

# Hoare Logic

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## Abstract

These theories contain a Hoare logic for a simple imperative programming language with while-loops, including a verification condition generator.

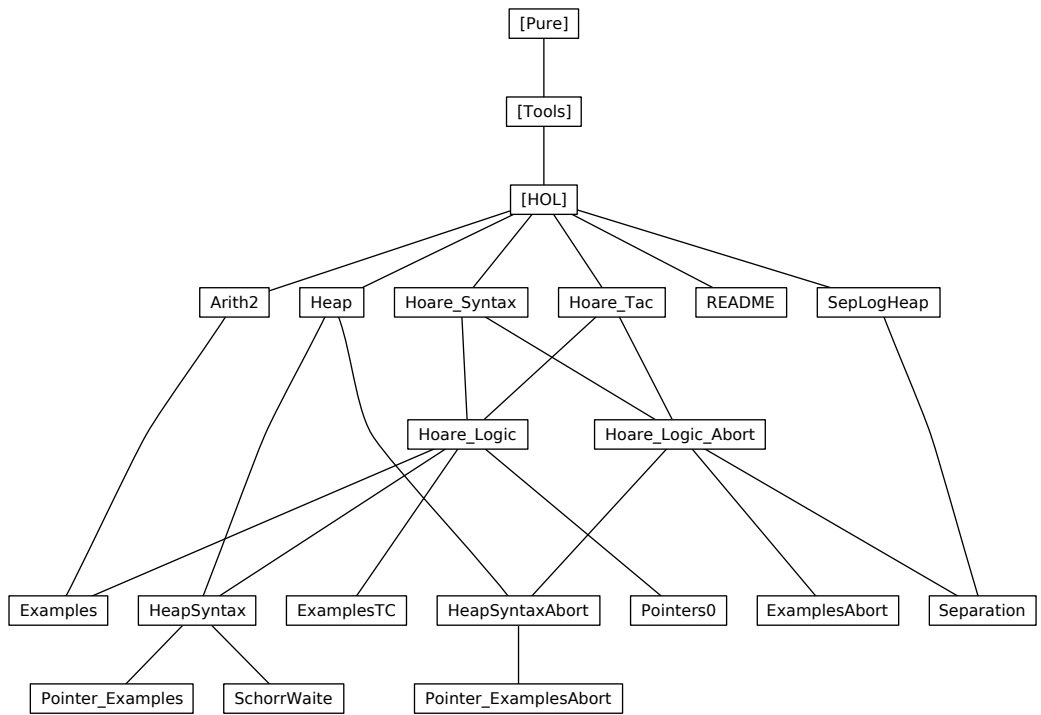
Special infrastructure for modelling and reasoning about pointer programs is provided, together with many examples, including Schorr-Waite. See [1, 2] for an excellent exposition.

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# 1 Concrete syntax for Hoare logic, with translations for variables

```
theory Hoare-Syntax
  imports Main
begin
```

```
syntax
```

```
-assign :: idt ⇒ 'b ⇒ 'com ((2- :=/ -) [70, 65] 61)
-Seq :: 'com ⇒ 'com ⇒ 'com ((-;/ -) [61,60] 60)
-Cond :: 'bexp ⇒ 'com ⇒ 'com ⇒ 'com ((IF -/ THEN - / ELSE -/ FI) [0,0,0] 61)
-While :: 'bexp ⇒ 'assn ⇒ 'var ⇒ 'com ⇒ 'com ((WHILE -/ INV {-} / VAR {-} //DO - /OD) [0,0,0,0] 61)
```

The `VAR {-}` syntax supports two variants:

- `VAR {x = t}` where  $t::nat$  is the decreasing expression, the variant, and  $x$  a variable that can be referred to from inner annotations. The  $x$  can be necessary for nested loops, e.g. to prove that the inner loops do not mess with  $t$ .
- `VAR {t}` where the variable is omitted because it is not needed.

```
syntax
```

```
-While0 :: 'bexp ⇒ 'assn ⇒ 'com ⇒ 'com ((1WHILE -/ INV {-} //DO - /OD) [0,0,0] 61)
```

The `-While0` syntax is translated into the `-While` syntax with the trivial variant 0. This is ok because partial correctness proofs do not make use of the variant.

```
syntax
```

```
-hoare-vars :: [idts, 'assn, 'com, 'assn] ⇒ bool (VARS -// {-} // - // {-} [0,0,55,0] 50)
```

```
-hoare-vars-tc :: [idts, 'assn, 'com, 'assn] ⇒ bool (VARS -// [-] // - // [-] [0,0,55,0] 50)
```

```
syntax (output)
```

```
-hoare :: ['assn, 'com, 'assn] ⇒ bool (({-} // - // {-}) [0,55,0] 50)
```

```
-hoare-tc :: ['assn, 'com, 'assn] ⇒ bool (([-] // - // [-]) [0,55,0] 50)
```

Completeness requires(?) the ability to refer to an outer variant in an inner invariant. Thus we need to abstract over a variable equated with the variant, the  $x$  in `VAR {x = t}`. But the  $x$  should only occur in invariants. To enforce this, syntax translations in `hoare-syntax.ML` separate the program from its annotations and only the latter are abstracted over over  $x$ . (Thus  $x$  can also occur in inner variants, but that neither helps nor hurts.)

```
datatype 'a anno =
  Abasic |
```

```
Aseq 'a anno 'a anno |  
Acond 'a anno 'a anno |  
Awhile 'a set 'a ⇒ nat nat ⇒ 'a anno
```

⟨ML⟩

end

## 2 Hoare logic VCG tactic

```
theory Hoare-Tac
```

```
  imports Main
```

```
begin
```

```
context
```

```
begin
```

```
qualified named-theorems BasicRule
```

```
qualified named-theorems SkipRule
```

```
qualified named-theorems AbortRule
```

```
qualified named-theorems SeqRule
```

```
qualified named-theorems CondRule
```

```
qualified named-theorems WhileRule
```

```
qualified named-theorems BasicRuleTC
```

```
qualified named-theorems SkipRuleTC
```

```
qualified named-theorems SeqRuleTC
```

```
qualified named-theorems CondRuleTC
```

```
qualified named-theorems WhileRuleTC
```

```
lemma Compl-Collect:  $\neg(\text{Collect } b) = \{x. \neg(b\ x)\}$   
  ⟨proof⟩
```

⟨ML⟩

end

end

## 3 Hoare logic

```
theory Hoare-Logic
```

```
  imports Hoare-Syntax Hoare-Tac
```

```
begin
```

### 3.1 Sugared semantic embedding of Hoare logic

Strictly speaking a shallow embedding (as implemented by Norbert Galm following Mike Gordon) would suffice. Maybe the datatype `com` comes in useful later.

**type-synonym**  $'a \text{ bexp} = 'a \text{ set}$

**type-synonym**  $'a \text{ assn} = 'a \text{ set}$

**datatype**  $'a \text{ com} =$

$\text{Basic } 'a \Rightarrow 'a$   
 $| \text{Seq } 'a \text{ com } 'a \text{ com}$   
 $| \text{Cond } 'a \text{ bexp } 'a \text{ com } 'a \text{ com}$   
 $| \text{While } 'a \text{ bexp } 'a \text{ com}$

**abbreviation**  $\text{annskip} \text{ (SKIP) where } \text{SKIP} == \text{Basic id}$

**type-synonym**  $'a \text{ sem} = 'a \Rightarrow \text{bool}$

**inductive**  $\text{Sem} :: 'a \text{ com} \Rightarrow 'a \text{ sem}$

**where**

$\text{Sem} (\text{Basic } f) s (f s)$   
 $| \text{Sem } c1 s s'' \Longrightarrow \text{Sem } c2 s'' s' \Longrightarrow \text{Sem} (\text{Seq } c1 c2) s s'$   
 $| s \in b \Longrightarrow \text{Sem } c1 s s' \Longrightarrow \text{Sem} (\text{Cond } b c1 c2) s s'$   
 $| s \notin b \Longrightarrow \text{Sem } c2 s s' \Longrightarrow \text{Sem} (\text{Cond } b c1 c2) s s'$   
 $| s \notin b \Longrightarrow \text{Sem} (\text{While } b c) s s$   
 $| s \in b \Longrightarrow \text{Sem } c s s'' \Longrightarrow \text{Sem} (\text{While } b c) s'' s' \Longrightarrow$   
 $\text{Sem} (\text{While } b c) s s'$

**definition**  $\text{Valid} :: 'a \text{ bexp} \Rightarrow 'a \text{ com} \Rightarrow 'a \text{ anno} \Rightarrow 'a \text{ bexp} \Rightarrow \text{bool}$

**where**  $\text{Valid } p c a q \equiv \forall s s'. \text{Sem } c s s' \longrightarrow s \in p \longrightarrow s' \in q$

**definition**  $\text{ValidTC} :: 'a \text{ bexp} \Rightarrow 'a \text{ com} \Rightarrow 'a \text{ anno} \Rightarrow 'a \text{ bexp} \Rightarrow \text{bool}$

**where**  $\text{ValidTC } p c a q \equiv \forall s. s \in p \longrightarrow (\exists t. \text{Sem } c s t \wedge t \in q)$

**inductive-cases**  $[\text{elim!}]$ :

$\text{Sem} (\text{Basic } f) s s' \text{Sem} (\text{Seq } c1 c2) s s'$   
 $\text{Sem} (\text{Cond } b c1 c2) s s'$

**lemma**  $\text{Sem-deterministic}$ :

**assumes**  $\text{Sem } c s s1$

**and**  $\text{Sem } c s s2$

**shows**  $s1 = s2$

$\langle \text{proof} \rangle$

**lemma**  $\text{tc-implies-pc}$ :

$\text{ValidTC } p c a q \Longrightarrow \text{Valid } p c a q$

$\langle \text{proof} \rangle$

**lemma**  $\text{tc-extract-function}$ :

$ValidTC\ p\ c\ a\ q \implies \exists f . \forall s . s \in p \longrightarrow f\ s \in q$   
 ⟨proof⟩

**lemma** *SkipRule*:  $p \subseteq q \implies Valid\ p\ (Basic\ id)\ a\ q$   
 ⟨proof⟩

**lemma** *BasicRule*:  $p \subseteq \{s. f\ s \in q\} \implies Valid\ p\ (Basic\ f)\ a\ q$   
 ⟨proof⟩

**lemma** *SeqRule*:  $Valid\ P\ c1\ a1\ Q \implies Valid\ Q\ c2\ a2\ R \implies Valid\ P\ (Seq\ c1\ c2)\ (Aseq\ a1\ a2)\ R$   
 ⟨proof⟩

**lemma** *CondRule*:  
 $p \subseteq \{s. (s \in b \longrightarrow s \in w) \wedge (s \notin b \longrightarrow s \in w')\}$   
 $\implies Valid\ w\ c1\ a1\ q \implies Valid\ w'\ c2\ a2\ q \implies Valid\ p\ (Cond\ b\ c1\ c2)\ (Acond\ a1\ a2)\ q$   
 ⟨proof⟩

**lemma** *While-aux*:  
**assumes**  $Sem\ (While\ b\ c)\ s\ s'$   
**shows**  $\forall s\ s'. Sem\ c\ s\ s' \longrightarrow s \in I \wedge s \in b \longrightarrow s' \in I \implies s \in I \implies s' \in I \wedge s' \notin b$   
 ⟨proof⟩

**lemma** *WhileRule*:  
 $p \subseteq i \implies Valid\ (i \cap b)\ c\ (A\ 0)\ i \implies i \cap (-b) \subseteq q \implies Valid\ p\ (While\ b\ c)\ (Awhile\ i\ v\ A)\ q$   
 ⟨proof⟩

**lemma** *SkipRuleTC*:  
**assumes**  $p \subseteq q$   
**shows**  $ValidTC\ p\ (Basic\ id)\ a\ q$   
 ⟨proof⟩

**lemma** *BasicRuleTC*:  
**assumes**  $p \subseteq \{s. f\ s \in q\}$   
**shows**  $ValidTC\ p\ (Basic\ f)\ a\ q$   
 ⟨proof⟩

**lemma** *SeqRuleTC*:  
**assumes**  $ValidTC\ p\ c1\ a1\ q$   
**and**  $ValidTC\ q\ c2\ a2\ r$   
**shows**  $ValidTC\ p\ (Seq\ c1\ c2)\ (Aseq\ a1\ a2)\ r$   
 ⟨proof⟩

**lemma** *CondRuleTC*:  
**assumes**  $p \subseteq \{s. (s \in b \longrightarrow s \in w) \wedge (s \notin b \longrightarrow s \in w')\}$



```

    and ValidTC w c1 a1 q
    and ValidTC w' c2 a2 q
  shows ValidTC p (Cond b c1 c2) (Acond a1 a2) q
<proof>

```

**lemma** *WhileRuleTC*:

```

  assumes  $p \subseteq i$ 
    and  $\bigwedge n::nat . ValidTC (i \cap b \cap \{s . v s = n\}) c (A n) (i \cap \{s . v s < n\})$ 
    and  $i \cap uminus b \subseteq q$ 
  shows ValidTC p (While b c) (Awhile i v ( $\lambda n. A n$ )) q
<proof>

```

### 3.1.1 Concrete syntax

<ML>

### 3.1.2 Proof methods: VCG

```

declare BasicRule [Hoare-Tac.BasicRule]
  and SkipRule [Hoare-Tac.SkipRule]
  and SeqRule [Hoare-Tac.SeqRule]
  and CondRule [Hoare-Tac.CondRule]
  and WhileRule [Hoare-Tac.WhileRule]

```

```

declare BasicRuleTC [Hoare-Tac.BasicRuleTC]
  and SkipRuleTC [Hoare-Tac.SkipRuleTC]
  and SeqRuleTC [Hoare-Tac.SeqRuleTC]
  and CondRuleTC [Hoare-Tac.CondRuleTC]
  and WhileRuleTC [Hoare-Tac.WhileRuleTC]

```

<ML>

end

## 4 More arithmetic

```

theory Arith2
  imports Main
begin

```

```

definition cd :: [nat, nat, nat]  $\Rightarrow$  bool
  where  $cd\ x\ m\ n \iff x\ dvd\ m \wedge x\ dvd\ n$ 

```

```

definition gcd :: [nat, nat]  $\Rightarrow$  nat
  where  $gcd\ m\ n = (SOME\ x. cd\ x\ m\ n \ \& \ (\forall\ y. (cd\ y\ m\ n) \longrightarrow y \leq x))$ 

```

```

primrec fac :: nat  $\Rightarrow$  nat
  where
    fac 0 = Suc 0

```

|  $fac (Suc\ n) = Suc\ n * fac\ n$

#### 4.1 cd

**lemma** *cd-nnn*:  $0 < n \implies cd\ n\ n\ n$   
*<proof>*

**lemma** *cd-le*:  $[| cd\ x\ m\ n; 0 < m; 0 < n |] \implies x \leq m \ \& \ x \leq n$   
*<proof>*

**lemma** *cd-swap*:  $cd\ x\ m\ n = cd\ x\ n\ m$   
*<proof>*

**lemma** *cd-diff-l*:  $n \leq m \implies cd\ x\ m\ n = cd\ x\ (m-n)\ n$   
*<proof>*

**lemma** *cd-diff-r*:  $m \leq n \implies cd\ x\ m\ n = cd\ x\ m\ (n-m)$   
*<proof>*

#### 4.2 gcd

**lemma** *gcd-nnn*:  $0 < n \implies n = gcd\ n\ n$   
*<proof>*

**lemma** *gcd-swap*:  $gcd\ m\ n = gcd\ n\ m$   
*<proof>*

**lemma** *gcd-diff-l*:  $n \leq m \implies gcd\ m\ n = gcd\ (m-n)\ n$   
*<proof>*

**lemma** *gcd-diff-r*:  $m \leq n \implies gcd\ m\ n = gcd\ m\ (n-m)$   
*<proof>*

#### 4.3 pow

**lemma** *sq-pow-div2* [*simp*]:  
 $m \bmod 2 = 0 \implies ((n::nat)*n)^{(m \div 2)} = n^m$   
*<proof>*

end

## 5 Various examples

**theory** *Examples*  
**imports** *Hoare-Logic Arith2*  
**begin**

## 5.1 Arithmetic

### 5.1.1 Multiplication by successive addition

**lemma** *multiply-by-add*: VARS  $m\ s\ a\ b$

$\{a=A \wedge b=B\}$   
 $m := 0; s := 0;$   
*WHILE*  $m \neq a$   
*INV*  $\{s=m*b \wedge a=A \wedge b=B\}$   
*DO*  $s := s+b; m := m+(1::nat)$  *OD*  
 $\{s = A*B\}$   
*<proof>*

**lemma** *multiply-by-add-time*: VARS  $m\ s\ a\ b\ t$

$\{a=A \wedge b=B \wedge t=0\}$   
 $m := 0; t := t+1; s := 0; t := t+1;$   
*WHILE*  $m \neq a$   
*INV*  $\{s=m*b \wedge a=A \wedge b=B \wedge t = 2*m + 2\}$   
*DO*  $s := s+b; t := t+1; m := m+(1::nat); t := t+1$  *OD*  
 $\{s = A*B \wedge t = 2*A + 2\}$   
*<proof>*

**lemma** *multiply-by-add2*: VARS  $M\ N\ P :: int$

$\{m=M \wedge n=N\}$   
*IF*  $M < 0$  *THEN*  $M := -M; N := -N$  *ELSE SKIP FI*;  
 $P := 0;$   
*WHILE*  $0 < M$   
*INV*  $\{0 \leq M \wedge (\exists p. p = (\text{if } m < 0 \text{ then } -m \text{ else } m) \ \& \ p*N = m*n \ \& \ P = (p-M)*N)\}$   
*DO*  $P := P+N; M := M - 1$  *OD*  
 $\{P = m*n\}$   
*<proof>*

**lemma** *multiply-by-add2-time*: VARS  $M\ N\ P\ t :: int$

$\{m=M \wedge n=N \wedge t=0\}$   
*IF*  $M < 0$  *THEN*  $M := -M; t := t+1; N := -N; t := t+1$  *ELSE SKIP FI*;  
 $P := 0; t := t+1;$   
*WHILE*  $0 < M$   
*INV*  $\{0 \leq M \ \& \ (\exists p. p = (\text{if } m < 0 \text{ then } -m \text{ else } m) \ \& \ p*N = m*n \ \& \ P = (p-M)*N \ \& \ t \geq 0 \ \& \ t \leq 2*(p-M)+3)\}$   
*DO*  $P := P+N; t := t+1; M := M - 1; t := t+1$  *OD*  
 $\{P = m*n \ \& \ t \leq 2*abs\ m + 3\}$   
*<proof>*

### 5.1.2 Euclid's algorithm for GCD

**lemma** *Euclid-GCD*: VARS  $a\ b$

$\{0 < A \ \& \ 0 < B\}$   
 $a := A; b := B;$   
*WHILE*  $a \neq b$

$INV \{0 < a \ \& \ 0 < b \ \& \ gcd \ A \ B = gcd \ a \ b\}$   
 $DO \ IF \ a < b \ THEN \ b := b - a \ ELSE \ a := a - b \ FI \ OD$   
 $\{a = gcd \ A \ B\}$   
 $\langle proof \rangle$

**lemma** *Euclid-GCD-time*:  $VARs \ a \ b \ t$   
 $\{0 < A \ \& \ 0 < B \ \& \ t = 0\}$   
 $a := A; \ t := t + 1; \ b := B; \ t := t + 1;$   
 $WHILE \ a \neq b$   
 $INV \{0 < a \ \& \ 0 < b \ \& \ gcd \ A \ B = gcd \ a \ b \ \& \ a \leq A \ \& \ b \leq B \ \& \ t \leq max \ A \ B - max \ a \ b + 2\}$   
 $DO \ IF \ a < b \ THEN \ b := b - a; \ t := t + 1 \ ELSE \ a := a - b; \ t := t + 1 \ FI \ OD$   
 $\{a = gcd \ A \ B \ \& \ t \leq max \ A \ B + 2\}$   
 $\langle proof \rangle$

### 5.1.3 Dijkstra's extension of Euclid's algorithm for simultaneous GCD and SCM

From E.W. Dijkstra. Selected Writings on Computing, p 98 (EWD474), where it is given without the invariant. Instead of defining *scm* explicitly we have used the theorem  $scm \ x \ y = x * y / gcd \ x \ y$  and avoided division by multiplying with  $gcd \ x \ y$ .

**lemmas** *distrib* =  
 $diff-mult-distrib \ diff-mult-distrib2 \ add-mult-distrib \ add-mult-distrib2$

**lemma** *gcd-scm*:  $VARs \ a \ b \ x \ y$   
 $\{0 < A \ \& \ 0 < B \ \& \ a = A \ \& \ b = B \ \& \ x = B \ \& \ y = A\}$   
 $WHILE \ a \neq b$   
 $INV \{0 < a \ \& \ 0 < b \ \& \ gcd \ A \ B = gcd \ a \ b \ \& \ 2 * A * B = a * x + b * y\}$   
 $DO \ IF \ a < b \ THEN \ (b := b - a; \ x := x + y) \ ELSE \ (a := a - b; \ y := y + x) \ FI \ OD$   
 $\{a = gcd \ A \ B \ \& \ 2 * A * B = a * (x + y)\}$   
 $\langle proof \rangle$

### 5.1.4 Power by iterated squaring and multiplication

**lemma** *power-by-mult*:  $VARs \ a \ b \ c$   
 $\{a = A \ \& \ b = B\}$   
 $c := (1 :: nat);$   
 $WHILE \ b \neq 0$   
 $INV \{A \wedge^b = c * a \wedge^b\}$   
 $DO \ WHILE \ b \ mod \ 2 = 0$   
 $INV \{A \wedge^b = c * a \wedge^b\}$   
 $DO \ a := a * a; \ b := b \ div \ 2 \ OD;$   
 $c := c * a; \ b := b - 1$   
 $OD$   
 $\{c = A \wedge^B\}$   
 $\langle proof \rangle$

### 5.1.5 Factorial

**lemma** *factorial*: VARS  $a\ b$

$\{a=A\}$   
 $b := 1;$   
WHILE  $a > 0$   
INV  $\{fac\ A = b * fac\ a\}$   
DO  $b := b*a; a := a - 1$  OD  
 $\{b = fac\ A\}$   
*<proof>*

**lemma** *factorial-time*: VARS  $a\ b\ t$

$\{a=A \ \& \ t=0\}$   
 $b := 1; t := t+1;$   
WHILE  $a > 0$   
INV  $\{fac\ A = b * fac\ a \ \& \ a \leq A \ \& \ t = 2*(A-a)+1\}$   
DO  $b := b*a; t := t+1; a := a - 1; t := t+1$  OD  
 $\{b = fac\ A \ \& \ t = 2*A + 1\}$   
*<proof>*

**lemma** [*simp*]:  $1 \leq i \implies fac\ (i - Suc\ 0) * i = fac\ i$

*<proof>*

**lemma** *factorial2*: VARS  $i\ f$

$\{True\}$   
 $i := (1::nat); f := 1;$   
WHILE  $i \leq n$  INV  $\{f = fac(i - 1) \ \& \ 1 \leq i \ \& \ i \leq n+1\}$   
DO  $f := f*i; i := i+1$  OD  
 $\{f = fac\ n\}$   
*<proof>*

**lemma** *factorial2-time*: VARS  $i\ f\ t$

$\{t=0\}$   
 $i := (1::nat); t := t+1; f := 1; t := t+1;$   
WHILE  $i \leq n$  INV  $\{f = fac(i - 1) \ \& \ 1 \leq i \ \& \ i \leq n+1 \ \& \ t = 2*(i-1)+2\}$   
DO  $f := f*i; t := t+1; i := i+1; t := t+1$  OD  
 $\{f = fac\ n \ \& \ t = 2*n+2\}$   
*<proof>*

### 5.1.6 Square root

**lemma** *sqrt*: VARS  $r\ x$

$\{True\}$   
 $r := (0::nat);$   
WHILE  $(r+1)*(r+1) \leq X$   
INV  $\{r*r \leq X\}$   
DO  $r := r+1$  OD  
 $\{r*r \leq X \ \& \ X < (r+1)*(r+1)\}$   
*<proof>*

**lemma** *sqrt-time*: *VAR*S  $r\ t$   
 $\{t=0\}$   
 $r := (0::nat); t := t+1;$   
**WHILE**  $(r+1)*(r+1) \leq X$   
**INV**  $\{r*r \leq X \ \& \ t = r+1\}$   
**DO**  $r := r+1; t := t+1$  **OD**  
 $\{r*r \leq X \ \& \ X < (r+1)*(r+1) \ \& \ (t-1)*(t-1) \leq X\}$   
 $\langle proof \rangle$

**lemma** *sqrt-without-multiplication*: *VAR*S  $u\ w\ r$   
 $\{x=X\}$   
 $u := 1; w := 1; r := (0::nat);$   
**WHILE**  $w \leq X$   
**INV**  $\{u = r+r+1 \ \& \ w = (r+1)*(r+1) \ \& \ r*r \leq X\}$   
**DO**  $r := r + 1; w := w + u + 2; u := u + 2$  **OD**  
 $\{r*r \leq X \ \& \ X < (r+1)*(r+1)\}$   
 $\langle proof \rangle$

## 5.2 Lists

**lemma** *imperative-reverse*: *VAR*S  $y\ x$   
 $\{x=X\}$   
 $y := [];$   
**WHILE**  $x \sim = []$   
**INV**  $\{rev(x)@y = rev(X)\}$   
**DO**  $y := (hd\ x \ \# \ y); x := tl\ x$  **OD**  
 $\{y=rev(X)\}$   
 $\langle proof \rangle$

**lemma** *imperative-reverse-time*: *VAR*S  $y\ x\ t$   
 $\{x=X \ \& \ t=0\}$   
 $y := []; t := t+1;$   
**WHILE**  $x \sim = []$   
**INV**  $\{rev(x)@y = rev(X) \ \& \ t = 2*(length\ y) + 1\}$   
**DO**  $y := (hd\ x \ \# \ y); t := t+1; x := tl\ x; t := t+1$  **OD**  
 $\{y=rev(X) \ \& \ t = 2*length\ X + 1\}$   
 $\langle proof \rangle$

**lemma** *imperative-append*: *VAR*S  $x\ y$   
 $\{x=X \ \& \ y=Y\}$   
 $x := rev(x);$   
**WHILE**  $x \sim = []$   
**INV**  $\{rev(x)@y = X@Y\}$   
**DO**  $y := (hd\ x \ \# \ y);$   
 $x := tl\ x$   
**OD**  
 $\{y = X@Y\}$   
 $\langle proof \rangle$

**lemma** *imperative-append-time-no-rev*: *VAR*S  $x\ y\ t$

```

{x=X & y=Y}
x := rev(x); t := 0;
WHILE x~=[]
INV {rev(x)@y = X@Y & length x ≤ length X & t = 2 * (length X - length x)}
DO y := (hd x # y); t := t+1;
  x := tl x; t := t+1
OD
{y = X@Y & t = 2 * length X}
⟨proof⟩

```

### 5.3 Arrays

#### 5.3.1 Search for a key

**lemma** *zero-search*: VARS  $A\ i$

```

{True}
i := 0;
WHILE i < length A & A!i ≠ key
INV {∀j. j < i → A!j ≠ key}
DO i := i+1 OD
{(i < length A → A!i = key) &
 (i = length A → (∀j. j < length A → A!j ≠ key))}
⟨proof⟩

```

**lemma** *zero-search-time*: VARS  $A\ i\ t$

```

{t=0}
i := 0; t := t+1;
WHILE i < length A ∧ A!i ≠ key
INV {(∀j. j < i → A!j ≠ key) ∧ i ≤ length A ∧ t = i+1}
DO i := i+1; t := t+1 OD
{(i < length A → A!i = key) ∧
 (i = length A → (∀j. j < length A → A!j ≠ key)) ∧ t ≤ length A + 1}
⟨proof⟩

```

The *partition* procedure for quicksort.

- $A$  is the array to be sorted (modelled as a list).
- Elements of  $A$  must be of class order to infer at the end that the elements between  $u$  and  $l$  are equal to pivot.

Ambiguity warnings of parser are due to  $:=$  being used both for assignment and list update.

**lemma** *lem*:  $m - \text{Suc } 0 < n \implies m < \text{Suc } n$   
⟨proof⟩

**lemma** *Partition*:  
 $[[ \text{leq} == \lambda A\ i. \forall k. k < i \rightarrow A!k \leq \text{pivot};$   
 $\text{geq} == \lambda A\ i. \forall k. i < k \wedge k < \text{length } A \rightarrow \text{pivot} \leq A!k ]]$   $\implies$

```

VARs A u l
{0 < length(A::('a::order)list)}
l := 0; u := length A - Suc 0;
WHILE l ≤ u
  INV {leq A l ∧ geq A u ∧ u < length A ∧ l ≤ length A}
  DO WHILE l < length A ∧ A!l ≤ pivot
    INV {leq A l & geq A u ∧ u < length A ∧ l ≤ length A}
    DO l := l+1 OD;
    WHILE 0 < u & pivot ≤ A!u
      INV {leq A l & geq A u ∧ u < length A ∧ l ≤ length A}
      DO u := u - 1 OD;
      IF l ≤ u THEN A := A[l := A!u, u := A!l] ELSE SKIP FI
    OD
  {leq A u & (∀ k. u < k ∧ k < l → A!k = pivot) ∧ geq A l}
— expand and delete abbreviations first
⟨proof⟩

```

**end**

## 6 Hoare Logic with an Abort statement for modelling run time errors

```

theory Hoare-Logic-Abort
  imports Hoare-Syntax Hoare-Tac
begin

type-synonym 'a bexp = 'a set
type-synonym 'a assn = 'a set
type-synonym 'a var = 'a ⇒ nat

datatype 'a com =
  Basic 'a ⇒ 'a
  | Abort
  | Seq 'a com 'a com
  | Cond 'a bexp 'a com 'a com
  | While 'a bexp 'a com

abbreviation annskip (SKIP) where SKIP == Basic id

type-synonym 'a sem = 'a option => 'a option => bool

inductive Sem :: 'a com ⇒ 'a sem
where
  Sem (Basic f) None None
  | Sem (Basic f) (Some s) (Some (f s))
  | Sem Abort s None
  | Sem c1 s s'' ⇒ Sem c2 s'' s' ⇒ Sem (Seq c1 c2) s s'
  | Sem (Cond b c1 c2) None None

```



$| s \in b \implies \text{Sem } c1 \text{ (Some } s) s' \implies \text{Sem (Cond } b \text{ } c1 \text{ } c2) \text{ (Some } s) s'$   
 $| s \notin b \implies \text{Sem } c2 \text{ (Some } s) s' \implies \text{Sem (Cond } b \text{ } c1 \text{ } c2) \text{ (Some } s) s'$   
 $| \text{Sem (While } b \text{ } c) \text{ None None}$   
 $| s \notin b \implies \text{Sem (While } b \text{ } c) \text{ (Some } s) \text{ (Some } s)$   
 $| s \in b \implies \text{Sem } c \text{ (Some } s) s'' \implies \text{Sem (While } b \text{ } c) s'' s' \implies$   
 $\quad \text{Sem (While } b \text{ } c) \text{ (Some } s) s'$

**inductive-cases** [elim!]:

$\text{Sem (Basic } f) s s' \text{ Sem (Seq } c1 \text{ } c2) s s'$   
 $\text{Sem (Cond } b \text{ } c1 \text{ } c2) s s'$

**lemma** *Sem-deterministic*:

**assumes**  $\text{Sem } c \text{ } s \text{ } s1$   
**and**  $\text{Sem } c \text{ } s \text{ } s2$   
**shows**  $s1 = s2$

*<proof>*

**definition** *Valid* ::  $'a \text{ bexp} \implies 'a \text{ com} \implies 'a \text{ anno} \implies 'a \text{ bexp} \implies \text{bool}$

**where**  $\text{Valid } p \text{ } c \text{ } a \text{ } q \equiv \forall s s'. \text{Sem } c \text{ } s \text{ } s' \longrightarrow s \in \text{Some } ' p \longrightarrow s' \in \text{Some } ' q$

**definition** *ValidTC* ::  $'a \text{ bexp} \implies 'a \text{ com} \implies 'a \text{ anno} \implies 'a \text{ bexp} \implies \text{bool}$

**where**  $\text{ValidTC } p \text{ } c \text{ } a \text{ } q \equiv \forall s. s \in p \longrightarrow (\exists t. \text{Sem } c \text{ (Some } s) \text{ (Some } t) \wedge t \in q)$

**lemma** *tc-implies-pc*:

$\text{ValidTC } p \text{ } c \text{ } a \text{ } q \implies \text{Valid } p \text{ } c \text{ } a \text{ } q$   
*<proof>*

**lemma** *tc-extract-function*:

$\text{ValidTC } p \text{ } c \text{ } a \text{ } q \implies \exists f. \forall s. s \in p \longrightarrow f s \in q$   
*<proof>*

The proof rules for partial correctness

**lemma** *SkipRule*:  $p \subseteq q \implies \text{Valid } p \text{ (Basic id) } a \text{ } q$

*<proof>*

**lemma** *BasicRule*:  $p \subseteq \{s. f s \in q\} \implies \text{Valid } p \text{ (Basic } f) \text{ } a \text{ } q$

*<proof>*

**lemma** *SeqRule*:  $\text{Valid } P \text{ } c1 \text{ } a1 \text{ } Q \implies \text{Valid } Q \text{ } c2 \text{ } a2 \text{ } R \implies \text{Valid } P \text{ (Seq } c1 \text{ } c2)$

$(\text{Aseq } a1 \text{ } a2) \text{ } R$

*<proof>*

**lemma** *CondRule*:

$p \subseteq \{s. (s \in b \longrightarrow s \in w) \wedge (s \notin b \longrightarrow s \in w')\}$   
 $\implies \text{Valid } w \text{ } c1 \text{ } a1 \text{ } q \implies \text{Valid } w' \text{ } c2 \text{ } a2 \text{ } q \implies \text{Valid } p \text{ (Cond } b \text{ } c1 \text{ } c2) \text{ (Acond } a1$   
 $a2) \text{ } q$

*<proof>*

**lemma** *While-aux*:

**assumes** *Sem* (*While* *b c*) *s s'*

**shows**  $\forall s s'. \text{Sem } c \ s \ s' \longrightarrow s \in \text{Some } 'I \cap b \longrightarrow s' \in \text{Some } 'I \Longrightarrow$   
 $s \in \text{Some } 'I \Longrightarrow s' \in \text{Some } '(I \cap -b)$

*<proof>*

**lemma** *WhileRule*:

$p \subseteq i \Longrightarrow \text{Valid } (i \cap b) \ c \ (A \ 0) \ i \Longrightarrow i \cap (-b) \subseteq q \Longrightarrow \text{Valid } p \ (\text{While } b \ c)$   
 $(\text{Awhile } i \ v \ A) \ q$

*<proof>*

**lemma** *AbortRule*:  $p \subseteq \{s. \text{False}\} \Longrightarrow \text{Valid } p \ \text{Abort } a \ q$

*<proof>*

The proof rules for total correctness

**lemma** *SkipRuleTC*:

**assumes**  $p \subseteq q$

**shows**  $\text{ValidTC } p \ (\text{Basic } id) \ a \ q$

*<proof>*

**lemma** *BasicRuleTC*:

**assumes**  $p \subseteq \{s. f \ s \in q\}$

**shows**  $\text{ValidTC } p \ (\text{Basic } f) \ a \ q$

*<proof>*

**lemma** *SeqRuleTC*:

**assumes**  $\text{ValidTC } p \ c1 \ a1 \ q$

**and**  $\text{ValidTC } q \ c2 \ a2 \ r$

**shows**  $\text{ValidTC } p \ (\text{Seq } c1 \ c2) \ (\text{Aseq } a1 \ a2) \ r$

*<proof>*

**lemma** *CondRuleTC*:

**assumes**  $p \subseteq \{s. (s \in b \longrightarrow s \in w) \wedge (s \notin b \longrightarrow s \in w')\}$

**and**  $\text{ValidTC } w \ c1 \ a1 \ q$

**and**  $\text{ValidTC } w' \ c2 \ a2 \ q$

**shows**  $\text{ValidTC } p \ (\text{Cond } b \ c1 \ c2) \ (\text{Acons } a1 \ a2) \ q$

*<proof>*

**lemma** *WhileRuleTC*:

**assumes**  $p \subseteq i$

**and**  $\bigwedge n::nat. \text{ValidTC } (i \cap b \cap \{s. v \ s = n\}) \ c \ (A \ n) \ (i \cap \{s. v \ s < n\})$

**and**  $i \cap \text{uminus } b \subseteq q$

**shows**  $\text{ValidTC } p \ (\text{While } b \ c) \ (\text{Awhile } i \ v \ A) \ q$

*<proof>*

## 6.1 Concrete syntax

*<ML>*

**syntax**

*-guarded-com* ::  $bool \Rightarrow 'a \ \text{com} \Rightarrow 'a \ \text{com} \ ((2- \rightarrow / -) \ 71)$

-array-update :: 'a list ⇒ nat ⇒ 'a ⇒ 'a com ((2-[-] :=/ -) [70, 65] 61)

**translations**

$P \rightarrow c \Leftrightarrow \text{IF } P \text{ THEN } c \text{ ELSE CONST Abort FI}$

$a[i] := v \rightarrow (i < \text{CONST length } a) \rightarrow (a := \text{CONST list-update } a \ i \ v)$

— reverse translation not possible because of duplicate  $a$

Note: there is no special syntax for guarded array access. Thus you must write  $j < \text{length } a \rightarrow a[i] := a!j$ .

## 6.2 Proof methods: VCG

**declare** *BasicRule* [*Hoare-Tac.BasicRule*]

**and** *SkipRule* [*Hoare-Tac.SkipRule*]

**and** *AbortRule* [*Hoare-Tac.AbortRule*]

**and** *SeqRule* [*Hoare-Tac.SeqRule*]

**and** *CondRule* [*Hoare-Tac.CondRule*]

**and** *WhileRule* [*Hoare-Tac.WhileRule*]

**declare** *BasicRuleTC* [*Hoare-Tac.BasicRuleTC*]

**and** *SkipRuleTC* [*Hoare-Tac.SkipRuleTC*]

**and** *SeqRuleTC* [*Hoare-Tac.SeqRuleTC*]

**and** *CondRuleTC* [*Hoare-Tac.CondRuleTC*]

**and** *WhileRuleTC* [*Hoare-Tac.WhileRuleTC*]

⟨*ML*⟩

**end**

## 7 Some small examples for programs that may abort

**theory** *ExamplesAbort*

**imports** *Hoare-Logic-Abort*

**begin**

**lemma** *VARs*  $x \ y \ z :: \text{nat}$

$\{y = z \ \& \ z \neq 0\} \ z \neq 0 \rightarrow x := y \ \text{div} \ z \ \{x = 1\}$

⟨*proof*⟩

**lemma**

*VARs*  $a \ i \ j$

$\{k \leq \text{length } a \ \& \ i < k \ \& \ j < k\} \ j < \text{length } a \rightarrow a[i] := a!j \ \{\text{True}\}$

⟨*proof*⟩

**lemma** *VARs*  $(a :: \text{int list}) \ i$

$\{\text{True}\}$

$i := 0;$

**WHILE**  $i < \text{length } a$

```

INV {i <= length a}
DO a[i] := 7; i := i+1 OD
{True}
⟨proof⟩

```

end

## 8 Examples using Hoare Logic for Total Correctness

```

theory ExamplesTC
  imports Hoare-Logic
begin

```

This theory demonstrates a few simple partial- and total-correctness proofs. The first example is taken from HOL/Hoare/Examples.thy written by N. Galm. We have added the invariant  $m \leq a$ .

```

lemma multiply-by-add: VARS m s a b
  {a=A ∧ b=B}
  m := 0; s := 0;
  WHILE m ≠ a
  INV {s=m*b ∧ a=A ∧ b=B ∧ m≤a}
  DO s := s+b; m := m+(1::nat) OD
  {s = A*B}
  ⟨proof⟩

```

Here is the total-correctness proof for the same program. It needs the additional invariant  $m \leq a$ .

```

lemma multiply-by-add-tc: VARS m s a b
  [a=A ∧ b=B]
  m := 0; s := 0;
  WHILE m ≠ a
  INV {s=m*b ∧ a=A ∧ b=B ∧ m≤a}
  VAR {a-m}
  DO s := s+b; m := m+(1::nat) OD
  [s = A*B]
  ⟨proof⟩

```

Next, we prove partial correctness of a program that computes powers.

```

lemma power: VARS (p::int) i
  { True }
  p := 1;
  i := 0;
  WHILE i < n
  INV { p = xi ∧ i ≤ n }
  DO p := p * x;
    i := i + 1
  OD

```

$\{ p = x \hat{n} \}$   
 $\langle proof \rangle$

Here is its total-correctness proof.

**lemma** *power-tc: VARS (p::int) i*  
 $[ True ]$   
 $p := 1;$   
 $i := 0;$   
**WHILE**  $i < n$   
  **INV**  $\{ p = x \hat{i} \wedge i \leq n \}$   
  **VAR**  $\{ n - i \}$   
  **DO**  $p := p * x;$   
   $i := i + 1$   
**OD**  
 $[ p = x \hat{n} ]$   
 $\langle proof \rangle$

The last example is again taken from HOL/Hoare/Examples.thy. We have modified it to integers so it requires precondition  $0 \leq x$ .

**lemma** *sqrt-tc: VARS r*  
 $[ 0 \leq (x::int) ]$   
 $r := 0;$   
**WHILE**  $(r+1)*(r+1) \leq x$   
  **INV**  $\{ r*r \leq x \}$   
  **VAR**  $\{ nat (x-r) \}$   
  **DO**  $r := r+1$  **OD**  
 $[ r*r \leq x \wedge x < (r+1)*(r+1) ]$   
 $\langle proof \rangle$

A total-correctness proof allows us to extract a function for further use. For every input satisfying the precondition the function returns an output satisfying the postcondition.

**lemma** *sqrt-exists:*  
 $0 \leq (x::int) \implies \exists r'. r'*r' \leq x \wedge x < (r'+1)*(r'+1)$   
 $\langle proof \rangle$

**definition** *sqrt (x::int)  $\equiv$  (SOME r'. r'\*r'  $\leq$  x  $\wedge$  x < (r'+1)\*(r'+1))*

**lemma** *sqrt-function:*  
**assumes**  $0 \leq (x::int)$   
  **and**  $r' = sqrt\ x$   
**shows**  $r'*r' \leq x \wedge x < (r'+1)*(r'+1)$   
 $\langle proof \rangle$

Nested loops!

**lemma** *VARS (i::nat) j*  
 $[ True ]$   
**WHILE**  $0 < i$   
  **INV**  $\{ True \}$

```

VAR { z = i }
DO i := i - 1; j := i;
  WHILE 0 < j
  INV { z = i+1 }
  VAR { j }
  DO j := j - 1 OD
OD
[ i ≤ 0 ]
⟨proof⟩

```

end

## 9 Alternative pointers

```

theory Pointers0
  imports Hoare-Logic
begin

```

### 9.1 References

```

class ref =
  fixes Null :: 'a

```

### 9.2 Field access and update

```

syntax
-fassign :: 'a::ref => id => 'v => 's com
  ((2-^.- := / -) [70,1000,65] 61)
-faccess :: 'a::ref => ('a::ref ⇒ 'v) => 'v
  (-^.- [65,1000] 65)

```

#### translations

```

p^f := e => f := CONST fun-upd f p e
p^f    => f p

```

An example due to Suzuki:

```

lemma VARS v n
  {distinct[w,x,y,z]}
  w^v := (1::int); w^n := x;
  x^v := 2; x^n := y;
  y^v := 3; y^n := z;
  z^v := 4; x^n := z
  {w^n.n^v = 4}
⟨proof⟩

```

### 9.3 The heap

#### 9.3.1 Paths in the heap

```

primrec Path :: ('a::ref ⇒ 'a) ⇒ 'a ⇒ 'a list ⇒ 'a ⇒ bool

```

**where**

$Path\ h\ x\ []\ y = (x = y)$   
 $| Path\ h\ x\ (a\#\!as)\ y = (x \neq Null \wedge x = a \wedge Path\ h\ (h\ a)\ as\ y)$

**lemma** *[iff]*:  $Path\ h\ Null\ xs\ y = (xs = [] \wedge y = Null)$   
*<proof>*

**lemma** *[simp]*:  $a \neq Null \implies Path\ h\ a\ as\ z =$   
 $(as = [] \wedge z = a \vee (\exists bs. as = a\#\!bs \wedge Path\ h\ (h\ a)\ bs\ z))$   
*<proof>*

**lemma** *[simp]*:  $\bigwedge x. Path\ f\ x\ (as\@\!bs)\ z = (\exists y. Path\ f\ x\ as\ y \wedge Path\ f\ y\ bs\ z)$   
*<proof>*

**lemma** *[simp]*:  $\bigwedge x. u \notin set\ as \implies Path\ (f(u := v))\ x\ as\ y = Path\ f\ x\ as\ y$   
*<proof>*

### 9.3.2 Lists on the heap

**Relational abstraction definition**  $List :: ('a::ref \Rightarrow 'a) \Rightarrow 'a \Rightarrow 'a\ list \Rightarrow bool$

**where**  $List\ h\ x\ as = Path\ h\ x\ as\ Null$

**lemma** *[simp]*:  $List\ h\ x\ [] = (x = Null)$   
*<proof>*

**lemma** *[simp]*:  $List\ h\ x\ (a\#\!as) = (x \neq Null \wedge x = a \wedge List\ h\ (h\ a)\ as)$   
*<proof>*

**lemma** *[simp]*:  $List\ h\ Null\ as = (as = [])$   
*<proof>*

**lemma** *List-Ref**[simp]*:  
 $a \neq Null \implies List\ h\ a\ as = (\exists bs. as = a\#\!bs \wedge List\ h\ (h\ a)\ bs)$   
*<proof>*

**theorem** *notin-List-update**[simp]*:  
 $\bigwedge x. a \notin set\ as \implies List\ (h(a := y))\ x\ as = List\ h\ x\ as$   
*<proof>*

**declare** *fun-upd-apply**[simp del]**fun-upd-same**[simp]* *fun-upd-other**[simp]*

**lemma** *List-unique*:  $\bigwedge x\ bs. List\ h\ x\ as \implies List\ h\ x\ bs \implies as = bs$   
*<proof>*

**lemma** *List-unique1*:  $List\ h\ p\ as \implies \exists! as. List\ h\ p\ as$   
*<proof>*

**lemma** *List-app*:  $\bigwedge x. \text{List } h \ x \ (as@bs) = (\exists y. \text{Path } h \ x \ as \ y \wedge \text{List } h \ y \ bs)$   
 ⟨proof⟩

**lemma** *List-hd-not-in-tl*[simp]:  $\text{List } h \ (h \ a) \ as \Longrightarrow a \notin \text{set } as$   
 ⟨proof⟩

**lemma** *List-distinct*[simp]:  $\bigwedge x. \text{List } h \ x \ as \Longrightarrow \text{distinct } as$   
 ⟨proof⟩

### 9.3.3 Functional abstraction

**definition** *islist* ::  $('a::\text{ref} \Rightarrow 'a) \Rightarrow 'a \Rightarrow \text{bool}$   
 where  $\text{islist } h \ p \longleftrightarrow (\exists as. \text{List } h \ p \ as)$

**definition** *list* ::  $('a::\text{ref} \Rightarrow 'a) \Rightarrow 'a \Rightarrow 'a \ \text{list}$   
 where  $\text{list } h \ p = (\text{SOME } as. \text{List } h \ p \ as)$

**lemma** *List-conv-islist-list*:  $\text{List } h \ p \ as = (\text{islist } h \ p \wedge as = \text{list } h \ p)$   
 ⟨proof⟩

**lemma** [simp]:  $\text{islist } h \ \text{Null}$   
 ⟨proof⟩

**lemma** [simp]:  $a \neq \text{Null} \Longrightarrow \text{islist } h \ a = \text{islist } h \ (h \ a)$   
 ⟨proof⟩

**lemma** [simp]:  $\text{list } h \ \text{Null} = []$   
 ⟨proof⟩

**lemma** *list-Ref-conv*[simp]:  
 $\llbracket a \neq \text{Null}; \text{islist } h \ (h \ a) \rrbracket \Longrightarrow \text{list } h \ a = a \# \text{list } h \ (h \ a)$   
 ⟨proof⟩

**lemma** [simp]:  $\text{islist } h \ (h \ a) \Longrightarrow a \notin \text{set}(\text{list } h \ (h \ a))$   
 ⟨proof⟩

**lemma** *list-upd-conv*[simp]:  
 $\text{islist } h \ p \Longrightarrow y \notin \text{set}(\text{list } h \ p) \Longrightarrow \text{list } (h(y := q)) \ p = \text{list } h \ p$   
 ⟨proof⟩

**lemma** *islist-upd*[simp]:  
 $\text{islist } h \ p \Longrightarrow y \notin \text{set}(\text{list } h \ p) \Longrightarrow \text{islist } (h(y := q)) \ p$   
 ⟨proof⟩

## 9.4 Verifications

### 9.4.1 List reversal

A short but unreadable proof:

**lemma** *VARs tl*  $p \ q \ r$



$$\begin{array}{l}
\{List\ tl\ p\ Ps \wedge List\ tl\ q\ Qs \wedge set\ Ps \cap set\ Qs = \{\}\} \\
WHILE\ p \neq Null \\
INV\ \{\exists ps\ qs.\ List\ tl\ p\ ps \wedge List\ tl\ q\ qs \wedge set\ ps \cap set\ qs = \{\} \wedge \\
\quad rev\ ps\ @\ qs = rev\ Ps\ @\ Qs\} \\
DO\ r := p; p := p^.tl; r^.tl := q; q := r\ OD \\
\{List\ tl\ q\ (rev\ Ps\ @\ Qs)\} \\
\langle proof \rangle
\end{array}$$

A longer readable version:

**lemma** *VARS*  $tl\ p\ q\ r$   
 $\{List\ tl\ p\ Ps \wedge List\ tl\ q\ Qs \wedge set\ Ps \cap set\ Qs = \{\}\}$   
 $WHILE\ p \neq Null$   
 $INV\ \{\exists ps\ qs.\ List\ tl\ p\ ps \wedge List\ tl\ q\ qs \wedge set\ ps \cap set\ qs = \{\} \wedge$   
 $\quad rev\ ps\ @\ qs = rev\ Ps\ @\ Qs\}$   
 $DO\ r := p; p := p^.tl; r^.tl := q; q := r\ OD$   
 $\{List\ tl\ q\ (rev\ Ps\ @\ Qs)\}$   
 $\langle proof \rangle$

Finally, the functional version. A bit more verbose, but automatic!

**lemma** *VARS*  $tl\ p\ q\ r$   
 $\{islist\ tl\ p \wedge islist\ tl\ q \wedge$   
 $\quad Ps = list\ tl\ p \wedge Qs = list\ tl\ q \wedge set\ Ps \cap set\ Qs = \{\}\}$   
 $WHILE\ p \neq Null$   
 $INV\ \{islist\ tl\ p \wedge islist\ tl\ q \wedge$   
 $\quad set(list\ tl\ p) \cap set(list\ tl\ q) = \{\} \wedge$   
 $\quad rev(list\ tl\ p)\ @\ (list\ tl\ q) = rev\ Ps\ @\ Qs\}$   
 $DO\ r := p; p := p^.tl; r^.tl := q; q := r\ OD$   
 $\{islist\ tl\ q \wedge list\ tl\ q = rev\ Ps\ @\ Qs\}$   
 $\langle proof \rangle$

### 9.4.2 Searching in a list

What follows is a sequence of successively more intelligent proofs that a simple loop finds an element in a linked list.

We start with a proof based on the *List* predicate. This means it only works for acyclic lists.

**lemma** *VARS*  $tl\ p$   
 $\{List\ tl\ p\ Ps \wedge X \in set\ Ps\}$   
 $WHILE\ p \neq Null \wedge p \neq X$   
 $INV\ \{p \neq Null \wedge (\exists ps.\ List\ tl\ p\ ps \wedge X \in set\ ps)\}$   
 $DO\ p := p^.tl\ OD$   
 $\{p = X\}$   
 $\langle proof \rangle$

Using *Path* instead of *List* generalizes the correctness statement to cyclic lists as well:

**lemma** *VARS*  $tl\ p$   
 $\{Path\ tl\ p\ Ps\ X\}$

```

WHILE  $p \neq \text{Null} \wedge p \neq X$ 
INV  $\{\exists ps. \text{Path } tl \ p \ ps \ X\}$ 
DO  $p := p \hat{\cdot} tl$  OD
 $\{p = X\}$ 
⟨proof⟩

```

Now it dawns on us that we do not need the list witness at all — it suffices to talk about reachability, i.e. we can use relations directly.

```

lemma VARS  $tl \ p$ 
 $\{(p,X) \in \{(x,y). y = tl \ x \ \& \ x \neq \text{Null}\}^*\}$ 
WHILE  $p \neq \text{Null} \wedge p \neq X$ 
INV  $\{(p,X) \in \{(x,y). y = tl \ x \ \& \ x \neq \text{Null}\}^*\}$ 
DO  $p := p \hat{\cdot} tl$  OD
 $\{p = X\}$ 
⟨proof⟩

```

### 9.4.3 Merging two lists

This is still a bit rough, especially the proof.

```

fun merge :: 'a list * 'a list * ('a  $\Rightarrow$  'a  $\Rightarrow$  bool)  $\Rightarrow$  'a list where
merge( $x\#\#xs, y\#\#ys, f$ ) = (if  $f \ x \ y$  then  $x \ \# \ \text{merge}(xs, y\#\#ys, f)$ 
                             else  $y \ \# \ \text{merge}(x\#\#xs, ys, f)$ ) |
merge( $x\#\#xs, [], f$ ) =  $x \ \# \ \text{merge}(xs, [], f)$  |
merge( $[], y\#\#ys, f$ ) =  $y \ \# \ \text{merge}([], ys, f)$  |
merge( $[], [], f$ ) = []

```

```

lemma imp-disjCL: ( $P|Q \longrightarrow R$ ) = ( $(P \longrightarrow R) \wedge (\sim P \longrightarrow Q \longrightarrow R)$ )
⟨proof⟩

```

```

declare disj-not1[simp del] imp-disjL[simp del] imp-disjCL[simp]

```

```

lemma VARS  $hd \ tl \ p \ q \ r \ s$ 
 $\{\text{List } tl \ p \ Ps \ \wedge \ \text{List } tl \ q \ Qs \ \wedge \ \text{set } Ps \ \cap \ \text{set } Qs = \{\}\ \wedge$ 
 $(p \neq \text{Null} \vee q \neq \text{Null})\}$ 
IF if  $q = \text{Null}$  then True else  $p \sim = \text{Null} \ \& \ p \hat{\cdot} hd \leq q \hat{\cdot} hd$ 
THEN  $r := p; p := p \hat{\cdot} tl$  ELSE  $r := q; q := q \hat{\cdot} tl$  FI;
 $s := r$ ;
WHILE  $p \neq \text{Null} \vee q \neq \text{Null}$ 
INV  $\{\exists rs \ ps \ qs. \text{Path } tl \ r \ rs \ s \ \wedge \ \text{List } tl \ p \ ps \ \wedge \ \text{List } tl \ q \ qs \ \wedge$ 
 $\text{distinct}(s \ \# \ ps \ @ \ qs \ @ \ rs) \ \wedge \ s \neq \text{Null} \ \wedge$ 
 $\text{merge}(Ps, Qs, \lambda x \ y. \text{hd } x \leq \text{hd } y) =$ 
 $rs \ @ \ s \ \# \ \text{merge}(ps, qs, \lambda x \ y. \text{hd } x \leq \text{hd } y) \ \wedge$ 
 $(tl \ s = p \vee tl \ s = q)\}$ 
DO IF if  $q = \text{Null}$  then True else  $p \neq \text{Null} \ \wedge \ p \hat{\cdot} hd \leq q \hat{\cdot} hd$ 
THEN  $s \hat{\cdot} tl := p; p := p \hat{\cdot} tl$  ELSE  $s \hat{\cdot} tl := q; q := q \hat{\cdot} tl$  FI;
 $s := s \hat{\cdot} tl$ 
OD
 $\{\text{List } tl \ r \ (\text{merge}(Ps, Qs, \lambda x \ y. \text{hd } x \leq \text{hd } y))\}$ 
⟨proof⟩

```

#### 9.4.4 Storage allocation

**definition**  $new :: 'a \text{ set} \Rightarrow 'a::\text{ref}$   
**where**  $new\ A = (\text{SOME } a. a \notin A \ \& \ a \neq \text{Null})$

**lemma** *new-notin*:

$\llbracket \sim \text{finite}(\text{UNIV}::('a::\text{ref})\text{set}); \text{finite}(A::'a \text{ set}); B \subseteq A \rrbracket \Longrightarrow$   
 $new\ (A) \notin B \ \& \ new\ A \neq \text{Null}$   
*<proof>*

**lemma**  $\sim \text{finite}(\text{UNIV}::('a::\text{ref})\text{set}) \Longrightarrow$

$\text{VARS } xs \ \text{elem } next \ \text{alloc } p \ q$   
 $\{Xs = xs \wedge p = (\text{Null}::'a)\}$   
 $\text{WHILE } xs \neq []$   
 $\text{INV } \{islist\ next\ p \wedge \text{set}(\text{list } next\ p) \subseteq \text{set } alloc \wedge$   
 $\text{map } elem\ (\text{rev}(\text{list } next\ p)) \ @ \ xs = Xs\}$   
 $\text{DO } q := \text{new}(\text{set } alloc); \text{alloc} := q\#\text{alloc};$   
 $q^\wedge.\text{next} := p; q^\wedge.\text{elem} := \text{hd } xs; xs := \text{tl } xs; p := q$   
 $\text{OD}$   
 $\{islist\ next\ p \wedge \text{map } elem\ (\text{rev}(\text{list } next\ p)) = Xs\}$   
*<proof>*

**end**

## 10 Pointers, heaps and heap abstractions

See the paper by Mehta and Nipkow.

**theory** *Heap*  
**imports** *Main*  
**begin**

### 10.1 References

**datatype**  $'a \ \text{ref} = \text{Null} \mid \text{Ref } 'a$

**lemma** *not-Null-eq [iff]*:  $(x \neq \text{Null}) = (\exists y. x = \text{Ref } y)$   
*<proof>*

**lemma** *not-Ref-eq [iff]*:  $(\forall y. x \neq \text{Ref } y) = (x = \text{Null})$   
*<proof>*

**primrec**  $addr :: 'a \ \text{ref} \Rightarrow 'a$  **where**  
 $addr\ (\text{Ref } a) = a$

## 10.2 The heap

### 10.2.1 Paths in the heap

**primrec** *Path* :: ('a ⇒ 'a ref) ⇒ 'a ref ⇒ 'a list ⇒ 'a ref ⇒ bool **where**  
 $Path\ h\ x\ []\ y \longleftrightarrow x = y$   
 $| Path\ h\ x\ (a\#\ as)\ y \longleftrightarrow x = Ref\ a \wedge Path\ h\ (h\ a)\ as\ y$

**lemma** [*iff*]:  $Path\ h\ Null\ xs\ y = (xs = [] \wedge y = Null)$   
 $\langle proof \rangle$

**lemma** [*simp*]:  $Path\ h\ (Ref\ a)\ as\ z =$   
 $(as = [] \wedge z = Ref\ a \vee (\exists bs. as = a\#\ bs \wedge Path\ h\ (h\ a)\ bs\ z))$   
 $\langle proof \rangle$

**lemma** [*simp*]:  $\bigwedge x. Path\ f\ x\ (as\@\ bs)\ z = (\exists y. Path\ f\ x\ as\ y \wedge Path\ f\ y\ bs\ z)$   
 $\langle proof \rangle$

**lemma** *Path-upd*[*simp*]:  
 $\bigwedge x. u \notin set\ as \implies Path\ (f(u := v))\ x\ as\ y = Path\ f\ x\ as\ y$   
 $\langle proof \rangle$

**lemma** *Path-snoc*:  
 $Path\ (f(a := q))\ p\ as\ (Ref\ a) \implies Path\ (f(a := q))\ p\ (as\ @\ [a])\ q$   
 $\langle proof \rangle$

### 10.2.2 Non-repeating paths

**definition** *distPath* :: ('a ⇒ 'a ref) ⇒ 'a ref ⇒ 'a list ⇒ 'a ref ⇒ bool  
**where**  $distPath\ h\ x\ as\ y \longleftrightarrow Path\ h\ x\ as\ y \wedge distinct\ as$

The term  $distPath\ h\ x\ as\ y$  expresses the fact that a non-repeating path  $as$  connects location  $x$  to location  $y$  by means of the  $h$  field. In the case where  $x = y$ , and there is a cycle from  $x$  to itself,  $as$  can be both  $[]$  and the non-repeating list of nodes in the cycle.

**lemma** *neg-dP*:  $p \neq q \implies Path\ h\ p\ Ps\ q \implies distinct\ Ps \implies$   
 $\exists a\ Qs. p = Ref\ a \wedge Ps = a\#\ Qs \wedge a \notin set\ Qs$   
 $\langle proof \rangle$

**lemma** *neg-dP-disp*:  $\llbracket p \neq q; distPath\ h\ p\ Ps\ q \rrbracket \implies$   
 $\exists a\ Qs. p = Ref\ a \wedge Ps = a\#\ Qs \wedge a \notin set\ Qs$   
 $\langle proof \rangle$

### 10.2.3 Lists on the heap

**Relational abstraction definition** *List* :: ('a ⇒ 'a ref) ⇒ 'a ref ⇒ 'a list  
 $\implies bool$   
**where**  $List\ h\ x\ as = Path\ h\ x\ as\ Null$

**lemma** *[simp]*:  $List\ h\ x\ [] = (x = Null)$   
 $\langle proof \rangle$

**lemma** *[simp]*:  $List\ h\ x\ (a\#\!as) = (x = Ref\ a \wedge List\ h\ (h\ a)\ as)$   
 $\langle proof \rangle$

**lemma** *[simp]*:  $List\ h\ Null\ as = (as = [])$   
 $\langle proof \rangle$

**lemma** *List-Ref[simp]*:  $List\ h\ (Ref\ a)\ as = (\exists\ bs.\ as = a\#\!bs \wedge List\ h\ (h\ a)\ bs)$   
 $\langle proof \rangle$

**theorem** *notin-List-update[simp]*:  
 $\bigwedge x.\ a \notin set\ as \implies List\ (h(a := y))\ x\ as = List\ h\ x\ as$   
 $\langle proof \rangle$

**lemma** *List-unique*:  $\bigwedge x\ bs.\ List\ h\ x\ as \implies List\ h\ x\ bs \implies as = bs$   
 $\langle proof \rangle$

**lemma** *List-unique1*:  $List\ h\ p\ as \implies \exists! as.\ List\ h\ p\ as$   
 $\langle proof \rangle$

**lemma** *List-app*:  $\bigwedge x.\ List\ h\ x\ (as@bs) = (\exists y.\ Path\ h\ x\ as\ y \wedge List\ h\ y\ bs)$   
 $\langle proof \rangle$

**lemma** *List-hd-not-in-tl[simp]*:  $List\ h\ (h\ a)\ as \implies a \notin set\ as$   
 $\langle proof \rangle$

**lemma** *List-distinct[simp]*:  $\bigwedge x.\ List\ h\ x\ as \implies distinct\ as$   
 $\langle proof \rangle$

**lemma** *Path-is-List*:  
 $\llbracket Path\ h\ b\ Ps\ (Ref\ a); a \notin set\ Ps \rrbracket \implies List\ (h(a := Null))\ b\ (Ps\ @\ [a])$   
 $\langle proof \rangle$

#### 10.2.4 Functional abstraction

**definition** *islist* ::  $('a \Rightarrow 'a\ ref) \Rightarrow 'a\ ref \Rightarrow bool$   
**where**  $islist\ h\ p \longleftrightarrow (\exists as.\ List\ h\ p\ as)$

**definition** *list* ::  $('a \Rightarrow 'a\ ref) \Rightarrow 'a\ ref \Rightarrow 'a\ list$   
**where**  $list\ h\ p = (SOME\ as.\ List\ h\ p\ as)$

**lemma** *List-conv-islist-list*:  $List\ h\ p\ as = (islist\ h\ p \wedge as = list\ h\ p)$   
 $\langle proof \rangle$

**lemma** *[simp]*:  $islist\ h\ Null$   
 $\langle proof \rangle$

**lemma** [simp]:  $islist\ h\ (Ref\ a) = islist\ h\ (h\ a)$   
<proof>

**lemma** [simp]:  $list\ h\ Null = []$   
<proof>

**lemma** list-Ref-conv[simp]:  
 $islist\ h\ (h\ a) \implies list\ h\ (Ref\ a) = a \# list\ h\ (h\ a)$   
<proof>

**lemma** [simp]:  $islist\ h\ (h\ a) \implies a \notin set(list\ h\ (h\ a))$   
<proof>

**lemma** list-upd-conv[simp]:  
 $islist\ h\ p \implies y \notin set(list\ h\ p) \implies list\ (h(y := q))\ p = list\ h\ p$   
<proof>

**lemma** islist-upd[simp]:  
 $islist\ h\ p \implies y \notin set(list\ h\ p) \implies islist\ (h(y := q))\ p$   
<proof>

end

## 11 Heap syntax

**theory** HeapSyntax  
  **imports** Hoare-Logic Heap  
**begin**

### 11.1 Field access and update

**syntax**

-refupdate ::  $(a \Rightarrow b) \Rightarrow a\ ref \Rightarrow b \Rightarrow (a \Rightarrow b)$   
   $(-\)'((- \rightarrow -)') [1000,0] 900)$   
-fassign ::  $a\ ref \Rightarrow id \Rightarrow v \Rightarrow s\ com$   
   $((2-\hat{\cdot} := / -) [70,1000,65] 61)$   
-faccess ::  $a\ ref \Rightarrow (a\ ref \Rightarrow v) \Rightarrow v$   
   $(-\hat{\cdot} - [65,1000] 65)$

**translations**

$f(r \rightarrow v) == f(CONST\ addr\ r := v)$   
 $p\hat{\cdot}f := e \Rightarrow f := f(p \rightarrow e)$   
 $p\hat{\cdot}f \Rightarrow f(CONST\ addr\ p)$

**declare** fun-upd-apply[simp del] fun-upd-same[simp] fun-upd-other[simp]

An example due to Suzuki:

**lemma** VARS  $v\ n$   
 $\{w = Ref\ w0 \ \& \ x = Ref\ x0 \ \& \ y = Ref\ y0 \ \& \ z = Ref\ z0 \ \&$

```

    distinct[w0,x0,y0,z0]
    w.v := (1::int); w.n := x;
    x.v := 2; x.n := y;
    y.v := 3; y.n := z;
    z.v := 4; x.n := z
    {w.n.n.v = 4}
  <proof>

```

end

## 12 Examples of verifications of pointer programs

```

theory Pointer-Examples
  imports HeapSyntax
begin

```

axiomatization where *unproven*: *PROP A*

### 12.1 Verifications

#### 12.1.1 List reversal

A short but unreadable proof:

```

lemma VARS tl p q r
  {List tl p Ps ∧ List tl q Qs ∧ set Ps ∩ set Qs = {}}
  WHILE p ≠ Null
  INV {∃ ps qs. List tl p ps ∧ List tl q qs ∧ set ps ∩ set qs = {} ∧
      rev ps @ qs = rev Ps @ Qs}
  DO r := p; p := p.tl; r.tl := q; q := r OD
  {List tl q (rev Ps @ Qs)}
<proof>

```

And now with ghost variables *ps* and *qs*. Even “more automatic”.

```

lemma VARS next p ps q qs r
  {List next p Ps ∧ List next q Qs ∧ set Ps ∩ set Qs = {} ∧
   ps = Ps ∧ qs = Qs}
  WHILE p ≠ Null
  INV {List next p ps ∧ List next q qs ∧ set ps ∩ set qs = {} ∧
      rev ps @ qs = rev Ps @ Qs}
  DO r := p; p := p.next; r.next := q; q := r;
   qs := (hd ps) # qs; ps := tl ps OD
  {List next q (rev Ps @ Qs)}
<proof>

```

A longer readable version:

```

lemma VARS tl p q r
  {List tl p Ps ∧ List tl q Qs ∧ set Ps ∩ set Qs = {}}
  WHILE p ≠ Null
  INV {∃ ps qs. List tl p ps ∧ List tl q qs ∧ set ps ∩ set qs = {} ∧

```

$$\begin{aligned} & \text{rev } ps @ qs = \text{rev } Ps @ Qs \} \\ DO \ r := p; p := p.^{tl}; r.^{tl} := q; q := r \ OD \\ & \{List \ tl \ q \ (\text{rev } Ps @ Qs)\} \\ \langle proof \rangle \end{aligned}$$

Finally, the functional version. A bit more verbose, but automatic!

**lemma** *VARs*  $tl \ p \ q \ r$   
 $\{islist \ tl \ p \wedge islist \ tl \ q \wedge$   
 $Ps = list \ tl \ p \wedge Qs = list \ tl \ q \wedge set \ Ps \cap set \ Qs = \{\}\}$   
 $WHILE \ p \neq Null$   
 $INV \ \{islist \ tl \ p \wedge islist \ tl \ q \wedge$   
 $set(list \ tl \ p) \cap set(list \ tl \ q) = \{\} \wedge$   
 $rev(list \ tl \ p) @ (list \ tl \ q) = rev \ Ps @ Qs\}$   
 $DO \ r := p; p := p.^{tl}; r.^{tl} := q; q := r \ OD$   
 $\{islist \ tl \ q \wedge list \ tl \ q = rev \ Ps @ Qs\}$   
 $\langle proof \rangle$

### 12.1.2 Searching in a list

What follows is a sequence of successively more intelligent proofs that a simple loop finds an element in a linked list.

We start with a proof based on the *List* predicate. This means it only works for acyclic lists.

**lemma** *VARs*  $tl \ p$   
 $\{List \ tl \ p \ Ps \wedge X \in set \ Ps\}$   
 $WHILE \ p \neq Null \wedge p \neq Ref \ X$   
 $INV \ \{\exists ps. List \ tl \ p \ ps \wedge X \in set \ ps\}$   
 $DO \ p := p.^{tl} \ OD$   
 $\{p = Ref \ X\}$   
 $\langle proof \rangle$

Using *Path* instead of *List* generalizes the correctness statement to cyclic lists as well:

**lemma** *VARs*  $tl \ p$   
 $\{Path \ tl \ p \ Ps \ X\}$   
 $WHILE \ p \neq Null \wedge p \neq X$   
 $INV \ \{\exists ps. Path \ tl \ p \ ps \ X\}$   
 $DO \ p := p.^{tl} \ OD$   
 $\{p = X\}$   
 $\langle proof \rangle$

Now it dawns on us that we do not need the list witness at all — it suffices to talk about reachability, i.e. we can use relations directly. The first version uses a relation on 'a *ref*:

**lemma** *VARs*  $tl \ p$   
 $\{(p, X) \in \{(Ref \ x, tl \ x) \mid x. True\}^*\}$   
 $WHILE \ p \neq Null \wedge p \neq X$   
 $INV \ \{(p, X) \in \{(Ref \ x, tl \ x) \mid x. True\}^*\}$





**lemma** [simp]:  $(\{\} = A \cap (B \cup C)) = (\{\} = A \cap B \ \& \ \{\} = A \cap C)$   
 <proof>

**lemma** VARS hd tl p q r s  
 $\{List\ tl\ p\ Ps \wedge List\ tl\ q\ Qs \wedge set\ Ps \cap set\ Qs = \{\} \wedge$   
 $(p \neq Null \vee q \neq Null)\}$   
 IF cor  $(q = Null)$  (cand  $(p \neq Null)$   $(p \hat{.}hd \leq q \hat{.}hd)$ )  
 THEN  $r := p; p := p \hat{.}tl$  ELSE  $r := q; q := q \hat{.}tl$  FI;  
 $s := r;$   
 WHILE  $p \neq Null \vee q \neq Null$   
 INV  $\{\exists rs\ ps\ qs\ a. Path\ tl\ r\ rs\ s \wedge List\ tl\ p\ ps \wedge List\ tl\ q\ qs \wedge$   
 $distinct(a \# ps \ @ \ qs \ @ \ rs) \wedge s = Ref\ a \wedge$   
 $merge(Ps, Qs, \lambda x\ y. hd\ x \leq hd\ y) =$   
 $rs \ @ \ a \ \# \ merge(ps, qs, \lambda x\ y. hd\ x \leq hd\ y) \wedge$   
 $(tl\ a = p \vee tl\ a = q)\}$   
 DO IF cor  $(q = Null)$  (cand  $(p \neq Null)$   $(p \hat{.}hd \leq q \hat{.}hd)$ )  
 THEN  $s \hat{.}tl := p; p := p \hat{.}tl$  ELSE  $s \hat{.}tl := q; q := q \hat{.}tl$  FI;  
 $s := s \hat{.}tl$   
 OD  
 $\{List\ tl\ r\ (merge(Ps, Qs, \lambda x\ y. hd\ x \leq hd\ y))\}$   
 <proof>

And now with ghost variables:

**lemma** VARS elem next p q r s ps qs rs a  
 $\{List\ next\ p\ Ps \wedge List\ next\ q\ Qs \wedge set\ Ps \cap set\ Qs = \{\} \wedge$   
 $(p \neq Null \vee q \neq Null) \wedge ps = Ps \wedge qs = Qs\}$   
 IF cor  $(q = Null)$  (cand  $(p \neq Null)$   $(p \hat{.}elem \leq q \hat{.}elem)$ )  
 THEN  $r := p; p := p \hat{.}next; ps := tl\ ps$   
 ELSE  $r := q; q := q \hat{.}next; qs := tl\ qs$  FI;  
 $s := r; rs := []; a := addr\ s;$   
 WHILE  $p \neq Null \vee q \neq Null$   
 INV  $\{Path\ next\ r\ rs\ s \wedge List\ next\ p\ ps \wedge List\ next\ q\ qs \wedge$   
 $distinct(a \# ps \ @ \ qs \ @ \ rs) \wedge s = Ref\ a \wedge$   
 $merge(Ps, Qs, \lambda x\ y. elem\ x \leq elem\ y) =$   
 $rs \ @ \ a \ \# \ merge(ps, qs, \lambda x\ y. elem\ x \leq elem\ y) \wedge$   
 $(next\ a = p \vee next\ a = q)\}$   
 DO IF cor  $(q = Null)$  (cand  $(p \neq Null)$   $(p \hat{.}elem \leq q \hat{.}elem)$ )  
 THEN  $s \hat{.}next := p; p := p \hat{.}next; ps := tl\ ps$   
 ELSE  $s \hat{.}next := q; q := q \hat{.}next; qs := tl\ qs$  FI;  
 $rs := rs \ @ \ [a]; s := s \hat{.}next; a := addr\ s$   
 OD  
 $\{List\ next\ r\ (merge(Ps, Qs, \lambda x\ y. elem\ x \leq elem\ y))\}$   
 <proof>

The proof is a LOT simpler because it does not need instantiations anymore, but it is still not quite automatic, probably because of this wrong orientation business.

More of the previous proof without ghost variables can be automated, but the runtime goes up drastically. In general it is usually more efficient

to give the witness directly than to have it found by proof.

Now we try a functional version of the abstraction relation *Path*. Since the result is not that convincing, we do not prove any of the lemmas.

#### axiomatization

*ispath* :: ('a ⇒ 'a ref) ⇒ 'a ref ⇒ 'a ref ⇒ bool **and**  
*path* :: ('a ⇒ 'a ref) ⇒ 'a ref ⇒ 'a ref ⇒ 'a list

First some basic lemmas:

**lemma** [*simp*]: *ispath* *f* *p* *p*

⟨*proof*⟩

**lemma** [*simp*]: *path* *f* *p* *p* = []

⟨*proof*⟩

**lemma** [*simp*]: *ispath* *f* *p* *q* ⇒ *a* ∉ *set*(*path* *f* *p* *q*) ⇒ *ispath* (*f* (*a* := *r*)) *p* *q*

⟨*proof*⟩

**lemma** [*simp*]: *ispath* *f* *p* *q* ⇒ *a* ∉ *set*(*path* *f* *p* *q*) ⇒

*path* (*f* (*a* := *r*)) *p* *q* = *path* *f* *p* *q*

⟨*proof*⟩

Some more specific lemmas needed by the example:

**lemma** [*simp*]: *ispath* (*f* (*a* := *q*)) *p* (*Ref* *a*) ⇒ *ispath* (*f* (*a* := *q*)) *p* *q*

⟨*proof*⟩

**lemma** [*simp*]: *ispath* (*f* (*a* := *q*)) *p* (*Ref* *a*) ⇒

*path* (*f* (*a* := *q*)) *p* *q* = *path* (*f* (*a* := *q*)) *p* (*Ref* *a*) @ [*a*]

⟨*proof*⟩

**lemma** [*simp*]: *ispath* *f* *p* (*Ref* *a*) ⇒ *f* *a* = *Ref* *b* ⇒

*b* ∉ *set* (*path* *f* *p* (*Ref* *a*))

⟨*proof*⟩

**lemma** [*simp*]: *ispath* *f* *p* (*Ref* *a*) ⇒ *f* *a* = *Null* ⇒ *islist* *f* *p*

⟨*proof*⟩

**lemma** [*simp*]: *ispath* *f* *p* (*Ref* *a*) ⇒ *f* *a* = *Null* ⇒ *list* *f* *p* = *path* *f* *p* (*Ref* *a*) @

[*a*]

⟨*proof*⟩

**lemma** [*simp*]: *islist* *f* *p* ⇒ *distinct* (*list* *f* *p*)

⟨*proof*⟩

**lemma** *VARs* *hd* *tl* *p* *q* *r* *s*

{ *islist* *tl* *p* ∧ *Ps* = *list* *tl* *p* ∧ *islist* *tl* *q* ∧ *Qs* = *list* *tl* *q* ∧

*set* *Ps* ∩ *set* *Qs* = {} ∧

(*p* ≠ *Null* ∨ *q* ≠ *Null*) }

*IF* *cor* (*q* = *Null*) (*cand* (*p* ≠ *Null*) (*p*^*hd* ≤ *q*^*hd*))

*THEN* *r* := *p*; *p* := *p*^*tl* *ELSE* *r* := *q*; *q* := *q*^*tl* *FI*;

*s* := *r*;

*WHILE* *p* ≠ *Null* ∨ *q* ≠ *Null*

*INV* { ∃ *rs* *ps* *qs* *a*. *ispath* *tl* *r* *s* ∧ *rs* = *path* *tl* *r* *s* ∧

*islist* *tl* *p* ∧ *ps* = *list* *tl* *p* ∧ *islist* *tl* *q* ∧ *qs* = *list* *tl* *q* ∧

*distinct*(*a* # *ps* @ *qs* @ *rs*) ∧ *s* = *Ref* *a* ∧

*merge*(*Ps*, *Qs*, λ*x* *y*. *hd* *x* ≤ *hd* *y*) =

*rs* @ *a* # *merge*(*ps*, *qs*, λ*x* *y*. *hd* *x* ≤ *hd* *y*) ∧

```

      (tl a = p ∨ tl a = q)}
DO IF cor (q = Null) (cand (p ≠ Null) (p^.hd ≤ q^.hd))
  THEN s^.tl := p; p := p^.tl ELSE s^.tl := q; q := q^.tl FI;
  s := s^.tl
OD
{islist tl r & list tl r = (merge(Ps, Qs, λx y. hd x ≤ hd y))}
⟨proof⟩

```

The proof is automatic, but requires a number of special lemmas.

### 12.1.5 Cyclic list reversal

We consider two algorithms for the reversal of circular lists.

**lemma** *circular-list-rev-I*:

```

  VARS next root p q tmp
  {root = Ref r ∧ distPath next root (r#Ps) root}
  p := root; q := root^.next;
  WHILE q ≠ root
  INV {∃ ps qs. distPath next p ps root ∧ distPath next q qs root ∧
      root = Ref r ∧ r ∉ set Ps ∧ set ps ∩ set qs = {} ∧
      Ps = (rev ps) @ qs }
  DO tmp := q; q := q^.next; tmp^.next := p; p:=tmp OD;
  root^.next := p
  { root = Ref r ∧ distPath next root (r#rev Ps) root }
⟨proof⟩

```

In the beginning, we are able to assert *distPath next root as root*, with *as* set to [] or [r, a, b, c]. Note that *Path next root as root* would additionally give us an infinite number of lists with the recurring sequence [r, a, b, c].

The precondition states that there exists a non-empty non-repeating path  $r \# Ps$  from pointer *root* to itself, given that *root* points to location *r*. Pointers *p* and *q* are then set to *root* and the successor of *root* respectively. If  $q = root$ , we have circled the loop, otherwise we set the *next* pointer field of *q* to point to *p*, and shift *p* and *q* one step forward. The invariant thus states that *p* and *q* point to two disjoint lists *ps* and *qs*, such that  $Ps = rev\ ps \ @ \ qs$ . After the loop terminates, one extra step is needed to close the loop. As expected, the postcondition states that the *distPath* from *root* to itself is now  $r \# rev\ Ps$ .

It may come as a surprise to the reader that the simple algorithm for acyclic list reversal, with modified annotations, works for cyclic lists as well:

**lemma** *circular-list-rev-II*:

```

  VARS next p q tmp
  {p = Ref r ∧ distPath next p (r#Ps) p}
  q:=Null;
  WHILE p ≠ Null
  INV
  { ((q = Null) → (∃ ps. distPath next p (ps) (Ref r) ∧ ps = r#Ps)) ∧

```

$((q \neq \text{Null}) \longrightarrow (\exists ps\ qs. \text{distPath next } q (qs) (\text{Ref } r) \wedge \text{List next } p\ ps \wedge$   
 $\quad \text{set } ps \cap \text{set } qs = \{\} \wedge \text{rev } qs @ ps = Ps@[r])) \wedge$   
 $\neg (p = \text{Null} \wedge q = \text{Null}) \}$   
*DO*  $tmp := p; p := p.\hat{\text{next}}; tmp.\hat{\text{next}} := q; q := tmp$  *OD*  
 $\{q = \text{Ref } r \wedge \text{distPath next } q (r \# \text{rev } Ps) q\}$   
*<proof>*

### 12.1.6 Storage allocation

**definition**  $\text{new} :: 'a \text{ set} \Rightarrow 'a$   
**where**  $\text{new } A = (\text{SOME } a. a \notin A)$

**lemma** *new-notin*:

$\llbracket \sim \text{finite}(\text{UNIV}::'a \text{ set}); \text{finite}(A::'a \text{ set}); B \subseteq A \rrbracket \Longrightarrow \text{new } (A) \notin B$   
*<proof>*

**lemma**  $\sim \text{finite}(\text{UNIV}::'a \text{ set}) \Longrightarrow$

$\text{VARS } xs \text{ elem next alloc } p\ q$   
 $\{Xs = xs \wedge p = (\text{Null}::'a \text{ ref})\}$   
 $\text{WHILE } xs \neq []$   
 $\text{INV } \{\text{islist next } p \wedge \text{set}(\text{list next } p) \subseteq \text{set alloc} \wedge$   
 $\quad \text{map elem } (\text{rev}(\text{list next } p)) @ xs = Xs\}$   
 $\text{DO } q := \text{Ref}(\text{new}(\text{set alloc})); \text{alloc} := (\text{addr } q)\#\text{alloc};$   
 $\quad q.\hat{\text{next}} := p; q.\hat{\text{elem}} := \text{hd } xs; xs := \text{tl } xs; p := q$   
 $\text{OD}$   
 $\{\text{islist next } p \wedge \text{map elem } (\text{rev}(\text{list next } p)) = Xs\}$   
*<proof>*

**end**

## 13 Heap syntax (abort)

**theory** *HeapSyntaxAbort*  
**imports** *Hoare-Logic-Abort Heap*  
**begin**

### 13.1 Field access and update

Heap update  $p.\hat{h} := e$  is now guarded against  $p$  being `Null`. However,  $p$  may still be illegal, e.g. uninitialized or dangling. To guard against that, one needs a more detailed model of the heap where allocated and free addresses are distinguished, e.g. by making the heap a map, or by carrying the set of free addresses around. This is needed anyway as soon as we want to reason about storage allocation/deallocation.

**syntax**

```

-refupdate :: ('a => 'b) => 'a ref => 'b => ('a => 'b)
  (-/'((- -> -)') [1000,0] 900)
-fassign :: 'a ref => id => 'v => 's com
  ((2-^- := / -) [70,1000,65] 61)
-faccess :: 'a ref => ('a ref => 'v) => 'v
  (-^- [65,1000] 65)

```

**translations**

```

-refupdate f r v == f(CONST addr r := v)
p^.f := e => (p ≠ CONST Null) → (f := -refupdate f p e)
p^.f => f(CONST addr p)

```

**declare** *fun-upd-apply*[simp del] *fun-upd-same*[simp] *fun-upd-other*[simp]

An example due to Suzuki:

```

lemma VARS v n
  {w = Ref w0 & x = Ref x0 & y = Ref y0 & z = Ref z0 &
   distinct[w0,x0,y0,z0]}
  w^.v := (1::int); w^.n := x;
  x^.v := 2; x^.n := y;
  y^.v := 3; y^.n := z;
  z^.v := 4; x^.n := z
  {w^.n^.n^.v = 4}
⟨proof⟩

end

```

## 14 Examples of verifications of pointer programs

```

theory Pointer-ExamplesAbort
  imports HeapSyntaxAbort
begin

```

### 14.1 Verifications

#### 14.1.1 List reversal

Interestingly, this proof is the same as for the unguarded program:

```

lemma VARS tl p q r
  {List tl p Ps ∧ List tl q Qs ∧ set Ps ∩ set Qs = {}}
  WHILE p ≠ Null
  INV {∃ ps qs. List tl p ps ∧ List tl q qs ∧ set ps ∩ set qs = {}} ∧
      rev ps @ qs = rev Ps @ Qs}
  DO r := p; (p ≠ Null → p := p^.tl); r^.tl := q; q := r OD
  {List tl q (rev Ps @ Qs)}
⟨proof⟩

end

```

## 15 Proof of the Schorr-Waite graph marking algorithm

```
theory SchorrWaite
  imports HeapSyntax
begin
```

### 15.1 Machinery for the Schorr-Waite proof

#### definition

— Relations induced by a mapping  
 $rel :: ('a \Rightarrow 'a\ ref) \Rightarrow ('a \times 'a)\ set$   
**where**  $rel\ m = \{(x,y). m\ x = Ref\ y\}$

#### definition

$relS :: ('a \Rightarrow 'a\ ref)\ set \Rightarrow ('a \times 'a)\ set$   
**where**  $relS\ M = (\bigcup m \in M. rel\ m)$

#### definition

$addrs :: 'a\ ref\ set \Rightarrow 'a\ set$   
**where**  $addrs\ P = \{a. Ref\ a \in P\}$

#### definition

$reachable :: ('a \times 'a)\ set \Rightarrow 'a\ ref\ set \Rightarrow 'a\ set$   
**where**  $reachable\ r\ P = (r^* \text{ `` } addrs\ P)$

**lemmas**  $rel-defs = relS-def\ rel-def$

Rewrite rules for relations induced by a mapping

**lemma**  $self-reachable: b \in B \Longrightarrow b \in R^* \text{ `` } B$   
 $\langle proof \rangle$

**lemma**  $oneStep-reachable: b \in R \text{ `` } B \Longrightarrow b \in R^* \text{ `` } B$   
 $\langle proof \rangle$

**lemma**  $still-reachable: \llbracket B \subseteq Ra^* \text{ `` } A; \forall (x,y) \in Rb-Ra. y \in (Ra^* \text{ `` } A) \rrbracket \Longrightarrow Rb^* \text{ `` } B \subseteq Ra^* \text{ `` } A$   
 $\langle proof \rangle$

**lemma**  $still-reachable-eq: \llbracket A \subseteq Rb^* \text{ `` } B; B \subseteq Ra^* \text{ `` } A; \forall (x,y) \in Ra-Rb. y \in (Rb^* \text{ `` } B); \forall (x,y) \in Rb-Ra. y \in (Ra^* \text{ `` } A) \rrbracket \Longrightarrow Ra^* \text{ `` } A = Rb^* \text{ `` } B$   
 $\langle proof \rangle$

**lemma**  $reachable-null: reachable\ mS\ \{Null\} = \{\}$   
 $\langle proof \rangle$

**lemma**  $reachable-empty: reachable\ mS\ \{\} = \{\}$   
 $\langle proof \rangle$

**lemma** *reachable-union*:  $(\text{reachable } mS \ aS \cup \text{reachable } mS \ bS) = \text{reachable } mS \ (aS \cup bS)$   
 $\langle \text{proof} \rangle$

**lemma** *reachable-union-sym*:  $\text{reachable } r \ (\text{insert } a \ aS) = (r^* \ \text{“ } \text{addrs } \{a\} \text{”}) \cup \text{reachable } r \ aS$   
 $\langle \text{proof} \rangle$

**lemma** *rel-upd1*:  $(a,b) \notin \text{rel } (r(q:=t)) \implies (a,b) \in \text{rel } r \implies a=q$   
 $\langle \text{proof} \rangle$

**lemma** *rel-upd2*:  $(a,b) \notin \text{rel } r \implies (a,b) \in \text{rel } (r(q:=t)) \implies a=q$   
 $\langle \text{proof} \rangle$

### definition

— Restriction of a relation

$\text{restr} :: ('a \times 'a) \text{ set} \Rightarrow ('a \Rightarrow \text{bool}) \Rightarrow ('a \times 'a) \text{ set} \quad ((-/ | -) [50, 51] 50)$

**where**  $\text{restr } r \ m = \{(x,y). (x,y) \in r \wedge \neg m \ x\}$

Rewrite rules for the restriction of a relation

**lemma** *restr-identity[simp]*:

$(\forall x. \neg m \ x) \implies (R | m) = R$

$\langle \text{proof} \rangle$

**lemma** *restr-rtrancl[simp]*:  $\llbracket m \ l \rrbracket \implies (R | m)^* \ \text{“ } \{l\} = \{l\}$

$\langle \text{proof} \rangle$

**lemma** *[simp]*:  $\llbracket m \ l \rrbracket \implies (l,x) \in (R | m)^* = (l=x)$

$\langle \text{proof} \rangle$

**lemma** *restr-upd*:  $((\text{rel } (r \ (q := t)) | (m(q := \text{True}))) = ((\text{rel } (r)) | (m(q := \text{True}))))$

$\langle \text{proof} \rangle$

**lemma** *restr-un*:  $((r \cup s) | m) = (r | m) \cup (s | m)$

$\langle \text{proof} \rangle$

**lemma** *rel-upd3*:  $(a, b) \notin (r | (m(q := t))) \implies (a,b) \in (r | m) \implies a = q$

$\langle \text{proof} \rangle$

### definition

— A short form for the stack mapping function for List

$S :: ('a \Rightarrow \text{bool}) \Rightarrow ('a \Rightarrow 'a \ \text{ref}) \Rightarrow ('a \Rightarrow 'a \ \text{ref}) \Rightarrow ('a \Rightarrow 'a \ \text{ref})$

**where**  $S \ c \ l \ r = (\lambda x. \text{if } c \ x \ \text{then } r \ x \ \text{else } l \ x)$

Rewrite rules for Lists using S as their mapping

**lemma** *[rule-format,simp]*:

$\forall p. \ a \notin \text{set } \text{stack} \longrightarrow \text{List } (S \ c \ l \ r) \ p \ \text{stack} = \text{List } (S \ (c(a:=x)) \ (l(a:=y)) \ (r(a:=z))) \ p \ \text{stack}$



$\langle \text{proof} \rangle$

**lemma**  $[rule\text{-}format, simp]$ :

$\forall p. a \notin \text{set } \text{stack} \longrightarrow \text{List } (S \ c \ l \ (r(a:=z))) \ p \ \text{stack} = \text{List } (S \ c \ l \ r) \ p \ \text{stack}$   
 $\langle \text{proof} \rangle$

**lemma**  $[rule\text{-}format, simp]$ :

$\forall p. a \notin \text{set } \text{stack} \longrightarrow \text{List } (S \ c \ (l(a:=z)) \ r) \ p \ \text{stack} = \text{List } (S \ c \ l \ r) \ p \ \text{stack}$   
 $\langle \text{proof} \rangle$

**lemma**  $[rule\text{-}format, simp]$ :

$\forall p. a \notin \text{set } \text{stack} \longrightarrow \text{List } (S \ (c(a:=z)) \ l \ r) \ p \ \text{stack} = \text{List } (S \ c \ l \ r) \ p \ \text{stack}$   
 $\langle \text{proof} \rangle$

**primrec**

— Recursive definition of what it means for a the graph/stack structure to be reconstructible

$\text{stkOk} :: ('a \Rightarrow \text{bool}) \Rightarrow ('a \Rightarrow 'a \ \text{ref}) \Rightarrow ('a \Rightarrow 'a \ \text{ref}) \Rightarrow ('a \Rightarrow 'a \ \text{ref}) \Rightarrow ('a \Rightarrow 'a \ \text{ref}) \Rightarrow 'a \ \text{ref} \Rightarrow 'a \ \text{list} \Rightarrow \text{bool}$

**where**

$\text{stkOk}\text{-nil}: \text{stkOk } c \ l \ r \ iL \ iR \ t \ [] = \text{True}$

|  $\text{stkOk}\text{-cons}$ :

$\text{stkOk } c \ l \ r \ iL \ iR \ t \ (p \# \text{stk}) = (\text{stkOk } c \ l \ r \ iL \ iR \ (\text{Ref } p) \ (\text{stk})) \wedge$

$iL \ p = (\text{if } c \ p \ \text{then } l \ p \ \text{else } t) \wedge$

$iR \ p = (\text{if } c \ p \ \text{then } t \ \text{else } r \ p)$

Rewrite rules for  $\text{stkOk}$

**lemma**  $[simp]$ :  $\bigwedge t. \llbracket x \notin \text{set } xs; \text{Ref } x \neq t \rrbracket \Longrightarrow$

$\text{stkOk } (c(x := f)) \ l \ r \ iL \ iR \ t \ xs = \text{stkOk } c \ l \ r \ iL \ iR \ t \ xs$   
 $\langle \text{proof} \rangle$

**lemma**  $[simp]$ :  $\bigwedge t. \llbracket x \notin \text{set } xs; \text{Ref } x \neq t \rrbracket \Longrightarrow$

$\text{stkOk } c \ (l(x := g)) \ r \ iL \ iR \ t \ xs = \text{stkOk } c \ l \ r \ iL \ iR \ t \ xs$   
 $\langle \text{proof} \rangle$

**lemma**  $[simp]$ :  $\bigwedge t. \llbracket x \notin \text{set } xs; \text{Ref } x \neq t \rrbracket \Longrightarrow$

$\text{stkOk } c \ l \ (r(x := g)) \ iL \ iR \ t \ xs = \text{stkOk } c \ l \ r \ iL \ iR \ t \ xs$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{stkOk}\text{-r}\text{-rewrite}$   $[simp]$ :  $\bigwedge x. x \notin \text{set } xs \Longrightarrow$

$\text{stkOk } c \ l \ (r(x := g)) \ iL \ iR \ (\text{Ref } x) \ xs = \text{stkOk } c \ l \ r \ iL \ iR \ (\text{Ref } x) \ xs$   
 $\langle \text{proof} \rangle$

**lemma**  $[simp]$ :  $\bigwedge x. x \notin \text{set } xs \Longrightarrow$

$\text{stkOk } c \ (l(x := g)) \ r \ iL \ iR \ (\text{Ref } x) \ xs = \text{stkOk } c \ l \ r \ iL \ iR \ (\text{Ref } x) \ xs$   
 $\langle \text{proof} \rangle$

**lemma**  $[simp]$ :  $\bigwedge x. x \notin \text{set } xs \Longrightarrow$

$\text{stkOk } (c(x := g)) \ l \ r \ iL \ iR \ (\text{Ref } x) \ xs = \text{stkOk } c \ l \ r \ iL \ iR \ (\text{Ref } x) \ xs$

*<proof>*

## 15.2 The Schorr-Waite algorithm

**theorem** *SchorrWaiteAlgorithm*:

*VARS*  $c\ m\ l\ r\ t\ p\ q\ root$

$\{R = \text{reachable}(\text{relS}\{l, r\})\ \{\text{root}\} \wedge (\forall x. \neg m\ x) \wedge iR = r \wedge iL = l\}$

$t := \text{root};\ p := \text{Null};$

*WHILE*  $p \neq \text{Null} \vee t \neq \text{Null} \wedge \neg t^{\wedge}.m$

*INV*  $\{\exists\ \text{stack}.$

$\text{List}(S\ c\ l\ r)\ p\ \text{stack} \wedge$

— *i1*

$(\forall x \in \text{set}\ \text{stack}. m\ x) \wedge$

— *i2*

$R = \text{reachable}(\text{relS}\{l, r\})\ \{t, p\} \wedge$

— *i3*

$(\forall x. x \in R \wedge \neg m\ x \longrightarrow$

— *i4*

$x \in \text{reachable}(\text{relS}\{l, r\}|m)\ (\{t\} \cup \text{set}(\text{map}\ r\ \text{stack}))) \wedge$

$(\forall x. m\ x \longrightarrow x \in R) \wedge$

— *i5*

$(\forall x. x \notin \text{set}\ \text{stack} \longrightarrow r\ x = iR\ x \wedge l\ x = iL\ x) \wedge$

— *i6*

$(\text{stkOk}\ c\ l\ r\ iL\ iR\ t\ \text{stack})$

— *i7*

*DO IF*  $t = \text{Null} \vee t^{\wedge}.m$

*THEN IF*  $p^{\wedge}.c$

*THEN*  $q := t; t := p; p := p^{\wedge}.r; t^{\wedge}.r := q$

— *pop*

*ELSE*  $q := t; t := p^{\wedge}.r; p^{\wedge}.r := p^{\wedge}.l;$

— *swing*

$p^{\wedge}.l := q; p^{\wedge}.c := \text{True}$  *FI*

*ELSE*  $q := p; p := t; t := t^{\wedge}.l; p^{\wedge}.l := q;$

— *push*

$p^{\wedge}.m := \text{True}; p^{\wedge}.c := \text{False}$  *FI*

*OD*

$\{(\forall x. (x \in R) = m\ x) \wedge (r = iR \wedge l = iL)\}$

*(is Valid*

$\{(c, m, l, r, t, p, q, \text{root}).\ ?\text{Pre}\ c\ m\ l\ r\ \text{root}\}$

$(\text{Seq} - (\text{Seq} - (\text{While}\ \{(c, m, l, r, t, p, q, \text{root}).\ ?\text{whileB}\ m\ t\ p\} -)))$

$(\text{Aseq} - (\text{Aseq} - (\text{Awhile}\ \{(c, m, l, r, t, p, q, \text{root}).\ ?\text{inv}\ c\ m\ l\ r\ t\ p\} - -))) -$

*<proof>*

**end**

## 16 Heap abstractions for Separation Logic

(at the moment only Path and List)

**theory** *SepLogHeap*

**imports** *Main*

**begin**

**type-synonym** *heap* =  $(\text{nat} \Rightarrow \text{nat}\ \text{option})$

*Some* means allocated, *None* means free. Address 0 serves as the null reference.

### 16.1 Paths in the heap

**primrec** *Path* ::  $\text{heap} \Rightarrow \text{nat} \Rightarrow \text{nat}\ \text{list} \Rightarrow \text{nat} \Rightarrow \text{bool}$

**where**

$Path\ h\ x\ []\ y = (x = y)$   
 $| Path\ h\ x\ (a\#\!as)\ y = (x\neq 0 \wedge a=x \wedge (\exists b. h\ x = Some\ b \wedge Path\ h\ b\ as\ y))$

**lemma** *[iff]*:  $Path\ h\ 0\ xs\ y = (xs = [] \wedge y = 0)$   
*<proof>*

**lemma** *[simp]*:  $x\neq 0 \implies Path\ h\ x\ as\ z =$   
 $(as = [] \wedge z = x \vee (\exists y\ bs. as = x\#\!bs \wedge h\ x = Some\ y \ \&\ Path\ h\ y\ bs\ z))$   
*<proof>*

**lemma** *[simp]*:  $\bigwedge x. Path\ f\ x\ (as@bs)\ z = (\exists y. Path\ f\ x\ as\ y \wedge Path\ f\ y\ bs\ z)$   
*<proof>*

**lemma** *Path-upd[simp]*:

$\bigwedge x. u \notin set\ as \implies Path\ (f(u := v))\ x\ as\ y = Path\ f\ x\ as\ y$   
*<proof>*

## 16.2 Lists on the heap

**definition**  $List :: heap \Rightarrow nat \Rightarrow nat\ list \Rightarrow bool$   
**where**  $List\ h\ x\ as = Path\ h\ x\ as\ 0$

**lemma** *[simp]*:  $List\ h\ x\ [] = (x = 0)$   
*<proof>*

**lemma** *[simp]*:

$List\ h\ x\ (a\#\!as) = (x\neq 0 \wedge a=x \wedge (\exists y. h\ x = Some\ y \wedge List\ h\ y\ as))$   
*<proof>*

**lemma** *[simp]*:  $List\ h\ 0\ as = (as = [])$   
*<proof>*

**lemma** *List-non-null*:  $a\neq 0 \implies$

$List\ h\ a\ as = (\exists b\ bs. as = a\#\!bs \wedge h\ a = Some\ b \wedge List\ h\ b\ bs)$   
*<proof>*

**theorem** *notin-List-update[simp]*:

$\bigwedge x. a \notin set\ as \implies List\ (h(a := y))\ x\ as = List\ h\ x\ as$   
*<proof>*

**lemma** *List-unique*:  $\bigwedge x\ bs. List\ h\ x\ as \implies List\ h\ x\ bs \implies as = bs$   
*<proof>*

**lemma** *List-unique1*:  $List\ h\ p\ as \implies \exists! as. List\ h\ p\ as$   
*<proof>*

**lemma** *List-app*:  $\bigwedge x. List\ h\ x\ (as@bs) = (\exists y. Path\ h\ x\ as\ y \wedge List\ h\ y\ bs)$   
*<proof>*

**lemma** *List-hd-not-in-tl*[simp]:  $List\ h\ b\ as \implies h\ a = Some\ b \implies a \notin set\ as$   
 <proof>

**lemma** *List-distinct*[simp]:  $\bigwedge x. List\ h\ x\ as \implies distinct\ as$   
 <proof>

**lemma** *list-in-heap*:  $\bigwedge p. List\ h\ p\ ps \implies set\ ps \subseteq dom\ h$   
 <proof>

**lemma** *list-ortho-sum1*[simp]:  
 $\bigwedge p. \llbracket List\ h1\ p\ ps; dom\ h1 \cap dom\ h2 = \{\} \rrbracket \implies List\ (h1++h2)\ p\ ps$   
 <proof>

**lemma** *list-ortho-sum2*[simp]:  
 $\bigwedge p. \llbracket List\ h2\ p\ ps; dom\ h1 \cap dom\ h2 = \{\} \rrbracket \implies List\ (h1++h2)\ p\ ps$   
 <proof>

end

## 17 Separation logic

**theory** *Separation*

**imports** *Hoare-Logic-Abort SepLogHeap*

**begin**

The semantic definition of a few connectives:

**definition** *ortho* ::  $heap \Rightarrow heap \Rightarrow bool$  (**infix**  $\perp$  55)  
**where**  $h1 \perp h2 \iff dom\ h1 \cap dom\ h2 = \{\}$

**definition** *is-empty* ::  $heap \Rightarrow bool$   
**where**  $is\_empty\ h \iff h = Map.empty$

**definition** *singl*::  $heap \Rightarrow nat \Rightarrow nat \Rightarrow bool$   
**where**  $singl\ h\ x\ y \iff dom\ h = \{x\} \ \&\ h\ x = Some\ y$

**definition** *star*::  $(heap \Rightarrow bool) \Rightarrow (heap \Rightarrow bool) \Rightarrow (heap \Rightarrow bool)$   
**where**  $star\ P\ Q = (\lambda h. \exists h1\ h2. h = h1++h2 \wedge h1 \perp h2 \wedge P\ h1 \wedge Q\ h2)$

**definition** *wand*::  $(heap \Rightarrow bool) \Rightarrow (heap \Rightarrow bool) \Rightarrow (heap \Rightarrow bool)$   
**where**  $wand\ P\ Q = (\lambda h. \forall h'. h' \perp h \wedge P\ h' \longrightarrow Q(h++h'))$

This is what assertions look like without any syntactic sugar:

**lemma** *VARS*  $x\ y\ z\ w\ h$   
 $\{star\ (\%h. singl\ h\ x\ y)\ (\%h. singl\ h\ z\ w)\ h\}$   
 SKIP  
 $\{x \neq z\}$   
 <proof>

Now we add nice input syntax. To suppress the heap parameter of the connectives, we assume it is always called  $H$  and add/remove it upon parsing/printing. Thus every pointer program needs to have a program variable  $H$ , and assertions should not contain any locally bound  $H$ s - otherwise they may bind the implicit  $H$ .

**syntax**

```
-emp :: bool (emp)
-singl :: nat => nat => bool ([- ↦ -])
-star :: bool => bool => bool (infixl ** 60)
-wand :: bool => bool => bool (infixl -* 60)
```

⟨ML⟩

Now it looks much better:

```
lemma VARS H x y z w
  {[x↦y] ** [z↦w]}
  SKIP
  {x ≠ z}
  ⟨proof⟩
```

```
lemma VARS H x y z w
  {emp ** emp}
  SKIP
  {emp}
  ⟨proof⟩
```

But the output is still unreadable. Thus we also strip the heap parameters upon output:

⟨ML⟩

Now the intermediate proof states are also readable:

```
lemma VARS H x y z w
  {[x↦y] ** [z↦w]}
  y := w
  {x ≠ z}
  ⟨proof⟩
```

```
lemma VARS H x y z w
  {emp ** emp}
  SKIP
  {emp}
  ⟨proof⟩
```

So far we have unfolded the separation logic connectives in proofs. Here comes a simple example of a program proof that uses a law of separation logic instead.

```
lemma star-comm: P ** Q = Q ** P
```

$\langle proof \rangle$

**lemma** *VARs*  $H x y z w$

$\{P ** Q\}$

*SKIP*

$\{Q ** P\}$

$\langle proof \rangle$

**lemma** *VARs*  $H$

$\{p \neq 0 \wedge [p \mapsto x] ** List\ H\ q\ qs\}$

$H := H(p \mapsto q)$

$\{List\ H\ p\ (p \# qs)\}$

$\langle proof \rangle$

**lemma** *VARs*  $H p q r$

$\{List\ H\ p\ Ps ** List\ H\ q\ Qs\}$

*WHILE*  $p \neq 0$

*INV*  $\{\exists ps\ qs. (List\ H\ p\ ps ** List\ H\ q\ qs) \wedge rev\ ps\ @\ qs = rev\ Ps\ @\ Qs\}$

*DO*  $r := p; p := the(H\ p); H := H(r \mapsto q); q := r\ OD$

$\{List\ H\ q\ (rev\ Ps\ @\ Qs)\}$

$\langle proof \rangle$

**end**

## References

- [1] Farhad Mehta and Tobias Nipkow. Proving pointer programs in higher-order logic. In F. Baader, editor, *Automated Deduction — CADE-19*, volume 2741 of *LNCS*, pages 121–135. Springer, 2003.
- [2] Farhad Mehta and Tobias Nipkow. Proving pointer programs in higher-order logic. *Information and Computation*, 199:200–227, 2005.