A Short Note on Contextual Action Calculi.

Philippa Gardner and Robin Milner

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1 Introduction

In [1], Gardner introduces the *contextual action calculi* which are an adaptation of action calculi, presented as a quotient of a term algebra, to allow for a local account of names using contexts. This note is a purely technical account of the meta-theoretic results linking the action calculus $\mathrm{AC}(\mathcal{K})$ with its contextual counterpart $\mathrm{AC}_n(\mathcal{K})$. The aim is to prove the following result:

$$s = t \in \mathrm{AC} \text{ if and only if } fn(s,t) \vdash s = t \in \mathrm{AC}_n.$$

This result is proved by direct appeal to the molecular forms, which reinforces our intuition that molecular forms can be regarded as normal forms. In particular, we show that $fn(s,t) \vdash s = t \in \mathrm{AC}_n$ if and only if the corresponding molecular forms of s and t are equal, using an analogous proof to that found in [4]. We restrict our attention to the static part of the definition of action calculi, since the dynamic part requires further investigation.

Summary This note is entirely technical, and does not contribute either motivation or examples. It presumes some familiarity with [3] and [4]. In section 2, we give a brief summary of action calculi. Section 3 contains the definition of contextual action calculi and section 4 contains the proof of the above result.

2 Action Calculi

2.1 Action Structures

An action structure is a strict monoidal category with additional structure. Its objects form a monoid $(M, \otimes, \epsilon)$, and we call them *arities*; we let k, l, m, n range over arities. The actions of an action structure A are its arrows; $A(m,n)$ is the set of all actions $a : m \to n$ in A. We let $a, b, \ldots$ range over actions, and we write $a \cdot b : k \to m$ (rather than $b \circ a$) for the composite of $a : k \to l$ and $b : l \to m$. There are also identities $\mathbf{id}_m$ and tensor product $a \otimes b : k \otimes l \to m \otimes n$, for $a : k \to m$ and $b : l \to n$. Each action structure A presupposes a denumerable set X of names, to each of which is assigned an arity. We let x, y, z, w range over names, and write $x : k$ to mean that x has arity k. Unless otherwise stated, the set X of names will remain fixed, as will the monoid $(M, \otimes, \epsilon)$ and the assignment of arities to names.

The first element of additional structure in A is the abstraction operation $\mathbf{ab}_x$ for each name x; if $x : k$ and $a : m \to n$ then $\mathbf{ab}_x a : k \otimes m \to k \otimes n$. The abstractors are endofunctors. Action structures obey the following eight axioms, in which arities may be assigned in any way which makes sense:

$A1$	$a \cdot \mathbf{id} = a = \mathbf{id} \cdot a$	$A5$	$a \otimes (b \otimes c) = (a \otimes b) \otimes c$
$A2$	$a \otimes \mathbf{id}_\epsilon = a = \mathbf{id}_\epsilon \otimes a$	$A6$	$(a \cdot b) \otimes (c \cdot d) = (a \otimes c) \cdot (b \otimes d)$
$A3$	$\mathbf{id} \otimes \mathbf{id} = \mathbf{id}$	$A7$	$\mathbf{ab}_x \mathbf{id} = \mathbf{id}$
$A4$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$	$A8$	$\mathbf{ab}_x(a \cdot b) = (\mathbf{ab}_x a) \cdot (\mathbf{ab}_x b)$

The second element of extra structure in A is a preorder $\searrow$ on each $A(m, n)$, called the *reaction relation*; this relation is preserved by composition, product and abstraction. Also each **id** is minimal for $\searrow$: that is, if $\mathbf{id} \searrow a$ then **id** is a. This preorder represents the dynamics of the actions. For action calculi we shall assume that the monoid of arities is isomorphic to the free monoid generated by a set of *prime* arities. We shall allow p, q to range over primes. We also assume that the arity of every name is prime and moreover, that there are infinitely many names associated with every prime arity.

2.2 Action calculi

An action calculus $\mathrm{AC}(\mathcal{K})$ possesses a set $\mathcal{K}$ of controls, each equipped with an arity rule. Each $\mathrm{AC}(\mathcal{K})$ is determined by its set of controls $\mathcal{K}$, together with a reaction relation. Terms have the form $t : m \to n$, for $m, n \in (M, \otimes, \epsilon)$, where t is constructed from the basic operators $\mathbf{id}_m, \cdot, \otimes, \mathbf{ab}_x, \boldsymbol{\omega}_p$ and $\langle x \rangle$, and the controls $K \in \mathcal{K}$.

Definition 2.1 [Terms] The *set of terms* over $\mathcal{K}$, denoted by $\mathrm{T}(\mathcal{K})$, is generated from the following rules:

$$\mathbf{id}_m : m \to m \qquad\qquad \frac{s : k \to l \qquad t : l \to m}{s \cdot t : k \to m}$$

$$\frac{s : k \to m \qquad t : l \to n}{s \otimes t : k \otimes l \to m \otimes n} \qquad\qquad \frac{t : m \to n}{\mathbf{ab}_x(t) : p \otimes m \to p \otimes n}\; x : p$$

$$\boldsymbol{\omega}_p : p \to \epsilon \qquad\qquad \langle x \rangle : \epsilon \to p, \quad x : p$$

$$* \; \frac{t_1 : m_1 \to n_1 \quad \ldots \quad t_r : m_r \to n_r}{K(t_1, \ldots, t_r) : m \to n}\; \chi$$

where each control operator $K \in \mathcal{K}$ is accompanied by an arity rule $*$, such that side-condition χ may constrain the value of the integer r and the arities m_i, n_i, m and n. We shall omit the arity subscripts on the basic operators when they are apparent. The notions of *free name* and *bound name* are standard; $\mathbf{ab}_x$ binds x and $\langle x \rangle$ represents a free occurrence of x. The set of names free in t is denoted by $fn(t)$. We let $fn(s, t)$ denote the set $fn(s) \cup fn(t)$. We let $\mathrm{T}_{\{\vec{x}\}}(\mathcal{K})$ denote the set of terms whose free names are contained in $\{\vec{x}\}$. Given a possibly empty sequence of names $[x_1 : p_1, \ldots, x_r : p_r]$ denoted by $\vec{x}$, we use the notation $|\vec{x}|$ to denote $p_1 \otimes \ldots \otimes p_r$.

Definition 2.2 [Derived operations] We define an alternative form $(x)t$ of *abstraction*, the *permutations* $\mathbf{p}_{m,n}$, the *copying* operator $\mathbf{copy}_m$ and some other standard abbreviations as follows:

$$(x)t \stackrel{\text{def}}{=} \mathbf{ab}_x t \cdot (\boldsymbol{\omega} \otimes \mathbf{id})$$

$$\begin{aligned} (\vec{x})t &\stackrel{\text{def}}{=} (x_1)\cdots(x_r)t && (\vec{x} = [x_1, \cdots, x_r],\ \text{all distinct},\ r \geq 1) \\ \langle\vec{x}\rangle &\stackrel{\text{def}}{=} \langle x_1\rangle \otimes \cdots \otimes \langle x_r\rangle && (\vec{x} = [x_1, \cdots, x_r],\ r \geq 1) \end{aligned}$$

$$\mathbf{p}_{m,n} \stackrel{\text{def}}{=} (\vec{x},\vec{y})\langle\vec{x},\vec{y}\rangle, \qquad (|\vec{x}| = m, |\vec{y}| = n)$$

$$\mathbf{copy}_m \stackrel{\text{def}}{=} (\vec{x})\langle\vec{x},\vec{x}\rangle$$

$$\boldsymbol{\omega}_m \stackrel{\text{def}}{=} \begin{cases} \boldsymbol{\omega}_{p_1} \otimes \ldots \otimes \boldsymbol{\omega}_{p_r} & (m = p_1 \otimes \ldots \otimes p_r) \\ \mathbf{id}_\epsilon & (m = \epsilon) \end{cases}$$

We assume that $(\,)t$ denotes the term t and $\langle\,\rangle$ denotes the term $\mathbf{id}_\epsilon$. Notice that $\mathbf{p}_{m,n}$ and $\mathbf{copy}_m$ are defined using a *particular* names; with α-conversion, we shall be justified in choosing these names at will. Throughout this paper we shall adopt the convention that all names appearing in a vector within round brackets are distinct. It shall also be assumed that all terms and expressions used are well formed, and when they occur in definitions or equations, those occurring on each side have identical arities.

Definition 2.3 [The theory AC] The *equational theory* AC is the set of equations upon terms generated by the action structure axioms and the *concrete* axioms:

$$\begin{aligned} \gamma:&\quad (x)t = \boldsymbol{\omega} \otimes t && (x \notin fn(t)) \\ \delta:&\quad (x)(\langle x\rangle \otimes \mathbf{id}_m) = \mathbf{id}_{p\otimes m} && (x : p) \\ \zeta:&\quad \mathbf{p}_{k,m} \cdot (t \otimes s) = (s \otimes t) \cdot \mathbf{p}_{l,n} && (s : k \to l,\ t : m \to n) \\ \sigma:&\quad (\langle y\rangle \otimes \mathbf{id}_m) \cdot (x)t = \{y/x\}t && (t : m \to n) \end{aligned}$$

For a given set of controls $\mathcal{K}$, we write $s = t \in \mathrm{AC}$ if $s, t \in \mathrm{T}(\mathcal{K})$ and $s = t$ in the equational theory AC. We say that $s = t$ is an *axiom* of AC if it is an instance of the action structure axioms or the concrete axioms.

The *action calculus* $\mathrm{AC}(\mathcal{K})$ is defined to be the quotient $\mathrm{T}(\mathcal{K})/\mathrm{AC}$, together with a reaction relation on the equivalence classes. The equivalence classes of $\mathrm{T}(\mathcal{K})/\mathrm{AC}$ are called the *actions*. Typically, actions will be denoted by a, b, c. We let $t \in a$ denote that term t is in the equivalence class a. We view the quotient $\mathrm{T}(\mathcal{K})/\mathrm{AC}$ as the *static* part of the action calculus, and view the reaction relation as the *dynamic* part. As mentioned in the introduction, the definition of the dynamics is not yet definitive. We therefore concentrate on the static part of action calculi in this note, and omit the dynamics.

3 Contextual Action Calculi

The contextual action calculi are an adaptation of action calculi to incorporate a local account of names using contexts.

Definition 3.1 [The Theory AC_n] The *equational theory with names*, denoted by AC_n, is the set of equations upon terms generated by the following rules, where $\{\vec{x}\}$ denotes a set of distinct names[1]:

$$\{\vec{x}\} \vdash s = t \qquad\qquad s = t \text{ an axiom of AC}, \mathit{fn}(s,t) \subseteq \{\vec{x}\}$$

$$\{\vec{x}\} \vdash s = s \qquad\qquad \mathit{fn}(s) \subseteq \{\vec{x}\}$$

$$\frac{\{\vec{x}\} \vdash s = t}{\{\vec{x}\} \vdash t = s}$$

$$\frac{\{\vec{x}\} \vdash s = t \quad \{\vec{x}\} \vdash t = u}{\{\vec{x}\} \vdash s = u}$$

$$\frac{\{\vec{x}\} \vdash s = t}{\{\vec{x}\} \vdash u \cdot s = u \cdot t \qquad \{\vec{x}\} \vdash s \cdot u = t \cdot u} \qquad\qquad \mathit{fn}(u) \subseteq \{\vec{x}\}$$

$$\frac{\{\vec{x}\} \vdash s = t}{\{\vec{x}\} \vdash u \otimes s = u \otimes t \qquad \{\vec{x}\} \vdash s \otimes u = t \otimes u} \qquad\qquad \mathit{fn}(u) \subseteq \{\vec{x}\}$$

$$\frac{\{\vec{x}, y\} \vdash s = t}{\{\vec{x}\} \vdash (y)s = (y)t} \qquad\qquad y \notin \{\vec{x}\}$$

$$\frac{\{\vec{x}\} \vdash s_i = t_i, \quad i \in \{1, \ldots, n\}}{\{\vec{x}\} \vdash K(s_1, \ldots, s_r) = K(t_1, \ldots, t_r)}$$

For a given set $\mathcal{K}$ of controls, we write $\{\vec{x}\} \vdash s = t \in \mathrm{AC}_n$ if $s, t \in \mathrm{T}(\mathcal{K})$ and $\{\vec{x}\} \vdash s = t$ can be proved using the above rules.

Lemma 3.2 [Basic Lemmas] The following equations are provable in AC_n.

C1. $\mathit{fn}(s,t)/\{x\} \vdash (x)(s \cdot t) = (x)s \cdot t, \quad x \notin \mathit{fn}(t)$

α. $\mathit{fn}(t)/\{x\} \vdash (x)t = (y)\{y/x\}t, \quad y \notin \mathit{fn}(t)$

C2. $\mathit{fn}(t)/\{x\} \vdash (x)(t \otimes \mathbf{id}) = (x)t \otimes \mathbf{id}$

C3. $\mathit{fn}(s,t)/\{x\} \vdash (x)(s \otimes t) = (x)s \otimes t, \quad x \notin \mathit{fn}(t)$

C4. $\mathit{fn}(t)/\{x\} \vdash \mathbf{ab}_x t = (x)(\langle x \rangle \otimes t)$

C5. $\mathit{fn}(t) \vdash \mathbf{ab}_x t = \mathbf{id} \otimes t, \quad x \notin \mathit{fn}(t)$

C6. $\mathit{fn}(t)/\{\vec{x}, \vec{y}\} \vdash (\vec{x}, \vec{y})t = (\mathbf{p}_{m,n} \otimes \mathbf{id}) \cdot (\vec{y}, \vec{x})t, \quad |\vec{x}| = m, |\vec{y}| = n$

C7. $\emptyset \vdash (\vec{x})\langle \vec{x} \rangle = \mathbf{id}$

Proof

(C1)

$$\begin{aligned} \mathit{fn}(s,t)/\{\vec{x}\} \vdash (x)(s \cdot t) &= \mathbf{ab}_x(s \cdot t) \cdot (\boldsymbol{\omega} \otimes \mathbf{id}) \\ &= \mathbf{ab}_x(s) \cdot (x)t \\ &\overset{\gamma}{=} \mathbf{ab}_x(s) \cdot (w \otimes \mathbf{id}) \cdot t \\ &= (x)s \cdot t \end{aligned}$$

[1] We use a rule with two conclusions as shorthand for two rules with the same premises and one conclusion each.

$$(\alpha)\qquad \begin{aligned} fn(t)/\{x\} \vdash (y)\{y/x\}t &\overset{\sigma}{=} (y)(((\langle y\rangle \otimes \mathbf{id}) \cdot (x)t) \\ &\overset{\text{C1}}{=} (y)((\langle y\rangle \otimes \mathbf{id}) \cdot (x)t \\ &\overset{\delta}{=} (x)t \end{aligned}$$

$$(\text{C2})\qquad \begin{aligned} fn(t)/\{x\} \vdash (x)(t \otimes \mathbf{id}) &= (x)(\{x/x\}t \otimes \mathbf{id}) \\ &\overset{\sigma}{=} (x)(((\langle x\rangle \otimes \mathbf{id}) \cdot (x)t \otimes \mathbf{id}) \\ &= (x)(((\langle x\rangle \otimes \mathbf{id} \otimes \mathbf{id}) \cdot ((x)t \otimes \mathbf{id})) \\ &\overset{\text{C1}}{=} (x)((\langle x\rangle \otimes \mathbf{id} \otimes \mathbf{id}) \cdot ((x)t \otimes \mathbf{id}) \\ &\overset{\delta}{=} (x)t \otimes \mathbf{id} \end{aligned}$$

$$(\text{C3})\qquad \begin{aligned} fn(s,t)/\{x\} \vdash (x)(s \otimes t) &= (x)((s \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes t)) \\ &\overset{\text{C1}}{=} (x)(s \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes t) \\ &\overset{\text{C2}}{=} ((x)s \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes t) \\ &= (x)s \otimes t \end{aligned}$$

$$(\text{C4})\qquad \begin{aligned} fn(t)/\{x\} \vdash \mathbf{ab}_x t &\overset{\delta}{=} \mathbf{ab}_x t \cdot (x)(\langle x\rangle \otimes \mathbf{id}) \\ &= (x)(t \cdot (\langle x\rangle \otimes \mathbf{id})) \\ &= (x)(\langle x\rangle \otimes t) \end{aligned}$$

$$(\text{C5})\qquad \begin{aligned} fn(t) \vdash \mathbf{ab}_x t &\overset{\text{C4}}{=} (x)(\langle x\rangle \otimes t) \\ &= (x)(((\langle x\rangle \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes t)) \\ &\overset{\text{C1}}{=} (x)(\langle x\rangle \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes t) \\ &\overset{\delta}{=} \mathbf{id} \otimes t \end{aligned}$$

$$(\text{C6})\qquad \begin{aligned} fn(t)/\{\vec{x},\vec{y}\} \vdash (\mathbf{p}_{m,n} \otimes \mathbf{id}) \cdot (\vec{y},\vec{x})t &\overset{\text{C3}}{=} (\vec{x},\vec{y})(\langle \vec{y},\vec{x}\rangle \otimes \mathbf{id}) \cdot (\vec{y},\vec{x})t \\ &\overset{\text{C1}}{=} (\vec{x},\vec{y})(((\langle \vec{y},\vec{x}\rangle \otimes \mathbf{id}) \cdot (\vec{y},\vec{x})t) \\ &\overset{\sigma}{=} (\vec{x},\vec{y})t \end{aligned}$$

The proof of C7 proceeds by induction on the length of $\vec{x}$. The base case is easy using the δ axiom with $m = \epsilon$. The inductive step proceeds as follows.

$$(\text{C7})\qquad \begin{aligned} \emptyset \vdash (x,\vec{y})\langle x,\vec{y}\rangle &= (x)(\vec{y})(\langle \vec{y}\rangle \cdot (\langle x\rangle \otimes \mathbf{id})) \\ &\overset{\text{C1}}{=} (x)((\vec{y})\langle \vec{y}\rangle \cdot (\langle x\rangle \otimes \mathbf{id})) \\ &\overset{\text{IH}}{=} (x)(\langle x\rangle \otimes \mathbf{id}) \\ &\overset{\delta}{=} \mathbf{id} \end{aligned}$$

The equation C1 and α-conversion are used to prove the following proposition; the equations C2–C7 are required in section 4.

Proposition 3.3

1. $\{\vec{x}\} \vdash s = t \in \mathrm{AC}_n$ implies $fn(s,t) \subseteq \{\vec{x}\}$.
2. $\{\vec{x}, y\} \vdash s = t \in \mathrm{AC}_n$ and $z \notin \{\vec{x}\}$ imply $\{\vec{x}, z\} \vdash s\{z/y\} = t\{z/y\} \in \mathrm{AC}_n$.
3. $\{\vec{x}\} \vdash s = t \in \mathrm{AC}_n$ and $y \notin \{\vec{x}\}$ imply $\{\vec{x}, y\} \vdash s = t \in \mathrm{AC}_n$.

Proof The proofs of parts 1–3 use induction on the structure of derivations. The proof of part 1 is trivial. Part 2 requires α-conversion for the binding structural rule case. Part 3 requires part 2 and α-conversion for the binding structural rule case. $\square$

Proposition 3.4 $s = t \in \mathrm{AC}$ if and only if $\{\vec{x}\} \vdash s = t \in \mathrm{AC}_n$ for some distinct list of names $\vec{x}$.

Proof Both implications are easy. The implication from left to right requires proposition 3.3, part 3. $\square$

Remark It is non-trivial to prove the stronger result that $s = t \in \mathrm{AC}$ implies $fn(s,t) \vdash s = t \in \mathrm{AC}_n$. By proposition 3.4, an equivalent result is to show that strengthening holds in AC_n: that is, $\{\vec{x}, y\} \vdash s = t \in \mathrm{AC}_n$ and $y \notin fn(s,t)$ implies $\{\vec{x}\} \vdash s = t \in \mathrm{AC}_n$. The proofs of these results are given in section 4, and rely on the connection between $\mathrm{AC}_n(\mathcal{K})$ and the alternative presentation of action calculi using the molecular forms.

Remark Notice that $\{\vec{x}, y\} \vdash s = t : m \to n$ and $y \notin fn(s,t)$ implies $\{\vec{x}\} \vdash \boldsymbol{\omega} \otimes s = \boldsymbol{\omega} \otimes t : m \to n$ using the γ axiom. Assume that y has accompanying arity p. If $x_i : p$, then $\{\vec{x}\} \vdash s = t$ using the σ axiom. If $m = p \otimes n$, then $\{\vec{x}\} \vdash s = t$ using the fact the $\emptyset \vdash (\mathbf{copy} \otimes \mathbf{id}) \cdot (\boldsymbol{\omega} \otimes \mathbf{id}) = \mathbf{id}$. It seems that strengthening cannot in general be proved by such simple manipulations. We require a better understanding of the link between the arities accompanying variables and the arities accompanying terms, in order to make this statement precise.

The *contextual action calculus* $\mathrm{AC}_n(\mathcal{K})$ is defined to be the quotient $\mathrm{T}(\mathcal{K})/\mathrm{AC}_n$, together with a reaction relation on equivalence classes. The equivalence classes of $\mathrm{T}(\mathcal{K})/\mathrm{AC}_n$ are also called the actions. Again, we let a, b, c denote actions and let $t \in a$ denote that term t is in the equivalence class a. In [1], Gardner defines the notion of a reaction relation for a contextual action calculus. Since the aim of this note is to provide a link between the static part of an action calculus and its corresponding contextual action calculus, we do not reproduce the definition here.

4 Result

The aim of this section is to show that $s = t \in \mathrm{AC}$ implies $fn(s,t) \vdash s = t \in \mathrm{AC}_n$. This result is proved by direct appeal to the molecular form presentation of action calculi. We show that $fn(s,t) \vdash s = t \in \mathrm{AC}_n$ if and only if the corresponding molecular forms of s and t are equal, using an analogous proof to that found in [4]. The main result easily follows.

4.1 Molecular Forms

Definition 4.1 Let $\mathcal{K}$ denote a set of controls and their associated arity rules. The set of *molecular forms* over $\mathcal{K}$, denoted by $\mathrm{M}(\mathcal{K})$, consist of the actions, given by the abstract grammar

$$
\begin{array}{lll}
a & ::= & (\vec{x})\ \mu_1 \cdots \mu_r\ \langle \vec{u} \rangle \qquad\qquad (a : |\vec{x}| \to |\vec{u}|) \\
\mu & ::= & \langle \vec{v} \rangle\ K\vec{a}\,(\vec{y}) \qquad\qquad\quad (K\vec{a} : |\vec{v}| \to |\vec{y}|)
\end{array}
$$

where μ ranges over *molecules*, and the construction $K\vec{a}$ is determined by the arity rule accompanying K.

In both actions and molecules, whenever a vector $\vec{x}$ occurs in round brackets, its names (which by our convention must be distinct) are binding occurrences with scope extending to the right capturing *any* free occurrence of the names, including those occurring in actions contained within molecule constructions. Names which are not thus bound are free and alpha conversion of bound names is allowed. In a above, $\vec{x}$ are called the *imported names* and $\vec{u}$, the *exported* ones. The construct $\mu_1 \cdots \mu_r$, called the *body* of a, is a possibly empty *partial* sequence of molecules, where the commutation of two non-dependent molecules is allowed; that is, the commutation of two molecules is allowed provided neither binds a name occurring free in the other. We sometimes write $a = b \in M(\mathcal{K})$ if a can be obtained from b by α-conversion and the commutation of non-dependent molecules, and write $a =_\alpha b \in \mathrm{M}(\mathcal{K})$ if a is obtained solely from b by α-conversion.

We shall now define operators on molecular forms which correspond to the action calculi operators.

Definition 4.2 Assume $a = (\vec{u})\,\vec{\lambda}\,\langle \vec{v} \rangle$ and $b = (\vec{x})\,\vec{\mu}\,\langle \vec{y} \rangle$.

$$
\begin{array}{rcl}
\mathbf{id}_m & \stackrel{\text{def}}{=} & (\vec{x})\langle \vec{x} \rangle, \quad (|\vec{x}| = m) \\
a \cdot b & \stackrel{\text{def}}{=} & (\vec{u})\,\vec{\lambda}\,\sigma(\vec{\mu}\,\langle \vec{y} \rangle), \quad (\sigma = \{\vec{v}/\vec{x}\}) \\
a \otimes b & \stackrel{\text{def}}{=} & (\vec{u}\vec{x})\vec{\lambda}\,\vec{\mu}\,\langle \vec{v}\vec{y} \rangle \\
\mathbf{ab}_x a & \stackrel{\text{def}}{=} & (x\vec{u})\,\vec{\lambda}\,\langle x\vec{v} \rangle \\
\langle x \rangle & \stackrel{\text{def}}{=} & (\ )\langle x \rangle \\
\boldsymbol{\omega} & \stackrel{\text{def}}{=} & (x)\langle \rangle \\
K(\vec{a}) & \stackrel{\text{def}}{=} & (\vec{x})\ \langle \vec{x} \rangle K\vec{a}(\vec{y})\ \langle \vec{y} \rangle, \quad (\vec{x} \text{ and } \vec{y} \text{ are distinct and are not free in } \vec{a})
\end{array}
$$

where $\{\vec{v}/\vec{x}\}$ is simultaneaous syntactic substitution for the free occurrences of names in $\vec{x}$ by the corresponding ones in $\vec{v}$.

The molecular form presentation of action calculi consists of the set $\mathrm{M}(\mathcal{K})$ together with a reaction relation. Again, we omit the definition of the reaction reaction, since the dynamics are not considered in this note.

4.2 Translating molecular forms to terms

We define a translation $\widehat{(\_)} : \mathrm{M}(\mathcal{K}) \to T(\mathcal{K})$ and show that it preserves equalities: that is, we show that

$$a = b \in \mathrm{M}(\mathcal{K}) \text{ implies } fn(a) \vdash \widehat{a} = \widehat{b} \in \mathrm{AC}_n(\mathcal{K}).$$

This result means that the translation $\widehat{(\_)}$ yields a well-defined translation from $\mathrm{M}(\mathcal{K})$ to $\mathrm{T}(\mathcal{K})/\mathrm{AC}_n$.

Definition 4.3 The function $\widehat{(\_)} : \mathrm{M}(\mathcal{K}) \to T(\mathcal{K})$ is defined inductively on the number of molecules in an action as follows:

$$\widehat{(\vec{x})\vec{\mu}\langle\vec{y}\rangle} \stackrel{\text{def}}{=} \begin{cases} (\vec{x})\langle\vec{y}\rangle & \text{if } \vec{\mu} \text{ is empty} \\ (\vec{x})(\langle\vec{u}\rangle \cdot K(\widehat{\vec{a}}) \cdot \widehat{(\vec{v})\vec{\mu'}\langle\vec{y}\rangle}) & \text{if } \vec{\mu} = \mu\vec{\mu'} \text{ and } \mu = \langle\vec{u}\rangle K(\vec{a})(\vec{v}) \end{cases}$$

Proposition 4.4 Let $a, b \in \mathrm{M}(\mathcal{K})$.

1. $fn(\widehat{a}) = fn(a)$.
2. $a =_\alpha b$ implies $fn(a) \vdash \widehat{a} = \widehat{b} \in \mathrm{AC}_n$.
3. $(fn(a)/\{\vec{x}\}) \cup \{\vec{y}\} \vdash \widehat{\sigma a} = \sigma\widehat{a}$, where $\sigma = \{\vec{y}/\vec{x}\}$.
4. $a = b \in \mathrm{M}(\mathcal{K})$ implies $fn(a) \vdash \widehat{a} = \widehat{b} \in \mathrm{AC}_n(\mathcal{K})$.

Proof Part 1 is easily proved by induction on the number of molecules in a. Parts 2 and 3 are proved simultaneously by induction on the number of molecules in a using the α axiom. To prove part 4, it remains to show that the translations of actions which only differ by the commutation of non-dependent molecules are equal in AC_n with respect to the appropriate context of names. It is enough to show, for s of the form $\langle\vec{u}\rangle \cdot K(\vec{s_1}) \cdot (\vec{v})(\langle\vec{w}\rangle \cdot L(\vec{s_2}) \cdot (\vec{z})t)$ with $t : \epsilon \to m$, that we have

$$fn(s) \vdash s = \langle\vec{w}\rangle \cdot L(\vec{s_2}) \cdot (\vec{z})(\langle\vec{u}\rangle \cdot K(\vec{s_1}) \cdot (\vec{v})t) \in \mathrm{AC}_n$$

if $\vec{v} \notin \{\vec{w}\} \cup fn(\vec{s_2})$ and $\vec{z} \notin \{\vec{u}\} \cup fn(\vec{s_1})$.

$$\begin{array}{rcl} fn(s) \vdash s & = & \langle\vec{u}\rangle \cdot K(\vec{s_1}) \cdot (\vec{v})(\langle\vec{w}\rangle \cdot L(\vec{s_2}) \cdot (\vec{z})t) \\ & = & \langle\vec{u}\rangle \cdot K(\vec{s_1}) \cdot \mathbf{ab}_{\vec{v}}(\langle\vec{w}\rangle \cdot L(\vec{s_2})) \cdot (\vec{v}, \vec{z})t \\ & \stackrel{\text{C5}}{=} & \langle\vec{u}\rangle \cdot K(\vec{s_1}) \cdot (\mathbf{id} \otimes \langle\vec{w}\rangle \cdot L(\vec{s_2})) \cdot (\vec{v}, \vec{z})t \\ & = & (\langle\vec{u}\rangle \cdot K(\vec{s_1}) \otimes \langle\vec{w}\rangle \cdot L(\vec{s_2})) \cdot (\vec{v}, \vec{z})t \\ & \stackrel{\zeta}{=} & (\langle\vec{w}\rangle \cdot L(\vec{s_2}) \otimes \langle\vec{u}\rangle \cdot K(\vec{s_1})) \cdot \mathbf{p}_{|\vec{z}|,|\vec{v}|} \cdot (\vec{v}, \vec{z})t \\ & \stackrel{\text{C6}}{=} & (\langle\vec{w}\rangle \cdot L(\vec{s_2}) \otimes \langle\vec{u}\rangle \cdot K(\vec{s_1})) \cdot (\vec{z}, \vec{v})t \\ & \stackrel{\text{symmetry}}{=} & \langle\vec{w}\rangle \cdot L(\vec{s_2}) \cdot (\vec{z})(\langle\vec{u}\rangle \cdot K(\vec{s_1}) \cdot (\vec{v})t) \end{array}$$

Proposition 4.5 The translation $\widehat{(\_)}$ preserves the action calculi operations: that is,

1. $\emptyset \vdash \widehat{\mathbf{id}} = \mathbf{id}$
2. $fn(a, b) \vdash \widehat{a \cdot b} = \widehat{a} \cdot \widehat{b}$
3. $fn(a, b) \vdash \widehat{a \otimes b} = \widehat{a} \otimes \widehat{b}$
4. $fn(a)/\{x\} \vdash \widehat{\mathbf{ab}_x a} = \mathbf{ab}_x \widehat{a}$
5. $\{x\} \vdash \widehat{\langle x\rangle} = \langle x\rangle$
6. $\emptyset \vdash \widehat{\boldsymbol{\omega}} = \boldsymbol{\omega}$
7. $\bigcup_{i=1}^n fn(a_i) \vdash \widehat{K(\vec{a})} = K(\widehat{\vec{a}})$, where $\vec{a}$ is the list $[a_1, \ldots, a_n]$.

Proof Parts 1 and 5–7 are immediate: parts 1 and 7 require C7, part 7 requires weakening, and part 6 requires $\emptyset \vdash w = (\vec{x})\mathbf{id}_\epsilon$ which follows from the γ axiom. Part 4 is easy using part 3 and weakening:

$$\begin{array}{rcl} fn(a)/\{x\} \vdash \widehat{\mathbf{ab}_x(a)} & = & (x)\widehat{(\langle x\rangle \otimes a)} \\ & \stackrel{4.5,3}{=} & (x)(\widehat{\langle x\rangle} \otimes \widehat{a}) \\ & \stackrel{4.5,5}{=} & (x)(\langle x\rangle \otimes \widehat{a}) \\ & = & \mathbf{ab}_x(\widehat{a}) \end{array}$$

Parts 2 and 3 are the interesting cases.

2. Let $a = (\vec{x})\vec{\lambda}\langle\vec{y}\rangle$, $b = (\vec{u})\vec{\mu}\langle\vec{v}\rangle$ and let $b_0 = \vec{\mu}\langle\vec{v}\rangle$. Without loss of generality we may assume that $\{\vec{x}\} \cap fn(b) = \emptyset$. We prove part 2 by induction on the number of molecules in $\vec{\lambda}$. First assume that $\vec{\lambda}$ is empty.

$$\begin{array}{rcl} fn(a,b) \vdash (\vec{x})\widehat{\langle\vec{y}\rangle \cdot (\vec{u})}b_0 & = & \widehat{(\vec{x})\sigma b_0}, \quad \sigma = \{\vec{y}/\vec{u}\} \\ & = & (\vec{x})\widehat{\sigma b_0} \\ & \stackrel{4.4,3}{=} & (\vec{x})\sigma\widehat{b_0} \\ & \stackrel{\sigma}{=} & (\vec{x})(\langle\vec{y}\rangle \cdot (\vec{u})\widehat{b_0}) \\ & \stackrel{C1}{=} & (\vec{x})\langle\vec{y}\rangle \cdot (\vec{u})\widehat{b_0} \\ & = & \widehat{a} \cdot \widehat{b} \end{array}$$

To prove the inductive step, let $\vec{\lambda} = \lambda\vec{\lambda}'$, where $\lambda = \langle\vec{w}\rangle K(\vec{a})(\vec{z})$, and let $a' = (\vec{z})\vec{\lambda}'\langle\vec{y}\rangle$ so that $a = (\vec{x})\lambda\langle\vec{z}\rangle \cdot a'$.

$$\begin{array}{rcl} fn(a,b) \vdash \widehat{a \cdot b} & = & (\vec{x})\widehat{\vec{\lambda}\sigma(\vec{\mu}\langle\vec{v}\rangle)}, \quad \sigma = \{\vec{y}/\vec{u}\} \\ & = & (\vec{x})(\langle\vec{w}\rangle \cdot K(\widehat{\vec{a}}) \cdot (\vec{z})\widehat{\vec{\lambda}'\sigma(\vec{\mu}\langle\vec{v}\rangle)}) \\ & = & (\vec{x})(\langle\vec{w}\rangle \cdot K(\widehat{\vec{a}}) \cdot \widehat{a' \cdot b}) \\ & \stackrel{IH}{=} & (\vec{x})(\langle\vec{w}\rangle \cdot K(\widehat{\vec{a}}) \cdot \widehat{a'} \cdot \widehat{b}) \\ & \stackrel{C1}{=} & (\vec{x})(\langle\vec{w}\rangle \cdot K(\widehat{\vec{a}}) \cdot \widehat{a'}) \cdot \widehat{b} \\ & = & \widehat{a} \cdot \widehat{b} \end{array}$$

3. Part 3 is proved in stages. We first prove the following:

   (a) $fn(a) \cup \{\vec{z}\} \vdash \widehat{a \otimes \langle\vec{z}\rangle} = \widehat{a} \otimes \langle\vec{z}\rangle$

   (b) $fn(a) \cup \{\vec{z}\} \vdash \widehat{\langle\vec{z}\rangle \otimes a} = \langle\vec{z}\rangle \otimes \widehat{a}$

   Let $a = (\vec{x})\vec{\lambda}\langle\vec{y}\rangle$. Without loss of generality, using α we may assume that $\{\vec{x}\} \cap \{\vec{z}\} = \emptyset$. Part 3a is proved by induction on the number of molecules in a. First assume that $\vec{\lambda}$ is empty.

$$\begin{array}{rcl} fn(a) \cup \{\vec{z}\} \vdash (\vec{x})\widehat{\langle\vec{y}\rangle \otimes \langle\vec{z}\rangle} & = & \widehat{(\vec{x})\langle\vec{y},\vec{z}\rangle} \\ & = & (\vec{x})\langle\vec{y},\vec{z}\rangle \\ & \stackrel{C3}{=} & (\vec{x})\langle\vec{y}\rangle \otimes \langle\vec{z}\rangle \\ & = & \widehat{a} \otimes \langle\vec{z}\rangle \end{array}$$

For the induction step, let $\vec{\lambda} = \lambda\vec{\lambda'}$, where $\lambda = \langle\vec{u}\rangle K(\vec{a})(\vec{v})$, and let $a' = (\vec{v})\vec{\lambda'}\langle\vec{y}\rangle$.

$$\begin{aligned}
fn(a) \cup \{\vec{z}\} \vdash \widehat{a \otimes \langle\vec{z}\rangle} &= (\vec{x})\widehat{\vec{\lambda}\langle\vec{y},\vec{z}\rangle} \\
&= (\vec{x})(\langle\vec{u}\rangle \cdot K(\vec{a}) \cdot \widehat{a' \otimes \langle\vec{z}\rangle}) \\
&\overset{\text{IH}}{=} (\vec{x})(\langle\vec{u}\rangle \cdot K(\vec{a}) \cdot \widehat{a'} \otimes \langle\vec{z}\rangle) \\
&\overset{\text{C3}}{=} (\vec{x})(\langle\vec{u}\rangle \cdot K(\vec{a}) \cdot \widehat{a'}) \otimes \langle\vec{z}\rangle \\
&= \widehat{a} \otimes \langle\vec{z}\rangle
\end{aligned}$$

Part 3b follows in a similar fashion. Using parts 3a and 3b, it is easy to show by induction on a that

$$fn(a) \vdash \widehat{a \otimes \mathbf{id}} = \widehat{a} \otimes \mathbf{id}$$

$$fn(a) \vdash \widehat{\mathbf{id} \otimes a} = \mathbf{id} \otimes \widehat{a}$$

The result easily follows using the above results, part 2 and the fact that $a \otimes b = (a \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes a) \in \mathrm{M}(\mathcal{K})$. □

4.3 Translating terms to molecular forms

Definition 4.6 The function $[\![\_]\!] : \mathrm{T}(\mathcal{K}) \to \mathrm{M}(\mathcal{K})$ is defined inductively on the structure of terms as follows:

$$\begin{array}{ll}
[\![\mathbf{id}]\!] = (\vec{x})\langle\vec{x}\rangle & [\![\langle x\rangle]\!] = \langle x\rangle \\
[\![s \cdot t]\!] = [\![s]\!] \cdot [\![t]\!] & [\![\boldsymbol{\omega}]\!] = \boldsymbol{\omega} \\
[\![s \otimes t]\!] = [\![s]\!] \otimes [\![t]\!] & [\![K(\vec{t})]\!] = K([\![\vec{t}]\!]) \\
[\![\mathbf{ab}_x(t)]\!] = \mathbf{ab}_x([\![t]\!]) &
\end{array}$$

Proposition 4.7

1. $fn([\![t]\!]) \subseteq fn(t)$
2. The translation $[\![\_]\!]$ preserves the action calculi operations.
3. $\{\vec{x}\} \vdash s = t \in \mathrm{AC}_n$ implies $[\![s]\!] = [\![t]\!]$.

Proof Parts 1 and 2 are proved in [4]. Part 3 follows from proposition 3.4 and the corresponding result for AC which is proved in [4]. □

Proposition 4.8

1. $fn(t) \vdash t = \widehat{[\![t]\!]} \in \mathrm{AC}_n(\mathcal{K})$.
2. $a = [\![\widehat{a}]\!] \in \mathrm{M}(\mathcal{K})$.

Proof The proof of part 1 is straightforward, since $[\![t]\!]$ is defined inductively via the action calculi operators, and $\widehat{(\_)}$ preserves all the operators (proposition 4.5). Part 2 is proved in [4]. □

Corollary 4.9 $t = \widehat{[\![t]\!]} \in \mathrm{AC}(\mathcal{K})$

Proof Follows easily from proposition 4.8 and proposition 3.4. □

Proposition 4.10 $s = t \in \mathrm{AC}$ if and only if $fn(s,t) \vdash s = t \in \mathrm{AC}_n$.

Proof The implication from right to left follows from proposition 3.4. Assume $s = t \in \mathrm{AC}(\mathcal{K})$. Proposition 4.7 and 3.4 imply $[\![s]\!] = [\![t]\!] \in \mathrm{M}(\mathcal{K})$, and proposition 4.4 implies $fn([\![s]\!]) \vdash \widehat{[\![s]\!]} = \widehat{[\![t]\!]} \in \mathrm{AC}$. Proposition 4.8 implies $fn(s) \vdash s = \widehat{[\![s]\!]}$ and $fn(t) \vdash t = \widehat{[\![t]\!]}$. Hence, the result follows using weakening. $\square$

Corollary 4.11 $\{\vec{x}, y\} \vdash s = t \in \mathrm{AC}_n$ and $y \notin \{\vec{x}\}$ implies $\{\vec{x}\} \vdash s = t \in \mathrm{AC}_n$.

Proof By proposition 4.10, we have $s = t \in \mathrm{AC}$ and $fn(s,t) \vdash s = t \in \mathrm{AC}_n$. Proposition 3.3 implies $fn(s,t) \subseteq \{\vec{x}\}$. Hence, the result follows by weakening. $\square$

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