*[*simp*]: *set-pmf pmf-of-set* = *S*
⟨*proof*⟩

lemma *emeasure-pmf-of-set-space*[*simp*]: *emeasure pmf-of-set S* = *1*
⟨*proof*⟩

lemma *nn-integral-pmf-of-set*: *nn-integral* (*measure-pmf pmf-of-set*) *f* = *sum f S / card S*
⟨*proof*⟩

lemma *integral-pmf-of-set*: $integral^L$ (*measure-pmf pmf-of-set*) *f* = *sum f S / card S*
⟨*proof*⟩

lemma *emeasure-pmf-of-set*: *emeasure* (*measure-pmf pmf-of-set*) *A* = *card* (*S* ∩ *A*) / *card S*
⟨*proof*⟩

lemma *measure-pmf-of-set*: *measure* (*measure-pmf pmf-of-set*) *A* = *card* (*S* ∩ *A*) / *card S*
⟨*proof*⟩

end

lemma *pmf-expectation-bind-pmf-of-set*:
fixes $A :: 'a\ set$ **and** $f :: 'a \Rightarrow 'b\ pmf$
and $h :: 'b \Rightarrow 'c::\{banach, second\text{-}countable\text{-}topology\}$
assumes $A \neq \{\}$ *finite* A $\bigwedge x.\ x \in A \Longrightarrow$ *finite* (*set-pmf* $(f\ x)$)
shows *measure-pmf.expectation* (*pmf-of-set* $A \gg\!= f$) $h =$
$(\sum a{\in}A.$ *measure-pmf.expectation* $(f\ a)\ h\ /_R$ *real* (*card* A))
⟨*proof*⟩

lemma *map-pmf-of-set*:
assumes *finite* A $A \neq \{\}$
shows *map-pmf* f (*pmf-of-set* A) = *pmf-of-multiset* (*image-mset* f (*mset-set* A))
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *pmf-bind-pmf-of-set*:
assumes $A \neq \{\}$ *finite* A
shows *pmf* (*bind-pmf* (*pmf-of-set* A) f) $x =$
$(\sum xa{\in}A.$ *pmf* $(f\ xa)\ x)$ / *real-of-nat* (*card* A) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *pmf-of-set-singleton*: *pmf-of-set* $\{x\}$ = *return-pmf* x
⟨*proof*⟩

lemma *map-pmf-of-set-inj*:
assumes f: *inj-on* f A
and [*simp*]: $A \neq \{\}$ *finite* A
shows *map-pmf* f (*pmf-of-set* A) = *pmf-of-set* (f ‘ A) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *map-pmf-of-set-bij-betw*:
assumes *bij-betw* f A B $A \neq \{\}$ *finite* A
shows *map-pmf* f (*pmf-of-set* A) = *pmf-of-set* B
⟨*proof*⟩

Choosing an element uniformly at random from the union of a disjoint family of finite non-empty sets with the same size is the same as first choosing a set from the family uniformly at random and then choosing an element from the chosen set uniformly at random.

lemma *pmf-of-set-UN*:
assumes *finite* $(\bigcup(f\ `\ A))$ $A \neq \{\}$ $\bigwedge x.\ x \in A \Longrightarrow f\ x \neq \{\}$
$\bigwedge x.\ x \in A \Longrightarrow$ *card* $(f\ x) = n$ *disjoint-family-on* f A
shows *pmf-of-set* $(\bigcup(f\ `\ A)) =$ *do* $\{x \leftarrow$ *pmf-of-set* A; *pmf-of-set* $(f\ x)\}$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *bernoulli-pmf-half-conv-pmf-of-set*: *bernoulli-pmf (1 / 2) = pmf-of-set UNIV*
⟨*proof*⟩

20.6.5 Poisson Distribution

context
fixes *rate* :: *real* **assumes** *rate-pos*: *0 < rate*
begin

lift-definition *poisson-pmf* :: *nat pmf* **is** *λk. rate ^ k / fact k * exp (−rate)*
⟨*proof*⟩

lemma *pmf-poisson*[*simp*]: *pmf poisson-pmf k = rate ^ k / fact k * exp (−rate)*
⟨*proof*⟩

lemma *set-pmf-poisson*[*simp*]: *set-pmf poisson-pmf = UNIV*
⟨*proof*⟩

end

20.6.6 Binomial Distribution

context
fixes *n* :: *nat* **and** *p* :: *real* **assumes** *p-nonneg*: *0 ≤ p* **and** *p-le-1*: *p ≤ 1*
begin

lift-definition *binomial-pmf* :: *nat pmf* **is** *λk.* (*n choose k*) * *p^k* * (*1 − p*)^(*n − k*)
⟨*proof*⟩

lemma *pmf-binomial*[*simp*]: *pmf binomial-pmf k = (n choose k) * p^k * (1 − p)^(n − k)*
⟨*proof*⟩

lemma *set-pmf-binomial-eq*: *set-pmf binomial-pmf = (if p = 0 then {0} else if p = 1 then {n} else {.. n})*
⟨*proof*⟩

end

end

lemma *set-pmf-binomial-0*[*simp*]: *set-pmf (binomial-pmf n 0) = {0}*
⟨*proof*⟩

lemma *set-pmf-binomial-1*[*simp*]: *set-pmf (binomial-pmf n 1) = {n}*
⟨*proof*⟩

lemma *set-pmf-binomial*[*simp*]: *0 < p ⟹ p < 1 ⟹ set-pmf (binomial-pmf n p) = {..n}*

⟨*proof*⟩

lemma *finite-set-pmf-binomial-pmf* [*intro*]: $p \in \{0..1\} \Longrightarrow$ *finite* (*set-pmf* (*binomial-pmf n p*))
⟨*proof*⟩

lemma *expectation-binomial-pmf'*:
fixes $f :: nat \Rightarrow 'a :: \{banach, second\text{-}countable\text{-}topology\}$
assumes p: $p \in \{0..1\}$
shows *measure-pmf.expectation* (*binomial-pmf n p*) $f =$
$(\sum k \leq n.$ (*real* (*n choose k*) $*$ $p \hat{} k * (1 - p) \hat{} (n - k)) *_R f\ k)$
⟨*proof*⟩

lemma *integrable-binomial-pmf* [*simp*, *intro*]:
fixes $f :: nat \Rightarrow 'a :: \{banach, second\text{-}countable\text{-}topology\}$
assumes p: $p \in \{0..1\}$
shows *integrable* (*binomial-pmf n p*) f
⟨*proof*⟩

context includes *lifting-syntax*
begin

lemma *bind-pmf-parametric* [*transfer-rule*]:
(*rel-pmf A* ===> (*A* ===> *rel-pmf B*) ===> *rel-pmf B*) *bind-pmf bind-pmf*
⟨*proof*⟩

lemma *return-pmf-parametric* [*transfer-rule*]: (*A* ===> *rel-pmf A*) *return-pmf return-pmf*
⟨*proof*⟩

end

primrec *replicate-pmf* :: *nat* ⇒ *'a pmf* ⇒ *'a list pmf* **where**
replicate-pmf 0 - = return-pmf []
| *replicate-pmf* (*Suc n*) *p* = *do* {*x* ← *p*; *xs* ← *replicate-pmf n p*; *return-pmf* (*x*#*xs*)}

lemma *replicate-pmf-1*: *replicate-pmf 1 p = map-pmf* (λ*x*. [*x*]) *p*
⟨*proof*⟩

lemma *set-replicate-pmf*:
set-pmf (*replicate-pmf n p*) = {*xs*∈*lists* (*set-pmf p*). *length xs* = *n*}
⟨*proof*⟩

lemma *replicate-pmf-distrib*:
replicate-pmf (*m* + *n*) *p* =
do {*xs* ← *replicate-pmf m p*; *ys* ← *replicate-pmf n p*; *return-pmf* (*xs* @ *ys*)}
⟨*proof*⟩

lemma *power-diff'*:
assumes $b \leq a$
shows $x \mathbin{\hat{}} (a - b) = (\textit{if } x = 0 \wedge a = b \textit{ then } 1 \textit{ else } x \mathbin{\hat{}} a \mathbin{/} (x::'a::\textit{field}) \mathbin{\hat{}} b)$
⟨*proof*⟩

lemma *binomial-pmf-Suc*:
assumes $p \in \{0..1\}$
shows *binomial-pmf* (*Suc n*) *p* =
do {*b* ← *bernoulli-pmf p*;
k ← *binomial-pmf n p*;
return-pmf ((*if b then 1 else 0*) + *k*)} (**is** - = *?rhs*)
⟨*proof*⟩

lemma *binomial-pmf-0*: $p \in \{0..1\} \Longrightarrow$ *binomial-pmf 0 p* = *return-pmf 0*
⟨*proof*⟩

lemma *binomial-pmf-altdef*:
assumes $p \in \{0..1\}$
shows *binomial-pmf n p* = *map-pmf* (*length* ∘ *filter id*) (*replicate-pmf n* (*bernoulli-pmf p*))
⟨*proof*⟩

20.7 Negative Binomial distribution

The negative binomial distribution counts the number of times a weighted coin comes up tails before having come up heads n times. In other words: how many failures do we see before seeing the n-th success?

An alternative view is that the negative binomial distribution is the sum of n i.i.d. geometric variables (this is the definition that we use).

Note that there are sometimes different conventions for this distributions in the literature; for instance, sometimes the number of *attempts* is counted instead of the number of failures. This only shifts the entire distribution by a constant number and is thus not a big difference. I think that the convention we use is the most natural one since the support of the distribution starts at 0, whereas for the other convention it starts at n.

primrec *neg-binomial-pmf* :: *nat* ⇒ *real* ⇒ *nat pmf* **where**
neg-binomial-pmf 0 p = *return-pmf 0*
| *neg-binomial-pmf* (*Suc n*) *p* =
map-pmf (λ(*x*,*y*). (*x* + *y*)) (*pair-pmf* (*geometric-pmf p*) (*neg-binomial-pmf n p*))

lemma *neg-binomial-pmf-Suc-0* [*simp*]: *neg-binomial-pmf* (*Suc 0*) *p* = *geometric-pmf p*
⟨*proof*⟩

lemmas *neg-binomial-pmf-Suc* [*simp del*] = *neg-binomial-pmf.simps*(*2*)

lemma *neg-binomial-prob-1* [*simp*]: *neg-binomial-pmf n 1 = return-pmf 0*
⟨*proof*⟩

We can now show the aforementioned intuition about counting the failures before the n-th success with the following recurrence:

lemma *neg-binomial-pmf-unfold*:
assumes *p*: $p \in \{0<..1\}$
shows *neg-binomial-pmf (Suc n) p =*
do {b ← bernoulli-pmf p;
if b then neg-binomial-pmf n p else map-pmf Suc (neg-binomial-pmf (Suc n) p)}
(**is** *- = ?rhs*)
⟨*proof*⟩

Next, we show an explicit formula for the probability mass function of the negative binomial distribution:

lemma *pmf-neg-binomial*:
assumes *p*: $p \in \{0<..1\}$
shows *pmf (neg-binomial-pmf n p) k = real ((k + n − 1) choose k) ∗ p ^ n ∗ (1 − p) ^ k*
⟨*proof*⟩

lemma *gbinomial-0-left*: *0 gchoose k = (if k = 0 then 1 else 0)*
⟨*proof*⟩

The following alternative formula highlights why it is called 'negative binomial distribution':

lemma *pmf-neg-binomial′*:
assumes *p*: $p \in \{0<..1\}$
shows *pmf (neg-binomial-pmf n p) k = (−1) ^ k ∗ ((−real n) gchoose k) ∗ p ^ n ∗ (1 − p) ^ k*
⟨*proof*⟩

The cumulative distribution function of the negative binomial distribution can be expressed in terms of that of the 'normal' binomial distribution.

lemma *prob-neg-binomial-pmf-atMost*:
assumes *p*: $p \in \{0<..1\}$
shows *measure-pmf.prob (neg-binomial-pmf n p) {..k} =*
measure-pmf.prob (binomial-pmf (n + k) (1 − p)) {..k}
⟨*proof*⟩

lemma *prob-neg-binomial-pmf-lessThan*:
assumes *p*: $p \in \{0<..1\}$
shows *measure-pmf.prob (neg-binomial-pmf n p) {..<k} =*
measure-pmf.prob (binomial-pmf (n + k − 1) (1 − p)) {..<k}
⟨*proof*⟩

The expected value of the negative binomial distribution is $n(1-p)/p$:

lemma *nn-integral-neg-binomial-pmf-real*:
assumes *p*: $p \in \{0<..1\}$
shows *nn-integral* (*measure-pmf* (*neg-binomial-pmf n p*)) *of-nat* = *ennreal* ($n * (1 - p) / p$)
⟨*proof*⟩

lemma *integrable-neg-binomial-pmf-real*:
assumes *p*: $p \in \{0<..1\}$
shows *integrable* (*measure-pmf* (*neg-binomial-pmf n p*)) *real*
⟨*proof*⟩

lemma *expectation-neg-binomial-pmf*:
assumes *p*: $p \in \{0<..1\}$
shows *measure-pmf.expectation* (*neg-binomial-pmf n p*) *real* = $n * (1 - p) / p$
⟨*proof*⟩

20.8 PMFs from association lists

definition *pmf-of-list* :: ('*a* × *real*) *list* ⇒ '*a pmf* **where**
pmf-of-list xs = *embed-pmf* (λ*x*. *sum-list* (*map snd* (*filter* (λ*z*. *fst z* = *x*) *xs*)))

definition *pmf-of-list-wf* **where**
pmf-of-list-wf xs ⟷ (∀ *x*∈*set* (*map snd xs*) . $x \geq 0$) ∧ *sum-list* (*map snd xs*) = *1*

lemma *pmf-of-list-wfI*:
(⋀*x*. *x* ∈ *set* (*map snd xs*) ⟹ $x \geq 0$) ⟹ *sum-list* (*map snd xs*) = *1* ⟹ *pmf-of-list-wf xs*
⟨*proof*⟩

context
begin

private lemma *pmf-of-list-aux*:
assumes ⋀*x*. *x* ∈ *set* (*map snd xs*) ⟹ $x \geq 0$
assumes *sum-list* (*map snd xs*) = *1*
shows ($\int^+$ *x*. *ennreal* (*sum-list* (*map snd* [*z*←*xs* . *fst z* = *x*])) ∂*count-space UNIV*) = *1*
⟨*proof*⟩

lemma *pmf-pmf-of-list*:
assumes *pmf-of-list-wf xs*
shows *pmf* (*pmf-of-list xs*) *x* = *sum-list* (*map snd* (*filter* (λ*z*. *fst z* = *x*) *xs*))
⟨*proof*⟩

end

lemma *set-pmf-of-list*:

assumes *pmf-of-list-wf xs*
shows *set-pmf* (*pmf-of-list xs*) ⊆ *set* (*map fst xs*)
⟨*proof*⟩

lemma *finite-set-pmf-of-list*:
assumes *pmf-of-list-wf xs*
shows *finite* (*set-pmf* (*pmf-of-list xs*))
⟨*proof*⟩

lemma *emeasure-Int-set-pmf*:
emeasure (*measure-pmf p*) (*A* ∩ *set-pmf p*) = *emeasure* (*measure-pmf p*) *A*
⟨*proof*⟩

lemma *measure-Int-set-pmf*:
measure (*measure-pmf p*) (*A* ∩ *set-pmf p*) = *measure* (*measure-pmf p*) *A*
⟨*proof*⟩

lemma *measure-prob-cong-0*:
assumes ⋀*x*. *x* ∈ *A* − *B* ⟹ *pmf p x* = *0*
assumes ⋀*x*. *x* ∈ *B* − *A* ⟹ *pmf p x* = *0*
shows *measure* (*measure-pmf p*) *A* = *measure* (*measure-pmf p*) *B*
⟨*proof*⟩

lemma *emeasure-pmf-of-list*:
assumes *pmf-of-list-wf xs*
shows *emeasure* (*pmf-of-list xs*) *A* = *ennreal* (*sum-list* (*map snd* (*filter* (λ*x*. *fst x* ∈ *A*) *xs*)))
⟨*proof*⟩

lemma *measure-pmf-of-list*:
assumes *pmf-of-list-wf xs*
shows *measure* (*pmf-of-list xs*) *A* = *sum-list* (*map snd* (*filter* (λ*x*. *fst x* ∈ *A*) *xs*))
⟨*proof*⟩

lemma *sum-list-nonneg-eq-zero-iff*:
fixes *xs* :: *'a* :: *linordered-ab-group-add list*
shows (⋀*x*. *x* ∈ *set xs* ⟹ *x* ≥ *0*) ⟹ *sum-list xs* = *0* ⟷ *set xs* ⊆ {*0*}
⟨*proof*⟩

lemma *sum-list-filter-nonzero*:
sum-list (*filter* (λ*x*. *x* ≠ *0*) *xs*) = *sum-list xs*
⟨*proof*⟩

lemma *set-pmf-of-list-eq*:
assumes *pmf-of-list-wf xs* ⋀*x*. *x* ∈ *snd* ' *set xs* ⟹ *x* > *0*
shows *set-pmf* (*pmf-of-list xs*) = *fst* ' *set xs*

⟨*proof*⟩

lemma *pmf-of-list-remove-zeros*:
assumes *pmf-of-list-wf xs*
defines $xs' \equiv$ *filter* ($\lambda z.$ *snd* $z \neq 0$) *xs*
shows *pmf-of-list-wf xs' pmf-of-list xs' = pmf-of-list xs*
⟨*proof*⟩

end

21 Code generation for PMFs

theory *PMF-Impl*
imports *Probability-Mass-Function HOL−Library.AList-Mapping*
begin

21.1 General code generation setup

definition *pmf-of-mapping* :: ('*a*, *real*) *mapping* ⇒ '*a pmf* **where**
pmf-of-mapping m = embed-pmf (*Mapping.lookup-default 0 m*)

lemma *nn-integral-lookup-default*:
fixes *m* :: ('*a*, *real*) *mapping*
assumes *finite* (*Mapping.keys m*) *All-mapping m* (λ- *x. x* ≥ *0*)
shows *nn-integral* (*count-space UNIV*) (λ*k. ennreal* (*Mapping.lookup-default 0 m k*)) =
ennreal ($\sum$ *k*∈*Mapping.keys m. Mapping.lookup-default 0 m k*)
⟨*proof*⟩

lemma *pmf-of-mapping*:
assumes *finite* (*Mapping.keys m*) *All-mapping m* (λ- *p. p* ≥ *0*)
assumes ($\sum$ *x*∈*Mapping.keys m. Mapping.lookup-default 0 m x*) = *1*
shows *pmf* (*pmf-of-mapping m*) *x* = *Mapping.lookup-default 0 m x*
⟨*proof*⟩

lemma *pmf-of-set-pmf-of-mapping*:
assumes *A* ≠ {} *set xs* = *A distinct xs*
shows *pmf-of-set A* = *pmf-of-mapping* (*Mapping.tabulate xs* (λ-. *1 / real* (*length xs*)))
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lift-definition *mapping-of-pmf* :: '*a pmf* ⇒ ('*a*, *real*) *mapping* **is**
λ*p x. if pmf p x* = *0 then None else Some* (*pmf p x*) ⟨*proof*⟩

lemma *lookup-default-mapping-of-pmf*:
Mapping.lookup-default 0 (*mapping-of-pmf p*) *x* = *pmf p x*
⟨*proof*⟩

context
begin

interpretation *pmf-as-function* ⟨*proof*⟩

lemma *nn-integral-pmf-eq-1*: $(\int^+ x.\ ennreal\ (pmf\ p\ x)\ \partial count\text{-}space\ UNIV) = 1$
⟨*proof*⟩
end

lemma *pmf-of-mapping-mapping-of-pmf* [*code abstype*]:
$pmf\text{-}of\text{-}mapping\ (mapping\text{-}of\text{-}pmf\ p) = p$
⟨*proof*⟩

lemma *mapping-of-pmfI*:
assumes $\bigwedge x.\ x \in Mapping.keys\ m \Longrightarrow Mapping.lookup\ m\ x = Some\ (pmf\ p\ x)$
assumes $Mapping.keys\ m = set\text{-}pmf\ p$
shows $mapping\text{-}of\text{-}pmf\ p = m$
⟨*proof*⟩

lemma *mapping-of-pmfI'*:
assumes $\bigwedge x.\ x \in Mapping.keys\ m \Longrightarrow Mapping.lookup\text{-}default\ 0\ m\ x = pmf\ p\ x$

assumes $Mapping.keys\ m = set\text{-}pmf\ p$
shows $mapping\text{-}of\text{-}pmf\ p = m$
⟨*proof*⟩

lemma *return-pmf-code* [*code abstract*]:
$mapping\text{-}of\text{-}pmf\ (return\text{-}pmf\ x) = Mapping.update\ x\ 1\ Mapping.empty$
⟨*proof*⟩

lemma *pmf-of-set-code-aux*:
assumes $A \neq \{\}$ *set xs* $= A$ *distinct xs*
shows $mapping\text{-}of\text{-}pmf\ (pmf\text{-}of\text{-}set\ A) = Mapping.tabulate\ xs\ (\lambda\text{-}.\ 1\ /\ real\ (length\ xs))$
⟨*proof*⟩

definition *pmf-of-set-impl* **where**
$pmf\text{-}of\text{-}set\text{-}impl\ A = mapping\text{-}of\text{-}pmf\ (pmf\text{-}of\text{-}set\ A)$

lemma *pmf-of-set-impl-code-alt*:
assumes $A \neq \{\}$ *finite A*
shows $pmf\text{-}of\text{-}set\text{-}impl\ A =$
$(let\ p = 1\ /\ real\ (card\ A)$
$in\ \ Finite\text{-}Set.fold\ (\lambda x.\ Mapping.update\ x\ p)\ Mapping.empty\ A)$
⟨*proof*⟩

lemma *pmf-of-set-impl-code* [*code*]:
$pmf\text{-}of\text{-}set\text{-}impl\ (set\ xs) =$

(*if xs = [] then*
Code.abort (STR ''pmf-of-set of empty set'') (λ-. mapping-of-pmf (pmf-of-set (set xs)))
else let xs' = remdups xs; p = 1 / real (length xs') in
Mapping.tabulate xs' (λ-. p))
⟨*proof*⟩

lemma *pmf-of-set-code* [*code abstract*]:
mapping-of-pmf (pmf-of-set A) = pmf-of-set-impl A
⟨*proof*⟩

lemma *pmf-of-multiset-pmf-of-mapping*:
assumes *A ≠ {#} set xs = set-mset A distinct xs*
shows *mapping-of-pmf (pmf-of-multiset A) = Mapping.tabulate xs (λx. count A x / real (size A))*
⟨*proof*⟩

definition *pmf-of-multiset-impl* **where**
pmf-of-multiset-impl A = mapping-of-pmf (pmf-of-multiset A)

lemma *pmf-of-multiset-impl-code-alt*:
assumes *A ≠ {#}*
shows *pmf-of-multiset-impl A =*
(let p = 1 / real (size A)
in fold-mset (λx. Mapping.map-default x 0 ((+) p)) Mapping.empty A)
⟨*proof*⟩

lemma *pmf-of-multiset-impl-code* [*code*]:
pmf-of-multiset-impl (mset xs) =
(if xs = [] then
Code.abort (STR ''pmf-of-multiset of empty multiset'')
(λ-. mapping-of-pmf (pmf-of-multiset (mset xs)))
else let xs' = remdups xs; p = 1 / real (length xs) in
*Mapping.tabulate xs' (λx. real (count (mset xs) x) * p))*
⟨*proof*⟩

lemma *pmf-of-multiset-code* [*code abstract*]:
mapping-of-pmf (pmf-of-multiset A) = pmf-of-multiset-impl A
⟨*proof*⟩

lemma *bernoulli-pmf-code* [*code abstract*]:
mapping-of-pmf (bernoulli-pmf p) =
(if p ≤ 0 then Mapping.update False 1 Mapping.empty
else if p ≥ 1 then Mapping.update True 1 Mapping.empty
else Mapping.update False (1 − p) (Mapping.update True p Mapping.empty))
⟨*proof*⟩

lemma *pmf-code* [*code*]: *pmf p x = Mapping.lookup-default 0 (mapping-of-pmf p) x*
⟨*proof*⟩

lemma *set-pmf-code* [*code*]: *set-pmf p = Mapping.keys (mapping-of-pmf p)*
⟨*proof*⟩

lemma *keys-mapping-of-pmf* [*simp*]: *Mapping.keys (mapping-of-pmf p) = set-pmf p*
⟨*proof*⟩

definition *fold-combine-plus* **where**
fold-combine-plus = comm-monoid-set.F (Mapping.combine ((+) :: real ⇒ -)) Mapping.empty

context
begin

interpretation *fold-combine-plus*: *combine-mapping-abel-semigroup (+) :: real ⇒ -*
⟨*proof*⟩ **lemma** *lookup-default-fold-combine-plus*:
fixes *A* :: *'b set* **and** *f* :: *'b ⇒ ('a, real) mapping*
assumes *finite A*
shows *Mapping.lookup-default 0 (fold-combine-plus f A) x =*
(∑ y∈A. Mapping.lookup-default 0 (f y) x)
⟨*proof*⟩ **lemma** *keys-fold-combine-plus*:
finite A ⟹ Mapping.keys (fold-combine-plus f A) = (⋃ x∈A. Mapping.keys (f x))
⟨*proof*⟩ **lemma** *fold-combine-plus-code* [*code*]:
fold-combine-plus g (set xs) = foldr (λx. Mapping.combine (+) (g x)) (remdups xs) Mapping.empty
⟨*proof*⟩ **lemma** *lookup-default-0-map-values*:
assumes *f x 0 = 0*
shows *Mapping.lookup-default 0 (Mapping.map-values f m) x = f x (Mapping.lookup-default 0 m x)*
⟨*proof*⟩ **lemma** *mapping-of-bind-pmf*:
assumes *finite (set-pmf p)*
shows *mapping-of-pmf (bind-pmf p f) =*
fold-combine-plus (λx. Mapping.map-values (λ-. (∗) (pmf p x))
(mapping-of-pmf (f x))) (set-pmf p)
⟨*proof*⟩

lift-definition *bind-pmf-aux* :: *'a pmf ⇒ ('a ⇒ 'b pmf) ⇒ 'a set ⇒ ('b, real) mapping* **is**
λ(p :: 'a pmf) (f :: 'a ⇒ 'b pmf) (A::'a set) (x::'b).
if x ∈ (⋃ y∈A. set-pmf (f y)) then

Some (*measure-pmf.expectation p* (λ*y*. *indicator A y* ∗ *pmf* (*f y*) *x*))
else None ⟨*proof*⟩

lemma *keys-bind-pmf-aux* [*simp*]:
Mapping.keys (*bind-pmf-aux p f A*) = (⋃ *x*∈*A*. *set-pmf* (*f x*))
⟨*proof*⟩

lemma *lookup-default-bind-pmf-aux*:
Mapping.lookup-default 0 (*bind-pmf-aux p f A*) *x* =
(*if x* ∈ (⋃ *y*∈*A*. *set-pmf* (*f y*)) *then*
measure-pmf.expectation p (λ*y*. *indicator A y* ∗ *pmf* (*f y*) *x*) *else 0*)
⟨*proof*⟩

lemma *lookup-default-bind-pmf-aux′* [*simp*]:
Mapping.lookup-default 0 (*bind-pmf-aux p f* (*set-pmf p*)) *x* = *pmf* (*bind-pmf p f*)
x
⟨*proof*⟩

lemma *bind-pmf-aux-correct*:
mapping-of-pmf (*bind-pmf p f*) = *bind-pmf-aux p f* (*set-pmf p*)
⟨*proof*⟩

lemma *bind-pmf-aux-code-aux*:
assumes *finite A*
shows *bind-pmf-aux p f A* =
fold-combine-plus (λ*x*. *Mapping.map-values* (λ-. (∗) (*pmf p x*))
(*mapping-of-pmf* (*f x*))) *A* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *bind-pmf-aux-code* [*code*]:
bind-pmf-aux p f (*set xs*) =
fold-combine-plus (λ*x*. *Mapping.map-values* (λ-. (∗) (*pmf p x*))
(*mapping-of-pmf* (*f x*))) (*set xs*)
⟨*proof*⟩

lemmas *bind-pmf-code* [*code abstract*] = *bind-pmf-aux-correct*

end

hide-const (**open**) *fold-combine-plus*

lift-definition *cond-pmf-impl* :: *′a pmf* ⇒ *′a set* ⇒ (*′a*, *real*) *mapping option* **is**
λ*p A*. *if A* ∩ *set-pmf p* = {} *then None else*
Some (λ*x*. *if x* ∈ *A* ∩ *set-pmf p then Some* (*pmf p x* / *measure-pmf.prob p A*)
else None) ⟨*proof*⟩

lemma *cond-pmf-impl-code-alt*:
assumes *finite A*

shows *cond-pmf-impl p A = (*
let C = A ∩ set-pmf p;
prob = (∑ x∈C. pmf p x)
in if prob = 0 then
None
else
Some (Mapping.map-values (λ- y. y / prob)
(Mapping.filter (λk -. k ∈ C) (mapping-of-pmf p))))
⟨*proof*⟩

lemma *cond-pmf-impl-code* [*code*]:
cond-pmf-impl p (set xs) = (
let C = set xs ∩ set-pmf p;
prob = (∑ x∈C. pmf p x)
in if prob = 0 then
None
else
Some (Mapping.map-values (λ- y. y / prob)
(Mapping.filter (λk -. k ∈ C) (mapping-of-pmf p))))
⟨*proof*⟩

lemma *cond-pmf-code* [*code abstract*]:
mapping-of-pmf (cond-pmf p A) =
(case cond-pmf-impl p A of
None ⇒ Code.abort (STR ''cond-pmf with set of probability 0'')
(λ-. mapping-of-pmf (cond-pmf p A))
| Some m ⇒ m)
⟨*proof*⟩

lemma *binomial-pmf-code* [*code abstract*]:
mapping-of-pmf (binomial-pmf n p) = (
if p < 0 ∨ p > 1 then
Code.abort (STR ''binomial-pmf with invalid probability'')
(λ-. mapping-of-pmf (binomial-pmf n p))
else if p = 0 then Mapping.update 0 1 Mapping.empty
else if p = 1 then Mapping.update n 1 Mapping.empty
*else Mapping.tabulate [0..<Suc n] (λk. real (n choose k) * p ^ k * (1 − p) ^*
(n − k)))
⟨*proof*⟩

lemma *pred-pmf-code* [*code*]:
pred-pmf P p = (∀ x∈set-pmf p. P x)
⟨*proof*⟩

lemma *mapping-of-pmf-pmf-of-list*:
assumes *⋀x. x ∈ snd ' set xs ⟹ x > 0 sum-list (map snd xs) = 1*

shows *mapping-of-pmf* (*pmf-of-list xs*) =
Mapping.tabulate (*remdups* (*map fst xs*))
(λ*x*. *sum-list* (*map snd* (*filter* (λ*z*. *fst z* = *x*) *xs*)))
⟨*proof*⟩

lemma *mapping-of-pmf-pmf-of-list'*:
assumes *pmf-of-list-wf xs*
defines *xs'* ≡ *filter* (λ*z*. *snd z* ≠ *0*) *xs*
shows *mapping-of-pmf* (*pmf-of-list xs*) =
Mapping.tabulate (*remdups* (*map fst xs'*))
(λ*x*. *sum-list* (*map snd* (*filter* (λ*z*. *fst z* = *x*) *xs'*))) (**is** - = *?rhs*)
⟨*proof*⟩

lemma *pmf-of-list-wf-code* [*code*]:
pmf-of-list-wf xs ⟷ *list-all* (λ*z*. *snd z* ≥ *0*) *xs* ∧ *sum-list* (*map snd xs*) = *1*
⟨*proof*⟩

lemma *pmf-of-list-code* [*code abstract*]:
mapping-of-pmf (*pmf-of-list xs*) = (
if pmf-of-list-wf xs then
let xs' = *filter* (λ*z*. *snd z* ≠ *0*) *xs*
in Mapping.tabulate (*remdups* (*map fst xs'*))
(λ*x*. *sum-list* (*map snd* (*filter* (λ*z*. *fst z* = *x*) *xs'*)))
else
Code.abort (*STR ''Invalid list for pmf-of-list''*) (λ-. *mapping-of-pmf* (*pmf-of-list xs*)))
⟨*proof*⟩

lemma *mapping-of-pmf-eq-iff* [*simp*]:
mapping-of-pmf p = *mapping-of-pmf q* ⟷ *p* = (*q* :: *'a pmf*)
⟨*proof*⟩

21.2 Code abbreviations for integrals and probabilities

Integrals and probabilities are defined for general measures, so we cannot give any code equations directly. We can, however, specialise these constants them to PMFs, give code equations for these specialised constants, and tell the code generator to unfold the original constants to the specialised ones whenever possible.

definition *pmf-integral* **where**
pmf-integral p f = *lebesgue-integral* (*measure-pmf p*) (*f* :: - ⇒ *real*)

definition *pmf-set-integral* **where**
pmf-set-integral p f A = *lebesgue-integral* (*measure-pmf p*) (λ*x*. *indicator A x* ∗ *f x* :: *real*)

definition *pmf-prob* **where**
pmf-prob p A = *measure-pmf.prob p A*

lemma *pmf-prob-compl*: *pmf-prob p* $(-A)$ = *1* − *pmf-prob p A*
⟨*proof*⟩

lemma *pmf-integral-pmf-set-integral* [*code*]:
pmf-integral p f = *pmf-set-integral p f* (*set-pmf p*)
⟨*proof*⟩

lemma *pmf-prob-pmf-set-integral*:
pmf-prob p A = *pmf-set-integral p* (λ-. *1*) *A*
⟨*proof*⟩

lemma *pmf-set-integral-code-alt-finite*:
finite A ⟹ *pmf-set-integral p f A* = $(\sum x{\in}A.$ *pmf p x* ∗ *f x*)
⟨*proof*⟩

lemma *pmf-set-integral-code* [*code*]:
pmf-set-integral p f (*set xs*) = $(\sum x{\in}$*set xs*. *pmf p x* ∗ *f x*)
⟨*proof*⟩

lemma *pmf-prob-code-alt-finite*:
finite A ⟹ *pmf-prob p A* = $(\sum x{\in}A.$ *pmf p x*)
⟨*proof*⟩

lemma *pmf-prob-code* [*code*]:
pmf-prob p (*set xs*) = $(\sum x{\in}$*set xs*. *pmf p x*)
pmf-prob p (*List.coset xs*) = *1* − $(\sum x{\in}$*set xs*. *pmf p x*)
⟨*proof*⟩

lemma *pmf-prob-code-unfold* [*code-abbrev*]: *pmf-prob p* = *measure-pmf.prob p*
⟨*proof*⟩

lemma *pmf-integral-code-unfold* [*code-abbrev*]: *pmf-integral p* = *measure-pmf.expectation p*
⟨*proof*⟩

definition *pmf-of-alist xs* = *embed-pmf* (λ*x*. *case map-of xs x of Some p* ⇒ *p* | *None* ⇒ *0*)

lemma *pmf-of-mapping-Mapping* [*code-post*]:
pmf-of-mapping (*Mapping xs*) = *pmf-of-alist xs*
⟨*proof*⟩

instantiation *pmf* :: (*equal*) *equal*
begin

definition *equal-pmf p q* = (*mapping-of-pmf p* = *mapping-of-pmf* (*q* :: *'a pmf*))

instance ⟨*proof*⟩
end

definition *single* :: *'a* ⇒ *'a multiset* **where**
single s = {#*s*#}

instantiation *pmf* :: (*random*) *random*
begin

context
includes *state-combinator-syntax term-syntax*
begin

definition
pmfify :: (*'b*::*typerep multiset* × (*unit* ⇒ *Code-Evaluation.term*)) ⇒
'b × (*unit* ⇒ *Code-Evaluation.term*) ⇒
'b pmf × (*unit* ⇒ *Code-Evaluation.term*) **where**
[*code-unfold*]: *pmfify A x* =
Code-Evaluation.valtermify pmf-of-multiset {·}
(*Code-Evaluation.valtermify* (+) {·} *A* {·}
(*Code-Evaluation.valtermify single* {·} *x*))

definition
Quickcheck-Random.random i =
Quickcheck-Random.random i ∘→ (λ*A*.
Quickcheck-Random.random i ∘→ (λ*x*. *Pair* (*pmfify A x*)))

instance ⟨*proof*⟩

end

end

instantiation *pmf* :: (*full-exhaustive*) *full-exhaustive*
begin

definition *full-exhaustive-pmf* :: (*'a pmf* × (*unit* ⇒ *term*) ⇒ (*bool* × *term list*) *option*) ⇒ *natural* ⇒ (*bool* × *term list*) *option*
where
full-exhaustive-pmf f i =
Quickcheck-Exhaustive.full-exhaustive (λ*A*.
Quickcheck-Exhaustive.full-exhaustive (λ*x*. *f* (*pmfify A x*)) *i*) *i*

instance ⟨*proof*⟩

end

end

22 Finite Maps

theory *Fin-Map*
imports *HOL−Analysis.Finite-Product-Measure HOL−Library.Finite-Map*
begin

The *fmap* type can be instantiated to *polish-space*, needed for the proof of projective limit. *extensional* functions are used for the representation in order to stay close to the developments of (finite) products Pi_E and their sigma-algebra Pi_M.

type-notation *fmap* ((- $\Rightarrow_F$ /-) [*22*, *21*] *21*)

unbundle *fmap.lifting*

22.1 Domain and Application

lift-definition *domain*::('*i* $\Rightarrow_F$ '*a*) $\Rightarrow$ '*i set* **is** *dom* ⟨*proof*⟩

lemma *finite-domain*[*simp*, *intro*]: *finite* (*domain P*)
⟨*proof*⟩

lift-definition *proj* :: ('*i* $\Rightarrow_F$ '*a*) $\Rightarrow$ '*i* $\Rightarrow$ '*a* ('((-)')$_F$ [*0*] *1000*) **is**
λ*f x*. *if x* $\in$ *dom f then the* (*f x*) *else undefined* ⟨*proof*⟩

declare [[*coercion proj*]]

lemma *extensional-proj*[*simp*, *intro*]: $(P)_F \in$ *extensional* (*domain P*)
⟨*proof*⟩

lemma *proj-undefined*[*simp*, *intro*]: *i* $\notin$ *domain P* $\Longrightarrow$ *P i* = *undefined*
⟨*proof*⟩

lemma *finmap-eq-iff*: *P* = *Q* $\longleftrightarrow$ (*domain P* = *domain Q* $\wedge$ ($\forall$ *i*$\in$*domain P*. *P i* = *Q i*))
⟨*proof*⟩

22.2 Constructor of Finite Maps

lift-definition *finmap-of*::'*i set* $\Rightarrow$ ('*i* $\Rightarrow$ '*a*) $\Rightarrow$ ('*i* $\Rightarrow_F$ '*a*) **is**
λ*I f x*. *if x* $\in$ *I* $\wedge$ *finite I then Some* (*f x*) *else None*
⟨*proof*⟩

lemma *proj-finmap-of*[*simp*]:

assumes *finite inds*
shows $(\mathit{finmap\text{-}of}\ \mathit{inds}\ f)_F = \mathit{restrict}\ f\ \mathit{inds}$
⟨*proof*⟩

lemma *domain-finmap-of*[*simp*]:
assumes *finite inds*
shows *domain* (*finmap-of inds f*) = *inds*
⟨*proof*⟩

lemma *finmap-of-eq-iff*[*simp*]:
assumes *finite i finite j*
shows *finmap-of i m* = *finmap-of j n* ⟷ $i = j \wedge (\forall\, k{\in}i.\ m\ k{=}\ n\ k)$
⟨*proof*⟩

lemma *finmap-of-inj-on-extensional-finite*:
assumes *finite K*
assumes $S \subseteq \mathit{extensional}\ K$
shows *inj-on* (*finmap-of K*) *S*
⟨*proof*⟩

22.3 Product set of Finite Maps

This is *Pi* for Finite Maps, most of this is copied

definition $\mathit{Pi'} :: 'i\ \mathit{set} \Rightarrow ('i \Rightarrow 'a\ \mathit{set}) \Rightarrow ('i \Rightarrow_F 'a)\ \mathit{set}$ **where**
$\mathit{Pi'}\ I\ A = \{\ P.\ \mathit{domain}\ P = I \wedge (\forall\, i.\ i \in I \longrightarrow (P)_F\ i \in A\ i)\ \}$

syntax
$\mathit{-Pi'} :: [\mathit{pttrn}, 'a\ \mathit{set}, 'b\ \mathit{set}] => ('a => 'b)\ \mathit{set}$ $((3\Pi''\ \text{-}{\in}\text{-}./\ \text{-}) \quad 10)$
translations
$\Pi'\ x{\in}A.\ B == \mathit{CONST}\ \mathit{Pi'}\ A\ (\lambda x.\ B)$

22.3.1 Basic Properties of *Pi'*

lemma *Pi'-I*[*intro*!]: *domain f* = *A* ⟹ $(\bigwedge x.\ x \in A \Longrightarrow f\ x \in B\ x) \Longrightarrow f \in \mathit{Pi'}\ A\ B$
⟨*proof*⟩

lemma *Pi'-I'*[*simp*]: *domain f* = *A* ⟹ $(\bigwedge x.\ x \in A \longrightarrow f\ x \in B\ x) \Longrightarrow f \in \mathit{Pi'}\ A\ B$
⟨*proof*⟩

lemma *Pi'-mem*: $f{\in}\ \mathit{Pi'}\ A\ B \Longrightarrow x \in A \Longrightarrow f\ x \in B\ x$
⟨*proof*⟩

lemma *Pi'-iff*: $f \in \mathit{Pi'}\ I\ X \longleftrightarrow \mathit{domain}\ f = I \wedge (\forall\, i{\in}I.\ f\ i \in X\ i)$
⟨*proof*⟩

lemma *Pi'E* [*elim*]:

$f \in Pi'\ A\ B \Longrightarrow (f\ x \in B\ x \Longrightarrow domain\ f = A \Longrightarrow Q) \Longrightarrow (x \notin A \Longrightarrow Q) \Longrightarrow Q$
⟨*proof*⟩

lemma *in-Pi'-cong*:
$domain\ f = domain\ g \Longrightarrow (\bigwedge w.\ w \in A \Longrightarrow f\ w = g\ w) \Longrightarrow f \in Pi'\ A\ B \longleftrightarrow g \in Pi'\ A\ B$
⟨*proof*⟩

lemma *Pi'-eq-empty*[*simp*]:
assumes *finite A* **shows** $(Pi'\ A\ B) = \{\} \longleftrightarrow (\exists x{\in}A.\ B\ x = \{\})$
⟨*proof*⟩

lemma *Pi'-mono*: $(\bigwedge x.\ x \in A \Longrightarrow B\ x \subseteq C\ x) \Longrightarrow Pi'\ A\ B \subseteq Pi'\ A\ C$
⟨*proof*⟩

lemma *Pi-Pi'*: $finite\ A \Longrightarrow (Pi_E\ A\ B) = proj\ `\ Pi'\ A\ B$
⟨*proof*⟩

22.4 Topological Space of Finite Maps

instantiation *fmap* :: (*type*, *topological-space*) *topological-space*
begin

definition *open-fmap* :: $('a \Rightarrow_F 'b)\ set \Rightarrow bool$ **where**
[*code del*]: *open-fmap* = *generate-topology* $\{Pi'\ a\ b | a\ b.\ \forall i{\in}a.\ open\ (b\ i)\}$

lemma *open-Pi'I*: $(\bigwedge i.\ i \in I \Longrightarrow open\ (A\ i)) \Longrightarrow open\ (Pi'\ I\ A)$
⟨*proof*⟩

instance ⟨*proof*⟩

end

lemma *open-restricted-space*:
shows *open* $\{m.\ P\ (domain\ m)\}$
⟨*proof*⟩

lemma *closed-restricted-space*:
shows *closed* $\{m.\ P\ (domain\ m)\}$
⟨*proof*⟩

lemma *tendsto-proj*: $((\lambda x.\ x) \longrightarrow a)\ F \Longrightarrow ((\lambda x.\ (x)_F\ i) \longrightarrow (a)_F\ i)\ F$
⟨*proof*⟩

lemma *continuous-proj*:
shows *continuous-on s* $(\lambda x.\ (x)_F\ i)$
⟨*proof*⟩

instance *fmap* :: (*type*, *first-countable-topology*) *first-countable-topology*
⟨*proof*⟩

22.5 Metric Space of Finite Maps

instantiation *fmap* :: (*type*, *metric-space*) *dist*
begin

definition *dist-fmap* **where**
dist P Q = *Max* (*range* (λ*i*. *dist* ((*P*)$_F$ *i*) ((*Q*)$_F$ *i*))) + (*if domain P* = *domain Q then 0 else 1*)

instance ⟨*proof*⟩
end

instantiation *fmap* :: (*type*, *metric-space*) *uniformity-dist*
begin

definition [*code del*]:
(*uniformity* :: (('*a*, '*b*) *fmap* × ('*a* ⇒$_F$ '*b*)) *filter*) =
(*INF e*∈{*0* <..}. *principal* {(*x*, *y*). *dist x y* < *e*})

instance
⟨*proof*⟩
end

declare *uniformity-Abort*[**where** '*a*=('*a* ⇒$_F$ '*b*::*metric-space*), *code*]

instantiation *fmap* :: (*type*, *metric-space*) *metric-space*
begin

lemma *finite-proj-image'*: *x* ∉ *domain P* ⟹ *finite* ((*P*)$_F$ ' *S*)
⟨*proof*⟩

lemma *finite-proj-image*: *finite* ((*P*)$_F$ ' *S*)
⟨*proof*⟩

lemma *finite-proj-diag*: *finite* ((λ*i*. *d* ((*P*)$_F$ *i*) ((*Q*)$_F$ *i*)) ' *S*)
⟨*proof*⟩

lemma *dist-le-1-imp-domain-eq*:
shows *dist P Q* < *1* ⟹ *domain P* = *domain Q*
⟨*proof*⟩

lemma *dist-proj*:
shows *dist* ((*x*)$_F$ *i*) ((*y*)$_F$ *i*) ≤ *dist x y*
⟨*proof*⟩

lemma *dist-finmap-lessI*:

assumes *domain P = domain Q*
assumes *0 < e*
assumes $\bigwedge i.\ i \in domain\ P \Longrightarrow dist\ (P\ i)\ (Q\ i) < e$
shows *dist P Q < e*
⟨*proof*⟩

instance
⟨*proof*⟩

end

22.6 Complete Space of Finite Maps

lemma *tendsto-finmap*:
fixes *f*::*nat* ⇒ (*'i* ⇒$_F$ (*'a*::*metric-space*))
assumes *ind-f*: $\bigwedge n.\ domain\ (f\ n) = domain\ g$
assumes *proj-g*: $\bigwedge i.\ i \in domain\ g \Longrightarrow (\lambda n.\ (f\ n)\ i) \longrightarrow g\ i$
shows $f \longrightarrow g$
⟨*proof*⟩

instance *fmap* :: (*type*, *complete-space*) *complete-space*
⟨*proof*⟩

22.7 Second Countable Space of Finite Maps

instantiation *fmap* :: (*countable*, *second-countable-topology*) *second-countable-topology*
begin

definition *basis-proj*::*'b set set*
where *basis-proj* = (*SOME B. countable B* ∧ *topological-basis B*)

lemma *countable-basis-proj*: *countable basis-proj* **and** *basis-proj*: *topological-basis basis-proj*
⟨*proof*⟩

definition *basis-finmap*::(*'a* ⇒$_F$ *'b*) *set set*
where *basis-finmap* = {*Pi' I S*|*I S. finite I* ∧ (∀ *i* ∈ *I. S i* ∈ *basis-proj*)}

lemma *in-basis-finmapI*:
assumes *finite I* **assumes** $\bigwedge i.\ i \in I \Longrightarrow S\ i \in basis\text{-}proj$
shows *Pi' I S* ∈ *basis-finmap*
⟨*proof*⟩

lemma *basis-finmap-eq*:
assumes *basis-proj* ≠ {}
shows *basis-finmap* = (λ*f*. *Pi'* (*domain f*) (λ*i*. *from-nat-into basis-proj* ((*f*)$_F$ *i*))) ‘
(*UNIV*::(*'a* ⇒$_F$ *nat*) *set*) (**is** - = *?f* ‘ -)
⟨*proof*⟩

lemma *basis-finmap-eq-empty*: *basis-proj* = {} ⟹ *basis-finmap* = {*Pi'* {} *undefined*}
⟨*proof*⟩

lemma *countable-basis-finmap*: *countable basis-finmap*
⟨*proof*⟩

lemma *finmap-topological-basis*:
topological-basis basis-finmap
⟨*proof*⟩

lemma *range-enum-basis-finmap-imp-open*:
assumes $x \in$ *basis-finmap*
shows *open x*
⟨*proof*⟩

instance ⟨*proof*⟩

end

22.8 Polish Space of Finite Maps

instance *fmap* :: (*countable*, *polish-space*) *polish-space* ⟨*proof*⟩

22.9 Product Measurable Space of Finite Maps

definition *PiF I M* ≡
sigma (⋃ *J* ∈ *I*. (Π' *j*∈*J*. *space* (*M j*))) {(Π' *j*∈*J*. *X j*) |*X J*. *J* ∈ *I* ∧ *X* ∈ (Π *j*∈*J*. *sets* (*M j*))}

abbreviation
Pi_F *I M* ≡ *PiF I M*

syntax
-PiF :: *pttrn* ⇒ *'i set* ⇒ *'a measure* ⇒ (*'i* => *'a*) *measure* ((3Π_F -∈-./ -) 10)
translations
Π_F *x*∈*I*. *M* == *CONST PiF I* (%*x*. *M*)

lemma *PiF-gen-subset*: {(Π' *j*∈*J*. *X j*) |*X J*. *J* ∈ *I* ∧ *X* ∈ (Π *j*∈*J*. *sets* (*M j*))}
⊆
Pow (⋃ *J* ∈ *I*. (Π' *j*∈*J*. *space* (*M j*)))
⟨*proof*⟩

lemma *space-PiF*: *space* (*PiF I M*) = (⋃ *J* ∈ *I*. (Π' *j*∈*J*. *space* (*M j*)))
⟨*proof*⟩

lemma *sets-PiF*:
sets (*PiF I M*) = *sigma-sets* (⋃ *J* ∈ *I*. (Π' *j*∈*J*. *space* (*M j*)))
{(Π' *j*∈*J*. *X j*) |*X J*. *J* ∈ *I* ∧ *X* ∈ (Π *j*∈*J*. *sets* (*M j*))}
⟨*proof*⟩

lemma *sets-PiF-singleton*:
sets (*PiF* {*I*} *M*) = *sigma-sets* (Π′ *j*∈*I*. *space* (*M j*))
{(Π′ *j*∈*I*. *X j*) |*X*. *X* ∈ (Π *j*∈*I*. *sets* (*M j*))}
⟨*proof*⟩

lemma *in-sets-PiFI*:
assumes *X* = (*Pi*′ *J S*) *J* ∈ *I* ⋀*i*. *i*∈*J* ⟹ *S i* ∈ *sets* (*M i*)
shows *X* ∈ *sets* (*PiF I M*)
⟨*proof*⟩

lemma *product-in-sets-PiFI*:
assumes *J* ∈ *I* ⋀*i*. *i*∈*J* ⟹ *S i* ∈ *sets* (*M i*)
shows (*Pi*′ *J S*) ∈ *sets* (*PiF I M*)
⟨*proof*⟩

lemma *singleton-space-subset-in-sets*:
fixes *J*
assumes *J* ∈ *I*
assumes *finite J*
shows *space* (*PiF* {*J*} *M*) ∈ *sets* (*PiF I M*)
⟨*proof*⟩

lemma *singleton-subspace-set-in-sets*:
assumes *A*: *A* ∈ *sets* (*PiF* {*J*} *M*)
assumes *finite J*
assumes *J* ∈ *I*
shows *A* ∈ *sets* (*PiF I M*)
⟨*proof*⟩

lemma *finite-measurable-singletonI*:
assumes *finite I*
assumes ⋀*J*. *J* ∈ *I* ⟹ *finite J*
assumes *MN*: ⋀*J*. *J* ∈ *I* ⟹ *A* ∈ *measurable* (*PiF* {*J*} *M*) *N*
shows *A* ∈ *measurable* (*PiF I M*) *N*
⟨*proof*⟩

lemma *countable-finite-comprehension*:
fixes *f* :: *'a*::*countable set* ⇒ -
assumes ⋀*s*. *P s* ⟹ *finite s*
assumes ⋀*s*. *P s* ⟹ *f s* ∈ *sets M*
shows ⋃{*f s*|*s*. *P s*} ∈ *sets M*
⟨*proof*⟩

lemma *space-subset-in-sets*:
fixes *J*::*'a*::*countable set set*
assumes *J* ⊆ *I*
assumes ⋀*j*. *j* ∈ *J* ⟹ *finite j*
shows *space* (*PiF J M*) ∈ *sets* (*PiF I M*)

⟨*proof*⟩

lemma *subspace-set-in-sets*:
fixes $J::'a::countable\ set\ set$
assumes A: $A \in sets\ (PiF\ J\ M)$
assumes $J \subseteq I$
assumes $\bigwedge j.\ j \in J \Longrightarrow finite\ j$
shows $A \in sets\ (PiF\ I\ M)$
⟨*proof*⟩

lemma *countable-measurable-PiFI*:
fixes $I::'a::countable\ set\ set$
assumes MN: $\bigwedge J.\ J \in I \Longrightarrow finite\ J \Longrightarrow A \in measurable\ (PiF\ \{J\}\ M)\ N$
shows $A \in measurable\ (PiF\ I\ M)\ N$
⟨*proof*⟩

lemma *measurable-PiF*:
assumes f: $\bigwedge x.\ x \in space\ N \Longrightarrow domain\ (f\ x) \in I \wedge (\forall\, i \in domain\ (f\ x).\ (f\ x)\ i \in space\ (M\ i))$
assumes S: $\bigwedge J\ S.\ J \in I \Longrightarrow (\bigwedge i.\ i \in J \Longrightarrow S\ i \in sets\ (M\ i)) \Longrightarrow$
$f\ -`\ (Pi'\ J\ S) \cap space\ N \in sets\ N$
shows $f \in measurable\ N\ (PiF\ I\ M)$
⟨*proof*⟩

lemma *restrict-sets-measurable*:
assumes A: $A \in sets\ (PiF\ I\ M)$ **and** $J \subseteq I$
shows $A \cap \{m.\ domain\ m \in J\} \in sets\ (PiF\ J\ M)$
⟨*proof*⟩

lemma *measurable-finmap-of*:
assumes f: $\bigwedge i.\ (\exists\, x \in space\ N.\ i \in J\ x) \Longrightarrow (\lambda x.\ f\ x\ i) \in measurable\ N\ (M\ i)$
assumes J: $\bigwedge x.\ x \in space\ N \Longrightarrow J\ x \in I\ \bigwedge x.\ x \in space\ N \Longrightarrow finite\ (J\ x)$
assumes JN: $\bigwedge S.\ \{x.\ J\ x = S\} \cap space\ N \in sets\ N$
shows $(\lambda x.\ finmap\text{-}of\ (J\ x)\ (f\ x)) \in measurable\ N\ (PiF\ I\ M)$
⟨*proof*⟩

lemma *measurable-PiM-finmap-of*:
assumes *finite* J
shows $finmap\text{-}of\ J \in measurable\ (Pi_M\ J\ M)\ (PiF\ \{J\}\ M)$
⟨*proof*⟩

lemma *proj-measurable-singleton*:
assumes $A \in sets\ (M\ i)$
shows $(\lambda x.\ (x)_F\ i)\ -`\ A \cap space\ (PiF\ \{I\}\ M) \in sets\ (PiF\ \{I\}\ M)$
⟨*proof*⟩

lemma *measurable-proj-singleton*:
assumes $i \in I$
shows $(\lambda x.\ (x)_F\ i) \in measurable\ (PiF\ \{I\}\ M)\ (M\ i)$

⟨*proof*⟩

lemma *measurable-proj-countable*:
fixes I::'a::*countable set set*
assumes $y \in$ *space* $(M\ i)$
shows $(\lambda x.$ *if* $i \in$ *domain* x *then* $(x)_F\ i$ *else* $y) \in$ *measurable* $(PiF\ I\ M)\ (M\ i)$
⟨*proof*⟩

lemma *measurable-restrict-proj*:
assumes $J \in$ *II finite* J
shows *finmap-of* $J \in$ *measurable* $(PiM\ J\ M)\ (PiF\ II\ M)$
⟨*proof*⟩

lemma *measurable-proj-PiM*:
fixes $J\ K$ ::'a::*countable set* **and** I::'a *set set*
assumes *finite* J $J \in I$
assumes $x \in$ *space* $(PiM\ J\ M)$
shows *proj* $\in$ *measurable* $(PiF\ \{J\}\ M)\ (PiM\ J\ M)$
⟨*proof*⟩

lemma *space-PiF-singleton-eq-product*:
assumes *finite* I
shows *space* $(PiF\ \{I\}\ M) = (\Pi'\ i{\in}I.$ *space* $(M\ i))$
⟨*proof*⟩

adapted from *sets* $(Pi_M\ ?I\ ?M) =$ *sigma-sets* $(\Pi_E\ i{\in}?I.$ *space* $(?M\ i))\ \{\{f \in \Pi_E\ i{\in}?I.$ *space* $(?M\ i).\ f\ i \in A\} \mid i\ A.\ i \in ?I \wedge A \in$ *sets* $(?M\ i)\}$

lemma *sets-PiF-single*:
assumes *finite* I $I \neq \{\}$
shows *sets* $(PiF\ \{I\}\ M) =$
sigma-sets $(\Pi'\ i{\in}I.$ *space* $(M\ i))$
$\{\{f{\in}\Pi'\ i{\in}I.$ *space* $(M\ i).\ f\ i \in A\} \mid i\ A.\ i \in I \wedge A \in$ *sets* $(M\ i)\}$
(**is** - = *sigma-sets* $?\Omega\ ?R$)
⟨*proof*⟩

adapted from $(\bigwedge i.\ i \in ?I \Longrightarrow ?A\ i = ?B\ i) \Longrightarrow Pi_E\ ?I\ ?A = Pi_E\ ?I\ ?B$

lemma *Pi'-cong*:
assumes *finite* I
assumes $\bigwedge i.\ i \in I \Longrightarrow f\ i = g\ i$
shows $Pi'\ I\ f = Pi'\ I\ g$
⟨*proof*⟩

adapted from $[\![$*finite* $?I$; $\bigwedge i\ n\ m.\ [\![i \in ?I;\ n \leq m]\!] \Longrightarrow ?A\ n\ i \subseteq ?A\ m\ i]\!] \Longrightarrow (\bigcup_n\ Pi\ ?I\ (?A\ n)) = (\Pi\ i{\in}?I.\ \bigcup_n\ ?A\ n\ i)$

lemma *Pi'-UN*:
fixes A :: *nat* ⇒ 'i ⇒ 'a *set*
assumes *finite* I
assumes *mono*: $\bigwedge i\ n\ m.\ i \in I \Longrightarrow n \leq m \Longrightarrow A\ n\ i \subseteq A\ m\ i$

shows $(\bigcup n.\ Pi'\ I\ (A\ n)) = Pi'\ I\ (\lambda i.\ \bigcup n.\ A\ n\ i)$
⟨*proof*⟩

adapted from $[\![\bigwedge i.\ i \in ?I \Longrightarrow \exists\, S \subseteq ?E\ i.\ countable\ S \wedge ?\Omega\ i = \bigcup\ S;\ \bigwedge i.\ i \in ?I \Longrightarrow ?E\ i \subseteq Pow\ (?\Omega\ i);\ \bigwedge j.\ j \in ?J \Longrightarrow finite\ j;\ \bigcup\ ?J = ?I]\!] \Longrightarrow sets\ (Pi_M\ ?I\ (\lambda i.\ sigma\ (?\Omega\ i)\ (?E\ i))) = sets\ (sigma\ (Pi_E\ ?I\ ?\Omega)\ \{\{f \in Pi_E\ ?I\ ?\Omega.\ \forall\, i \in j.\ f\ i \in A\ i\}\ |A\ j.\ j \in ?J \wedge A \in Pi\ j\ ?E\})$

lemma *sigma-fprod-algebra-sigma-eq*:
fixes $E :: 'i \Rightarrow 'a\ set\ set$ **and** $S :: 'i \Rightarrow nat \Rightarrow 'a\ set$
assumes [*simp*]: *finite I* $I \neq \{\}$
and *S-union*: $\bigwedge i.\ i \in I \Longrightarrow (\bigcup j.\ S\ i\ j) = space\ (M\ i)$
and *S-in-E*: $\bigwedge i.\ i \in I \Longrightarrow range\ (S\ i) \subseteq E\ i$
assumes *E-closed*: $\bigwedge i.\ i \in I \Longrightarrow E\ i \subseteq Pow\ (space\ (M\ i))$
and *E-generates*: $\bigwedge i.\ i \in I \Longrightarrow sets\ (M\ i) = sigma\text{-}sets\ (space\ (M\ i))\ (E\ i)$
defines $P == \{\ Pi'\ I\ F \mid F.\ \forall\, i \in I.\ F\ i \in E\ i\ \}$
shows $sets\ (PiF\ \{I\}\ M) = sigma\text{-}sets\ (space\ (PiF\ \{I\}\ M))\ P$
⟨*proof*⟩

lemma *product-open-generates-sets-PiF-single*:
assumes $I \neq \{\}$
assumes [*simp*]: *finite I*
shows $sets\ (PiF\ \{I\}\ (\lambda\text{-}.\ borel::'b::second\text{-}countable\text{-}topology\ measure)) =$
$sigma\text{-}sets\ (space\ (PiF\ \{I\}\ (\lambda\text{-}.\ borel)))\ \{Pi'\ I\ F\ |F.\ (\forall\, i \in I.\ F\ i \in Collect\ open)\}$
⟨*proof*⟩

lemma *finmap-UNIV*[*simp*]: $(\bigcup J \in Collect\ finite.\ \Pi'\ j \in J.\ UNIV) = UNIV$ ⟨*proof*⟩

lemma *borel-eq-PiF-borel*:
shows $(borel :: ('i::countable \Rightarrow_F 'a::polish\text{-}space)\ measure) =$
$PiF\ (Collect\ finite)\ (\lambda\text{-}.\ borel :: 'a\ measure)$
⟨*proof*⟩

22.10 Isomorphism between Functions and Finite Maps

lemma *measurable-finmap-compose*:
shows $(\lambda m.\ compose\ J\ m\ f) \in measurable\ (PiM\ (f\ `\ J)\ (\lambda\text{-}.\ M))\ (PiM\ J\ (\lambda\text{-}.\ M))$
⟨*proof*⟩

lemma *measurable-compose-inv*:
assumes *inj*: $\bigwedge j.\ j \in J \Longrightarrow f'\ (f\ j) = j$
shows $(\lambda m.\ compose\ (f\ `\ J)\ m\ f') \in measurable\ (PiM\ J\ (\lambda\text{-}.\ M))\ (PiM\ (f\ `\ J)\ (\lambda\text{-}.\ M))$
⟨*proof*⟩

locale *function-to-finmap* =
fixes $J::'a\ set$ **and** $f :: 'a \Rightarrow 'b::countable$ **and** f'
assumes [*simp*]: *finite J*

assumes *inv*: $i \in J \Longrightarrow f'\ (f\ i) = i$
begin

to measure finmaps

definition $fm = (\mathit{finmap\text{-}of}\ (f\ `\ J))\ o\ (\lambda g.\ \mathit{compose}\ (f\ `\ J)\ g\ f')$

lemma *domain-fm*[*simp*]: $\mathit{domain}\ (\mathit{fm}\ x) = f\ `\ J$
⟨*proof*⟩

lemma *fm-restrict*[*simp*]: $\mathit{fm}\ (\mathit{restrict}\ y\ J) = \mathit{fm}\ y$
⟨*proof*⟩

lemma *fm-product*:
assumes $\bigwedge i.\ \mathit{space}\ (M\ i) = \mathit{UNIV}$
shows $\mathit{fm} -` Pi'\ (f\ `\ J)\ S \cap \mathit{space}\ (Pi_M\ J\ M) = (\Pi_E\ j \in J.\ S\ (f\ j))$
⟨*proof*⟩

lemma *fm-measurable*:
assumes $f\ `\ J \in N$
shows $\mathit{fm} \in \mathit{measurable}\ (Pi_M\ J\ (\lambda\text{-}.\ M))\ (Pi_F\ N\ (\lambda\text{-}.\ M))$
⟨*proof*⟩

lemma *proj-fm*:
assumes $x \in J$
shows $\mathit{fm}\ m\ (f\ x) = m\ x$
⟨*proof*⟩

lemma *inj-on-compose-f'*: $\mathit{inj\text{-}on}\ (\lambda g.\ \mathit{compose}\ (f\ `\ J)\ g\ f')\ (\mathit{extensional}\ J)$
⟨*proof*⟩

lemma *inj-on-fm*:
assumes $\bigwedge i.\ \mathit{space}\ (M\ i) = \mathit{UNIV}$
shows $\mathit{inj\text{-}on}\ \mathit{fm}\ (\mathit{space}\ (Pi_M\ J\ M))$
⟨*proof*⟩

to measure functions

definition $mf = (\lambda g.\ \mathit{compose}\ J\ g\ f)\ o\ \mathit{proj}$

lemma *mf-fm*:
assumes $x \in \mathit{space}\ (Pi_M\ J\ (\lambda\text{-}.\ M))$
shows $\mathit{mf}\ (\mathit{fm}\ x) = x$
⟨*proof*⟩

lemma *mf-measurable*:
assumes $\mathit{space}\ M = \mathit{UNIV}$
shows $\mathit{mf} \in \mathit{measurable}\ (\mathit{PiF}\ \{f\ `\ J\}\ (\lambda\text{-}.\ M))\ (\mathit{PiM}\ J\ (\lambda\text{-}.\ M))$
⟨*proof*⟩

lemma *fm-image-measurable*:

assumes *space M = UNIV*
assumes *X ∈ sets (Pi$_M$ J (λ-. M))*
shows *fm ' X ∈ sets (PiF {f ' J} (λ-. M))*
⟨*proof*⟩

lemma *fm-image-measurable-finite*:
assumes *space M = UNIV*
assumes *X ∈ sets (Pi$_M$ J (λ-. M::'c measure))*
shows *fm ' X ∈ sets (PiF (Collect finite) (λ-. M::'c measure))*
⟨*proof*⟩

measure on finmaps

definition *mapmeasure M N = distr M (PiF (Collect finite) N) (fm)*

lemma *sets-mapmeasure*[*simp*]: *sets (mapmeasure M N) = sets (PiF (Collect finite) N)*
⟨*proof*⟩

lemma *space-mapmeasure*[*simp*]: *space (mapmeasure M N) = space (PiF (Collect finite) N)*
⟨*proof*⟩

lemma *mapmeasure-PiF*:
assumes *s1*: *space M = space (Pi$_M$ J (λ-. N))*
assumes *s2*: *sets M = sets (Pi$_M$ J (λ-. N))*
assumes *space N = UNIV*
assumes *X ∈ sets (PiF (Collect finite) (λ-. N))*
shows *emeasure (mapmeasure M (λ-. N)) X = emeasure M ((fm −' X ∩ extensional J))*
⟨*proof*⟩

lemma *mapmeasure-PiM*:
fixes *N::'c measure*
assumes *s1*: *space M = space (Pi$_M$ J (λ-. N))*
assumes *s2*: *sets M = (Pi$_M$ J (λ-. N))*
assumes *N*: *space N = UNIV*
assumes *X*: *X ∈ sets M*
shows *emeasure M X = emeasure (mapmeasure M (λ-. N)) (fm ' X)*
⟨*proof*⟩

end

end

23 Projective Limit

theory *Projective-Limit*
imports
Fin-Map

Infinite-Product-Measure
HOL−Library.Diagonal-Subsequence
begin

23.1 Sequences of Finite Maps in Compact Sets

locale *finmap-seqs-into-compact* =
fixes K::$nat \Rightarrow (nat \Rightarrow_F 'a$::*metric-space*$) set$ **and** f::$nat \Rightarrow (nat \Rightarrow_F 'a)$ **and** M
assumes *compact*: $\bigwedge n.$ *compact* $(K\ n)$
assumes *f-in-K*: $\bigwedge n.\ K\ n \neq \{\}$
assumes *domain-K*: $\bigwedge n.\ k \in K\ n \Longrightarrow$ *domain* $k =$ *domain* $(f\ n)$
assumes *proj-in-K*:
$\bigwedge t\ n\ m.\ m \geq n \Longrightarrow t \in$ *domain* $(f\ n) \Longrightarrow (f\ m)_F\ t \in (\lambda k.\ (k)_F\ t)\ `\ K\ n$
begin

lemma *proj-in-K'*: $(\exists n.\ \forall m \geq n.\ (f\ m)_F\ t \in (\lambda k.\ (k)_F\ t)\ `\ K\ n)$
⟨*proof*⟩

lemma *proj-in-KE*:
obtains n **where** $\bigwedge m.\ m \geq n \Longrightarrow (f\ m)_F\ t \in (\lambda k.\ (k)_F\ t)\ `\ K\ n$
⟨*proof*⟩

lemma *compact-projset*:
shows *compact* $((\lambda k.\ (k)_F\ i)\ `\ K\ n)$
⟨*proof*⟩

end

lemma *compactE'*:
fixes $S :: 'a ::$ *metric-space set*
assumes *compact* $S\ \forall n{\geq}m.\ f\ n \in S$
obtains $l\ r$ **where** $l \in S$ *strict-mono* $(r$::$nat{\Rightarrow}nat)\ ((f \circ r) \longrightarrow l)$ *sequentially*
⟨*proof*⟩

sublocale *finmap-seqs-into-compact* $\subseteq$ *subseqs* $\lambda n\ s.\ (\exists l.\ (\lambda i.\ ((f\ o\ s)\ i)_F\ n) \longrightarrow l)$
⟨*proof*⟩

lemma (**in** *finmap-seqs-into-compact*) *diagonal-tendsto*: $\exists l.\ (\lambda i.\ (f\ (diagseq\ i))_F\ n) \longrightarrow l$
⟨*proof*⟩

23.2 Daniell-Kolmogorov Theorem

Existence of Projective Limit

locale *polish-projective* = *projective-family I P* λ-. *borel*::$'a$::*polish-space measure*
for I::$'i$ *set* **and** P
begin

lemma *emeasure-lim-emb*:
  assumes X: $J \subseteq I$ *finite* J $X \in$ *sets* $(\Pi_M\ i{\in}J.\ borel)$
  shows *lim* $(emb\ I\ J\ X) = P\ J\ X$
⟨*proof*⟩

lemma *measure-lim-emb*:
  $J \subseteq I \Longrightarrow$ *finite* $J \Longrightarrow X \in$ *sets* $(\Pi_M\ i{\in}J.\ borel) \Longrightarrow$ *measure lim* $(emb\ I\ J\ X)$ $=$ *measure* $(P\ J)\ X$
  ⟨*proof*⟩

end

hide-const (**open**) *PiF*
hide-const (**open**) Pi_F
hide-const (**open**) Pi'
hide-const (**open**) *finmap-of*
hide-const (**open**) *proj*
hide-const (**open**) *domain*
hide-const (**open**) *basis-finmap*

sublocale *polish-projective* $\subseteq$ P: *prob-space lim*
⟨*proof*⟩

locale *polish-product-prob-space* =
  product-prob-space λ-. *borel*::('*a*::*polish-space*) *measure* I **for** I::'*i set*

sublocale *polish-product-prob-space* $\subseteq$ P: *polish-projective* I $\lambda J.\ PiM\ J$ (λ-. *borel*::('*a*) *measure*)
  ⟨*proof*⟩

lemma (**in** *polish-product-prob-space*) *limP-eq-PiM*: *lim* $=$ *PiM* I (λ-. *borel*)
  ⟨*proof*⟩

end

24 Random Permutations

theory *Random-Permutations*
imports
  HOL−Combinatorics.Multiset-Permutations
  Probability-Mass-Function
begin

Choosing a set permutation (i.e. a distinct list with the same elements as the set) uniformly at random is the same as first choosing the first element of the list and then choosing the rest of the list as a permutation of the remaining set.

lemma *random-permutation-of-set*:
  assumes *finite A* $A \neq \{\}$
  shows *pmf-of-set* (*permutations-of-set A*) =
    do {
      $x \leftarrow$ *pmf-of-set A*;
      xs $\leftarrow$ *pmf-of-set* (*permutations-of-set* ($A - \{x\}$));
      return-pmf ($x \# xs$)
    } (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

A generic fold function that takes a function, an initial state, and a set and chooses a random order in which it then traverses the set in the same fashion as a left fold over a list. We first give a recursive definition.

function *fold-random-permutation* :: $('a \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'b \Rightarrow 'a$ *set* $\Rightarrow 'b$ *pmf*
where
  fold-random-permutation f x $\{\}$ = *return-pmf x*
| $\neg$*finite A* $\Longrightarrow$ *fold-random-permutation f x A* = *return-pmf x*
| *finite A* $\Longrightarrow$ $A \neq \{\} \Longrightarrow$
    fold-random-permutation f x A =
      pmf-of-set A $\gg\!=$ (λa. *fold-random-permutation f* (*f a x*) ($A - \{a\}$))
  ⟨*proof*⟩
termination ⟨*proof*⟩

We can now show that the above recursive definition is equivalent to choosing a random set permutation and folding over it (in any direction).

lemma *fold-random-permutation-foldl*:
  assumes *finite A*
  shows *fold-random-permutation f x A* =
    map-pmf (*foldl* ($\lambda x\ y$. *f y x*) *x*) (*pmf-of-set* (*permutations-of-set A*))
⟨*proof*⟩

lemma *fold-random-permutation-foldr*:
  assumes *finite A*
  shows *fold-random-permutation f x A* =
    map-pmf (λxs. *foldr f xs x*) (*pmf-of-set* (*permutations-of-set A*))
⟨*proof*⟩

lemma *fold-random-permutation-fold*:
  assumes *finite A*
  shows *fold-random-permutation f x A* =
    map-pmf (λxs. *fold f xs x*) (*pmf-of-set* (*permutations-of-set A*))
  ⟨*proof*⟩

lemma *fold-random-permutation-code* [*code*]:
  fold-random-permutation f x (*set xs*) =
    map-pmf (*foldl* ($\lambda x\ y$. *f y x*) *x*) (*pmf-of-set* (*permutations-of-set* (*set xs*)))
  ⟨*proof*⟩

We now introduce a slightly generalised version of the above fold operation

that does not simply return the result in the end, but applies a monadic bind to it. This may seem somewhat arbitrary, but it is a common use case, e.g. in the Social Decision Scheme of Random Serial Dictatorship, where voters narrow down a set of possible winners in a random order and the winner is chosen from the remaining set uniformly at random.

function *fold-bind-random-permutation*
  :: $('a \Rightarrow 'b \Rightarrow 'b) \Rightarrow ('b \Rightarrow 'c\ pmf) \Rightarrow 'b \Rightarrow 'a\ set \Rightarrow 'c\ pmf$ **where**
 fold-bind-random-permutation f g x $\{\}$ = *g x*
| ¬*finite A* ⟹ *fold-bind-random-permutation f g x A* = *g x*
| *finite A* ⟹ *A* ≠ $\{\}$ ⟹
    fold-bind-random-permutation f g x A =
      pmf-of-set A ⤜ (λ*a*. *fold-bind-random-permutation f g* (*f a x*) (*A* − {*a*}))
 ⟨*proof*⟩
termination ⟨*proof*⟩

We now show that the recursive definition is equivalent to a random fold followed by a monadic bind.

lemma *fold-bind-random-permutation-altdef* [*code*]:
 fold-bind-random-permutation f g x A = *fold-random-permutation f x A* ⤜ *g*
⟨*proof*⟩

We can now derive the following nice monadic representations of the combined fold-and-bind:

lemma *fold-bind-random-permutation-foldl*:
 assumes *finite A*
 shows *fold-bind-random-permutation f g x A* =
        do {*xs* ← *pmf-of-set* (*permutations-of-set A*); *g* (*foldl* (λ*x y*. *f y x*) *x xs*)}
 ⟨*proof*⟩

lemma *fold-bind-random-permutation-foldr*:
 assumes *finite A*
 shows *fold-bind-random-permutation f g x A* =
        do {*xs* ← *pmf-of-set* (*permutations-of-set A*); *g* (*foldr f xs x*)}
 ⟨*proof*⟩

lemma *fold-bind-random-permutation-fold*:
 assumes *finite A*
 shows *fold-bind-random-permutation f g x A* =
        do {*xs* ← *pmf-of-set* (*permutations-of-set A*); *g* (*fold f xs x*)}
 ⟨*proof*⟩

The following useful lemma allows us to swap partitioning a set w. r. t. a predicate and drawing a random permutation of that set.

lemma *partition-random-permutations*:
 assumes *finite A*
 shows *map-pmf* (*partition P*) (*pmf-of-set* (*permutations-of-set A*)) =
        pair-pmf (*pmf-of-set* (*permutations-of-set* {*x*∈*A*. *P x*}))

(*pmf-of-set* (*permutations-of-set* {*x*∈*A*. ¬*P x*})) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

end

25 Discrete subprobability distribution

theory *SPMF* **imports**
Probability-Mass-Function
HOL−Library.Complete-Partial-Order2
HOL−Library.Rewrite
begin

25.1 Auxiliary material

lemma *cSUP-singleton* [*simp*]: (*SUP x*∈{*x*}. *f x* :: - :: *conditionally-complete-lattice*) = *f x*
⟨*proof*⟩

25.1.1 More about extended reals

lemma [*simp*]:
shows *ennreal-max-0*: *ennreal* (*max 0 x*) = *ennreal x*
and *ennreal-max-0'*: *ennreal* (*max x 0*) = *ennreal x*
⟨*proof*⟩

lemma *e2ennreal-0* [*simp*]: *e2ennreal 0* = *0*
⟨*proof*⟩

lemma *enn2real-bot* [*simp*]: *enn2real* ⊥ = *0*
⟨*proof*⟩

lemma *continuous-at-ennreal*[*continuous-intros*]: *continuous F f* ⟹ *continuous F* (λ*x*. *ennreal* (*f x*))
⟨*proof*⟩

lemma *ennreal-Sup*:
assumes ∗: (*SUP a*∈*A*. *ennreal a*) ≠ ⊤
and *A* ≠ {}
shows *ennreal* (*Sup A*) = (*SUP a*∈*A*. *ennreal a*)
⟨*proof*⟩

lemma *ennreal-SUP*:
⟦ (*SUP a*∈*A*. *ennreal* (*f a*)) ≠ ⊤; *A* ≠ {} ⟧ ⟹ *ennreal* (*SUP a*∈*A*. *f a*) = (*SUP a*∈*A*. *ennreal* (*f a*))
⟨*proof*⟩

lemma *ennreal-lt-0*: *x* < *0* ⟹ *ennreal x* = *0*
⟨*proof*⟩

25.1.2 More about *'a option*

lemma *None-in-map-option-image* [*simp*]: *None* ∈ *map-option f* ‘ *A* ⟷ *None* ∈ *A*
⟨*proof*⟩

lemma *Some-in-map-option-image* [*simp*]: *Some x* ∈ *map-option f* ‘ *A* ⟷ (∃ *y*. *x* = *f y* ∧ *Some y* ∈ *A*)
⟨*proof*⟩

lemma *case-option-collapse*: *case-option x* (λ-. *x*) = (λ-. *x*)
⟨*proof*⟩

lemma *case-option-id*: *case-option None Some* = *id*
⟨*proof*⟩

inductive *ord-option* :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *'a option* ⇒ *'b option* ⇒ *bool*
  for *ord* :: *'a* ⇒ *'b* ⇒ *bool*
where
  None: *ord-option ord None x*
| *Some*: *ord x y* ⟹ *ord-option ord* (*Some x*) (*Some y*)

inductive-simps *ord-option-simps* [*simp*]:
  ord-option ord None x
  ord-option ord x None
  ord-option ord (*Some x*) (*Some y*)
  ord-option ord (*Some x*) *None*

inductive-simps *ord-option-eq-simps* [*simp*]:
  ord-option (=) *None y*
  ord-option (=) (*Some x*) *y*

lemma *ord-option-reflI*: (⋀*y*. *y* ∈ *set-option x* ⟹ *ord y y*) ⟹ *ord-option ord x x*
⟨*proof*⟩

lemma *reflp-ord-option*: *reflp ord* ⟹ *reflp* (*ord-option ord*)
⟨*proof*⟩

lemma *ord-option-trans*:
  ⟦ *ord-option ord x y*; *ord-option ord y z*;
    ⋀*a b c*. ⟦ *a* ∈ *set-option x*; *b* ∈ *set-option y*; *c* ∈ *set-option z*; *ord a b*; *ord b c* ⟧ ⟹ *ord a c* ⟧
  ⟹ *ord-option ord x z*
⟨*proof*⟩

lemma *transp-ord-option*: *transp ord* ⟹ *transp* (*ord-option ord*)
⟨*proof*⟩

lemma *antisymp-ord-option*: *antisymp ord* ⟹ *antisymp* (*ord-option ord*)

⟨*proof*⟩

lemma *ord-option-chainD*:
Complete-Partial-Order.chain (*ord-option ord*) *Y*
⟹ *Complete-Partial-Order.chain ord* {*x*. *Some x* ∈ *Y*}
⟨*proof*⟩

definition *lub-option* :: ('*a set* ⇒ '*b*) ⇒ '*a option set* ⇒ '*b option*
where *lub-option lub Y* = (*if Y* ⊆ {*None*} *then None else Some* (*lub* {*x*. *Some x* ∈ *Y*}))

lemma *map-lub-option*: *map-option f* (*lub-option lub Y*) = *lub-option* (*f* ∘ *lub*) *Y*
⟨*proof*⟩

lemma *lub-option-upper*:
assumes *Complete-Partial-Order.chain* (*ord-option ord*) *Y x* ∈ *Y*
and *lub-upper*: ⋀*Y x*. ⟦ *Complete-Partial-Order.chain ord Y*; *x* ∈ *Y* ⟧ ⟹ *ord x* (*lub Y*)
shows *ord-option ord x* (*lub-option lub Y*)
⟨*proof*⟩

lemma *lub-option-least*:
assumes *Y*: *Complete-Partial-Order.chain* (*ord-option ord*) *Y*
and *upper*: ⋀*x*. *x* ∈ *Y* ⟹ *ord-option ord x y*
assumes *lub-least*: ⋀*Y y*. ⟦ *Complete-Partial-Order.chain ord Y*; ⋀*x*. *x* ∈ *Y* ⟹ *ord x y* ⟧ ⟹ *ord* (*lub Y*) *y*
shows *ord-option ord* (*lub-option lub Y*) *y*
⟨*proof*⟩

lemma *lub-map-option*: *lub-option lub* (*map-option f* ' *Y*) = *lub-option* (*lub* ∘ (') *f*) *Y*
⟨*proof*⟩

lemma *ord-option-mono*: ⟦ *ord-option A x y*; ⋀*x y*. *A x y* ⟹ *B x y* ⟧ ⟹ *ord-option B x y*
⟨*proof*⟩

lemma *ord-option-mono'* [*mono*]:
(⋀*x y*. *A x y* ⟶ *B x y*) ⟹ *ord-option A x y* ⟶ *ord-option B x y*
⟨*proof*⟩

lemma *ord-option-compp*: *ord-option* (*A OO B*) = *ord-option A OO ord-option B*
⟨*proof*⟩

lemma *ord-option-inf*: *inf* (*ord-option A*) (*ord-option B*) = *ord-option* (*inf A B*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *ord-option-map2*: *ord-option ord x* (*map-option f y*) = *ord-option* (λ*x y*.

ord x (f y)) x y
⟨*proof*⟩

lemma *ord-option-map1*: *ord-option ord* (*map-option f x*) *y* = *ord-option* (λ*x y*. *ord* (*f x*) *y*) *x y*
⟨*proof*⟩

lemma *option-ord-Some1-iff*: *option-ord* (*Some x*) *y* ⟷ *y* = *Some x*
⟨*proof*⟩

25.1.3 A relator for sets that treats sets like predicates

context includes *lifting-syntax*
begin

definition *rel-pred* :: ('*a* ⇒ '*b* ⇒ *bool*) ⇒ '*a set* ⇒ '*b set* ⇒ *bool*
where *rel-pred R A B* = (*R* ===> (=)) (λ*x*. *x* ∈ *A*) (λ*y*. *y* ∈ *B*)

lemma *rel-predI*: (*R* ===> (=)) (λ*x*. *x* ∈ *A*) (λ*y*. *y* ∈ *B*) ⟹ *rel-pred R A B*
⟨*proof*⟩

lemma *rel-predD*: ⟦ *rel-pred R A B*; *R x y* ⟧ ⟹ *x* ∈ *A* ⟷ *y* ∈ *B*
⟨*proof*⟩

lemma *Collect-parametric*: ((*A* ===> (=)) ===> *rel-pred A*) *Collect Collect*
— Declare this rule as *transfer-rule* only locally because it blows up the search space for *transfer* (in combination with *Collect-transfer*)
⟨*proof*⟩

end

25.1.4 Monotonicity rules

lemma *monotone-gfp-eadd1*: *monotone* (≥) (≥) (λ*x*. *x* + *y* :: *enat*)
⟨*proof*⟩

lemma *monotone-gfp-eadd2*: *monotone* (≥) (≥) (λ*y*. *x* + *y* :: *enat*)
⟨*proof*⟩

lemma *mono2mono-gfp-eadd*[*THEN gfp.mono2mono2*, *cont-intro*, *simp*]:
shows *monotone-eadd*: *monotone* (*rel-prod* (≥) (≥)) (≥) (λ(*x*, *y*). *x* + *y* :: *enat*)
⟨*proof*⟩

lemma *eadd-gfp-partial-function-mono* [*partial-function-mono*]:
⟦ *monotone* (*fun-ord* (≥)) (≥) *f*; *monotone* (*fun-ord* (≥)) (≥) *g* ⟧
⟹ *monotone* (*fun-ord* (≥)) (≥) (λ*x*. *f x* + *g x* :: *enat*)
⟨*proof*⟩

lemma *mono2mono-ereal*[*THEN lfp.mono2mono*]:
shows *monotone-ereal*: *monotone* (≤) (≤) *ereal*

⟨*proof*⟩

lemma *mono2mono-ennreal*[*THEN lfp.mono2mono*]:
shows *monotone-ennreal*: *monotone* (≤) (≤) *ennreal*
⟨*proof*⟩

25.1.5 Bijections

lemma *bi-unique-rel-set-bij-betw*:
assumes *unique*: *bi-unique R*
and *rel*: *rel-set R A B*
shows ∃*f*. *bij-betw f A B* ∧ (∀ *x*∈*A*. *R x* (*f x*))
⟨*proof*⟩

lemma *bij-betw-rel-setD*: *bij-betw f A B* ⟹ *rel-set* (λ*x y*. *y* = *f x*) *A B*
⟨*proof*⟩

25.2 Subprobability mass function

type-synonym *'a spmf* = *'a option pmf*
translations (*type*) *'a spmf* ↽ (*type*) *'a option pmf*

definition *measure-spmf* :: *'a spmf* ⇒ *'a measure*
where *measure-spmf p* = *distr* (*restrict-space* (*measure-pmf p*) (*range Some*)) (*count-space UNIV*) *the*

abbreviation *spmf* :: *'a spmf* ⇒ *'a* ⇒ *real*
where *spmf p x* ≡ *pmf p* (*Some x*)

lemma *space-measure-spmf*: *space* (*measure-spmf p*) = *UNIV*
⟨*proof*⟩

lemma *sets-measure-spmf* [*simp*, *measurable-cong*]: *sets* (*measure-spmf p*) = *sets* (*count-space UNIV*)
⟨*proof*⟩

lemma *measure-spmf-not-bot* [*simp*]: *measure-spmf p* ≠ ⊥
⟨*proof*⟩

lemma *measurable-the-measure-pmf-Some* [*measurable*, *simp*]:
the ∈ *measurable* (*restrict-space* (*measure-pmf p*) (*range Some*)) (*count-space UNIV*)
⟨*proof*⟩

lemma *measurable-spmf-measure1*[*simp*]: *measurable* (*measure-spmf M*) *N* = *UNIV* → *space N*
⟨*proof*⟩

lemma *measurable-spmf-measure2*[*simp*]: *measurable N* (*measure-spmf M*) = *measurable N* (*count-space UNIV*)

⟨*proof*⟩

lemma *subprob-space-measure-spmf* [*simp*, *intro*!]: *subprob-space* (*measure-spmf p*)
⟨*proof*⟩

interpretation *measure-spmf*: *subprob-space measure-spmf p* **for** *p*
⟨*proof*⟩

lemma *finite-measure-spmf* [*simp*]: *finite-measure* (*measure-spmf p*)
⟨*proof*⟩

lemma *spmf-conv-measure-spmf*: *spmf p x* = *measure* (*measure-spmf p*) {*x*}
⟨*proof*⟩

lemma *emeasure-measure-spmf-conv-measure-pmf*:
emeasure (*measure-spmf p*) *A* = *emeasure* (*measure-pmf p*) (*Some* ‘ *A*)
⟨*proof*⟩

lemma *measure-measure-spmf-conv-measure-pmf*:
measure (*measure-spmf p*) *A* = *measure* (*measure-pmf p*) (*Some* ‘ *A*)
⟨*proof*⟩

lemma *emeasure-spmf-map-pmf-Some* [*simp*]:
emeasure (*measure-spmf* (*map-pmf Some p*)) *A* = *emeasure* (*measure-pmf p*) *A*
⟨*proof*⟩

lemma *measure-spmf-map-pmf-Some* [*simp*]:
measure (*measure-spmf* (*map-pmf Some p*)) *A* = *measure* (*measure-pmf p*) *A*
⟨*proof*⟩

lemma *nn-integral-measure-spmf*: (∫⁺ *x*. *f x* ∂*measure-spmf p*) = ∫⁺ *x*. *ennreal* (*spmf p x*) ∗ *f x* ∂*count-space UNIV*
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *integral-measure-spmf*:
assumes *integrable* (*measure-spmf p*) *f*
shows (∫ *x*. *f x* ∂*measure-spmf p*) = ∫ *x*. *spmf p x* ∗ *f x* ∂*count-space UNIV*
⟨*proof*⟩

lemma *emeasure-spmf-single*: *emeasure* (*measure-spmf p*) {*x*} = *spmf p x*
⟨*proof*⟩

lemma *measurable-measure-spmf*[*measurable*]:
(λ*x*. *measure-spmf* (*M x*)) ∈ *measurable* (*count-space UNIV*) (*subprob-algebra* (*count-space UNIV*))
⟨*proof*⟩

lemma *nn-integral-measure-spmf-conv-measure-pmf*:

assumes [*measurable*]: $f \in$ *borel-measurable* (*count-space UNIV*)
shows *nn-integral* (*measure-spmf p*) *f* = *nn-integral* (*restrict-space* (*measure-pmf p*) (*range Some*)) (*f* ∘ *the*)
⟨*proof*⟩

lemma *measure-spmf-in-space-subprob-algebra* [*simp*]:
measure-spmf p ∈ *space* (*subprob-algebra* (*count-space UNIV*))
⟨*proof*⟩

lemma *nn-integral-spmf-neq-top*: (∫⁺ *x*. *spmf p x* ∂*count-space UNIV*) ≠ ⊤
⟨*proof*⟩

lemma *SUP-spmf-neq-top'*: (*SUP p*∈*Y*. *ennreal* (*spmf p x*)) ≠ ⊤
⟨*proof*⟩

lemma *SUP-spmf-neq-top*: (*SUP i*. *ennreal* (*spmf* (*Y i*) *x*)) ≠ ⊤
⟨*proof*⟩

lemma *SUP-emeasure-spmf-neq-top*: (*SUP p*∈*Y*. *emeasure* (*measure-spmf p*) *A*) ≠ ⊤
⟨*proof*⟩

25.3 Support

definition *set-spmf* :: *'a spmf* ⇒ *'a set*
where *set-spmf p* = *set-pmf p* ⤜ *set-option*

lemma *set-spmf-rep-eq*: *set-spmf p* = {*x*. *measure* (*measure-spmf p*) {*x*} ≠ *0*}
⟨*proof*⟩

lemma *in-set-spmf*: *x* ∈ *set-spmf p* ⟷ *Some x* ∈ *set-pmf p*
⟨*proof*⟩

lemma *AE-measure-spmf-iff* [*simp*]: (*AE x in measure-spmf p*. *P x*) ⟷ (∀ *x*∈*set-spmf p*. *P x*)
⟨*proof*⟩

lemma *spmf-eq-0-set-spmf*: *spmf p x* = *0* ⟷ *x* ∉ *set-spmf p*
⟨*proof*⟩

lemma *in-set-spmf-iff-spmf*: *x* ∈ *set-spmf p* ⟷ *spmf p x* ≠ *0*
⟨*proof*⟩

lemma *set-spmf-return-pmf-None* [*simp*]: *set-spmf* (*return-pmf None*) = {}
⟨*proof*⟩

lemma *countable-set-spmf* [*simp*]: *countable* (*set-spmf p*)
⟨*proof*⟩

lemma *spmf-eqI*:
assumes $\bigwedge i.\ spmf\ p\ i = spmf\ q\ i$
shows $p = q$
⟨*proof*⟩

lemma *integral-measure-spmf-restrict*:
fixes $f :: 'a \Rightarrow 'b :: \{banach, second\text{-}countable\text{-}topology\}$ **shows**
$(\int x.\ f\ x\ \partial measure\text{-}spmf\ M) = (\int x.\ f\ x\ \partial restrict\text{-}space\ (measure\text{-}spmf\ M)\ (set\text{-}spmf\ M))$
⟨*proof*⟩

lemma *nn-integral-measure-spmf'*:
$(\int^+ x.\ f\ x\ \partial measure\text{-}spmf\ p) = \int^+ x.\ ennreal\ (spmf\ p\ x) * f\ x\ \partial count\text{-}space\ (set\text{-}spmf\ p)$
⟨*proof*⟩

25.4 Functorial structure

abbreviation *map-spmf* :: $('a \Rightarrow 'b) \Rightarrow 'a\ spmf \Rightarrow 'b\ spmf$
where $map\text{-}spmf\ f \equiv map\text{-}pmf\ (map\text{-}option\ f)$

context begin
⟨*ML*⟩

lemma *map-comp*: $map\text{-}spmf\ f\ (map\text{-}spmf\ g\ p) = map\text{-}spmf\ (f \circ g)\ p$
⟨*proof*⟩

lemma *map-id0*: $map\text{-}spmf\ id = id$
⟨*proof*⟩

lemma *map-id* [*simp*]: $map\text{-}spmf\ id\ p = p$
⟨*proof*⟩

lemma *map-ident* [*simp*]: $map\text{-}spmf\ (\lambda x.\ x)\ p = p$
⟨*proof*⟩

end

lemma *set-map-spmf* [*simp*]: $set\text{-}spmf\ (map\text{-}spmf\ f\ p) = f\ `\ set\text{-}spmf\ p$
⟨*proof*⟩

lemma *map-spmf-cong*:
$\llbracket\ p = q;\ \bigwedge x.\ x \in set\text{-}spmf\ q \Longrightarrow f\ x = g\ x\ \rrbracket$
$\Longrightarrow map\text{-}spmf\ f\ p = map\text{-}spmf\ g\ q$
⟨*proof*⟩

lemma *map-spmf-cong-simp*:
$\llbracket\ p = q;\ \bigwedge x.\ x \in set\text{-}spmf\ q =simp=> f\ x = g\ x\ \rrbracket$
$\Longrightarrow map\text{-}spmf\ f\ p = map\text{-}spmf\ g\ q$

⟨*proof*⟩

lemma *map-spmf-idI*: (⋀*x*. *x* ∈ *set-spmf p* ⟹ *f x* = *x*) ⟹ *map-spmf f p* = *p*
⟨*proof*⟩

lemma *emeasure-map-spmf*:
emeasure (*measure-spmf* (*map-spmf f p*)) *A* = *emeasure* (*measure-spmf p*) (*f* −‘ *A*)
⟨*proof*⟩

lemma *measure-map-spmf*: *measure* (*measure-spmf* (*map-spmf f p*)) *A* = *measure* (*measure-spmf p*) (*f* −‘ *A*)
⟨*proof*⟩

lemma *measure-map-spmf-conv-distr*:
measure-spmf (*map-spmf f p*) = *distr* (*measure-spmf p*) (*count-space UNIV*) *f*
⟨*proof*⟩

lemma *spmf-map-pmf-Some* [*simp*]: *spmf* (*map-pmf Some p*) *i* = *pmf p i*
⟨*proof*⟩

lemma *spmf-map-inj*: ⟦ *inj-on f* (*set-spmf M*); *x* ∈ *set-spmf M* ⟧ ⟹ *spmf* (*map-spmf f M*) (*f x*) = *spmf M x*
⟨*proof*⟩

lemma *spmf-map-inj'*: *inj f* ⟹ *spmf* (*map-spmf f M*) (*f x*) = *spmf M x*
⟨*proof*⟩

lemma *spmf-map-outside*: *x* ∉ *f* ‘ *set-spmf M* ⟹ *spmf* (*map-spmf f M*) *x* = *0*
⟨*proof*⟩

lemma *ennreal-spmf-map*: *ennreal* (*spmf* (*map-spmf f p*) *x*) = *emeasure* (*measure-spmf p*) (*f* −‘ {*x*})
⟨*proof*⟩

lemma *spmf-map*: *spmf* (*map-spmf f p*) *x* = *measure* (*measure-spmf p*) (*f* −‘ {*x*})
⟨*proof*⟩

lemma *ennreal-spmf-map-conv-nn-integral*:
ennreal (*spmf* (*map-spmf f p*) *x*) = *integral*^{N} (*measure-spmf p*) (*indicator* (*f* −‘ {*x*}))
⟨*proof*⟩

25.5 Monad operations

25.5.1 Return

abbreviation *return-spmf* :: *'a* ⇒ *'a spmf*
where *return-spmf x* ≡ *return-pmf* (*Some x*)

lemma *pmf-return-spmf*: *pmf* (*return-spmf x*) *y* = *indicator* {*y*} (*Some x*)
⟨*proof*⟩

lemma *measure-spmf-return-spmf*: *measure-spmf* (*return-spmf x*) = *Giry-Monad.return* (*count-space UNIV*) *x*
⟨*proof*⟩

lemma *measure-spmf-return-pmf-None* [*simp*]: *measure-spmf* (*return-pmf None*) = *null-measure* (*count-space UNIV*)
⟨*proof*⟩

lemma *set-return-spmf* [*simp*]: *set-spmf* (*return-spmf x*) = {*x*}
⟨*proof*⟩

25.5.2 Bind

definition *bind-spmf* :: *'a spmf* ⇒ (*'a* ⇒ *'b spmf*) ⇒ *'b spmf*
where *bind-spmf x f* = *bind-pmf x* (λ*a*. *case a of None* ⇒ *return-pmf None* | *Some a'* ⇒ *f a'*)

adhoc-overloading *Monad-Syntax.bind bind-spmf*

lemma *return-None-bind-spmf* [*simp*]: *return-pmf None* ⤜ (*f* :: *'a* ⇒ -) = *return-pmf None*
⟨*proof*⟩

lemma *return-bind-spmf* [*simp*]: *return-spmf x* ⤜ *f* = *f x*
⟨*proof*⟩

lemma *bind-return-spmf* [*simp*]: *x* ⤜ *return-spmf* = *x*
⟨*proof*⟩

lemma *bind-spmf-assoc* [*simp*]:
fixes *x* :: *'a spmf* **and** *f* :: *'a* ⇒ *'b spmf* **and** *g* :: *'b* ⇒ *'c spmf*
shows (*x* ⤜ *f*) ⤜ *g* = *x* ⤜ (λ*y*. *f y* ⤜ *g*)
⟨*proof*⟩

lemma *pmf-bind-spmf-None*: *pmf* (*p* ⤜ *f*) *None* = *pmf p None* + ∫ *x*. *pmf* (*f x*) *None* ∂*measure-spmf p*
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *spmf-bind*: *spmf* (*p* ⤜ *f*) *y* = ∫ *x*. *spmf* (*f x*) *y* ∂*measure-spmf p*
⟨*proof*⟩

lemma *ennreal-spmf-bind*: *ennreal* (*spmf* (*p* ⤜ *f*) *x*) = ∫⁺ *y*. *spmf* (*f y*) *x* ∂*measure-spmf p*
⟨*proof*⟩

lemma *measure-spmf-bind-pmf*: *measure-spmf* $(p \gg\!= f) =$ *measure-pmf* $p \gg\!=$ *measure-spmf* $\circ$ f
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *measure-spmf-bind*: *measure-spmf* $(p \gg\!= f) =$ *measure-spmf* $p \gg\!=$ *measure-spmf* $\circ$ f
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *map-spmf-bind-spmf*: *map-spmf f* (*bind-spmf p g*) = *bind-spmf p* (*map-spmf f* $\circ$ *g*)
⟨*proof*⟩

lemma *bind-map-spmf*: *map-spmf f p* $\gg\!=$ *g* = *p* $\gg\!=$ *g* $\circ$ *f*
⟨*proof*⟩

lemma *spmf-bind-leI*:
assumes $\bigwedge y.$ *y* $\in$ *set-spmf p* $\Longrightarrow$ *spmf* (*f y*) *x* $\leq$ *r*
and *0* $\leq$ *r*
shows *spmf* (*bind-spmf p f*) *x* $\leq$ *r*
⟨*proof*⟩

lemma *map-spmf-conv-bind-spmf*: *map-spmf f p* = (*p* $\gg\!=$ ($\lambda x.$ *return-spmf* (*f x*)))
⟨*proof*⟩

lemma *bind-spmf-cong*:
⟦ *p* = *q*; $\bigwedge x.$ *x* $\in$ *set-spmf q* $\Longrightarrow$ *f x* = *g x* ⟧
$\Longrightarrow$ *bind-spmf p f* = *bind-spmf q g*
⟨*proof*⟩

lemma *bind-spmf-cong-simp*:
⟦ *p* = *q*; $\bigwedge x.$ *x* $\in$ *set-spmf q* =*simp*=> *f x* = *g x* ⟧
$\Longrightarrow$ *bind-spmf p f* = *bind-spmf q g*
⟨*proof*⟩

lemma *set-bind-spmf*: *set-spmf* $(M \gg\!= f) =$ *set-spmf M* $\gg\!=$ (*set-spmf* $\circ$ *f*)
⟨*proof*⟩

lemma *bind-spmf-const-return-None* [*simp*]: *bind-spmf p* (λ-. *return-pmf None*) = *return-pmf None*
⟨*proof*⟩

lemma *bind-commute-spmf*:
bind-spmf p ($\lambda x.$ *bind-spmf q* (*f x*)) = *bind-spmf q* ($\lambda y.$ *bind-spmf p* ($\lambda x.$ *f x y*))
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

25.6 Relator

abbreviation *rel-spmf* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow 'a\ spmf \Rightarrow 'b\ spmf \Rightarrow bool$
where *rel-spmf R* $\equiv$ *rel-pmf* (*rel-option R*)

lemma *rel-pmf-mono*:
$[\![rel\text{-}pmf\ A\ f\ g;\ \bigwedge x\ y.\ A\ x\ y \Longrightarrow B\ x\ y\ ]\!] \Longrightarrow rel\text{-}pmf\ B\ f\ g$
⟨*proof*⟩

lemma *rel-spmf-mono*:
$[\![rel\text{-}spmf\ A\ f\ g;\ \bigwedge x\ y.\ A\ x\ y \Longrightarrow B\ x\ y\ ]\!] \Longrightarrow rel\text{-}spmf\ B\ f\ g$
⟨*proof*⟩

lemma *rel-spmf-mono-strong*:
$[\![\ rel\text{-}spmf\ A\ f\ g;\ \bigwedge x\ y.\ [\![\ A\ x\ y;\ x \in set\text{-}spmf\ f;\ y \in set\text{-}spmf\ g\ ]\!] \Longrightarrow B\ x\ y\ ]\!] \Longrightarrow$
rel-spmf B f g
⟨*proof*⟩

lemma *rel-spmf-reflI*: $(\bigwedge x.\ x \in set\text{-}spmf\ p \Longrightarrow P\ x\ x) \Longrightarrow rel\text{-}spmf\ P\ p\ p$
⟨*proof*⟩

lemma *rel-spmfI* [*intro?*]:
$[\![\ \bigwedge x\ y.\ (x,\ y) \in set\text{-}spmf\ pq \Longrightarrow P\ x\ y;\ map\text{-}spmf\ fst\ pq = p;\ map\text{-}spmf\ snd\ pq = q\ ]\!]$
$\Longrightarrow rel\text{-}spmf\ P\ p\ q$
⟨*proof*⟩

lemma *rel-spmfE* [*elim?*, *consumes 1*, *case-names rel-spmf*]:
assumes *rel-spmf P p q*
obtains *pq* **where**
$\bigwedge x\ y.\ (x,\ y) \in set\text{-}spmf\ pq \Longrightarrow P\ x\ y$
$p = map\text{-}spmf\ fst\ pq$
$q = map\text{-}spmf\ snd\ pq$
⟨*proof*⟩

lemma *rel-spmf-simps*:
$rel\text{-}spmf\ R\ p\ q \longleftrightarrow (\exists\, pq.\ (\forall\, (x,\ y) \in set\text{-}spmf\ pq.\ R\ x\ y) \wedge map\text{-}spmf\ fst\ pq = p \wedge map\text{-}spmf\ snd\ pq = q)$
⟨*proof*⟩

lemma *spmf-rel-map*:
shows *spmf-rel-map1*: $\bigwedge R\ f\ x.\ rel\text{-}spmf\ R\ (map\text{-}spmf\ f\ x) = rel\text{-}spmf\ (\lambda x.\ R\ (f\ x))\ x$
and *spmf-rel-map2*: $\bigwedge R\ x\ g\ y.\ rel\text{-}spmf\ R\ x\ (map\text{-}spmf\ g\ y) = rel\text{-}spmf\ (\lambda x\ y.\ R\ x\ (g\ y))\ x\ y$
⟨*proof*⟩

lemma *spmf-rel-conversep*: $rel\text{-}spmf\ R^{-1-1} = (rel\text{-}spmf\ R)^{-1-1}$
⟨*proof*⟩

lemma *spmf-rel-eq*: *rel-spmf* (=) = (=)
⟨*proof*⟩

context includes *lifting-syntax*
begin

lemma *bind-spmf-parametric* [*transfer-rule*]:
(*rel-spmf A* ===> (*A* ===> *rel-spmf B*) ===> *rel-spmf B*) *bind-spmf bind-spmf*
⟨*proof*⟩

lemma *return-spmf-parametric*: (*A* ===> *rel-spmf A*) *return-spmf return-spmf*
⟨*proof*⟩

lemma *map-spmf-parametric*: ((*A* ===> *B*) ===> *rel-spmf A* ===> *rel-spmf B*) *map-spmf map-spmf*
⟨*proof*⟩

lemma *rel-spmf-parametric*:
((*A* ===> *B* ===> (=)) ===> *rel-spmf A* ===> *rel-spmf B* ===> (=)) *rel-spmf rel-spmf*
⟨*proof*⟩

lemma *set-spmf-parametric* [*transfer-rule*]:
(*rel-spmf A* ===> *rel-set A*) *set-spmf set-spmf*
⟨*proof*⟩

lemma *return-spmf-None-parametric*:
(*rel-spmf A*) (*return-pmf None*) (*return-pmf None*)
⟨*proof*⟩

end

lemma *rel-spmf-bindI*:
⟦ *rel-spmf R p q*; ⋀*x y*. *R x y* ⟹ *rel-spmf P* (*f x*) (*g y*) ⟧
⟹ *rel-spmf P* (*p* ⤜ *f*) (*q* ⤜ *g*)
⟨*proof*⟩

lemma *rel-spmf-bind-reflI*:
(⋀*x*. *x* ∈ *set-spmf p* ⟹ *rel-spmf P* (*f x*) (*g x*)) ⟹ *rel-spmf P* (*p* ⤜ *f*) (*p* ⤜ *g*)
⟨*proof*⟩

lemma *rel-pmf-return-pmfI*: *P x y* ⟹ *rel-pmf P* (*return-pmf x*) (*return-pmf y*)
⟨*proof*⟩

context includes *lifting-syntax*
begin

We do not yet have a relator for *'a measure*, so we combine *Sigma-Algebra.measure*

and *measure-pmf*

lemma *measure-pmf-parametric*:
(*rel-pmf A* ===> *rel-pred A* ===> (=)) (λ*p*. *measure* (*measure-pmf p*)) (λ*q*. *measure* (*measure-pmf q*))
⟨*proof*⟩

lemma *measure-spmf-parametric*:
(*rel-spmf A* ===> *rel-pred A* ===> (=)) (λ*p*. *measure* (*measure-spmf p*)) (λ*q*. *measure* (*measure-spmf q*))
⟨*proof*⟩

end

25.7 From *'a pmf* to *'a spmf*

definition *spmf-of-pmf* :: *'a pmf* ⇒ *'a spmf*
where *spmf-of-pmf* = *map-pmf Some*

lemma *set-spmf-spmf-of-pmf* [*simp*]: *set-spmf* (*spmf-of-pmf p*) = *set-pmf p*
⟨*proof*⟩

lemma *spmf-spmf-of-pmf* [*simp*]: *spmf* (*spmf-of-pmf p*) *x* = *pmf p x*
⟨*proof*⟩

lemma *pmf-spmf-of-pmf-None* [*simp*]: *pmf* (*spmf-of-pmf p*) *None* = *0*
⟨*proof*⟩

lemma *emeasure-spmf-of-pmf* [*simp*]: *emeasure* (*measure-spmf* (*spmf-of-pmf p*)) *A* = *emeasure* (*measure-pmf p*) *A*
⟨*proof*⟩

lemma *measure-spmf-spmf-of-pmf* [*simp*]: *measure-spmf* (*spmf-of-pmf p*) = *measure-pmf p*
⟨*proof*⟩

lemma *map-spmf-of-pmf* [*simp*]: *map-spmf f* (*spmf-of-pmf p*) = *spmf-of-pmf* (*map-pmf f p*)
⟨*proof*⟩

lemma *rel-spmf-spmf-of-pmf* [*simp*]: *rel-spmf R* (*spmf-of-pmf p*) (*spmf-of-pmf q*) = *rel-pmf R p q*
⟨*proof*⟩

lemma *spmf-of-pmf-return-pmf* [*simp*]: *spmf-of-pmf* (*return-pmf x*) = *return-spmf x*
⟨*proof*⟩

lemma *bind-spmf-of-pmf* [*simp*]: *bind-spmf* (*spmf-of-pmf p*) *f* = *bind-pmf p f*
⟨*proof*⟩

lemma *set-spmf-bind-pmf*: *set-spmf* (*bind-pmf p f*) = *Set.bind* (*set-pmf p*) (*set-spmf* ∘ *f*)
⟨*proof*⟩

lemma *spmf-of-pmf-bind*: *spmf-of-pmf* (*bind-pmf p f*) = *bind-pmf p* (λ*x*. *spmf-of-pmf* (*f x*))
⟨*proof*⟩

lemma *bind-pmf-return-spmf*: *p* ⤜ (λ*x*. *return-spmf* (*f x*)) = *spmf-of-pmf* (*map-pmf f p*)
⟨*proof*⟩

25.8 Weight of a subprobability

abbreviation *weight-spmf* :: *'a spmf* ⇒ *real*
where *weight-spmf p* ≡ *measure* (*measure-spmf p*) (*space* (*measure-spmf p*))

lemma *weight-spmf-def*: *weight-spmf p* = *measure* (*measure-spmf p*) *UNIV*
⟨*proof*⟩

lemma *weight-spmf-le-1*: *weight-spmf p* ≤ *1*
⟨*proof*⟩

lemma *weight-return-spmf* [*simp*]: *weight-spmf* (*return-spmf x*) = *1*
⟨*proof*⟩

lemma *weight-return-pmf-None* [*simp*]: *weight-spmf* (*return-pmf None*) = *0*
⟨*proof*⟩

lemma *weight-map-spmf* [*simp*]: *weight-spmf* (*map-spmf f p*) = *weight-spmf p*
⟨*proof*⟩

lemma *weight-spmf-of-pmf* [*simp*]: *weight-spmf* (*spmf-of-pmf p*) = *1*
⟨*proof*⟩

lemma *weight-spmf-nonneg*: *weight-spmf p* ≥ *0*
⟨*proof*⟩

lemma (**in** *finite-measure*) *integrable-weight-spmf* [*simp*]:
(λ*x*. *weight-spmf* (*f x*)) ∈ *borel-measurable M* ⟹ *integrable M* (λ*x*. *weight-spmf* (*f x*))
⟨*proof*⟩

lemma *weight-spmf-eq-nn-integral-spmf*: *weight-spmf p* = ∫⁺ *x*. *spmf p x* ∂*count-space UNIV*
⟨*proof*⟩

lemma *weight-spmf-eq-nn-integral-support*:

weight-spmf p = $\int^+$ *x*. *spmf p x* ∂*count-space* (*set-spmf p*)
⟨*proof*⟩

lemma *pmf-None-eq-weight-spmf*: *pmf p None* = *1* − *weight-spmf p*
⟨*proof*⟩

lemma *weight-spmf-conv-pmf-None*: *weight-spmf p* = *1* − *pmf p None*
⟨*proof*⟩

lemma *weight-spmf-le-0*: *weight-spmf p* ≤ *0* ⟷ *weight-spmf p* = *0*
⟨*proof*⟩

lemma *weight-spmf-lt-0*: ¬ *weight-spmf p* < *0*
⟨*proof*⟩

lemma *spmf-le-weight*: *spmf p x* ≤ *weight-spmf p*
⟨*proof*⟩

lemma *weight-spmf-eq-0*: *weight-spmf p* = *0* ⟷ *p* = *return-pmf None*
⟨*proof*⟩

lemma *weight-bind-spmf*: *weight-spmf* (*x* ⤜ *f*) = *lebesgue-integral* (*measure-spmf x*) (*weight-spmf* ∘ *f*)
⟨*proof*⟩

lemma *rel-spmf-weightD*: *rel-spmf A p q* ⟹ *weight-spmf p* = *weight-spmf q*
⟨*proof*⟩

lemma *rel-spmf-bij-betw*:
assumes *f*: *bij-betw f* (*set-spmf p*) (*set-spmf q*)
and *eq*: ⋀*x*. *x* ∈ *set-spmf p* ⟹ *spmf p x* = *spmf q* (*f x*)
shows *rel-spmf* (λ*x y*. *f x* = *y*) *p q*
⟨*proof*⟩

25.9 From density to spmfs

context fixes *f* :: *'a* ⇒ *real* **begin**

definition *embed-spmf* :: *'a spmf*
where *embed-spmf* = *embed-pmf* (λ*x*. *case x of None* ⇒ *1* − *enn2real* (∫$^+$ *x*. *ennreal* (*f x*) ∂*count-space UNIV*) | *Some x'* ⇒ *max 0* (*f x'*))

context
assumes *prob*: (∫$^+$ *x*. *ennreal* (*f x*) ∂*count-space UNIV*) ≤ *1*
begin

lemma *nn-integral-embed-spmf-eq-1*:
(∫$^+$ *x*. *ennreal* (*case x of None* ⇒ *1* − *enn2real* (∫$^+$ *x*. *ennreal* (*f x*) ∂*count-space UNIV*) | *Some x'* ⇒ *max 0* (*f x'*)) ∂*count-space UNIV*) = *1*

(**is** *?lhs* = - **is** $(\int^+ x.\ ?f\ x\ \partial ?M)$ = -)
⟨*proof*⟩

lemma *pmf-embed-spmf-None*: *pmf embed-spmf None* = *1* − *enn2real* $(\int^+ x.$ *ennreal* $(f\ x)$ *∂count-space UNIV*$)$
⟨*proof*⟩

lemma *spmf-embed-spmf* [*simp*]: *spmf embed-spmf x* = *max 0* $(f\ x)$
⟨*proof*⟩

end

end

lemma *embed-spmf-K-0*[*simp*]: *embed-spmf* $(\lambda$-. *0*$)$ = *return-pmf None* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

25.10 Ordering on spmfs

rel-pmf does not preserve a ccpo structure. Counterexample by Saheb-Djahromi: Take prefix order over *bool llist* and the set *range* $(\lambda n ::$ *nat*. *uniform* $($*llist-n n*$))$ where *llist-n* is the set of all *llist*s of length n and *uniform* returns a uniform distribution over the given set. The set forms a chain in *ord-pmf lprefix*, but it has not an upper bound. Any upper bound may contain only infinite lists in its support because otherwise it is not greater than the *n+1*-st element in the chain where n is the length of the finite list. Moreover its support must contain all infinite lists, because otherwise there is a finite list all of whose finite extensions are not in the support - a contradiction to the upper bound property. Hence, the support is uncountable, but pmf's only have countable support.

However, if all chains in the ccpo are finite, then it should preserve the ccpo structure.

abbreviation *ord-spmf* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a$ *spmf* $\Rightarrow 'a$ *spmf* $\Rightarrow$ *bool*
where *ord-spmf ord* ≡ *rel-pmf* $($*ord-option ord*$)$

locale *ord-spmf-syntax* **begin**
notation *ord-spmf* (**infix** $\sqsubseteq_1$ *60*)
end

lemma *ord-spmf-map-spmf1*: *ord-spmf R* $($*map-spmf f p*$)$ = *ord-spmf* $(\lambda x.\ R\ (f\ x))$ *p*
⟨*proof*⟩

lemma *ord-spmf-map-spmf2*: *ord-spmf R p* $($*map-spmf f q*$)$ = *ord-spmf* $(\lambda x\ y.\ R\ x\ (f\ y))$ *p q*
⟨*proof*⟩

lemma *ord-spmf-map-spmf12*: *ord-spmf R* (*map-spmf f p*) (*map-spmf f q*) = *ord-spmf* (λ*x y*. *R* (*f x*) (*f y*)) *p q*
⟨*proof*⟩

lemmas *ord-spmf-map-spmf = ord-spmf-map-spmf1 ord-spmf-map-spmf2 ord-spmf-map-spmf12*

context fixes *ord* :: *'a* ⇒ *'a* ⇒ *bool* (**structure**) **begin**
interpretation *ord-spmf-syntax* ⟨*proof*⟩

lemma *ord-spmfI*:
⟦ ⋀*x y*. (*x*, *y*) ∈ *set-spmf pq* ⟹ *ord x y*; *map-spmf fst pq* = *p*; *map-spmf snd pq* = *q* ⟧
⟹ *p* ⊑ *q*
⟨*proof*⟩

lemma *ord-spmf-None* [*simp*]: *return-pmf None* ⊑ *x*
⟨*proof*⟩

lemma *ord-spmf-reflI*: (⋀*x*. *x* ∈ *set-spmf p* ⟹ *ord x x*) ⟹ *p* ⊑ *p*
⟨*proof*⟩

lemma *rel-spmf-inf*:
assumes *p* ⊑ *q*
and *q* ⊑ *p*
and *refl*: *reflp ord*
and *trans*: *transp ord*
shows *rel-spmf* (*inf ord ord*$^{-1-1}$) *p q*
⟨*proof*⟩

end

lemma *ord-spmf-return-spmf2*: *ord-spmf R p* (*return-spmf y*) ⟷ (∀ *x*∈*set-spmf p*. *R x y*)
⟨*proof*⟩

lemma *ord-spmf-mono*: ⟦ *ord-spmf A p q*; ⋀*x y*. *A x y* ⟹ *B x y* ⟧ ⟹ *ord-spmf B p q*
⟨*proof*⟩

lemma *ord-spmf-compp*: *ord-spmf* (*A OO B*) = *ord-spmf A OO ord-spmf B*
⟨*proof*⟩

lemma *ord-spmf-bindI*:
assumes *pq*: *ord-spmf R p q*
and *fg*: ⋀*x y*. *R x y* ⟹ *ord-spmf P* (*f x*) (*g y*)
shows *ord-spmf P* (*p* ⤜ *f*) (*q* ⤜ *g*)
⟨*proof*⟩

lemma *ord-spmf-bind-reflI*:
(⋀*x*. *x* ∈ *set-spmf p* ⟹ *ord-spmf R* (*f x*) (*g x*))
⟹ *ord-spmf R* (*p* ⤜ *f*) (*p* ⤜ *g*)
⟨*proof*⟩

lemma *ord-pmf-increaseI*:
assumes *le*: ⋀*x*. *spmf p x* ≤ *spmf q x*
and *refl*: ⋀*x*. *x* ∈ *set-spmf p* ⟹ *R x x*
shows *ord-spmf R p q*
⟨*proof*⟩

lemma *ord-spmf-eq-leD*:
assumes *ord-spmf* (=) *p q*
shows *spmf p x* ≤ *spmf q x*
⟨*proof*⟩

lemma *ord-spmf-eqD-set-spmf*: *ord-spmf* (=) *p q* ⟹ *set-spmf p* ⊆ *set-spmf q*
⟨*proof*⟩

lemma *ord-spmf-eqD-emeasure*:
ord-spmf (=) *p q* ⟹ *emeasure* (*measure-spmf p*) *A* ≤ *emeasure* (*measure-spmf q*) *A*
⟨*proof*⟩

lemma *ord-spmf-eqD-measure-spmf*: *ord-spmf* (=) *p q* ⟹ *measure-spmf p* ≤ *measure-spmf q*
⟨*proof*⟩

25.11 CCPO structure for the flat ccpo *ord-option* (=)

context fixes *Y* :: *'a spmf set* **begin**

definition *lub-spmf* :: *'a spmf*
where *lub-spmf* = *embed-spmf* (λ*x*. *enn2real* (*SUP p* ∈ *Y*. *ennreal* (*spmf p x*)))
— We go through *ennreal* to have a sensible definition even if *Y* is empty.

lemma *lub-spmf-empty* [*simp*]: *SPMF.lub-spmf* {} = *return-pmf None*
⟨*proof*⟩

context assumes *chain*: *Complete-Partial-Order.chain* (*ord-spmf* (=)) *Y* **begin**

lemma *chain-ord-spmf-eqD*: *Complete-Partial-Order.chain* (≤) ((λ*p x*. *ennreal* (*spmf p x*)) ‘ *Y*)
(**is** *Complete-Partial-Order.chain* - (*?f* ‘ -))
⟨*proof*⟩

lemma *ord-spmf-eq-pmf-None-eq*:
assumes *le*: *ord-spmf* (=) *p q*
and *None*: *pmf p None* = *pmf q None*

shows $p = q$
⟨*proof*⟩

lemma *ord-spmf-eqD-pmf-None*:
assumes *ord-spmf* (=) *x y*
shows *pmf x None* ≥ *pmf y None*
⟨*proof*⟩

Chains on $'a$ *spmf* maintain countable support. Thanks to Johannes Hölzl for the proof idea.

lemma *spmf-chain-countable*: *countable* (⋃ *p*∈*Y*. *set-spmf p*)
⟨*proof*⟩

lemma *lub-spmf-subprob*: (∫⁺ *x*. (*SUP p* ∈ *Y*. *ennreal* (*spmf p x*)) ∂*count-space UNIV*) ≤ *1*
⟨*proof*⟩

lemma *spmf-lub-spmf*:
assumes *Y* ≠ {}
shows *spmf lub-spmf x* = (*SUP p* ∈ *Y*. *spmf p x*)
⟨*proof*⟩

lemma *ennreal-spmf-lub-spmf*: *Y* ≠ {} ⟹ *ennreal* (*spmf lub-spmf x*) = (*SUP p*∈*Y*. *ennreal* (*spmf p x*))
⟨*proof*⟩

lemma *lub-spmf-upper*:
assumes *p*: *p* ∈ *Y*
shows *ord-spmf* (=) *p lub-spmf*
⟨*proof*⟩

lemma *lub-spmf-least*:
assumes *z*: ⋀*x*. *x* ∈ *Y* ⟹ *ord-spmf* (=) *x z*
shows *ord-spmf* (=) *lub-spmf z*
⟨*proof*⟩

lemma *set-lub-spmf*: *set-spmf lub-spmf* = (⋃ *p*∈*Y*. *set-spmf p*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *emeasure-lub-spmf*:
assumes *Y*: *Y* ≠ {}
shows *emeasure* (*measure-spmf lub-spmf*) *A* = (*SUP y*∈*Y*. *emeasure* (*measure-spmf y*) *A*)
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *measure-lub-spmf*:
assumes *Y*: *Y* ≠ {}
shows *measure* (*measure-spmf lub-spmf*) *A* = (*SUP y*∈*Y*. *measure* (*measure-spmf*

y) *A*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *weight-lub-spmf*:
assumes *Y*: *Y* ≠ {}
shows *weight-spmf lub-spmf* = (*SUP y*∈*Y*. *weight-spmf y*)
⟨*proof*⟩

lemma *measure-spmf-lub-spmf*:
assumes *Y*: *Y* ≠ {}
shows *measure-spmf lub-spmf* = (*SUP p*∈*Y*. *measure-spmf p*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

end

end

lemma *partial-function-definitions-spmf*: *partial-function-definitions* (*ord-spmf* (=)) *lub-spmf*
(**is** *partial-function-definitions ?R* -)
⟨*proof*⟩

lemma *ccpo-spmf*: *class.ccpo lub-spmf* (*ord-spmf* (=)) (*mk-less* (*ord-spmf* (=)))
⟨*proof*⟩

interpretation *spmf*: *partial-function-definitions ord-spmf* (=) *lub-spmf*
rewrites *lub-spmf* {} ≡ *return-pmf None*
⟨*proof*⟩

⟨*ML*⟩

declare *spmf.leq-refl*[*simp*]
declare *admissible-leI*[*OF ccpo-spmf*, *cont-intro*]

abbreviation *mono-spmf* ≡ *monotone* (*fun-ord* (*ord-spmf* (=))) (*ord-spmf* (=))

lemma *lub-spmf-const* [*simp*]: *lub-spmf* {*p*} = *p*
⟨*proof*⟩

lemma *bind-spmf-mono′*:
assumes *fg*: *ord-spmf* (=) *f g*
and *hk*: ⋀*x* :: *′a*. *ord-spmf* (=) (*h x*) (*k x*)
shows *ord-spmf* (=) (*f* ⤜ *h*) (*g* ⤜ *k*)
⟨*proof*⟩

lemma *bind-spmf-mono* [*partial-function-mono*]:
assumes *mf*: *mono-spmf B* **and** *mg*: ⋀*y*. *mono-spmf* (λ*f*. *C y f*)
shows *mono-spmf* (λ*f*. *bind-spmf* (*B f*) (λ*y*. *C y f*))
⟨*proof*⟩

lemma *monotone-bind-spmf1*: *monotone* (*ord-spmf* (=)) (*ord-spmf* (=)) (λ*y*. *bind-spmf y g*)
⟨*proof*⟩

lemma *monotone-bind-spmf2*:
assumes *g*: ⋀*x*. *monotone ord* (*ord-spmf* (=)) (λ*y*. *g y x*)
shows *monotone ord* (*ord-spmf* (=)) (λ*y*. *bind-spmf p* (*g y*))
⟨*proof*⟩

lemma *bind-lub-spmf*:
assumes *chain*: *Complete-Partial-Order.chain* (*ord-spmf* (=)) *Y*
shows *bind-spmf* (*lub-spmf Y*) *f* = *lub-spmf* ((λ*p*. *bind-spmf p f*) ‘ *Y*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *map-lub-spmf*:
Complete-Partial-Order.chain (*ord-spmf* (=)) *Y*
⟹ *map-spmf f* (*lub-spmf Y*) = *lub-spmf* (*map-spmf f* ‘ *Y*)
⟨*proof*⟩

lemma *mcont-bind-spmf1*: *mcont lub-spmf* (*ord-spmf* (=)) *lub-spmf* (*ord-spmf* (=)) (λ*y*. *bind-spmf y f*)
⟨*proof*⟩

lemma *bind-lub-spmf2*:
assumes *chain*: *Complete-Partial-Order.chain ord Y*
and *g*: ⋀*y*. *monotone ord* (*ord-spmf* (=)) (*g y*)
shows *bind-spmf x* (λ*y*. *lub-spmf* (*g y* ‘ *Y*)) = *lub-spmf* ((λ*p*. *bind-spmf x* (λ*y*. *g y p*)) ‘ *Y*)
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *mcont-bind-spmf* [*cont-intro*]:
assumes *f*: *mcont luba orda lub-spmf* (*ord-spmf* (=)) *f*
and *g*: ⋀*y*. *mcont luba orda lub-spmf* (*ord-spmf* (=)) (*g y*)
shows *mcont luba orda lub-spmf* (*ord-spmf* (=)) (λ*x*. *bind-spmf* (*f x*) (λ*y*. *g y x*))
⟨*proof*⟩

lemma *bind-pmf-mono* [*partial-function-mono*]:
(⋀*y*. *mono-spmf* (λ*f*. *C y f*)) ⟹ *mono-spmf* (λ*f*. *bind-pmf p* (λ*x*. *C x f*))
⟨*proof*⟩

lemma *map-spmf-mono* [*partial-function-mono*]: *mono-spmf B* ⟹ *mono-spmf* (λ*g*. *map-spmf f* (*B g*))
⟨*proof*⟩

lemma *mcont-map-spmf* [*cont-intro*]:

mcont luba orda lub-spmf (*ord-spmf* (=)) *g*
⟹ *mcont luba orda lub-spmf* (*ord-spmf* (=)) (λ*x*. *map-spmf f* (*g x*))
⟨*proof*⟩

lemma *monotone-set-spmf*: *monotone* (*ord-spmf* (=)) (⊆) *set-spmf*
⟨*proof*⟩

lemma *cont-set-spmf*: *cont lub-spmf* (*ord-spmf* (=)) *Union* (⊆) *set-spmf*
⟨*proof*⟩

lemma *mcont2mcont-set-spmf*[*THEN mcont2mcont*, *cont-intro*]:
shows *mcont-set-spmf*: *mcont lub-spmf* (*ord-spmf* (=)) *Union* (⊆) *set-spmf*
⟨*proof*⟩

lemma *monotone-spmf*: *monotone* (*ord-spmf* (=)) (≤) (λ*p*. *spmf p x*)
⟨*proof*⟩

lemma *cont-spmf*: *cont lub-spmf* (*ord-spmf* (=)) *Sup* (≤) (λ*p*. *spmf p x*)
⟨*proof*⟩

lemma *mcont-spmf*: *mcont lub-spmf* (*ord-spmf* (=)) *Sup* (≤) (λ*p*. *spmf p x*)
⟨*proof*⟩

lemma *cont-ennreal-spmf*: *cont lub-spmf* (*ord-spmf* (=)) *Sup* (≤) (λ*p*. *ennreal* (*spmf p x*))
⟨*proof*⟩

lemma *mcont2mcont-ennreal-spmf* [*THEN mcont2mcont*, *cont-intro*]:
shows *mcont-ennreal-spmf*: *mcont lub-spmf* (*ord-spmf* (=)) *Sup* (≤) (λ*p*. *ennreal* (*spmf p x*))
⟨*proof*⟩

lemma *nn-integral-map-spmf* [*simp*]: *nn-integral* (*measure-spmf* (*map-spmf f p*)) *g* = *nn-integral* (*measure-spmf p*) (*g* ∘ *f*)
⟨*proof*⟩

25.11.1 Admissibility of *rel-spmf*

lemma *rel-spmf-measureD*:
assumes *rel-spmf R p q*
shows *measure* (*measure-spmf p*) *A* ≤ *measure* (*measure-spmf q*) {*y*. ∃ *x*∈*A*. *R x y*} (**is** *?lhs* ≤ *?rhs*)
⟨*proof*⟩

locale *rel-spmf-characterisation* =
assumes *rel-pmf-measureI*:
⋀(*R* :: *'a option* ⇒ *'b option* ⇒ *bool*) *p q*.
(⋀*A*. *measure* (*measure-pmf p*) *A* ≤ *measure* (*measure-pmf q*) {*y*. ∃ *x*∈*A*. *R x y*})

$\Longrightarrow$ *rel-pmf R p q*
— This assumption is shown to hold in general in the AFP entry *MFMC-Countable*.
begin

context fixes $R :: 'a \Rightarrow 'b \Rightarrow bool$ **begin**

lemma *rel-spmf-measureI*:
assumes *eq1*: $\bigwedge A.$ *measure* (*measure-spmf p*) $A \leq$ *measure* (*measure-spmf q*) $\{y. \exists x \in A.\ R\ x\ y\}$
assumes *eq2*: *weight-spmf q* $\leq$ *weight-spmf p*
shows *rel-spmf R p q*
⟨*proof*⟩

lemma *admissible-rel-spmf*:
ccpo.admissible (*prod-lub lub-spmf lub-spmf*) (*rel-prod* (*ord-spmf* (=)) (*ord-spmf* (=))) (*case-prod* (*rel-spmf R*))
(**is** *ccpo.admissible ?lub ?ord ?P*)
⟨*proof*⟩

lemma *admissible-rel-spmf-mcont* [*cont-intro*]:
⟦ *mcont lub ord lub-spmf* (*ord-spmf* (=)) *f*; *mcont lub ord lub-spmf* (*ord-spmf* (=)) *g* ⟧
$\Longrightarrow$ *ccpo.admissible lub ord* ($\lambda x.$ *rel-spmf R* (*f x*) (*g x*))
⟨*proof*⟩

context includes *lifting-syntax*
begin

lemma *fixp-spmf-parametric'*:
assumes *f*: $\bigwedge x.$ *monotone* (*ord-spmf* (=)) (*ord-spmf* (=)) *F*
and *g*: $\bigwedge x.$ *monotone* (*ord-spmf* (=)) (*ord-spmf* (=)) *G*
and *param*: (*rel-spmf R* ===> *rel-spmf R*) *F G*
shows (*rel-spmf R*) (*ccpo.fixp lub-spmf* (*ord-spmf* (=)) *F*) (*ccpo.fixp lub-spmf* (*ord-spmf* (=)) *G*)
⟨*proof*⟩

lemma *fixp-spmf-parametric*:
assumes *f*: $\bigwedge x.$ *mono-spmf* ($\lambda f.$ *F f x*)
and *g*: $\bigwedge x.$ *mono-spmf* ($\lambda f.$ *G f x*)
and *param*: ((*A* ===> *rel-spmf R*) ===> *A* ===> *rel-spmf R*) *F G*
shows (*A* ===> *rel-spmf R*) (*spmf.fixp-fun F*) (*spmf.fixp-fun G*)
⟨*proof*⟩

end

end

end

25.12 Restrictions on spmfs

definition *restrict-spmf* :: *'a spmf* ⇒ *'a set* ⇒ *'a spmf* (**infixl** ↾ *110*)
where $p \restriction A = \textit{map-pmf}\ (\lambda x.\ x \gg\!= (\lambda y.\ \textit{if}\ y \in A\ \textit{then Some}\ y\ \textit{else None}))\ p$

lemma *set-restrict-spmf* [*simp*]: $\textit{set-spmf}\ (p \restriction A) = \textit{set-spmf}\ p \cap A$
⟨*proof*⟩

lemma *restrict-map-spmf*: $\textit{map-spmf}\ f\ p \restriction A = \textit{map-spmf}\ f\ (p \restriction (f -` A))$
⟨*proof*⟩

lemma *restrict-restrict-spmf* [*simp*]: $p \restriction A \restriction B = p \restriction (A \cap B)$
⟨*proof*⟩

lemma *restrict-spmf-empty* [*simp*]: $p \restriction \{\} = \textit{return-pmf None}$
⟨*proof*⟩

lemma *restrict-spmf-UNIV* [*simp*]: $p \restriction \textit{UNIV} = p$
⟨*proof*⟩

lemma *spmf-restrict-spmf-outside* [*simp*]: $x \notin A \Longrightarrow \textit{spmf}\ (p \restriction A)\ x = 0$
⟨*proof*⟩

lemma *emeasure-restrict-spmf* [*simp*]:
$\textit{emeasure}\ (\textit{measure-spmf}\ (p \restriction A))\ X = \textit{emeasure}\ (\textit{measure-spmf}\ p)\ (X \cap A)$
⟨*proof*⟩

lemma *measure-restrict-spmf* [*simp*]:
$\textit{measure}\ (\textit{measure-spmf}\ (p \restriction A))\ X = \textit{measure}\ (\textit{measure-spmf}\ p)\ (X \cap A)$
⟨*proof*⟩

lemma *spmf-restrict-spmf*: $\textit{spmf}\ (p \restriction A)\ x = (\textit{if}\ x \in A\ \textit{then spmf}\ p\ x\ \textit{else}\ 0)$
⟨*proof*⟩

lemma *spmf-restrict-spmf-inside* [*simp*]: $x \in A \Longrightarrow \textit{spmf}\ (p \restriction A)\ x = \textit{spmf}\ p\ x$
⟨*proof*⟩

lemma *pmf-restrict-spmf-None*: $\textit{pmf}\ (p \restriction A)\ \textit{None} = \textit{pmf}\ p\ \textit{None} + \textit{measure}\ (\textit{measure-spmf}\ p)\ (-A)$
⟨*proof*⟩

lemma *restrict-spmf-trivial*: $(\bigwedge x.\ x \in \textit{set-spmf}\ p \Longrightarrow x \in A) \Longrightarrow p \restriction A = p$
⟨*proof*⟩

lemma *restrict-spmf-trivial'*: $\textit{set-spmf}\ p \subseteq A \Longrightarrow p \restriction A = p$
⟨*proof*⟩

lemma *restrict-return-spmf*: $\textit{return-spmf}\ x \restriction A = (\textit{if}\ x \in A\ \textit{then return-spmf}\ x\ \textit{else return-pmf None})$
⟨*proof*⟩

lemma *restrict-return-spmf-inside* [*simp*]: $x \in A \Longrightarrow$ *return-spmf* $x \upharpoonright A =$ *return-spmf* x
⟨*proof*⟩

lemma *restrict-return-spmf-outside* [*simp*]: $x \notin A \Longrightarrow$ *return-spmf* $x \upharpoonright A =$ *return-pmf None*
⟨*proof*⟩

lemma *restrict-spmf-return-pmf-None* [*simp*]: *return-pmf None* $\upharpoonright A =$ *return-pmf None*
⟨*proof*⟩

lemma *restrict-bind-pmf*: *bind-pmf* p $g \upharpoonright A = p \gg\!= (\lambda x.\ g\ x \upharpoonright A)$
⟨*proof*⟩

lemma *restrict-bind-spmf*: *bind-spmf* p $g \upharpoonright A = p \gg\!= (\lambda x.\ g\ x \upharpoonright A)$
⟨*proof*⟩

lemma *bind-restrict-pmf*: *bind-pmf* $(p \upharpoonright A)$ $g = p \gg\!= (\lambda x.$ *if* $x \in$ *Some* ‘ A *then* $g\ x$ *else* g *None*)
⟨*proof*⟩

lemma *bind-restrict-spmf*: *bind-spmf* $(p \upharpoonright A)$ $g = p \gg\!= (\lambda x.$ *if* $x \in A$ *then* $g\ x$ *else return-pmf None*)
⟨*proof*⟩

lemma *spmf-map-restrict*: *spmf* (*map-spmf fst* $(p \upharpoonright (snd -‘ \{y\}))) x =$ *spmf* p (x, y)
⟨*proof*⟩

lemma *measure-eqI-restrict-spmf*:
assumes *rel-spmf* R (*restrict-spmf* p A) (*restrict-spmf* q B)
shows *measure* (*measure-spmf* p) $A =$ *measure* (*measure-spmf* q) B
⟨*proof*⟩

25.13 Subprobability distributions of sets

definition *spmf-of-set* :: *'a set* ⇒ *'a spmf*
where
spmf-of-set $A =$ (*if finite* $A \wedge A \neq \{\}$ *then spmf-of-pmf* (*pmf-of-set* A) *else return-pmf None*)

lemma *spmf-of-set*: *spmf* (*spmf-of-set* A) $x =$ *indicator* $A\ x$ / *card* A
⟨*proof*⟩

lemma *pmf-spmf-of-set-None* [*simp*]: *pmf* (*spmf-of-set* A) *None* = *indicator* $\{A.$ *infinite* $A \vee A = \{\}\}$ A
⟨*proof*⟩

lemma *set-spmf-of-set*: *set-spmf* (*spmf-of-set A*) = (*if finite A then A else* {})
⟨*proof*⟩

lemma *set-spmf-of-set-finite* [*simp*]: *finite A* ⟹ *set-spmf* (*spmf-of-set A*) = *A*
⟨*proof*⟩

lemma *spmf-of-set-singleton*: *spmf-of-set* {*x*} = *return-spmf x*
⟨*proof*⟩

lemma *map-spmf-of-set-inj-on* [*simp*]:
inj-on f A ⟹ *map-spmf f* (*spmf-of-set A*) = *spmf-of-set* (*f* ‘ *A*)
⟨*proof*⟩

lemma *spmf-of-pmf-pmf-of-set* [*simp*]:
⟦ *finite A*; *A* ≠ {} ⟧ ⟹ *spmf-of-pmf* (*pmf-of-set A*) = *spmf-of-set A*
⟨*proof*⟩

lemma *weight-spmf-of-set*:
weight-spmf (*spmf-of-set A*) = (*if finite A* ∧ *A* ≠ {} *then 1 else 0*)
⟨*proof*⟩

lemma *weight-spmf-of-set-finite* [*simp*]: ⟦ *finite A*; *A* ≠ {} ⟧ ⟹ *weight-spmf* (*spmf-of-set A*) = *1*
⟨*proof*⟩

lemma *weight-spmf-of-set-infinite* [*simp*]: *infinite A* ⟹ *weight-spmf* (*spmf-of-set A*) = *0*
⟨*proof*⟩

lemma *measure-spmf-spmf-of-set*:
measure-spmf (*spmf-of-set A*) = (*if finite A* ∧ *A* ≠ {} *then measure-pmf* (*pmf-of-set A*) *else null-measure* (*count-space UNIV*))
⟨*proof*⟩

lemma *emeasure-spmf-of-set*:
emeasure (*measure-spmf* (*spmf-of-set S*)) *A* = *card* (*S* ∩ *A*) / *card S*
⟨*proof*⟩

lemma *measure-spmf-of-set*:
measure (*measure-spmf* (*spmf-of-set S*)) *A* = *card* (*S* ∩ *A*) / *card S*
⟨*proof*⟩

lemma *nn-integral-spmf-of-set*: *nn-integral* (*measure-spmf* (*spmf-of-set A*)) *f* = *sum f A* / *card A*
⟨*proof*⟩

lemma *integral-spmf-of-set*: $integral^L$ (*measure-spmf* (*spmf-of-set A*)) *f* = *sum f A* / *card A*

⟨*proof*⟩

notepad begin — *pmf-of-set* is not fully parametric.
⟨*proof*⟩
end

lemma *rel-pmf-of-set-bij*:
assumes *f*: *bij-betw f A B*
and *A*: $A \neq \{\}$ *finite A*
and *B*: $B \neq \{\}$ *finite B*
and *R*: $\bigwedge x.\ x \in A \Longrightarrow R\ x\ (f\ x)$
shows *rel-pmf R* (*pmf-of-set A*) (*pmf-of-set B*)
⟨*proof*⟩

lemma *rel-spmf-of-set-bij*:
assumes *f*: *bij-betw f A B*
and *R*: $\bigwedge x.\ x \in A \Longrightarrow R\ x\ (f\ x)$
shows *rel-spmf R* (*spmf-of-set A*) (*spmf-of-set B*)
⟨*proof*⟩

context includes *lifting-syntax*
begin

lemma *rel-spmf-of-set*:
assumes *bi-unique R*
shows (*rel-set R* ===> *rel-spmf R*) *spmf-of-set spmf-of-set*
⟨*proof*⟩

end

lemma *map-mem-spmf-of-set*:
assumes *finite B B* $\neq \{\}$
shows *map-spmf* ($\lambda x.\ x \in A$) (*spmf-of-set B*) = *spmf-of-pmf* (*bernoulli-pmf* (*card* ($A \cap B$) / *card B*))
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

abbreviation *coin-spmf* :: *bool spmf*
where *coin-spmf* ≡ *spmf-of-set UNIV*

lemma *map-eq-const-coin-spmf*: *map-spmf* ((=) *c*) *coin-spmf* = *coin-spmf*
⟨*proof*⟩

lemma *bind-coin-spmf-eq-const*: *coin-spmf* ⤜ (λx :: *bool*. *return-spmf* (*b* = *x*))
= *coin-spmf*
⟨*proof*⟩

lemma *bind-coin-spmf-eq-const′*: *coin-spmf* ⤜ (λx :: *bool*. *return-spmf* (*x* = *b*))
= *coin-spmf*

⟨*proof*⟩

25.14 Losslessness

definition *lossless-spmf* :: *'a spmf* ⇒ *bool*
where *lossless-spmf p* ⟷ *weight-spmf p = 1*

lemma *lossless-iff-pmf-None*: *lossless-spmf p* ⟷ *pmf p None = 0*
⟨*proof*⟩

lemma *lossless-return-spmf* [*iff*]: *lossless-spmf* (*return-spmf x*)
⟨*proof*⟩

lemma *lossless-return-pmf-None* [*iff*]: ¬ *lossless-spmf* (*return-pmf None*)
⟨*proof*⟩

lemma *lossless-map-spmf* [*simp*]: *lossless-spmf* (*map-spmf f p*) ⟷ *lossless-spmf p*
⟨*proof*⟩

lemma *lossless-bind-spmf* [*simp*]:
lossless-spmf (*p* ⤜ *f*) ⟷ *lossless-spmf p* ∧ (∀ *x*∈*set-spmf p*. *lossless-spmf* (*f x*))
⟨*proof*⟩

lemma *lossless-weight-spmfD*: *lossless-spmf p* ⟹ *weight-spmf p = 1*
⟨*proof*⟩

lemma *lossless-iff-set-pmf-None*:
lossless-spmf p ⟷ *None* ∉ *set-pmf p*
⟨*proof*⟩

lemma *lossless-spmf-of-set* [*simp*]: *lossless-spmf* (*spmf-of-set A*) ⟷ *finite A* ∧ *A* ≠ {}
⟨*proof*⟩

lemma *lossless-spmf-spmf-of-spmf* [*simp*]: *lossless-spmf* (*spmf-of-pmf p*)
⟨*proof*⟩

lemma *lossless-spmf-bind-pmf* [*simp*]:
lossless-spmf (*bind-pmf p f*) ⟷ (∀ *x*∈*set-pmf p*. *lossless-spmf* (*f x*))
⟨*proof*⟩

lemma *lossless-spmf-conv-spmf-of-pmf*: *lossless-spmf p* ⟷ (∃ *p'*. *p = spmf-of-pmf p'*)
⟨*proof*⟩

lemma *spmf-False-conv-True*: *lossless-spmf p* ⟹ *spmf p False = 1 − spmf p True*
⟨*proof*⟩

lemma *spmf-True-conv-False*: *lossless-spmf p* $\Longrightarrow$ *spmf p True* = *1* − *spmf p False*
⟨*proof*⟩

lemma *bind-eq-return-spmf*:
bind-spmf p f = *return-spmf x* $\longleftrightarrow$ (∀ *y*∈*set-spmf p*. *f y* = *return-spmf x*) ∧ *lossless-spmf p*
⟨*proof*⟩

lemma *rel-spmf-return-spmf2*:
rel-spmf R p (*return-spmf x*) $\longleftrightarrow$ *lossless-spmf p* ∧ (∀ *a*∈*set-spmf p*. *R a x*)
⟨*proof*⟩

lemma *rel-spmf-return-spmf1*:
rel-spmf R (*return-spmf x*) *p* $\longleftrightarrow$ *lossless-spmf p* ∧ (∀ *a*∈*set-spmf p*. *R x a*)
⟨*proof*⟩

lemma *rel-spmf-bindI1*:
assumes *f*: ⋀*x*. *x* ∈ *set-spmf p* $\Longrightarrow$ *rel-spmf R* (*f x*) *q*
and *p*: *lossless-spmf p*
shows *rel-spmf R* (*bind-spmf p f*) *q*
⟨*proof*⟩

lemma *rel-spmf-bindI2*:
⟦ ⋀*x*. *x* ∈ *set-spmf q* $\Longrightarrow$ *rel-spmf R p* (*f x*); *lossless-spmf q* ⟧
$\Longrightarrow$ *rel-spmf R p* (*bind-spmf q f*)
⟨*proof*⟩

25.15 Scaling

definition *scale-spmf* :: *real* ⇒ *'a spmf* ⇒ *'a spmf*
where
scale-spmf r p = *embed-spmf* (λ*x*. *min* (*inverse* (*weight-spmf p*)) (*max 0 r*) ∗ *spmf p x*)

lemma *scale-spmf-le-1*:
(∫$^+$ *x*. *min* (*inverse* (*weight-spmf p*)) (*max 0 r*) ∗ *spmf p x* ∂*count-space UNIV*) ≤ *1* (**is** *?lhs* ≤ -)
⟨*proof*⟩

lemma *spmf-scale-spmf*: *spmf* (*scale-spmf r p*) *x* = *max 0* (*min* (*inverse* (*weight-spmf p*)) *r*) ∗ *spmf p x* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *real-inverse-le-1-iff*: **fixes** *x* :: *real*
shows ⟦ *0* ≤ *x*; *x* ≤ *1* ⟧ $\Longrightarrow$ *1* / *x* ≤ *1* $\longleftrightarrow$ *x* = *1* ∨ *x* = *0*
⟨*proof*⟩

lemma *spmf-scale-spmf'*: *r* ≤ *1* $\Longrightarrow$ *spmf* (*scale-spmf r p*) *x* = *max 0 r* ∗ *spmf p*

x
⟨*proof*⟩

lemma *scale-spmf-neg*: $r \leq 0 \Longrightarrow$ *scale-spmf r p* = *return-pmf None*
⟨*proof*⟩

lemma *scale-spmf-return-None* [*simp*]: *scale-spmf r* (*return-pmf None*) = *return-pmf None*
⟨*proof*⟩

lemma *scale-spmf-conv-bind-bernoulli*:
assumes $r \leq 1$
shows *scale-spmf r p* = *bind-pmf* (*bernoulli-pmf r*) (λ*b*. *if b then p else return-pmf None*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *nn-integral-spmf*: $(\int^+ x.$ *spmf p x* ∂*count-space A*) = *emeasure* (*measure-spmf p*) *A*
⟨*proof*⟩

lemma *measure-spmf-scale-spmf*: *measure-spmf* (*scale-spmf r p*) = *scale-measure* (*min* (*inverse* (*weight-spmf p*)) *r*) (*measure-spmf p*)
⟨*proof*⟩

lemma *measure-spmf-scale-spmf'*:
$r \leq 1 \Longrightarrow$ *measure-spmf* (*scale-spmf r p*) = *scale-measure r* (*measure-spmf p*)
⟨*proof*⟩

lemma *scale-spmf-1* [*simp*]: *scale-spmf 1 p* = *p*
⟨*proof*⟩

lemma *scale-spmf-0* [*simp*]: *scale-spmf 0 p* = *return-pmf None*
⟨*proof*⟩

lemma *bind-scale-spmf*:
assumes *r*: $r \leq 1$
shows *bind-spmf* (*scale-spmf r p*) *f* = *bind-spmf p* (λ*x*. *scale-spmf r* (*f x*))
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *scale-bind-spmf*:
assumes $r \leq 1$
shows *scale-spmf r* (*bind-spmf p f*) = *bind-spmf p* (λ*x*. *scale-spmf r* (*f x*))
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *bind-spmf-const*: *bind-spmf p* (λ*x*. *q*) = *scale-spmf* (*weight-spmf p*) *q* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *map-scale-spmf*: *map-spmf f (scale-spmf r p) = scale-spmf r (map-spmf f p)* (**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *set-scale-spmf*: *set-spmf (scale-spmf r p) = (if r > 0 then set-spmf p else {})*
⟨*proof*⟩

lemma *set-scale-spmf'* [*simp*]: *0 < r ⟹ set-spmf (scale-spmf r p) = set-spmf p*
⟨*proof*⟩

lemma *rel-spmf-scaleI*:
assumes *r > 0 ⟹ rel-spmf A p q*
shows *rel-spmf A (scale-spmf r p) (scale-spmf r q)*
⟨*proof*⟩

lemma *weight-scale-spmf*: *weight-spmf (scale-spmf r p) = min 1 (max 0 r * weight-spmf p)*
⟨*proof*⟩

lemma *weight-scale-spmf'* [*simp*]:
⟦ *0 ≤ r*; *r ≤ 1* ⟧ *⟹ weight-spmf (scale-spmf r p) = r * weight-spmf p*
⟨*proof*⟩

lemma *pmf-scale-spmf-None*:
*pmf (scale-spmf k p) None = 1 − min 1 (max 0 k * (1 − pmf p None))*
⟨*proof*⟩

lemma *scale-scale-spmf*:
*scale-spmf r (scale-spmf r' p) = scale-spmf (r * max 0 (min (inverse (weight-spmf p)) r')) p*
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *scale-scale-spmf'* [*simp*]:
⟦ *0 ≤ r*; *r ≤ 1*; *0 ≤ r'*; *r' ≤ 1* ⟧
*⟹ scale-spmf r (scale-spmf r' p) = scale-spmf (r * r') p*
⟨*proof*⟩

lemma *scale-spmf-eq-same*: *scale-spmf r p = p ⟷ weight-spmf p = 0 ∨ r = 1 ∨ r ≥ 1 ∧ weight-spmf p = 1*
(**is** *?lhs ⟷ ?rhs*)
⟨*proof*⟩

lemma *map-const-spmf-of-set*:
⟦ *finite A*; *A ≠ {}* ⟧ *⟹ map-spmf (λ-. c) (spmf-of-set A) = return-spmf c*
⟨*proof*⟩

25.16 Conditional spmfs

lemma *set-pmf-Int-Some*: *set-pmf p* ∩ *Some* ‘ *A* = {} ⟷ *set-spmf p* ∩ *A* = {}
⟨*proof*⟩

lemma *measure-spmf-zero-iff*: *measure* (*measure-spmf p*) *A* = *0* ⟷ *set-spmf p* ∩ *A* = {}
⟨*proof*⟩

definition *cond-spmf* :: *'a spmf* ⇒ *'a set* ⇒ *'a spmf*
where *cond-spmf p A* = (*if set-spmf p* ∩ *A* = {} *then return-pmf None else cond-pmf p* (*Some* ‘ *A*))

lemma *set-cond-spmf* [*simp*]: *set-spmf* (*cond-spmf p A*) = *set-spmf p* ∩ *A*
⟨*proof*⟩

lemma *cond-map-spmf* [*simp*]: *cond-spmf* (*map-spmf f p*) *A* = *map-spmf f* (*cond-spmf p* (*f* −‘ *A*))
⟨*proof*⟩

lemma *spmf-cond-spmf* [*simp*]:
spmf (*cond-spmf p A*) *x* = (*if x* ∈ *A then spmf p x* / *measure* (*measure-spmf p*) *A else 0*)
⟨*proof*⟩

lemma *bind-eq-return-pmf-None*:
bind-spmf p f = *return-pmf None* ⟷ (∀ *x*∈*set-spmf p*. *f x* = *return-pmf None*)
⟨*proof*⟩

lemma *return-pmf-None-eq-bind*:
return-pmf None = *bind-spmf p f* ⟷ (∀ *x*∈*set-spmf p*. *f x* = *return-pmf None*)
⟨*proof*⟩

25.17 Product spmf

definition *pair-spmf* :: *'a spmf* ⇒ *'b spmf* ⇒ (*'a* × *'b*) *spmf*
where *pair-spmf p q* = *bind-pmf* (*pair-pmf p q*) (λ*xy*. *case xy of* (*Some x*, *Some y*) ⇒ *return-spmf* (*x*, *y*) | - ⇒ *return-pmf None*)

lemma *map-fst-pair-spmf* [*simp*]: *map-spmf fst* (*pair-spmf p q*) = *scale-spmf* (*weight-spmf q*) *p*
⟨*proof*⟩

lemma *map-snd-pair-spmf* [*simp*]: *map-spmf snd* (*pair-spmf p q*) = *scale-spmf* (*weight-spmf p*) *q*
⟨*proof*⟩

lemma *set-pair-spmf* [*simp*]: *set-spmf* (*pair-spmf p q*) = *set-spmf p* × *set-spmf q*
⟨*proof*⟩

lemma *spmf-pair* [*simp*]: *spmf* (*pair-spmf p q*) (*x*, *y*) = *spmf p x* ∗ *spmf q y* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *pair-map-spmf2*: *pair-spmf p* (*map-spmf f q*) = *map-spmf* (*apsnd f*) (*pair-spmf p q*)
⟨*proof*⟩

lemma *pair-map-spmf1*: *pair-spmf* (*map-spmf f p*) *q* = *map-spmf* (*apfst f*) (*pair-spmf p q*)
⟨*proof*⟩

lemma *pair-map-spmf*: *pair-spmf* (*map-spmf f p*) (*map-spmf g q*) = *map-spmf* (*map-prod f g*) (*pair-spmf p q*)
⟨*proof*⟩

lemma *pair-spmf-alt-def*: *pair-spmf p q* = *bind-spmf p* (λ*x*. *bind-spmf q* (λ*y*. *return-spmf* (*x*, *y*)))
⟨*proof*⟩

lemma *weight-pair-spmf* [*simp*]: *weight-spmf* (*pair-spmf p q*) = *weight-spmf p* ∗ *weight-spmf q*
⟨*proof*⟩

lemma *pair-scale-spmf1*:
r ≤ *1* ⟹ *pair-spmf* (*scale-spmf r p*) *q* = *scale-spmf r* (*pair-spmf p q*)
⟨*proof*⟩

lemma *pair-scale-spmf2*:
r ≤ *1* ⟹ *pair-spmf p* (*scale-spmf r q*) = *scale-spmf r* (*pair-spmf p q*)
⟨*proof*⟩

lemma *pair-spmf-return-None1* [*simp*]: *pair-spmf* (*return-pmf None*) *p* = *return-pmf None*
⟨*proof*⟩

lemma *pair-spmf-return-None2* [*simp*]: *pair-spmf p* (*return-pmf None*) = *return-pmf None*
⟨*proof*⟩

lemma *pair-spmf-return-spmf1*: *pair-spmf* (*return-spmf x*) *q* = *map-spmf* (*Pair x*) *q*
⟨*proof*⟩

lemma *pair-spmf-return-spmf2*: *pair-spmf p* (*return-spmf y*) = *map-spmf* (λ*x*. (*x*, *y*)) *p*
⟨*proof*⟩

lemma *pair-spmf-return-spmf* [*simp*]: *pair-spmf* (*return-spmf x*) (*return-spmf y*)

= *return-spmf* (*x*, *y*)
⟨*proof*⟩

lemma *rel-pair-spmf-prod*:
rel-spmf (*rel-prod A B*) (*pair-spmf p q*) (*pair-spmf p′ q′*) ⟷
rel-spmf A (*scale-spmf* (*weight-spmf q*) *p*) (*scale-spmf* (*weight-spmf q′*) *p′*) ∧
rel-spmf B (*scale-spmf* (*weight-spmf p*) *q*) (*scale-spmf* (*weight-spmf p′*) *q′*)
(**is** *?lhs* ⟷ *?rhs* **is** - ⟷ *?A* ∧ *?B* **is** - ⟷ *rel-spmf* - *?p ?p′* ∧ *rel-spmf* - *?q ?q′*)
⟨*proof*⟩

lemma *pair-pair-spmf*:
pair-spmf (*pair-spmf p q*) *r* = *map-spmf* (λ(*x*, (*y*, *z*)). ((*x*, *y*), *z*)) (*pair-spmf p* (*pair-spmf q r*))
⟨*proof*⟩

lemma *pair-commute-spmf*:
pair-spmf p q = *map-spmf* (λ(*y*, *x*). (*x*, *y*)) (*pair-spmf q p*)
⟨*proof*⟩

25.18 Assertions

definition *assert-spmf* :: *bool* ⇒ *unit spmf*
where *assert-spmf b* = (*if b then return-spmf* () *else return-pmf None*)

lemma *assert-spmf-simps* [*simp*]:
assert-spmf True = *return-spmf* ()
assert-spmf False = *return-pmf None*
⟨*proof*⟩

lemma *in-set-assert-spmf* [*simp*]: *x* ∈ *set-spmf* (*assert-spmf p*) ⟷ *p*
⟨*proof*⟩

lemma *set-spmf-assert-spmf-eq-empty* [*simp*]: *set-spmf* (*assert-spmf b*) = {} ⟷ ¬ *b*
⟨*proof*⟩

lemma *lossless-assert-spmf* [*iff*]: *lossless-spmf* (*assert-spmf b*) ⟷ *b*
⟨*proof*⟩

25.19 Try

definition *try-spmf* :: *'a spmf* ⇒ *'a spmf* ⇒ *'a spmf* (*TRY - ELSE - [0,60] 59*)
where *try-spmf p q* = *bind-pmf p* (λ*x*. *case x of None* ⇒ *q* | *Some y* ⇒ *return-spmf y*)

lemma *try-spmf-lossless* [*simp*]:
assumes *lossless-spmf p*
shows *TRY p ELSE q* = *p*
⟨*proof*⟩

lemma *try-spmf-return-spmf1*: *TRY return-spmf x ELSE q = return-spmf x*
⟨*proof*⟩

lemma *try-spmf-return-None* [*simp*]: *TRY return-pmf None ELSE q = q*
⟨*proof*⟩

lemma *try-spmf-return-pmf-None2* [*simp*]: *TRY p ELSE return-pmf None = p*
⟨*proof*⟩

lemma *map-try-spmf*: *map-spmf f (try-spmf p q) = try-spmf (map-spmf f p) (map-spmf f q)*
⟨*proof*⟩

lemma *try-spmf-bind-pmf*: *TRY (bind-pmf p f) ELSE q = bind-pmf p (λx. TRY (f x) ELSE q)*
⟨*proof*⟩

lemma *try-spmf-bind-spmf-lossless*:
lossless-spmf p ⟹ TRY (bind-spmf p f) ELSE q = bind-spmf p (λx. TRY (f x) ELSE q)
⟨*proof*⟩

lemma *try-spmf-bind-out*:
lossless-spmf p ⟹ bind-spmf p (λx. TRY (f x) ELSE q) = TRY (bind-spmf p f) ELSE q
⟨*proof*⟩

lemma *lossless-try-spmf* [*simp*]:
lossless-spmf (TRY p ELSE q) ⟷ lossless-spmf p ∨ lossless-spmf q
⟨*proof*⟩

context includes *lifting-syntax*
begin

lemma *try-spmf-parametric* [*transfer-rule*]:
(rel-spmf A ===> rel-spmf A ===> rel-spmf A) try-spmf try-spmf
⟨*proof*⟩

end

lemma *try-spmf-cong*:
⟦ p = p'; ¬ lossless-spmf p' ⟹ q = q' ⟧ ⟹ TRY p ELSE q = TRY p' ELSE q'
⟨*proof*⟩

lemma *rel-spmf-try-spmf*:
⟦ rel-spmf R p p'; ¬ lossless-spmf p' ⟹ rel-spmf R q q' ⟧
⟹ rel-spmf R (TRY p ELSE q) (TRY p' ELSE q')
⟨*proof*⟩

lemma *spmf-try-spmf*:
spmf (*TRY p ELSE q*) *x* = *spmf p x* + *pmf p None* ∗ *spmf q x*
⟨*proof*⟩

lemma *try-scale-spmf-same* [*simp*]: *lossless-spmf p* ⟹ *TRY scale-spmf k p ELSE p* = *p*
⟨*proof*⟩

lemma *pmf-try-spmf-None* [*simp*]: *pmf* (*TRY p ELSE q*) *None* = *pmf p None* ∗ *pmf q None* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *try-bind-spmf-lossless2*:
lossless-spmf q ⟹ *TRY* (*bind-spmf p f*) *ELSE q* = *TRY* (*p* ⤜ (λ*x*. *TRY* (*f x*) *ELSE q*)) *ELSE q*
⟨*proof*⟩

lemma *try-bind-spmf-lossless2′*:
fixes *f* :: *′a* ⇒ *′b spmf* **shows**
⟦ *NO-MATCH* (λ*x* :: *′a*. *try-spmf* (*g x* :: *′b spmf*) (*h x*)) *f*; *lossless-spmf q* ⟧
⟹ *TRY* (*bind-spmf p f*) *ELSE q* = *TRY* (*p* ⤜ (λ*x* :: *′a*. *TRY* (*f x*) *ELSE q*)) *ELSE q*
⟨*proof*⟩

lemma *try-bind-assert-spmf*:
TRY (*assert-spmf b* ⤜ *f*) *ELSE q* = (*if b then TRY* (*f* ()) *ELSE q else q*)
⟨*proof*⟩

25.20 Miscellaneous

lemma assumes *rel-spmf* (λ*x y*. *bad1 x* = *bad2 y* ∧ (¬ *bad2 y* ⟶ *A x* ⟷ *B y*)) *p q* (**is** *rel-spmf ?A - -*)
shows *fundamental-lemma-bad*: *measure* (*measure-spmf p*) {*x*. *bad1 x*} = *measure* (*measure-spmf q*) {*y*. *bad2 y*} (**is** *?bad*)
and *fundamental-lemma*: |*measure* (*measure-spmf p*) {*x*. *A x*} − *measure* (*measure-spmf q*) {*y*. *B y*}| ≤
measure (*measure-spmf p*) {*x*. *bad1 x*} (**is** *?fundamental*)
⟨*proof*⟩

end

26 Indexed products of PMFs

theory *Product-PMF*
imports *Probability-Mass-Function Independent-Family*
begin

Conflicting notation from *HOL−Analysis.Infinite-Sum*

no-notation *Infinite-Sum.abs-summable-on* (**infixr** *abs'-summable'-on 46*)

26.1 Preliminaries

lemma *pmf-expectation-eq-infsetsum*: *measure-pmf.expectation p f = infsetsum* (*λx. pmf p x * f x*) *UNIV*
⟨*proof*⟩

lemma *measure-pmf-prob-product*:
assumes *countable A countable B*
shows *measure-pmf.prob* (*pair-pmf M N*) (*A × B*) = *measure-pmf.prob M A * measure-pmf.prob N B*
⟨*proof*⟩

26.2 Definition

In analogy to Pi_M, we define an indexed product of PMFs. In the literature, this is typically called taking a vector of independent random variables. Note that the components do not have to be identically distributed.

The operation takes an explicit index set A and a function f that maps each element from A to a PMF and defines the product measure $\bigotimes_{i \in A} f(i)$, which is represented as a (*'a ⇒ 'b*) *pmf*.

Note that unlike Pi_M, this only works for *finite* index sets. It could be extended to countable sets and beyond, but the construction becomes somewhat more involved.

definition *Pi-pmf* :: *'a set ⇒ 'b ⇒* (*'a ⇒ 'b pmf*) *⇒* (*'a ⇒ 'b*) *pmf* **where**
Pi-pmf A dflt p =
embed-pmf (*λf. if* (*∀ x. x ∉ A ⟶ f x = dflt*) *then* ∏ *x∈A. pmf* (*p x*) (*f x*) *else 0*)

A technical subtlety that needs to be addressed is this: Intuitively, the functions in the support of a product distribution have domain A. However, since HOL is a total logic, these functions must still return *some* value for inputs outside A. The product measure Pi_M simply lets these functions return *undefined* in these cases. We chose a different solution here, which is to supply a default value *dflt* that is returned in these cases.

As one possible application, one could model the result of n different independent coin tosses as *Pi-pmf* {*0::'a..<n*} *False* (*λ-. bernoulli-pmf* (*1 / 2*)). This returns a function of type *nat ⇒ bool* that maps every natural number below n to the result of the corresponding coin toss, and every other natural number to *False*.

lemma *pmf-Pi*:
assumes *A*: *finite A*
shows *pmf* (*Pi-pmf A dflt p*) *f =*
(*if* (*∀ x. x ∉ A ⟶ f x = dflt*) *then* ∏ *x∈A. pmf* (*p x*) (*f x*) *else 0*)
⟨*proof*⟩

lemma *Pi-pmf-cong*:
assumes $A = A'$ *dflt* = *dflt'* $\bigwedge x.\ x \in A \Longrightarrow f\ x = f'\ x$
shows *Pi-pmf A dflt f* = *Pi-pmf A' dflt' f'*
⟨*proof*⟩

lemma *pmf-Pi'*:
assumes *finite A* $\bigwedge x.\ x \notin A \Longrightarrow f\ x =$ *dflt*
shows *pmf* (*Pi-pmf A dflt p*) *f* = $(\prod x \in A.$ *pmf* (*p x*) (*f x*))
⟨*proof*⟩

lemma *pmf-Pi-outside*:
assumes *finite A* $\exists\, x.\ x \notin A \wedge f\ x \neq$ *dflt*
shows *pmf* (*Pi-pmf A dflt p*) *f* = *0*
⟨*proof*⟩

lemma *pmf-Pi-empty* [*simp*]: *Pi-pmf* {} *dflt p* = *return-pmf* (λ-. *dflt*)
⟨*proof*⟩

lemma *set-Pi-pmf-subset*: *finite A* ⟹ *set-pmf* (*Pi-pmf A dflt p*) ⊆ {*f*. $\forall\, x.\ x \notin A \longrightarrow f\ x =$ *dflt*}
⟨*proof*⟩

26.3 Dependent product sets with a default

The following describes a dependent product of sets where the functions are required to return the default value *dflt* outside their domain, in analogy to Pi_E, which uses *undefined*.

definition *PiE-dflt*
where *PiE-dflt A dflt B* = {*f*. $\forall\, x.\ (x \in A \longrightarrow f\ x \in B\ x) \wedge (x \notin A \longrightarrow f\ x =$ *dflt*)}

lemma *restrict-PiE-dflt*: (λ*h*. *restrict h A*) ‘ *PiE-dflt A dflt B* = *PiE A B*
⟨*proof*⟩

lemma *dflt-image-PiE*: (λ*h x*. *if x* ∈ *A then h x else dflt*) ‘ *PiE A B* = *PiE-dflt A dflt B*
(**is** *?f* ‘ *?X* = *?Y*)
⟨*proof*⟩

lemma *finite-PiE-dflt* [*intro*]:
assumes *finite A* $\bigwedge x.\ x \in A \Longrightarrow$ *finite* (*B x*)
shows *finite* (*PiE-dflt A d B*)
⟨*proof*⟩

lemma *card-PiE-dflt*:
assumes *finite A* $\bigwedge x.\ x \in A \Longrightarrow$ *finite* (*B x*)
shows *card* (*PiE-dflt A d B*) = $(\prod x \in A.$ *card* (*B x*))
⟨*proof*⟩

lemma *PiE-dflt-empty-iff* [*simp*]: *PiE-dflt A dflt B* = {} ⟷ (∃ *x*∈*A*. *B x* = {})
⟨*proof*⟩

lemma *set-Pi-pmf-subset'*:
assumes *finite A*
shows *set-pmf* (*Pi-pmf A dflt p*) ⊆ *PiE-dflt A dflt* (*set-pmf* ∘ *p*)
⟨*proof*⟩

lemma *set-Pi-pmf*:
assumes *finite A*
shows *set-pmf* (*Pi-pmf A dflt p*) = *PiE-dflt A dflt* (*set-pmf* ∘ *p*)
⟨*proof*⟩

The probability of an independent combination of events is precisely the product of the probabilities of each individual event.

lemma *measure-Pi-pmf-PiE-dflt*:
assumes [*simp*]: *finite A*
shows *measure-pmf.prob* (*Pi-pmf A dflt p*) (*PiE-dflt A dflt B*) =
(∏ *x*∈*A*. *measure-pmf.prob* (*p x*) (*B x*))
⟨*proof*⟩

lemma *measure-Pi-pmf-Pi*:
fixes *t*::*nat*
assumes [*simp*]: *finite A*
shows *measure-pmf.prob* (*Pi-pmf A dflt p*) (*Pi A B*) =
(∏ *x*∈*A*. *measure-pmf.prob* (*p x*) (*B x*)) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

26.4 Common PMF operations on products

Pi-pmf distributes over the 'bind' operation in the Giry monad:

lemma *Pi-pmf-bind*:
assumes *finite A*
shows *Pi-pmf A d* (λ*x*. *bind-pmf* (*p x*) (*q x*)) =
do {*f* ← *Pi-pmf A d' p*; *Pi-pmf A d* (λ*x*. *q x* (*f x*))} (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *Pi-pmf-return-pmf* [*simp*]:
assumes *finite A*
shows *Pi-pmf A dflt* (λ*x*. *return-pmf* (*f x*)) = *return-pmf* (λ*x*. *if x* ∈ *A then f x else dflt*)
⟨*proof*⟩

Analogously any componentwise mapping can be pulled outside the product:

lemma *Pi-pmf-map*:
assumes [*simp*]: *finite A* **and** *f dflt* = *dflt'*
shows *Pi-pmf A dflt'* (λ*x*. *map-pmf f* (*g x*)) = *map-pmf* (λ*h*. *f* ∘ *h*) (*Pi-pmf A dflt g*)

⟨*proof*⟩

We can exchange the default value in a product of PMFs like this:

lemma *Pi-pmf-default-swap*:
assumes *finite A*
shows *map-pmf* (λ*f x*. *if x* ∈ *A then f x else dflt'*) (*Pi-pmf A dflt p*) =
Pi-pmf A dflt' p (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

The following rule allows reindexing the product:

lemma *Pi-pmf-bij-betw*:
assumes *finite A bij-betw h A B* ⋀*x*. *x* ∉ *A* ⟹ *h x* ∉ *B*
shows *Pi-pmf A dflt* (λ-. *f*) = *map-pmf* (λ*g*. *g* ∘ *h*) (*Pi-pmf B dflt* (λ-. *f*))
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

A product of uniform random choices is again a uniform distribution.

lemma *Pi-pmf-of-set*:
assumes *finite A* ⋀*x*. *x* ∈ *A* ⟹ *finite* (*B x*) ⋀*x*. *x* ∈ *A* ⟹ *B x* ≠ {}
shows *Pi-pmf A d* (λ*x*. *pmf-of-set* (*B x*)) = *pmf-of-set* (*PiE-dflt A d B*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

26.5 Merging and splitting PMF products

The following lemma shows that we can add a single PMF to a product:

lemma *Pi-pmf-insert*:
assumes *finite A x* ∉ *A*
shows *Pi-pmf* (*insert x A*) *dflt p* = *map-pmf* (λ(*y*,*f*). *f*(*x*:=*y*)) (*pair-pmf* (*p x*) (*Pi-pmf A dflt p*))
⟨*proof*⟩

lemma *Pi-pmf-insert'*:
assumes *finite A x* ∉ *A*
shows *Pi-pmf* (*insert x A*) *dflt p* =
do {*y* ← *p x*; *f* ← *Pi-pmf A dflt p*; *return-pmf* (*f*(*x* := *y*))}
⟨*proof*⟩

lemma *Pi-pmf-singleton*:
Pi-pmf {*x*} *dflt p* = *map-pmf* (λ*a b*. *if b* = *x then a else dflt*) (*p x*)
⟨*proof*⟩

Projecting a product of PMFs onto a component yields the expected result:

lemma *Pi-pmf-component*:
assumes *finite A*
shows *map-pmf* (λ*f*. *f x*) (*Pi-pmf A dflt p*) = (*if x* ∈ *A then p x else return-pmf dflt*)
⟨*proof*⟩

We can take merge two PMF products on disjoint sets like this:

lemma *Pi-pmf-union*:
assumes *finite A finite B* $A \cap B = \{\}$
shows *Pi-pmf* $(A \cup B)$ *dflt p* =
map-pmf $(\lambda(f,g)\ x.$ *if* $x \in A$ *then f x else g x*)
(*pair-pmf* (*Pi-pmf A dflt p*) (*Pi-pmf B dflt p*)) (**is** - = *map-pmf* (*?h A*) (*?q A*))
⟨*proof*⟩

We can also project a product to a subset of the indices by mapping all the other indices to the default value:

lemma *Pi-pmf-subset*:
assumes *finite A* $A' \subseteq A$
shows *Pi-pmf A' dflt p* = *map-pmf* $(\lambda f\ x.$ *if* $x \in A'$ *then f x else dflt*) (*Pi-pmf A dflt p*)
⟨*proof*⟩

lemma *Pi-pmf-subset'*:
fixes $f :: 'a \Rightarrow 'b$ *pmf*
assumes *finite A* $B \subseteq A$ $\bigwedge x.\ x \in A - B \Longrightarrow f\ x =$ *return-pmf dflt*
shows *Pi-pmf A dflt f* = *Pi-pmf B dflt f*
⟨*proof*⟩

lemma *Pi-pmf-if-set*:
assumes *finite A*
shows *Pi-pmf A dflt* $(\lambda x.$ *if b x then f x else return-pmf dflt*) =
Pi-pmf $\{x \in A.\ b\ x\}$ *dflt f*
⟨*proof*⟩

lemma *Pi-pmf-if-set'*:
assumes *finite A*
shows *Pi-pmf A dflt* $(\lambda x.$ *if b x then return-pmf dflt else f x*) =
Pi-pmf $\{x \in A.\ \neg b\ x\}$ *dflt f*
⟨*proof*⟩

Lastly, we can delete a single component from a product:

lemma *Pi-pmf-remove*:
assumes *finite A*
shows *Pi-pmf* $(A - \{x\})$ *dflt p* = *map-pmf* $(\lambda f.\ f(x := dflt))$ (*Pi-pmf A dflt p*)
⟨*proof*⟩

26.6 Additional properties

lemma *nn-integral-prod-Pi-pmf*:
assumes *finite A*
shows *nn-integral* (*Pi-pmf A dflt p*) $(\lambda y.\ \prod x \in A.\ f\ x\ (y\ x)) = (\prod x \in A.$ *nn-integral* $(p\ x)\ (f\ x))$

⟨*proof*⟩

lemma *integrable-prod-Pi-pmf*:
fixes $f :: 'a \Rightarrow 'b \Rightarrow 'c :: \{$*real-normed-field*, *second-countable-topology*, *banach*$\}$
assumes *finite* A **and** $\bigwedge x.\ x \in A \Longrightarrow$ *integrable* (*measure-pmf* $(p\ x)$) $(f\ x)$
shows *integrable* (*measure-pmf* (*Pi-pmf* A *dflt* p)) $(\lambda h.\ \prod x{\in}A.\ f\ x\ (h\ x))$
⟨*proof*⟩

lemma *expectation-prod-Pi-pmf*:
fixes $f :: - \Rightarrow - \Rightarrow$ *real*
assumes *finite* A
assumes $\bigwedge x.\ x \in A \Longrightarrow$ *integrable* (*measure-pmf* $(p\ x)$) $(f\ x)$
assumes $\bigwedge x\ y.\ x \in A \Longrightarrow y \in$ *set-pmf* $(p\ x) \Longrightarrow f\ x\ y \geq 0$
shows *measure-pmf.expectation* (*Pi-pmf* A *dflt* p) $(\lambda y.\ \prod x{\in}A.\ f\ x\ (y\ x)) =$
$(\prod x{\in}A.$ *measure-pmf.expectation* $(p\ x)\ (\lambda v.\ f\ x\ v))$
⟨*proof*⟩

lemma *indep-vars-Pi-pmf*:
assumes *fin*: *finite* I
shows *prob-space.indep-vars* (*measure-pmf* (*Pi-pmf* I *dflt* p))
$(\lambda\text{-}.$ *count-space UNIV*$)\ (\lambda x\ f.\ f\ x)\ I$
⟨*proof*⟩

lemma
fixes $h :: 'a ::$ *comm-monoid-add* $\Rightarrow 'b{::}\{$*banach*, *second-countable-topology*$\}$
assumes *fin*: *finite* I
assumes *integrable*: $\bigwedge i.\ i \in I \Longrightarrow$ *integrable* (*measure-pmf* $(D\ i)$) h
shows *integrable-sum-Pi-pmf*: *integrable* (*Pi-pmf* I *dflt* D) $(\lambda g.\ \sum i{\in}I.\ h\ (g\ i))$
and *expectation-sum-Pi-pmf*:
measure-pmf.expectation (*Pi-pmf* I *dflt* D) $(\lambda g.\ \sum i{\in}I.\ h\ (g\ i)) =$
$(\sum i{\in}I.$ *measure-pmf.expectation* $(D\ i)\ h)$
⟨*proof*⟩

26.7 Applications

Choosing a subset of a set uniformly at random is equivalent to tossing a fair coin independently for each element and collecting all the elements that came up heads.

lemma *pmf-of-set-Pow-conv-bernoulli*:
assumes *finite* $(A :: 'a\ set)$
shows *map-pmf* $(\lambda b.\ \{x{\in}A.\ b\ x\})$ (*Pi-pmf* A P $(\lambda\text{-}.$ *bernoulli-pmf* $(1/2)))$ =
pmf-of-set $(Pow\ A)$
⟨*proof*⟩

A binomial distribution can be seen as the number of successes in n independent coin tosses.

lemma *binomial-pmf-altdef'*:
fixes $A :: 'a\ set$

assumes *finite A* **and** *card A* = *n* **and** *p*: *p* ∈ {*0..1*}
shows *binomial-pmf n p* =
map-pmf (λ*f*. *card* {*x*∈*A*. *f x*}) (*Pi-pmf A dflt* (λ-. *bernoulli-pmf p*)) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

end

27 Hoeffding's Lemma and Hoeffding's Inequality

theory *Hoeffding*
imports *Product-PMF Independent-Family*
begin

Hoeffding's inequality shows that a sum of bounded independent random variables is concentrated around its mean, with an exponential decay of the tail probabilities.

27.1 Hoeffding's Lemma

lemma *convex-on-exp*:
fixes *l* :: *real*
assumes *l* ≥ *0*
shows *convex-on UNIV* (λ*x*. *exp*(*l*∗*x*))
⟨*proof*⟩

lemma *mult-const-minus-self-real-le*:
fixes *x* :: *real*
shows *x* ∗ (*c* − *x*) ≤ *c*^{2} / *4*
⟨*proof*⟩

lemma *Hoeffdings-lemma-aux*:
fixes *h p* :: *real*
assumes *h* ≥ *0* **and** *p* ≥ *0*
defines *L* ≡ (λ*h*. −*h* ∗ *p* + *ln* (*1* + *p* ∗ (*exp h* − *1*)))
shows *L h* ≤ *h*^{2} / *8*
⟨*proof*⟩

locale *interval-bounded-random-variable* = *prob-space* +
fixes *f* :: *'a* ⇒ *real* **and** *a b* :: *real*
assumes *random-variable* [*measurable*]: *random-variable borel f*
assumes *AE-in-interval*: *AE x in M*. *f x* ∈ {*a..b*}
begin

lemma *integrable* [*intro*]: *integrable M f*
⟨*proof*⟩

We first show Hoeffding's lemma for distributions whose expectation is 0. The general case will easily follow from this later.

lemma *Hoeffdings-lemma-nn-integral-0*:
assumes $l > 0$ **and** *E0*: *expectation* $f = 0$
shows *nn-integral* M $(\lambda x.\ exp\ (l * f\ x)) \leq ennreal\ (exp\ (l^2 * (b - a)^2\ /\ 8))$
⟨*proof*⟩

context
begin

interpretation *shift*: *interval-bounded-random-variable* M $\lambda x.\ f\ x - \mu$ $a - \mu$ $b - \mu$
rewrites $b - \mu - (a - \mu) \equiv b - a$
⟨*proof*⟩

lemma *expectation-shift*: *expectation* $(\lambda x.\ f\ x - expectation\ f) = 0$
⟨*proof*⟩

lemmas *Hoeffdings-lemma-nn-integral* = *shift.Hoeffdings-lemma-nn-integral-0*[*OF - expectation-shift*]

end

end

27.2 Hoeffding's Inequality

Consider n independent real random variables $X_1, \ldots, X_n$ that each almost surely lie in a compact interval $[a_i, b_i]$. Hoeffding's inequality states that the distribution of the sum of the X_i is tightly concentrated around the sum of the expected values: the probability of it being above or below the sum of the expected values by more than some ε decreases exponentially with ε.

locale *indep-interval-bounded-random-variables* = *prob-space* +
fixes $I :: 'b\ set$ **and** $X :: 'b \Rightarrow 'a \Rightarrow real$
fixes $a\ b :: 'b \Rightarrow real$
assumes *fin*: *finite* I
assumes *indep*: *indep-vars* $(\lambda\text{-}.\ borel)\ X\ I$
assumes *AE-in-interval*: $\bigwedge i.\ i \in I \Longrightarrow AE\ x\ in\ M.\ X\ i\ x \in \{a\ i..b\ i\}$
begin

lemma *random-variable* [*measurable*]:
assumes *i*: $i \in I$
shows *random-variable borel* $(X\ i)$
⟨*proof*⟩

lemma *bounded-random-variable* [*intro*]:
assumes *i*: $i \in I$
shows *interval-bounded-random-variable* M $(X\ i)$ $(a\ i)$ $(b\ i)$

⟨*proof*⟩

end

locale *Hoeffding-ineq* = *indep-interval-bounded-random-variables* +
fixes μ :: *real*
defines $\mu \equiv (\sum i{\in}I.\ expectation\ (X\ i))$
begin

theorem *Hoeffding-ineq-ge*:
assumes $\varepsilon \geq 0$
assumes $(\sum i{\in}I.\ (b\ i - a\ i)^2) > 0$
shows *prob* $\{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) \geq \mu + \varepsilon\} \leq exp\ (-2 * \varepsilon^2\ /\ (\sum i{\in}I.\ (b\ i - a\ i)^2))$
proof (*cases* $\varepsilon = 0$)
case [*simp*]: *True*
have *prob* $\{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) \geq \mu + \varepsilon\} \leq 1$
by *simp*
thus *?thesis* **by** *simp*
next
case *False*
with ‹$\varepsilon \geq 0$› **have** ε: $\varepsilon > 0$
by *auto*

define d **where** $d = (\sum i{\in}I.\ (b\ i - a\ i)^2)$
define l :: *real* **where** $l = 4 * \varepsilon\ /\ d$
have d: $d > 0$
using *assms* **by** (*simp add*: *d-def*)
have l: $l > 0$
using ε d **by** (*simp add*: *l-def*)
define μ' **where** $\mu' = (\lambda i.\ expectation\ (X\ i))$

have $\{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) \geq \mu + \varepsilon\} = \{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) - \mu \geq \varepsilon\}$
by (*simp add*: *algebra-simps*)
hence *ennreal* (*prob* $\{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) \geq \mu + \varepsilon\}$) = *emeasure M* ...
by (*simp add*: *emeasure-eq-measure*)
also have ... $\leq$ *ennreal* (*exp* $(-l{*}\varepsilon)) * (\int^+ x{\in}space\ M.\ exp\ (l * ((\sum i{\in}I.\ X\ i\ x) - \mu))\ \partial M)$
by (*intro Chernoff-ineq-nn-integral-ge l*) *auto*
also have $(\lambda x.\ (\sum i{\in}I.\ X\ i\ x) - \mu) = (\lambda x.\ (\sum i{\in}I.\ X\ i\ x - \mu'\ i))$
by (*simp add*: *μ-def sum-subtractf μ'-def*)
also have $(\int^+ x{\in}space\ M.\ exp\ (l * ((\sum i{\in}I.\ X\ i\ x - \mu'\ i)))\ \partial M) =$
$(\int^+ x.\ (\prod i{\in}I.\ ennreal\ (exp\ (l * (X\ i\ x - \mu'\ i))))\ \partial M)$
by (*intro nn-integral-cong*)
(*simp-all add*: *sum-distrib-left ring-distribs exp-diff exp-sum fin prod-ennreal*)
also have ... = $(\prod i{\in}I.\ \int^+ x.\ ennreal\ (exp\ (l * (X\ i\ x - \mu'\ i)))\ \partial M)$
by (*intro indep-vars-nn-integral fin indep-vars-compose2*[*OF indep*]) *auto*

also have *ennreal* (*exp* (−*l* ∗ *ε*)) ∗ ... ≤
ennreal (*exp* (−*l* ∗ *ε*)) ∗ (∏ *i*∈*I*. *ennreal* (*exp* (*l*² ∗ (*b i* − *a i*)² / *8*)))
proof (*intro mult-left-mono prod-mono-ennreal*)
fix *i* **assume** *i*: *i* ∈ *I*
from *i* **interpret** *interval-bounded-random-variable M X i a i b i* **..**
show (∫⁺*x*. *ennreal* (*exp* (*l* ∗ (*X i x* − *μ' i*))) ∂*M*) ≤ *ennreal* (*exp* (*l*² ∗ (*b i* − *a i*)² / *8*))
unfolding *μ'-def* **by** (*rule Hoeffdings-lemma-nn-integral*) *fact+*
qed *auto*
also have ... = *ennreal* (*exp* (−*l*∗*ε*) ∗ (∏ *i*∈*I*. *exp* (*l*² ∗ (*b i* − *a i*)² / *8*)))
by (*simp add*: *prod-ennreal prod-nonneg flip*: *ennreal-mult*)
also have *exp* (−*l*∗*ε*) ∗ (∏ *i*∈*I*. *exp* (*l*² ∗ (*b i* − *a i*)² / *8*)) = *exp* (*d* ∗ *l*² / *8* − *l* ∗ *ε*)
by (*simp add*: *exp-diff exp-minus sum-divide-distrib sum-distrib-left*
sum-distrib-right exp-sum fin divide-simps mult-ac d-def)
also have *d* ∗ *l*² / *8* − *l* ∗ *ε* = −*2* ∗ *ε*² / *d*
using *d* **by** (*simp add*: *l-def field-simps power2-eq-square*)
finally show *?thesis*
by (*subst* (*asm*) *ennreal-le-iff*) (*simp-all add*: *d-def*)
qed

corollary *Hoeffding-ineq-le*:
assumes *ε*: *ε* ≥ *0*
assumes (∑ *i*∈*I*. (*b i* − *a i*)²) > *0*
shows *prob* {*x*∈*space M*. (∑ *i*∈*I*. *X i x*) ≤ *μ* − *ε*} ≤ *exp* (−*2* ∗ *ε*² / (∑ *i*∈*I*. (*b i* − *a i*)²))
⟨*proof*⟩

corollary *Hoeffding-ineq-abs-ge*:
assumes *ε*: *ε* ≥ *0*
assumes (∑ *i*∈*I*. (*b i* − *a i*)²) > *0*
shows *prob* {*x*∈*space M*. |(∑ *i*∈*I*. *X i x*) − *μ*| ≥ *ε*} ≤ *2* ∗ *exp* (−*2* ∗ *ε*² / (∑ *i*∈*I*. (*b i* − *a i*)²))
⟨*proof*⟩

end

27.3 Hoeffding's inequality for i.i.d. bounded random variables

If we have n even identically-distributed random variables, the statement of Hoeffding's lemma simplifies a bit more: it shows that the probability that the average of the X_i is more than ε above the expected value is no greater than $e^{\frac{-2ny^2}{(b-a)^2}}$.

This essentially gives us a more concrete version of the weak law of large numbers: the law states that the probability vanishes for $n \to \infty$ for any $\varepsilon > 0$. Unlike Hoeffding's inequality, it does not assume the variables to have

bounded support, but it does not provide concrete bounds.

locale *iid-interval-bounded-random-variables* = *prob-space* +
  fixes I :: *'b set* **and** X :: *'b* ⇒ *'a* ⇒ *real* **and** Y :: *'a* ⇒ *real*
  fixes a b :: *real*
  assumes *fin*: *finite I*
  assumes *indep*: *indep-vars* (λ-. *borel*) X I
  assumes *distr-X*: $i \in I \Longrightarrow$ *distr M borel* (X i) = *distr M borel* Y
  assumes *rv-Y* [*measurable*]: *random-variable borel Y*
  assumes *AE-in-interval*: *AE x in M*. $Y\ x \in \{a..b\}$
begin

lemma *random-variable* [*measurable*]:
  assumes *i*: $i \in I$
  shows *random-variable borel* (X i)
  ⟨*proof*⟩

sublocale X: *indep-interval-bounded-random-variables M I X* λ-. a λ-. b
⟨*proof*⟩

lemma *expectation-X* [*simp*]:
  assumes *i*: $i \in I$
  shows *expectation* (X i) = *expectation* Y
⟨*proof*⟩

end

locale *Hoeffding-ineq-iid* = *iid-interval-bounded-random-variables* +
  fixes μ :: *real*
  defines $\mu \equiv$ *expectation* Y
begin

sublocale X: *Hoeffding-ineq M I X* λ-. a λ-. b *real* (*card I*) ∗ μ
  ⟨*proof*⟩

corollary
  assumes ε: $\varepsilon \geq 0$
  assumes $a < b$ $I \neq \{\}$
  defines $n \equiv$ *card I*
  shows *Hoeffding-ineq-ge*:
        prob $\{x \in$ *space M*. $(\sum i \in I.\ X\ i\ x) \geq n * \mu + \varepsilon\} \leq$
          exp $(-2 * \varepsilon^2 \ /\ (n * (b - a)^2))$ (**is** *?le*)
    and *Hoeffding-ineq-le*:
        prob $\{x \in$ *space M*. $(\sum i \in I.\ X\ i\ x) \leq n * \mu - \varepsilon\} \leq$
          exp $(-2 * \varepsilon^2 \ /\ (n * (b - a)^2))$ (**is** *?ge*)
    and *Hoeffding-ineq-abs-ge*:
        prob $\{x \in$ *space M*. $|(\sum i \in I.\ X\ i\ x) - n * \mu| \geq \varepsilon\} \leq$
          $2 *$ *exp* $(-2 * \varepsilon^2 \ /\ (n * (b - a)^2))$ (**is** *?abs-ge*)
⟨*proof*⟩

lemma
assumes ε: $\varepsilon \geq 0$
assumes $a < b$ $I \neq \{\}$
defines $n \equiv$ *card* I
shows *Hoeffding-ineq-ge'*:
$prob\ \{x \in space\ M.\ (\sum i \in I.\ X\ i\ x) / n \geq \mu + \varepsilon\} \leq$
$exp\ (-2 * n * \varepsilon^2 / (b - a)^2)$ (**is** *?ge*)
and *Hoeffding-ineq-le'*:
$prob\ \{x \in space\ M.\ (\sum i \in I.\ X\ i\ x) / n \leq \mu - \varepsilon\} \leq$
$exp\ (-2 * n * \varepsilon^2 / (b - a)^2)$ (**is** *?le*)
and *Hoeffding-ineq-abs-ge'*:
$prob\ \{x \in space\ M.\ |(\sum i \in I.\ X\ i\ x) / n - \mu| \geq \varepsilon\} \leq$
$2 * exp\ (-2 * n * \varepsilon^2 / (b - a)^2)$ (**is** *?abs-ge*)
⟨*proof*⟩

end

27.4 Hoeffding's Inequality for the Binomial distribution

We can now apply Hoeffding's inequality to the Binomial distribution, which can be seen as the sum of n i.i.d. coin flips (the support of each of which is contained in $[0, 1]$).

locale *binomial-distribution* =
fixes n :: *nat* **and** p :: *real*
assumes *p*: $p \in \{0..1\}$
begin

context
fixes *coins* :: (*nat* ⇒ *bool*) *pmf* **and** μ
assumes *n*: $n > 0$
defines *coins* ≡ *Pi-pmf* {..<*n*} *False* (λ-. *bernoulli-pmf p*)
begin

lemma *coins-component*:
assumes *i*: $i < n$
shows *distr* (*measure-pmf coins*) *borel* (λ*f*. *if f i then 1 else 0*) =
distr (*measure-pmf* (*bernoulli-pmf p*)) *borel* (λ*b*. *if b then 1 else 0*)
⟨*proof*⟩

lemma *prob-binomial-pmf-conv-coins*:
measure-pmf.prob (*binomial-pmf n p*) {*x*. *P* (*real x*)} =
measure-pmf.prob coins {*x*. *P* (∑ *i*<*n*. *if x i then 1 else 0*)}
⟨*proof*⟩

interpretation *Hoeffding-ineq-iid*
coins {..<*n*} λ*i f*. *if f i then 1 else 0* λ*f*. *if f 0 then 1 else 0 0 1 p*
⟨*proof*⟩

corollary
fixes ε :: *real*
assumes ε: $\varepsilon \geq 0$
shows *prob-ge*: *measure-pmf.prob* (*binomial-pmf n p*) $\{x.\ x \geq n * p + \varepsilon\} \leq exp\ (-2 * \varepsilon^2 / n)$
and *prob-le*: *measure-pmf.prob* (*binomial-pmf n p*) $\{x.\ x \leq n * p - \varepsilon\} \leq exp\ (-2 * \varepsilon^2 / n)$
and *prob-abs-ge*:
measure-pmf.prob (*binomial-pmf n p*) $\{x.\ |x - n * p| \geq \varepsilon\} \leq 2 * exp\ (-2 * \varepsilon^2 / n)$
⟨*proof*⟩

corollary
fixes ε :: *real*
assumes ε: $\varepsilon \geq 0$
shows *prob-ge'*: *measure-pmf.prob* (*binomial-pmf n p*) $\{x.\ x / n \geq p + \varepsilon\} \leq exp\ (-2 * n * \varepsilon^2)$
and *prob-le'*: *measure-pmf.prob* (*binomial-pmf n p*) $\{x.\ x / n \leq p - \varepsilon\} \leq exp\ (-2 * n * \varepsilon^2)$
and *prob-abs-ge'*:
measure-pmf.prob (*binomial-pmf n p*) $\{x.\ |x / n - p| \geq \varepsilon\} \leq 2 * exp\ (-2 * n * \varepsilon^2)$
⟨*proof*⟩

end

end

27.5 Tail bounds for the negative binomial distribution

Since the tail probabilities of a negative Binomial distribution are equal to the tail probabilities of some Binomial distribution, we can obtain tail bounds for the negative Binomial distribution through the Hoeffding tail bounds for the Binomial distribution.

context
fixes *p q* :: *real*
assumes *p*: $p \in \{0<..<1\}$
defines $q \equiv 1 - p$
begin

lemma *prob-neg-binomial-pmf-ge-bound*:
fixes *n* :: *nat* **and** k :: *real*
defines $\mu \equiv real\ n * q / p$
assumes k: $k \geq 0$
shows *measure-pmf.prob* (*neg-binomial-pmf n p*) $\{x.\ real\ x \geq \mu + k\}$
$\leq exp\ (-2 * p\ \hat{}\ 3 * k^2 / (n + p * k))$
⟨*proof*⟩

lemma *prob-neg-binomial-pmf-le-bound*:
fixes n :: *nat* **and** k :: *real*
defines $\mu \equiv real\ n * q\ /\ p$
assumes *k*: $k \geq 0$
shows *measure-pmf.prob* (*neg-binomial-pmf n p*) $\{x.\ real\ x \leq \mu - k\}$
$\leq exp\ (-2 * p\ \hat{}\ 3 * k^2\ /\ (n - p * k))$
⟨*proof*⟩

Due to the function $exp(-l/x)$ being concave for $x \geq \frac{l}{2}$, the above two bounds can be combined into the following one for moderate values of k. (cf. https://math.stackexchange.com/questions/1565559)

lemma *prob-neg-binomial-pmf-abs-ge-bound*:
fixes n :: *nat* **and** k :: *real*
defines $\mu \equiv real\ n * q\ /\ p$
assumes $k \geq 0$ **and** *n-ge*: $n \geq p * k * (p^2 * k + 1)$
shows *measure-pmf.prob* (*neg-binomial-pmf n p*) $\{x.\ |real\ x - \mu| \geq k\} \leq$
$2 * exp\ (-2 * p\ \hat{}\ 3 * k\ \hat{}\ 2\ /\ n)$
⟨*proof*⟩

end

end

theory *Stream-Space*
imports
Infinite-Product-Measure
HOL−Library.Stream
HOL−Library.Linear-Temporal-Logic-on-Streams
begin

lemma *stream-eq-Stream-iff*: $s = x\ \#\#\ t \longleftrightarrow (shd\ s = x \wedge stl\ s = t)$
⟨*proof*⟩

lemma *Stream-snth*: $(x\ \#\#\ s)\ !!\ n = (case\ n\ of\ 0 \Rightarrow x \mid Suc\ n \Rightarrow s\ !!\ n)$
⟨*proof*⟩

definition *to-stream* :: $(nat \Rightarrow 'a) \Rightarrow 'a\ stream$ **where**
to-stream X = *smap X nats*

lemma *to-stream-nat-case*: *to-stream* $(case\text{-}nat\ x\ X) = x\ \#\#\ to\text{-}stream\ X$
⟨*proof*⟩

lemma *to-stream-in-streams*: *to-stream* $X \in streams\ S \longleftrightarrow (\forall n.\ X\ n \in S)$
⟨*proof*⟩

definition *stream-space* :: $'a\ measure \Rightarrow 'a\ stream\ measure$ **where**
stream-space M =
distr $(\Pi_M\ i \in UNIV.\ M)$ (*vimage-algebra* (*streams* (*space M*)) *snth* $(\Pi_M\ i \in UNIV.$

M)) *to-stream*

lemma *space-stream-space*: *space* (*stream-space M*) = *streams* (*space M*)
⟨*proof*⟩

lemma *streams-stream-space*[*intro*]: *streams* (*space M*) ∈ *sets* (*stream-space M*)
⟨*proof*⟩

lemma *stream-space-Stream*:
x ## ω ∈ *space* (*stream-space M*) ⟷ *x* ∈ *space M* ∧ ω ∈ *space* (*stream-space M*)
⟨*proof*⟩

lemma *stream-space-eq-distr*: *stream-space M* = *distr* (Π$_M$ *i*∈*UNIV*. *M*) (*stream-space M*) *to-stream*
⟨*proof*⟩

lemma *sets-stream-space-cong*[*measurable-cong*]:
sets M = *sets N* ⟹ *sets* (*stream-space M*) = *sets* (*stream-space N*)
⟨*proof*⟩

lemma *measurable-snth-PiM*: (λω *n*. ω !! *n*) ∈ *measurable* (*stream-space M*) (Π$_M$ *i*∈*UNIV*. *M*)
⟨*proof*⟩

lemma *measurable-snth*[*measurable*]: (λω. ω !! *n*) ∈ *measurable* (*stream-space M*) *M*
⟨*proof*⟩

lemma *measurable-shd*[*measurable*]: *shd* ∈ *measurable* (*stream-space M*) *M*
⟨*proof*⟩

lemma *measurable-stream-space2*:
assumes *f-snth*: ⋀*n*. (λ*x*. *f x* !! *n*) ∈ *measurable N M*
shows *f* ∈ *measurable N* (*stream-space M*)
⟨*proof*⟩

lemma *measurable-stream-coinduct*[*consumes 1*, *case-names shd stl*, *coinduct set*: *measurable*]:
assumes *F f*
assumes *h*: ⋀*f*. *F f* ⟹ (λ*x*. *shd* (*f x*)) ∈ *measurable N M*
assumes *t*: ⋀*f*. *F f* ⟹ *F* (λ*x*. *stl* (*f x*))
shows *f* ∈ *measurable N* (*stream-space M*)
⟨*proof*⟩

lemma *measurable-sdrop*[*measurable*]: *sdrop n* ∈ *measurable* (*stream-space M*) (*stream-space M*)
⟨*proof*⟩

lemma *measurable-stl*[*measurable*]: $(\lambda\omega.\ stl\ \omega) \in measurable\ (stream\text{-}space\ M)\ (stream\text{-}space\ M)$
⟨*proof*⟩

lemma *measurable-to-stream*[*measurable*]: $to\text{-}stream \in measurable\ (\Pi_M\ i{\in}UNIV.\ M)\ (stream\text{-}space\ M)$
⟨*proof*⟩

lemma *measurable-Stream*[*measurable* (*raw*)]:
assumes *f*[*measurable*]: $f \in measurable\ N\ M$
assumes *g*[*measurable*]: $g \in measurable\ N\ (stream\text{-}space\ M)$
shows $(\lambda x.\ f\ x\ \#\#\ g\ x) \in measurable\ N\ (stream\text{-}space\ M)$
⟨*proof*⟩

lemma *measurable-smap*[*measurable*]:
assumes *X*[*measurable*]: $X \in measurable\ N\ M$
shows $smap\ X \in measurable\ (stream\text{-}space\ N)\ (stream\text{-}space\ M)$
⟨*proof*⟩

lemma *measurable-stake*[*measurable*]:
$stake\ i \in measurable\ (stream\text{-}space\ (count\text{-}space\ UNIV))\ (count\text{-}space\ (UNIV :: 'a{::}countable\ list\ set))$
⟨*proof*⟩

lemma *measurable-shift*[*measurable*]:
assumes *f*: $f \in measurable\ N\ (stream\text{-}space\ M)$
assumes [*measurable*]: $g \in measurable\ N\ (stream\text{-}space\ M)$
shows $(\lambda x.\ stake\ n\ (f\ x)\ @{-}\ g\ x) \in measurable\ N\ (stream\text{-}space\ M)$
⟨*proof*⟩

lemma *measurable-case-stream-replace*[*measurable* (*raw*)]:
$(\lambda x.\ f\ x\ (shd\ (g\ x))\ (stl\ (g\ x))) \in measurable\ M\ N \Longrightarrow (\lambda x.\ case\text{-}stream\ (f\ x)\ (g\ x)) \in measurable\ M\ N$
⟨*proof*⟩

lemma *measurable-ev-at*[*measurable*]:
assumes [*measurable*]: $Measurable.pred\ (stream\text{-}space\ M)\ P$
shows $Measurable.pred\ (stream\text{-}space\ M)\ (ev\text{-}at\ P\ n)$
⟨*proof*⟩

lemma *measurable-alw*[*measurable*]:
$Measurable.pred\ (stream\text{-}space\ M)\ P \Longrightarrow Measurable.pred\ (stream\text{-}space\ M)\ (alw\ P)$
⟨*proof*⟩

lemma *measurable-ev*[*measurable*]:
$Measurable.pred\ (stream\text{-}space\ M)\ P \Longrightarrow Measurable.pred\ (stream\text{-}space\ M)\ (ev\ P)$
⟨*proof*⟩

lemma *measurable-until*:
assumes [*measurable*]: *Measurable.pred* (*stream-space M*) φ *Measurable.pred* (*stream-space M*) ψ
shows *Measurable.pred* (*stream-space M*) (φ *until* ψ)
⟨*proof*⟩

lemma *measurable-holds* [*measurable*]: *Measurable.pred M P* $\Longrightarrow$ *Measurable.pred* (*stream-space M*) (*holds P*)
⟨*proof*⟩

lemma *measurable-hld*[*measurable*]: **assumes** [*measurable*]: *t* $\in$ *sets M* **shows** *Measurable.pred* (*stream-space M*) (*HLD t*)
⟨*proof*⟩

lemma *measurable-nxt*[*measurable* (*raw*)]:
Measurable.pred (*stream-space M*) *P* $\Longrightarrow$ *Measurable.pred* (*stream-space M*) (*nxt P*)
⟨*proof*⟩

lemma *measurable-suntil*[*measurable*]:
assumes [*measurable*]: *Measurable.pred* (*stream-space M*) *Q Measurable.pred* (*stream-space M*) *P*
shows *Measurable.pred* (*stream-space M*) (*Q suntil P*)
⟨*proof*⟩

lemma *measurable-szip*:
(λ(*ω1*, *ω2*). *szip ω1 ω2*) $\in$ *measurable* (*stream-space M* $\bigotimes_M$ *stream-space N*) (*stream-space* (*M* $\bigotimes_M$ *N*))
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-space-stream-space*: *prob-space* (*stream-space M*)
⟨*proof*⟩

lemma (**in** *prob-space*) *nn-integral-stream-space*:
assumes [*measurable*]: *f* $\in$ *borel-measurable* (*stream-space M*)
shows ($\int^+ X.$ *f X* ∂*stream-space M*) = ($\int^+ x.$ ($\int^+ X.$ *f* (*x* ## *X*) ∂*stream-space M*) ∂*M*)
⟨*proof*⟩

lemma (**in** *prob-space*) *emeasure-stream-space*:
assumes *X*[*measurable*]: *X* $\in$ *sets* (*stream-space M*)
shows *emeasure* (*stream-space M*) *X* = ($\int^+ t.$ *emeasure* (*stream-space M*) {*x*∈*space* (*stream-space M*). *t* ## *x* ∈ *X* } ∂*M*)
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-stream-space*:
assumes *P*[*measurable*]: {*x*∈*space* (*stream-space M*). *P x*} $\in$ *sets* (*stream-space M*)

shows $\mathcal{P}(x \text{ in stream-space } M.\ P\ x) = (\int^+ t.\ \mathcal{P}(x \text{ in stream-space } M.\ P\ (t\ \#\#\ x))\ \partial M)$
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-stream-space*:
assumes [*measurable*]: *Measurable.pred* (*stream-space M*) *P*
shows (*AE X in stream-space M. P X*) = (*AE x in M. AE X in stream-space M. P* (*x ## X*))
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-stream-all*:
assumes [*measurable*]: *Measurable.pred M P* **and** *P*: *AE x in M. P x*
shows *AE x in stream-space M. stream-all P x*
⟨*proof*⟩

lemma *streams-sets*:
assumes *X*[*measurable*]: *X* ∈ *sets M* **shows** *streams X* ∈ *sets* (*stream-space M*)
⟨*proof*⟩

lemma *sets-stream-space-in-sets*:
assumes *space*: *space N* = *streams* (*space M*)
assumes *sets*: $\bigwedge i.$ (λ*x. x* !! *i*) ∈ *measurable N M*
shows *sets* (*stream-space M*) ⊆ *sets N*
⟨*proof*⟩

lemma *sets-stream-space-eq*: *sets* (*stream-space M*) =
sets (*SUP i*∈*UNIV. vimage-algebra* (*streams* (*space M*)) (λ*s. s* !! *i*) *M*)
⟨*proof*⟩

lemma *sets-restrict-stream-space*:
assumes *S*[*measurable*]: *S* ∈ *sets M*
shows *sets* (*restrict-space* (*stream-space M*) (*streams S*)) = *sets* (*stream-space* (*restrict-space M S*))
⟨*proof*⟩

primrec *sstart* :: *'a set* ⇒ *'a list* ⇒ *'a stream set* **where**
sstart S [] = *streams S*
| [*simp del*]: *sstart S* (*x # xs*) = (##) *x* ‘ *sstart S xs*

lemma *in-sstart*[*simp*]: *s* ∈ *sstart S* (*x # xs*) ⟷ *shd s* = *x* ∧ *stl s* ∈ *sstart S xs*
⟨*proof*⟩

lemma *sstart-in-streams*: *xs* ∈ *lists S* ⟹ *sstart S xs* ⊆ *streams S*
⟨*proof*⟩

lemma *sstart-eq*: *x* ∈ *streams S* ⟹ *x* ∈ *sstart S xs* = (∀ *i*<*length xs. x* !! *i* = *xs* ! *i*)
⟨*proof*⟩

lemma *sstart-sets*: *sstart S xs* ∈ *sets* (*stream-space* (*count-space UNIV*))
⟨*proof*⟩

lemma *sigma-sets-singletons*:
assumes *countable S*
shows *sigma-sets S* ((λ*s*. {*s*})‘*S*) = *Pow S*
⟨*proof*⟩

lemma *sets-count-space-eq-sigma*:
countable S ⟹ *sets* (*count-space S*) = *sets* (*sigma S* ((λ*s*. {*s*})‘*S*))
⟨*proof*⟩

lemma *sets-stream-space-sstart*:
assumes *S*[*simp*]: *countable S*
shows *sets* (*stream-space* (*count-space S*)) = *sets* (*sigma* (*streams S*) (*sstart S*‘*lists S* ∪ {{}}))
⟨*proof*⟩

lemma *Int-stable-sstart*: *Int-stable* (*sstart S*‘*lists S* ∪ {{}})
⟨*proof*⟩

lemma *stream-space-eq-sstart*:
assumes *S*[*simp*]: *countable S*
assumes *P*: *prob-space M prob-space N*
assumes *ae*: *AE x in M*. *x* ∈ *streams S AE x in N*. *x* ∈ *streams S*
assumes *sets-M*: *sets M* = *sets* (*stream-space* (*count-space UNIV*))
assumes *sets-N*: *sets N* = *sets* (*stream-space* (*count-space UNIV*))
assumes ∗: ⋀*xs*. *xs* ≠ [] ⟹ *xs* ∈ *lists S* ⟹ *emeasure M* (*sstart S xs*) = *emeasure N* (*sstart S xs*)
shows *M* = *N*
⟨*proof*⟩

lemma *sets-sstart*[*measurable*]: *sstart* Ω *xs* ∈ *sets* (*stream-space* (*count-space UNIV*))
⟨*proof*⟩

primrec *scylinder* :: *'a set* ⇒ *'a set list* ⇒ *'a stream set*
where
scylinder S [] = *streams S*
| *scylinder S* (*A* # *As*) = {ω∈*streams S*. *shd* ω ∈ *A* ∧ *stl* ω ∈ *scylinder S As*}

lemma *scylinder-streams*: *scylinder S xs* ⊆ *streams S*
⟨*proof*⟩

lemma *sets-scylinder*: (∀ *x*∈*set xs*. *x* ∈ *sets S*) ⟹ *scylinder* (*space S*) *xs* ∈ *sets* (*stream-space S*)
⟨*proof*⟩

lemma *stream-space-eq-scylinder*:
assumes *P*: *prob-space M prob-space N*

assumes *Int-stable G* **and** *S*: *sets S = sets* (*sigma* (*space S*) *G*)
and *C*: *countable C C* ⊆ *G* ⋃ *C = space S* **and** *G*: *G* ⊆ *Pow* (*space S*)
assumes *sets-M*: *sets M = sets* (*stream-space S*)
assumes *sets-N*: *sets N = sets* (*stream-space S*)
assumes ∗: ⋀*xs*. *xs* ≠ [] ⟹ *xs* ∈ *lists G* ⟹ *emeasure M* (*scylinder* (*space S*) *xs*) = *emeasure N* (*scylinder* (*space S*) *xs*)
shows *M = N*
⟨*proof*⟩

lemma *stream-space-coinduct*:
fixes *R* :: *'a stream measure* ⇒ *'a stream measure* ⇒ *bool*
assumes *R A B*
assumes *R*: ⋀*A B*. *R A B* ⟹ ∃ *K*∈*space* (*prob-algebra M*).
∃ *A'*∈*M* →$_M$ *prob-algebra* (*stream-space M*). ∃ *B'*∈*M* →$_M$ *prob-algebra* (*stream-space M*).
(*AE y in K*. *R* (*A' y*) (*B' y*) ∨ *A' y = B' y*) ∧
A = do { *y* ← *K*; *ω* ← *A' y*; *return* (*stream-space M*) (*y ## ω*) } ∧
B = do { *y* ← *K*; *ω* ← *B' y*; *return* (*stream-space M*) (*y ## ω*) }
shows *A = B*
⟨*proof*⟩

end

theory *Tree-Space*
imports *HOL−Analysis.Analysis HOL−Library.Tree*
begin

lemma *countable-lfp*:
assumes *step*: ⋀*Y*. *countable Y* ⟹ *countable* (*F Y*)
and *cont*: *Order-Continuity.sup-continuous F*
shows *countable* (*lfp F*)
⟨*proof*⟩

lemma *countable-lfp-apply*:
assumes *step*: ⋀*Y x*. (⋀*x*. *countable* (*Y x*)) ⟹ *countable* (*F Y x*)
and *cont*: *Order-Continuity.sup-continuous F*
shows *countable* (*lfp F x*)
⟨*proof*⟩

inductive-set *trees* :: *'a set* ⇒ *'a tree set* **for** *S* :: *'a set* **where**
[*intro*!]: *Leaf* ∈ *trees S*
| *l* ∈ *trees S* ⟹ *r* ∈ *trees S* ⟹ *v* ∈ *S* ⟹ *Node l v r* ∈ *trees S*

lemma *Node-in-trees-iff*[*simp*]: *Node l v r* ∈ *trees S* ⟷ (*l* ∈ *trees S* ∧ *v* ∈ *S* ∧ *r* ∈ *trees S*)
⟨*proof*⟩

lemma *trees-sub-lfp*: *trees S* ⊆ *lfp* (λ*T*. *T* ∪ {*Leaf*} ∪ (⋃ *l*∈*T*. (⋃ *v*∈*S*. (⋃ *r*∈*T*.

$\{Node\ l\ v\ r\}))))$
⟨*proof*⟩

lemma *countable-trees*: *countable A* ⟹ *countable* (*trees A*)
⟨*proof*⟩

lemma *trees-UNIV*[*simp*]: *trees UNIV* = *UNIV*
⟨*proof*⟩

instance *tree* :: (*countable*) *countable*
⟨*proof*⟩

lemma *map-in-trees*[*intro*]: (⋀*x*. *x* ∈ *set-tree t* ⟹ *f x* ∈ *S*) ⟹ *map-tree f t* ∈ *trees S*
⟨*proof*⟩

primrec *trees-cyl* :: *'a set tree* ⇒ *'a tree set* **where**
trees-cyl Leaf = {*Leaf*}
| *trees-cyl* (*Node l v r*) = (⋃*l'*∈*trees-cyl l*. (⋃*v'*∈*v*. (⋃*r'*∈*trees-cyl r*. {*Node l' v' r'*})))

definition *tree-sigma* :: *'a measure* ⇒ *'a tree measure*
where
tree-sigma M = *sigma* (*trees* (*space M*)) (*trees-cyl* ‘ *trees* (*sets M*))

lemma *Node-in-trees-cyl*: *Node l' v' r'* ∈ *trees-cyl t* ⟷
(∃ *l v r*. *t* = *Node l v r* ∧ *l'* ∈ *trees-cyl l* ∧ *r'* ∈ *trees-cyl r* ∧ *v'* ∈ *v*)
⟨*proof*⟩

lemma *trees-cyl-sub-trees*:
assumes *t* ∈ *trees A A* ⊆ *Pow B* **shows** *trees-cyl t* ⊆ *trees B*
⟨*proof*⟩

lemma *trees-cyl-sets-in-space*: *trees-cyl* ‘ *trees* (*sets M*) ⊆ *Pow* (*trees* (*space M*))
⟨*proof*⟩

lemma *space-tree-sigma*: *space* (*tree-sigma M*) = *trees* (*space M*)
⟨*proof*⟩

lemma *sets-tree-sigma-eq*: *sets* (*tree-sigma M*) = *sigma-sets* (*trees* (*space M*)) (*trees-cyl* ‘ *trees* (*sets M*))
⟨*proof*⟩

lemma *Leaf-in-space-tree-sigma* [*measurable*, *simp*, *intro*]: *Leaf* ∈ *space* (*tree-sigma M*)
⟨*proof*⟩

lemma *Leaf-in-tree-sigma* [*measurable*, *simp*, *intro*]: {*Leaf*} ∈ *sets* (*tree-sigma M*)
⟨*proof*⟩

lemma *trees-cyl-map-treeI*: $t \in$ *trees-cyl* (*map-tree* ($\lambda x.\ A$) t) **if** $*$: $t \in$ *trees* A
⟨*proof*⟩

lemma *trees-cyl-map-in-sets*:
($\bigwedge x.\ x \in$ *set-tree* $t \Longrightarrow f\ x \in$ *sets* M) $\Longrightarrow$ *trees-cyl* (*map-tree* $f\ t$) $\in$ *sets* (*tree-sigma* M)
⟨*proof*⟩

lemma *Node-in-tree-sigma*:
assumes L: $X \in$ *sets* ($M \bigotimes_M$ (*tree-sigma* $M \bigotimes_M$ *tree-sigma* M))
shows $\{$*Node* $l\ v\ r \mid l\ v\ r.\ (v,\ l,\ r) \in X\} \in$ *sets* (*tree-sigma* M)
⟨*proof*⟩

lemma *measurable-left*[*measurable*]: *left* $\in$ *tree-sigma* $M \to_M$ *tree-sigma* M
⟨*proof*⟩

lemma *measurable-right*[*measurable*]: *right* $\in$ *tree-sigma* $M \to_M$ *tree-sigma* M
⟨*proof*⟩

lemma *measurable-value'*: *value* $\in$ *restrict-space* (*tree-sigma* M) ($-\{$*Leaf*$\}$) $\to_M$ M
⟨*proof*⟩

lemma *measurable-value*[*measurable* (*raw*)]:
assumes $f \in X \to_M$ *tree-sigma* M
and $\bigwedge x.\ x \in$ *space* $X \Longrightarrow f\ x \neq$ *Leaf*
shows ($\lambda\omega.$ *value* ($f\ \omega$)) $\in X \to_M M$
⟨*proof*⟩

lemma *measurable-Node* [*measurable*]:
($\lambda(l,x,r).$ *Node* $l\ x\ r$) $\in$ *tree-sigma* $M \bigotimes_M M \bigotimes_M$ *tree-sigma* $M \to_M$ *tree-sigma* M
⟨*proof*⟩

lemma *measurable-Node'* [*measurable* (*raw*)]:
assumes [*measurable*]: $l \in B \to_M$ *tree-sigma* A
assumes [*measurable*]: $x \in B \to_M A$
assumes [*measurable*]: $r \in B \to_M$ *tree-sigma* A
shows ($\lambda y.$ *Node* ($l\ y$) ($x\ y$) ($r\ y$)) $\in B \to_M$ *tree-sigma* A
⟨*proof*⟩

lemma *measurable-rec-tree*[*measurable* (*raw*)]:
assumes t: $t \in B \to_M$ *tree-sigma* M
assumes l: $l \in B \to_M A$
assumes n: ($\lambda(x,\ l,\ v,\ r,\ al,\ ar).\ n\ x\ l\ v\ r\ al\ ar$) $\in$
($B \bigotimes_M$ *tree-sigma* $M \bigotimes_M M \bigotimes_M$ *tree-sigma* $M \bigotimes_M A \bigotimes_M A$) $\to_M A$ (**is** *?N* $\in$ *?M* $\to_M A$)

shows $(\lambda x.\ \textit{rec-tree}\ (l\ x)\ (n\ x)\ (t\ x)) \in B \to_M A$
⟨*proof*⟩

lemma *measurable-case-tree* [*measurable* (*raw*)]:
assumes $t \in B \to_M \textit{tree-sigma}\ M$
assumes $l \in B \to_M A$
assumes $(\lambda(x,\ l,\ v,\ r).\ n\ x\ l\ v\ r)$
$\in B \bigotimes_M \textit{tree-sigma}\ M \bigotimes_M M \bigotimes_M \textit{tree-sigma}\ M \to_M A$
shows $(\lambda x.\ \textit{case-tree}\ (l\ x)\ (n\ x)\ (t\ x)) \in B \to_M (A :: 'a\ \textit{measure})$
⟨*proof*⟩

hide-const (**open**) *left*
hide-const (**open**) *right*

end

28 Conditional Expectation

theory *Conditional-Expectation*
imports *Probability-Measure*
begin

28.1 Restricting a measure to a sub-sigma-algebra

definition *subalgebra*::'*a measure* ⇒ '*a measure* ⇒ *bool* **where**
subalgebra M F = ((*space F* = *space M*) ∧ (*sets F* ⊆ *sets M*))

lemma *sub-measure-space*:
assumes *i*: *subalgebra M F*
shows *measure-space* (*space M*) (*sets F*) (*emeasure M*)
⟨*proof*⟩

definition *restr-to-subalg*::'*a measure* ⇒ '*a measure* ⇒ '*a measure* **where**
restr-to-subalg M F = *measure-of* (*space M*) (*sets F*) (*emeasure M*)

lemma *space-restr-to-subalg*:
space (*restr-to-subalg M F*) = *space M*
⟨*proof*⟩

lemma *sets-restr-to-subalg* [*measurable-cong*]:
assumes *subalgebra M F*
shows *sets* (*restr-to-subalg M F*) = *sets F*
⟨*proof*⟩

lemma *emeasure-restr-to-subalg*:
assumes *subalgebra M F*
A ∈ *sets F*
shows *emeasure* (*restr-to-subalg M F*) *A* = *emeasure M A*
⟨*proof*⟩

lemma *null-sets-restr-to-subalg*:
assumes *subalgebra M F*
shows *null-sets* (*restr-to-subalg M F*) = *null-sets M* ∩ *sets F*
⟨*proof*⟩

lemma *AE-restr-to-subalg*:
assumes *subalgebra M F*
AE x in (*restr-to-subalg M F*). *P x*
shows *AE x in M*. *P x*
⟨*proof*⟩

lemma *AE-restr-to-subalg2*:
assumes *subalgebra M F*
AE x in M. *P x* **and** [*measurable*]: *P* ∈ *measurable F* (*count-space UNIV*)
shows *AE x in* (*restr-to-subalg M F*). *P x*
⟨*proof*⟩

lemma *prob-space-restr-to-subalg*:
assumes *subalgebra M F*
prob-space M
shows *prob-space* (*restr-to-subalg M F*)
⟨*proof*⟩

lemma *finite-measure-restr-to-subalg*:
assumes *subalgebra M F*
finite-measure M
shows *finite-measure* (*restr-to-subalg M F*)
⟨*proof*⟩

lemma *measurable-in-subalg*:
assumes *subalgebra M F*
f ∈ *measurable F N*
shows *f* ∈ *measurable* (*restr-to-subalg M F*) *N*
⟨*proof*⟩

lemma *measurable-in-subalg′*:
assumes *subalgebra M F*
f ∈ *measurable* (*restr-to-subalg M F*) *N*
shows *f* ∈ *measurable F N*
⟨*proof*⟩

lemma *measurable-from-subalg*:
assumes *subalgebra M F*
f ∈ *measurable F N*
shows *f* ∈ *measurable M N*
⟨*proof*⟩

The following is the direct transposition of `nn_integral_subalgebra` (from

`Nonnegative_Lebesgue_Integration`) in the current notations, with the removal of the useless assumption $f \geq 0$.

lemma *nn-integral-subalgebra2*:
assumes *subalgebra M F* **and** [*measurable*]: $f \in$ *borel-measurable F*
shows $(\int^+ x.\ f\ x\ \partial(\textit{restr-to-subalg}\ M\ F)) = (\int^+ x.\ f\ x\ \partial M)$
⟨*proof*⟩

The following is the direct transposition of `integral_subalgebra` (from `Bochner_Integration`) in the current notations.

lemma *integral-subalgebra2*:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes *subalgebra M F* **and**
[*measurable*]: $f \in$ *borel-measurable F*
shows $(\int x.\ f\ x\ \partial(\textit{restr-to-subalg}\ M\ F)) = (\int x.\ f\ x\ \partial M)$
⟨*proof*⟩

lemma *integrable-from-subalg*:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes *subalgebra M F*
integrable (*restr-to-subalg M F*) *f*
shows *integrable M f*
⟨*proof*⟩

lemma *integrable-in-subalg*:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes [*measurable*]: *subalgebra M F*
$f \in$ *borel-measurable F*
integrable M f
shows *integrable* (*restr-to-subalg M F*) *f*
⟨*proof*⟩

28.2 Nonnegative conditional expectation

The conditional expectation of a function f, on a measure space M, with respect to a sub sigma algebra F, should be a function g which is F-measurable whose integral on any F-set coincides with the integral of f. Such a function is uniquely defined almost everywhere. The most direct construction is to use the measure fdM, restrict it to the sigma-algebra F, and apply the Radon-Nikodym theorem to write it as $gdM_{|F}$ for some F-measurable function g. Another classical construction for L^2 functions is done by orthogonal projection on F-measurable functions, and then extending by density to L^1. The Radon-Nikodym point of view avoids the L^2 machinery, and works for all positive functions.

In this paragraph, we develop the definition and basic properties for nonnegative functions, as the basics of the general case. As in the definition of integrals, the nonnegative case is done with ennreal-valued functions, without any integrability assumption.

definition *nn-cond-exp* :: *'a measure* ⇒ *'a measure* ⇒ (*'a* ⇒ *ennreal*) ⇒ (*'a* ⇒ *ennreal*)
where
nn-cond-exp M F f =
(*if f* ∈ *borel-measurable M* ∧ *subalgebra M F*
then RN-deriv (*restr-to-subalg M F*) (*restr-to-subalg* (*density M f*) *F*)
else (λ-. *0*))

lemma
shows *borel-measurable-nn-cond-exp* [*measurable*]: *nn-cond-exp M F f* ∈ *borel-measurable F*
and *borel-measurable-nn-cond-exp2* [*measurable*]: *nn-cond-exp M F f* ∈ *borel-measurable M*
⟨*proof*⟩

The good setting for conditional expectations is the situation where the subalgebra F gives rise to a sigma-finite measure space. To see what goes wrong if it is not sigma-finite, think of $\mathbb{R}$ with the trivial sigma-algebra $\{\emptyset, \mathbb{R}\}$. In this case, conditional expectations have to be constant functions, so they have integral 0 or ∞. This means that a positive integrable function can have no meaningful conditional expectation.

locale *sigma-finite-subalgebra* =
fixes *M F*::*'a measure*
assumes *subalg*: *subalgebra M F*
and *sigma-fin-subalg*: *sigma-finite-measure* (*restr-to-subalg M F*)

lemma *sigma-finite-subalgebra-is-sigma-finite*:
assumes *sigma-finite-subalgebra M F*
shows *sigma-finite-measure M*
⟨*proof*⟩

sublocale *sigma-finite-subalgebra* ⊆ *sigma-finite-measure*
⟨*proof*⟩

Conditional expectations are very often used in probability spaces. This is a special case of the previous one, as we prove now.

locale *finite-measure-subalgebra* = *finite-measure* +
fixes *F*::*'a measure*
assumes *subalg*: *subalgebra M F*

lemma *finite-measure-subalgebra-is-sigma-finite*:
assumes *finite-measure-subalgebra M F*
shows *sigma-finite-subalgebra M F*
⟨*proof*⟩

sublocale *finite-measure-subalgebra* ⊆ *sigma-finite-subalgebra*
⟨*proof*⟩

context *sigma-finite-subalgebra*
begin

The next lemma is arguably the most fundamental property of conditional expectation: when computing an expectation against an F-measurable function, it is equivalent to work with a function or with its F-conditional expectation.

This property (even for bounded test functions) characterizes conditional expectations, as the second lemma below shows. From this point on, we will only work with it, and forget completely about the definition using Radon-Nikodym derivatives.

lemma *nn-cond-exp-intg*:
assumes [*measurable*]: $f \in$ *borel-measurable* F $g \in$ *borel-measurable* M
shows $(\int^+ x.\ f\ x * \textit{nn-cond-exp}\ M\ F\ g\ x\ \partial M) = (\int^+ x.\ f\ x * g\ x\ \partial M)$
⟨*proof*⟩

lemma *nn-cond-exp-charact*:
assumes $\bigwedge A.\ A \in \textit{sets}\ F \Longrightarrow (\int^+ x \in A.\ f\ x\ \partial M) = (\int^+ x \in A.\ g\ x\ \partial M)$ **and**
[*measurable*]: $f \in$ *borel-measurable* M $g \in$ *borel-measurable* F
shows *AE x in M. g x = nn-cond-exp M F f x*
⟨*proof*⟩

lemma *nn-cond-exp-F-meas*:
assumes $f \in$ *borel-measurable* F
shows *AE x in M. f x = nn-cond-exp M F f x*
⟨*proof*⟩

lemma *nn-cond-exp-prod*:
assumes [*measurable*]: $f \in$ *borel-measurable* F $g \in$ *borel-measurable* M
shows *AE x in M. f x * nn-cond-exp M F g x = nn-cond-exp M F* $(\lambda x.\ f\ x * g\ x)\ x$
⟨*proof*⟩

lemma *nn-cond-exp-sum*:
assumes [*measurable*]: $f \in$ *borel-measurable* M $g \in$ *borel-measurable* M
shows *AE x in M. nn-cond-exp M F f x + nn-cond-exp M F g x = nn-cond-exp M F* $(\lambda x.\ f\ x + g\ x)\ x$
⟨*proof*⟩

lemma *nn-cond-exp-cong*:
assumes *AE x in M. f x = g x*
and [*measurable*]: $f \in$ *borel-measurable* M $g \in$ *borel-measurable* M
shows *AE x in M. nn-cond-exp M F f x = nn-cond-exp M F g x*
⟨*proof*⟩

lemma *nn-cond-exp-mono*:
assumes *AE x in M. f x* $\leq$ *g x*
and [*measurable*]: $f \in$ *borel-measurable* M $g \in$ *borel-measurable* M

shows *AE x in M. nn-cond-exp M F f x ≤ nn-cond-exp M F g x*
⟨*proof*⟩

lemma *nested-subalg-is-sigma-finite*:
assumes *subalgebra M G subalgebra G F*
shows *sigma-finite-subalgebra M G*
⟨*proof*⟩

lemma *nn-cond-exp-nested-subalg*:
assumes *subalgebra M G subalgebra G F*
and [*measurable*]: *f ∈ borel-measurable M*
shows *AE x in M. nn-cond-exp M F f x = nn-cond-exp M F (nn-cond-exp M G f) x*
⟨*proof*⟩

end

28.3 Real conditional expectation

Once conditional expectations of positive functions are defined, the definition for real-valued functions follows readily, by taking the difference of positive and negative parts. One could also define a conditional expectation of vector-space valued functions, as in `Bochner_Integral`, but since the real-valued case is the most important, and quicker to formalize, I concentrate on it. (It is also essential for the case of the most general Pettis integral.)

definition *real-cond-exp* :: *'a measure ⇒ 'a measure ⇒ ('a ⇒ real) ⇒ ('a ⇒ real)*
where
real-cond-exp M F f =
(λx. enn2real(nn-cond-exp M F (λx. ennreal (f x)) x) − enn2real(nn-cond-exp M F (λx. ennreal (−f x)) x))

lemma
shows *borel-measurable-cond-exp* [*measurable*]: *real-cond-exp M F f ∈ borel-measurable F*
and *borel-measurable-cond-exp2* [*measurable*]: *real-cond-exp M F f ∈ borel-measurable M*
⟨*proof*⟩

context *sigma-finite-subalgebra*
begin

lemma *real-cond-exp-abs*:
assumes [*measurable*]: *f ∈ borel-measurable M*
shows *AE x in M. abs(real-cond-exp M F f x) ≤ nn-cond-exp M F (λx. ennreal (abs(f x))) x*
⟨*proof*⟩

The next lemma shows that the conditional expectation is an F-measurable function whose average against an F-measurable function f coincides with the average of the original function against f. It is obtained as a consequence of the same property for the positive conditional expectation, taking the difference of the positive and the negative part. The proof is given first assuming $f \geq 0$ for simplicity, and then extended to the general case in the subsequent lemma. The idea of the proof is essentially trivial, but the implementation is slightly tedious as one should check all the integrability properties of the different parts, and go back and forth between positive integral and signed integrals, and between real-valued functions and ennreal-valued functions.

Once this lemma is available, we will use it to characterize the conditional expectation, and never come back to the original technical definition, as we did in the case of the nonnegative conditional expectation.

lemma *real-cond-exp-intg-fpos*:
  assumes *integrable M* (λ*x*. *f x* ∗ *g x*) **and** *f-pos*[*simp*]: ⋀*x*. *f x* ≥ *0* **and**
      [*measurable*]: *f* ∈ *borel-measurable F g* ∈ *borel-measurable M*
  shows *integrable M* (λ*x*. *f x* ∗ *real-cond-exp M F g x*)
    (∫ *x*. *f x* ∗ *real-cond-exp M F g x* ∂*M*) = (∫ *x*. *f x* ∗ *g x* ∂*M*)
⟨*proof*⟩

lemma *real-cond-exp-intg*:
  assumes *integrable M* (λ*x*. *f x* ∗ *g x*) **and**
      [*measurable*]: *f* ∈ *borel-measurable F g* ∈ *borel-measurable M*
  shows *integrable M* (λ*x*. *f x* ∗ *real-cond-exp M F g x*)
    (∫ *x*. *f x* ∗ *real-cond-exp M F g x* ∂*M*) = (∫ *x*. *f x* ∗ *g x* ∂*M*)
⟨*proof*⟩

lemma *real-cond-exp-intA*:
  assumes [*measurable*]: *integrable M f A* ∈ *sets F*
  shows (∫ *x* ∈ *A*. *f x* ∂*M*) = (∫ *x* ∈ *A*. *real-cond-exp M F f x* ∂*M*)
⟨*proof*⟩

lemma *real-cond-exp-int* [*intro*]:
  assumes *integrable M f*
  shows *integrable M* (*real-cond-exp M F f*) (∫ *x*. *real-cond-exp M F f x* ∂*M*) =
(∫ *x*. *f x* ∂*M*)
⟨*proof*⟩

lemma *real-cond-exp-charact*:
  assumes ⋀*A*. *A* ∈ *sets F* ⟹ (∫ *x* ∈ *A*. *f x* ∂*M*) = (∫ *x* ∈ *A*. *g x* ∂*M*)
    and [*measurable*]: *integrable M f integrable M g*
      g ∈ *borel-measurable F*
  shows *AE x in M*. *real-cond-exp M F f x* = *g x*
⟨*proof*⟩

lemma *real-cond-exp-F-meas* [*intro*, *simp*]:

assumes *integrable M f*
f ∈ borel-measurable F
shows *AE x in M. real-cond-exp M F f x = f x*
⟨*proof*⟩

lemma *real-cond-exp-mult*:
assumes [*measurable*]:*f ∈ borel-measurable F g ∈ borel-measurable M integrable M* (λ*x. f x * g x*)
shows *AE x in M. real-cond-exp M F* (λ*x. f x * g x*) *x = f x * real-cond-exp M F g x*
⟨*proof*⟩

lemma *real-cond-exp-add* [*intro*]:
assumes [*measurable*]: *integrable M f integrable M g*
shows *AE x in M. real-cond-exp M F* (λ*x. f x + g x*) *x = real-cond-exp M F f x + real-cond-exp M F g x*
⟨*proof*⟩

lemma *real-cond-exp-cong*:
assumes *ae*: *AE x in M. f x = g x* **and** [*measurable*]: *f ∈ borel-measurable M g ∈ borel-measurable M*
shows *AE x in M. real-cond-exp M F f x = real-cond-exp M F g x*
⟨*proof*⟩

lemma *real-cond-exp-cmult* [*intro*, *simp*]:
fixes *c*::*real*
assumes *integrable M f*
shows *AE x in M. real-cond-exp M F* (λ*x. c * f x*) *x = c * real-cond-exp M F f x*
⟨*proof*⟩

lemma *real-cond-exp-cdiv* [*intro*, *simp*]:
fixes *c*::*real*
assumes *integrable M f*
shows *AE x in M. real-cond-exp M F* (λ*x. f x / c*) *x = real-cond-exp M F f x / c*
⟨*proof*⟩

lemma *real-cond-exp-diff* [*intro*, *simp*]:
assumes [*measurable*]: *integrable M f integrable M g*
shows *AE x in M. real-cond-exp M F* (λ*x. f x − g x*) *x = real-cond-exp M F f x − real-cond-exp M F g x*
⟨*proof*⟩

lemma *real-cond-exp-pos* [*intro*]:
assumes *AE x in M. f x ≥ 0* **and** [*measurable*]: *f ∈ borel-measurable M*
shows *AE x in M. real-cond-exp M F f x ≥ 0*
⟨*proof*⟩

lemma *real-cond-exp-mono*:
assumes *AE x in M. f x ≤ g x* **and** [*measurable*]: *integrable M f integrable M g*
shows *AE x in M. real-cond-exp M F f x ≤ real-cond-exp M F g x*
⟨*proof*⟩

lemma (**in** −) *measurable-P-restriction* [*measurable* (*raw*)]:
assumes [*measurable*]: *Measurable.pred M P A ∈ sets M*
shows *{x ∈ A. P x} ∈ sets M*
⟨*proof*⟩

lemma *real-cond-exp-gr-c*:
assumes [*measurable*]: *integrable M f*
and *AE*: *AE x in M. f x > c*
shows *AE x in M. real-cond-exp M F f x > c*
⟨*proof*⟩

lemma *real-cond-exp-less-c*:
assumes [*measurable*]: *integrable M f*
and *AE x in M. f x < c*
shows *AE x in M. real-cond-exp M F f x < c*
⟨*proof*⟩

lemma *real-cond-exp-ge-c*:
assumes [*measurable*]: *integrable M f*
and *AE x in M. f x ≥ c*
shows *AE x in M. real-cond-exp M F f x ≥ c*
⟨*proof*⟩

lemma *real-cond-exp-le-c*:
assumes [*measurable*]: *integrable M f*
and *AE x in M. f x ≤ c*
shows *AE x in M. real-cond-exp M F f x ≤ c*
⟨*proof*⟩

lemma *real-cond-exp-mono-strict*:
assumes *AE x in M. f x < g x* **and** [*measurable*]: *integrable M f integrable M g*
shows *AE x in M. real-cond-exp M F f x < real-cond-exp M F g x*
⟨*proof*⟩

lemma *real-cond-exp-nested-subalg* [*intro*, *simp*]:
assumes *subalgebra M G subalgebra G F*
and [*measurable*]: *integrable M f*
shows *AE x in M. real-cond-exp M F (real-cond-exp M G f) x = real-cond-exp M F f x*
⟨*proof*⟩

lemma *real-cond-exp-sum* [*intro*, *simp*]:
fixes *f*::*'b ⇒ 'a ⇒ real*
assumes [*measurable*]: *⋀i. integrable M (f i)*

shows *AE x in M. real-cond-exp M F* ($\lambda x. \sum i{\in}I. f\ i\ x$) *x* = ($\sum i{\in}I.$ *real-cond-exp M F* (*f i*) *x*)
⟨*proof*⟩

Jensen's inequality, describing the behavior of the integral under a convex function, admits a version for the conditional expectation, as follows.

theorem *real-cond-exp-jensens-inequality*:
fixes *q* :: *real* ⇒ *real*
assumes *X*: *integrable M X AE x in M. X x* ∈ *I*
assumes *I*: *I* = {*a* <..< *b*} ∨ *I* = {*a* <..} ∨ *I* = {..< *b*} ∨ *I* = *UNIV*
assumes *q*: *integrable M* (λ*x. q* (*X x*)) *convex-on I q q* ∈ *borel-measurable borel*
shows *AE x in M. real-cond-exp M F X x* ∈ *I*
AE x in M. q (*real-cond-exp M F X x*) ≤ *real-cond-exp M F* (λ*x. q* (*X x*)) *x*
⟨*proof*⟩

Jensen's inequality does not imply that $q(E(X|F))$ is integrable, as it only proves an upper bound for it. Indeed, this is not true in general, as the following counterexample shows:

on $[1, \infty)$ with Lebesgue measure, let F be the sigma-algebra generated by the intervals $[n, n+1)$ for integer n. Let $q(x) = -\sqrt{x}$ for $x \geq 0$. Define X which is equal to $1/n$ over $[n, n+1/n)$ and 2^{-n} on $[n+1/n, n+1)$. Then X is integrable as $\sum 1/n^2 < \infty$, and $q(X)$ is integrable as $\sum 1/n^{3/2} < \infty$. On the other hand, $E(X|F)$ is essentially equal to $1/n^2$ on $[n, n+1)$ (we neglect the term 2^{-n}, we only put it there because X should take its values in $I = (0, \infty)$). Hence, $q(E(X|F))$ is equal to $-1/n$ on $[n, n+1)$, hence it is not integrable.

However, this counterexample is essentially the only situation where this function is not integrable, as shown by the next lemma.

lemma *integrable-convex-cond-exp*:
fixes *q* :: *real* ⇒ *real*
assumes *X*: *integrable M X AE x in M. X x* ∈ *I*
assumes *I*: *I* = {*a* <..< *b*} ∨ *I* = {*a* <..} ∨ *I* = {..< *b*} ∨ *I* = *UNIV*
assumes *q*: *integrable M* (λ*x. q* (*X x*)) *convex-on I q q* ∈ *borel-measurable borel*
assumes *H*: *emeasure M* (*space M*) = ∞ ⟹ *0* ∈ *I*
shows *integrable M* (λ*x. q* (*real-cond-exp M F X x*))
⟨*proof*⟩

end

end

theory *Essential-Supremum*
imports *HOL−Analysis.Analysis*
begin

lemma *ae-filter-eq-bot-iff*: *ae-filter M* = *bot* ⟷ *emeasure M* (*space M*) = *0*

⟨*proof*⟩

29 The essential supremum

In this paragraph, we define the essential supremum and give its basic properties. The essential supremum of a function is its maximum value if one is allowed to throw away a set of measure 0. It is convenient to define it to be infinity for non-measurable functions, as it allows for neater statements in general. This is a prerequisiste to define the space L^∞.

definition *esssup*::*'a measure* ⇒ (*'a* ⇒ *'b*::{*second-countable-topology*, *dense-linorder*, *linorder-topology*, *complete-linorder*}) ⇒ *'b*
where *esssup M f* = (*if f* ∈ *borel-measurable M then Limsup* (*ae-filter M*) *f else top*)

lemma *esssup-non-measurable*: *f* ∉ *M* →$_M$ *borel* ⟹ *esssup M f* = *top*
⟨*proof*⟩

lemma *esssup-eq-AE*:
assumes *f*: *f* ∈ *M* →$_M$ *borel* **shows** *esssup M f* = *Inf* {*z*. *AE x in M*. *f x* ≤ *z*}
⟨*proof*⟩

lemma *esssup-eq*: *f* ∈ *M* →$_M$ *borel* ⟹ *esssup M f* = *Inf* {*z*. *emeasure M* {*x* ∈ *space M*. *f x* > *z*} = *0*}
⟨*proof*⟩

lemma *esssup-zero-measure*:
emeasure M {*x* ∈ *space M*. *f x* > *esssup M f*} = *0*
⟨*proof*⟩

lemma *esssup-AE*: *AE x in M*. *f x* ≤ *esssup M f*
⟨*proof*⟩

lemma *esssup-pos-measure*:
f ∈ *borel-measurable M* ⟹ *z* < *esssup M f* ⟹ *emeasure M* {*x* ∈ *space M*. *f x* > *z*} > *0*
⟨*proof*⟩

lemma *esssup-I* [*intro*]: *f* ∈ *borel-measurable M* ⟹ *AE x in M*. *f x* ≤ *c* ⟹ *esssup M f* ≤ *c*
⟨*proof*⟩

lemma *esssup-AE-mono*: *f* ∈ *borel-measurable M* ⟹ *AE x in M*. *f x* ≤ *g x* ⟹ *esssup M f* ≤ *esssup M g*
⟨*proof*⟩

lemma *esssup-mono*: *f* ∈ *borel-measurable M* ⟹ (⋀*x*. *f x* ≤ *g x*) ⟹ *esssup M f* ≤ *esssup M g*
⟨*proof*⟩

lemma *esssup-AE-cong*:
f ∈ *borel-measurable M* ⟹ *g* ∈ *borel-measurable M* ⟹ *AE x in M*. *f x* = *g x* ⟹ *esssup M f* = *esssup M g*
⟨*proof*⟩

lemma *esssup-const*: *emeasure M* (*space M*) ≠ *0* ⟹ *esssup M* (λ*x*. *c*) = *c*
⟨*proof*⟩

lemma *esssup-cmult*: **assumes** *c* > (*0*::*real*) **shows** *esssup M* (λ*x*. *c* ∗ *f x*::*ereal*) = *c* ∗ *esssup M f*
⟨*proof*⟩

lemma *esssup-add*:
esssup M (λ*x*. *f x* + *g x*::*ereal*) ≤ *esssup M f* + *esssup M g*
⟨*proof*⟩

lemma *esssup-zero-space*:
emeasure M (*space M*) = *0* ⟹ *f* ∈ *borel-measurable M* ⟹ *esssup M f* = (−∞::*ereal*)
⟨*proof*⟩

end

30 Stopping times

theory *Stopping-Time*
imports *HOL−Analysis.Analysis*
begin

30.1 Stopping Time

This is also called strong stopping time. Then stopping time is T with alternative is *T x* < *t* measurable.

definition *stopping-time* :: ('*t*::*linorder* ⇒ '*a measure*) ⇒ ('*a* ⇒ '*t*) ⇒ *bool*
where
stopping-time F T = (∀ *t*. *Measurable.pred* (*F t*) (λ*x*. *T x* ≤ *t*))

lemma *stopping-time-cong*: (⋀*t x*. *x* ∈ *space* (*F t*) ⟹ *T x* = *S x*) ⟹ *stopping-time F T* = *stopping-time F S*
⟨*proof*⟩

lemma *stopping-timeD*: *stopping-time F T* ⟹ *Measurable.pred* (*F t*) (λ*x*. *T x* ≤ *t*)
⟨*proof*⟩

lemma *stopping-timeD2*: *stopping-time F T* ⟹ *Measurable.pred* (*F t*) (λ*x*. *t* < *T x*)

⟨*proof*⟩

lemma *stopping-timeI*[*intro?*]: $(\bigwedge t.\ \textit{Measurable.pred}\ (F\ t)\ (\lambda x.\ T\ x \leq t)) \Longrightarrow \textit{stopping-time}\ F\ T$
⟨*proof*⟩

lemma *measurable-stopping-time*:
fixes $T :: 'a \Rightarrow 't{::}\{\textit{linorder-topology}, \textit{second-countable-topology}\}$
assumes T: *stopping-time F T*
and M: $\bigwedge t.\ \textit{sets}\ (F\ t) \subseteq \textit{sets}\ M\ \bigwedge t.\ \textit{space}\ (F\ t) = \textit{space}\ M$
shows $T \in M \rightarrow_M \textit{borel}$
⟨*proof*⟩

lemma *stopping-time-const*: *stopping-time F* $(\lambda x.\ c)$
⟨*proof*⟩

lemma *stopping-time-min*:
stopping-time F T $\Longrightarrow$ *stopping-time F S* $\Longrightarrow$ *stopping-time F* $(\lambda x.\ \textit{min}\ (T\ x)\ (S\ x))$
⟨*proof*⟩

lemma *stopping-time-max*:
stopping-time F T $\Longrightarrow$ *stopping-time F S* $\Longrightarrow$ *stopping-time F* $(\lambda x.\ \textit{max}\ (T\ x)\ (S\ x))$
⟨*proof*⟩

31 Filtration

locale *filtration* =
fixes $\Omega :: 'a\ \textit{set}$ **and** $F :: 't{::}\{\textit{linorder-topology}, \textit{second-countable-topology}\} \Rightarrow 'a\ \textit{measure}$
assumes *space-F*: $\bigwedge i.\ \textit{space}\ (F\ i) = \Omega$
assumes *sets-F-mono*: $\bigwedge i\ j.\ i \leq j \Longrightarrow \textit{sets}\ (F\ i) \leq \textit{sets}\ (F\ j)$
begin

31.1 σ-algebra of a Stopping Time

definition *pre-sigma* :: $('a \Rightarrow 't) \Rightarrow 'a\ \textit{measure}$
where
pre-sigma T = *sigma* $\Omega\ \{A.\ \forall t.\ \{\omega{\in}A.\ T\ \omega \leq t\} \in \textit{sets}\ (F\ t)\}$

lemma *space-pre-sigma*: *space* (*pre-sigma T*) $= \Omega$
⟨*proof*⟩

lemma *measure-pre-sigma*[*simp*]: *emeasure* (*pre-sigma T*) $= (\lambda\text{-}.\ 0)$
⟨*proof*⟩

lemma *sigma-algebra-pre-sigma*:
assumes *T*: *stopping-time F T*

shows *sigma-algebra* Ω $\{A.\ \forall t.\ \{\omega{\in}A.\ T\ \omega \leq t\} \in sets\ (F\ t)\}$
⟨*proof*⟩

lemma *sets-pre-sigma*: *stopping-time F T* $\Longrightarrow$ *sets* (*pre-sigma T*) = $\{A.\ \forall t.\ \{\omega{\in}A.\ T\ \omega \leq t\} \in sets\ (F\ t)\}$
⟨*proof*⟩

lemma *sets-pre-sigmaI*: *stopping-time F T* $\Longrightarrow$ ($\bigwedge t.\ \{\omega{\in}A.\ T\ \omega \leq t\} \in sets\ (F\ t)$) $\Longrightarrow$ $A \in sets\ (pre\text{-}sigma\ T)$
⟨*proof*⟩

lemma *pred-pre-sigmaI*:
assumes *T*: *stopping-time F T*
shows ($\bigwedge t.$ *Measurable.pred* (*F t*) ($\lambda\omega.$ *P* $\omega \wedge$ *T* $\omega \leq t$)) $\Longrightarrow$ *Measurable.pred* (*pre-sigma T*) *P*
⟨*proof*⟩

lemma *sets-pre-sigmaD*: *stopping-time F T* $\Longrightarrow$ $A \in sets\ (pre\text{-}sigma\ T) \Longrightarrow \{\omega{\in}A.\ T\ \omega \leq t\} \in sets\ (F\ t)$
⟨*proof*⟩

lemma *stopping-time-le-const*: *stopping-time F T* $\Longrightarrow$ $s \leq t \Longrightarrow$ *Measurable.pred* (*F t*) ($\lambda\omega.$ *T* $\omega \leq s$)
⟨*proof*⟩

lemma *measurable-stopping-time-pre-sigma*:
assumes *T*: *stopping-time F T* **shows** $T \in pre\text{-}sigma\ T \rightarrow_M borel$
⟨*proof*⟩

lemma *mono-pre-sigma*:
assumes *T*: *stopping-time F T* **and** *S*: *stopping-time F S*
and *le*: $\bigwedge\omega.\ \omega \in \Omega \Longrightarrow T\ \omega \leq S\ \omega$
shows *sets* (*pre-sigma T*) $\subseteq$ *sets* (*pre-sigma S*)
⟨*proof*⟩

lemma *stopping-time-less-const*:
assumes *T*: *stopping-time F T* **shows** *Measurable.pred* (*F t*) ($\lambda\omega.$ *T* $\omega < t$)
⟨*proof*⟩

lemma *stopping-time-eq-const*: *stopping-time F T* $\Longrightarrow$ *Measurable.pred* (*F t*) ($\lambda\omega.$ *T* $\omega = t$)
⟨*proof*⟩

lemma *stopping-time-less*:
assumes *T*: *stopping-time F T* **and** *S*: *stopping-time F S*
shows *Measurable.pred* (*pre-sigma T*) ($\lambda\omega.$ *T* $\omega <$ *S* ω)
⟨*proof*⟩

end

lemma *stopping-time-SUP-enat*:
fixes $T :: nat \Rightarrow ('a \Rightarrow enat)$
shows $(\bigwedge i.\ stopping\text{-}time\ F\ (T\ i)) \Longrightarrow stopping\text{-}time\ F\ (SUP\ i.\ T\ i)$
⟨*proof*⟩

lemma *less-eSuc-iff*: $a < eSuc\ b \longleftrightarrow (a \leq b \wedge a \neq \infty)$
⟨*proof*⟩

lemma *stopping-time-Inf-enat*:
fixes $F :: enat \Rightarrow 'a\ measure$
assumes F: *filtration* $\Omega\ F$
assumes P: $\bigwedge i.\ Measurable.pred\ (F\ i)\ (P\ i)$
shows $stopping\text{-}time\ F\ (\lambda\omega.\ Inf\ \{i.\ P\ i\ \omega\})$
⟨*proof*⟩

lemma *stopping-time-Inf-nat*:
fixes $F :: nat \Rightarrow 'a\ measure$
assumes F: *filtration* $\Omega\ F$
assumes P: $\bigwedge i.\ Measurable.pred\ (F\ i)\ (P\ i)$ **and** wf: $\bigwedge i\ \omega.\ \omega \in \Omega \Longrightarrow \exists n.\ P\ n\ \omega$
shows $stopping\text{-}time\ F\ (\lambda\omega.\ Inf\ \{i.\ P\ i\ \omega\})$
⟨*proof*⟩

end

theory *Probability*
imports
Central-Limit-Theorem
Discrete-Topology
PMF-Impl
Projective-Limit
Random-Permutations
SPMF
Product-PMF
Hoeffding
Stream-Space
Tree-Space
Conditional-Expectation
Essential-Supremum
Stopping-Time
begin

end