A Representation Result for Free Cocompletions

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Abstract

Given a class F of weights, one can consider the construction that takes a small category $\mathbb{C}$ to the free cocompletion of $\mathbb{C}$ under weighted colimits, for which the weight lies in F. Provided these free F-cocompletions are small, this construction generates a 2-monad on **Cat**, or more generally on $\mathcal{V}$-**Cat** for monoidal biclosed complete and cocomplete $\mathcal{V}$. We develop the notion of a dense 2-monad on $\mathcal{V}$-**Cat** and characterise free F-cocompletions by dense KZ-monads on $\mathcal{V}$-**Cat**. We prove various corollaries about the structure of such 2-monads and their Kleisli 2-categories, as needed for the use of open maps in giving an axiomatic study of bisimulation in concurrency. This requires the introduction of the concept of a pseudo-commutativity for a strong 2-monad on a symmetric monoidal 2-category, and a characterisation of it in terms of structure on the Kleisli 2-category.

1 Introduction

Given a class of small categories S, one can consider the construction that takes a small category $\mathbb{C}$ to the free cocompletion of $\mathbb{C}$ under colimits of di-

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agrams with shape in the class S. More subtly, and more naturally from the perspective of enriched category theory, for any monoidal biclosed complete and cocomplete $\mathcal{V}$, given a class F of weights, one can consider the construction that takes a small $\mathcal{V}$-category $\mathbb{C}$ to the free cocompletion of $\mathbb{C}$ under weighted colimits for which the weight lies in F. Provided that these S- or F-cocompletions are small, this construction generates a 2-monad on **Cat**, or more generally on $\mathcal{V}$-**Cat**. We give an axiomatic characterisation of those 2-monads that arise in this way.

Although our characterisation and the results leading up to it hold for monoidal biclosed complete and cocomplete $\mathcal{V}$, we shall only state them for symmetric monoidal closed complete and cocomplete $\mathcal{V}$, because otherwise the swapping between $\mathcal{V}$-categories and $\mathcal{V}_t$-categories, where $\mathcal{V}_t = \mathcal{V}$ but has the tensor product switched, becomes notationally tiresome. Some later results only hold for symmetric or even cartesian closed $\mathcal{V}$: where that is so, we state it explicitly.

The notion of KZ-monad was introduced as a property of 2-monads given by cocompletions under classes of colimits [12,11,17,15]. But it is not sufficient to characterise them. For instance, consider the 2-monad T on **Cat** that sends every category to $\mathbf{1}$, the category with one object and one arrow only. It is a KZ-monad, but it is not given by a free cocompletion as there are many categories $\mathbb{C}$ that do not embed fully into $\mathbf{1}$. So we need an additional condition. The condition we introduce is that of density of a 2-monad. We say that a 2-monad on $\mathcal{V}$-**Cat** is dense if both the unit $\eta_{\mathbb{C}} : \mathbb{C} \longrightarrow T\mathbb{C}$ and the $\mathcal{V}$-functor induced by the unit $\tilde{\eta}_{\mathbb{C}} : T\mathbb{C} \longrightarrow [\mathbb{C}^{\mathrm{op}}, \mathcal{V}]$ are fully faithful, coherently with respect to the action of T on $\mathcal{V}$-functors. It follows that the coherence holds for $\mathcal{V}$-natural transformations too. The central result of this paper is the proof that a 2-monad on $\mathcal{V}$-**Cat** is a dense KZ-monad if and only if there is a class F of weights such that T is the 2-monad whose algebras are small F-cocomplete $\mathcal{V}$-categories.

En route to that result, we characterise dense KZ-monads as those 2-monads for which the $\mathcal{V}$-functors $\eta_{\mathbb{C}}$ and $\tilde{\eta}_{\mathbb{C}}$ are always fully faithful and such that, for any $\mathcal{V}$-functor $H : \mathbb{C} \longrightarrow T\mathbb{D}$, the lifting of H to $T\mathbb{C}$ given by the Kleisli construction is the restriction, up to coherent isomorphism, of $\mathrm{Lan}_{\mathrm{y}_{\mathbb{C}}}(\tilde{\eta}_{\mathbb{D}} H)$, where $\mathrm{y}_{\mathbb{C}} : \mathbb{C}^{\mathrm{op}} \to [\mathbb{C}^{\mathrm{op}}, \mathcal{V}]$ is the Yoneda embedding.

We further show consequences of the definition, or alternatively of the characterisation, of dense KZ-monads: it follows that T sends finite coproducts to finite products if finite coproducts exist coherently in each $T\mathbb{C}$; in fact, the Kleisli 2-category has finite coproducts and they agree with finite products. Moreover, under a mild extra condition, every dense KZ-monad is pseudo-commutative, or equivalently, the Kleisli category is equivalent to a **Gray**-monoid, with structure induced by that in $\mathcal{V}$-**Cat**.

The structures developed here are not purely of abstract interest, but have been used in the study of concurrency. They are at the heart of attempts to model bisimulation by open maps [6,3]. Given a monad T on **Cat**, one may define a notion of open map within any category of the form $T\mathbb{C}$, with openness with respect to maps in $\mathbb{C}$. This notion of open map agrees with the notion of functional bisimulation, and can be extended, by considering spans of epimorphic open maps, to account for bisimulation in general as in [13]. One can prove, for a dense KZ-monad T, that all functors in the Kleisli category for T preserve open maps. The above structures allow one to account for the preservation of bisimulation by the various constructors, such as prefixing, nondeterministic sum, and a parallel operator, in a process algebra such as **CCS**. See [3], which contains an early version of some of the results of this paper, for more detail.

The paper is organised as follows. In Section 2, we recall the definition of weighted colimit in a $\mathcal{V}$-category for monoidal biclosed complete and cocomplete $\mathcal{V}$. In Section 3, we motivate and define the notion of dense KZ-monad, and characterise it in terms of a condition on the Kleisli 2-category. In Section 4, we give our characterisation theorem. In Section 5, we develop the notion of a pseudo-commutativity for a strong 2-monad on a symmetric monoidal 2-category, and we characterise it in terms of a **Gray**-monoid structure on the Kleisli 2-category. Finally, in Section 6, we deduce further properties of dense KZ-monads, requiring some further conditions.

2 Weighted colimits and free cocompletions

In this section, we recall some definitions and results associated with weighted colimits and free cocompletions. The standard reference for these, in the case of enrichment over symmetric monoidal closed $\mathcal{V}$, is Kelly's book [9]. For the case of monoidal biclosed $\mathcal{V}$, some more delicacy is needed, and the definitions for such $\mathcal{V}$, more generally for enrichment in a biclosed bicategory $\mathcal{W}$, are in [4]. For ease of exposition, we shall write in terms of symmetric $\mathcal{V}$.

Definition 1 *A* weight *is a small $\mathcal{V}$-category $\mathbb{D}$ together with a $\mathcal{V}$-functor*

$$f : \mathbb{D}^{\mathrm{op}} \longrightarrow \mathcal{V} \ .$$

Definition 2 *Let $f : \mathbb{D}^{\mathrm{op}} \longrightarrow \mathcal{V}$ be a weight and let $g : \mathbb{D} \longrightarrow \mathbb{C}$ be a $\mathcal{V}$-functor. An f-*weighted colimit *of g is an object $\mathrm{colim}(f, g)$ of $\mathbb{C}$ together with, for each object X of $\mathbb{C}$, an isomorphism between $[\mathbb{D}^{\mathrm{op}}, \mathcal{V}](f-, \mathbb{C}(g-, X))$ and $\mathbb{C}(\mathrm{colim}(f, g), X)$, $\mathcal{V}$-naturally in X.*

This definition can equally be expressed as asking for the existence of a colimiting cylinder [9] (rather than cone), such that composition with that cylinder yields the above natural isomorphism for every X. Here, a cylinder is just an object of $[\mathbb{D}^{\mathrm{op}}, \mathcal{V}](f-, \mathbb{C}(g-, X))$, extending the notion of cone, which is essentially the case where f is the constant at the unit of the tensor 1: it is not quite that simple, mainly because for arbitrary $\mathcal{V}$, the unit need not be the terminal object, and consequently constant $\mathcal{V}$-functors need not exist. See [9] for more detail on that. Nevertheless, weighted colimits do include all conical colimits. They also include all tensors, which one obtains by considering $\mathbb{D} = 1$. For the case of $\mathcal{V} = \mathbf{Set}$, *i.e.*, for ordinary categories, tensors are copowers, so the concept of weighted colimit loses much of its force. But that is not true for $\mathcal{V}$-categories in general, for instance for $\mathcal{V} = \mathbf{Poset}$ or $\mathcal{V} = \mathbf{Cat}$. And even in the case of ordinary categories, our characterisation here of free cocompletions under weighted colimits cannot be expressed simply in terms of free cocompletions under conical colimits, as we shall see.

Definition 3 *Given a class F of weights, a locally small $\mathcal{V}$-category with all f-weighted colimits for all f in F is called* F-cocomplete. *A $\mathcal{V}$-functor between F-cocomplete $\mathcal{V}$-categories that preserves all f-weighted colimits for all f in F is called* F-cocontinuous.

Definition 4 *Given a class F of weights, a* 2-monad for F-cocompletions *is a 2-monad T on $\mathcal{V}$-$\mathbf{Cat}$ for which, for every small $\mathcal{V}$-category $\mathbb{C}$, the unit $\eta_{\mathbb{C}} : \mathbb{C} \longrightarrow T\mathbb{C}$ exhibits $T\mathbb{C}$ as the free F-cocompletion of $\mathbb{C}$,* i.e., *for any locally small F-cocomplete $\mathcal{V}$-category $\mathbb{D}$, composition with $\eta_{\mathbb{C}}$ gives an equivalence between the category of F-cocontinuous $\mathcal{V}$-functors from $T\mathbb{C}$ to $\mathbb{D}$ and that of $\mathcal{V}$-functors from $\mathbb{C}$ to $\mathbb{D}$.*

Letting F-$\mathbf{Coc}$ denote the 2-category of small F-cocomplete $\mathcal{V}$-categories, F-cocontinuous $\mathcal{V}$-functors, and all $\mathcal{V}$-natural transformations, an immediate consequence of the definition is

Proposition 5 *If T is a 2-monad for F-cocompletions, then T provides a left biadjoint to the forgetful 2-functor $U : F$-$\mathbf{Coc} \longrightarrow \mathcal{V}$-$\mathbf{Cat}$.*

The corollary does not have a converse. For instance, if $\mathcal{V} = \mathbf{Set}$ and F consists of all possible weights, then F-$\mathbf{Coc}$ is biequivalent to the 2-category of complete lattices, as every small cocomplete category is a preordered set; and a left biadjoint to U is not what we mean by the free cocompletion of a small category. If F is the class of all weights, then there is no 2-monad for free cocompletions, precisely because the free cocompletion of a small category is not small. However, it follows from the corollary that if it exists, a 2-monad for F-cocompletions is unique up to equivalence.

Note that we have been careful to consider F-cocontinuous $\mathcal{V}$-functors, not $\mathcal{V}$-functors that strictly preserve F-weighted colimits. The situation we consider is the natural one, but we remark that our results here do not restrict well, in that the 2-monads we consider do not act as free F-cocompletions with respect to $\mathcal{V}$-functors that strictly preserve F-weighted colimits. For an account of some of the relevant issues, and for our 2-categorical notation, see [2].

A characterisation of 2-monads for F-cocompletions is the main result of this paper. The central background result we use, shown for symmetric $\mathcal{V}$ in [9], extending routinely to non-symmetric $\mathcal{V}$ by the work of [4], is

Theorem 6 *The free F-cocompletion of a small $\mathcal{V}$-category $\mathbb{C}$ is given by the closure of the representables in $[\mathbb{C}^{\mathrm{op}}, \mathcal{V}]$ under F-weighted colimits.*

3 Dense KZ-monads

In this section, we motivate and define the notion of dense KZ-monad, and we characterise them in terms of their Kleisli categories. In order to fix notation, we recall

Definition 7 *A 2-monad on a 2-category $\mathcal{K}$ consists of a 2-functor*

$$T : \mathcal{K} \longrightarrow \mathcal{K} \ ,$$

together with 2-natural transformations $\eta : id \Rightarrow T$ and $\mu : T^2 \Rightarrow T$, satisfying the usual three axioms treated as axioms about 2-natural transformations rather than ordinary natural transformations.

Definition 8 *Given a 2-monad T on $\mathcal{K}$, the* Kleisli 2-category of T, *$Kl(T)$, has as objects all objects C of $\mathcal{K}$, and arrows those arrows $H : TC \longrightarrow TD$ that respect μ_C and μ_D, with composition given by composition in $\mathcal{K}$. An arrow in $Kl(T)$ may equivalently be described as any arrow in $\mathcal{K}$ from C to TD. The 2-cells are those 2-cells in $\mathcal{K}$ that respect μ.*

Considerable detail of 2-monads and the category theoretic constructions associated with them appears in [10].

Definition 9 *A* KZ-monad *is a 2-monad for which $\mu : T^2 \Rightarrow T$ is left adjoint to ηT with counit of the adjunction given by the identity. It is equivalent to ask that μ be right adjoint to $T\eta$ with unit of the adjunction given by the identity (see [12]).*

The notion of KZ-monad was introduced to study particular features of 2-monads given by free cocompletions under classes of colimits [12]. But they do not characterise such free cocompletions as the following example shows.

Example 10 *Observe that by Theorem 6 the functors $\eta_{\mathbb{C}} : \mathbb{C} \to T\mathbb{C}$ in a free cocompletion (*cf. *Definition 4) are full and faithful. Consider then the 2-*

monad on **Cat** *that sends every category to the one object one arrow category* **1**. *It is a KZ-monad trivially, but it does not give free cocompletions under a class of colimits because* $\mathbb{C}$ *typically is not a full subcategory of* **1**.

So we need to consider a condition that is stronger than that of being a *KZ*-monad.

NOTATION:

- Given a 2-monad T on $\mathcal{V}$-**Cat**, let $\tilde{\eta}_{\mathbb{C}} : T\mathbb{C} \longrightarrow [\mathbb{C}^{\mathrm{op}}, \mathcal{V}]$ denote the $\mathcal{V}$-functor that sends an object X to the $\mathcal{V}$-functor

$$T\mathbb{C}(\eta_{\mathbb{C}}-, X) : \mathbb{C}^{\mathrm{op}} \longrightarrow \mathcal{V} \ .$$

- Given $\mathcal{V}$-functors $H : \mathcal{C} \longrightarrow \mathcal{D}$ and $J : \mathcal{C} \longrightarrow \mathcal{C}'$, the *left Kan extension of H along J* is given by a $\mathcal{V}$-functor $\mathrm{Lan}_J H : \mathcal{C}' \longrightarrow \mathcal{D}$ and a $\mathcal{V}$-natural transformation $\alpha : H \Rightarrow (\mathrm{Lan}_J H)J$ that is universal among such $\mathcal{V}$-natural transformations, *i.e.*, given any $\mathcal{V}$-functor $K : \mathcal{C}' \longrightarrow \mathcal{D}$ and any $\mathcal{V}$-natural transformation $\beta : H \Rightarrow KJ$, there exists a unique $\mathcal{V}$-natural transformation $\gamma : \mathrm{Lan}_J H \Rightarrow K$ making the evident triangle commute.

By the usual abuse of language we will often address the $\mathcal{V}$-functor $\mathrm{Lan}_J(H)$ as the left Kan extension of H along J. If it exists, a left Kan extension is unique up to coherent isomorphism. If J is fully faithful and a left Kan extension exists, then α is necessarily an isomorphism. The left Kan extension always exists if $\mathcal{C}$ is a small $\mathcal{V}$-category and $\mathcal{D}$ is cocomplete (see [9] for more detail on left Kan extensions).

Definition 11 *A 2-monad T on $\mathcal{V}$-**Cat** is* dense *if for every small $\mathcal{V}$-category $\mathbb{C}$, the $\mathcal{V}$-functors $\eta_{\mathbb{C}} : \mathbb{C} \longrightarrow T\mathbb{C}$ and $\tilde{\eta}_{\mathbb{C}} : T\mathbb{C} \longrightarrow [\mathbb{C}^{\mathrm{op}}, \mathcal{V}]$ are fully faithful, and for any $H : \mathbb{C} \longrightarrow \mathbb{D}$, the $\mathcal{V}$-functor*

$$\mathrm{Lan}_{\mathrm{y}_{\mathbb{C}}}(\mathrm{y}_{\mathbb{D}} H) : [\mathbb{C}^{\mathrm{op}}, \mathcal{V}] \longrightarrow [\mathbb{D}^{\mathrm{op}}, \mathcal{V}]$$

restricts to $TH : T\mathbb{C} \longrightarrow T\mathbb{D}$ up to coherent isomorphism, where $\mathrm{y}_{\mathbb{C}}$ and $\mathrm{y}_{\mathbb{D}}$ are the Yoneda embeddings.

It follows from our conditions that the behaviour of T on 2-cells is given by the restriction of that determined by left Kan extension on $[\mathbb{C}^{\mathrm{op}}, \mathcal{V}]$, *cf.* the proof of Corollary 13.

For 2-monads given by free cocompletions, $\eta_{\mathbb{C}} : \mathbb{C} \longrightarrow T\mathbb{C}$ and $\tilde{\eta}_{\mathbb{C}} : T\mathbb{C} \longrightarrow [\mathbb{C}^{\mathrm{op}}, \mathcal{V}]$ are fully faithful by Theorem 6, and for every $\mathcal{V}$-functor $H : \mathbb{C} \longrightarrow \mathbb{D}$, the $\mathcal{V}$-functor $\mathrm{Lan}_{\mathrm{y}_{\mathbb{C}}}(\mathrm{y}_{\mathbb{D}} H) : [\mathbb{C}^{\mathrm{op}}, \mathcal{V}] \longrightarrow [\mathbb{D}^{\mathrm{op}}, \mathcal{V}]$ restricts to $TH : T\mathbb{C} \longrightarrow T\mathbb{D}$ up to coherent isomorphism (see [9]).

Theorem 12 *Let T be a 2-monad on $\mathcal{V}$-**Cat** for which $\eta_{\mathbb{C}} : \mathbb{C} \longrightarrow T\mathbb{C}$ and $\tilde{\eta}_{\mathbb{C}}$:*

$T\mathbb{C} \longrightarrow [\mathbb{C}^{\mathrm{op}}, \mathcal{V}]$ *are fully faithful for every* $\mathbb{C}$. *Then* T *is a dense* KZ*-monad if and only if every* $\mathcal{V}$*-functor* $H : T\mathbb{C} \longrightarrow T\mathbb{D}$ *in* $Kl(T)$ *is the restriction of* $\mathrm{Lan}_{\mathrm{y}_{\mathbb{C}}}(\tilde{\eta}_{\mathbb{D}} H \eta_{\mathbb{C}}) : [\mathbb{C}^{op}, \mathcal{V}] \longrightarrow [\mathbb{D}^{\mathrm{op}}, \mathcal{V}]$ *up to coherent isomorphism.*

Proof: Suppose T is dense and KZ, and let $H : T\mathbb{C} \longrightarrow T\mathbb{D}$ be a $\mathcal{V}$-functor in $Kl(T)$. Then $H = \mu_{\mathbb{D}} K$ where $K = H\eta_{\mathbb{C}} : \mathbb{C} \longrightarrow T\mathbb{D}$. Using the density condition applied to K and the definition of KZ-monad, and the fact that left Kan extensions into cocomplete $\mathcal{V}$-categories (such as $[\mathbb{D}^{\mathrm{op}}, \mathcal{V}]$) are colimits, so are preserved by $\mathcal{V}$-functors with right $\mathcal{V}$-adjoints, gives the result.

For the converse, given $L : \mathbb{C} \longrightarrow \mathbb{D}$, let $H = TL$. By $\mathcal{V}$-naturality of η and since $\tilde{\eta}_{\mathbb{D}}\eta_{\mathbb{D}}$ is isomorphic to $\mathrm{y}_{\mathbb{D}} : \mathbb{D} \longrightarrow [\mathbb{D}^{\mathrm{op}}, \mathcal{V}]$ by full faithfulness of $\eta_{\mathbb{D}}$, it follows that $\tilde{\eta}_{\mathbb{D}} H \eta_{\mathbb{C}} = \mathrm{y}_{\mathbb{D}} L$, so T is dense.

To see that T is KZ, first observe that $\mu_{\mathbb{C}} : T^2\mathbb{C} \longrightarrow T\mathbb{C}$ is a $\mathcal{V}$-functor in $Kl(T)$. So, up to isomorphism, $\mu_{\mathbb{C}}$ is the restriction of

$$\mathrm{Lan}_{\mathrm{y}_{T\mathbb{C}}}(\tilde{\eta}_{\mathbb{C}} \mu_{\mathbb{C}} \eta_{T\mathbb{C}}) = \mathrm{Lan}_{\mathrm{y}_{T\mathbb{C}}} \tilde{\eta}_{\mathbb{C}} \ .$$

But, by full faithfulness of $\tilde{\eta}_{\mathbb{C}}$, the $\mathcal{V}$-functor $\eta_{T\mathbb{C}} : T\mathbb{C} \longrightarrow T^2\mathbb{C}$ is the restriction of the $\mathcal{V}$-functor sending $K \in [\mathbb{C}^{\mathrm{op}}, \mathcal{V}]$ to $[\mathbb{C}^{\mathrm{op}}, \mathcal{V}](\tilde{\eta}_{\mathbb{C}}-, K)$, but this latter $\mathcal{V}$-functor is the right $\mathcal{V}$-adjoint of $\mathrm{Lan}_{\mathrm{y}_{T\mathbb{C}}} \tilde{\eta}_{\mathbb{C}}$. Since $\tilde{\eta}_{\mathbb{C}}$ and $\tilde{\eta}_{T\mathbb{C}}$ are both fully faithful, it follows that $\mu_{\mathbb{C}}$ is left $\mathcal{V}$-adjoint to $\eta_{T\mathbb{C}}$. $\square$

Corollary 13 *For any dense* KZ*-monad on* $\mathcal{V}$-**Cat**, *every* H *in* $Kl(T)$ *is a left Kan extension of* $H\eta_{\mathbb{C}} : \mathbb{C} \longrightarrow T\mathbb{D}$ *along* $\eta_{\mathbb{C}} : \mathbb{C} \longrightarrow T\mathbb{C}$.

Proof: Given any $K : T\mathbb{C} \longrightarrow T\mathbb{D}$, it follows by full faithfulness of $\tilde{\eta}_{\mathbb{C}} : T\mathbb{C} \longrightarrow [\mathbb{C}^{\mathrm{op}}, \mathcal{V}]$ that $\tilde{\eta}_{\mathbb{D}} K$ is isomorphic to $\mathrm{Lan}_{\tilde{\eta}_{\mathbb{C}}}(\tilde{\eta}_{\mathbb{D}} K)\tilde{\eta}_{\mathbb{C}}$. Moreover, since $\eta_{\mathbb{C}}$ is fully faithful, the Yoneda embedding $\mathrm{y}_{\mathbb{C}} : \mathbb{C} \longrightarrow [\mathbb{C}^{\mathrm{op}}, \mathcal{V}]$ is isomorphic to $\tilde{\eta}_{\mathbb{C}}\eta_{\mathbb{C}}$. So any $\mathcal{V}$-natural transformation $\alpha : H\eta_{\mathbb{C}} \Rightarrow K\eta_{\mathbb{C}}$, induces a $\mathcal{V}$-natural transformation from $\tilde{\eta}_{\mathbb{D}} H \eta_{\mathbb{C}} : \mathbb{C} \longrightarrow [\mathbb{D}^{\mathrm{op}}, \mathcal{V}]$ to $\mathrm{Lan}_{\tilde{\eta}_{\mathbb{C}}}(\tilde{\eta}_{\mathbb{D}} K)\mathrm{y}_{\mathbb{C}}$, hence by definition of left Kan extension, a $\mathcal{V}$-natural transformation

$$\bar{\alpha} : \mathrm{Lan}_{\mathrm{y}_{\mathbb{C}}}(\tilde{\eta}_{\mathbb{D}} H \eta_{\mathbb{C}}) \Rightarrow \mathrm{Lan}_{\tilde{\eta}_{\mathbb{C}}}(\tilde{\eta}_{\mathbb{D}} K) \ .$$

The result follows immediately from Theorem 12, the commutativity up to coherent isomorphism for K, and full faithfulness of $\tilde{\eta}_{\mathbb{D}}$. $\square$

Corollary 14 *Given a dense* KZ*-monad* T *on* $\mathcal{V}$-**Cat**, *for every* $\mathbb{C}$, $\mu_{\mathbb{C}}$ *is the left Kan extension of* $id_{T\mathbb{C}}$ *along* $\eta_{T\mathbb{C}} : T\mathbb{C} \longrightarrow T^2\mathbb{C}$. *Moreover, it is the restriction of* $[\eta_{\mathbb{C}}-, \mathcal{V}] : [T\mathbb{C}^{\mathrm{op}}, \mathcal{V}] \longrightarrow [\mathbb{C}^{\mathrm{op}}, \mathcal{V}]$.

4 The representation theorem

For a dense KZ-monad T, it is not true that there exists a class S of small $\mathcal{V}$-categories such that T is the 2-monad for S-cocompletions. This is the case even for $\mathcal{V} = \mathbf{Set}$, *i.e.*, even for ordinary categories.

Example 15 *Let T be the 2-monad on* **Cat** *such that $T\mathbf{0} = \mathbf{0}$, where $\mathbf{0}$ is the empty category and $T\mathbb{C}$ is given by freely adding an initial object to $\mathbb{C}$. It is routine to check the 2-monad axioms and to verify that T is dense and KZ. However, it is not a free cocompletion. Suppose otherwise. Let S be a class of small categories such that $T\mathbb{C}$ is the free S-cocompletion of $\mathbb{C}$. Since $T\mathbf{0}$ is empty, every category in S must be nonempty. Now consider the transfinite construction of the free S-cocompletion [8]. Since each category in S is nonempty, every colimit that is introduced has a map into it from an already existing object, so ultimately from an object in $\mathbb{C}$. So at no point in the transfinite induction do we introduce an initial object, a contradiction.*

We now state our representation theorem. The above counterexample means we need to consider weighted colimits rather than restricting ourselves to conical ones. Moreover, we need a little caution about the precise statement here, because the class F is typically not small. So the assumption that T is a 2-monad is important to ensure that the free F-cocompletion of a small $\mathcal{V}$-category is small.

Theorem 16 *A 2-monad T on $\mathcal{V}$-***Cat** *is a dense KZ-monad if and only if there is a class F of weights for which T is the 2-monad for F-cocompletions.*

Proof: The forward direction of this largely follows immediately from Theorem 6. It is routine to verify from the universal property and construction in [9] that the 2-monad is a KZ-monad.

For the converse, given T, define F to be the disjoint union over all small $\mathcal{V}$-categories $\mathbb{D}$ of $\{\tilde{\eta}_{\mathbb{D}}(X) : \mathbb{D}^{\mathrm{op}} \longrightarrow \mathcal{V} | X \in T\mathbb{D}\}$. We must show for every small $\mathcal{V}$-category $\mathbb{C}$, that $T\mathbb{C}$ is the closure in $[\mathbb{C}^{\mathrm{op}}, \mathcal{V}]$ of the representables under colimits of diagrams with weight in F. Trivially, $T\mathbb{C}$ is contained in the latter $\mathcal{V}$-category, as (the image under $\tilde{\eta}_{\mathbb{D}}$ of) every object of $T\mathbb{C}$ lies in F, and is the colimit of representables determined by itself as a weight. So it remains to show for any small $\mathcal{V}$-category $\mathbb{D}$, any X in $T\mathbb{D}$, and any $g : \mathbb{D} \longrightarrow T\mathbb{C}$, that the weighted colimit $\mathrm{colim}(\tilde{\eta}_{\mathbb{D}}X, \tilde{\eta}_{\mathbb{C}}g)$ in $[\mathbb{C}^{\mathrm{op}}, \mathcal{V}]$ lies in $T\mathbb{C}$. But this colimit is $(\mathrm{Lan}_{\mathrm{y}_{\mathbb{D}}}(\tilde{\eta}_{\mathbb{C}}g))\tilde{\eta}_{\mathbb{D}}X$, and $\mathrm{Lan}_{\mathrm{y}_{\mathbb{D}}}(\tilde{\eta}_{\mathbb{C}}g)$ restricts to the lifting of g. So the colimit does lie in $T\mathbb{C}$ and is given by the lifting of g. $\square$

We can gain a little more out of this with care. Given a small class F of weights, such as the class of weights for all finite colimits, one can consider the 2-monad T_F for F-cocompletions. One can then use the construction of the Theorem 16 to obtain another class of weights F' such that T is the 2-monad for F'-cocompletions. But by construction, F' will be large. But we can characterise F' directly in terms of F. Albert and Kelly [1] defined a notion of closed class of weights.

Definition 17 *A class F of weights is* closed *if for any weight $f : \mathbb{D}^{\mathrm{op}} \to \mathcal{V}$, if every F-cocomplete $\mathcal{V}$-category has f-weighted colimits, and if every F-cocontinuous $\mathcal{V}$-functor preserves f-weighted colimits, then f lies in F.*

Any class of weights F has a closure $\bar{F}$. For instance, if F includes weights for finite products and equalisers, then $\bar{F}$ includes weights for all finite limits. Albert and Kelly showed

Proposition 18 *A weight $f : \mathbb{D}^{\mathrm{op}} \longrightarrow \mathcal{V}$ is in $\bar{F}$ if and only if it is in the closure of the representables in $[\mathbb{D}^{\mathrm{op}}, \mathcal{V}]$ under F-weighted colimits.*

So, it follows from the construction of the Theorem 16 that we have

Corollary 19 *The class of weights given by the construction of the Theorem 16 is always closed. Moreover for any class F of weights for which F-cocompletions of small $\mathcal{V}$-categories are small, the class of weights given by the construction of the Theorem 16 is the closure $\bar{F}$ of F.*

5 Pseudo-commutativity

In order to study the properties of dense KZ-monads, we need a notion of pseudo-commutativity of a strong 2-monad on a symmetric monoidal 2-category, and we need a result characterising pseudo-commutative 2-monads in terms of the Kleisli construction. So we develop such definitions and results in this section. A notion of pseudo-commutativity appeared in Kelly's [7], but was not developed in general. We have the same notion but more compact axioms.

Definition 20 *Let $\mathcal{K}$ be a symmetric monoidal 2-category with symmetry c. A* pseudo-commutativity *for a strong 2-monad T on $\mathcal{V}$ with strength t consists of, for every pair of objects (C, D), an isomorphic 2-cell $\chi_{C,D}$ between the two maps from $TC \otimes TD$ to $T(C \otimes D)$ induced by the strength, such that χ is natural in C and D and satisfies three coherence conditions: $T(c_{C,D})\chi_{C,D} = (\chi_{D,C})^{-1}c_{TC,TD}$, coherence with respect to μ_C, and $\chi_{C,D}(\eta_C \otimes id_{TD}) = id_{t_{C,D}}$ (coherence for D being a consequence).*

A pseudo-commutative *monad is a strong 2-monad together with a pseudo-commutativity.*

In general, a strong monad T on a symmetric monoidal category is commutative if and only if $Kl(T)$ is symmetric monoidal, with tensor product given by $T(C \otimes D)$. We generalise that result. Modulo a straightforward generalisation of the usual coherence relating monoidal and strict monoidal categories, the notion we need here is that of **Gray**-monoid. The category 2-**Cat** of small 2-categories and 2-functors has a symmetric monoidal closed structure for which the closed structure is given by the 2-category $Ps(\mathbb{C}, \mathbb{D})$ of 2-functors from $\mathbb{C}$ to $\mathbb{D}$ and pseudo-natural transformations between them. The category 2-**Cat** together with this symmetric monoidal closed structure is called **Gray** [5].

Definition 21 *A* **Gray**-*monoid is a monoid in the symmetric monoidal closed category* **Gray**, i.e., *a 2-category $\mathcal{K}$ together with a 2-functor*

$$\cdot : \mathcal{K} \otimes \mathcal{K} \longrightarrow \mathcal{K} \ ,$$

subject to the monoid laws.

Spelling this out, we have

Proposition 22 *A* **Gray**-*monoid consists of*

- *a 2-category $\mathcal{K}$,*
- *for each object X of $\mathcal{K}$, 2-functors $h_X : \mathcal{K} \longrightarrow \mathcal{K}$ and $k_X : \mathcal{K} \longrightarrow \mathcal{K}$, such that for every pair of objects (X, Y), $h_X Y = k_Y X$, which we denote by $X \otimes Y$,*
- *for each pair of maps $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$, a natural isomorphism between the two induced maps from $X \otimes Y$ to $X' \otimes Y'$*

such that the isomorphisms respect the 2-functorial structure of h_X and k_X, and all the data respects associativity and units for $\otimes$.

In the sequel we will write $X \otimes g$ for $h_X(g)$ and $f \otimes Y$ for $k_Y(f)$.

Definition 23 *A* symmetry *for a* **Gray**-*monoid consists of, for each X and Y, an isomorphism $c_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$, such that $c^2 = id$ and $(id_Y \otimes c_{X,Z})(c_{X,Y} \otimes id_Z) = c_{X,Y \otimes Z}$, and such that c respects all the* **Gray**-*structure. A* symmetric **Gray**-*monoid is a* **Gray**-*monoid together with a symmetry on it.*

We want to characterise a pseudo-commutativity of a strong monad T by the structure of a **Gray**-monoid on $Kl(T)$. That involves some delicate coherence. So we first recall a characterisation of a strength for a monad on a symmetric monoidal category in terms of structure on the Kleisli category [14]. A strict premonoidal category is exactly a **Gray**-monoid but with no two-dimensional data (see [14] for more details about premonoidal categories). In the following result, the notion of strict symmetric premonoidal functor is not entirely obvious, as it includes the axiom that the functor preserves central maps, as we shall define in our setting shortly.

Theorem 24 (Power-Robinson [14]) *Let $\mathcal{C}$ be a symmetric monoidal category, and let T be a monad on it. Then, to give a strength for T is to give a symmetric premonoidal structure on $Kl(T)$ such that the canonical functor $J : \mathcal{C} \longrightarrow Kl(T)$ is a strict symmetric premonoidal functor.*

We generalise the constructions of this result.

Definition 25 *A 1-cell $f : X \longrightarrow X'$ in a* **Gray***-monoid is* central *if for every 1-cell $g : Y \longrightarrow Y'$, the isomorphic 2-cell between $(X' \otimes g)(f \otimes Y)$ and $(f \otimes Y')(X \otimes g)$ is the identity, and the dual.*

Theorem 26 *Given a strong 2-monad T on a strict symmetric monoidal 2-category $\mathcal{K}$, to give a pseudo-commutativity for T is to give $Kl(T)$ the structure of a symmetric* **Gray***-monoid such that the canonical 2-functor $J : \mathcal{K} \longrightarrow Kl(T)$ strictly preserves the strict symmetric monoidal structure, with each 1-cell in $\mathcal{K}$ sent to a central 1-cell in $Kl(T)$.*

Proof: Given a pseudo-commutativity for T, and given $f : TX \longrightarrow TX'$ and $g : TY \longrightarrow TY'$ in $Kl(T)$, a study of the construction of the two composites from $T(X \otimes Y)$ to $T(X' \otimes Y')$ in $Kl(T)$ yields the desired isomorphism, and the coherence laws follow directly from the coherence for pseudo-commutativity.

Conversely, to obtain a pseudo-commutativity from a symmetric **Gray**-monoid structure on $Kl(T)$, consider the components $\varepsilon_X : T^2X \longrightarrow TX$ and $\varepsilon_Y : T^2Y \longrightarrow TY$ of the counit ε of the adjunction between $\mathcal{K}$ and $Kl(T)$, and use the isomorphism determined by them. The coherence follows directly.

It is routine to verify that these constructions are mutually inverse. $\square$

We have glossed over a coherence issue here by assuming that $\mathcal{K}$ is strict symmetric monoidal. If $\mathcal{K}$ is symmetric monoidal, we cannot obtain a **Gray**-monoid structure on $Kl(T)$ because the monoidal structure is not strict on the objects of $Kl(T)$. The most natural structure to consider in this regard is that of a monoidal bicategory, or a one object tricategory [5]. In *loc. cit.* there is a coherence theorem that says every monoidal bicategory is equivalent to a **Gray**-monoid. But although the structure on $Kl(T)$ is more general than that of a **Gray**-monoid, it is not as complicated as the structure of a monoidal bicategory, as the monoidal structure of $Kl(T)$ involves coherent isomorphisms rather than coherent equivalences, and several commutativities hold here but do not hold in an arbitrary monoidal bicategory. The structure does not seem important enough to merit its own name, but we can at least say what structure $Kl(T)$ has.

Theorem 27 *To give a pseudo-commutativity for a strong* 2*-monad* T *on a symmetric monoidal* 2*-category* $\mathcal{K}$ *is equivalent to the assertion that the Kleisli* 2*-category* $Kl(T)$ *has the following structure:*

- *for each object* X*,* 2*-functors* $h_X : Kl(T) \longrightarrow Kl(T)$ *and* $k_X : Kl(T) \longrightarrow Kl(T)$ *such that* $h_X Y = k_Y X$*, extending* $\otimes$ *on* $\mathcal{K}$*,*
- *for each pair of maps* $f : TX \longrightarrow TX'$ *and* $g : TY \longrightarrow TY'$ *in* $Kl(T)$*, a natural isomorphism between the two induced maps from* $T(X \otimes Y)$ *to* $T(X' \otimes Y')$*, extending the commutativity on* $\mathcal{K}$*,*

such that

- *the isomorphisms respect the* 2*-functorial structure of* h_X *and* k_X*,*
- *all* 1*-cells in* $\mathcal{K}$ *are sent to central* 1*-cells in* $Kl(T)$*,*
- *all the data respects the associativity and units for* $\otimes$*, meaning that* $h_X h_{X'}$ *is isomorphic to* $h_{X \otimes X'}$*, the dual for* k*, that* $h_X k_Y$ *is isomorphic to* $k_Y h_X$*, and that* h_I *is isomorphic to the identity and dually for* k_I*, all with the isomorphisms on objects determined by those of* $\mathcal{K}$*, and*
- *the symmetry* c *of* $\mathcal{K}$ *extends to being* 2*-natural on* $Kl(T)$*.*

6 Properties of dense KZ-monads

Now that we have the notion of pseudo-commutativity, we can investigate further properties of dense KZ-monads. For our first result, we need care with the precise statement for two reasons. In general, to give a strength to a monad on a symmetric monoidal closed $\mathcal{V}$ is equivalent to giving a $\mathcal{V}$-enrichment of the monad. So every 2-monad on **Cat** has a unique strength associated with it. If $\mathcal{V}$ is symmetric monoidal closed, complete and cocomplete, $\mathcal{V}$-**Cat** acquires a symmetric monoidal closed structure as a 2-category [9], but a 2-monad on $\mathcal{V}$-**Cat** is not a $\mathcal{V}$-**Cat**-enriched monad on $\mathcal{V}$-**Cat**, so there is no reason in general to expect it to have a strength. In particular cases, it does have a strength, for instance if $\mathcal{V}$ is the arrow category and so $\mathcal{V}$-**Cat** is **Poset**. We resolve this by restricting attention to $\mathcal{V}$-**Cat**-enriched monads on $\mathcal{V}$-**Cat**. If $\mathcal{V} =$ **Cat**, this reduces to 2-monads on **Cat**.

Example 28 *Let* F *be any class of weights such that the* F*-cocompletion of any small* $\mathcal{V}$*-category is small. Then the induced* 2*-monad acquires an enrichment in* $\mathcal{V}$*-***Cat***. This follows by close inspection of left Kan extensions in enriched category theory [9].*

Second, there is delicacy with coherence. In principle, we should like to prove that dense KZ-monads are always commutative. But we cannot prove precisely that, but can only prove the result up to canonical isomorphism. Here, we need the definition of pseudo-commutativity we developed in Section 5. By the usual constructions involving change of universe, the definitions and results extend from small to large categories. So assuming that, we have

Theorem 29 *Every $\mathcal{V}$-**Cat**-monad T on $\mathcal{V}$-**Cat** whose underlying 2-monad is dense KZ has a pseudo-commutativity.*

Proof: This is most easily proved by use of Theorem 16. The two $\mathcal{V}$-functors from $T\mathbb{C} \otimes T\mathbb{D}$ to $T(\mathbb{C} \otimes \mathbb{D})$ send (f, g) to $\mathrm{colim}(f, \mathrm{colim}(g, \mathrm{y}_{\mathbb{C}\otimes\mathbb{D}}))$ and to $\mathrm{colim}(g, \mathrm{colim}(f, \mathrm{y}_{\mathbb{C}\otimes\mathbb{D}}))$. Since colimits commute with colimits, and because comparison maps are unique, we have a pseudo-commutativity. $\square$

Corollary 30 *If T is a $\mathcal{V}$-**Cat**-monad on $\mathcal{V}$-**Cat** whose underlying 2-monad is dense KZ, then $Kl(T)$ is equivalent to a symmetric **Gray**-monoid with monoidal operation given by $T(C \otimes D)$.*

A detailed description of the structure of $Kl(T)$ rather than that of a 2-category equivalent to $Kl(T)$ is given by Theorem 27.

We now consider constructions of the form $T\mathbb{C} \times T\mathbb{D}$. Here, we can make a slightly stronger statement by an axiomatic proof than we can by using an explicit construction of an equivalent version of T. The latter would yield an equivalence of $\mathcal{V}$-categories, whereas axiomatically we can obtain an isomorphism, avoiding unpleasant coherence issues. We need to assume $\mathcal{V}$ cartesian closed here, as we use constant $\mathcal{V}$-functors.

Theorem 31 *Let $\mathcal{V}$ be cartesian closed and let T be a dense KZ-monad on $\mathcal{V}$-**Cat**. Suppose every $T\mathbb{C}$ has and every TH and $\mu_{\mathbb{C}}$ strictly preserve finite coproducts. Then $T\mathbb{C} \times T\mathbb{D}$ is isomorphic to $T(\mathbb{C}+\mathbb{D})$ 2-naturally in $\mathbb{C}$ and $\mathbb{D}$ and coherently with respect to the associative, commutative, and unit structures of binary product and coproduct.*

Proof: First observe that every $\mathcal{V}$-functor in $Kl(T)$ strictly preserves finite coproducts. Using the universal property of the Kleisli construction, and using the universal properties of products and coproducts, and in order to define a $\mathcal{V}$-functor $H : T(\mathbb{C} + \mathbb{D}) \longrightarrow T\mathbb{C} \times T\mathbb{D}$, we give $\mathcal{V}$-functors from $\mathbb{C}$ to each of $T\mathbb{C}$ and $T\mathbb{D}$ and from $\mathbb{D}$ to each of $T\mathbb{C}$ and $T\mathbb{D}$. We define them by $\eta_{\mathbb{C}}$ and the constant at the initial object of $T\mathbb{D}$, and by duality. In more detail, let

$$h_0 : \mathbb{C} + \mathbb{D} \to T\mathbb{C}$$

be defined as the copairing $[\eta_{\mathbb{C}}, 0]$, where $0 : \mathbb{D} \to T\mathbb{C}$ is the constant functor which always picks the initial object of $T\mathbb{C}$. Similarly define $h_1 = [0, \eta_{\mathbb{D}}]$. Define $H_0 = \mu_{\mathbb{C}} T h_0$ and $H_1 = \mu_{\mathbb{D}} T h_1$. Finally define $H = \langle H_0, H_1 \rangle$.

We define $K : T\mathbb{C} \times T\mathbb{D} \longrightarrow T(\mathbb{C}+\mathbb{D})$ by sending (X, Y) to $(Ti_0)X + (Ti_1)Y$, where $i_0 : \mathbb{C} \longrightarrow \mathbb{C}+\mathbb{D}$ and $i_1 : \mathbb{C} \longrightarrow \mathbb{C}+\mathbb{D}$ are the left and right coprojections respectively. So $K = +_{T(\mathbb{C}+\mathbb{D})}(Ti_0 \times Ti_1)$.

We must show that H and K are mutually inverse. They are obviously 2-natural in $\mathbb{C}$ and $\mathbb{D}$. To see that $KH = id_{T(\mathbb{C}+\mathbb{D})}$, first see that $KH\eta_{\mathbb{C}+\mathbb{D}} = \eta_{\mathbb{C}+\mathbb{D}}$, which may be checked on each component. That follows routinely since Ti_0 strictly preserves the initial object, and dually. Now, using the explicit coherence assumption of the theorem and some routine manipulation of diagrams, the commutativity extends to $T(\mathbb{C}+\mathbb{D})$.

To see that $HK = id_{T\mathbb{C}\times T\mathbb{D}}$, we must verify that $\pi_0 HK = \pi_0$ and $\pi_1 HK = \pi_1$, where π_0 and π_1 are the first and second projections from $T\mathbb{C}\times T\mathbb{D}$ respectively. By definition, $\pi_0 H$ and $\pi_1 H$ both lie in $Kl(T)$, so preserve finite coproducts strictly. Restricting our attention to π_0, the other case being dual, it suffices to show that $(\pi_0 H \times \pi_0 H)(Ti_0 \times Ti_1)$ sends (X, Y) to $(X, 0)$, where 0 is the initial object of $T\mathbb{C}$. Again, this amounts to two commutativities.

For the first, observe that $\pi_0 H = \mu_{\mathbb{C}} T(\eta_{\mathbb{C}}, 0)$, so precomposing with Ti_0 yields the identity since $(\eta_{\mathbb{C}}, 0)i_o = \eta_{\mathbb{C}}$ and by one of the monadic unit laws, giving the desired commutativity.

For the second, by a similar calculation, it suffices to show that the lifting of the constant $\mathcal{V}$-functor $0 : \mathbb{D} \longrightarrow T\mathbb{C}$ to $T\mathbb{D}$ is the constant $\mathcal{V}$-functor at the initial object 0 of $T\mathbb{C}$. But by Theorem 12, the lifting is given by the left Kan extension of $0 : \mathbb{D} \longrightarrow T\mathbb{C}$ along $\eta_{\mathbb{C}} : \mathbb{C} \longrightarrow T\mathbb{C}$; and one can check by calculation that that is necessarily the constant at 0. □

Corollary 32 *Under the hypotheses of the Theorem 31, $Kl(T)$ has finite products and finite coproducts, both given by $T(\mathbb{C}+\mathbb{D})$.*

Proof: The projection $\mathcal{V}$-functors are given by the span

$$T\mathbb{C} \xleftarrow{\quad H_0 \quad} T(\mathbb{C}+\mathbb{D}) \xrightarrow{\quad H_1 \quad} T\mathbb{D}$$

defined in the proof of Theorem 31. One can verify that for any other span $T\mathbb{C} \overset{F}{\leftarrow} T\mathbb{A} \overset{G}{\rightarrow} T\mathbb{D}$ in $Kl(T)$ the composite $K\langle F, G\rangle$ is in $Kl(T)$ too. □

Remark: It is worth noticing that in order to avoid having to do the explicit verification that the composite $K\langle F, G\rangle$ is an arrow of $Kl(T)$ one can appeal to a general bicategorical product/coproduct coincidence result [16]. In fact the projection functor $\pi_0 : T\mathbb{C}\times T\mathbb{D} \to T\mathbb{C}$ has a left adjoint given by $\langle id_{\mathbb{C}}, 0\rangle$. Since $K = +_{T(\mathbb{C}+\mathbb{D})}(T(i_0) \times T(i_1)) : T\mathbb{C} \times T\mathbb{D} \to T(\mathbb{C}+\mathbb{D})$ which was defined in the proof of Theorem 31 is an isomorphism with inverse $H = \langle H_0, H_1\rangle$, one has that $H_0 = \pi_0 H$ is right adjoint to $T(i_0) = K\langle id_{\mathbb{C}}, 0\rangle$. Similarly H_1 is proved to be right adjoint to $T(i_1)$. Since $T(\mathbb{C}) \xrightarrow{T(i_0)} T(\mathbb{C}+\mathbb{D}) \xleftarrow{T(i_1)} T(\mathbb{D})$ is a coproduct diagram, by the bicategorical product/coproduct coincidence of [16] one deduces that $T\mathbb{C} \xleftarrow{H_0} T(\mathbb{C}+\mathbb{D}) \xrightarrow{H_1} T\mathbb{D}$ is a bicategorical product. Since we have already proved that $T(\mathbb{C}+\mathbb{D})$ is isomorphic (as opposed to equivalent) to $T\mathbb{C}\times T\mathbb{D}$ and the latter is a product in $\mathcal{V}$-**Cat** in the strict categorical sense, one can conclude that $T(\mathbb{C}+\mathbb{D})$ is a product in the strict categorical sense too.

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