Action Calculi VII : Closed action calculi

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DRAFT Comments are welcome.

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1 Introduction

Action calculi provide a unifying framework for representing a variety of models of interaction, such as Petri nets and the π-calculus. Two presentations of action calculi exist [4]: a direct construction of the syntactic objects called *molecular forms*, and the quotient of a term algebra whose constructors include the operations of action structures. These two presentations are isomorphic [6]. Even though the term algebra provides an abstract account of action calculi, it still requires the use of names. We introduce a name-free account of action calculi, called the *closed action calculi*, and prove that there is a strong correspondence between them. Our results show that, although names play an important presentational role, they are in fact not essential.

Each action calculus $\mathrm{AC}(\mathcal{K})$ is presented as a quotient of a set of terms, generated by the set $\mathcal{K}$ of controls, over an equational theory AC. Associated with each action calculus is a set of reaction rules which determine its reaction relation; these rules provide the meaning of the control set $\mathcal{K}$. Every action calculus $\mathrm{AC}(\mathcal{K})$ has a corresponding closed action calculus $\mathrm{CAC}(\mathcal{K}')$. Given a term in $\mathrm{AC}(\mathcal{K})$ with free names in $\{\vec{x}\}$, we define its closure $[\![t]\!]_{\vec{x}}$ in $\mathrm{CAC}(\mathcal{K}')$. To show that derivations in AC correspond to derivations in the closed world, we must choose $\vec{x}$ so that $[\![\_]\!]_{\vec{x}}$ is defined for every term in the derivation. We therefore introduce an adaptation of AC, called AC_n, which provides a local account of names using contexts. The partial function $[\![\_]\!]_{\vec{x}}$ preserves the equalities in AC_n. We also define an equality-preserving function $\langle\!\langle \_ \rangle\!\rangle$ the other way, and show that

$$\begin{array}{rcl} \langle\!\langle [\![t]\!]_{\vec{x}} \rangle\!\rangle & =_{\emptyset} & (\vec{x})t \in \mathrm{AC}_n, \quad \text{and} \\ [\![\langle\!\langle t \rangle\!\rangle]\!]_{[\,]} & = & t \in \ \mathrm{CAC}\ . \end{array}$$

These results justify our intuition that $[\![\_]\!]_{\vec{x}}$ can be viewed as a closure function.

The reaction relations for action calculi have not yet been fully explored, and so we do not give a general account connecting the reaction relations in $\mathrm{AC}(\mathcal{K})$ and $\mathrm{CAC}(\mathcal{K}')$. We look at two examples, $\mathrm{AC}(\boldsymbol{\nu}, \mathbf{out}, \mathbf{box})$ which corresponds to (part of) the π-calculus and $\mathrm{AC}(\ulcorner\_\urcorner, \mathbf{ap})$ which adds higher-order features, to illustrate that such a correspondence is possible in these cases. We also show that our ideas easily extend to Milner's reflexive action calculi [7].

Summary We give an overview of action calculi in section 2 to make the paper self-contained. Familiarity with action calculi is assumed. In section 3, we introduce the corresponding closed action calculi. Sections 4 contains the definitions of $[\![\_]\!]_{\vec{x}}$ and $\langle\!\langle \_ \rangle\!\rangle$ respectively, and the associated results outlined above. Section 5 shows how these results easily extend to reflexive action calculi. Section 6 explores the dynamics of the action calculi $\mathrm{AC}(\boldsymbol{\nu}, \mathbf{out}, \mathbf{box})$ and $\mathrm{AC}(\ulcorner\_\urcorner, \mathbf{ap})$.

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2 Action Calculi

We give a brief account of action calculi presented as the quotient of a term algebra [4]. Some familiarity with action structures and action calculi will be assumed. We also require a non-standard presentation of action calculi, via an equational theory which accounts for free names [1]. Such a presentation is non-trivial, since a derivation may use free names which do not occur in the conclusion. This presentation is used to link action calculi with their corresponding closed action calculi, defined in section 3.

2.1 Action Structures

We begin by reviewing the definition of action structures, introduced in [3]. We recall that an action structure is a strict monoidal category with additional structure. Its objects form a monoid $(M, \otimes, \varepsilon)$, and we call them *arities*; we let k, l, m, n range over arities. The actions of an action structure A are its arrows; $A(m, n)$ is the set of all actions $a : m \to n$ in A. We let $a, b, \ldots$ range over actions, and we write $a \cdot b : k \to m$ (rather than $b \circ a$) for the composite of $a : k \to l$ and $b : l \to m$. There are also identities $\mathbf{id}_m$ and tensor product $a \otimes b : k \otimes l \to m \otimes n$, for $a : k \to m$ and $b : l \to n$. Each action structure A presupposes a denumerable set X of names, to each of which is assigned an arity. We let x, y, z, w range over names, and write $x : k$ to mean that x has arity k. Unless otherwise stated, the set X of names will remain fixed, as will the monoid $(M, \otimes, \epsilon)$ and the assignment of arities to names.

The first element of additional structure in A is the abstraction operation $\mathbf{ab}_x$ for each name x; if $x : k$ and $a : m \to n$ then $\mathbf{ab}_x a : k \otimes m \to k \otimes n$. The abstractors are endofunctors. Action structures obey the following eight axioms, in which arities may be assigned in any way which makes sense:

$A1$	$a \cdot \mathbf{id} = a = \mathbf{id} \cdot a$	$A5$	$a \otimes (b \otimes c) = (a \otimes b) \otimes c$
$A2$	$a \otimes \mathbf{id}_\epsilon = a = \mathbf{id}_\epsilon \otimes a$	$A6$	$(a \cdot b) \otimes (c \cdot d) = (a \otimes c) \cdot (b \otimes d)$
$A3$	$\mathbf{id} \otimes \mathbf{id} = \mathbf{id}$	$A7$	$\mathbf{ab}_x \mathbf{id} = \mathbf{id}$
$A4$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$	$A8$	$\mathbf{ab}_x(a \cdot b) = (\mathbf{ab}_x a) \cdot (\mathbf{ab}_x b)$

The second element of extra structure in A is a preorder $\searrow$ on each $A(m, n)$, called the *reaction relation*; this relation is preserved by composition, product and abstraction. Also each $\mathbf{id}$ is minimal for $\searrow$: that is, if $\mathbf{id} \searrow a$ then $\mathbf{id}$ is a. This preorder represents the dynamics of the actions. For action calculi we shall assume that the monoid of arities is isomorphic to the free monoid generated by a set of *prime* arities. We shall allow p, q to range over primes. We also assume that the arity of every name is prime and moreover, that there are infinitely many names associated with every prime arity.

2.2 Action calculi

We are now ready to define an action calculus as a quotient of a term algebra. An action calculus $\mathrm{AC}(\mathcal{K})$ possesses a set $\mathcal{K}$ of controls, each equipped with an arity rule. Each $\mathrm{AC}(\mathcal{K})$ is determined by its set of controls $\mathcal{K}$, together with a set of reaction rules which define its dynamics. Terms have the form $t : m \to n$, for $m, n \in (M, \otimes, \varepsilon)$, where t is constructed from the basic operators $\mathbf{id}_m, \cdot, \otimes, \mathbf{ab}_x, \boldsymbol{\omega}_p$ and $\langle x \rangle$, and the controls $K \in \mathcal{K}$.

2.1 Definition [Terms] The *set of terms* over $\mathcal{K}$, denoted by $\mathrm{T}(\mathcal{K})$, is generated from the following rules:

$$\mathbf{id}_m : m \to m \qquad\qquad \frac{s : k \to l \qquad t : l \to m}{s \cdot t : k \to m}$$

$$\frac{s : k \to m \qquad t : l \to n}{s \otimes t : k \otimes l \to m \otimes n} \qquad\qquad \frac{t : m \to n}{\mathbf{ab}_x(t) : p \otimes m \to p \otimes n}\ x : p$$

$$\boldsymbol{\omega}_p : p \to \epsilon \qquad\qquad \langle x \rangle : \varepsilon \to p, \qquad x : p$$

$$* \ \frac{t_1 : m_1 \to n_1 \quad \ldots \quad t_r : m_r \to n_r}{K(t_1, \ldots, t_r) : m \to n}\ \chi$$

where each control operator $K \in \mathcal{K}$ is accompanied by an arity rule $*$, such that side-condition χ may constrain the value of the integer r and the arities m_i, n_i, m and n. Terms of the form $K(t_1, \ldots, t_r)$ are called *control terms.* We shall omit the arity subscripts on the basic operators when they are apparent. The notions of *free name* and *bound name* are standard; $\mathbf{ab}_x$ binds x and $\langle x \rangle$ represents a free occurrence of x. The set of names free in t is denoted by $fn(t)$. We let $fn(s, t)$ denote the set $fn(s) \cup fn()$. We let $\mathrm{T}_{\{\vec{x}\}}(\mathcal{K})$ denote the set of terms whose free names are contained in $\{\vec{x}\}$. Given a possibly empty sequence of names $[x_1 : p_1, \ldots, x_r : p_r]$ denoted by $\vec{x}$, we use the notation $|\vec{x}|$ to denote $p_1 \otimes \ldots \otimes p_r$.

2.2 Definition [Derived operations] We define an alternative form $(x)t$ of *abstraction*, the *permutations* $\mathbf{p}_{m,n}$, the *copying* operator $\mathbf{copy}_m$ and some other standard abbreviations as follows:

$$(x)t \stackrel{def}{=} \mathbf{ab}_x t \cdot (\boldsymbol{\omega} \otimes \mathbf{id})$$

$$(\vec{x})t \stackrel{def}{=} (x_1) \cdots (x_r)t \qquad (\vec{x} = [x_1, \cdots, x_r],\ \text{all distinct},\ r \geq 1)$$

$$\langle \vec{x} \rangle \stackrel{def}{=} \langle x_1 \rangle \otimes \cdots \otimes \langle x_r \rangle \qquad (\vec{x} = [x_1, \cdots, x_r],\ r \geq 1)$$

$$\mathbf{p}_{m,n} \stackrel{def}{=} (\vec{x}, \vec{y})\langle \vec{x}, \vec{y} \rangle, \qquad (|\vec{x}| = m, |\vec{y}| = n)$$

$$\mathbf{copy}_m \stackrel{def}{=} (\vec{x})\langle \vec{x}, \vec{x} \rangle$$

$$\boldsymbol{\omega}_m \stackrel{def}{=} \begin{cases} \boldsymbol{\omega}_{p_1} \otimes \ldots \otimes \boldsymbol{\omega}_{p_r} & (m = p_1 \otimes \ldots \otimes p_r) \\ \mathbf{id}_\epsilon & (m = \epsilon) \end{cases}$$

We assume that $(\ )t$ denotes the term t and $\langle\ \rangle$ denotes the term $\mathbf{id}_\epsilon$. Notice that $\mathbf{p}_{m,n}$ and $\mathbf{copy}_m$ are defined using a *particular* names; with α-conversion, we shall be justified in choosing these names at will. Throughout this paper we shall adopt the convention that all names appearing in a vector within round brackets are distinct. It shall also be assumed that all terms and expressions used are well formed, and when they occur in definitions or equations, those occurring on each side have identical arities.

2.3 Definition [The theory AC] The *equational theory* AC is the set of equations upon terms generated by the action structure axioms and the *concrete* axioms:

$$\begin{array}{lll}
\gamma : & (x)t = \boldsymbol{\omega} \otimes t & (x \notin fn(t)) \\
\delta : & (x)(\langle x \rangle \otimes \mathbf{id}_m) = \mathbf{id}_{p \otimes m} & (x : p) \\
\zeta : & \mathbf{p}_{k,m} \cdot (t \otimes s) = (s \otimes t) \cdot \mathbf{p}_{l,n} & (s : k \to l,\ t : m \to n) \\
\sigma : & (\langle y \rangle \otimes \mathbf{id}_m) \cdot (x)t = \{y/x\}t & (t : m \to n)
\end{array}$$

For a given set of controls $\mathcal{K}$, we write $s = t \in \mathrm{AC}$ if $s, t \in \mathrm{T}(\mathcal{K})$ and $s = t$ in the equational theory AC. We say that $s = t$ is an *axiom* of AC if it is an instance of the action structure axioms or the concrete axioms.

2.4 Lemma [Basic lemmas] The following are provable in AC if $x \notin fn(t)$:

1. $(x)(s \cdot t) = (x)s \cdot t$
2. $(x)(s \otimes t) = (x)s \otimes t$
3. $(x)(t \otimes s) = t \otimes (x)s, \qquad (t : \varepsilon \to m)$

α. $(y)t = (x)t\{x/y\}$

4. $(\vec{x})(\langle \vec{x} \rangle \otimes \mathbf{id}) = \mathbf{id}$
5. $\mathbf{p}_{m,n} \cdot \mathbf{p}_{n,m} = \mathbf{id}$
6. $(\langle \vec{x} \rangle \otimes \mathbf{id}) \cdot (\vec{y})t = t\{x_1/y_1\} \ldots \{x_r/y_r\}, \qquad (\vec{x} = [x_1, \ldots, x_r], \vec{y} = [y_1, \ldots, y_r], r \geq 1)$
7. $\mathbf{copy}_m \cdot (\boldsymbol{\omega}_m \otimes \mathbf{id}) = \mathbf{id}$
8. $\mathbf{copy}_m \cdot (\mathbf{copy}_m \otimes \mathbf{id}) = \mathbf{copy}_m \cdot (\mathbf{id} \otimes \mathbf{copy}_m)$
9. $\mathbf{copy}_m \cdot \mathbf{p}_{m,m} = \mathbf{copy}_m$

The *action calculus* $\mathrm{AC}(\mathcal{K})$ is defined to be the quotient $\mathrm{T}(\mathcal{K})/\mathrm{AC}$, together with a reaction relation on equivalence classes defined below. The equivalence classes of $\mathrm{T}(\mathcal{K})/\mathrm{AC}$ are called the *actions*. Typically, actions will be denoted by a, b, c. We let $t \in a$ denote that term t is in the equivalence class a. We view the quotient $\mathrm{T}(\mathcal{K})/\mathrm{AC}$ as the *static* part of the action calculus, and view the reaction relation as the *dynamic* part.

In previous work, the reaction relation has been typically defined directly on actions. We generate such a reaction relation by first defining a relation on terms. Such an approach is necessary for this paper, since the connection between the open and closed world is given via the terms. [2]

2.5 Definition The reaction relation $\triangleright \in \mathrm{T}(\mathcal{K}) \times \mathrm{T}(\mathcal{K})$ is a binary relation which is closed under composition, tensor, abstraction and the equalities given by AC, and which is generated from a finite set of reaction rules $\mathcal{R} \subseteq \mathrm{T}(\mathcal{K}) \times \mathrm{T}(\mathcal{K})$ such that

1. $s \triangleright t \in \mathcal{R}$ implies $fn(t) \subseteq fn(s)$;

[2]It should be noted that the dynamics of action calculi are not yet fully understood, and so definition 2.5 should not be regarded as definitive.

2. there is no $t \in T(\mathcal{K})$ such that $\mathbf{id} \rhd t$[3].

We let $\rhd^*$ denote the reflexive, transitive closure of $\rhd$.

2.6 DEFINITION The reaction relation $\searrow \in \mathrm{T}(\mathcal{K})/\mathrm{AC} \times \mathrm{T}(\mathcal{K})/\mathrm{AC}$ is defined by

$$a \searrow b \text{ if and only if } s \rhd^* t \text{ for some } s \in a \text{ and } t \in b.$$

Notice that the above definition is equivalent to requiring that $s \rhd^* t$ for every $s \in a$ and $t \in b$.

2.7 PROPOSITION $\mathrm{AC}(\mathcal{K})$ is an action structure.

We give two examples of action calculi: $\mathrm{AC}(\boldsymbol{\nu}, \mathbf{out}, \mathbf{box})$, which corresponds to (part of) Milner's π-calculus [2], and $\mathrm{AC}(\ulcorner \_ \urcorner, \mathbf{ap})$, which is introduced in [5] as an interim step to the higher-order action calculi. In section 6, we use these examples to explore the correspondence between the dynamics of action calculi and their corresponding closed action calculi.

2.8 EXAMPLE Consider the monoid $(N, +, 0)$, given by the set of natural numbers N with addition for the monoidal tensor and 0 for the unit. The action calculus $\mathrm{PIC} = \mathrm{AC}(\boldsymbol{\nu}, \mathbf{out}, \mathbf{box})$, has the arity rules

$$\boldsymbol{\nu} : 0 \to 1$$

$$\mathbf{out} : 1 \otimes m \to 0$$

$$\frac{t : m \to n}{\mathbf{box}(t) : 1 \to n}$$

and the reaction rule

$$(\langle x \rangle \otimes \mathbf{id}_m) \cdot \mathbf{out} \otimes \langle x \rangle \cdot \mathbf{box}(t) \searrow t$$

This action calculus essentially corresponds to Milner's π-calculus [2]. Sometimes the presentation of the π-calculus as an action calculus also uses the control

$$\mathbf{in} : 1 \to m$$

and the reaction rule

$$(\langle x \rangle \otimes \mathbf{id}_m) \cdot \mathbf{out} \otimes \langle x \rangle \cdot \mathbf{in} \searrow \mathbf{id}_m$$

We choose not to include the control **in**, since it can be mimiced by the term $\mathbf{box}(\mathbf{id}_m)$. We use PIC as a motivating example throughout this paper.

2.9 EXAMPLE Let $M^{\Rightarrow} = (M, \otimes, \Rightarrow, \varepsilon)$ denote a monoid $(M, \otimes, \varepsilon)$ with the binary operator $\Rightarrow$ on arities freely added. If $(M, \otimes, \varepsilon)$ has the set of primes P, then $M^{\Rightarrow}$ has the set of primes $P \cup \{m \Rightarrow n : m, n \in M^{\Rightarrow}\}$. We define an action calculus $\mathrm{AC}(\ulcorner \_ \urcorner, \mathbf{ap})$ with arity rules

$$\frac{t : m \to n}{\ulcorner t \urcorner : \varepsilon \to m \Rightarrow n}$$

$$\mathbf{ap} : (m \Rightarrow n) \otimes n \to n$$

[3] It also seems reasonable to require that $s \rhd t$ implies s and t are not members of the same action a, and that $\searrow$ describes the interaction of controls, rather than affecting the static part of the term. More investigation regarding the dynamics is required.

and the reaction rules

$$\begin{array}{ll}(\ulcorner t \urcorner \otimes \mathbf{id}) \cdot (x)s \searrow s\{t/x\} & \qquad (x : m \Rightarrow n) \\ (\ulcorner t \urcorner \otimes \mathbf{id}) \cdot \mathbf{ap} \searrow t & \end{array}$$

where $s\{t/x\}$ denotes the substitution of any free occurrence of x by term t. Milner defines this action calculus in [5] as an interim step to the higher-order action calculi. Higher-order action calculi are action calculi with extra structure given by treating the operators $\ulcorner\_\urcorner$ and **ap** as basic operators, rather than control operators.

Recall from the introduction the we translate terms in $\mathrm{T}(\mathcal{K})$ to terms in the closed world using a function $[\![.]\!]_{\vec{x}}$, which is defined for those terms whose free names are contained in $\{\vec{x}\}$. To show that derivations in AC correspond to derivations in the closed world, we must choose $\vec{x}$ so that $[\![.]\!]_{\vec{x}}$ is defined for every term in the derivation. We therefore introduce an adaptation of AC which incorporates this information regarding free names.

2.10 Definition The *equational theory with names*, denoted by AC_n, is defined by the following rules, where $\{\vec{x}\}$ denotes a set of distinct names[4]:

$$s =_{\{\vec{x}\}} t \qquad\qquad s = t \text{ an axiom of AC}, \mathit{fn}(s) \cup \mathit{fn}(t) \subseteq \{\vec{x}\}$$

$$s =_{\{\vec{x}\}} s \qquad\qquad \mathit{fn}(s) \subseteq \{\vec{x}\}$$

$$\frac{s =_{\{\vec{x}\}} t}{t =_{\{\vec{x}\}} s}$$

$$\frac{s =_{\{\vec{x}\}} t \quad t =_{\{\vec{x}\}} u}{s =_{\{\vec{x}\}} u}$$

$$\frac{s =_{\{\vec{x}\}} t}{u \cdot s =_{\{\vec{x}\}} u \cdot t \qquad s \cdot u =_{\{\vec{x}\}} t \cdot u} \qquad\qquad \mathit{fn}(u) \subseteq \{\vec{x}\}$$

$$\frac{s =_{\{\vec{x}\}} t}{u \otimes s =_{\{\vec{x}\}} u \otimes t \qquad s \otimes u =_{\{\vec{x}\}} t \otimes u} \qquad\qquad \mathit{fn}(u) \subseteq \{\vec{x}\}$$

$$\frac{s =_{\{\vec{x},y\}} t}{(y)s =_{\{\vec{x}\}} (y)t} \qquad\qquad y \notin \{\vec{x}\}$$

$$\frac{s_i =_{\{\vec{x}\}} t_i}{K(s_1, \ldots, s_r) =_{\{\vec{x}\}} K(t_1, \ldots, t_r)}$$

For a given set $\mathcal{K}$ of controls, we write $s =_{\{\vec{x}\}} t \in \mathrm{AC}_n$ if $s, t \in \mathrm{T}(\mathcal{K})$ and $s =_{\{\vec{x}\}} t$ can be proved using the above rules.

2.11 Proposition

1. $s =_{\{\vec{x}\}} t \in \mathrm{AC}_n$ implies $\mathit{fn}(s) \cup \mathit{fn}(t) \subseteq \{\vec{x}\}$.
2. $s =_{\{\vec{x},y\}} t \in \mathrm{AC}_n$ and $z \notin \{\vec{x}\}$ imply $s\{z/y\} =_{\{\vec{x},z\}} t\{z/y\} \in \mathrm{AC}_n$.

[4] We use a rule with two conclusions as shorthand for two rules with the same premises and one conclusion each.

3. $s =_{\{\vec{x}\}} t \in \mathrm{AC}_n$ and $y \notin \{\vec{x}\}$ imply $s =_{\{\vec{x},y\}} t \in \mathrm{AC}_n$.

4. $s =_{\{\vec{x},y\}} t \in \mathrm{AC}_n$ and $y \notin fn(s,t)$ imply $s =_{\{\vec{x}\}} t \in \mathrm{AC}_n$.

Remark The proofs of parts 1 to 3 are easy. The proof of part 4 is less straightforward, and relies on the connection between $\mathrm{AC}_n(\mathcal{K})$ and the alternative presentation of actions using the molecular forms. See [1] for a detailed proof.

2.12 Proposition $s = t \in \mathrm{AC}$ if and only if $s =_{fn(s,t)} t \in \mathrm{AC}_n$.

Proof It is easy to prove that

$$s = t \in \mathrm{AC} \text{ if and only if } s =_{\{\vec{x}\}} t \in \mathrm{AC}_n, \text{ for some set } \{\vec{x}\} \text{ of distinct names.}$$

The implication from left to right requires proposition 2.11, part 3. The result then follows using proposition 2.11, part 4. □

Remark Given $a \in \mathrm{AC}(\mathcal{K})$ and set of distinct names $\{\vec{x}\}$, we can always find a term $t \in a$ whose free names are not included in $\{\vec{x}\}$. To find such a term, first choose $t \in a$ with $fn(t) \subseteq \{\vec{x}\}$. Let y, z be distinct names of the same arity which do not appear in $\{\vec{x}\}$. Using the σ axiom, we have $(\langle z \rangle \otimes \mathbf{id}) \cdot (y)t = t$, and so $(\langle z \rangle \otimes \mathbf{id}) \cdot (y)t$ is an element of a whose free names are not contained in $\{\vec{x}\}$. In section 4, we show that there is a strong correspondence between the equalities in AC_n and equalities in the closed world. The correspondence between $\mathrm{AC}(\mathcal{K})$ and the closed world therefore has a similar flavour to the connection between AC and AC_n given in proposition 2.12.

We also link the reaction relation for AC and the following reaction relation for AC_n.

2.13 Definition The reaction relation $\triangleright_{\{\vec{x}\}} \subseteq \mathrm{T}(\mathcal{K}) \times \mathrm{T}(\mathcal{K})$ is a binary relation generated by a finite set of reaction rules $\mathcal{R} \subseteq \mathrm{T}(\mathcal{K}) \times \mathrm{T}(\mathcal{K})$, where $s \triangleright t \in \mathcal{R}$ implies $fn(t) \subseteq fn(s)$, using the rules

$$s \triangleright_{\{\vec{x}\}} t \qquad\qquad fn(s) \subseteq \{\vec{x}\}, (s,t) \in \mathcal{R}$$

$$\frac{u =_{\{\vec{x}\}} s \quad s \triangleright_{\{\vec{x}\}} t \quad t =_{\{\vec{x}\}} v}{u \triangleright_{\{\vec{x}\}} v}$$

$$\frac{s \triangleright_{\{\vec{x},y\}} t}{(y)s \triangleright_{\{\vec{x}\}} (y)t}$$

$$\frac{s \triangleright_{\{\vec{x}\}} t}{u \otimes s \triangleright_{\{\vec{x}\}} u \otimes t \qquad s \otimes u \triangleright_{\{\vec{x}\}} t \otimes u} \qquad u \in \mathrm{T}_{\{\vec{x}\}}(\mathcal{K})$$

$$\frac{s \triangleright_{\{\vec{x}\}} t}{u \cdot s \triangleright_{\{\vec{x}\}} u \cdot t \qquad s \cdot u \triangleright_{\{\vec{x}\}} t \cdot u} \qquad u \in \mathrm{T}_{\{\vec{x}\}}(\mathcal{K})$$

We also require that there is no $t \in T_{\{\vec{x}\}}(\mathcal{K})$ such that $\mathbf{id} \triangleright_{\{\vec{x}\}} t$ for every set $\{\vec{x}\}$ of distinct variables[5]. We let $\triangleright^*_{\{\vec{x}\}}$ denote the reflexive, transitive closure of $\triangleright_{\{\vec{x}\}}$.

[5]Again we expect that $s \triangleright_{\{\vec{x}\}} t$ does not hold when $s =_{\{\vec{x}\}} t$, and that $\triangleright_{\{\vec{x}\}}$ describes the interaction of controls, rather than affecting the static part of the term

2.14 PROPOSITION

1. $s \rhd_{\{\vec{x}\}} t$ implies $fn(s,t) \subseteq \{\vec{x}\}$.

2. $s \rhd_{\{\vec{x},y\}} t$ and $z \notin \{\vec{x}\}$ implies $s\{z/y\} \rhd_{\{\vec{x},z\}} t\{z/y\}$.

3. $s \rhd_{\{\vec{x}\}} t$ and $y \notin \{\vec{x}\}$ implies $s \rhd_{\{\vec{x},y\}} t$.

2.15 PROPOSITION $s \rhd t \in \mathrm{AC}$ if and only if $s \rhd_{\{\vec{x}\}} t \in \mathrm{AC}_n$ for some set $\{\vec{x}\}$ of distinct names.

Remark The stronger result that $s \rhd t \in \mathrm{AC}$ implies $s \rhd_{fn(s,t)} t \in \mathrm{AC}_n$ probably does not hold in general. For example, consider the reaction relation generated by the reaction rule

$$\langle x \rangle \cdot \omega \otimes K \rhd \mathbf{id}.$$

We have $K \rhd_{\{x\}} \mathbf{id}$, but it looks as if $K \rhd_{\emptyset} \mathbf{id}$ does not hold (not proved). There is a simple condition on rection rules under which the stronger result holds. This condition is satisfied by all the examples studied by Milner in [4] and [5], including the action calculi PIC and $\mathrm{AC}(\ulcorner . \urcorner, \mathbf{ap})$ given in examples 2.8 and 2.9 respectively. We do not however explore this connection between the dynamics of $\mathrm{AC}(\mathcal{K})$ and $\mathrm{AC}_n(\mathcal{K})$, since the dynamics of action calculi are not yet fully understood.

3 Closed Action Calculi

Using an analogous approach to the definition of action calculi, we define a closed action calculus as a quotient of a term algebra constructed from an underlying monoid $(M, \otimes, \epsilon)$. In particular, given an action calculus $\mathrm{AC}(\mathcal{K})$, we distinguish the corresponding closed action calculus $\mathrm{CAC}(\mathcal{K}_m)$. Section 4 contains the translations and results which explain the correspondence between $\mathrm{AC}(\mathcal{K})$ and $\mathrm{CAC}(\mathcal{K}_m)$.

A closed action calculus $\mathrm{CAC}(\mathcal{K})$ possess a set $\mathcal{K}$ of controls, each equipped with an arity rule. Each $\mathrm{CAC}(\mathcal{K})$ is determined by its controls, together with a set of reaction rules which define its dynamics. The closed terms have the form $t : m \to n$, for $m, n \in (M, \otimes, \epsilon)$, where t is constructed from the basic operators $\mathbf{id}_m, \boldsymbol{\omega}_m, \Delta_m, \mathbf{i}_{m,n}, \cdot, \otimes$ and the controls $K \in \mathcal{K}$. The operators Δ_m and $\mathbf{i}_{m,n}$ correspond to the copying and permutation operators given in definition 2.2, as is apparent from the axioms accompanying these operators. This correspondence is expressed formally in section 4. The other operators are self-explanatory.

3.1 DEFINITION [Closed Terms] The *set of closed terms* over $\mathcal{K}$, denoted by $\mathrm{CT}(\mathcal{K})$, is generated from the following rules:

$$\mathbf{id}_m : m \to m \qquad\qquad \frac{s : k \to l \qquad t : l \to m}{s \cdot t : k \to m}$$

$$\frac{s : k \to m \qquad t : l \to n}{s \otimes t : k \otimes l \to m \otimes n} \qquad\qquad \Delta_m : m \to m \otimes m$$

$$\mathbf{i}_{m,n} : m \otimes n \to n \otimes m \qquad\qquad \boldsymbol{\omega}_m : m \to \epsilon$$

$$* \ \frac{t_1 : m_1 \to n_1 \quad \ldots \quad t_r : m_r \to n_r}{K(t_1, \ldots, t_r) : m \to n} \ \chi$$

where each control $K \in \mathcal{K}$ is accompanied by an arity rule $*$, such that side-condition χ may constrict the value of the integer r and the arities m_i, n_i, m and n. We shall omit the arity subscripts on the basic operators when they are apparent.

3.2 Definition [The Theory CAC] The equational theory CAC is the set of equations upon terms generated by the action structure axioms A1–A8 from section 2, and the following:

$$
\begin{array}{llcl}
\text{(B1)} & \Delta_m \cdot (\boldsymbol{\omega}_m \otimes \mathbf{id}) & = & \mathbf{id} \\
\text{(B2)} & \Delta_m \cdot \mathbf{i}_{m,m} & = & \Delta_m \\
\text{(B3)} & \mathbf{i}_{k,m} \cdot (s \otimes t) & = & (t \otimes s) \cdot \mathbf{i}_{l,n} \\
\text{(B4)} & \mathbf{i}_{m,n} \cdot \mathbf{i}_{n,m} & = & \mathbf{id} \\
\text{(B5)} & \mathbf{i}_{m \otimes n,k} & = & (\mathbf{id} \otimes \mathbf{i}_{n,k}) \cdot (\mathbf{i}_{m,n} \otimes \mathbf{id}) \\
\text{(B6)} & \boldsymbol{\omega}_{m \otimes n} & = & \boldsymbol{\omega}_m \otimes \boldsymbol{\omega}_n \\
\text{(B7)} & \Delta_{m \otimes n} & = & (\Delta_m \otimes \Delta_n) \cdot (\mathbf{id} \otimes \mathbf{i}_{m,n} \otimes \mathbf{id}) \\
\text{(B8)} & \Delta_m \cdot (\Delta_m \otimes \mathbf{id}) & = & \Delta_m \cdot (\mathbf{id} \otimes \Delta_m)
\end{array}
$$

For a given set of controls $\mathcal{K}$, we write $s = t \in \text{CAC}$ if $s, t \in \text{CT}(\mathcal{K})$ and $s = t$ is in the equational theory CAC.

Remark We have chosen to define $\mathbf{id}_m$, $\boldsymbol{\omega}_m$, Δ_m and $\mathbf{i}_{m,n}$ for arbitrary arities and include the axioms B5–B7. Since arities can be uniquely factorized into primes, an alternative approach is to restrict the definitions to prime arities, remove B5–B7 and define the composite cases in terms of the prime cases and the other operators. This alternative approach is used in the definition of action calculi, since names are forced to have prime arity.

3.3 Lemma [Basic Lemmas] The following are provable in CAC:

$$
\begin{array}{llcl}
\text{(C1)} & \mathbf{i}_{\epsilon,m} & = & \mathbf{id} \quad = \quad \mathbf{i}_{m,\epsilon} \\
 & \boldsymbol{\omega}_\epsilon & = & \mathbf{id} \\
 & \Delta_\epsilon & = & \mathbf{id} \\
\text{(C2)} & \mathbf{i}_{k,m \otimes n} & = & (\mathbf{i}_{k,m} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{k,n}) \\
\text{(C3)} & (\Delta_m \otimes \mathbf{id}) \cdot (\boldsymbol{\omega}_m \otimes t) & = & t \\
 & (\mathbf{id} \otimes \Delta_m) \cdot (t \otimes \boldsymbol{\omega}_m) & = & t \\
\text{(C4)} & \Delta_{m \otimes k \otimes n} \cdot (\boldsymbol{\omega}_m \otimes \mathbf{id} \otimes \boldsymbol{\omega}_n \otimes \mathbf{id}) & = & (\mathbf{id} \otimes \Delta_k \otimes \mathbf{id}) \cdot (\mathbf{i}_{m,k} \otimes \mathbf{id} \otimes \mathbf{id}) \\
 & \Delta_{m \otimes k \otimes n} \cdot (\mathbf{id} \otimes \boldsymbol{\omega}_m \otimes \mathbf{id} \otimes \boldsymbol{\omega}_n) & = & (\mathbf{id} \otimes \Delta_k \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{id} \otimes \mathbf{i}_{k,n}) \\
\text{(C5)} & \Delta_m \cdot \Delta_{m \otimes m} & = & \Delta_m \cdot (\Delta_m \otimes \Delta_m) \\
\text{(C6)} & \Delta_{m \otimes k \otimes n} \cdot (\mathbf{id} \otimes \Delta_k \otimes \mathbf{id} \otimes \mathbf{id} \otimes \Delta_k \otimes \mathbf{id}) & = & (\mathbf{id} \otimes \Delta_k \otimes \mathbf{id}) \cdot \Delta_{m \otimes k \otimes k \otimes n} \\
\text{(C7)} & \mathbf{i}_{m,n} \cdot \Delta_{n \otimes m} & = & \Delta_{m \otimes n} \cdot (\mathbf{i}_{m,n} \otimes \mathbf{i}_{m,n}) \\
\text{(C8)} & (\mathbf{id} \otimes \mathbf{i}_{n,k}) \cdot \Delta_{m \otimes k \otimes n} & = & \Delta_{m \otimes k \otimes n} \cdot (\mathbf{id} \otimes \mathbf{i}_{n,k} \otimes \mathbf{id} \otimes \mathbf{i}_{n,k})
\end{array}
$$

Proof See appendix. □

The *closed* action calculus $\text{CAC}(\mathcal{K})$ is defined to be the quotient $\text{CT}(\mathcal{K})/$ CAC, together with a reaction relation which we define below. We call the elements of $\text{CT}(\mathcal{K})/\text{CAC}$ the *closed actions*. Typically, closed actions will be denoted by a, b, c; it should be clear from the context when a denotes an action or a closed action. We also let $t \in a$ denote that closed term t inhabits the equivalence class denoted by a.

3.4 DEFINITION

1. The reaction relation $\rhd \in \mathrm{CT}(\mathcal{K}) \times \mathrm{CT}(\mathcal{K})$ is a binary relation generated by a finite set of reduction rules $\mathcal{R} \subseteq \mathrm{CT}(\mathcal{K}) \times \mathrm{CT}(\mathcal{K})$, which is closed under composition and tensor, such that there is no $t \in \mathrm{CT}(\mathcal{K})$ with $\mathbf{id} \rhd t$.

   We let $\rhd^*$ denote the reflexive, transitive closure of $\rhd$.

2. The reaction relation $\searrow \in \mathrm{CT}(\mathcal{K})/\mathrm{CAC} \times \mathrm{CT}(\mathcal{K})/\mathrm{CAC}$ is defined as follows:

$$a \searrow b \text{ if and only if } s \rhd^* t \text{ for some } s \in a \text{ and } t \in b.$$

Given an action calculus $\mathrm{AC}(\mathcal{K})$, we distinguish the corresponding closed action calculus $\mathrm{CAC}(\mathcal{K}_m)$. The free names of an action calculus provide an interface between the terms inside controls and the rest of the term. For example, using the action calculus $\mathrm{AC}(\boldsymbol{\nu}, \mathbf{out}, \mathbf{box})$, given in example 2.8, we have

$$(\langle z, z\rangle \otimes \mathbf{id}) \cdot (x, y)\mathbf{box}(\langle x, y\rangle) = \mathbf{box}(\langle z, z\rangle).$$

In order to mimic this behaviour in the closed world, we declare for each $K \in \mathcal{K}$ the controls $K_n \in \mathcal{K}_m$ for every $n \in (M, \otimes, \mathbf{id})$. The purpose of the index n is to record the fact that terms inside the control K_n have been closed with respect to some sequence of names $\vec{x}$, where $|\vec{x}| = n$. For example, if we close the term $\mathbf{box}(\langle x, y\rangle)$ using sequence $[x : p, y : q]$ we obtain the closed term $\mathbf{box}_{p\otimes q}(\mathbf{id})$. If we close the same term using sequence $[y : q, x : p]$, we obtain the closed term $\mathbf{box}_{q\otimes p}(\mathbf{i}_{q,p})$. Intuitively, these two closed terms should be connected since they have come from the same term $\mathbf{box}(\langle x, y\rangle)$. This intuition is captured by adding extra equalities to link controls with related indexing. For example, the controls $\mathbf{box}_{p\otimes q}$ and $\mathbf{box}_{q\otimes p}$ are connected by the equality

$$(\mathbf{i}_{p,q} \otimes \mathbf{id}) \cdot \mathbf{box}_{q\otimes p}(t) = \mathbf{box}_{p\otimes q}((\mathbf{i}_{p,q} \otimes \mathbf{id}) \cdot t),$$

which results in $\mathbf{box}_{p\otimes q}(\mathbf{id})$ and $(\mathbf{i}_{p,q} \otimes \mathbf{id}) \cdot \mathbf{box}_{q\otimes p}(\mathbf{i}_{q,p})$ being equal. Using these extra equalities on the indexed controls, we obtain a tight correspondence between the equational theories AC_n and CAC.

3.5 DEFINITION The *closed action calculus for* $\mathrm{AC}(\mathcal{K})$, denoted by $\mathrm{CAC}(\mathcal{K}_m)$, is a closed action calculus with the same underlying monoid as $\mathrm{AC}(\mathcal{K})$, such that

$$\mathcal{K}_m = \{K_n : n \in (M, \otimes, \epsilon) \text{ and } K \in \mathcal{K}\}$$

and if the arity rule accompanying K is

$$\frac{t_1 : m_1 \to n_1 \quad \ldots \quad t_r : m_r \to n_r}{K(t_1, \ldots, t_r) : k \to l}\chi$$

then the arity rule accompanying K_p for each $p \in (M, \otimes \epsilon)$ is

$$\frac{t_1 : p \otimes m_1 \to n_1 \quad \ldots \quad t_r : p \otimes m_r \to n_r}{K_p(t_1, \ldots, t_r) : p \otimes k \to l}\chi$$

We also require that the following *control axioms* hold:

D1 $\boldsymbol{\omega}_q \otimes K_p(t_1, \ldots, t_r) = K_{q\otimes p}(\boldsymbol{\omega}_q \otimes t_1, \ldots, \boldsymbol{\omega}_q \otimes t_r)$

D2 $(\mathbf{id} \otimes \mathbf{i}_{p,q} \otimes \mathbf{id}) \cdot K_{k\otimes q\otimes p}(t_1, \ldots, t_r) = K_{k\otimes p\otimes q}((\mathbf{id} \otimes \mathbf{i}_{p,q} \otimes \mathbf{id}) \cdot t_1, \ldots, (\mathbf{id} \otimes \mathbf{i}_{p,q} \otimes \mathbf{id}) \cdot t_r)$

D3 $(\mathbf{id} \otimes \Delta_p \otimes \mathbf{id}) \cdot K_{k\otimes p\otimes p}(t_1, \ldots, t_r) = K_{k\otimes p}((\mathbf{id} \otimes \Delta_p \otimes \mathbf{id}) \cdot t_1, \ldots, (\mathbf{id} \otimes \Delta_p \otimes \mathbf{id}) \cdot t_r)$

We intend that for the reaction relation accompanying AC($\mathcal{K}$), there is a corresponding relation in CAC($\mathcal{K}_m$). The dynamics for action calculi have not yet been fully explored, and so a general account connecting the dynamics of action calculi and their corresponding closed action calculi is not possible. We explore the dynamics of AC($\boldsymbol{\nu}$, **out**, **box**) and AC($\ulcorner\_\urcorner$, **ap**), and their corresponding closed action calculi in section 6.

4 Translations

This section contains the formal justification for introducing the closed action calculi. We define the *closure functions* $[\![\_]\!]_{\vec{x}} : \mathrm{T}_{\{\vec{x}\}}(\mathcal{K}) \to \mathrm{CT}(\mathcal{K}_m)$ and the function $\langle\!\langle \_ \rangle\!\rangle : \mathrm{CT}(\mathcal{K}_m) \to T_{\emptyset}(\mathcal{K})$, and show that they preserve the equalities given by AC_n and CAC. We also show that $\langle\!\langle [\![t]\!]_{\vec{x}} \rangle\!\rangle =_{\emptyset} (\vec{x})t \in \mathrm{AC}_n$ and $[\![\langle\!\langle t \rangle\!\rangle]\!]_{\vec{x}} = \boldsymbol{\omega}_{|\vec{x}|} \otimes t \in \mathrm{CAC}$, which illustrates the close correspondence between AC_n and CAC.

4.1 Action Calculi to Closed Action Calculi

As the name suggests, we intuitively regard the closure function $[\![\_]\!]_{\vec{x}}$ as closing up the terms in $\mathrm{T}_{\{\vec{x}\}}(\mathcal{K})$ using the sequence of variables $\vec{x}$. Recall from the previous section that we use the arity indexing of controls in $\mathcal{K}_m$ to record the information that terms inside controls have been closed using a sequence of variables of the appropriate arity.

4.1 Definition We define the *closure functions* $[\![\_]\!]_{\vec{x}} : T_{\{\vec{x}\}}(\mathcal{K}) \to \mathrm{CT}(\mathcal{K}_m)$, for each distinct list of names $\vec{x} = [x_1 : p_1, \ldots, x_r : p_r]$, inductively on the structure of terms in $\mathrm{T}_{\{\vec{x}\}}(K)$ as follows:

$$
\begin{array}{rcl}
[\![\mathbf{id}]\!]_{\vec{x}} & = & \boldsymbol{\omega}_{|\vec{x}|} \otimes \mathbf{id} \\
[\![s \cdot t]\!]_{\vec{x}} & = & (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id}_{|\vec{x}|} \otimes [\![s]\!]_{\vec{x}}) \cdot [\![t]\!]_{\vec{x}} \\
[\![s \otimes t]\!]_{\vec{x}} & = & (\Delta_{\vec{x}} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{|\vec{x}|,k} \otimes \mathbf{id}) \cdot ([\![s]\!]_{\vec{x}} \otimes [\![t]\!]_{\vec{x}}) \\
[\![\mathbf{ab}_x(t)]\!]_{\vec{x}} & = & (\mathbf{id} \otimes \Delta_p \otimes \mathbf{id}) \cdot (\mathbf{i}_{|\vec{x}|,p} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes [\![t\{y/x\}]\!]_{\vec{x},y}), \quad y \notin \{\vec{x}\} \\
[\![\langle x \rangle]\!]_{\vec{x}} & = & \boldsymbol{\omega}_{p_i \otimes \ldots \otimes p_{i-1}} \otimes \mathbf{id}_{p_i} \otimes \boldsymbol{\omega}_{p_{i+1} \otimes \ldots \otimes p_r}, \quad x = x_i \\
[\![\boldsymbol{\omega}_p]\!]_{\vec{x}} & = & \boldsymbol{\omega}_{|\vec{x}|} \otimes \boldsymbol{\omega}_p \\
[\![K(t_1, \ldots, t_r)]\!]_{\vec{x}} & = & K_{|\vec{x}|}([\![t_1]\!]_{\vec{x}}, \ldots, [\![t_r]\!]_{\vec{x}})
\end{array}
$$

Whenever we write $[\![\_]\!]_{\vec{x}}$, we assume that $\vec{x}$ is a list of distinct names. We shall often wish to distinguish a particular name in such a list. We therefore write $\vec{x}, y, \vec{z}$ to denote a sequence of distinct names with the name y distinguished.

Remark We intuitively regard the operators Δ_m and $\mathbf{i}_{m,n}$ in CAC as having the same role as the operators $\mathbf{copy}_m$ and $\mathbf{p}_{m,n}$ in AC. It is not difficult to show that $[\![\mathbf{copy}_m]\!]_{\vec{x}} = \boldsymbol{\omega}_{|\vec{x}|} \otimes \Delta_m \in \mathrm{CAC}$ and $[\![\mathbf{p}_{m,n}]\!]_{\vec{x}} = \boldsymbol{\omega}_{|\vec{x}|} \otimes \mathbf{i}_{m,n} \in \mathrm{CAC}$, which justifies this intuition.

In the definition of $[\![\_]\!]_{\vec{x}}$, the abstraction case is perhaps the most confusing. The idea of viewing the behaviour of $[\![\_]\!]_{\vec{x}}$ as the closure of a term using $\vec{x}$ becomes clearer when we use the alternative form of abstraction, given by

$$(x)t \stackrel{def}{=} \mathbf{ab}_x(t) \cdot (w_p \otimes \mathbf{id}_m), \quad x : p$$

The following proposition shows that $[\![(x)t]\!]_{\vec{x}} = [\![t\{y/x\}]\!]_{\vec{x},y}$, for $y \notin \{\vec{x}\}$.

4.2 Proposition

$$[\![\mathbf{ab}_x(t) \cdot (\boldsymbol{\omega}_p \otimes \mathbf{id})]\!]_{\vec{x}} = [\![t\{y/x\}]\!]_{\vec{x},y} \in \mathrm{CAC}, \text{for some } y : p \notin \{\vec{x}\}.$$

Proof

$$\begin{array}{cl}
\multicolumn{2}{l}{[\![\mathbf{ab}_x(t) \cdot (\boldsymbol{\omega}_p \otimes \mathbf{id})]\!]_{\vec{x}}} \\
= & (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes [\![\mathbf{ab}_x(t)]\!]_{\vec{x}}) \cdot [\![\boldsymbol{\omega}_p \otimes \mathbf{id}]\!]_{\vec{x}} \\
= & (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{id} \otimes \Delta_p \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{|\vec{x}|,p} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{id} \otimes [\![t\{y/x\}]\!]_{\vec{x},y}) \\
 & (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{|\vec{x}|,p} \otimes \mathbf{id}) \cdot (\boldsymbol{\omega}_{|\vec{x}|} \otimes \boldsymbol{\omega}_p \otimes \boldsymbol{\omega}_{|\vec{x}|} \otimes \mathbf{id}) \\
\stackrel{C3,\ B3,\ C1}{=} & (\Delta_{|\vec{x}|} \otimes \Delta_p \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{|\vec{x}|,p} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{id} \otimes [\![t\{y/x\}]\!]_{\vec{x},y}) \cdot (\boldsymbol{\omega}_{|\vec{x}|} \otimes \boldsymbol{\omega}_p \otimes \mathbf{id}) \\
\stackrel{B3,\ C1}{=} & (\Delta_{|\vec{x}|} \otimes \Delta_p \otimes \mathbf{id}) \cdot (\boldsymbol{\omega}_{|\vec{x}|} \otimes \mathbf{id} \otimes \boldsymbol{\omega}_p \otimes \mathbf{id} \otimes \mathbf{id}) \cdot [\![t\{y/x\}]\!]_{\vec{x},y} \\
\stackrel{C3}{=} & [\![t\{y/x\}]\!]_{\vec{x},y}
\end{array}$$

Notice that $[\![\mathbf{ab}_x(t)]\!]_{\vec{x}}$ and $[\![(x)t]\!]_{\vec{x}}$ are defined using a chosen $y \notin \{\vec{x}\}$. The next lemma shows that this choice of y is not important.

4.3 Lemma

$$[\![t]\!]_{\vec{x},u,\vec{y}} = [\![t\{v/u\}]\!]_{\vec{x},v,\vec{y}} \in \mathrm{CAC}, \text{ if } u : p \text{ and } v : p.$$

Proof The proof is by induction on the structure of t. When t is $\mathbf{id}, \langle z \rangle$ or $\boldsymbol{\omega}$, the result is immediate. When t has the form $t_1 \cdot t_2$, $t_1 \otimes t_2$ or $K(t_1, \ldots, t_r)$, the result is immediate using the induction hypothesis. The only non-trivial case is when t has the form $(z)s$.

$$\begin{array}{rcll}
[\![(z)s]\!]_{\vec{x},u,\vec{y}} & = & [\![s\{w/z\}]\!]_{\vec{x},u,\vec{y},w} & \\
 & \stackrel{IH}{=} & [\![s\{w'/z\}]\!]_{\vec{x},u,\vec{y},w'} & \quad w' \notin \{\vec{x}, u, \vec{y}, v\} \\
 & \stackrel{IH}{=} & [\![s\{w'/z\}\{v/u\}]\!]_{\vec{x},v,\vec{y},w'} & \\
 & = & [\![(w')(s\{w'/z\}\{v/u\})]\!]_{\vec{x},v,\vec{y}} & \\
 & = & [\![((z)s)\{v/u\}]\!]_{\vec{x},v,\vec{y}} &
\end{array}$$

The following three lemmas illustrate the connection between the closure functions $[\![\_]\!]_{\vec{x}}$ and $[\![\_]\!]_{\vec{y}}$, when $\{\vec{x}\} \subseteq \{\vec{y}\}$. They are proved by induction on the structure of term t. In each proof, the interesting case is when t has the form $K(t_1, \ldots, t_r)$, since this case shows that the proofs rely directly on the control axioms D1–D3 introduced in section 3. The full proofs can be found in the appendix.

4.4 Lemma

$$[\![t]\!]_{y,\vec{x}} = \boldsymbol{\omega}_p \otimes [\![t]\!]_{\vec{x}} \in \mathrm{CAC}, \text{when } y : p \notin fn(t).$$

Proof

$$\begin{array}{rcl} [\![K(t_1,\ldots,t_r)]\!]_{y,\vec{x}} & = & K_{p\otimes|\vec{x}|}([\![t_1]\!]_{y,\vec{x}},\ldots,[\![t_r]\!]_{y,\vec{x}}) \\ & \stackrel{IH}{=} & K_{p\otimes|\vec{x}|}(\boldsymbol{\omega}_p \otimes [\![t_1]\!]_{\vec{x}},\ldots,\boldsymbol{\omega}_p \otimes [\![t_r]\!]_{\vec{x}}) \\ & \stackrel{D1}{=} & \boldsymbol{\omega}_p \otimes K_{|\vec{x}|}([\![t_1]\!]_{\vec{x}},\ldots,[\![t_r]\!]_{\vec{x}}) \\ & = & \boldsymbol{\omega}_p \otimes [\![K(t_1,\ldots,t_r)]\!]_{\vec{x}} \end{array}$$

4.5 Lemma

$$[\![t]\!]_{\vec{x},\vec{y},\vec{z}} = (\mathbf{id} \otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|} \otimes \mathbf{id}) \cdot [\![t]\!]_{\vec{x},\vec{z},\vec{y}} \in \text{CAC}$$

Proof

$$\begin{array}{l} [\![K(t_1,\ldots,t_r)_{\vec{x},\vec{y},\vec{z}}]\!] \\ \quad = \quad K_{|\vec{x}|\otimes|\vec{y}|\otimes|\vec{z}|}([\![t_1]\!]_{\vec{x},\vec{y},\vec{z}},\ldots,[\![t_r]\!]_{\vec{x},\vec{y},\vec{z}}) \\ \quad \stackrel{IH}{=} \quad K_{|\vec{x}|\otimes|\vec{y}|\otimes|\vec{z}|}((\mathbf{id} \otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|} \otimes \mathbf{id}) \cdot [\![t_1]\!]_{\vec{x},\vec{z},\vec{y}},\ldots,(\mathbf{id} \otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|} \otimes \mathbf{id}) \cdot [\![t_r]\!]_{\vec{x},\vec{z},\vec{y}}) \\ \quad \stackrel{D2}{=} \quad (\mathbf{id} \otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|} \otimes \mathbf{id}) \cdot K_{|\vec{x}|\otimes|\vec{z}|\otimes|\vec{y}|}([\![t_1]\!]_{\vec{x},\vec{z},\vec{y}},\ldots,[\![t_r]\!]_{\vec{x},\vec{z},\vec{y}}) \end{array}$$

4.6 Lemma

$$(\mathbf{id} \otimes \Delta_p \otimes \mathbf{id} \otimes \mathbf{id}) \cdot [\![t]\!]_{\vec{x},u,v,\vec{y}} = [\![t\{u/v\}]\!]_{\vec{x},u,\vec{y}} \in \text{CAC}, \text{ for } u:p \text{ and } v:p$$

Proof

$$\begin{array}{l} (\mathbf{id} \otimes \Delta_p \otimes \mathbf{id}) \cdot [\![K(t_1,\ldots,t_r)]\!]_{\vec{x},u,v} \\ \quad = \quad (\mathbf{id} \otimes \Delta_p \otimes \mathbf{id}) \cdot K_{|\vec{x}|\otimes p\otimes p}([\![t_1]\!]_{\vec{x},u,v},\ldots,[\![t_r]\!]_{\vec{x},u,v}) \\ \quad \stackrel{D3}{=} \quad K_{|\vec{x}|\otimes p}((\mathbf{id} \otimes \Delta_p \otimes \mathbf{id}) \cdot [\![t_1]\!]_{\vec{x},u,v},\ldots,(\mathbf{id} \otimes \Delta_p \otimes \mathbf{id}) \cdot [\![t_r]\!]_{\vec{x},u,v}) \\ \quad \stackrel{IH}{=} \quad K_{|\vec{x}|\otimes p}([\![t_1\{u/v\}]\!]_{\vec{x},u},\ldots,[\![t_r\{u/v\}]\!]_{\vec{x},u}) \end{array}$$

The following lemma consists of technical results used to simplify the proof that the translation $[\![\_]\!]_{\vec{x}}$ preserves equality.

4.7 Lemma The following hold in CAC.

1. $[\![s \otimes t]\!]_{\vec{x}} = [\![s]\!]_{\vec{x}} \otimes [\![t]\!]_{[\,]}$, if $fn(t) = \emptyset$
2. $[\![t \otimes s]\!]_{\vec{x}} = (\mathbf{i}_{|\vec{x}|,m} \otimes \mathbf{id}) \cdot ([\![t]\!]_{[\,]} \otimes [\![s]\!]_{\vec{x}})$, if $fn(t) = \emptyset$ and $t : m \to n$.
3. $[\![s \cdot t]\!]_{\vec{x}} = [\![s]\!]_{\vec{x}} \cdot [\![t]\!]_{[\,]}$, if $fn(t) = \emptyset$
4. $[\![t \cdot s]\!]_{\vec{x}} = (\mathbf{id} \otimes [\![t]\!]_{[\,]}) \cdot [\![s]\!]_{\vec{x}}$, if $fn(t) = \emptyset$
5. $[\![\mathbf{p}_{m,n}]\!]_{\vec{x}} = \boldsymbol{\omega}_{|\vec{x}|} \otimes \mathbf{i}_{m,n}$
6. $[\![\langle\vec{x}\rangle]\!]_{\vec{x}} = \mathbf{id}_{|\vec{x}|}$, if $|\vec{x}| = m \neq \varepsilon$.

Proof See appendix. □

We are now able to prove that the closure functions preserve the equalities given by AC_n. Lemma 4.5, 4.6 and 4.7 are used to show that the axioms γ and σ are preserved under translation.

4.8 Theorem

$$s =_{\{\vec{x}\}} t \in \mathrm{AC}_n(\mathcal{K}) \text{ implies } [\![s]\!]_{\vec{x}} = [\![t]\!]_{\vec{x}} \in \mathrm{CAC}.$$

Proof The proof that the translation $[\![\_]\!]_{\vec{x}}$ preserves the basic axioms involves simple equational reasoning using the axioms A1–A6 and B1–B8. (The proof can be found in the appendix.) It is easy to prove that the reflexive, symmetric and transitive rules, and the structural rules are preserved under translation; the structural rule for abstraction requires lemma 4.3. Here we prove that the concrete axioms are preserved.

The proof that the axiom γ is preserved requires lemmas 4.4 and 4.5.

$$\begin{aligned}
[\![\boldsymbol{\omega}_p \otimes s]\!]_{\vec{x}} &\stackrel{4.7}{=} (\mathbf{i}_{|\vec{x}|,p} \otimes \mathbf{id}) \cdot (\boldsymbol{\omega}_p \otimes [\![s]\!]_{\vec{x}}) \\
&\stackrel{4.4}{=} (\mathbf{i}_{|\vec{x}|,p} \otimes \mathbf{id}) \cdot [\![s]\!]_{y,\vec{x}}, \quad y \notin \{\vec{x}\} \\
&\stackrel{4.5}{=} [\![s]\!]_{\vec{x},y} \\
&= [\![(y)s]\!]_{\vec{x}}
\end{aligned}$$

The proof that the axiom δ is preserved is easy.

$$\begin{aligned}
[\![(x)(\langle x\rangle \otimes \mathbf{id})]\!]_{\vec{x}} &= [\![\langle y\rangle \otimes \mathbf{id}]\!]_{\vec{x},y} \quad y \notin \{\vec{x}\} \\
&\stackrel{4.7}{=} \boldsymbol{\omega}_{|\vec{x}|} \otimes \mathbf{id} \otimes \mathbf{id} \\
&= [\![\mathbf{id}]\!]_{\vec{x}}
\end{aligned}$$

Proving that σ is preserved requires lemmas 4.5 and 4.6. Let $\vec{x}$ be $\vec{y}, u : p, \vec{z}$ such that $|\vec{y}| = k$ and $|\vec{z}| = l$.

$$\begin{aligned}
&[\![(\langle u\rangle \otimes \mathbf{id}) \cdot (x)s]\!]_{\vec{y},u,\vec{z}} \\
&\stackrel{4.7,\ C1}{=} (\Delta_{k\otimes p\otimes l} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \boldsymbol{\omega}_k \otimes \mathbf{id} \otimes \boldsymbol{\omega}_l \otimes \mathbf{id}) \cdot [\![s\{v/x\}]\!]_{\vec{y},u,\vec{z},v} \\
&\stackrel{C4}{=} (\mathbf{id} \otimes \Delta_p \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{id} \otimes \mathbf{i}_{p,l} \otimes \mathbf{id}) \cdot [\![s\{v/x\}]\!]_{\vec{y},u,\vec{z},v} \\
&\stackrel{C3,4.5}{=} (\mathbf{id} \otimes \Delta_p \otimes \mathbf{id} \otimes \mathbf{id}) \cdot [\![s\{v/x\}]\!]_{\vec{y},u,v,\vec{z}} \\
&\stackrel{4.6}{=} [\![s\{v/x\}\{u/v\}]\!]_{\vec{y},u,\vec{z}} \\
&= [\![s\{u/x\}]\!]_{\vec{y},u,\vec{z}}
\end{aligned}$$

The proof that ζ is preserved by the translation involves simple equational reasoning.

$$\begin{aligned}
&[\![\mathbf{i}_{k,m} \cdot (t \otimes s)]\!]_{\vec{x}} \\
&\stackrel{4.7,\ C1}{=} (\mathbf{id} \otimes \mathbf{i}_{k,m}) \cdot (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{|\vec{x}|,m} \otimes \mathbf{id}) \cdot ([\![t]\!]_{\vec{x}} \otimes [\![s]\!]_{\vec{x}}) \\
&\stackrel{B3,B5,B4}{=} (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{|\vec{x}|\otimes k,m}) \cdot ([\![t]\!]_{\vec{x}} \otimes [\![s]\!]_{\vec{x}}) \cdot \mathbf{i}_{n,l} \cdot \mathbf{i}_{l,n} \\
&\stackrel{B3}{=} (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{|\vec{x}|\otimes k,m}) \cdot \mathbf{i}_{|\vec{x}|\otimes m,|\vec{x}|\otimes k} \cdot ([\![s]\!]_{\vec{x}} \otimes [\![t]\!]_{\vec{x}}) \cdot \mathbf{i}_{l,n} \\
&\stackrel{B5,\ B4}{=} (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{i}_{|\vec{x}|,|\vec{x}|\otimes k} \otimes \mathbf{id}) \cdot ([\![s]\!]_{\vec{x}} \otimes [\![t]\!]_{\vec{x}}) \cdot \mathbf{i}_{l,n} \\
&\stackrel{C2}{=} (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{i}_{|\vec{x}|,|\vec{x}|} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{|\vec{x}|,k} \otimes \mathbf{id}) \cdot ([\![s]\!]_{\vec{x}} \otimes [\![t]\!]_{\vec{x}}) \cdot \mathbf{i}_{l,n} \\
&\stackrel{B2}{=} [\![s \otimes t]\!]_{\vec{x}} \cdot \mathbf{i}_{l,n} \\
&\stackrel{4.7,\ C1}{=} [\![(s \otimes t) \cdot \mathbf{i}_{l,n}]\!]_{\vec{x}}
\end{aligned}$$

4.9 COROLLARY $s = t \in \mathrm{AC}$ implies $[\![s]\!]_{fn(s,t)} = [\![t]\!]_{fn(s,t)} \in \mathrm{CAC}$.

4.2 Closed Action Calculi to Action Calculi

In this section, we define the translation $\langle\!\langle \_ \rangle\!\rangle : \mathrm{CT}(\mathcal{K}_m) \to \mathrm{T}_\emptyset(\mathcal{K})$ and show that it preserves the equalities given by CAC and the control axioms. We also show that this translation, together with the closure functions defined in the previous section, yield a tight correspondence between the static parts of $\mathrm{AC}(\mathcal{K})$ and $\mathrm{CAC}(\mathcal{K}_m)$, in the sense that $\langle\!\langle [\![t]\!]_{\vec{x}} \rangle\!\rangle =_\emptyset (\vec{x})t \in \mathrm{AC}_n$ and $[\![\langle\!\langle t \rangle\!\rangle]\!]_{\vec{x}} = \boldsymbol{\omega}_{|\vec{x}|} \otimes t \in \mathrm{CAC}$.

Using the intuition that the operators Δ_m and $\mathbf{i}_{m,n}$ in CAC play essentially the same role as the operators $\mathbf{copy}_m$ and $\mathbf{p}_{m,n}$ in AC, we view the translation $\langle\!\langle \_ \rangle\!\rangle$ as the identity function in all cases, except the control case. Recall that the indexing in $\mathcal{K}_m$ is used to record the information that the terms inside the controls have been closed using a sequence of variables of the appropriate arity. We use this information during translation in an essential way to incorporate free variables inside the controls.

4.10 DEFINITION The translation $\langle\!\langle \_ \rangle\!\rangle : \mathrm{CT}(\mathcal{K}_m) \to \mathrm{T}_\emptyset(\mathcal{K})$ is defined inductively on the structure of closed terms as follows:

$$\begin{array}{rcl}
\langle\!\langle \mathbf{id} \rangle\!\rangle & = & \mathbf{id} \\
\langle\!\langle s \cdot t \rangle\!\rangle & = & \langle\!\langle s \rangle\!\rangle \cdot \langle\!\langle t \rangle\!\rangle \\
\langle\!\langle s \otimes t \rangle\!\rangle & = & \langle\!\langle s \rangle\!\rangle \otimes \langle\!\langle t \rangle\!\rangle \\
\langle\!\langle \Delta_m \rangle\!\rangle & = & \mathbf{copy}_m \\
\langle\!\langle \mathbf{i}_{m,n} \rangle\!\rangle & = & \mathbf{p}_{m,n} \\
\langle\!\langle \boldsymbol{\omega} \rangle\!\rangle & = & \boldsymbol{\omega} \\
\langle\!\langle K_m(t_1, \ldots, t_r) \rangle\!\rangle & = & (\vec{x})K((\langle\vec{x}\rangle \otimes \mathbf{id}) \cdot \langle\!\langle s \rangle\!\rangle, \ldots, (\langle\vec{x}\rangle \otimes \mathbf{id}) \cdot \langle\!\langle t_r \rangle\!\rangle), \\
 & & \text{where } m = p_1 \otimes \ldots \otimes p_s \text{ and } \vec{x} \text{ is } [x_1 : p_1, \ldots, x : s : p_s].
\end{array}$$

The proof that $\langle\!\langle \_ \rangle\!\rangle$ preserves equality is easier that the corresponding proof for $[\![\_]\!]$. This is due to the fact that we can intuitively view $\langle\!\langle \_ \rangle\!\rangle$ as the identity mapping, except when mapping the control terms.

4.11 THEOREM

$$s = t \in \mathrm{CAC} \text{ implies } \langle\!\langle s \rangle\!\rangle =_\emptyset \langle\!\langle t \rangle\!\rangle \in \mathrm{AC}_n$$

Proof The proof that the axioms A1–A6 and B1–B8 are preserved under translation is easy, since $\langle\!\langle \_ \rangle\!\rangle$ preserves the structure of these axioms and the corresponding equalities hold in AC_n. The proof that the structural rules, and the reflexive, symmetric and transitive rules are preserved is trivial using the induction hypothesis. The interesting cases are the axioms D1–D3.

$$\begin{array}{l}
\langle\!\langle K_{q\otimes p}(\boldsymbol{\omega}_q \otimes t_1, \ldots, \boldsymbol{\omega}_q \otimes t_r) \rangle\!\rangle \\
\quad =_\emptyset \quad (\vec{y}, \vec{x})K((\langle\vec{y},\vec{x}\rangle \otimes \mathbf{id}) \cdot (\boldsymbol{\omega} \otimes \langle\!\langle t_1 \rangle\!\rangle), \ldots, (\langle\vec{y},\vec{x}\rangle \otimes \mathbf{id}) \cdot (\boldsymbol{\omega} \otimes \langle\!\langle t_r \rangle\!\rangle)), \quad |\vec{y}| = q, |\vec{x}| = p \\
\quad \overset{\gamma;\sigma}{=}_\emptyset \quad (\vec{y}, \vec{x})K((\langle\vec{x}\rangle \otimes \mathbf{id}) \cdot \langle\!\langle t_1 \rangle\!\rangle, \ldots, (\langle\vec{x}\rangle \otimes \mathbf{id}) \cdot \langle\!\langle t_r \rangle\!\rangle) \\
\quad =_\emptyset \quad (\vec{y})\langle\!\langle K_p(t_1, \ldots, t_r) \rangle\!\rangle \\
\quad \overset{\gamma}{=}_\emptyset \quad \boldsymbol{\omega}_q \otimes \langle\!\langle K_p(t_1, \ldots, t_r) \rangle\!\rangle
\end{array}$$

$$
\begin{aligned}
&\langle\!\langle K_{k\otimes p\otimes q}((\mathbf{id}\otimes\mathbf{i}_{p,q}\otimes\mathbf{id})\cdot t_1,\ldots,(\mathbf{id}\otimes\mathbf{i}_{p,q}\otimes\mathbf{id})\cdot t_r)\rangle\!\rangle\\
&\quad =_\emptyset \quad (\vec{x},\vec{y},\vec{z})K(((\langle\vec{x},\vec{y},\vec{z}\rangle\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{p}_{p,q}\otimes\mathbf{id})\cdot\langle\!\langle t_1\rangle\!\rangle,\ldots,((\langle\vec{x},\vec{y},\vec{z}\rangle\otimes\mathbf{id})\cdot\\
&\qquad\quad (\mathbf{id}\otimes\mathbf{p}_{p,q}\otimes\mathbf{id})\cdot\langle\!\langle t_r\rangle\!\rangle)),\quad |\vec{x}|=k,|\vec{y}|=p,|\vec{z}|=q\\
&\quad \overset{\sigma}{=}_\emptyset \quad (\vec{x},\vec{y},\vec{z})K(((\langle\vec{x},\vec{z},\vec{y}\rangle\otimes\mathbf{id})\cdot\langle\!\langle t_1\rangle\!\rangle,\ldots,((\langle\vec{x},\vec{z},\vec{y}\rangle\otimes\mathbf{id})\cdot\langle\!\langle t_r\rangle\!\rangle)\\
&\quad \overset{\sigma}{=}_\emptyset \quad (\vec{u},\vec{v},\vec{w})(((\langle\vec{u},\vec{w},\vec{v}\rangle\otimes\mathbf{id})\cdot(\vec{x},\vec{z},\vec{y})K(((\langle\vec{x},\vec{z},\vec{y}\rangle\otimes\mathbf{id})\cdot\langle\!\langle t_1\rangle\!\rangle,\ldots,((\langle\vec{x},\vec{z},\vec{y}\rangle\otimes\mathbf{id})\cdot\langle\!\langle t_r\rangle\!\rangle)\\
&\quad \overset{2.4,\sigma}{=}_\emptyset \quad (\mathbf{id}\otimes\mathbf{p}_{p,q}\otimes\mathbf{id})\cdot\langle\!\langle K_{k\otimes q\otimes p}(t_1,\ldots,t_r)\rangle\!\rangle\\
&\quad =_\emptyset \quad \langle\!\langle(\mathbf{id}\otimes\mathbf{i}_{p,q}\otimes\mathbf{id})\cdot K_{k\otimes q\otimes p}(t_1,\ldots,t_r)\rangle\!\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\langle\!\langle K_{k\otimes p}((\mathbf{id}\otimes\Delta_p\otimes\mathbf{id})\cdot t_1,\ldots,(\mathbf{id}\otimes\Delta_p\otimes\mathbf{id})\cdot t_r)\rangle\!\rangle\\
&\quad =_\emptyset \quad (\vec{x},\vec{y})K(((\langle\vec{x},\vec{y}\rangle\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{copy}_p\otimes\mathbf{id})\cdot\langle\!\langle t_1\rangle\!\rangle,\ldots,((\langle\vec{x},\vec{y}\rangle\otimes\mathbf{id})\cdot\\
&\qquad\quad (\mathbf{id}\otimes\mathbf{copy}_p\otimes\mathbf{id})\cdot\langle\!\langle t_r\rangle\!\rangle),\quad |\vec{x}|=k,|\vec{y}|=p\\
&\quad \overset{\sigma}{=}_\emptyset \quad (\vec{x},\vec{y})K(((\langle\vec{x},\vec{y},\vec{y}\rangle\otimes\mathbf{id})\cdot\langle\!\langle t_1\rangle\!\rangle,\ldots,((\langle\vec{x},\vec{y},\vec{y}\rangle\otimes\mathbf{id})\cdot\langle\!\langle t_r\rangle\!\rangle)\\
&\quad \overset{\sigma}{=}_\emptyset \quad (\vec{x},\vec{y})(((\langle\vec{x},\vec{y},\vec{y}\rangle\otimes\mathbf{id})\cdot(\vec{u},\vec{v},\vec{w})K(((\langle\vec{u},\vec{v},\vec{w}\rangle\otimes\mathbf{id})\cdot\langle\!\langle t_1\rangle\!\rangle,\ldots,((\langle\vec{u},\vec{v},\vec{w}\rangle\otimes\mathbf{id})\cdot\langle\!\langle t_r\rangle\!\rangle))\\
&\quad \overset{2.4,\sigma}{=}_\emptyset \quad (\mathbf{id}\otimes\mathbf{copy}_p\otimes\mathbf{id})\cdot\langle\!\langle K_{k\otimes p\otimes p}(t_1,\ldots,t_r)\rangle\!\rangle\\
&\quad =_\emptyset \quad \langle\!\langle(\mathbf{id}\otimes\Delta_p\otimes\mathbf{id})\cdot K_{k\otimes p\otimes p}(t_1,\ldots,t_r)\rangle\!\rangle
\end{aligned}
$$

4.12 Theorem

$$
\begin{array}{lrcl}
(1) & \langle\!\langle[\![t]\!]_{\vec{x}}\rangle\!\rangle & =_\emptyset & (\vec{x})t\in \mathrm{AC}_n,\ \text{if } fn(t)\subseteq\{\vec{x}\}.\\
(2) & [\![\langle\!\langle s\rangle\!\rangle]\!]_{\vec{x}} & = & \boldsymbol{\omega}_{|\vec{x}|}\otimes s\in \mathrm{CAC}.
\end{array}
$$

Proof Both parts are proved by induction on the structure of t.

1. When t is $\mathbf{id}$, $\langle x\rangle$ or $\boldsymbol{\omega}$, the result follows easily using the axiom γ. The case when t is $(x)s$ follows immediately from the induction hypothesis and α. We prove in detail the cases when t has the form $K(t_1,\ldots,t_r)$, $t_1\cdot t$ or $t_1\otimes t_2$.

$$
\begin{aligned}
\langle\!\langle[\![K(t_1,\ldots,t_r)]\!]_{\vec{x}}\rangle\!\rangle &=_\emptyset \langle\!\langle K_{|\vec{x}|}([\![t_1]\!]_{\vec{x}},\ldots,[\![t_r]\!]_{\vec{x}})\rangle\!\rangle\\
&\overset{IH}{=}_\emptyset (\vec{x})K(((\langle\vec{x}\rangle\otimes\mathbf{id})\cdot(\vec{x})t_1,\ldots,((\langle\vec{x}\rangle\otimes\mathbf{id})\cdot(\vec{x})t_r)\\
&\overset{\sigma}{=}_\emptyset (\vec{x})K(t_1,\ldots,t_r)
\end{aligned}
$$

$$
\begin{aligned}
\langle\!\langle[\![t_1\cdot t_2]\!]_{\vec{x}}\rangle\!\rangle &=_\emptyset \langle\!\langle(\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes[\![t_1]\!]_{\vec{x}})\cdot[\![t_2]\!]_{\vec{x}}\rangle\!\rangle\\
&\overset{IH}{=}_\emptyset (\mathbf{copy}_{|\vec{x}|}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes(\vec{x})t_1)\cdot(\vec{x})t_2\\
&\overset{\sigma}{=}_\emptyset (\vec{y})(((\langle\vec{y},\vec{y}\rangle\otimes\mathbf{id})\cdot(\mathbf{id}\otimes(\vec{x})t_1)\cdot(\vec{x})t_2)\\
&\overset{\sigma}{=}_\emptyset (\vec{y})(((\langle\vec{y}\rangle\otimes t_1\{\vec{y}/\vec{x}\}).(\vec{x})t_2)\\
&\overset{\sigma}{=}_\emptyset (\vec{y})(t_1\{\vec{y}/\vec{x}\}\cdot t_1\{\vec{y}/\vec{x}\})\\
&\overset{\alpha}{=}_\emptyset (\vec{x})(t_1\cdot t_2)
\end{aligned}
$$

$$
\begin{aligned}
\langle\!\langle[\![t_1\otimes t_2]\!]_{\vec{x}}\rangle\!\rangle &=_\emptyset \langle\!\langle(\Delta_{\vec{x}}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{\vec{x},k}\otimes\mathbf{id})\cdot([\![t_1]\!]_{\vec{x}}\otimes[\![t_2]\!]_{\vec{x}})\rangle\!\rangle\\
&\overset{IH}{=}_\emptyset (\mathbf{copy}_{|\vec{x}|}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{p}_{|\vec{x}|,k}\otimes\mathbf{id})\cdot((\vec{x})t_1\otimes(\vec{x})t_2)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{2.4}{=}_{\emptyset} (\vec{y})((\langle\vec{y}\rangle \otimes \mathbf{id} \otimes \langle\vec{y}\rangle \otimes \mathbf{id}) \cdot ((\vec{x})t_1 \otimes (\vec{x})t_2)) \\
&\overset{\sigma}{=}_{\emptyset} (\vec{y})(t_1\{\vec{y}/\vec{x}\} \otimes t_2\{\vec{y}/\vec{x}\}) \\
&\overset{\alpha}{=}_{\emptyset} (\vec{x})(t_1 \otimes t_2)
\end{aligned}
$$

2. When s is $\mathbf{id}$ or $\boldsymbol{\omega}$, the result follows trivially. When s is $\mathbf{i}_{m,n}$, the result follows trivially from lemma 4.7. We present the remaining cases for s.

$$
\begin{aligned}
[\![\langle\!\langle \Delta_m \rangle\!\rangle]\!]_{\vec{x}} &\overset{4.7}{=} \boldsymbol{\omega}_{|\vec{x}|} \otimes [\![\mathbf{copy}_m]\!]_{[\,]} \\
&= \boldsymbol{\omega}_{|\vec{x}|} \otimes [\![\langle\vec{x}\rangle \otimes \langle\vec{x}\rangle]\!]_{\vec{x}} \\
&\overset{C1,4.7}{=} \boldsymbol{\omega}_{|\vec{x}|} \otimes \Delta_{|\vec{x}|} \cdot (\mathbf{id} \otimes \mathbf{id}) \\
&= \boldsymbol{\omega}_{|\vec{x}|} \otimes \Delta_{|\vec{x}|}
\end{aligned}
$$

$$
\begin{aligned}
[\![\langle\!\langle s_1 \cdot s_2 \rangle\!\rangle]\!]_{\vec{x}} &\overset{IH}{=} (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \boldsymbol{\omega}_{|\vec{x}|} \otimes s_1) \cdot (\boldsymbol{\omega}_{|\vec{x}|} \otimes s_2) \\
&\overset{C3}{=} \boldsymbol{\omega}_{|\vec{x}|} \otimes s_1 \cdot s_2
\end{aligned}
$$

$$
\begin{aligned}
[\![\langle\!\langle s_1 \otimes s_2 \rangle\!\rangle]\!]_{|x} &\overset{IH}{=} (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{|\vec{x}|,k} \otimes \mathbf{id}) \cdot (\boldsymbol{\omega}_{|\vec{x}|} \otimes s_1 \otimes \boldsymbol{\omega}_{|\vec{x}|} \otimes s_2) \\
&\overset{B3,\ C1}{=} (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\boldsymbol{\omega}_{|\vec{x}|} \otimes \boldsymbol{\omega}_{|\vec{x}|} \otimes s_1 \otimes s_2) \\
&\overset{C3}{=} \boldsymbol{\omega}_{|\vec{x}|} \otimes s_1 \otimes s_2
\end{aligned}
$$

$$
\begin{aligned}
[\![\langle\!\langle K_m(s_1,\ldots,s_r) \rangle\!\rangle]\!]_{\vec{x}} &= [\![(\vec{y})K(((\langle\vec{y}\rangle \otimes \mathbf{id}) \cdot \langle\!\langle s_1 \rangle\!\rangle, \ldots, (\langle\vec{y}\rangle \otimes \mathbf{id}) \cdot \langle\!\langle s_r \rangle\!\rangle)]\!]_{\vec{x}}, \quad |\vec{y}| = m \\
&= [\![K(((\langle\vec{z}\rangle \otimes \mathbf{id}) \cdot \langle\!\langle s_1 \rangle\!\rangle, \ldots, (\langle\vec{z}\rangle \otimes \mathbf{id}) \cdot \langle\!\langle s_r \rangle\!\rangle)]\!]_{\vec{x},\vec{z}} \\
&= K_{|\vec{x}| \otimes m}([\![(\langle z\rangle \otimes \mathbf{id}) \cdot \langle\!\langle s_1 \rangle\!\rangle]\!]_{\vec{x},\vec{z}}, \ldots, [\![(\langle z\rangle \otimes \mathbf{id}) \cdot \langle\!\langle s_r \rangle\!\rangle]\!]_{\vec{x},\vec{z}}) \\
&\overset{4.7}{=} K_{|\vec{x}| \otimes m}([\![\langle z\rangle \otimes \mathbf{id}]\!]_{\vec{x},\vec{z}} \cdot s_1, \ldots, [\![\langle z\rangle \otimes \mathbf{id}]\!]_{\vec{x},\vec{z}} \cdot s_r) \\
&\overset{C1,IH}{=} K_{|\vec{x}| \otimes k}(\boldsymbol{\omega}_{|\vec{x}|} \otimes s_1, \ldots, \boldsymbol{\omega}_{|\vec{x}|} \otimes s_r) \\
&\overset{D1}{=} \boldsymbol{\omega}_{|\vec{x}|} \otimes K_m(s_1,\ldots,s_r)
\end{aligned}
$$

4.13 COROLLARY

1. $[\![s_1]\!]_{\vec{x}} = [\![s_2]\!]_{\vec{x}} \in \mathrm{CAC}$ implies $s_1 =_{\{\vec{x}\}} s_2 \in \mathrm{AC}_n$, when $fn(s_1, s_2) \subseteq \{\vec{x}\}$.
2. $\langle\!\langle t_1 \rangle\!\rangle =_{\emptyset} \langle\!\langle t_2 \rangle\!\rangle \in \mathrm{AC}_n$ implies $t_1 = t_2 \in \mathrm{CAC}$.

Proof

(1)
$$
\begin{aligned}
[\![s_1]\!]_{\vec{x}} = [\![s_2]\!]_{\vec{x}} &\overset{4.11}{\Rightarrow} \langle\!\langle [\![s_1]\!]_{\vec{x}} \rangle\!\rangle =_{\emptyset} \langle\!\langle [\![s_2]\!]_{\vec{x}} \rangle\!\rangle \\
&\overset{4.12}{\Rightarrow} (\vec{x})s_1 =_{\emptyset} (\vec{x})s_2 \\
&\Rightarrow (\langle\vec{x}\rangle \otimes \mathbf{id}) \cdot (\vec{x})s_1 =_{\{\vec{x}\}} (\langle\vec{x}\rangle \otimes \mathbf{id}) \cdot (\vec{x})s_2 \\
&\overset{\sigma}{\Rightarrow} s_1 =_{\{\vec{x}\}} s_2
\end{aligned}
$$

(2)
$$
\begin{aligned}
\langle\!\langle t_1 \rangle\!\rangle =_{\emptyset} \langle\!\langle t_2 \rangle\!\rangle &\overset{4.8}{\Rightarrow} [\![\langle\!\langle t_1 \rangle\!\rangle]\!]_{[\,]} = [\![\langle\!\langle t_2 \rangle\!\rangle]\!]_{[\,]} \\
&\overset{4.12,C1}{\Rightarrow} s = t
\end{aligned}
$$

5 Adding Reflexion

In this section, we look at the *reflexive* action calculi, introduced by Milner in [7], and show that adding reflexion to the corresponding closed action calculi is straightforward. Reflexive calculi are action calculi with extra structure given by a *reflexive* operator $\uparrow_p$, with the accompanying arity rule

$$\frac{t : p \otimes m \to p \otimes n}{\uparrow_p(t) : m \to n}$$

and additional axioms to account for the behaviour of $\uparrow_p$. The closed reflexive action calculi are also constructed using a reflexive operator, with the same accompanying arity rule and a similar set of axioms to describe the operator. This treatment of adding extra structure in the closed world to mimic the extra structure in the open world is very different to the approach taken if we had declared $\uparrow_p$ as a control. In the closed world, controls should be indexed with closure information; it looks as if structural operators need not.

The set of *reflexive terms* over $\mathcal{K}$, denoted by $\mathrm{RT}(\mathcal{K})$, are constructed using the operators of action calculi with their associated arity rules, plus the reflexion operator $\uparrow_p$ with its arity rule given above. The equational theory RAC is given by the set of equations upon reflexive terms generated by the action structure axioms, the axioms for AC and the following additional axioms which account for the behaviour of $\uparrow_p$:

$$\begin{array}{llcll}
(\rho\,1) & \mathbf{id} & = & \uparrow_p \mathbf{i}_{p,p} & \\
(\rho\,2) & \uparrow_p t \otimes \mathbf{id} & = & \uparrow_p (t \otimes \mathbf{id}) & \\
(\rho\,3) & \uparrow_p s \cdot t & = & \uparrow_p (s \cdot (\mathbf{id}_p \otimes t)) & \\
(\rho\,4) & s \cdot \uparrow_p t & = & \uparrow_p ((\mathbf{id}_p \otimes s) \cdot t) & \\
(\rho\,5) & (x) \uparrow_p t & = & \uparrow_p ((\mathbf{i}_{p,q} \otimes \mathbf{id}) \cdot (x)t), & (x : q) \\
(\rho\,6) & \uparrow_q \uparrow_p t & = & \uparrow_p \uparrow_q ((\mathbf{i}_{q,p} \otimes \mathbf{id}) \cdot t \cdot (\mathbf{i}_{p,q} \otimes \mathbf{id})) &
\end{array}$$

For a given set of controls $\mathcal{K}$, we write $s = t \in \mathrm{RAC}$ if $s, t \in \mathrm{RT}(\mathcal{K})$ and $s = t$ in the equational theory RAC. We call $\rho_1, \ldots, \rho_6$ the reflexion axioms. It is natural to define the iterated reflexion as follows:

$$\uparrow_p t \stackrel{def}{=} \uparrow_{p_r} \ldots \uparrow_{p_1} t, \quad m = p_1 \otimes \ldots p_r$$

Notice that adding reflexion to an arbitrary action calculus brings with it the notion of *restriction*, given by the simple definition

$$\boldsymbol{\nu} \stackrel{def}{=} \uparrow_p ((x)\langle x, x\rangle).$$

5.1 Lemma [Basic Lemmas] The following are provable in RAC.

1. $s \cdot t = \uparrow_m (\mathbf{i}_{m,k} \cdot (s \otimes t)), \qquad s : k \to m, t : m \to n$
2. $(s \otimes \mathbf{id}_k) \cdot t = \uparrow_m (s \otimes t), \qquad s : \varepsilon \to m, t : m \otimes k \to n$
3. $\uparrow_m s \otimes t = \uparrow_m (s \otimes t)$
4. $\uparrow_m s \cdot t = \uparrow_m (s \cdot (\mathbf{id}_m \otimes t))$
5. $s \cdot \uparrow_m t = \uparrow_m ((\mathbf{id}_m \otimes s) \cdot t)$

6. $(\vec{x})\uparrow_m t = \uparrow_m((\mathbf{i}_{m,n} \otimes \mathbf{id}) \cdot (\vec{x})t) \qquad (\vec{x} : n)$

7. $\uparrow_n \uparrow_m t = \uparrow_m \uparrow_n((\mathbf{i}_{n,m} \otimes \mathbf{id}) \cdot t \cdot (\mathbf{i}_{m,n} \otimes \mathbf{id}))$

Proof These results are proved in [7]. $\square$

The reflexive equational theory with names, denoted by RAC_n, is defined using the rules given in the definition of AC_n (definition 2.10), where in this case the axioms include the appropriate reflexion axioms given above.

The *closed reflexive terms over* $\mathcal{K}$, denoted by $\mathrm{CRT}(\mathcal{K})$, are defined in an analogous way using the operators of closed action calculi and the reflexive operator $\uparrow_p$ with the same arity rule as before. The equational theory CRAC is given by the set of equations upon closed reflexive terms generated by the action structure axioms A1–A6, the CAC axioms B1–B8 and the following additional axioms which account for the behaviour of $\uparrow_p$ in the closed setting:

$$
\begin{array}{lrcl}
\text{(R1)} & \mathbf{id} & = & \uparrow_m \mathbf{i}_{m,m} \\
\text{(R2)} & \uparrow_m t \otimes \mathbf{id} & = & \uparrow_m(t \otimes \mathbf{id}) \\
\text{(R3)} & \uparrow_m s \cdot t & = & \uparrow_m(s \cdot (\mathbf{id}_m \otimes t)) \\
\text{(R4)} & s \cdot \uparrow_m t & = & \uparrow_m((\mathbf{id}_m \otimes s) \cdot t) \\
\text{(R5)} & \uparrow_n \uparrow_m t & = & \uparrow_m \uparrow_n((\mathbf{i}_{n,m} \otimes \mathbf{id}) \cdot t \cdot (\mathbf{i}_{m,n} \otimes \mathbf{id})) \\
\text{(R6)} & \uparrow_{m \otimes n} t & = & \uparrow_n \uparrow_m t
\end{array}
$$

We also call R1–R6 the reflexion axioms. It should be clear from the context whether the reflexion axioms refer to axioms in CRAC or RAC. We have chosen to define $\uparrow_m$ for arbitrary arities m, and include the axiom R6. An alternative approach would be to define reflexivity initially for the prime arities, remove axiom R6 and define the composite cases in terms of the prime cases and the other operators.

Given a reflexive action calculus $\mathrm{RAC}(\mathcal{K})$, the corresponding closed reflexive action calculus $\mathrm{CRAC}(\mathcal{K}_m)$ is defined in analogous way to definition 3.5, in the sense that $\mathcal{K}_m$ is the set defined in definition 3.5 and we require the additional axioms D1–D3 on closed reflexive terms. We let $\mathrm{CRT}(\mathcal{K}_m)$ denote the set of closed reflexive terms generated by the control set $\mathcal{K}_m$.

The functions $[\![\_]\!]_{\vec{x}} : \mathrm{T}_{\{\vec{x}\}}(\mathcal{K}) \to \mathrm{CT}(\mathcal{K}_m)$ and $\langle\_\rangle : \mathrm{CT}(\mathcal{K}_m) \to \mathrm{T}_\emptyset(\mathcal{K})$ are easily extended to account for this extra operator. We define the translations $[\![\_]\!]^r_{\vec{x}} : \mathrm{RT}_{\{\vec{x}\}}(\mathcal{K}) \to \mathrm{CRT}(\mathcal{K}_m)$ and $\langle\_\rangle^r : \mathrm{CRT}(\mathcal{K}_m) \to \mathrm{RT}_\emptyset(\mathcal{K})$ by induction on the structure of terms as follows:

$$
\begin{array}{rcl}
[\![\uparrow_p t]\!]^r_{\vec{x}} & = & \uparrow_p((\mathbf{i}_{p,|\vec{x}|} \otimes \mathbf{id}) \cdot [\![t]\!]_{\vec{x}}), \\
\langle \uparrow_p t \rangle^r & = & \uparrow_p(\langle t \rangle),
\end{array}
$$

and the other cases are the same as those given in definition 4.1 and 4.10 for $[\![\_]\!]$ and $\langle\_\rangle$ respectively. In order to show that the function $[\![\_]\!]^r_{\vec{x}}$ preserves equality, we must check that the lemmas 4.3, 4.4, 4.5 and 4.6 hold in the reflexion case, and prove that the axioms describing the behaviour of reflexion are preserved.

5.2 Lemma The following equalities hold in CRAC.

1. $[\![t]\!]^r_{\vec{x},u,\vec{y}} = [\![t\{v/u\}]\!]^r_{\vec{x},v,\vec{y}}$, if $u : p$ and $v : p$.

2. $[\![t]\!]_{y,\vec{x}} = \boldsymbol{\omega}_p \otimes [\![t]\!]_{\vec{x}}$, when $y : p \notin fn(t)$.

3. $[\![t]\!]^r_{\vec{x},\vec{y},\vec{z}} = (\mathbf{id} \otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|} \otimes \mathbf{id}) \cdot [\![t]\!]^r_{\vec{x},\vec{z},\vec{y}}$

4. $[\![t\{u/v\}]\!]^r_{\vec{x},u,\vec{y}} = (\mathbf{id} \otimes \Delta_p \otimes \mathbf{id} \otimes \mathbf{id}) \cdot [\![t]\!]^r_{\vec{x},u,v,\vec{y}}$, if $u : p$ and $v : p$.

Proof For each part of the lemma, the proof proceeds by induction on the structure of t. The only case we need check is when t has the form $\uparrow_p s$, since the other cases follow immediately from lemmas 4.3, 4.4, 4.5 and 4.6.

$$\begin{array}{rcl}
[\![\uparrow_m s]\!]^r_{\vec{x},u,\vec{y}} & = & \uparrow_m((\mathbf{i}_{m,|\vec{x}|\otimes p\otimes|\vec{y}|} \otimes \mathbf{id}) \cdot [\![s]\!]^r_{\vec{x},u,\vec{y}}) \\
& \overset{IH}{=} & \uparrow_m((\mathbf{i}_{m,|\vec{x}|\otimes p\otimes|\vec{y}|} \otimes \mathbf{id}) \cdot [\![s\{v/u\}]\!]^r_{\vec{x},v,\vec{y}}) \\
& = & [\![(\uparrow_m s)\{v/u\}]\!]^r_{\vec{x},v,\vec{y}}
\end{array}$$

$$\begin{array}{rcl}
[\![\uparrow_m s]\!]^r_{y,\vec{x}} & = & \uparrow_m((\mathbf{i}_{m,p\otimes|\vec{x}|} \otimes \mathbf{id}) \cdot [\![s]\!]^r_{y,|\vec{x}|}) \\
& \overset{IH}{=} & \uparrow_m((\mathbf{i}_{m,p\otimes|\vec{x}|} \otimes \mathbf{id}) \cdot (\boldsymbol{\omega} \otimes [\![s]\!]^r_{|\vec{x}|}) \\
& \overset{B3}{=} & \uparrow_m((\mathbf{id}_m \otimes \boldsymbol{\omega}_p \otimes \mathbf{id}) \cdot (\mathbf{i}_{m,|\vec{x}|} \otimes \mathbf{id}) \cdot [\![s]\!]^r_{|\vec{x}|}) \\
& \overset{R4}{=} & (\boldsymbol{\omega}_p \otimes \mathbf{id}) \cdot \uparrow_m((\mathbf{i}_{m,|\vec{x}|} \otimes \mathbf{id}) \cdot [\![s]\!]^r_{|\vec{x}|}) \\
& = & \boldsymbol{\omega}_p \otimes [\![\uparrow_m s]\!]^r_{|\vec{x}|}
\end{array}$$

$$\begin{array}{rcl}
[\![\uparrow_m s]\!]^r_{\vec{x},\vec{y},\vec{z}} & = & \uparrow_m((\mathbf{i}_{m,|\vec{x}|\otimes|\vec{y}|\otimes|\vec{z}|} \otimes \mathbf{id}) \cdot [\![s]\!]^r_{\vec{x},\vec{y},\vec{z}}) \\
& \overset{IH}{=} & \uparrow_m((\mathbf{i}_{m,|\vec{x}|\otimes|\vec{y}|\otimes|\vec{z}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|} \otimes \mathbf{id}) \cdot [\![s]\!]^r_{\vec{x},\vec{z},\vec{y}}) \\
& \overset{C2}{=} & \uparrow_m((\mathbf{i}_{m,|\vec{x}|} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{m,|\vec{y}|\otimes|\vec{z}|} \otimes \mathbf{id}) \cdot \\
& & (\mathbf{id} \otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|} \otimes \mathbf{id}) \cdot [\![s]\!]^r_{\vec{x},\vec{z},\vec{y}}) \\
& \overset{B3}{=} & \uparrow_m((\mathbf{i}_{m,|\vec{x}|} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{id} \otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|} \otimes \mathbf{id}) \cdot \\
& & (\mathbf{id} \otimes \mathbf{i}_{m,|\vec{z}|\otimes|\vec{y}|} \otimes \mathbf{id}) \cdot [\![s]\!]^r_{\vec{x},\vec{z},\vec{y}}) \\
& \overset{C2}{=} & \uparrow_m((\mathbf{id} \otimes \mathbf{id} \otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|} \otimes \mathbf{id}) \cdot (\mathbf{i}_{m,|\vec{x}|\otimes|\vec{z}|\otimes|\vec{y}|} \otimes \mathbf{id}) \cdot [\![s]\!]^r_{\vec{x},\vec{z},\vec{y}}) \\
& \overset{R4}{=} & (\mathbf{id} \otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|} \otimes \mathbf{id}) \cdot \uparrow_m((\mathbf{i}_{m,|\vec{x}|\otimes|\vec{z}|\otimes|\vec{y}|} \otimes \mathbf{id}) \cdot [\![s]\!]^r_{\vec{x},\vec{z},\vec{y}}) \\
& = & (\mathbf{id} \otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|} \otimes \mathbf{id}) \cdot [\![\uparrow_m s]\!]^r_{\vec{x},\vec{z},\vec{y}})
\end{array}$$

$$\begin{array}{rl}
& (\mathbf{id} \otimes \Delta_p \otimes \mathbf{id} \otimes \mathbf{id}) \cdot [\![\uparrow_m s]\!]^r_{\vec{x},u,v,\vec{y}} \\
= & (\mathbf{id} \otimes \Delta_p \otimes \mathbf{id} \otimes \mathbf{id}) \cdot \uparrow_m((\mathbf{i}_{m,|\vec{x}|\otimes p\otimes p\otimes|\vec{y}|} \otimes \mathbf{id}) \cdot [\![s]\!]^r_{|x,u,v,\vec{y}}) \\
\overset{R4}{=} & \uparrow_m((\mathbf{id} \otimes \mathbf{id} \otimes \Delta_p \otimes \mathbf{id}) \cdot (\mathbf{i}_{m,|\vec{x}|\otimes p\otimes p\otimes|\vec{y}|} \otimes \mathbf{id}) \cdot [\![s]\!]^r_{\vec{x},u,v,\vec{y}}) \\
\overset{B3}{=} & \uparrow_m((\mathbf{i}_{m,|\vec{x}|\otimes p\otimes|\vec{y}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \Delta_p \otimes \mathbf{id} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot [\![s]\!]^r_{\vec{x},u,v,\vec{y}}) \\
\overset{IH}{=} & \uparrow_m((\mathbf{i}_{m,|\vec{x}|\otimes p\otimes|\vec{y}|} \otimes \mathbf{id})[\![s\{u/v\}]\!]_{\vec{x},u,\vec{y}}) \\
= & [\![(\uparrow_m s)\{u/v\}]\!]_{\vec{x},u,\vec{y}}
\end{array}$$

Notice that the equalities in lemma 4.7 also hold for the functions $[\![\_]\!]^r_{\vec{x}}$. The following technical result is also used to prove that $[\![\_]\!]^r_{\vec{x}}$ preserves the equalities in RAC_n.

5.3 Lemma [Basic Lemma]

$$\mathbf{id}_k \otimes \uparrow_m t = \uparrow_m((\mathbf{i}_{m,k} \otimes \mathbf{id}_l) \cdot (\mathbf{id}_k \otimes t) \cdot (\mathbf{i}_{k,m} \otimes \mathbf{id}_l)) \in \mathrm{CRAC}.$$

Proof

$$\begin{aligned}
\mathbf{id}_k \otimes {\uparrow_m} t &\overset{B3,\ B4}{=} \mathbf{i}_{k,l} \cdot ({\uparrow_m} t \otimes \mathbf{id}_k) \cdot \mathbf{i}_{n,k} \qquad (t : l \to n)\\
&\overset{R2,\ R3,\ R4}{=} {\uparrow_m}((\mathbf{id} \otimes \mathbf{i}_{k,l}) \cdot (t \otimes \mathbf{id}_k) \cdot (\mathbf{id} \otimes \mathbf{i}_{n,k}))\\
&\overset{B3,\ B4}{=} {\uparrow_m}((\mathbf{id} \otimes \mathbf{i}_{k,l}) \cdot \mathbf{i}_{m\otimes l,k} \cdot (\mathbf{id}_k \otimes t) \cdot \mathbf{i}_{k,m\otimes n} \cdot (\mathbf{id} \otimes \mathbf{i}_{n,k}))\\
&\overset{B5,\ C2}{=} {\uparrow_m}((\mathbf{i}_{m,k} \otimes \mathbf{id}_l) \cdot (\mathbf{id}_k \otimes t) \cdot (\mathbf{i}_{k,m} \otimes \mathbf{id}_l))
\end{aligned}$$

5.4 Theorem

$$s =_{\{\vec{x}\}} t \in \mathrm{RAC}_n \text{ implies } [\![s]\!]^r_{\vec{x}} = [\![t]\!]^r_{\vec{x}} \in \mathrm{CRAC}.$$

Proof The proof proceeds by induction on the structure of $s =_{\{\vec{x}\}} t \in \mathrm{RAC}$. We only need to consider the cases when $s =_{\{\vec{x}\}} t$ is a reflexion axiom, since the other cases follow from theorem 4.8.

$$\begin{aligned}
[\![{\uparrow_p} \mathbf{i}_{p,p}]\!]^r_{\vec{x}} &= {\uparrow_p}((\mathbf{i}_{p,|\vec{x}|} \otimes \mathbf{id}) \cdot [\![\mathbf{i}_{p,p}]\!]^r_{\vec{x}})\\
&\overset{4.7}{=} {\uparrow_p}((\mathbf{i}_{p,|\vec{x}|} \otimes \mathbf{id}) \cdot (\boldsymbol{\omega}_{|\vec{x}|} \otimes \mathbf{i}_{p,p}))\\
&\overset{B3,\ C1}{=} {\uparrow_p}((\mathbf{id} \otimes \boldsymbol{\omega}_{|\vec{x}|} \otimes \mathbf{id}) \cdot \mathbf{i}_{p,p})\\
&\overset{R4}{=} (\boldsymbol{\omega}_{|\vec{x}|} \otimes \mathbf{id}) \cdot {\uparrow_p} \mathbf{i}_{p,p}\\
&\overset{R1}{=} \boldsymbol{\omega}_{|\vec{x}|} \otimes \mathbf{id}\\
&= [\![\mathbf{id}]\!]^r_{\vec{x}}
\end{aligned}$$

$$\begin{aligned}
[\![{\uparrow_p} t \otimes \mathbf{id}]\!]^r_{\vec{x}} &\overset{4.7}{=} {\uparrow_p}((\mathbf{i}_{p,|\vec{x}|} \otimes \mathbf{id}) \cdot [\![t]\!]^r_{\vec{x}}) \otimes \mathbf{id}\\
&\overset{R2}{=} {\uparrow_p}((\mathbf{i}_{p,|\vec{x}|} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot ([\![t]\!]^r_{\vec{x}} \otimes \mathbf{id}))\\
&\overset{4.7}{=} [\![{\uparrow_p}(t \otimes \mathbf{id})]\!]^r_{\vec{x}}
\end{aligned}$$

$$\begin{aligned}
&[\![{\uparrow_p}(s \cdot (\mathbf{id}_p \otimes t))]\!]^r_{\vec{x}}\\
&\overset{4.7,\ C1}{=} {\uparrow_p}((\mathbf{i}_{p,|\vec{x}|} \otimes \mathbf{id}) \cdot (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes [\![s]\!]^r_{\vec{x}}) \cdot (\mathbf{i}_{|\vec{x}|,p} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes [\![t]\!]^r_{\vec{x}}))\\
&\overset{B3}{=} {\uparrow_p}((\mathbf{id} \otimes \Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{i}_{p,|\vec{x}|\otimes|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes [\![s]\!]^r_{\vec{x}}) \cdot (\mathbf{i}_{|\vec{x}|,p} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes [\![t]\!]^r_{\vec{x}}))\\
&\overset{R4,\ R3}{=} (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot {\uparrow_p}((\mathbf{i}_{p,|\vec{x}|\otimes|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes [\![s]\!]^r_{\vec{x}}) \cdot (\mathbf{i}_{|\vec{x}|,p} \otimes \mathbf{id})) \cdot [\![t]\!]^r_{|\vec{x}|}\\
&\overset{C2}{=} (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot {\uparrow_p}((\mathbf{i}_{p,|\vec{x}|} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes (\mathbf{i}_{p,|\vec{x}|} \otimes \mathbf{id}) \cdot [\![s]\!]^r_{\vec{x}}) \cdot (\mathbf{i}_{|\vec{x}|,p} \otimes \mathbf{id})) \cdot [\![t]\!]^r_{|\vec{x}|}\\
&\overset{5.3}{=} (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes {\uparrow_p}((\mathbf{i}_{p,|\vec{x}|} \otimes \mathbf{id}) \cdot [\![s]\!]^r_{\vec{x}})) \cdot [\![t]\!]^r_{|\vec{x}|}\\
&= [\![{\uparrow_p} s \cdot t]\!]_{\vec{x}}
\end{aligned}$$

$$\begin{aligned}
&[\![{\uparrow_p}((\mathbf{id} \otimes s) \cdot t)]\!]^r_{\vec{x}}\\
&\overset{4.7}{=} {\uparrow_p}((\mathbf{i}_{p,|\vec{x}|} \otimes \mathbf{id}) \cdot (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{|\vec{x}|,p} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{id} \otimes [\![s]\!]^r_{\vec{x}}) \cdot [\![t]\!]^r_{\vec{x}})\\
&\overset{B3}{=} {\uparrow_p}((\mathbf{id} \otimes \Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{i}_{p,|\vec{x}|\otimes|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{|\vec{x}|,p} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{id} \otimes [\![s]\!]^r_{\vec{x}}) \cdot [\![t]\!]^r_{\vec{x}})
\end{aligned}$$

$$
\begin{aligned}
&\overset{C2,\ B4,\ R4}{=} && (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot \uparrow_p((\mathbf{i}_{p,|\vec{x}|} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{id} \otimes [\![s]\!]^r_{\vec{x}}) \cdot [\![t]\!]^r_{\vec{x}}) \\
&\overset{B3,\ R4}{=} && (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes [\![s]\!]^r_{\vec{x}}) \cdot \uparrow_p((\mathbf{i}_{p,|\vec{x}|} \otimes \mathbf{id}) \cdot [\![t]\!]^r_{\vec{x}}) \\
&= && [\![s \cdot \uparrow_p t]\!]^r_{\vec{x}}
\end{aligned}
$$

$$
\begin{aligned}
[\![\uparrow_p((\mathbf{i}_{p,q} \otimes \mathbf{id}) \cdot (x)t]\!]^r_{\vec{x}} &\overset{4.7}{=} \uparrow_p((\mathbf{i}_{p,|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{p,q} \otimes \mathbf{id}) \cdot [\![t\{y/x\}]\!]^r_{\vec{x},y}) \\
&\overset{C2}{=} \uparrow_p((\mathbf{i}_{p,|\vec{x}|\otimes q} \otimes \mathbf{id}) \cdot [\![t\{y/x\}^r_{\vec{x},y}]\!]) \\
&= [\![\uparrow_p(t\{y/x\})]\!]^r_{\vec{x},y} \\
&= [\![(x)\uparrow_p t]\!]^r_{\vec{x}}
\end{aligned}
$$

$$
\begin{aligned}
&[\![\uparrow_p\uparrow_q((\mathbf{i}_{q,p} \otimes \mathbf{id}) \cdot t \cdot (\mathbf{i}_{p,q} \otimes \mathbf{id}))]\!]^r_{\vec{x}} \\
&\overset{4.7,\ C1}{=} \uparrow_p((\mathbf{i}_{p,|\vec{x}|} \otimes \mathbf{id}) \cdot \uparrow_q((\mathbf{i}_{q,|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{q,p} \otimes \mathbf{id}) \cdot [\![t]\!]^r_{\vec{x}} \cdot (\mathbf{i}_{p,q} \otimes \mathbf{id}))) \\
&\overset{R4}{=} \uparrow_p\uparrow_q((\mathbf{id} \otimes \mathbf{i}_{p,|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{i}_{q,|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{q,p} \otimes \mathbf{id}) \cdot [\![t]\!]^r_{\vec{x}} \cdot (\mathbf{i}_{p,q} \otimes \mathbf{id})) \\
&\overset{B5}{=} \uparrow_p\uparrow_q((\mathbf{i}_{q\otimes p,|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{q,p} \otimes \mathbf{id}) \cdot [\![t]\!]^r_{\vec{x}} \cdot (\mathbf{i}_{p,q} \otimes \mathbf{id})) \\
&\overset{B3}{=} \uparrow_p\uparrow_q((\mathbf{i}_{q,p} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\mathbf{i}_{p\otimes q,|\vec{x}|} \otimes \mathbf{id}) \cdot [\![t]\!]^r_{\vec{x}} \cdot (\mathbf{i}_{p,q} \otimes \mathbf{id})) \\
&\overset{R5}{=} \uparrow_q\uparrow_p((\mathbf{i}_{p\otimes q,|\vec{x}|} \otimes \mathbf{id}) \cdot [\![t]\!]^r_{\vec{x}}) \\
&\overset{B5,\ R4}{=} \uparrow_q((\mathbf{i}_{q,|\vec{x}|} \otimes \mathbf{id}) \cdot \uparrow_p((\mathbf{i}_{p,|\vec{x}|} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot [\![t]\!]^r_{\vec{x}})) \\
&= [\![\uparrow_q\uparrow_p t]\!]^r_{\vec{x}}
\end{aligned}
$$

5.5 Theorem

$$s = t \in \text{CRAC implies } \langle\!\langle s\rangle\!\rangle^r =_\emptyset \langle\!\langle t\rangle\!\rangle^r \in \text{RAC}_n.$$

Proof The proof follows by induction on the structure of $s = t \in$ CRAC. We only need to look at the reflexion axioms, since the other cases follow from the proof of theorem 4.11. However, it is easy to see that the reflexion axioms are preserved by $\langle\!\langle \_ \rangle\!\rangle$, since the structure of these axioms is preserved under translation and the corresponding axioms hold in RAC. □

5.6 Lemma For $t \in \text{RT}(\mathcal{K})$ and $s \in \text{CRT}(\mathcal{K}_m)$, we have

1. $\langle\!\langle [\![t]\!]^r_{\vec{x}} \rangle\!\rangle^r =_\emptyset (\vec{x})t \in$ RAC.
2. $[\![\langle\!\langle s\rangle\!\rangle^r]\!]^r_{\vec{x}} = \boldsymbol{\omega}_{|\vec{x}|} \otimes s \in$ CRAC.

Proof In both parts, the proof proceeds by induction on the structure of t. It is enough to look at the case when t is $\uparrow_m s$, since the other cases are covered by lemma 4.12.

$$
\begin{aligned}
\langle\!\langle [\![\uparrow_m s]\!]^r_{\vec{x}} \rangle\!\rangle^r &=_\emptyset \langle\!\langle \uparrow_m((\mathbf{i}_{m,|\vec{x}|} \otimes \mathbf{id}) \cdot [\![s]\!]^r_{\vec{x}}) \rangle\!\rangle^r \\
&=_\emptyset \uparrow_m((\mathbf{p}_{m,|\vec{x}|} \otimes \mathbf{id}) \cdot \langle\!\langle [\![s]\!]^r_{\vec{x}} \rangle\!\rangle^r) \\
&\overset{IH}{=}_\emptyset \uparrow_m((\mathbf{i}_{m,|\vec{x}|} \otimes \mathbf{id}) \cdot (\vec{x})s) \\
&\overset{\rho 5}{=}_\emptyset (\vec{x})\uparrow_m s
\end{aligned}
$$

$$\begin{aligned}
[\![\langle\!\langle \uparrow_m s\rangle\!\rangle^r]\!]^r_{\vec{x}} &\stackrel{IH}{=} \uparrow_m((\mathbf{i}_{m,|\vec{x}|}\otimes \mathbf{id})\cdot(\boldsymbol{\omega}_{|\vec{x}|}\otimes s))\\
&\stackrel{B3,\ C1}{=} \uparrow_m((\mathbf{id}\otimes\boldsymbol{\omega}_{|\vec{x}|}\otimes\mathbf{id})\cdot s)\\
&\stackrel{\rho 4}{=} \boldsymbol{\omega}_{|\vec{x}|}\otimes\uparrow_m s
\end{aligned}$$

Remark In the appendix to [7], Jensen has argued for including the reflexion axiom

$$(\rho\,7) \qquad\qquad \uparrow_m \mathbf{id}_m \quad = \quad \mathbf{id}_\varepsilon,$$

with the motivation that $\uparrow_m \mathbf{id}_m$ is inactive and inaccessible. If we incorporate the same axiom into CRAC, we obtain analogous results to theorems 5.4 and 5.5.

6 Dynamics

The dynamics of action calculi have not yet been fully explored, and so we do not give a general account connecting the dynamics of an action calculus with its corresponding closed action calculus. In this section, we explore two examples, AC($\boldsymbol{\nu}$, **out**,**box**) and AC($\ulcorner\_\urcorner$, **ap**), to illustrate that such a correspondence is possible in these cases. These examples also highlight the difficulties in providing a more general account, since the first example is straightforward whereas the second requires more care.

6.1 Comparison between AC($\boldsymbol{\nu}$, out, box) and CAC($\boldsymbol{\nu}_m$, out$_m$, box$_m$)

The action calculus AC($\boldsymbol{\nu}$, **out**, **box**) is the action calculus given in example 2.8. The corresponding closed action calculus CAC($\boldsymbol{\nu}_m, \mathbf{out}_m, \mathbf{box}_m$) is given by definition 3.5 and the reaction rule

$$(\Delta_{k\otimes 1}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{k\otimes 1,m})\cdot(\mathbf{out}_k\otimes\mathbf{box}_k(t))\rhd(\mathbf{id}\otimes\boldsymbol{\omega}_1\otimes\mathbf{id})\cdot t$$

This example is straightforward in the sense that one reaction rule in the open world corresponds to one reaction rule in the closed world. The translations $[\![\_]\!]_{\vec{x}} : \mathrm{T}_{\{\vec{x}\}}(\boldsymbol{\nu},\mathbf{out},\mathbf{box}) \to \mathrm{CT}(\boldsymbol{\nu}_m,\mathbf{out}_m,\mathbf{box}_m)$ and $\langle\!\langle\_\rangle\!\rangle : \mathrm{CT}(\boldsymbol{\nu}_m,\mathbf{out}_m,\mathbf{box}_m)\to \mathrm{T}_{\emptyset}(\boldsymbol{\nu},\mathbf{out},\mathbf{box})$ preserve the reaction relations as the following lemma states.

6.1 Lemma

1. $s \rhd_{\{\vec{x}\}} t \in \mathrm{AC}_n$ implies $[\![s]\!]_{\vec{x}} \rhd [\![t]\!]_{\vec{x}} \in \mathrm{CAC}$.
2. $s \rhd t \in \mathrm{CAC}$ implies $\langle\!\langle s\rangle\!\rangle \rhd_{\emptyset} \langle\!\langle t\rangle\!\rangle \in \mathrm{AC}_n$.

Proof

1. By theorem 4.8, it is enough to prove the result when $s \rhd_{\{\vec{x}\}} t \in \mathrm{AC}_n$ is an instance of a reaction rule. Let $\vec{x} = \vec{u}, y, \vec{v}$ such that $|\vec{x}| = k, |y| = 1, |\vec{u}| = p$ and $|\vec{v}| = q$.

$$\begin{aligned}
&[\![(\langle y\rangle\otimes\mathbf{id}_m)\cdot\mathbf{out}\otimes\langle y\rangle\cdot\mathbf{box}(t)]\!]_{\vec{x}}\\
&\stackrel{C1}{=} (\Delta_k\otimes\mathbf{id}_m)\cdot(\mathbf{id}\otimes\mathbf{i}_{k,m})\cdot([\![(\langle y\rangle\otimes\mathbf{id}_m)\cdot\mathbf{out}]\!]_{\vec{x}}\otimes[\![\langle y\rangle\cdot\mathbf{box}(t)]\!]_{\vec{x}})\\
&\stackrel{C1,4.7}{=} (\Delta_k\otimes\mathbf{id}_m)\cdot(\mathbf{id}\otimes\mathbf{i}_{k,m})\cdot(\Delta_k\otimes\mathbf{id}_m\otimes\Delta_k)\cdot(\mathbf{id}\otimes[\![\langle y\rangle]\!]_{\vec{x}}\otimes\mathbf{id}_m\otimes\mathbf{id}\otimes[\![\langle y\rangle]\!]_{\vec{x}})\cdot\\
&\qquad(\mathbf{out}_k\otimes\mathbf{box}_k([\![t]\!]_{\vec{x}}))\\
&= (\Delta_k\otimes\mathbf{id}_m)\cdot(\mathbf{id}\otimes\mathbf{i}_{k,m})\cdot(\Delta_k\otimes\mathbf{id}_m\otimes\Delta_k)\cdot
\end{aligned}$$

$$
\begin{aligned}
& (\mathbf{id} \otimes \boldsymbol{\omega}_p \otimes \mathbf{id} \otimes \boldsymbol{\omega}_q \otimes \mathbf{id}_m \otimes \mathbf{id} \otimes \boldsymbol{\omega}_p \otimes \mathbf{id} \otimes \boldsymbol{\omega}_q) \cdot (\mathbf{out}_k \otimes \mathbf{box}_k([\![t]\!]_{\vec{x}})) \\
\overset{C4}{=}\; & (\Delta_k \otimes \mathbf{id}_m) \cdot (\mathbf{id} \otimes \mathbf{i}_{k,m}) \cdot (\mathbf{id} \otimes \Delta_1 \otimes \mathbf{id} \otimes \mathbf{id}_m \otimes \mathbf{id} \otimes \Delta_1 \otimes \mathbf{id}) \cdot \\
& (\mathbf{id} \otimes \mathbf{id} \otimes \mathbf{i}_{1,q} \otimes \mathbf{id}_m \otimes \mathbf{id} \otimes \mathbf{id} \otimes \mathbf{i}_{1,q}) \cdot (\mathbf{out}_k \otimes \mathbf{box}_k([\![t]\!]_{\vec{x}})) \\
\overset{B3,C6}{=}\; & (\mathbf{id} \otimes \Delta_1 \otimes \mathbf{id} \otimes \mathbf{id}_m) \cdot (\Delta_{p\otimes 1 \otimes 1 \otimes q} \otimes \mathbf{id}_m) \cdot (\mathbf{id} \otimes \mathbf{i}_{p\otimes 1\otimes 1\otimes q,m}) \cdot \\
& (\mathbf{id} \otimes \mathbf{id} \otimes \mathbf{i}_{1,q} \otimes \mathbf{id}_m \otimes \mathbf{id} \otimes \mathbf{id} \otimes \mathbf{i}_{1,q}) \cdot (\mathbf{out}_k \otimes \mathbf{box}_k([\![t]\!]_{\vec{x}})) \\
\overset{B3}{=}\; & (\mathbf{id} \otimes \Delta_1 \otimes \mathbf{id} \otimes \mathbf{id}_m) \cdot (\Delta_{p\otimes 1 \otimes 1 \otimes q} \otimes \mathbf{id}_m) \cdot (\mathbf{id} \otimes \mathbf{id} \otimes \mathbf{i}_{1,q} \otimes \mathbf{id} \otimes \mathbf{id} \otimes \mathbf{i}_{1,q} \otimes \mathbf{id}_m) \cdot \\
& (\mathbf{id} \otimes \mathbf{i}_{q\otimes 1\otimes q\otimes 1,m}) \cdot (\mathbf{out}_k \otimes \mathbf{box}_k([\![t]\!]_{\vec{x}})) \\
\overset{C8}{=}\; & (\mathbf{id} \otimes \Delta_1 \otimes \mathbf{id} \otimes \mathbf{id}_m) \cdot (\mathbf{id} \otimes \mathbf{i}_{1,q} \otimes \mathbf{id}_m) \cdot (\Delta_{k\otimes 1} \otimes \mathbf{id}_m) \cdot (\mathbf{id} \otimes \mathbf{i}_{k\otimes 1,m}) \cdot \\
& (\mathbf{out}_k \otimes \mathbf{box}_k([\![t]\!]_{\vec{x}})) \\
\triangleright\; & (\mathbf{id} \otimes \Delta_1 \otimes \mathbf{id} \otimes \mathbf{id}_m) \cdot (\mathbf{id} \otimes \mathbf{i}_{1,q} \otimes \mathbf{id}_m) \cdot (\mathbf{id} \otimes \boldsymbol{\omega}_1 \otimes \mathbf{id}_m) \cdot [\![t]\!]_{\vec{x}} \\
\overset{B3,C1}{=}\; & (\mathbf{id} \otimes \Delta_1 \otimes \mathbf{id} \otimes \mathbf{id}_m) \cdot (\mathbf{id} \otimes \boldsymbol{\omega}_1 \otimes \mathbf{id} \otimes \mathbf{id}_m) \cdot [\![t]\!]_{\vec{x}} \\
\overset{C3}{=}\; & [\![t]\!]_{\vec{x}}
\end{aligned}
$$

2. By theorem 4.11, it is enough to prove the result when $s \triangleright t \in \mathrm{CAC}$ is an instance of a reaction rule.

$$
\begin{aligned}
& \langle\!\langle (\Delta_{k\otimes 1} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{k\otimes 1,m}) \cdot (\mathbf{out}_k \otimes \mathbf{box}_k(t)) \rangle\!\rangle \\
& =_\emptyset \quad (\mathbf{copy}_{k\otimes 1} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{p}_{k\otimes 1,m}) \cdot ((\vec{x})\mathbf{out} \otimes (\vec{y})\mathbf{box}((\langle \vec{y} \rangle \otimes \mathbf{id}) \cdot \langle t \rangle)) \\
& \overset{2.4,\zeta,\sigma}{=} \quad (\vec{u}, v, \vec{w})\langle \vec{u}, v, \vec{w}, \vec{u}, v \rangle \cdot ((\vec{x})\mathbf{out} \otimes (\vec{y})\mathbf{box}((\langle \vec{y} \rangle \otimes \mathbf{id}) \cdot \langle t \rangle)) \\
& =_\emptyset \quad (\vec{u}, v, \vec{w})(\langle v, \vec{w} \rangle \cdot (\langle \vec{u} \rangle \otimes \mathbf{id}) \cdot (\vec{x})\mathbf{out} \otimes \langle v \rangle \cdot ((\langle \vec{w} \rangle \otimes \mathbf{id}) \cdot (\vec{y})\mathbf{box}((\langle \vec{y} \rangle \otimes \mathbf{id}) \cdot \langle\!\langle t \rangle\!\rangle)) \\
& \overset{\sigma}{=}_\emptyset \quad (\vec{u}, v, \vec{w})(\langle \vec{w} \rangle \cdot (\langle v \rangle \otimes \mathbf{id}) \cdot \mathbf{out} \otimes \langle v \rangle \cdot \mathbf{box}((\langle \vec{u} \rangle \otimes \mathbf{id}) \cdot \langle\!\langle t \rangle\!\rangle)) \\
& \overset{2.4}{=} \quad (\vec{u}, v, \vec{w})(\langle \vec{w} \rangle \cdot (((\langle v \rangle \otimes \mathbf{id}) \cdot \mathbf{out} \otimes \langle v \rangle \cdot \mathbf{box}((\langle \vec{u} \rangle \otimes \mathbf{id}) \cdot \langle\!\langle t \rangle\!\rangle))) \\
& \triangleright \quad (\vec{u}, v, \vec{w})((\langle \vec{u} \rangle \otimes \langle \vec{w} \rangle) \cdot \langle\!\langle t \rangle\!\rangle) \\
& =_\emptyset \quad (\mathbf{id} \otimes \boldsymbol{\omega}_1 \otimes \mathbf{id}) \cdot \langle\!\langle t \rangle\!\rangle
\end{aligned}
$$

6.2 Corollary

1. $[\![s]\!]_{\vec{x}} \triangleright t \in \mathrm{CAC}$ implies $s \triangleright_{\{\vec{x}\}} s' \in \mathrm{AC}$ and $[\![s']\!]_{\vec{x}} = t \in \mathrm{CAC}$.
2. $\langle\!\langle s \rangle\!\rangle \triangleright_\emptyset t \in \mathrm{AC}$ implies $s \triangleright s' \in \mathrm{CAC}$ and $\langle\!\langle s' \rangle\!\rangle =_\emptyset t \in \mathrm{AC}$.

Proof Part 1 is easy to prove using lemma 6.1 and theorem 4.12 with $s' = (\langle \vec{x} \rangle \otimes \mathbf{id}) \cdot t$. Part 2 follows immediately from lemma 6.1 and theorem 4.12.

6.2 Comparison between $\mathrm{AC}(\ulcorner\_\urcorner, \mathbf{ap})$ and $\mathrm{CAC}(\ulcorner\_\urcorner_m, \mathbf{ap}_m)$

The action calculus $\mathrm{AC}(\ulcorner\_\urcorner, \mathbf{ap})$ is defined in example 2.9. Its corresponding closed action calculus $\mathrm{CAC}(\ulcorner\_\urcorner_m, \mathbf{ap}_m)$ is given by definition 3.5 and the reaction rules

$$\ulcorner t \urcorner_k \cdot \Delta_{m \Rightarrow n} \quad \triangleright \quad \Delta_k \cdot (\ulcorner t \urcorner_k \otimes \ulcorner t \urcorner_k) \tag{1}$$

$$\ulcorner t \urcorner_k \cdot \boldsymbol{\omega}_{m \Rightarrow n} \quad \triangleright \quad \boldsymbol{\omega}_k \tag{2}$$

$$(\ulcorner t \urcorner_k \otimes \mathbf{id}) \cdot \mathbf{ap}_k \quad \triangleright \quad t \tag{3}$$

The reaction rules 1 and 2 are used to mimic in the closed world the substitution of a term t for a name in the open world. This example is not as straightforward as the previous example, since two reaction rules in the closed world correspond to one reaction rule in the open world. In general, we can have an arbitrary number of reaction rules in the closed world corresponding to one reaction rule in the open world. For example, given the action calculus $\mathrm{AC}(\ulcorner\_\urcorner, \mathbf{ap}, \mathcal{K})$, for an arbitrary control set $\mathcal{K}$, then the corresponding closed action calculus would contain a reaction rule

$$(\Delta_k \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \ulcorner b \urcorner_k \otimes \mathbf{id}) \cdot K_{k \otimes m \Rightarrow n}(t_1, \ldots, t_r) \rhd$$
$$K_k((\Delta_k \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \ulcorner b \urcorner_k \otimes \mathbf{id}) \cdot t_1, \ldots, (\Delta_k \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \ulcorner b \urcorner_k \otimes \mathbf{id}) \cdot t_r)$$

for each $K \in \mathcal{K}$.

6.3 Lemma $(\Delta_{|\vec{y}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \ulcorner [\![t]\!]_{\vec{y}} \urcorner_{|\vec{y}|} \otimes \mathbf{id}) \cdot [\![s\{u/x\}]\!]_{\vec{y},u} \rhd^* [\![s\{t/x\}]\!]_{\vec{y}} \in \mathrm{CAC}$, for $x : m \Rightarrow n$.

Proof The proof proceeds by induction on the structure of s. First we prove the result for $x \notin fn(s)$: this deals with the cases $s = \mathbf{id}, \boldsymbol{\omega}$ and $\langle z \rangle$ for $z \neq x$. Let $|\vec{y}| = k$. When $x \notin fn(s)$, the proof uses the reduction rule 2.

$$\begin{array}{cl}
\multicolumn{2}{l}{(\Delta_k \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \ulcorner [\![t]\!]_{\vec{y}} \urcorner_k \otimes \mathbf{id}) \cdot [\![s\{u/x\}]\!]_{\vec{y},u}} \\
\stackrel{4.4,4.5}{=} & (\Delta_k \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \ulcorner [\![t]\!]_{\vec{y}} \urcorner_k \otimes \mathbf{id}) \cdot (\mathbf{i}_{|\vec{y}|, m \Rightarrow n} \otimes \mathbf{id}) \cdot (\boldsymbol{\omega}_{m \Rightarrow n} \otimes [\![s]\!]_{\vec{y}}) \\
\stackrel{B3}{=} & (\Delta_k \otimes \mathbf{id}) \cdot (\mathbf{i}_{|\vec{y}|, k} \otimes \mathbf{id}) \cdot (\ulcorner [\![t]\!]_{\vec{y}} \urcorner_k \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\boldsymbol{\omega}_{m \Rightarrow n} \otimes [\![s]\!]_{\vec{y}}) \\
\stackrel{B2}{=} & (\Delta_k \otimes \mathbf{id}) \cdot (\ulcorner [\![t]\!]_{\vec{y}} \urcorner_k \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\boldsymbol{\omega}_{m \Rightarrow n} \otimes [\![s]\!]_{\vec{y}}) \\
\rhd & (\Delta_{|\vec{y}|} \otimes \mathbf{id}) \cdot (w_k \otimes [\![s]\!]_{\vec{y}}) \\
= & [\![s]\!]_{\vec{y}}
\end{array}$$

When $s = \langle x \rangle$, the result is immediate. When $s = s_1 \cdot s_2$, we use the reduction rule 1.

$$\begin{array}{cl}
\multicolumn{2}{l}{(\Delta_k \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \ulcorner [\![t]\!]_{\vec{y}} \urcorner_k \otimes \mathbf{id}) \cdot [\![s_1\{t/x\} \cdot s_2\{t/x\}]\!]_{\vec{y},u}} \\
= & (\Delta_k \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \ulcorner [\![t]\!]_{\vec{y}} \urcorner_k \otimes \mathbf{id}) \cdot (\Delta_{k \otimes (m \Rightarrow n)} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes [\![s_1\{u/x\}]\!]_{\vec{y},u}) \cdot [\![s_2\{u/x\}]\!]_{\vec{y},u} \\
\stackrel{B7}{=} & (\Delta_k \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \ulcorner [\![t]\!]_{\vec{y}} \urcorner_k \otimes \mathbf{id}) \cdot (\Delta_k \otimes \Delta_{m \Rightarrow n} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{k, m \Rightarrow n} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot \\
 & (\mathbf{id} \otimes [\![s_1\{u/x\}]\!]_{\vec{y},u}) \cdot [\![s_2\{u/x\}]\!]_{\vec{y},u} \\
\rhd & (\Delta_k \otimes \mathbf{id}) \cdot (\Delta_k \otimes \Delta_k \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \ulcorner [\![t]\!]_{\vec{y}} \urcorner_k \otimes \ulcorner [\![t]\!]_{\vec{y}} \urcorner_k \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{k, m \Rightarrow n} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot \\
 & (\mathbf{id} \otimes [\![s_1\{u/x\}]\!]_{\vec{y},u}) \cdot [\![s_2\{u/x\}]\!]_{\vec{y},u} \\
= & (\Delta_k \otimes \mathbf{id}) \cdot (\Delta_k \otimes \Delta_k \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{k,k} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \ulcorner [\![t]\!]_{\vec{y}} \urcorner_k \otimes \mathbf{id} \otimes \ulcorner [\![t]\!]_{\vec{y}} \urcorner_k \otimes \mathbf{id}) \cdot \\
 & (\mathbf{id} \otimes [\![s_1\{u/x\}]\!]_{\vec{y},u}) \cdot [\![s_2\{u/x\}]\!]_{\vec{y},u} \\
\stackrel{B7,C5}{=} & (\Delta_k \otimes \mathbf{id}) \cdot (\Delta_k \otimes \Delta_k \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \ulcorner [\![t]\!]_{\vec{y}} \urcorner_k \otimes \mathbf{id} \otimes \ulcorner [\![t]\!]_{\vec{y}} \urcorner_k \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes [\![s_1\{u/x\}]\!]_{\vec{y},u}) \cdot \\
 & [\![s_2\{u/x\}]\!]_{\vec{y},u} \\
= & (\Delta_k \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes ((\Delta_k \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \ulcorner [\![t]\!]_{\vec{y}} \urcorner_k \otimes \mathbf{id}) \cdot [\![s_1\{u/x\}]\!]_{\vec{y},u})) \cdot (\Delta_k \otimes \mathbf{id}) \cdot \\
 & (\mathbf{id} \otimes \ulcorner [\![t]\!]_{\vec{y}} \urcorner_k \otimes \mathbf{id}) \cdot [\![s_2\{u/x\}]\!]_{\vec{y},u} \\
\stackrel{IH}{\rhd^*} & (\Delta_k \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes [\![s_1\{u/x\}]\!]_{\vec{y}}) \cdot [\![s_2\{u/x\}]\!]_{\vec{y}}
\end{array}$$

The case when $s = s_1 \otimes s_2$ follows in a similar fashion. $\square$

We use the above lemma to illustrate the connection between the dynamics of AC($\ulcorner \_ \urcorner$, $\mathbf{ap}$) and CAC($\ulcorner \_ \urcorner_m$, $\mathbf{ap}_m$). This connection is not strong as the previous example, since one reduction step in AC($\ulcorner \_ \urcorner$, $\mathbf{ap}$) corresponds to many reduction steps in CAC($\ulcorner \_ \urcorner_m$, $\mathbf{ap}_m$).

6.4 Lemma

1. $s \rhd_{\vec{x}} t \in \mathrm{AC}_n$ implies $[\![s]\!]_{\vec{x}} \rhd^* [\![t]\!]_{\vec{x}} \in \mathrm{CAC}$.
2. $s \rhd t \in \mathrm{CAC}$ implies $\langle\!\langle s \rangle\!\rangle \rhd_{\emptyset} \langle\!\langle t \rangle\!\rangle \in \mathrm{AC}_n$.

Proof

1. By theorem 4.8, it's enough to prove the result when $s \rhd_{\vec{x}} t \in \mathrm{AC}_n$ is an instance of a reaction rule.

$$\begin{aligned} [\![(\ulcorner t \urcorner \otimes \mathbf{id}) \cdot (x)s]\!]_{\vec{y}} &\overset{4.7}{=} (\Delta_k \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \ulcorner [\![t]\!]_{\vec{y}} \urcorner_k \otimes \mathbf{id}) \cdot [\![s\{u/x\}]\!]_{\vec{y},u} \\ &\overset{6.3}{\rhd^*} [\![s\{t/x\}]\!]_{\vec{y}} \end{aligned}$$

$$\begin{aligned} [\![(\ulcorner t \urcorner \otimes \mathbf{id}) \cdot \mathbf{ap}]\!]_{\vec{y}} &\overset{4.7}{=} (\ulcorner [\![t]\!]_{\vec{y}} \urcorner_k \otimes \mathbf{id}) \cdot \mathbf{ap}_k \\ &\rhd [\![t]\!]_{\vec{y}} \end{aligned}$$

2. By theorem 4.11, it is enough to prove the result when $s \rhd t \in \mathrm{CAC}$ is an instance of a reaction rule.

$$\begin{aligned} \langle\!\langle \ulcorner t \urcorner_k \cdot \Delta_{m \Rightarrow n} \rangle\!\rangle &=_{\emptyset} (\vec{x})\ulcorner(\langle\vec{x}\rangle \otimes \mathbf{id}) \cdot [\![t]\!]\urcorner \cdot \mathbf{copy}_{m \Rightarrow n}, \quad |\vec{x}| = k \\ &\overset{2.4}{=}_{\emptyset} (\vec{x})(\ulcorner(\langle\vec{x}\rangle \otimes \mathbf{id}) \cdot [\![t]\!]\urcorner \cdot (\vec{y})\langle\vec{y},\vec{y}\rangle) \\ &\rhd (\vec{x})(\ulcorner(\langle\vec{x}\rangle \otimes \mathbf{id}) \cdot [\![t]\!]\urcorner \otimes \ulcorner(\langle\vec{x}\rangle \otimes \mathbf{id}) \cdot [\![t]\!]\urcorner) \\ &\overset{\sigma}{=}_{\emptyset} (\vec{x})(\langle\vec{x}\rangle \cdot (\vec{y})\ulcorner(\langle\vec{y}\rangle \otimes \mathbf{id}) \cdot [\![t]\!]\urcorner \otimes \langle\vec{x}\rangle \cdot (\vec{z})\ulcorner(\langle\vec{z}\rangle \otimes \mathbf{id}) \cdot [\![t]\!]\urcorner) \\ &\overset{2.4}{=}_{\emptyset} (\vec{x})\langle\vec{x},\vec{x}\rangle \cdot ((\vec{y})\ulcorner(\langle\vec{y}\rangle \otimes \mathbf{id}) \cdot [\![t]\!]\urcorner \otimes (\vec{z})\ulcorner(\langle\vec{z}\rangle \otimes \mathbf{id}) \cdot [\![t]\!]\urcorner) \\ &=_{\emptyset} \langle\!\langle \Delta_k \cdot (\ulcorner t \urcorner_k \otimes \ulcorner t \urcorner_k) \rangle\!\rangle \end{aligned}$$

$$\begin{aligned} \langle\!\langle \ulcorner t \urcorner_k \cdot \boldsymbol{\omega}_{m \Rightarrow n} \rangle\!\rangle &\overset{2.4}{=}_{\emptyset} (\vec{x})(\ulcorner(\langle\vec{x}\rangle \otimes \mathbf{id}) \cdot [\![t]\!]\urcorner \cdot \boldsymbol{\omega}_{m \Rightarrow n}), \quad |\vec{x}| = k \\ &\overset{\gamma}{=}_{\emptyset} (\vec{x})(\ulcorner(\langle\vec{x}\rangle \otimes \mathbf{id}) \cdot [\![t]\!]\urcorner \cdot (z)\mathbf{id}_{\varepsilon}) \\ &\rhd (\vec{x})\mathbf{id}_{\varepsilon} \\ &\overset{\gamma}{=}_{\emptyset} \boldsymbol{\omega}_k \end{aligned}$$

$$\begin{aligned} \langle\!\langle (\ulcorner t \urcorner_k \otimes \mathbf{id}) \cdot \mathbf{ap}_k \rangle\!\rangle &=_{\emptyset} ((\vec{x})\ulcorner(\langle\vec{x}\rangle \otimes \mathbf{id}) \cdot [\![t]\!]\urcorner \otimes \mathbf{id}) \cdot (\vec{y})\mathbf{ap}, \quad |\vec{x}| = |\vec{y}| = k \\ &\overset{2.4}{=}_{\emptyset} (\vec{x})((\ulcorner(\langle\vec{x}\rangle \otimes \mathbf{id}) \cdot [\![t]\!]\urcorner \otimes \mathbf{id}) \cdot (\vec{y})\mathbf{ap}) \\ &\rhd (\vec{x})((\langle\vec{x}\rangle \otimes \mathbf{id}) \cdot \langle\!\langle t \rangle\!\rangle) \\ &\overset{\delta}{=}_{\emptyset} \langle\!\langle t \rangle\!\rangle \end{aligned}$$

7 Concluding Remarks

The original motivation for studying closed action calculi arose from looking at Milner's higher-order action calculi [5]. Milner shows that HAC($\emptyset$), the higher-order action calculus with no controls, plus one extra axiom yields a cartesian closed category (CCC). In fact, there is a natural connection between terms in HAC($\emptyset$) and terms of the simply-typed λ-calculus. This connection does not, however, yield functors between the corresponding CCCs. Using closed action calculi, the structural operators of CAC($\emptyset$) give rise to a cartesian category. It is easy to add extra structural operators, analogous to currying and evaluation, to obtain a CCC. There seems to be a strong correspondence between CAC($\emptyset$) with this extra structure and simply-typed λ-calculus, which yields an equivalence between the corresponding CCCs. One direction for further research is to investigate whether there is a direct type-theoretic understanding of the higher-order action calculus HAC($\emptyset$).

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A Appendix

A.1 LEMMA [Basic Lemmas] The following are provable in CAC:

$$
\begin{array}{llcl}
\text{(C1)} & \mathbf{i}_{\epsilon,m} & = & \mathbf{id} \;=\; \mathbf{i}_{m,\epsilon} \\
 & \boldsymbol{\omega}_{\epsilon} & = & \mathbf{id} \\
 & \Delta_{\epsilon} & = & \mathbf{id} \\
\text{(C2)} & \mathbf{i}_{k,m\otimes n} & = & (\mathbf{i}_{k,m} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{k,n}) \\
\text{(C3)} & (\Delta_m \otimes \mathbf{id}) \cdot (\boldsymbol{\omega}_m \otimes t) & = & t \\
 & (\mathbf{id} \otimes \Delta_m) \cdot (t \otimes \boldsymbol{\omega}_m) & = & t \\
\text{(C4)} & \Delta_{m\otimes k\otimes n} \cdot (\boldsymbol{\omega}_m \otimes \mathbf{id} \otimes \boldsymbol{\omega}_n \otimes \mathbf{id}) & = & (\mathbf{id} \otimes \Delta_k \otimes \mathbf{id}) \cdot (\mathbf{i}_{m,k} \otimes \mathbf{id} \otimes \mathbf{id}) \\
 & \Delta_{m\otimes k\otimes n} \cdot (\mathbf{id} \otimes \boldsymbol{\omega}_m \otimes \mathbf{id} \otimes \boldsymbol{\omega}_n) & = & (\mathbf{id} \otimes \Delta_k \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{id} \otimes \mathbf{i}_{k,n})
\end{array}
$$

$$
\begin{array}{llcl}
\text{(C5)} & \Delta_m \cdot \Delta_{m\otimes m} & = & \Delta_m \cdot (\Delta_m \otimes \Delta_m) \\
\text{(C6)} & \Delta_{m\otimes k\otimes n} \cdot (\mathbf{id} \otimes \Delta_k \otimes \mathbf{id} \otimes \mathbf{id} \otimes \Delta_k \otimes \mathbf{id}) & = & (\mathbf{id} \otimes \Delta_k \otimes \mathbf{id}) \cdot \Delta_{m\otimes k\otimes k\otimes n} \\
\text{(C7)} & \mathbf{i}_{m,n} \cdot \Delta_{n\otimes m} & = & \Delta_{m\otimes n} \cdot (\mathbf{i}_{m,n} \otimes \mathbf{i}_{m,n}) \\
\text{(C8)} & (\mathbf{id} \otimes \mathbf{i}_{n,k}) \cdot \Delta_{m\otimes k\otimes n} & = & \Delta_{m\otimes n\otimes k} \cdot (\mathbf{id} \otimes \mathbf{i}_{n,k} \otimes \mathbf{id} \otimes \mathbf{i}_{n,k})
\end{array}
$$

Proof

(C1) We show that $\mathbf{id}_n \otimes \mathbf{i}_{\epsilon,m} = \mathbf{id}$. Putting $n = \epsilon$ yields the equality for $\mathbf{i}_{\epsilon,m}$.

$$
\begin{array}{rcl}
\mathbf{id}_n \otimes \mathbf{i}_{\epsilon,m} & \stackrel{B4}{=} & (\mathbf{id}_n \otimes \mathbf{i}_{\epsilon,m}) \cdot (\mathbf{i}_{n,m} \otimes \mathbf{id}_\epsilon) \cdot \mathbf{i}_{m,n} \\
& \stackrel{B5}{=} & \mathbf{i}_{n,m} \cdot \mathbf{i}_{m,n} \\
& \stackrel{B4}{=} & \mathbf{id}
\end{array}
$$

The equality for $\mathbf{i}_{m,\epsilon}$ follows trivially from the above result using B4.

To prove the equality for Δ_ϵ, it is enough to show that $\mathbf{id}_m = \mathbf{id}_m \otimes \Delta_\epsilon$.

$$
\begin{array}{rcl}
\mathbf{id}_m & \stackrel{B1}{=} & \Delta_{m\otimes\epsilon} \cdot (\boldsymbol{\omega}_m \otimes \mathbf{id}_m) \\
& \stackrel{B7}{=} & (\Delta_m \otimes \Delta_\epsilon) \cdot (\mathbf{id} \otimes \mathbf{i}_{m,\epsilon} \otimes \mathbf{id}) \cdot (\boldsymbol{\omega}_m \otimes \mathbf{id}_m) \\
& \stackrel{C1}{=} & (\Delta_m \otimes \Delta_\epsilon) \cdot (\boldsymbol{\omega}_m \otimes \mathbf{id}_m \otimes \mathbf{id}_\epsilon) \\
& = & (\Delta_m \cdot (\boldsymbol{\omega}_m \otimes \mathbf{id}_m)) \otimes \Delta_\epsilon \\
& \stackrel{B1}{=} & \mathbf{id}_m \otimes \Delta_\epsilon
\end{array}
$$

$$
\begin{array}{rcl}
\boldsymbol{\omega}_\epsilon & = & \boldsymbol{\omega}_\epsilon \otimes \mathbf{id}_\epsilon \\
& \stackrel{C1}{=} & \Delta_\epsilon \cdot (\boldsymbol{\omega}_\epsilon \otimes \mathbf{id}_\epsilon) \\
& \stackrel{B1}{=} & \mathbf{id}_\epsilon
\end{array}
$$

(C2)

$$
\begin{array}{rcl}
\mathbf{i}_{k,m\otimes n} & \stackrel{B4}{=} & \mathbf{i}_{k,m\otimes n} \cdot (\mathbf{id} \otimes \mathbf{i}_{n,k}) \cdot (\mathbf{i}_{m,k} \otimes \mathbf{id}) \cdot (\mathbf{i}_{k,m} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{k,n}) \\
& \stackrel{B5}{=} & \mathbf{i}_{k,m\otimes n} \cdot \mathbf{i}_{m\otimes n,k} \cdot (\mathbf{i}_{k,m} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{k,n}) \\
& \stackrel{B4}{=} & (\mathbf{i}_{k,m} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{k,n})
\end{array}
$$

(C3)

$$
\begin{array}{rcl}
(\Delta_m \otimes \mathbf{id}) \cdot (\boldsymbol{\omega}_m \otimes t) & = & (\Delta_m \otimes \mathbf{id}) \cdot (\boldsymbol{\omega}_m \otimes \mathbf{id}) \cdot t \\
& \stackrel{B1}{=} & t
\end{array}
$$

$$
\begin{array}{rcl}
(\mathbf{id} \otimes \Delta_m) \cdot (t \otimes \boldsymbol{\omega}_m) & = & (\mathbf{id} \otimes \Delta_m) \cdot (\mathbf{id} \otimes \boldsymbol{\omega}_m) \cdot t \\
& \stackrel{B2}{=} & (\mathbf{id} \otimes \Delta_m) \cdot (\mathbf{id} \otimes \mathbf{i}_{m,m}) \cdot (\mathbf{id} \otimes \boldsymbol{\omega}_m) \cdot t \\
& \stackrel{B3,C1}{=} & (\mathbf{id} \otimes \Delta_m) \cdot (\mathbf{id} \otimes \boldsymbol{\omega}_m \otimes \mathbf{id}) \cdot t \\
& \stackrel{B1}{=} & t
\end{array}
$$

(C4)

$$
\begin{aligned}
&\Delta_{m\otimes k\otimes n}\cdot(\boldsymbol{\omega}_m\otimes\mathbf{id}\otimes\boldsymbol{\omega}_k\otimes\mathbf{id})\\
&\stackrel{B7}{=}\quad(\Delta_m\otimes\Delta_k\otimes\Delta_n)\cdot(\mathbf{id}\otimes\mathbf{i}_{m,k}\otimes\mathbf{id}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{m\otimes k,n}\otimes\mathbf{id})\cdot(\boldsymbol{\omega}_m\otimes\mathbf{id}\otimes w_n\otimes\mathbf{id})\\
&\stackrel{B3,\ C1}{=}\quad(\Delta_m\otimes\Delta_k\otimes\Delta_n)\cdot(\boldsymbol{\omega}_m\otimes\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{id}\otimes\boldsymbol{\omega}_n\otimes\mathbf{id})\cdot(\mathbf{i}_{m,k}\otimes\mathbf{id}\otimes\mathbf{id})\\
&\stackrel{C3}{=}\quad(\mathbf{id}\otimes\Delta_k\otimes\mathbf{id})\cdot(\mathbf{i}_{m,k}\otimes\mathbf{id}\otimes\mathbf{id})
\end{aligned}
$$

The other case is similar.

(C5)

$$
\begin{aligned}
\Delta_m\cdot\Delta_{m\otimes m}&\stackrel{B7}{=}\Delta_m\cdot(\Delta_m\otimes\Delta_m)\cdot(\mathbf{id}\otimes\mathbf{i}_{m,m}\otimes\mathbf{id})\\
&=\Delta_m\cdot(\mathbf{id}\otimes\Delta_m)\cdot(\Delta_m\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{m,m}\otimes\mathbf{id})\\
&\stackrel{B8}{=}\Delta_m\cdot(\Delta_m\otimes\mathbf{id})\cdot(\Delta_m\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{m,m}\otimes\mathbf{id})\\
&\stackrel{B8}{=}\Delta_m\cdot(\Delta_m\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\Delta_m\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{m,m}\otimes\mathbf{id})\\
&\stackrel{B8,B2}{=}\Delta_m\cdot(\mathbf{id}\otimes\Delta_m)\cdot(\mathbf{id}\otimes\Delta_m)\\
&\stackrel{B8}{=}\Delta_m\cdot(\Delta_m\otimes\Delta_m)
\end{aligned}
$$

(C6)

$$
\begin{aligned}
&\Delta_{m\otimes k\otimes n}\cdot(\mathbf{id}\otimes\Delta_k\otimes\mathbf{id}\otimes\mathbf{id}\otimes\Delta_k\otimes\mathbf{id})\\
&\stackrel{B7}{=}\ (\Delta_m\otimes\Delta_k\otimes\Delta_n)\cdot(\mathbf{id}\otimes\mathbf{i}_{m,k}\otimes\mathbf{id}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{m\otimes k,n}\otimes\mathbf{id})\cdot\\
&\qquad(\mathbf{id}\otimes\Delta_k\otimes\mathbf{id}\otimes\mathbf{id}\otimes\Delta_k\otimes\mathbf{id})\\
&\stackrel{B3}{=}\ (\Delta_m\otimes\Delta_k\otimes\Delta_n)\cdot(\mathbf{id}\otimes\mathbf{id}\otimes\Delta_k\otimes\Delta_k\otimes\mathbf{id}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{m,k\otimes k}\otimes\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{id})\cdot\\
&\qquad(\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{i}_{m\otimes k\otimes k,n}\otimes\mathbf{id})\\
&\stackrel{C5}{=}\ (\mathbf{id}\otimes\Delta_k\otimes\mathbf{id})\cdot(\Delta_m\otimes\Delta_{k\otimes k}\otimes\Delta_n)\cdot(\mathbf{id}\otimes\mathbf{i}_{m,k\otimes k}\otimes\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{id})\cdot\\
&\qquad(\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{i}_{m\otimes k\otimes k,n}\otimes\mathbf{id})\\
&=\ (\mathbf{id}\otimes\Delta_k\otimes\mathbf{id})\cdot\Delta_{m\otimes k\otimes k\otimes n}
\end{aligned}
$$

(C7)

$$
\begin{aligned}
\mathbf{i}_{m,n}\cdot\Delta_{n\otimes m}&\stackrel{B7}{=}\mathbf{i}_{m,n}\cdot(\Delta_n\otimes\Delta_m)\cdot(\mathbf{id}\otimes\mathbf{i}_{n,m}\otimes\mathbf{id})\\
&\stackrel{B3}{=}(\Delta_m\otimes\Delta_n)\cdot\mathbf{i}_{m\otimes m,n\otimes n}\cdot(\mathbf{id}\otimes\mathbf{i}_{n,m}\otimes\mathbf{id})\\
&\stackrel{B5}{=}(\Delta_m\otimes\Delta_n)\cdot(\mathbf{id}\otimes\mathbf{i}_{m,n\otimes n})\cdot(\mathbf{i}_{m,n\otimes n}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{n,m}\otimes\mathbf{id})\\
&\stackrel{C2}{=}(\Delta_m\otimes\Delta_n)\cdot(\mathbf{id}\otimes\mathbf{i}_{m,n}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{i}_{m,n})\cdot\\
&\qquad(\mathbf{i}_{m,n}\otimes\mathbf{id}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{m,n}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{n,m}\otimes\mathbf{id})\\
&\stackrel{B7,B4}{=}\Delta_{m\otimes n}\cdot(\mathbf{i}_{m,n}\otimes\mathbf{i}_{m,n})
\end{aligned}
$$

(C8)

$$
(\mathbf{id}\otimes\mathbf{i}_{n,k})\cdot\Delta_{m\otimes k\otimes n}\quad\stackrel{B7}{=}\quad(\mathbf{id}\otimes\mathbf{i}_{n,k})\cdot(\Delta_m\otimes\Delta_{k\otimes n})\cdot(\mathbf{id}\otimes\mathbf{i}_{m,k\otimes n}\otimes\mathbf{id})
$$

$$
\begin{aligned}
&\overset{C7}{=} (\Delta_m \otimes \Delta_{n\otimes k}) \cdot (\mathbf{id} \otimes \mathbf{i}_{n,k} \otimes \mathbf{i}_{n,k}) \cdot (\mathbf{id} \otimes \mathbf{i}_{m,k\otimes n} \otimes \mathbf{id}) \\
&\overset{B3}{=} (\Delta_m \otimes \Delta_{n\otimes k}) \cdot (\mathbf{id} \otimes \mathbf{i}_{m,n\otimes k} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{n,k} \otimes \mathbf{id} \otimes \mathbf{i}_{n,k}) \\
&\overset{B7}{=} \Delta_{m\otimes n\otimes k} \cdot (\mathbf{id} \otimes \mathbf{i}_{n,k} \otimes \mathbf{id} \otimes \mathbf{i}_{n,k})
\end{aligned}
$$

A.2 Lemma

$$[\![t]\!]_{y,\vec{x}} = \boldsymbol{\omega}_p \otimes [\![t]\!]_{\vec{x}} \in \mathrm{CAC}, \text{when } y : p \notin fn(t).$$

Proof The proof proceeds by induction on the structure of t. When t is $\mathbf{id}, \langle x\rangle$ or $\boldsymbol{\omega}$, the result is obtained easily using the definition of $[\![\_]\!]_{y,\vec{x}}$ and the assumption that $y \notin fn(t)$. When t is $(x)s$, the result follows immediately using the induction hypothesis. The proof when t has the form $K(t_1, \ldots, t_r)$ is interesting because it requires the use of axiom D1.

$$
\begin{aligned}
[\![K(t_1,\ldots,t_r)]\!]_{y,\vec{x}} &= K_{p\otimes|\vec{x}|}([\![t_1]\!]_{y,\vec{x}}, \ldots, [\![t_r]\!]_{y,\vec{x}}) \\
&\overset{IH}{=} K_{p\otimes|\vec{x}|}(\boldsymbol{\omega}_p \otimes [\![t_1]\!]_{\vec{x}}, \ldots, \boldsymbol{\omega}_p \otimes [\![t_r]\!]_{\vec{x}}) \\
&\overset{D1}{=} \boldsymbol{\omega}_p \otimes K_{|\vec{x}|}([\![t_1]\!]_{\vec{x}}, \ldots, [\![t_r]\!]_{\vec{x}}) \\
&= \boldsymbol{\omega}_p \otimes [\![K(t_1,\ldots,t_r)]\!]_{\vec{x}}
\end{aligned}
$$

The other cases are presented for completeness. Their proofs require simple equational reasoning and the induction hypothesis.

$$
\begin{aligned}
[\![t_1 \cdot t_2]\!]_{y,\vec{x}} &= (\Delta_{p\otimes|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes [\![t_1]\!]_{y,\vec{x}}) \cdot [\![t_2]\!]_{y,\vec{x}} \\
&\overset{IH}{=} (\Delta_{p\otimes|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \boldsymbol{\omega}_p \otimes [\![t_1]\!]_{|\vec{x}|}) \cdot (\boldsymbol{\omega}_p \otimes [\![t_2]\!]_{\vec{x}}) \\
&\overset{B7,B3}{=} (\Delta_p \otimes \Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \boldsymbol{\omega}_p \otimes \mathbf{id} \otimes [\![t_1]\!]_{\vec{x}}) \cdot (\boldsymbol{\omega}_p \otimes [\![t_2]\!]_{\vec{x}}) \\
&\overset{C3}{=} (\boldsymbol{\omega}_p \otimes \Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id}_{|\vec{x}|} \otimes [\![t_1]\!]_{\vec{x}}) \cdot [\![t_2]\!]_{\vec{x}} \\
&= \boldsymbol{\omega}_p \otimes [\![t_1 \cdot t_2]\!]_{\vec{x}}
\end{aligned}
$$

$$
\begin{aligned}
[\![t_1 \otimes t_2]\!]_{y,\vec{x}} &= (\Delta_{p\otimes|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{p\otimes|\vec{x}|,k} \otimes \mathbf{id}) \cdot ([\![t_1]\!]_{y,\vec{x}} \otimes [\![t_2]\!]_{y,\vec{x}}) \\
&\overset{IH}{=} (\Delta_{p\otimes|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{p\otimes|\vec{x}|,k} \otimes \mathbf{id}) \cdot (\boldsymbol{\omega}_p \otimes [\![t_1]\!]_{\vec{x}} \otimes \boldsymbol{\omega}_p \otimes [\![t_2]\!]_{\vec{x}}) \\
&\overset{B7,B5}{=} (\Delta_p \otimes \Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{p,|\vec{x}|} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{id} \otimes \mathbf{i}_{|\vec{x}|,k} \otimes \mathbf{id}) \cdot \\
&\qquad (\mathbf{id} \otimes \mathbf{i}_{p,k} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\boldsymbol{\omega}_p \otimes [\![t_1]\!]_{\vec{x}} \otimes \boldsymbol{\omega}_p \otimes [\![t_2]\!]_{\vec{x}}) \\
&\overset{B3}{=} (\Delta_p \otimes \Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{id} \otimes \mathbf{i}_{|\vec{x}|,k} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{p,|\vec{x}|} \otimes \mathbf{id} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot \\
&\qquad (\mathbf{id} \otimes \mathbf{i}_{p,k} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\boldsymbol{\omega}_p \otimes [\![t_1]\!]_{\vec{x}} \otimes \boldsymbol{\omega}_p \otimes [\![t_2]\!]_{\vec{x}}) \\
&\overset{C2}{=} (\Delta_p \otimes \Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{id} \otimes \mathbf{i}_{|\vec{x}|,k} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{p,|\vec{x}|\otimes k} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot \\
&\qquad (\boldsymbol{\omega}_p \otimes [\![t_1]\!]_{\vec{x}} \otimes \boldsymbol{\omega}_p \otimes [\![t_2]\!]_{\vec{x}}) \\
&\overset{B3,\ C1}{=} (\Delta_p \otimes \Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{id} \otimes \mathbf{i}_{|\vec{x}|,k} \otimes \mathbf{id}) \cdot (\boldsymbol{\omega}_p \otimes \boldsymbol{\omega}_p \otimes [\![t_1]\!]_{\vec{x}} \otimes [\![t_2]\!]_{\vec{x}}) \\
&\overset{C3}{=} \boldsymbol{\omega}_p \otimes [\![t_1 \otimes t_2]\!]_{\vec{x}}
\end{aligned}
$$

A.3 Lemma

$$[\![t]\!]_{\vec{x},\vec{y},\vec{z}} = (\mathbf{id} \otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|} \otimes \mathbf{id}) \cdot [\![t]\!]_{\vec{x},\vec{z},\vec{y}} \in \mathrm{CAC}$$

Proof The proof follows by induction on the structure of t. When t is $\mathbf{id}$ or $\boldsymbol{\omega}$, the result is immediate. When t is $\langle u\rangle$, the result is easily proved by observing where u occurs in $\vec{x}, \vec{y}, \vec{z}$. When t is $(u)s$, the result follows immediately from the induction hypothesis. When t has shape $K(t_1, \ldots, t_r)$, the proof is more interesting since it requires D2.

$$
\begin{aligned}
&[\![K(t_1, \ldots, t_r)_{\vec{x},\vec{y},\vec{z}}]\!] \\
&\quad = \quad K_{|\vec{x}|\otimes|\vec{y}|\otimes|\vec{z}|}([\![t_1]\!]_{\vec{x},\vec{y},\vec{z}}, \ldots, [\![t_r]\!]_{\vec{x},\vec{y},\vec{z}}) \\
&\quad \stackrel{IH}{=} \quad K_{|\vec{x}|\otimes|\vec{y}|\otimes|\vec{z}|}((\mathbf{id}\otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|}\otimes \mathbf{id})\cdot[\![t_1]\!]_{\vec{x},\vec{z},\vec{y}}, \ldots, (\mathbf{id}\otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|}\otimes \mathbf{id})\cdot[\![t_r]\!]_{\vec{x},\vec{z},\vec{y}}) \\
&\quad \stackrel{D2}{=} \quad (\mathbf{id}\otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|}\otimes \mathbf{id})\cdot K_{|\vec{x}|\otimes|\vec{z}|\otimes|\vec{y}|}([\![t_1]\!]_{\vec{x},\vec{z},\vec{y}}, \ldots, [\![t_r]\!]_{\vec{x},\vec{z},\vec{y}})
\end{aligned}
$$

The cases when t has the form $t_1 \cdot t_2$ and $t_1 \otimes t_2$ require simple equational reasoning and the induction hypothesis; they are presented here for completeness.

$$
\begin{aligned}
&[\![t_1\cdot t_2]\!]_{\vec{x},\vec{y},\vec{z}} \\
&\quad = \quad (\Delta_{|\vec{x}|\otimes|\vec{y}|\otimes|\vec{z}|}\otimes \mathbf{id})\cdot(\mathbf{id}\otimes[\![t_1]\!]_{\vec{x},\vec{y},\vec{z}})\cdot[\![t_2]\!]_{\vec{x},\vec{y},\vec{z}} \\
&\quad \stackrel{IH,\ B3}{=} \quad (\Delta_{|\vec{x}|\otimes|\vec{y}|\otimes|\vec{z}|}\otimes \mathbf{id})\cdot(\mathbf{id}\otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|}\otimes \mathbf{id}\otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|}\otimes \mathbf{id})\cdot(\mathbf{id}\otimes[\![t_1]\!]_{\vec{x},\vec{z},\vec{y}})\cdot[\![t_2]\!]_{\vec{x},\vec{z},\vec{y}} \\
&\quad \stackrel{B7}{=} \quad (\Delta_{|\vec{x}|}\otimes\Delta_{|\vec{y}|\otimes|\vec{z}|}\otimes \mathbf{id})\cdot(\mathbf{id}\otimes \mathbf{i}_{|\vec{x}|,|\vec{y}|\otimes|\vec{z}|}\otimes \mathbf{id}\otimes \mathbf{id})\cdot(\mathbf{id}\otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|}\otimes \mathbf{id}\otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|}\otimes \mathbf{id})\cdot \\
&\qquad\qquad (\mathbf{id}\otimes[\![t_1]\!]_{\vec{x},\vec{z},\vec{y}})\cdot[\![t_2]\!]_{\vec{x},\vec{z},\vec{y}} \\
&\quad \stackrel{B3}{=} \quad (\Delta_{|\vec{x}|}\otimes\Delta_{|\vec{y}|\otimes|\vec{z}|}\otimes \mathbf{id})\cdot(\mathbf{id}\otimes \mathbf{id}\otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|}\otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|}\otimes \mathbf{id})\cdot(\mathbf{id}\otimes \mathbf{i}_{|\vec{x}|,|\vec{z}|\otimes|\vec{y}|}\otimes \mathbf{id}\otimes \mathbf{id})\cdot \\
&\qquad\qquad (\mathbf{id}\otimes[\![t_1]\!]_{\vec{x},\vec{z},\vec{y}})\cdot[\![t_2]\!]_{\vec{x},\vec{z},\vec{y}} \\
&\quad \stackrel{C7}{=} \quad (\mathbf{id}\otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|}\otimes \mathbf{id})\cdot(\Delta_{|\vec{x}|}\otimes\Delta_{|\vec{z}|\otimes|\vec{y}|}\otimes \mathbf{id})\cdot(\mathbf{id}\otimes \mathbf{i}_{|\vec{x}|,|\vec{z}|\otimes|\vec{y}|}\otimes \mathbf{id}\otimes \mathbf{id})\cdot(\mathbf{id}\otimes[\![t_1]\!]_{\vec{x},\vec{z},\vec{y}})\cdot \\
&\qquad\qquad [\![t_2]\!]_{\vec{x},\vec{z},\vec{y}} \\
&\quad \stackrel{B7}{=} \quad (\mathbf{id}\otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|}\otimes \mathbf{id})\cdot(\Delta_{|\vec{x}|\otimes|\vec{z}|\otimes|\vec{y}|}\otimes \mathbf{id})\cdot(\mathbf{id}\otimes[\![t_1]\!]_{\vec{x},\vec{z},\vec{y}})\cdot[\![t_2]\!]_{\vec{x},\vec{z},\vec{y}} \\
&\quad = \quad (\mathbf{id}\otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|}\otimes \mathbf{id})\cdot[\![t_1\cdot t_2]\!]_{\vec{x},\vec{z},\vec{y}}
\end{aligned}
$$

$$
\begin{aligned}
&[\![t_1\otimes t_2]\!]_{\vec{x},\vec{y},\vec{z}} \\
&\quad = \quad (\Delta_{|\vec{x}|\otimes|\vec{y}|\otimes|\vec{z}|}\otimes \mathbf{id})\cdot(\mathbf{id}\otimes \mathbf{i}_{|\vec{x}|\otimes|\vec{y}|\otimes|\vec{z}|,k}\otimes \mathbf{id})\cdot([\![t_1]\!]_{\vec{x},\vec{y},\vec{z}}\otimes[\![t_2]\!]_{\vec{x},\vec{y},\vec{z}}) \\
&\quad \stackrel{IH}{=} \quad (\Delta_{|\vec{x}|\otimes|\vec{y}|\otimes|\vec{z}|}\otimes \mathbf{id})\cdot(\mathbf{id}\otimes \mathbf{i}_{|\vec{x}|\otimes|\vec{y}|\otimes|\vec{z}|,k}\otimes \mathbf{id})\cdot(\mathbf{id}\otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|}\otimes \mathbf{id}\otimes \mathbf{id}\otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|}\otimes \mathbf{id})\cdot \\
&\qquad\qquad ([\![t_1]\!]_{\vec{x},\vec{z},\vec{y}}\otimes[\![t_2]\!]_{\vec{x},\vec{z},\vec{y}}) \\
&\quad = \quad (\Delta_{|\vec{x}|\otimes|\vec{y}|\otimes|\vec{z}|}\otimes \mathbf{id})\cdot(\mathbf{id}\otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|}\otimes \mathbf{id}\otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|}\otimes \mathbf{id}\otimes \mathbf{id})\cdot(\mathbf{id}\otimes \mathbf{i}_{|\vec{x}|\otimes|\vec{z}|\otimes|\vec{y}|,k}\otimes \mathbf{id})\cdot \\
&\qquad\qquad ([\![t_1]\!]_{\vec{x},\vec{z},\vec{y}}\otimes[\![t_2]\!]_{\vec{x},\vec{z},\vec{y}}) \\
&\quad \stackrel{as\ above}{=} \quad (\mathbf{id}\otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|}\otimes \mathbf{id})\cdot(\Delta_{|\vec{x}|\otimes|\vec{z}|\otimes|\vec{y}|}\otimes \mathbf{id})\cdot(\mathbf{id}\otimes \mathbf{i}_{|\vec{x}|\otimes|\vec{z}|\otimes|\vec{y}|,k}\otimes \mathbf{id})\cdot([\![t_1]\!]_{\vec{x},\vec{z},\vec{y}}\otimes[\![t_2]\!]_{\vec{x},\vec{z},\vec{y}}) \\
&\quad = \quad (\mathbf{id}\otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|}\otimes \mathbf{id})\cdot[\![t_1\otimes t_2]\!]_{\vec{x},\vec{z},\vec{y}}
\end{aligned}
$$

A.4 Lemma

$$(\mathbf{id}\otimes\Delta_p\otimes \mathbf{id}\otimes \mathbf{id})\cdot[\![t]\!]_{\vec{x},u,v,\vec{y}} = [\![t\{u/v\}]\!]_{\vec{x},u,\vec{y}} \in \mathrm{CAC},\ \text{for } u:p \text{ and } v:p$$

Proof We prove the result when $\vec{y}$ denotes the empty list. It is easy to prove the more general case using lemma A.3. The proof proceeds by induction on the structure of t. When t is $\mathbf{id}, \langle z\rangle$ or $\boldsymbol{\omega}$, the result is immediate. When t is $(z)s$, the result follows immediately using the induction hypothesis. When t has form $K(t_1,\ldots,t_r)$, the proof requires the axiom D3.

$$\begin{array}{rl}
& (\mathbf{id}\otimes\Delta_p\otimes\mathbf{id})\cdot[\![K(t_1,\ldots,t_r)]\!]_{\vec{x},u,v} \\
= & (\mathbf{id}\otimes\Delta_p\otimes\mathbf{id})\cdot K_{|\vec{x}|\otimes p\otimes p}([\![t_1]\!]_{\vec{x},u,v},\ldots,[\![t_r]\!]_{\vec{x},u,v}) \\
\stackrel{D3}{=} & K_{|\vec{x}|\otimes p}((\mathbf{id}\otimes\Delta_p\otimes\mathbf{id})\cdot[\![t_1]\!]_{\vec{x},u,v},\ldots,(\mathbf{id}\otimes\Delta_p\otimes\mathbf{id})\cdot[\![t_r]\!]_{\vec{x},u,v}) \\
\stackrel{IH}{=} & K_{|\vec{x}|\otimes p}([\![t_1\{u/v\}]\!]_{\vec{x},u},\ldots,[\![t_r\{u/v\}]\!]_{\vec{x},u})
\end{array}$$

The other cases are presented for completeness. The proofs involve simple equational reasoning and the induction hypothesis.

$$\begin{array}{rl}
& (\mathbf{id}\otimes\Delta_p\otimes\mathbf{id})\cdot[\![t_1\cdot t_2]\!]_{\vec{x},u,v} \\
= & (\mathbf{id}\otimes\Delta_p\otimes\mathbf{id})\cdot(\Delta_{|\vec{x}|\otimes p\otimes p}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes[\![t_1]\!]_{\vec{x},u,v})\cdot[\![t_2]\!]_{\vec{x},u,v} \\
\stackrel{B7}{=} & (\mathbf{id}\otimes\Delta_p\otimes\mathbf{id})\cdot(\Delta_{|\vec{x}|}\otimes\Delta_{p\otimes p}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,p\otimes p}\otimes\mathbf{id}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes[\![t_1]\!]_{\vec{x},u,v})\cdot[\![t_2]\!]_{\vec{x},u,v} \\
\stackrel{C5}{=} & (\Delta_{|\vec{x}|}\otimes\Delta_p\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\Delta_p\otimes\Delta_p\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,p\otimes p}\otimes\mathbf{id}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes[\![t_1]\!]_{\vec{x},u,v})\cdot[\![t_2]\!]_{\vec{x},u,v} \\
\stackrel{B3}{=} & (\Delta_{|\vec{x}|}\otimes\Delta_p\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,p}\otimes\mathbf{id}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\Delta_p\otimes\mathbf{id}\otimes\Delta_p\otimes\mathbf{id})\cdot(\mathbf{id}\otimes[\![t_1]\!]_{\vec{x},u,v})\cdot[\![t_2]\!]_{\vec{x},u,v} \\
\stackrel{B7}{=} & (\Delta_{|\vec{x}|\otimes p}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\Delta_p\otimes\mathbf{id}\otimes\Delta_p\otimes\mathbf{id})\cdot(\mathbf{id}\otimes[\![t_1]\!]_{\vec{x},x,y})\cdot[\![t_2]\!]_{\vec{x},x,y} \\
\stackrel{IH}{=} & (\Delta_{|\vec{x}|\otimes p}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes[\![t_1\{u/v\}]\!]_{\vec{x},u})\cdot[\![t_2\{u/v\}]\!]_{\vec{x},u} \\
= & [\![(t_1\otimes t_2)\{u/v\}]\!]_{\vec{x},u}
\end{array}$$

$$\begin{array}{rl}
& (\mathbf{id}\otimes\Delta_p\otimes\mathbf{id})\cdot[\![t_1\otimes t_2]\!]_{\vec{x},u,v} \\
= & (\mathbf{id}\otimes\Delta_p\otimes\mathbf{id})\cdot(\Delta_{|\vec{x}|\otimes p\otimes p}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|\otimes p\otimes p,k}\otimes\mathbf{id})\cdot([\![t_1]\!]_{\vec{x},u,v}\otimes[\![t_2]\!]_{\vec{x},u,v}) \\
\stackrel{as\ above}{=} & (\Delta_{|\vec{x}|\otimes p}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\Delta_p\otimes\mathbf{id}\otimes\Delta_p\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|\otimes p\otimes p,k}\otimes\mathbf{id})\cdot([\![t_1]\!]_{\vec{x},u,v}\otimes[\![t_2]\!]_{\vec{x},u,v}) \\
\stackrel{B3}{=} & (\Delta_{|\vec{x}|\otimes p}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|\otimes p,k}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\Delta_p\otimes\mathbf{id}\otimes\mathbf{id}\otimes\Delta_p\otimes\mathbf{id})\cdot([\![t_1]\!]_{\vec{x},u,v}\otimes[\![t_2]\!]_{\vec{x},u,v}) \\
\stackrel{IH}{=} & (\Delta_{|\vec{x}|\otimes p}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|\otimes p,k}\otimes\mathbf{id})\cdot([\![t_1\{u/v\}]\!]_{\vec{x},u}\otimes[\![t_2\{u/v\}]\!]_{\vec{x},u}) \\
= & [\![(t_1\otimes t_2)\{u/v\}]\!]_{\vec{x},u}
\end{array}$$

A.5 Lemma The following hold in CAC.

1. $[\![s\otimes t]\!]_{\vec{x}} = [\![s]\!]_{\vec{x}}\otimes[\![t]\!]_{[\,]}$, if $fn(t)=\emptyset$
2. $[\![t\otimes s]\!]_{\vec{x}} = (\mathbf{i}_{|\vec{x}|,m}\otimes\mathbf{id})\cdot([\![t]\!]_{[\,]}\otimes[\![s]\!]_{\vec{x}})$, if $fn(t)=\emptyset$ and $t: m\to n$.
3. $[\![s\cdot t]\!]_{\vec{x}} = [\![s]\!]_{\vec{x}}\cdot[\![t]\!]_{[\,]}$, if $fn(t)=\emptyset$
4. $[\![t\cdot s]\!]_{\vec{x}} = (\mathbf{id}\otimes[\![t]\!]_{[\,]})\cdot[\![s]\!]_{\vec{x}}$, if $fn(t)=\emptyset$
5. $[\![\mathbf{p}_{m,n}]\!]_{\vec{x}} = \boldsymbol{\omega}_{|\vec{x}|}\otimes\mathbf{i}_{m,n}$
6. $[\![\langle\vec{x}\rangle]\!]_{\vec{x}} = \mathbf{id}_{|\vec{x}|}$, if $|\vec{x}| = m\neq\varepsilon$.

Proof

$$\begin{array}{llll}
(1) & [\![s\otimes t]\!]_{\vec{x}} & \stackrel{4.4}{=} & (\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,k}\otimes\mathbf{id})\cdot([\![s]\!]_{\vec{x}}\otimes\boldsymbol{\omega}_{|\vec{x}|}\otimes[\![t]\!]_{[\,]}) \\
& & \stackrel{B2}{=} & (\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\mathbf{i}_{|\vec{x}|,|\vec{x}|}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,k}\otimes\mathbf{id})\cdot([\![s]\!]_{\vec{x}}\otimes\boldsymbol{\omega}_{|\vec{x}|}\otimes[\![t]\!]_{[\,]})
\end{array}$$

$$
\begin{aligned}
&\overset{C2}{=} && (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{i}_{|\vec{x}|,|\vec{x}|\otimes k} \otimes \mathbf{id}) \cdot ([\![s]\!]_{\vec{x}} \otimes \boldsymbol{\omega}_{|\vec{x}|} \otimes [\![t]\!]_{[\,]}) \\
&\overset{B3,\ C1}{=} && (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\boldsymbol{\omega}_{|\vec{x}|} \otimes [\![s]\!]_{\vec{x}} \otimes [\![t]\!]_{[\,]}) \\
&\overset{C3}{=} && [\![s]\!]_{\vec{x}} \otimes [\![t]\!]_{[\,]}
\end{aligned}
$$

$$
\begin{aligned}
(2) \qquad [\![t \otimes s]\!]_{\vec{x}} &\overset{4.4}{=} (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{|\vec{x}|,m} \otimes \mathbf{id}) \cdot (\boldsymbol{\omega}_{|\vec{x}|} \otimes [\![t]\!]_{[\,]} \otimes [\![s]\!]_{\vec{x}}) \\
&\overset{C3}{=} (\mathbf{i}_{|\vec{x}|,m} \otimes \mathbf{id}) \cdot ([\![t]\!]_{[\,]} \otimes [\![s]\!]_{\vec{x}})
\end{aligned}
$$

$$
\begin{aligned}
(3) \qquad [\![s \cdot t]\!]_{\vec{x}} &\overset{4.4}{=} (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes [\![s]\!]_{\vec{x}}) \cdot (\boldsymbol{\omega}_{|\vec{x}|} \otimes [\![t]\!]_{[\,]}) \\
&\overset{C3}{=} [\![s]\!]_{\vec{x}} \cdot [\![t]\!]_{[\,]}
\end{aligned}
$$

$$
\begin{aligned}
(4) \qquad [\![t \cdot s]\!]_{\vec{x}} &\overset{4.4}{=} (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \boldsymbol{\omega}_{|\vec{x}|} \otimes [\![t]\!]_{[\,]}) \cdot [\![s]\!]_{\vec{x}} \\
&\overset{C3}{=} (\mathbf{id} \otimes [\![t]\!]_{[\,]}) \cdot [\![s]\!]_{\vec{x}}
\end{aligned}
$$

We prove (5) by simultaneous induction on the lengths of the prime factorizations of m and n. When m is p and n is q for primes p and q, then $\mathbf{i}_{p,q}$ has the form $(x)(y)(\langle y\rangle \otimes \langle x\rangle)$, and the result can be proved as follows:

$$
\begin{aligned}
[\![(x)(y)(\langle y\rangle \otimes \langle x\rangle)]\!]_{\vec{x}} &= [\![\langle v\rangle \otimes \langle u\rangle]\!]_{\vec{x},u,v} \\
&\overset{C1}{=} \Delta_{|\vec{x}|\otimes p\otimes q} \cdot (\boldsymbol{\omega}_{|\vec{x}|} \otimes \boldsymbol{\omega}_p \otimes \mathbf{id} \otimes \boldsymbol{\omega}_{|\vec{x}|} \otimes \mathbf{id} \otimes \boldsymbol{\omega}_q) \\
&\overset{B7,B3,C1}{=} (\Delta_{|\vec{x}|} \otimes \Delta_{p\otimes q}) \cdot (\boldsymbol{\omega}_{|\vec{x}|} \otimes \boldsymbol{\omega}_{|\vec{x}|} \otimes \boldsymbol{\omega}_p \otimes \mathbf{id} \otimes \mathbf{id} \otimes \boldsymbol{\omega}_q) \\
&\overset{C3,B7}{=} (\boldsymbol{\omega}_{|\vec{x}|} \otimes \Delta_p \otimes \Delta_q) \cdot (\mathbf{id} \otimes \mathbf{i}_{p,q} \otimes \mathbf{id}) \cdot (\boldsymbol{\omega}_p \otimes \mathbf{id} \otimes \mathbf{id} \otimes \boldsymbol{\omega}_q) \\
&\overset{B3,C3}{=} \boldsymbol{\omega}_{|\vec{x}|} \otimes \mathbf{i}_{p,q}
\end{aligned}
$$

When $m = m_1 \otimes m_2$, then using the induction hypothesis we have

$$
\begin{aligned}
[\![\mathbf{i}_{m_1\otimes m_2,n}]\!]_{\vec{x}} &\overset{B5}{=} [\![(\mathbf{id} \otimes \mathbf{i}_{m_2,n}) \cdot (\mathbf{i}_{m_1,n} \otimes \mathbf{id})]\!]_{\vec{x}} \\
&\overset{part\ 3}{=} [\![\mathbf{id} \otimes \mathbf{i}_{m_2,n}]\!]_{\vec{x}} \cdot [\![\mathbf{i}_{m_1,n} \otimes \mathbf{id}]\!]_{[\,]} \\
&\overset{part\ 1}{=} ([\![\mathbf{id}]\!]_{\vec{x}} \otimes [\![\mathbf{i}_{m_2,n}]\!]_{[\,]}) \cdot ([\![\mathbf{i}_{m_1,n}]\!]_{[\,]} \otimes [\![\mathbf{id}]\!]_{[\,]}) \\
&\overset{IH,\ C1}{=} (\boldsymbol{\omega}_{|\vec{x}|} \otimes \mathbf{id} \otimes \mathbf{i}_{m_2,n}) \cdot (\mathbf{i}_{m_1,n} \otimes \mathbf{id}) \\
&\overset{B5}{=} \boldsymbol{\omega}_{|\vec{x}|} \otimes \mathbf{i}_{m_1\otimes m_2,n}
\end{aligned}
$$

When $n = n_1 \otimes n_2$, then the result follows in a similar fashion to the above.

We prove 6 by induction on the length of $\vec{x}$. The base case follows easily by the definition of $[\![\_]\!]_{\vec{x}}$. The inductive case is given below for $\vec{x} = \vec{y}, z : p$.

$$
\begin{aligned}
[\![\langle \vec{y}, z\rangle]\!]_{\vec{y},z} &\overset{C1}{=} \Delta_{|\vec{y}|\otimes p} \cdot (\mathbf{id} \otimes \boldsymbol{\omega}_p \otimes \boldsymbol{\omega}_{|\vec{y}|} \otimes \mathbf{id}) \\
&\overset{B7}{=} (\Delta_{|\vec{y}|} \otimes \Delta_p) \cdot (\mathbf{id} \otimes \mathbf{i}_{|\vec{y}|,p} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \boldsymbol{\omega}_p \otimes \boldsymbol{\omega}_{|\vec{y}|} \otimes \mathbf{id}) \\
&\overset{B3,\ C1}{=} (\Delta_{|\vec{y}|} \otimes \Delta_p) \cdot (\mathbf{id} \otimes \boldsymbol{\omega}_{|\vec{y}|} \otimes \boldsymbol{\omega}_p \otimes \mathbf{id}) \\
&\overset{C3}{=} \mathbf{id}
\end{aligned}
$$

A.6 Theorem

$$s =_{\{\vec{x}\}} t \in \mathrm{AC}_n(\mathcal{K}) \text{ implies } [\![s]\!]_{\vec{x}} = [\![t]\!]_{\vec{x}} \in \mathrm{CAC}.$$

Proof We prove here that the translation $[\![\_]\!]_{\vec{x}}$ preserves the basic axioms A1–A8. the proof involves simple equational reasoning. It's easy to show that the axioms A1–A3 are preserved using lemma 4.7.

$$\begin{aligned}
[\![s\cdot(t\cdot s)]\!]_{\vec{x}} &= (\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes[\![s]\!]_{\vec{x}})\cdot(\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes[\![t]\!]_{\vec{x}})\cdot[\![u]\!]_{\vec{x}}\\
&= (\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{id}\otimes[\![s]\!]_{\vec{x}})\cdot(\mathbf{id}\otimes[\![t]\!]_{\vec{x}})\cdot[\![u]\!]_{\vec{x}}\\
&\overset{B5}{=} (\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{id}\otimes[\![s]\!]_{\vec{x}})\cdot(\mathbf{id}\otimes[\![t]\!]_{\vec{x}})\cdot[\![u]\!]_{\vec{x}}\\
&= (\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes[\![s\cdot t]\!]_{\vec{x}})\cdot[\![u]\!]_{\vec{x}}\\
&= [\![(s\cdot t)\cdot u]\!]_{\vec{x}}
\end{aligned}$$

$$\begin{aligned}
&[\![s\otimes(t\otimes u)]\!]_{\vec{x}}\\
&= (\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,m}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,n}\otimes\mathbf{id})\cdot([\![s]\!]_{\vec{x}}\otimes[\![t]\!]_{\vec{x}}\otimes[\![u]\!]_{\vec{x}})\\
&\overset{B3}{=} (\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\Delta_{|\vec{x}|}\otimes\mathbf{id}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|\otimes|\vec{x}|,m}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,n}\otimes\mathbf{id})\cdot\\
&\quad([\![s]\!]_{\vec{x}}\otimes[\![t]\!]_{\vec{x}}\otimes[\![u]\!]_{\vec{x}})\\
&\overset{B8,B5}{=} (\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\Delta_{|\vec{x}|}\otimes\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,m}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,m}\otimes\mathbf{id}\otimes\mathbf{id})\cdot\\
&\quad(\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,n}\otimes\mathbf{id})\cdot([\![s]\!]_{\vec{x}}\otimes[\![t]\!]_{\vec{x}}\otimes[\![u]\!]_{\vec{x}})\\
&\overset{B3}{=} (\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\Delta_{|\vec{x}|}\otimes\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,m}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,n}\otimes\mathbf{id})\cdot\\
&\quad(\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,m}\otimes\mathbf{id}\otimes\mathbf{id})\cdot([\![s]\!]_{\vec{x}}\otimes[\![t]\!]_{\vec{x}}\otimes[\![u]\!]_{\vec{x}})\\
&\overset{C2}{=} (\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\Delta_{|\vec{x}|}\otimes\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,m\otimes n}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,m}\otimes\mathbf{id}\otimes\mathbf{id})\cdot\\
&\quad([\![s]\!]_{\vec{x}}\otimes[\![t]\!]_{\vec{x}}\otimes[\![u]\!]_{\vec{x}})\\
&\overset{B3}{=} (\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,m\otimes n}\otimes\mathbf{id})\cdot(\Delta_{|\vec{x}|}\otimes\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,m}\otimes\mathbf{id}\otimes\mathbf{id})\cdot\\
&\quad([\![s]\!]_{\vec{x}}\otimes[\![t]\!]_{\vec{x}}\otimes[\![u]\!]_{\vec{x}})\\
&= [\![(s\otimes t)\otimes u]\!]_{\vec{x}}
\end{aligned}$$

$$\begin{aligned}
&[\![(s\otimes u)\cdot(t\otimes v)]\!]_{\vec{x}}\\
&= (\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,m}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes[\![s]\!]_{\vec{x}}\otimes[\![u]\!]_{\vec{x}})\cdot(\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot\\
&\quad(\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,n}\otimes\mathbf{id})\cdot([\![t]\!]_{\vec{x}}\otimes[\![u]\!]_{\vec{x}})\\
&= (\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\Delta_{|\vec{x}|}\otimes\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,m}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,|\vec{x}|\otimes m}\otimes\mathbf{id})\cdot\\
&\quad(\mathbf{id}\otimes[\![s]\!]_{\vec{x}}\otimes\mathbf{id}\otimes[\![u]\!]_{\vec{x}})\cdot([\![t]\!]_{\vec{x}}\otimes[\![u]\!]_{\vec{x}})\\
&\overset{C2}{=} (\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\Delta_{|\vec{x}|}\otimes\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,m}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,|\vec{x}|}\otimes\mathbf{id}\otimes\mathbf{id})\cdot\\
&\quad(\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,m}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes[\![s]\!]_{\vec{x}}\otimes\mathbf{id}\otimes[\![u]\!]_{\vec{x}})\cdot([\![t]\!]_{\vec{x}}\otimes[\![u]\!]_{\vec{x}})\\
&\overset{B7}{=} (\Delta_{|\vec{x}|}\otimes\mathbf{id})\cdot(\Delta_{|\vec{x}|\otimes|\vec{x}|}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,m}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{id}\otimes\mathbf{i}_{|\vec{x}|,m}\otimes\mathbf{id})\cdot\\
&\quad(\mathbf{id}\otimes[\![s]\!]_{\vec{x}}\otimes\mathbf{id}\otimes[\![u]\!]_{\vec{x}})\cdot([\![t]\!]_{\vec{x}}\otimes[\![u]\!]_{\vec{x}})
\end{aligned}$$

$$
\begin{aligned}
&\overset{B5,\,C5}{=} && (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\Delta_{|\vec{x}|} \otimes \Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{|\vec{x}| \otimes |\vec{x}|, m} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes [\![s]\!]_{\vec{x}} \otimes \mathbf{id} \otimes [\![u]\!]_{\vec{x}}) \cdot ([\![t]\!]_{\vec{x}} \otimes [\![u]\!]_{\vec{x}}) \\
&= && (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{|\vec{x}|, m} \otimes \mathbf{id}) \cdot (\Delta_{|\vec{x}|} \otimes \mathbf{id} \otimes \Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes [\![s]\!]_{\vec{x}} \otimes \mathbf{id} \otimes [\![u]\!]_{\vec{x}}) \cdot ([\![t]\!]_{\vec{x}} \otimes [\![u]\!]_{\vec{x}}) \\
&= && (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{|\vec{x}|, m} \otimes \mathbf{id}) \cdot ([\![s \cdot t]\!]_{\vec{x}} \otimes [\![u \cdot v]\!]_{\vec{x}}) \\
&= && [\![(s \cdot t) \otimes (u \cdot v)]\!]_{\vec{x}}
\end{aligned}
$$

$$
\begin{aligned}
[\![\mathbf{ab}_y \mathbf{id}]\!]_{\vec{x}} &= (\mathbf{id} \otimes \Delta_p \otimes \mathbf{id}) \cdot (\mathbf{i}_{|\vec{x}|, p} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \boldsymbol{\omega}_{|\vec{x}| \otimes p} \otimes \mathbf{id}), \qquad y : p \\
&\overset{B3,\,C1}{=} (\mathbf{id} \otimes \Delta_p \otimes \mathbf{id}) \cdot (\boldsymbol{\omega}_{|\vec{x}|} \otimes \mathbf{id} \otimes \boldsymbol{\omega}_p \otimes \mathbf{id}) \\
&\overset{C3}{=} \boldsymbol{\omega}_{|\vec{x}|} \otimes \mathbf{id} \otimes \mathbf{id} \\
&= [\![\mathbf{id}]\!]_{\vec{x}|}
\end{aligned}
$$

$$
\begin{aligned}
&[\![\mathbf{ab}_x (s \cdot t)]\!]_{\vec{x}} \\
&= (\mathbf{id} \otimes \Delta_p \otimes \mathbf{id}) \cdot (\mathbf{i}_{|\vec{x}|, p} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \Delta_{|\vec{x}| \otimes p} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{id} \otimes [\![s\{y/x\}]\!]_{\vec{x},y}) \cdot \\
&\quad (\mathbf{id} \otimes [\![t\{y/x\}]\!]_{\vec{x},y}) \\
&\overset{B3,\,B7}{=} (\mathbf{id} \otimes \Delta_p \otimes \mathbf{id}) \cdot (\Delta_{|\vec{x}|} \otimes \mathbf{id} \otimes \Delta_p \otimes \mathbf{id}) \cdot (\mathbf{i}_{|\vec{x}| \otimes |\vec{x}|, p} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{id} \otimes \mathbf{i}_{|\vec{x}|, p} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot \\
&\quad (\mathbf{id} \otimes \mathbf{id} \otimes [\![s\{y/x\}]\!]_{\vec{x},y}) \cdot (\mathbf{id} \otimes [\![t\{y/x\}]\!]_{\vec{x},y}) \\
&\overset{B5,\,C2}{=} (\mathbf{id} \otimes \Delta_p \otimes \mathbf{id}) \cdot (\Delta_{|\vec{x}|} \otimes \mathbf{id} \otimes \Delta_p \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{|\vec{x}|, p \otimes p} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\mathbf{i}_{|\vec{x}|, p} \otimes \mathbf{id} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot \\
&\quad (\mathbf{id} \otimes \mathbf{id} \otimes [\![s\{y/x\}]\!]_{\vec{x},y}) \cdot (\mathbf{id} \otimes [\![t\{y/x\}]\!]_{\vec{x},y}) \\
&\overset{B8}{=} (\Delta_{|\vec{x}|} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \Delta_p \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \Delta_p \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{|\vec{x}|, p \otimes p} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot \\
&\quad (\mathbf{i}_{|\vec{x}|, p} \otimes \mathbf{id} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{id} \otimes [\![s\{y/x\}]\!]_{\vec{x},y}) \cdot (\mathbf{id} \otimes [\![t\{y/x\}]\!]_{\vec{x},y}) \\
&\overset{B3}{=} (\Delta_{|\vec{x}|} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \Delta_p \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{|\vec{x}|, p} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{id} \otimes [\![s\{y/x\}]\!]_{\vec{x},y}) \cdot \\
&\quad (\mathbf{id} \otimes \Delta_p \otimes \mathbf{id}) \cdot (\mathbf{i}_{|\vec{x}|, p} \otimes \mathbf{id} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes [\![t\{y/x\}]\!]_{\vec{x},y}) \\
&= [\![\mathbf{ab}_x s \cdot \mathbf{ab}_x t]\!]_{\vec{x}}
\end{aligned}
$$