Action Calculi V : Reflexive Molecular forms

Robin Milner (with Appendix by Ole Jensen)
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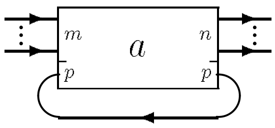
1 Introduction

The key property of action calculi [4] is that each calculus $\mathsf{AC}(\mathcal{K})$ can be presented in two ways: as the quotient of a term algebra by some simple axioms, and as an algebra of *molecular forms*. Molecular forms can be seen as normal forms for the term algebra under the axioms; but they are also important as they suggest a modular style of programming and system description. In [6] the isomorphism between the two presentations is proved in detail.

The present paper adapts that proof to admit *reflexion* into action calculi, in the form of a unary reflexion operator $\uparrow_p$ for each prime arity p. The other operators, the arity monoid $(M, \otimes, \epsilon)$ and the name-set X remain unchanged. Here is the arity rule for $\uparrow_p$, and a diagram to suggest the meaning of reflexion:

$$\frac{a : p \otimes m \to p \otimes n}{\uparrow_p a : m \to n}$$

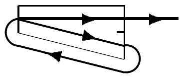


This operator was first studied axiomatically by Stefănescu in 1986 [7]. His work was in the context of Elgot's iterative theories; he called the operator "feedback", and pointed out some of its advantages compared with Elgot's iteration operator. Feedback is also treated in the recent book of Bloom and Ésik [2]. The work reported below was done in ignorance of Stefănescu's results, but the results are essentially the same. The main difference is that the algebra of action calculi includes an abstraction operator, so an additional axiom is needed to express how feedback relates to this operator. I am grateful to Gheorghe Stefănescu for pointing out this close correspondence to me, including the fact that Theorem 6.1 below, which asserts the completeness of the axiomatisation, is (modulo the abstraction operator) essentially anticipated by the main theorem in his paper [7].

This correspondence is strengthened by the fact that the intended interpretations of feedback in iterative theories and in action calculi is somewhat different. In the former it represents control flow in flowcharts; in the latter it plays a more static rôle since the dynamics of action calculi is provided by other constructions. In my view, this variety of interpretation shows that Stefănescu's feedback is an important abstract notion. Here I have adopted his symbol $\uparrow$ for it, but retain the term "reflexion" as a synonym for "feedback".

Reflexion has several intriguing properties in action structures. It was introduced in [3], Part II, in the context of an action structure for the π-calculus, where it was shown to have two virtues; it simplifies the graphical presentation of the synchronous π-calculus, and allows a natural bisimulation congruence for that calculus. More generally, in an arbitrary action calculus reflexion brings with it the notion of *restriction*, the action $\boldsymbol{\nu} : \epsilon \to p$ which appears to be a natural ingredient in many action calculi (not only for the π-calculus, but also for Petri nets – see [4]). In fact restriction has a strikingly simple definition in terms of reflexion, and its meaning is indicated by an equally simple diagram:

$$\boldsymbol{\nu} \stackrel{\text{def}}{=} \uparrow (x)\langle xx\rangle \ .$$



Finally, the modification of molecular forms needed for reflexion is simple and beautiful: the *body* of an action, previously a *partial sequence* of molecules (where each molecule may bind any other occurring later in the sequence), now becomes a *multiset* of molecules in which each member may bind any of the others.

I shall not argue further in this paper for the notion of reflexion, nor provide examples of its use; the purpose here is just to place reflexion on a firm technical footing. This consists of three parts: the extra equational axioms which characterise reflexion, the precise definition of reflexive molecular forms, and the detailed proof that the algebra of the latter is indeed isomorphic, in the strongest appropriate sense, to the quotient of the term algebra by the former.

The paper rests heavily upon the corresponding exercise for ordinary action calculi, *Action calculi IV: Molecular forms* [6], whose definitions and proofs are extended or modified here. I do not give a free-standing presentation here; it is more illuminating to see the incremental change exhibited clearly. In fact each section here corresponds closely with the like numbered section (up to Section 6) of the previous paper. A reference to the latter, e.g. to its Proposition 5.3, is written IV–5.3.

For the future, it will be important to study the addition of reflexion to *higher-order* action calculi [5], where it may be the best way to introduce recursion.

This work continues to build on the important insight of Berry and Boudol, in their Chemical Abstract Machine [1]. In fact, the correspondence of action calculi with the CHAM is even clearer with reflexion, since the body of an action is now a multiset – which is exactly what a *solution* is in the CHAM. The difference is therefore clearer, too. In action calculi we give structure to the solution, in the form of a bonding of molecules which is distinct from their dynamic behaviour. Indeed, a large purpose of action structures and action calculi is to explain what kind of structure this should be. In particular, reflexion is a structural notion, not a dynamic one; the dynamics of each (reflexive) action calculus is represented by *reaction rules* –another close correspondence with the CHAM– which operate within this structure.

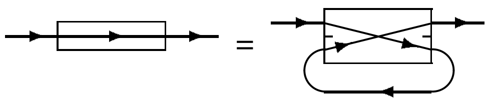
2 Terms

The reflexive terms $\mathcal{T}^r(\mathcal{K})$ are just $\mathcal{T}(\mathcal{K},\uparrow)$, where $\mathcal{T}(\mathcal{K})$ is as defined in Section IV–2. However, the new operator $\uparrow$ will not be treated dynamically, as are the controls $\mathcal{K}$ in an action calculus. Instead it is characterized by adding equations for $\uparrow$ to the theory AC, yielding a new theory AC^r as follows:

The theory AC^r

$\boldsymbol{\gamma}$:	$(x)t = \boldsymbol{\omega} \otimes t$	(x not free in t)
$\boldsymbol{\delta}$:	$(x)(\langle x\rangle \otimes \mathbf{id}_m) = \mathbf{id}_{p\otimes m}$	$(x : p)$
$\boldsymbol{\zeta}$:	$\mathbf{i}_{km} \cdot (t \otimes s) = (s \otimes t) \cdot \mathbf{i}_{\ell n}$	$(s : k \to \ell,\ t : m \to n)$
$\boldsymbol{\sigma}$:	$(\langle y\rangle \otimes \mathbf{id}_m) \cdot (x)t = \{y/x\}t$	$(t : m \to n)$
$\boldsymbol{\rho_1}$:	$\mathbf{id}_p = \uparrow_p \mathbf{i}_{pp}$	
$\boldsymbol{\rho_2}$:	$\uparrow_p t \otimes \mathbf{id} = \uparrow_p (t \otimes \mathbf{id})$	
$\boldsymbol{\rho_3}$:	$\uparrow_p s \cdot t = \uparrow_p (s \cdot (\mathbf{id}_p \otimes t))$	
$\boldsymbol{\rho_4}$:	$s \cdot \uparrow_p t = \uparrow_p ((\mathbf{id}_p \otimes s) \cdot t)$	
$\boldsymbol{\rho_5}$:	$(x)\uparrow_p t = \uparrow_p ((\mathbf{i}_{pq} \otimes \mathbf{id}) \cdot (x)t)$	$(x : q)$
$\boldsymbol{\rho_6}$:	$\uparrow_q \uparrow_p t = \uparrow_p \uparrow_q ((\mathbf{i}_{qp} \otimes \mathbf{id}) \cdot t \cdot (\mathbf{i}_{pq} \otimes \mathbf{id}))$.	

The axioms $\boldsymbol{\rho}$ all indicate how reflexion in the argument(s) of an operator can be pulled outside the operator. This is perhaps an odd way to describe $\boldsymbol{\rho_1}$, since $\mathbf{id}_p$ –being a nullary operator– has no arguments! In fact $\boldsymbol{\rho_1}$ has a different significance from $\boldsymbol{\rho_2}$–$\boldsymbol{\rho_6}$, given simply by the diagram



It is worth noting that $\boldsymbol{\rho_1}$ is independent of the other axioms, since any instance of any of the other axioms must have equally many occurrences of $\uparrow$ on either side.

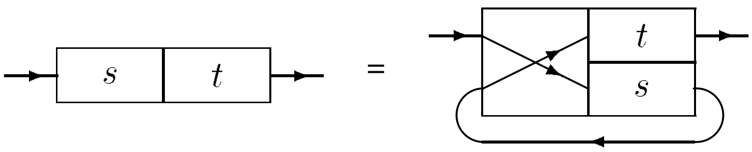
It is natural to define iterated reflexion thus:

$$\uparrow_m a \stackrel{\text{def}}{=} \uparrow_{p_r} \cdots \uparrow_{p_1} a \qquad (m = p_1 \otimes \cdots \otimes p_r).$$

Proposition 2.1 below asserts a strong generalisation of $\boldsymbol{\rho_1}$:

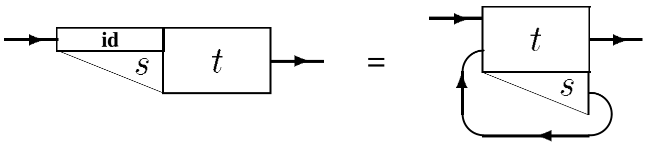
$$s \cdot t = \uparrow_m (\mathbf{i}_{mk} \cdot (s \otimes t)) \qquad (s : k \to m,\ t : m \to n)\,.$$

This shows that, in the presence of reflexion, the composition of arbitary actions can be defined by tensor product and composition with an isomorphism. Again, there is a simple diagrammatic intuition:



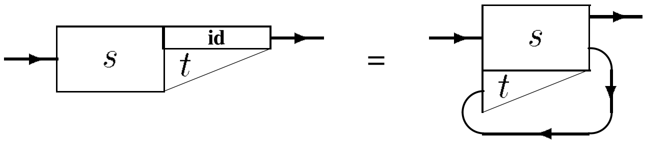
Two more particular consequences of $\boldsymbol{\rho_1}$, which we shall use later, are

$$(s \otimes \mathbf{id}_k) \cdot t = \uparrow_m (s \otimes t) \qquad (s : \epsilon \to m,\ t : m \otimes k \to n)\ ,$$



and its dual

$$s \cdot (t \otimes \mathbf{id}_n) = \uparrow_m (s \otimes t) \qquad (s : k \to m \otimes n,\ t : m \to \epsilon)\ .$$



The proposition also asserts a generalisation of $\boldsymbol{\rho_2}$ with an arbitrary term in place of **id**; thus all reflexions (except those inside $\mathcal{K}$-constructions) can be pulled outermost.

In fact, the proposition asserts more general forms of all the equations $\boldsymbol{\rho}$, with the prime arities p, q replaced by arbitrary arities m, n.

2.1 Proposition The following are provable in AC^r:

(1a)	$s \cdot t = \uparrow_m (\mathbf{i}_{mk} \cdot (s \otimes t))$	$(s : k \to m,\ t : m \to n)$
(1b)	$(s \otimes \mathbf{id}_k) \cdot t = \uparrow_m (s \otimes t)$	$(s : \epsilon \to m,\ t : m \otimes k \to n)$
(1c)	$s \cdot (t \otimes \mathbf{id}_n) = \uparrow_m (s \otimes t)$	$(s : k \to m \otimes n,\ t : m \to \epsilon)$

(2) $\uparrow_m s \otimes t = \uparrow_m (s \otimes t)$
(3) $\uparrow_m s \cdot t = \uparrow_m (s \cdot (\mathbf{id}_m \otimes t))$
(4) $s \cdot \uparrow_m t = \uparrow_m ((\mathbf{id}_m \otimes s) \cdot t)$
(5) $(\vec{x}) \uparrow_m t = \uparrow_m ((\mathbf{i}_{mn} \otimes \mathbf{id}) \cdot (\vec{x}) t)$ $\qquad (\vec{x} : n)$
(6) $\uparrow_n \uparrow_m t = \uparrow_m \uparrow_n ((\mathbf{i}_{nm} \otimes \mathbf{id}) \cdot t \cdot (\mathbf{i}_{mn} \otimes \mathbf{id}))\ .$

Proof Often the proof is by induction on the number of prime factors in m; in those cases the basis is obvious since $\uparrow_\epsilon t$ is just t, and we just prove the induction step. We prove (2), (3) and (4) first, since (1a) and (1b) depend upon them.

(2) First,

$$\begin{array}{rcll}
\Uparrow_p s \otimes t & = & (\Uparrow_p s \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes t) & \\
& = & \Uparrow_p (s \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes t) & \boldsymbol{\rho_2} \\
& = & \Uparrow_p ((s \otimes \mathbf{id}) \cdot (\mathbf{id}_p \otimes \mathbf{id} \otimes t)) & \boldsymbol{\rho_3} \\
& = & \Uparrow_p (s \otimes t)\ . &
\end{array}$$

Now for (2), by induction:

$$\begin{array}{rcll}
\Uparrow_{m\otimes p} s \otimes t & = & \Uparrow_p \Uparrow_m s \otimes t & \\
& = & \Uparrow_p (\Uparrow_m s \otimes t) & \text{above} \\
& = & \Uparrow_p \Uparrow_m (s \otimes t) & \text{induction} \\
& = & \Uparrow_{m\otimes p} (s \otimes t)\ . &
\end{array}$$

(3)

$$\begin{array}{rcll}
\Uparrow_{m\otimes p} s \cdot t & = & \Uparrow_p \Uparrow_m s \cdot t & \\
& = & \Uparrow_p (\Uparrow_m s \cdot (\mathbf{id}_p \otimes t)) & \boldsymbol{\rho_3} \\
& = & \Uparrow_p \Uparrow_m (s \cdot (\mathbf{id}_m \otimes \mathbf{id}_p \otimes t)) & \text{induction} \\
& = & \Uparrow_{m\otimes p} (s \cdot (\mathbf{id}_{m\otimes p} \otimes t))\ . &
\end{array}$$

(4) Similar.

(1a) First, we prove $\mathbf{id}_m = \Uparrow_m \mathbf{i}_{mm}$ by induction on m:

$$\begin{array}{rcll}
\Uparrow_{m\otimes p} \mathbf{i}_{m\otimes p, m\otimes p} & = & \Uparrow_p \Uparrow_m ((\mathbf{id}_m \otimes \mathbf{i}_{pm} \otimes \mathbf{id}_p) \cdot (\mathbf{i}_{mm} \otimes \mathbf{i}_{pp}) \cdot & \\
& & \qquad (\mathbf{id}_m \otimes \mathbf{i}_{mp} \otimes \mathbf{id}_p)) & \mathsf{AC} \\
& = & \Uparrow_p ((\mathbf{i}_{pm} \otimes \mathbf{id}_p) \cdot \Uparrow_m (\mathbf{i}_{mm} \otimes \mathbf{i}_{pp}) \cdot (\mathbf{i}_{mp} \otimes \mathbf{id}_p)) & (3,4) \\
& = & \Uparrow_p ((\mathbf{i}_{pm} \otimes \mathbf{id}_p) \cdot (\Uparrow_m \mathbf{i}_{mm} \otimes \mathbf{i}_{pp}) \cdot (\mathbf{i}_{mp} \otimes \mathbf{id}_p)) & (2) \\
& = & \Uparrow_p ((\mathbf{i}_{pm} \otimes \mathbf{id}_p) \cdot (\mathbf{id}_m \otimes \mathbf{i}_{pp}) \cdot (\mathbf{i}_{mp} \otimes \mathbf{id}_p)) & \text{induction} \\
& = & \Uparrow_p ((\mathbf{id}_p \otimes \mathbf{i}_{mp}) \cdot (\mathbf{i}_{pp} \otimes \mathbf{id}_m) \cdot (\mathbf{id}_p \otimes \mathbf{i}_{pm})) & \mathsf{AC} \\
& = & \mathbf{i}_{mp} \cdot (\Uparrow_p \mathbf{i}_{pp} \otimes \mathbf{id}_m) \cdot \mathbf{i}_{pm} & (2,3,4) \\
& = & \mathbf{i}_{mp} \cdot \mathbf{id}_{p\otimes m} \cdot \mathbf{i}_{pm} & \boldsymbol{\rho_1} \\
& = & \mathbf{id}_{m\otimes p}\ . &
\end{array}$$

Now the main result is simple:

$$\begin{array}{rcll}
s \cdot t & = & s \cdot \mathbf{id}_m \cdot t & \\
& = & s \cdot \Uparrow_m \mathbf{i}_{mm} \cdot t & \text{above} \\
& = & \Uparrow_m ((\mathbf{id}_m \otimes s) \cdot \mathbf{i}_{mm} \cdot (\mathbf{id}_m \otimes t)) & (3,4) \\
& = & \Uparrow_m (\mathbf{i}_{mk} \cdot (s \otimes \mathbf{id}_m) \cdot (\mathbf{id}_m \otimes t)) & \boldsymbol{\zeta} \\
& = & \Uparrow_m (\mathbf{i}_{mk} \cdot (s \otimes t))\ . &
\end{array}$$

(1b)

$$\begin{array}{rcll}
(s \otimes \mathbf{id}_k) \cdot t & = & (s \cdot \Uparrow_m \mathbf{i}_{mm} \otimes \mathbf{id}_k) \cdot t & (1a) \\
& = & (\Uparrow_m ((\mathbf{id}_m \otimes s) \cdot \mathbf{i}_{mm}) \otimes \mathbf{id}_k) \cdot t & (4) \\
& = & \Uparrow_m (s \otimes \mathbf{id}_m \otimes \mathbf{id}_k) \cdot t & (2) \\
& = & \Uparrow_m ((s \otimes \mathbf{id}_{m\otimes k}) \cdot (\mathbf{id}_m \otimes t)) & (3) \\
& = & \Uparrow_m (s \otimes t)\ . &
\end{array}$$

(1c) Similar.

(5) First, for single $x : q$, by induction on m:

$$\begin{array}{rcll} (x)\uparrow_{m\otimes p} t & = & \uparrow_p((\mathbf{i}_{mq} \otimes \mathbf{id}) \cdot (x)\uparrow_m t) & \boldsymbol{\rho}_5 \\ & = & \uparrow_p((\mathbf{i}_{pq} \otimes \mathbf{id}) \cdot \uparrow_m((\mathbf{i}_{mq} \otimes \mathbf{id}) \cdot (x)t) & \text{induction} \\ & = & \uparrow_p \uparrow_m((\mathbf{id}_m \otimes \mathbf{i}_{pq} \otimes \mathbf{id}) \cdot (\mathbf{i}_{mq} \otimes \mathbf{id}) \cdot (x)t) & (3,4) \\ & = & \uparrow_{m\otimes p}((\mathbf{i}_{m\otimes p,q} \otimes \mathbf{id}) \cdot (x)t)\ . & \mathsf{AC} \end{array}$$

Then a similar induction on n proves the result for $\vec{x} : n$.

(6) First, for prime $n = q$, by induction on m:

$$\begin{array}{rcll} \uparrow_q \uparrow_{m\otimes p} t & = & \uparrow_q \uparrow_p \uparrow_m t & \\ & = & \uparrow_p \uparrow_q((\mathbf{i}_{qp} \otimes \mathbf{id}) \cdot \uparrow_m t \cdot (\mathbf{i}_{pq} \otimes \mathbf{id})) & \boldsymbol{\rho}_6 \\ & = & \uparrow_p \uparrow_q \uparrow_m((\mathbf{id}_m \otimes \mathbf{i}_{qp} \otimes \mathbf{id}) \cdot t \cdot (\mathbf{id}_m \otimes \mathbf{i}_{pq} \otimes \mathbf{id})) & (3,4) \\ & = & \uparrow_p \uparrow_m \uparrow_q((\mathbf{i}_{qm} \otimes \mathbf{id}) \cdot (\mathbf{id}_m \otimes \mathbf{i}_{qp} \otimes \mathbf{id}) \cdot t \cdot & \\ & & \qquad (\mathbf{id}_m \otimes \mathbf{i}_{pq} \otimes \mathbf{id}) \cdot (\mathbf{i}_{mq} \otimes \mathbf{id})) & \text{induction} \\ & = & \uparrow_{m\otimes p} \uparrow_q((\mathbf{i}_{q,m\otimes p} \otimes \mathbf{id}) \cdot t \cdot (\mathbf{i}_{m\otimes p,q} \otimes \mathbf{id}))\ . & \mathsf{AC} \end{array}$$

Now the full result is proved by a similar induction on n. ■

3 Molecular forms

The molecular forms $\mathcal{M}^r(\mathcal{K})$ are, with one essential difference, just those of $\mathcal{M}(\mathcal{K},\boldsymbol{\nu})$, where $\mathcal{M}(\mathcal{K})$ is as defined in Section IV–3. The actions a and molecules μ take the form

$$\begin{array}{rcl} a & = & (\vec{x})\, \mu_1 \cdots \mu_r\, \langle \vec{u} \rangle \\ \mu & = & \langle \vec{u} \rangle\, K \vec{c}\, (\vec{y})\ , \end{array}$$

where $\vec{c}$ is a tuple of actions, and $K \in \mathcal{K} \cup \{\boldsymbol{\nu}\}$. A restriction molecule $\langle\ \rangle \boldsymbol{\nu}(y)$ will be written $(\boldsymbol{\nu} y)$. The essential difference from $\mathcal{M}(\mathcal{K},\boldsymbol{\nu})$ is that the body $\vec{\mu}$ of the action a is now a *multiset*, not a partial sequence; in fact the scope of every name bound by a molecule in $\vec{\mu}$ is now the whole of a, so that permutation of molecules has no effect upon meaning. The names bound by the molecules $\vec{\mu}$, and those in $\vec{x}$, must all be distinct. We do not distinguish actions which differ only by change of bound names. Note that the scope of every name is the smallest subaction containing its binding occurrence.

The action structure operations $\mathbf{id}^{\mathcal{M}}$, $\cdot^{\mathcal{M}}$, $\otimes^{\mathcal{M}}$ and $\mathbf{ab}_x^{\mathcal{M}}$, the data $\langle x \rangle^{\mathcal{M}}$ and the discard $\boldsymbol{\omega}^{\mathcal{M}}$ are defined over just as they were for $\mathcal{M}(\mathcal{K},\boldsymbol{\nu})$; so are the control operations $K^{\mathcal{M}}$ for $K \in \mathcal{K} \cup \{\boldsymbol{\nu}\}$. Note that, as there, we use the superscript ${}^{\mathcal{M}}$ to distinguish the operations over molecular forms from the operators of the term algebra. It remains to describe the reflexion operation $\uparrow^{\mathcal{M}}$ on molecular forms; in doing so we shall see that the restriction operation $\boldsymbol{\nu}^{\mathcal{M}} : \epsilon \to p$, instead of being a free construction like the other control operations, is actually definable by using reflexion.

The essential effect of applying $\uparrow^{\mathcal{M}}$ to an action $a = (z\vec{x})\, \vec{\mu}\, \langle y \vec{u} \rangle$ is as follows: If $y = z$ then the molecule $(\boldsymbol{\nu} z)$ is added to a, otherwise z is replaced everywhere by y. This replacement cannot be represented by the ordinary substitution $\{y/z\}$, because y may be a bound name of a.

To define $\uparrow^{\mathcal{M}}$ formally we need a notion called reflexive substitution. Since this notion does *not* preserve alpha-conversion we shall work for the moment with the *syntactic forms* of molecules, without alpha-conversion or permutation.

We use ${}^{y/z}a$ to mean the *literal replacement* of y for z in (the syntactic form) a, and $(\boldsymbol{\nu} z)a$ to mean the introduction of the molecule $(\boldsymbol{\nu} z)$ into a, where in neither case is z bound in a. For example ${}^{y/z}((y)\langle z\rangle) = (y)\langle y\rangle$. We shall also use the simultaneous versions ${}^{\vec{y}/\vec{z}}$ and $(\boldsymbol{\nu}\vec{z})$, with obvious meaning.

3.1 Definition The *reflexive substitution* operation ${}^*\{{}^{y}/_{z}\}$ on actions is defined as follows, where it is assumed that z is not bound in a:

$$ {}^*\{{}^{y}/_{z}\}a \stackrel{\text{def}}{=} \begin{cases} {}^{y/z}a & (y \neq z) \\ (\boldsymbol{\nu} z)a & (y = z)\ . \end{cases} \qquad \blacksquare $$

Though reflexive substitution does not preserve alpha-conversion, it enjoys some properties which we shall need later:

3.2 Proposition

(1) Let $(z')a'$ be an alpha-variant of $(z)a$, with y not bound in either. Then ${}^*\{{}^{y}/_{z'}\}a'$ is an alpha-variant of ${}^*\{{}^{y}/_{z}\}a$.

(2) Let $(z')a'$ be an alpha-variant of $(z)a$, with y a bound name in $(z)a$ and y' its variant in $(z')a'$. Then ${}^*\{{}^{y'}/_{z'}\}a'$ is an alpha-variant of ${}^*\{{}^{y}/_{z}\}a$.

(3) If z is not bound in a, and neither z nor any bound name of a is in $\vec{w}$ or $\vec{x}$, then the following hold as identities of syntactic forms:

$$ {}^{\vec{w}/\vec{x}}({}^*\{{}^{y}/_{z}\}a) = {}^*\{{}^{{}^{\vec{w}/\vec{x}}y}/_{z}\}\,{}^{\vec{w}/\vec{x}}a\ ,\quad (\boldsymbol{\nu}\vec{x})({}^*\{{}^{y}/_{z}\}a) = {}^*\{{}^{y}/_{z}\}(\boldsymbol{\nu}\vec{x})a\ . \qquad \blacksquare $$

We now define the operation of reflexion $\uparrow^{\mathcal{M}}$ upon molecular forms as follows:

3.3 Definition Let $a = (z\vec{x})\,\vec{\mu}\,\langle y\vec{u}\rangle$; then

$$ \uparrow^{\mathcal{M}} a \stackrel{\text{def}}{=} {}^*\{{}^{y}/_{z}\}(\vec{x})\vec{\mu}\langle\vec{u}\rangle\ . \qquad \blacksquare $$

We can now justify our claim that the nullary control operation $\boldsymbol{\nu}$, whose molecular form is $\boldsymbol{\nu} = (\boldsymbol{\nu} x)\langle x\rangle$, is definable from reflexion. In fact it is tight reflexive knot, as suggested by the diagram in Section 1; for it is immediate from our definition of reflexion that

$$ \boldsymbol{\nu} = \uparrow^{\mathcal{M}} (x)\langle xx\rangle\ . $$

From Proposition 3.2(1),(2) we deduce that $\uparrow^{\mathcal{M}}$ preserves alpha-conversion; it also clearly preserves permutation of molecules. Therefore it is a well-defined operation on molecular forms, and we are again free to consider molecular forms only up to alpha-conversion and permutation of molecules.

An important property of $\uparrow^{\mathcal{M}}$ is:

3.4 Proposition Reflexion preserves ordinary substitution, i.e. $\uparrow^{\mathcal{M}} \sigma a = \sigma \uparrow^{\mathcal{M}} a$.

Proof By alpha-conversion we may assume that no bound name of a is involved in σ, so that σ can operate as a literal substitution. Now use Proposition 3.2(3). ■

We shall have reason to use both an iterated form of reflexion and a simultaneous version of reflexive substitution, as follows:

3.5 Definition The *iterated reflexion* operator $\uparrow_k^{\mathcal{M}}$, for $k = p_1 \otimes \cdots \otimes p_r$, is given by

$$\uparrow_k^{\mathcal{M}} a \stackrel{\text{def}}{=} \uparrow_{p_r}^{\mathcal{M}} \cdots \uparrow_{p_1}^{\mathcal{M}} a \ .$$

(Note that if $r = 0$ then $k = \epsilon$ and $\uparrow_k^{\mathcal{M}} a = a$.) The *simultaneous reflexive substitution* ${}^*\{\vec{y}/\vec{z}\}$ is given recursively in terms of the single form by

$$\begin{aligned} {}^*\{\ \}a &\stackrel{\text{def}}{=} a \\ {}^*\{z/w, \vec{y}/\vec{x}\}a &\stackrel{\text{def}}{=} {}^*\{{}^{z/w}\vec{y}/\vec{x}\}\,{}^*\{z/w\}a \ . \end{aligned}$$

■

The following summarises what we need to know about these two concepts. The first part justifies calling the substitution ${}^*\{\vec{y}/\vec{z}\}$ "simultaneous", and the second part shows that it underlies iterated reflexion just as the single version underlies single reflexion.

3.6 Proposition

(1) The reflexive substitution ${}^*\{\vec{y}/\vec{z}\}a$ is unaffected (up to alpha-conversion and permutation of molecules) by permutation of the substitution elements y_i/z_i.

(2) If $\vec{y}, \vec{z} : k$ and $a = (\vec{z}\vec{x})\,\vec{\mu}\,\langle\vec{y}\vec{u}\rangle$ then $\uparrow_k^{\mathcal{M}} a \ = {}^*\{\vec{y}/\vec{z}\}\,(\vec{x})\vec{\mu}\langle\vec{u}\rangle$.

Proof Both parts can be proved by induction on r, the number of elements in the substitution ${}^*\{\vec{y}/\vec{z}\}$. In (2) take $k = p_1 \otimes \cdots \otimes p_r$. ■

We now assert

3.7 Proposition $\mathcal{M}^r(\mathcal{K})$ is an action structure, and satisfies the axioms of AC^r.

Proof The proofs for the basic action structure axioms and those of AC are just as for $\mathcal{M}(\mathcal{K})$; see also Proposition IV–3.1. Reflexive substitution is not involved, since it is only used by $\uparrow^{\mathcal{M}}$.

For the axioms $\boldsymbol{\rho}$, the properties of reflexive substitution are needed. Consider $\boldsymbol{\rho}_6$; we have to check that

$$\uparrow_{p'}^{\mathcal{M}} \uparrow_p^{\mathcal{M}} (zz'\vec{x})\,\vec{\mu}\,\langle yy'\vec{u}\rangle \ = \uparrow_p^{\mathcal{M}} \uparrow_{p'}^{\mathcal{M}} (z'z\vec{x})\,\vec{\mu}\,\langle y'y\vec{u}\rangle \ ,$$

where $y, z : p$ and $y', z' : p'$. This follows directly from Proposition 3.6(1). ■

4 Translating reflexive molecular forms to terms

We shall now define the translation $\widehat{(-)} : \mathcal{M}^r(\mathcal{K}) \to \mathcal{T}^r(\mathcal{K})$. Note first that $\boldsymbol{\nu}$ is not a member of $\mathcal{K}$; but in $\mathcal{T}^r(\mathcal{K})$ we *define* $\boldsymbol{\nu} \stackrel{\text{def}}{=} \uparrow(x)\langle xx\rangle$. The translation, which differs from that in Section IV–4, is defined inductively on syntactic molecular forms, as follows. Let

$$a \;=\; (\vec{x})\,[\,\langle\vec{v_1}\rangle K_1\vec{c_1}(\vec{y_1}),\;\ldots,\;\langle\vec{v_r}\rangle K_r\vec{c_r}(\vec{y_r})\,]\,\langle\vec{u}\rangle\;,$$

where $K_i \in \mathcal{K} \cup \{\boldsymbol{\nu}\}$ and $\vec{y_i} : k_i$. Then

$$\widehat{a} \stackrel{\text{def}}{=} \uparrow_{k_r}\cdots\uparrow_{k_1}(\vec{y_1}\cdots\vec{y_r}\vec{x})(\langle\vec{v_1}\rangle\cdot K_1\widehat{\vec{c_1}}\otimes\cdots\otimes\langle\vec{v_r}\rangle\cdot K_r\widehat{\vec{c_r}}\otimes\langle\vec{u}\rangle)\;.$$

Note the leading abstractions $(\vec{y_i})$, to give these names scope over the whole action; the export of each molecule is linked to the names $\vec{y_i}$ by the reflexion $\uparrow_{k_i}$.

We can immediately assert

4.1 Proposition The translation $\widehat{(-)} : \mathcal{M}^r(\mathcal{K}) \to \mathcal{T}^r(\mathcal{K})$ preserves free names, alpha-convertibility and substitution. Further, the translation $\widehat{(-)} : \mathcal{M}^r(\mathcal{K}) \to \mathcal{T}^r(\mathcal{K})/\mathsf{AC}^r$ is well-defined.

Proof For the first part we proceed as in Proposition IV–4.2. It is a straightforward induction that $\mathrm{fn}\,\widehat{a} = \mathrm{fn}\,a$. Then we prove simultaneously by induction that the translation preserves substitution and alpha-convertibility. To be precise: $\sigma\widehat{a}$ is an alpha-variant of $\widehat{\sigma a}$, and if a' is an alpha-variant of a then $\widehat{a'}$ is an alpha-variant of $\widehat{a}$.

For the second part, we have to show that if a' is a variant of a by permutation of molecules and alpha-conversion, then $\mathsf{AC}^r \vdash \widehat{a'} = \widehat{a}$. One can deal with the two kinds of variation separately.

For permutation only, consider the form of $\widehat{a}$ shown in the definition; it is clear that the effect of permutation is just to permute the reflexion operations $\uparrow_{k_i}$, the name-vectors $\vec{y_i}$ and the factors $\langle\vec{v_i}\rangle\cdot K_i\widehat{\vec{c_i}}$ of the tensor product; the rules $\boldsymbol{\zeta}$ and $\boldsymbol{\rho_6}$ will handle this.

For alpha-conversion, another induction suffices; in the inductive step one uses the first part of the proposition. ■

4.2 Proposition The translation $\widehat{(-)} : \mathcal{M}^r(\mathcal{K}) \to \mathcal{T}^r(\mathcal{K})/\mathsf{AC}^r$ preserves the action calculus operations, and also reflexion and restriction.

Proof It is immediate that $\widehat{\mathbf{id}^{\mathcal{M}}} = \mathbf{id}$, $\widehat{\langle x\rangle^{\mathcal{M}}} = \langle x\rangle$ and $\widehat{\boldsymbol{\omega}^{\mathcal{M}}} = \boldsymbol{\omega}$. It remains to show that the translation preserves product, composition, abstraction, reflexion and restriction. Throughout the proof we suppose that

$$\begin{aligned} a &= (\vec{x})\,\vec{\lambda}\,\langle\vec{u}\rangle \;:\; m \to m' \\ b &= (\vec{y})\,\vec{\mu}\,\langle\vec{v}\rangle \;:\; n \to n'\;, \end{aligned}$$

where no name bound in one occurs in the other. We also suppose that the names bound by $\vec{\lambda}$ and $\vec{\mu}$ are $\vec{z} : h$ and $\vec{w} : k$ respectively, so that the translations of a and b are

$$\begin{aligned} \widehat{a} &= \uparrow_h(\vec{z}\vec{x})(s\otimes\langle\vec{u}\rangle) \\ \widehat{b} &= \uparrow_k(\vec{w}\vec{y})(t\otimes\langle\vec{v}\rangle) \end{aligned}$$

where the terms s and t arise from translating $\vec{\lambda}$ and $\vec{\mu}$. Since the translation preserves free names (from an earlier proposition) we know that s does not contain $\vec{w}, \vec{y}$ free, and t does not contain $\vec{z}, \vec{x}$ free; this justifies some steps in the cases concerning product and composition.

The proofs indicate all uses of $\boldsymbol{\rho_1}$–$\boldsymbol{\rho_6}$, via parts (1–6) of Proposition 2.1. Manipulations by AC are often abbreviated.

Product:

$$
\begin{array}{rcll}
\widehat{a \otimes^{\mathcal{M}} b} & = & \uparrow_{h\otimes k}(\vec{z}\vec{w}\vec{x}\vec{y})(s \otimes t \otimes \langle \vec{u}\vec{v}\rangle) & \\
& = & \uparrow_k \uparrow_h(\vec{z}\vec{w}\vec{x}\vec{y})(s \otimes t \otimes \langle \vec{u}\vec{v}\rangle) & \\
& = & \uparrow_h \uparrow_k(\vec{w}\vec{z}\vec{x}\vec{y})(t \otimes s \otimes \langle \vec{u}\vec{v}\rangle) & (6) \\
& = & \uparrow_h \uparrow_k\big((\mathbf{id}_k \otimes \mathbf{i}_{h\otimes m,n}) \cdot & \\
& & \quad (\vec{w}\vec{y}\vec{z}\vec{x})(t \otimes \langle \vec{v}\rangle \otimes s \otimes \langle \vec{u}\rangle) \cdot (\mathbf{id}_k \otimes \mathbf{i}_{n',h\otimes m'})\big) & \\
& = & \uparrow_h\big(\mathbf{i}_{h\otimes m,n}) \cdot (\widehat{b} \otimes (\vec{z}\vec{x})(s \otimes \langle \vec{u}\rangle)) \cdot \mathbf{i}_{n',h\otimes m'}\big) & (2,3,4) \\
& = & \uparrow_h\big((\vec{z}\vec{x})(s \otimes \langle \vec{u}\rangle) \otimes \widehat{b}\big) & \\
& = & \widehat{a} \otimes \widehat{b}\,. & (2)
\end{array}
$$

Composition, with $m' = n$:

$$
\begin{array}{rcll}
\widehat{a \cdot^{\mathcal{M}} b} & = & \uparrow_{h\otimes k}(\vec{z}\vec{w}\vec{x})(s \otimes \{\vec{u}/\vec{y}\}(t \otimes \langle \vec{v}\rangle)) & \\
& = & \uparrow_k \uparrow_h(\vec{z}\vec{w}\vec{x})\big((s \otimes \langle \vec{u}\rangle) \cdot (\mathbf{id}_h \otimes (\vec{y})(t \otimes \langle \vec{v}\rangle))\big) & \\
& = & \uparrow_k \uparrow_h(\vec{z}\vec{w}\vec{x})\big((s \otimes \langle \vec{u}\rangle) \cdot \mathbf{i}_{hn} \cdot & \\
& & \qquad ((\vec{y})(t \otimes \langle \vec{v}\rangle) \otimes \mathbf{id}_h) \cdot \mathbf{i}_{k\otimes n',h}\big) & \\
& = & \uparrow_h \uparrow_k(\vec{w}\vec{z}\vec{x})\big((s \otimes \langle \vec{u}\rangle) \cdot \mathbf{i}_{hn} \cdot & \\
& & \qquad ((\vec{y})(t \otimes \langle \vec{v}\rangle) \otimes \mathbf{id}_h) \cdot (\mathbf{id}_k \otimes \mathbf{i}_{n'h})\big) & (6) \\
& = & \uparrow_h \uparrow_k\big((\mathbf{id}_k \otimes (\vec{z}\vec{x})(s \otimes \langle \vec{u}\rangle) \cdot \mathbf{i}_{hn}) \cdot & \\
& & \qquad ((\vec{w}\vec{y})(t \otimes \langle \vec{v}\rangle) \otimes \mathbf{id}_h) \cdot (\mathbf{id}_k \otimes \mathbf{i}_{n'h})\big) & \\
& = & \uparrow_h\big((\vec{z}\vec{x})(s \otimes \langle \vec{u}\rangle) \cdot \mathbf{i}_{hn} \cdot (\widehat{b} \otimes \mathbf{id}_h) \cdot \mathbf{i}_{n'h}\big) & (2,3,4) \\
& = & \uparrow_h\big((\vec{z}\vec{x})(s \otimes \langle \vec{u}\rangle) \cdot (\mathbf{id}_h \otimes \widehat{b})\big) & \\
& = & \widehat{a} \cdot \widehat{b}\,. & (3)
\end{array}
$$

Abstraction, with $y : p$ not among $\vec{x}$ or $\vec{z}$:

$$
\begin{array}{rcll}
\widehat{(y)^{\mathcal{M}} a} & = & \uparrow_h(\vec{z}y\vec{x})(s \otimes \langle \vec{u}\rangle) & \\
& = & \uparrow_h\big((\mathbf{i}_{hp} \otimes \mathbf{id}) \cdot (y\vec{z}\vec{x})(s \otimes \langle \vec{u}\rangle)\big) & \\
& = & (y)\uparrow_h(\vec{z}\vec{x})(s \otimes \langle \vec{u}\rangle) & (5) \\
& = & (y)\widehat{a}\,. &
\end{array}
$$

Hence also $\widehat{\mathbf{ab}_y^{\mathcal{M}} a} = \mathbf{ab}_y \widehat{a}$, using $\mathbf{ab}_y^{\mathcal{M}} a = (y)^{\mathcal{M}}(\langle y\rangle^{\mathcal{M}} \otimes^{\mathcal{M}} a)$.

Reflexion; $c = (y\vec{x})\,\vec{\lambda}\,\langle v\vec{u}\rangle$ with $y:p$, $v:p$ and $\vec{x}, \vec{\lambda}, \vec{u}$ as before.

Case $y = v$, with $\uparrow_p^{\mathcal{M}} c = (\vec{x})\,(\boldsymbol{\nu} y)\vec{\lambda}\,\langle\vec{u}\rangle$:

$$\begin{array}{rcll}
\widehat{\uparrow_p^{\mathcal{M}} c} & = & \uparrow_{p\otimes h}(y\vec{z}\vec{x})(\boldsymbol{\nu} \otimes s \otimes \langle\vec{u}\rangle) & \\
& = & \uparrow_h \uparrow_p\big(\boldsymbol{\nu} \otimes (y\vec{z}\vec{x})(s \otimes \langle\vec{u}\rangle)\big) & \\
& = & \uparrow_h\big((\boldsymbol{\nu} \otimes \mathbf{id}) \cdot (y\vec{z}\vec{x})(s \otimes \langle\vec{u}\rangle)\big) & (1b) \\
& = & \uparrow_h\big((\uparrow_p (y)\langle yy\rangle \otimes \mathbf{id}) \cdot (y\vec{z}\vec{x})(s \otimes \langle\vec{u}\rangle)\big) & \\
& = & \uparrow_h \uparrow_p (y\vec{z}\vec{x})(\langle y\rangle \otimes s \otimes \langle\vec{u}\rangle) & (2,3) \\
& = & \uparrow_p \uparrow_h (\vec{z}y\vec{x})(s \otimes \langle y\vec{u}\rangle) & (6) \\
& = & \uparrow_p \widehat{c}\,. &
\end{array}$$

Case $y \neq v$, with $\uparrow_p^{\mathcal{M}} c = (\vec{x})\,{}^{v}\!/\!{}_{y}(\vec{\lambda}\,\langle\vec{u}\rangle)$:

$$\begin{array}{rcll}
\widehat{\uparrow_p^{\mathcal{M}}\, c} & = & \uparrow_h(\vec{z}\vec{x})\{{}^{v}\!/\!{}_{y}\}(s \otimes \langle\vec{u}\rangle) & \\
& = & \uparrow_h(\vec{z}\vec{x})\big(\langle v\rangle \cdot (y)(s \otimes \langle\vec{u}\rangle)\big) & \\
& = & \uparrow_h(\vec{z}\vec{x})\uparrow_p\big(\langle v\rangle \otimes (y)(s \otimes \langle\vec{u}\rangle)\big) & (1b) \\
& = & \uparrow_h(\vec{z}\vec{x})\uparrow_p(y)(\langle v\rangle \otimes s \otimes \langle\vec{u}\rangle) & \\
& = & \uparrow_h \uparrow_p(\mathbf{i}_{p,h\otimes m} \cdot (\vec{z}\vec{x}y)(\langle v\rangle \otimes s \otimes \langle\vec{u}\rangle)) & (5) \\
& = & \uparrow_h \uparrow_p(y\vec{z}\vec{x})(\langle v\rangle \otimes s \otimes \langle\vec{u}\rangle) & \\
& = & \uparrow_p \uparrow_h(\vec{z}y\vec{x})(s \otimes \langle v\vec{u}\rangle) & (6) \\
& = & \uparrow_p \widehat{c}\,. &
\end{array}$$

Restriction:

$$\begin{array}{rcll}
\widehat{\boldsymbol{\nu}^{\mathcal{M}}} & = & \widehat{(\boldsymbol{\nu} x)\langle x\rangle} & \\
& = & \uparrow_p(x)(\boldsymbol{\nu} \otimes \langle x\rangle) & \\
& = & \uparrow_p(\boldsymbol{\nu} \otimes \mathbf{id}) & \\
& = & \boldsymbol{\nu}\,. & (1b)
\end{array}$$

■

5 Translating reflexive terms to molecular forms

We now define a translation $[\![-]\!] : \mathcal{T}^r(\mathcal{K}) \to \mathcal{M}^r(\mathcal{K})$ inductively on the structure of terms. It is exactly the translation from $\mathcal{T}(\mathcal{K})$ to $\mathcal{M}(\mathcal{K})$ given in Section IV–5, with an extra case for reflexion. We give it in full for ease of reference:

$$\begin{array}{rclcrcl}
[\![\mathbf{id}]\!] & \stackrel{\text{def}}{=} & (x)\langle x\rangle & & [\![\langle x\rangle]\!] & \stackrel{\text{def}}{=} & \langle x\rangle^{\mathcal{M}} \\
[\![s \cdot t]\!] & \stackrel{\text{def}}{=} & [\![s]\!] \cdot^{\mathcal{M}} [\![t]\!] & & [\![\boldsymbol{\omega}]\!] & \stackrel{\text{def}}{=} & \boldsymbol{\omega}^{\mathcal{M}} \\
[\![s \otimes t]\!] & \stackrel{\text{def}}{=} & [\![s]\!] \otimes^{\mathcal{M}} [\![t]\!] & & [\![K(\vec{t})]\!] & \stackrel{\text{def}}{=} & K^{\mathcal{M}}[\![\vec{t}]\!] \\
[\![\mathbf{ab}_x t]\!] & \stackrel{\text{def}}{=} & \mathbf{ab}_x^{\mathcal{M}}[\![t]\!] & & [\![\uparrow t]\!] & \stackrel{\text{def}}{=} & \uparrow [\![t]\!]\,.
\end{array}$$

The following, as before, is immediate:

5.1 Proposition (1) $\mathrm{fn}\,\widehat{t} \subseteq \mathrm{fn}\,t$. (2) The translation $[\![-]\!]$ preserves the action structure operations, data, discard, substitution, reflexion and restriction.

Proof The first part is a simple induction. So is the second part (except for restriction) because the translation is inductively defined. Using this, we then have for restriction:

$$[\![\boldsymbol{\nu}]\!] = [\![\uparrow(x)\langle xx\rangle]\!] = \uparrow^{\mathcal{M}}(x)\langle xx\rangle = \boldsymbol{\nu}\ . \qquad \blacksquare$$

The essential property of the translation is then

5.2 Proposition The translation $[\![-]\!]$ is a well-defined function from $\mathcal{T}^r(\mathcal{K})/\mathsf{AC}^r$ to $\mathcal{M}^r(\mathcal{K})$; that is, if $\mathsf{AC}^r \vdash s = t$ then $[\![s]\!] = [\![t]\!]$.

Proof The proof is as for Proposition IV–5.3, except that we must now prove the result also when $\vdash s = t$ is an instance of one of the six reflexion axioms $\boldsymbol{\rho_1}$–$\boldsymbol{\rho_6}$. But it is again easy because of the inductive definition of $[\![\ ]\!]$, and the fact that molecular forms satisfy these axioms (Proposition 3.7). Take for example $\boldsymbol{\rho_5}$; for $x : q$ we have

$$[\![(x)\uparrow_p t]\!] = (x)^{\mathcal{M}}\uparrow_p^{\mathcal{M}}[\![t]\!] = \uparrow_p^{\mathcal{M}}((\mathbf{i}_{pq}^{\mathcal{M}} \otimes^{\mathcal{M}} \mathbf{id}^{\mathcal{M}}) \cdot^{\mathcal{M}} (x)^{\mathcal{M}}[\![t]\!]) = [\![\uparrow_p((\mathbf{i}_{pq} \otimes \mathbf{id}) \cdot (x)t)]\!]\ . \quad \blacksquare$$

6 The reflexive action calculus $\mathsf{AC}^r(\mathcal{K})$

We are now ready to prove the main result.

6.1 Theorem The translations $\widehat{(-)}$ and $[\![-]\!]$ between $\mathcal{M}^r(\mathcal{K})$ and $\mathcal{T}^r(\mathcal{K})/\mathsf{AC}^r$ constitute a static isomorphism of action structures, which moreover preserves the control operations $K \in \mathcal{K}$, the data $\langle x\rangle$, the discard $\boldsymbol{\omega}$, the substitutions $\{\vec{y}/\vec{x}\}$, the reflexion operation $\uparrow$ and restriction $\boldsymbol{\nu}$.

Proof We have shown preservation of all operations in both directions; all that remains is to prove that the two translations are indeed mutually inverse.

As before, it is straightforward to prove that $\mathsf{AC}^r \vdash \widehat{[\![t]\!]} = t$ for all terms t, since $[\![-]\!]$ is defined inductively via the action structure operations, the controls, data, discard and reflexion, and $\widehat{(-)}$ has been shown to preserve all of these.

To prove that $[\![\widehat{a}]\!] = a$ for all molecular forms a, we proceed inductively on the structure of a. Let

$$a \;=\; (\vec{x})\,[\,\langle\vec{v_1}\rangle K_1\vec{c_1}(\vec{y_1}),\ \ldots,\ \langle\vec{v_r}\rangle K_r\vec{c_r}(\vec{y_r})\,]\,\langle\vec{u}\rangle\ ,$$

where $K_i \in \mathcal{K} \cup \{\boldsymbol{\nu}\}$ and $\vec{y_i} : k_i$. Let $k = k_1 \otimes \cdots \otimes k_r$. Then as defined in Section 4

$$\widehat{a} \;=\; \uparrow_k(\vec{y_1}\cdots\vec{y_r}\vec{x})(\langle\vec{v_1}\rangle\cdot K_1\widehat{\vec{c_1}} \otimes \cdots \otimes \langle\vec{v_r}\rangle\cdot K_r\widehat{\vec{c_r}} \otimes \langle\vec{u}\rangle)\ ,$$

and by the inductive definition of $[\![-]\!]$, together with the inductive assumption that $[\![\widehat{\vec{c_i}}]\!] = \vec{c_i}$, we obtain

$$[\![\widehat{a}]\!] \;=\; \uparrow_k^{\mathcal{M}}(\vec{y_1}\cdots\vec{y_r}\vec{x})^{\mathcal{M}}(b_1 \otimes^{\mathcal{M}} \cdots \otimes^{\mathcal{M}} b_r \otimes^{\mathcal{M}} \langle\vec{u}\rangle^{\mathcal{M}})$$

where $b_i = \langle\vec{v_i}\rangle^{\mathcal{M}} \cdot^{\mathcal{M}} K_i^{\mathcal{M}}\vec{c_i}$. Note that this is an expression in molecular forms. Using the action calculus operations over molecular forms defined in Section IV–3, this becomes (using new names $\vec{z_1}, \ldots, \vec{z_r}$)

$$[\![\hat{a}]\!] \;=\; \uparrow_k^{\mathcal{M}} (\vec{y_1} \cdots \vec{y_r}\vec{x}) \,[\, \langle\vec{v_1}\rangle K_1\vec{c_1}(\vec{z_1}) \,, \;\ldots\;, \langle\vec{v_r}\rangle K_r\vec{c_r}(\vec{z_r}) \,]\, \langle\vec{z_1} \cdots \vec{z_r}\vec{u}\rangle \;.$$

Now, applying the reflexion yields by Proposition 3.6(2)

$$[\![\hat{a}]\!] \;=\; {}^{*}\{\vec{z}/\vec{y}\}\,(\vec{x}) \,[\, \langle\vec{v_1}\rangle K_1\vec{c_1}(\vec{z_1}) \,, \;\ldots\;, \langle\vec{v_r}\rangle K_r\vec{c_r}(\vec{z_r}) \,]\, \langle\vec{u}\rangle \;;$$

then by the definition of reflexive substitution and a final alpha-conversion we obtain as required

$$[\![\hat{a}]\!] \;=\; a \;. \qquad \blacksquare$$

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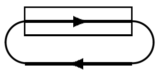
APPENDIX A: Reflecting the identity

Consider the term $\uparrow_p \mathbf{id}_p$ whose diagrammatic representation is



This term is inactive and inaccessible, so there seems to be no point in distinguishing it from $\mathbf{id}_\epsilon$. However, these two terms are not equal in the theory AC^r of Section 2.

In this section we look at the theory $\mathsf{AC}^{r\epsilon}$ obtained from AC^r by adding the axiom

$$\boxed{\boldsymbol{\rho_0} : \quad \uparrow_p \mathbf{id}_p = \mathbf{id}_\epsilon \ .}$$

The following proposition states one consequence of adding $\boldsymbol{\rho_0}$: The action of providing a fresh name and then immediately discarding it is the same as doing nothing.

A.1 Proposition The identity $\boldsymbol{\nu} \cdot \boldsymbol{\omega} = \mathbf{id}_\epsilon$ is provable in $\mathsf{AC}^{r\epsilon}$.

Proof

$$\begin{array}{rcll} \boldsymbol{\nu}_p \cdot \boldsymbol{\omega}_p & = & \uparrow_p (x)\langle xx\rangle \cdot \boldsymbol{\omega}_p & \\ & = & \uparrow_p \big((x)\langle xx\rangle \cdot (\mathbf{id}_p \otimes \boldsymbol{\omega}_p)\big) & \boldsymbol{\rho_3} \\ & = & \uparrow_p \mathbf{id}_p & \mathsf{AC} \\ & = & \mathbf{id}_\epsilon . & \boldsymbol{\rho_0} \end{array}$$

■

Which changes must we make to our notion of reflexive molecular form to match the extra axiom? As we shall see, we simply need one more rule of equality, in addition to alpha-conversion and permutation of molecules. The new rule has to do with *redundant* $\boldsymbol{\nu}$-molecules; we say that a molecule $\boldsymbol{\nu}(z)$ is redundant in a if it creates no bindings, i.e., if z does not occur free in a. The rule which we shall adopt about redundant $\boldsymbol{\nu}$-molecules is that they do not matter: Two molecular forms differing only by the addition or removal of redundant $\boldsymbol{\nu}$-molecules are considered equal.

Let us denote by $\mathcal{M}^{r\epsilon}$ the molecular forms of $\mathcal{M}^r$ under this additional rule of equality. The action structure operations and data, discard and controls are unchanged. The reflection operator $\uparrow^{\mathcal{M}}$ is also unchanged, but we can simplify the definition of the reflective substitution ${}^*\{y/z\}$, in terms of which reflection is defined. The definition of ${}^*\{y/z\}$ in Section 3 distinguishes between two cases, $y \neq z$ and $y = z$, to achieve that a restriction molecule is added only when necessary. In the presence of the new equality rule the following simpler definition is equivalent:

$${}^*\{y/z\}a \stackrel{\text{def}}{=} (\boldsymbol{\nu} z)^{y/z} a \ .$$

The results of the previous sections about $\mathcal{M}^r$ and its relationship with AC^r have analogues for $\mathcal{M}^{r\epsilon}$ and $\mathsf{AC}^{r\epsilon}$: $\mathcal{M}^{r\epsilon}(\mathcal{K})$ is an action structure and satisfies the axioms of $\mathsf{AC}^{r\epsilon}$; the translations $\widehat{(-)}$ and $[\![-]\!]$ (without change) are well-defined translations between $\mathcal{M}^{r\epsilon}(\mathcal{K})$ and $\mathcal{T}^r(\mathcal{K})/\mathsf{AC}^{r\epsilon}$, and they constitute a static isomorphism of action structures.

The two essential properties necessary for these analogous results to hold are stated and proved below. We must check that the axiom $\boldsymbol{\rho_0}$ holds for $\mathcal{M}^{r\epsilon}$, and that any two molecular forms differing only by redundant restriction molecules translate into terms which are equal under $\mathsf{AC}^{r\epsilon}$.

A.2 Proposition

(1) $\boldsymbol{\uparrow}_p^{\mathcal{M}}\mathbf{id}_p^{\mathcal{M}} = \mathbf{id}_\epsilon^{\mathcal{M}}$

(2) $\widehat{(\boldsymbol{\nu} z)a} = \widehat{a}$ $\qquad$ (x not free in a).

Proof

(1) Assuming $x : p$,

$$\boldsymbol{\uparrow}_p^{\mathcal{M}}\mathbf{id}_p^{\mathcal{M}} = \boldsymbol{\uparrow}_p^{\mathcal{M}}(x)\langle x\rangle = {}^*\{x/x\}(x)\langle x\rangle = (\boldsymbol{\nu} x)(\,)\langle\,\rangle = (\,)\langle\,\rangle = \mathbf{id}_\epsilon^{\mathcal{M}}.$$

(2) Let

$$a \;=\; (\vec{x})\,[\,\langle\vec{v_1}\rangle K_1\vec{c_1}(\vec{y_1}),\ \ldots,\ \langle\vec{v_r}\rangle K_r\vec{c_r}(\vec{y_r})\,]\,\langle\vec{u}\rangle\ ,$$

where $K_i \in \mathcal{K}\cup\{\boldsymbol{\nu}\}$. Then

$$(\boldsymbol{\nu} z)a \;=\; (\vec{x})\,[\,\boldsymbol{\nu}(z),\ \langle\vec{v_1}\rangle K_1\vec{c_1}(\vec{y_1}),\ \ldots,\ \langle\vec{v_r}\rangle K_r\vec{c_r}(\vec{y_r})\,]\,\langle\vec{u}\rangle\ .$$

Assuming arities $\vec{y_i} : k_i$, $\vec{x} : m$ and $z : p$ we then have

$$\begin{array}{lll}
\multicolumn{3}{l}{\widehat{(\boldsymbol{\nu} z)a}} \\
= & \boldsymbol{\uparrow}_{k_r}\cdots\boldsymbol{\uparrow}_{k_1}\boldsymbol{\uparrow}_p & \\
 & (z\vec{y_1}\cdots\vec{y_r}\vec{x})(\boldsymbol{\nu}\otimes\langle\vec{v_1}\rangle\cdot K_1\widehat{\vec{c_1}}\otimes\cdots\otimes\langle\vec{v_r}\rangle\cdot K_r\widehat{\vec{c_r}}\otimes\langle\vec{u}\rangle) & \\
= & \boldsymbol{\uparrow}_{k_r}\cdots\boldsymbol{\uparrow}_{k_1}\boldsymbol{\uparrow}_p & \\
 & (\boldsymbol{\omega}\otimes(\vec{y_1}\cdots\vec{y_r}\vec{x})(\boldsymbol{\nu}\otimes\langle\vec{v_1}\rangle\cdot K_1\widehat{\vec{c_1}}\otimes\cdots\otimes\langle\vec{v_r}\rangle\cdot K_r\widehat{\vec{c_r}}\otimes\langle\vec{u}\rangle)) & \boldsymbol{\gamma} \\
= & \boldsymbol{\uparrow}_{k_r}\cdots\boldsymbol{\uparrow}_{k_1}\boldsymbol{\uparrow}_p & \\
 & (\boldsymbol{\omega}\otimes\boldsymbol{\nu}\otimes(\vec{y_1}\cdots\vec{y_r}\vec{x})(\langle\vec{v_1}\rangle\cdot K_1\widehat{\vec{c_1}}\otimes\cdots\langle\vec{v_r}\rangle\cdot K_r\widehat{\vec{c_r}}\otimes\langle\vec{u}\rangle)) & \mathsf{AC} \\
= & \boldsymbol{\uparrow}_{k_r}\cdots\boldsymbol{\uparrow}_{k_1} & \\
 & ((\boldsymbol{\nu}\otimes(\vec{y_1}\cdots\vec{y_r}\vec{x})(\langle\vec{v_1}\rangle\cdot K_1\widehat{\vec{c_1}}\otimes\cdots\langle\vec{v_r}\rangle\cdot K_r\widehat{\vec{c_r}}\otimes\langle\vec{u}\rangle))\cdot & \\
 & \;(\boldsymbol{\omega}\otimes\mathbf{id}_{k_1\otimes\cdots\otimes k_r\otimes m})) & 2.1(1c) \\
= & \boldsymbol{\uparrow}_{k_r}\cdots\boldsymbol{\uparrow}_{k_1} & \\
 & ((\boldsymbol{\nu}\cdot\boldsymbol{\omega})\otimes(\vec{y_1}\cdots\vec{y_r}\vec{x})(\langle\vec{v_1}\rangle\cdot K_1\widehat{\vec{c_1}}\otimes\cdots\langle\vec{v_r}\rangle\cdot K_r\widehat{\vec{c_r}}\otimes\langle\vec{u}\rangle)) & \mathsf{AC} \\
= & \boldsymbol{\uparrow}_{k_r}\cdots\boldsymbol{\uparrow}_{k_1} & \\
 & (\vec{y_1}\cdots\vec{y_r}\vec{x})(\langle\vec{v_1}\rangle\cdot K_1\widehat{\vec{c_1}}\otimes\cdots\langle\vec{v_r}\rangle\cdot K_r\widehat{\vec{c_r}}\otimes\langle\vec{u}\rangle) & 7.1 \\
= & \widehat{a}. &
\end{array}$$

■