Can Punctured Rate-1/2 Turbo Codes Achieve a Lower Error Floor than their Rate-1/3 Parent Codes?

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***Abstract*— In this paper we concentrate on rate-1/3 systematic parallel concatenated convolutional codes and their rate-1/2 punctured child codes. Assuming maximum-likelihood decoding over an additive white Gaussian channel, we demonstrate that a rate-1/2 non-systematic child code can exhibit a lower error floor than that of its rate-1/3 parent code, if a particular condition is met. However, assuming iterative decoding, convergence of the non-systematic code towards low bit-error rates is problematic. To alleviate this problem, we propose rate-1/2 partially-systematic codes that can still achieve a lower error floor than that of their rate-1/3 parent codes. Results obtained from extrinsic information transfer charts and simulations support our conclusion.**

I. Introduction

A punctured convolutional code is obtained by the periodic elimination of symbols from the output of a low-rate parent convolutional code. Extensive analyses on the structure and performance of punctured convolutional codes has shown that their performance is always inferior than the performance of their low-rate parent codes (e.g. see [1], [2]).

The performance of punctured parallel concatenated convolutional codes (PCCCs), also known as punctured turbo codes, has also been investigated. Design considerations have been derived by analytical [3]–[5] as well as simulation-based approaches [6]–[8], while upper bounds to the bit error probability (BEP) were evaluated in [5], [9]. The most recent papers [7]–[9] demonstrate that puncturing both systematic and parity outputs of a rate-1/3 turbo code results in better high-rate turbo codes, in terms of BEP performance, than puncturing only the parity outputs of the original turbo code.

The aim of this paper is to explore whether rate-1/2 punctured turbo codes can eventually achieve better performance than their parent rate-1/3 systematic turbo codes on additive white Gaussian (AWGN) channels. Assuming maximum-likelihood (ML) decoding, we demonstrate that, contrary to punctured convolutional codes, punctured turbo codes yielding lower error floors than that of their parent code can be constructed. Nevertheless, we cannot be conclusive when suboptimal iterative decoding is used. For this reason, we also study the convergence behavior of iterative decoding and we investigate whether the performance of the proposed rate-1/2 punctured turbo codes converges towards the theoretical error floor, at low bit error probabilities.

This work was supported by EPSRC under Grant GR/S46437/01.

II. Performance Evaluation of Rate-1/3 Turbo Codes

Turbo codes, in the form of symmetric rate-1/3 PCCCs, consist of two identical rate-1/2 recursive systematic convolutional encoders separated by an interleaver of size N [10]. The information bits are input to the first constituent convolutional encoder, while an interleaved version of the information bits are input to the second convolutional encoder. The output of the turbo encoder consists of the systematic bits of the first encoder, which are identical to the information bits, the parity check bits of the first encoder and the parity check bits of the second encoder.

It was shown in [11] and [12] that the performance of a PCCC can be obtained from the input-redundancy weight enumerating functions (IRWEFs) of the terminated constituent recursive convolutional codes. The IRWEF for the case of a convolutional code $\mathcal{C}$ assumes the form

$$A^{\mathcal{C}}(W,Z) = \sum_{w}\sum_{j} A^{\mathcal{C}}_{w,j} W^{w} Z^{j}, \tag{1}$$

where $A^{\mathcal{C}}_{w,j}$ denotes the number of codeword sequences having parity check weight j, which were generated by an input sequence of weight w. The overall output weight of the codeword sequence, for the case of a systematic code, is $w+j$.

The conditional weight enumerating function (CWEF), $A^{\mathcal{C}}(w,Z)$, provides all codeword sequences generated by an input sequence of weight w. Consequently, the relationship between the CWEF and the IRWEF is

$$A^{\mathcal{C}}(W,Z) = \sum_{w} A^{\mathcal{C}}(w,Z) W^{w}. \tag{2}$$

A relationship between the CWEF of a PCCC, $\mathcal{P}$, and the CWEF of $\mathcal{C}$, which is one of the two identical constituent codes, can be easily derived only if we assume the use of a uniform interleaver, an abstract probabilistic concept introduced in [12]. In particular, if N is the size of the uniform interleaver and $A^{\mathcal{C}}(w,Z)$ is the CWEF of the constituent code, the CWEF of the PCCC, $A^{\mathcal{P}}(w,Z)$, is equal to

$$A^{\mathcal{P}}(w,Z) = \frac{\left[A^{\mathcal{C}}(w,Z)\right]^2}{\binom{N}{w}}. \tag{3}$$

The IRWEF of $\mathcal{P}$, $A^{\mathcal{P}}(W,Z)$, can be then computed from the CWEF, $A^{\mathcal{P}}(w,Z)$, in a manner identical to (2).

The input-output weight enumerating function (IOWEF) provides the number of codeword sequences generated by an input sequence of weight w, whose overall output weight is d, in contrast with the IRWEF, which only considers the output parity check weight j. If $\mathcal{P}$ is a systematic PCCC, the corresponding IOWEF assumes the form

$$B^{\mathcal{P}}(W,D) = \sum_{w}\sum_{d} B^{\mathcal{P}}_{w,d} W^w D^d, \tag{4}$$

where the coefficients $B^{\mathcal{P}}_{w,d}$ can be derived from the coefficients $A^{\mathcal{P}}_{w,z}$ of the IRWEF, based on the expressions

$$B^{\mathcal{P}}_{w,d} = A^{\mathcal{P}}_{w,j}, \quad \text{and} \quad d = w + j. \tag{5}$$

The IOWEF coefficients $B^{\mathcal{P}}_{w,d}$ can be used to determine a tight upper bound on the BEP for ML soft decoding for the case of an AWGN channel, as follows

$$P_B \leq \frac{1}{N}\sum_{d}\sum_{w} w B^{\mathcal{P}}_{w,d} Q\left(\sqrt{\frac{2R_{\mathcal{P}}E_b}{N_0}\cdot d}\right), \tag{6}$$

where $R_{\mathcal{P}}$ is the rate of the turbo code, which in our case is equal to $1/3$. The upper bound can be rewritten as

$$P_B \leq \sum_{w} P(w), \tag{7}$$

where $P(w)$ is the contribution to the overall BEP of all error events having information weight w, and is defined as

$$P(w) = \sum_{d} \frac{w}{N} B^{\mathcal{P}}_{w,d} Q\left(\sqrt{\frac{2R_{\mathcal{P}}E_b}{N_0}\cdot d}\right). \tag{8}$$

Benedetto *et al.* showed in [12] that the upper bound on the BEP of a PCCC using a uniform interleaver of size N coincides with the average of the upper bounds obtainable from the whole class of deterministic interleavers of size N. For small values of N, the upper bound can be very loose compared with the actual performance of turbo codes using specific deterministic interleavers. However, for $N \geq 1000$, it has been observed that randomly generated interleavers generally perform better than deterministic interleaver designs [13]. Consequently, the upper bound provides a good indication of the actual error rate performance of a PCCC, when long interleavers are considered.

III. Performance Evaluation of Rate-1/2 Punctured Non-Systematic PCCCs

Rates higher than 1/3 can be achieved by puncturing the output of a rate-1/3 turbo encoder. Punctured codes are classified as systematic (S), partially systematic (PS) or non-systematic (NS) depending on whether all, some or none of their systematic bits are transmitted [7]. In this section we concentrate specifically on rate-1/2 NS-PCCCs, because their weight enumerating functions can be easily related to the weight enumerating functions of their parent codes, as it will now be demonstrated.

A symmetric rate-1/2 NS-PCCC, $\mathcal{P}'$, can be obtained by puncturing the systematic output of a rate-1/3 PCCC, $\mathcal{P}$, which consists of two identical rate-1/2 recursive systematic convolutional codes. However, $\mathcal{P}'$ can also be seen as a PCCC constructed using two identical rate-1 non-systematic convolutional codes, each one of which has been obtained by puncturing the systematic bits of a rate-1/2 systematic convolutional code, identical to the one used in $\mathcal{P}$. If $\mathcal{C}'$ is the punctured rate-1 non-systematic convolutional code and $\mathcal{C}$ is the parent rate-1/2 systematic convolutional code, their IRWEFs, $A^{\mathcal{C}'}(W,Z)$ and $A^{\mathcal{C}}(W,Z)$ respectively, are identical, i.e.,

$$A^{\mathcal{C}'}(W,Z) = A^{\mathcal{C}}(W,Z), \tag{9}$$

since, by definition, the IRWEF does not provide information about the weight of the systematic bits of a codeword. Thus, puncturing of the systematic bits of $\mathcal{C}$ will not cause any change in its IRWEF.

Either by applying the same reasoning or by considering (2) and (3), we find that the IRWEF of the rate-1/2 NS-PCCC, $A^{\mathcal{P}'}(W,Z)$, is identical to the IRWEF of its parent code, $A^{\mathcal{P}}(W,Z)$, i.e.,

$$A^{\mathcal{P}'}(W,Z) = A^{\mathcal{P}}(W,Z). \tag{10}$$

Puncturing has an effect only when calculating the IOWEF of $\mathcal{P}'$, $B^{\mathcal{P}'}(W,D)$. We use the notation d' to denote the overall weight of a codeword sequence after puncturing as opposed to d, which refers to the overall weight of the same codeword sequence before puncturing. Therefore the IOWEF of $\mathcal{P}'$ can be expressed as

$$B^{\mathcal{P}'}(W,D) = \sum_{w}\sum_{d'} B^{\mathcal{P}'}_{w,d'} W^w D^{d'}. \tag{11}$$

Since all systematic bits are punctured, the weight w of the information bits does not contribute to the overall weight d' of the punctured codeword sequences, and hence it follows that

$$B^{\mathcal{P}'}_{w,d'} = A^{\mathcal{P}}_{w,j}, \quad \text{and} \quad d' = j. \tag{12}$$

From (5) and (12) we find that the relationship between the IOWEF coefficients, $B^{\mathcal{P}'}_{w,d'}$ and $B^{\mathcal{P}}_{w,d}$, is as follows

$$B^{\mathcal{P}'}_{w,d'} = B^{\mathcal{P}}_{w,(d'+w)} \quad \text{or} \quad B^{\mathcal{P}'}_{w,(d-w)} = B^{\mathcal{P}}_{w,d} \tag{13}$$

since

$$d = d' + w. \tag{14}$$

This is equivalent to saying that if all information sequences of weight w are input to both $\mathcal{P}$ and $\mathcal{P}'$, the overall weight of the generated codeword sequences follows the same distribution in both cases, but is shifted by w in the case of $\mathcal{P}'$.

The upper bound on the BEP for ML soft decoding for the case of an AWGN channel takes the form

$$P_B \leq \sum_{w} P'(w), \tag{15}$$

where $P'(w)$ is given by

$$P'(w) = \sum_{d'} \frac{w}{N} B^{\mathcal{P}'}_{w,d'} Q\left(\sqrt{\frac{2R_{\mathcal{P}'}E_b}{N_0}\cdot d'}\right) \tag{16}$$

and $R_{\mathcal{P}'} = 1/2$. Taking into account (13) and (14), we can rewrite (16) as a function of d and $B^{\mathcal{P}}_{w,d}$, as follows

$$P'(w) = \sum_{d} \frac{w}{N} B^{\mathcal{P}}_{w,d} Q\left(\sqrt{\frac{2R_{\mathcal{P}'}E_b}{N_0} \cdot (d-w)}\right) \tag{17}$$

IV. Performance Analysis

Before continuing to the derivation of a condition which needs to be met so that a rate-1/2 NS-PCCC can achieve a better ML bound than its parent rate-1/3 PCCC, we first enumerate a number of results, derived and justified in [14]:

1) The minimum information weight $w_{\min}$ for recursive convolutional encoders is $w_{\min} = 2$.
2) For recursive constituent codes and long interleavers, the contribution to the overall BEP of all error events with odd information weight is negligible. Furthermore, as the interleaver size increases, the contribution to the overall BEP of all error events with information weight $w_{\min}$ is dominant.
3) The upper bound of a PCCC which uses recursive constituent codes, depends on its free effective distance $d_{\text{free.eff}}$, which corresponds to the minimum overall output weight when the information weight is $w_{\min}$.

It is also straightforward to verify that, although puncturing of the systematic bits of a rate-1/3 PCCC affects the upper bound on the BEP, the previously highlighted trends still apply.

Based on (7) and (15), the rate-1/2 NS-PCCC, $\mathcal{P}'$, will achieve a better bound than its parent, if

$$\sum_{w \geq 2} P'(w) < \sum_{w \geq 2} P(w). \tag{18}$$

For large interleaver sizes, the dominant terms will be $P'(2)$ and $P(2)$, thus (18) reduces to $P'(2) < P(2)$, or equivalently

$$\sum_{d \geq d_{\text{free.eff}}} Q\left(f'(d)\right) < \sum_{d \geq d_{\text{free.eff}}} Q\left(f(d)\right), \tag{19}$$

where $f'(d)$ and $f(d)$ are defined as

$$\begin{aligned} f'(d) &= \sqrt{\frac{2R_{\mathcal{P}'}E_b}{N_0} \cdot (d-2)}, \\ f(d) &= \sqrt{\frac{2R_{\mathcal{P}}E_b}{N_0} \cdot d}, \end{aligned} \tag{20}$$

according to (8) and (17).

Function $Q(\xi)$ is a monotonically decreasing function of ξ, where ξ is a real number. Consequently, if ξ_1 and ξ_2 are real numbers with $\xi_1 > \xi_2$, it follows that $Q(\xi_1) < Q(\xi_2)$, and vice versa, i.e,

$$Q(\xi_1) < Q(\xi_2) \Leftrightarrow \xi_1 > \xi_2. \tag{21}$$

Therefore, inequality (19) is satisfied if

$$f'(d) > f(d), \quad \text{for every } d \geq d_{\text{free.eff}}. \tag{22}$$

Both $f'(d)$ and $f(d)$ are monotonically increasing functions, therefore (22) holds true if only $f'(d_{\text{free.eff}}) > f(d_{\text{free.eff}})$, or

$$d_{\text{free.eff}} > \frac{2R_{\mathcal{P}'}}{R_{\mathcal{P}'} - R_{\mathcal{P}}}. \tag{23}$$

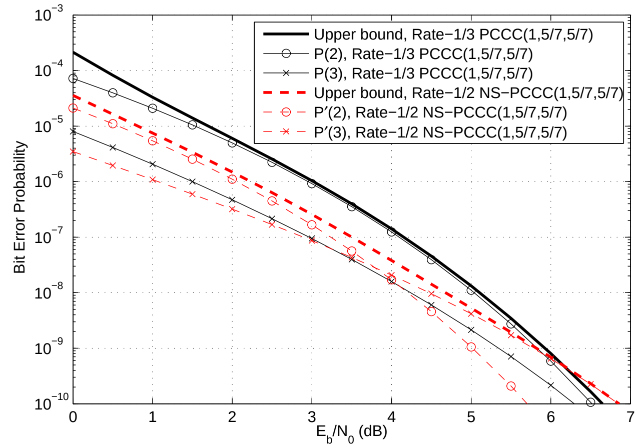


Fig. 1. Upper bounds and contributions to the BEP of all error events with information weight of 2 and 3, for the rate-1/2 NS-PCCC(1,5/7,5/7) and its parent rate-1/3 PCCC. The interleaver size is 1,000.

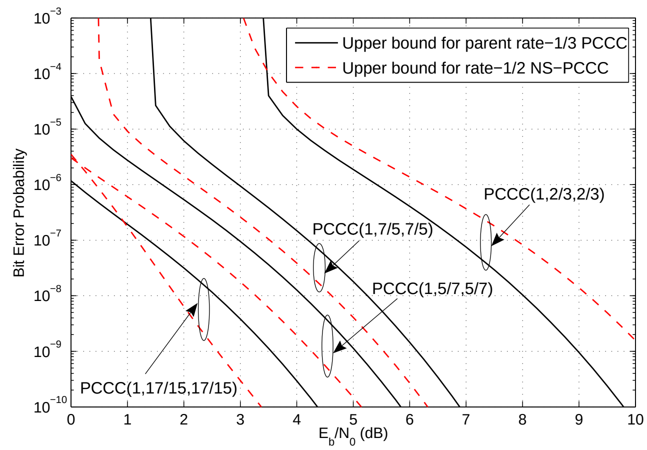
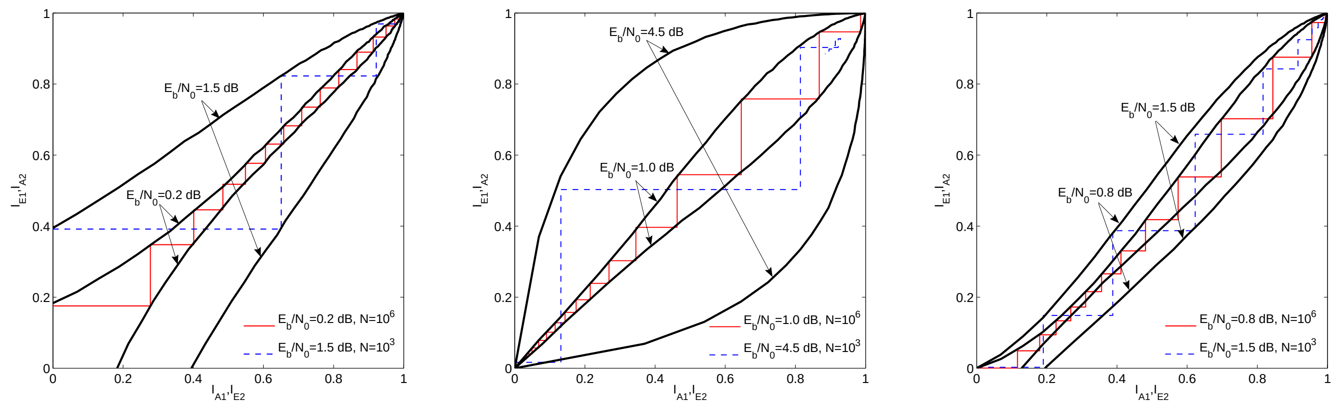


Fig. 2. Comparisson of upper bounds for various rate-1/2 NS-PCCCs and their parent rate-1/3 PCCCs. The interleaver size is 10,000.

After substituting $R_{\mathcal{P}'}$ and $R_{\mathcal{P}}$ with $1/2$ and $1/3$ respectively, we find that a rate-1/2 NS-PCCC can achieve a lower upper bound on the BEP, over an AWGN channel, than its rate-1/3 parent PCCC only if the effective free distance of the parent PCCC, $d_{\text{free.eff}}$, meets the condition

$$d_{\text{free.eff}} > 6. \tag{24}$$

The contributions of the error events with weight 2 and 3, for the previous case, are examined in Fig.1. We observe that up to a certain value of E_b/N_0, the rate-1/2 PCCC(1,5/7,5/7) exhibits a lower upper bound than its parent code. Note that for this range of E_b/N_0 values, the inequality $P'(2) < P(2)$ is satisfied. Although $P(3)$ and $P'(3)$ do not significantly affect the bounds at low E_b/N_0 values, they play an important role at higher E_b/N_0 values. However, for a larger interleaver size (e.g., $N = 10,000$), $P'(2)$ is dominant, as explained previously. More specifically, $P'(2)$ determines the upper bound of the rate-1/2 NS-PCCC(1,5/7,5/7), which is always lower than the upper bound of its parent rate-1/3 PCCC for the range of E_b/N_0 values investigated, as we observe in Fig.2.



(a) Rate-1/3 PCCC(1,5/7,5/7) (b) Rate-1/2 NS-PCCC(1,5/7,5/7) (c) Rate-1/2 PS-PCCC(1,5/7,5/7)

Fig. 3. Extrinsic information transfer characteristics of iterative decoding for various turbo codes using an interleaver size of 10^6 bits. Two decoding averaged trajectories for each case are also plotted; one for an interleaver size of 10^3 bits and the other for an interleaver size of 10^6 bits.

In Fig.2, we also observe that in all cases except for PCCC(1,2/3,2/3), the rate-1/2 NS-PCCCs achieve better bounds than their parent PCCCs. The reason is that the free effective distance of the parent rate-1/3 PCCC(1,2/3,2/3) is $d_{\text{free.eff}} = 4$, thus condition (24) is not met. For all other turbo codes investigated in Fig.2, $d_{\text{free.eff}} > 6$ holds true.

V. Convergence Considerations

The upper bound on the BEP for ML soft decoding provides an accurate estimate of the suboptimal iterative decoder performance at high E_b/N_0 values, for an increasing number of iterations [12]. Since the performance of rate-1/3 PCCCs gradually converges to the ML bound, this bound can be used to predict the BEP error floor region of the corresponding code. However, when puncturing occurs, we need to explore whether the performance of the iterative decoder eventually converges to the ML bound. For this reason, we use extrinsic information transfer (EXIT) chart analysis [15], which can accurately predict the convergence behavior of the iterative decoder for very large interleaver sizes (e.g., $N = 10^6$ bits).

An iterative decoder consists of two soft-input/soft-output decoders. Each decoder uses the received systematic and parity bits as well as a-priori knowledge from the previous decoder to produce extrinsic information on the systematic bits. Ten Brink described the decoding algorithm process using EXIT chart analysis [15]. To this end, the information content of the a-priori knowledge is measured using the mutual information I_A between the information bits at the transmitter and the a-priori input to the constituent decoder. Mutual information I_E is also used to quantify the extrinsic output. The extrinsic information transfer characteristics are then defined as a function of I_A and E_b/N_0, i.e., $I_E = T(I_A, E_b/N_0)$. By plotting the mutual information transfer characteristics of both constituent decoders in a single EXIT chart, evolution of the iterative decoding process can be visualised.

During the first iteration, the first decoder does not have any a-priori knowledge, thus $I_{A_1,1} = 0$, while the second decoder uses the extrinsic output $I_{E_1,1} = T(0, E_b/N_0)$ of the first decoder as a-priori knowledge, i.e., $I_{A_2,1} = I_{E_1,1}$. The extrinsic output of the second decoder, $I_{E_2,1} = T(I_{A_2,1}, E_b/N_0)$, is forwarded to the first decoder to become a-priori knowledge during the next iteration, i.e., $I_{A_1,2} = I_{E_2,1}$, and so on. Note that convergence to the $(I_A, I_E) = (1,1)$ point, i.e., towards low BEPs, occurs if the transfer characteristics do not cross.

As an example, we consider the rate-1/3 systematic PCCC(1,5/7,5/7) to be the parent code. Fig.3(a) shows the transfer characteristics of the constituent decoders for the parent PCCC using an interleaver of size $N = 10^6$. We see that for $E_b/N_0 = 0.2$ dB, the decoder characteristics do not intersect and the averaged decoding trajectory [15] manages to go through a narrow tunnel. Fig.3(b) depicts the decoder characteristics for the rate-1/2 NS-PCCC employing an interleaver of the same size. For $E_b/N_0 = 1.0$ dB, the averaged trajectory just manages to pass through a narrow opening, which appears close to the starting point $(0,0)$. Therefore, for long interleavers, the performance of the suboptimal iterative decoder for the rate-1/2 NS-PCCC converges towards the error floor region, defined by the upper bound, and eventually outperforms its rate-1/3 parent PCCC. However, convergence begins at a higher E_b/N_0 value and a larger number of iterations is required.

In Fig.3(a) and Fig.3(b) the averaged trajectories for the more practical interleaver size of 1,000 bits are also depicted. For $E_b/N_0 = 1.5$ dB the trajectory for rate-1/3 PCCC quickly converges towards low BEPs. However, the trajectory for rate-1/2 NS-PCCC dies away after 2 iterations, even for an E_b/N_0 value of 4.5 dB, due to the increasing correlations of extrinsic information. We attribute this problem to the absence of received systematic bits, which causes erroneous decisions. As a result, error propagation prohibits the iterative decoder from converging. Thus, for small and more practical interleaver sizes, the performance of rate-1/2 NS-PCCC does not gradually approach the upper bound for ML decoding.

To alleviate this problem, the turbo encoder could send some systematic bits, while keeping the rate equal to 1/2 by puncturing some parity bits. In [9] we have presented a technique for deriving good punctured codes and we have identified a rate-1/2 PS-PCCC that achieves the second best performance bound for ML decoding, after the rate-1/2 NS-PCCC. Fig.3(c) shows the transfer characteristics of the constituent decoders as well as the averaged trajectory for $N=10^6$. A comparison with Fig.3(b) for the NS-PCCC case, reveals that a lower E_b/N_0 is required and less iterations are needed, in order for the rate-1/2 PS-PCCC to converge. Furthermore, the trajectory for an interleaver size of 1,000 bits reaches the top corner of the EXIT chart for the same E_b/N_0 as the parent rate-1/3 PCCC, guaranteeing that the iterative decoder will converge towards low BEP values.

VI. Simulation Results

The process for selecting rate-1/2 PS-PCCCs that lead to superior performance than their parent rate-1/3 PCCCs can be summarized in two steps. We first implement the technique we have proposed in [9] to derive good punctured PCCCs that exhibit low error floors. We then use EXIT charts and averaged trajectories to identify the punctured PCCC whose performance converges towards the error floor region, when iterative decoding is used.

We consider PCCC(1,5/7,5/7) and PCCC(1,7/5,7/5) to demonstrate the effectiveness of this process. The iterative decoder applies the BCJR algorithm [16] and performance is plotted after 10 iterations. A random interleaver of size 1,000 bits is used. In Fig.4 we see that the performance of both parent rate-1/3 PCCCs coincides with the corresponding upper bounds for ML decoding, at high E_b/N_0 values. As expected, the performance of iterative decoding for rate-1/2 NS-PCCCs does not converge towards low BEPs, thus they do not outperform their parent codes, although they exhibit a lower upper bound. Nevertheless, rate-1/2 PS-PCCCs that achieve a lower error floor than their parent rate-1/3 PCCCs, can be found based on [9]. Note that, in both cases, the encoder of the selected rate-1/2 PS-PCCC transmits 7 parity bits and only 1 systematic bit for every 4 input information bits.

VII. Conclusion

In this paper we have demonstrated that, if a certain condition is met over the AWGN channel, puncturing of the systematic output of a rate-1/3 turbo code using a random interleaver leads to a rate-1/2 non-systematic turbo code that achieves a better performance than its parent code, when ML decoding is used. In the case of iterative decoding, the absence of systematic bits makes convergence towards low bit-error rates difficult for the rate-1/2 non-systematic turbo decoder. Nevertheless, we can assist convergence by transmitting some systematic bits, while keeping the rate 1/2 by puncturing more parity bits. Thus, we can combine the techniques described in [9] and [15] to identify good puncturing patterns, improve bandwidth efficiency by reducing the rate of a PCCC from 1/3 to 1/2 and, at the same time, achieve a lower error floor.

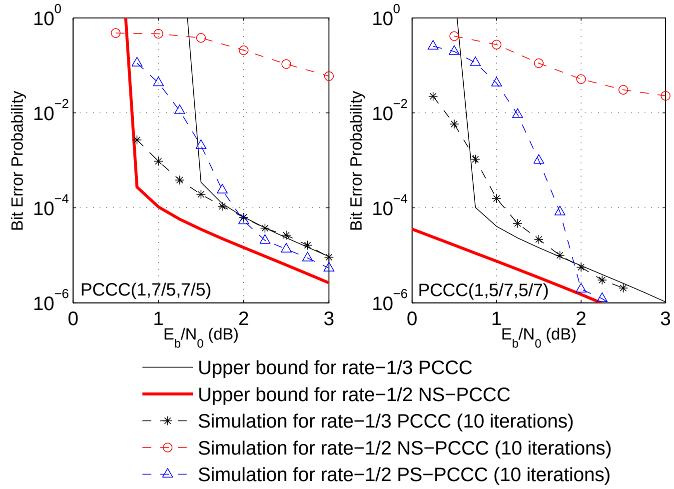


Fig. 4. Comparison between the upper bounds and simulation results for PCCCs using an interleaver size of 1,000 bits.

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