Elementary control structures[1]

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1 Introduction

In [HP], it was shown that the category of control structures satisfying a mild condition is equivalent to the category of what were called abstract control structures. Abstract control structures do not have an explicit set of names, and they fit more naturally into the usual use of category theory in computer science, particularly in denotational semantics. Here, we go one step further. We drop the one (mild) categorically unnatural condition on abstract control structures, and prove an equivalence with what we call elementary control structures.

Elementary control structures fit immediately into the general semantic theory of "notions of computation" of [PR], for which the simplest situation studied has a base category B with finite products and an identity on objects strict symmetric monoidal functor into a symmetric monoidal category C. That paper includes a mild generalisation of that setting, in order to incorporate the start of a study of adding state to traditional denotational semantics. So, the equivalence we prove here proves that control structures fit naturally and simply into the broader study of denotational semantics.

In Section 2, we review some basic definitions from [HP] that are needed to explain its central definition, that of abstract control structure, and the mild generalisation we consider. In Section 3, we give the mild generalisation, defining what we call fibrational control structures. In Section 4, we define elementary control structures and prove the two notions equivalent. Finally, in Section 5, we outline the way this fits into the general theory of [PR]. To get a feel for how that works, it may be useful to see how nondeterminism is modelled in Power and Robinson's setting, as detailed in [AP].

I gratefully acknowledge the influence of Philippa Gardner's work [G] on a "closed" version of Robin Milner's action calculi, which amounts to something like a type theoretic version of the following construction and proof, and inspiration by excellent questions and remarks by Michael Fourman and Robin Milner.

2 The "swap" construction

We first give an explicit description of the free category with strictly associative finite products on a set P. The idea is that P will be the set of primes used in the definition of control structure, and our main constructions are based upon the free category with strictly associative finite products built from it.

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2.1 Proposition Given a set P, the free category P with strictly associative finite products on P may be described as follows:

- an object of P is a finite sequence $(p_1, \cdots, p_m)$ of elements of P
- an arrow from $(p_1, \cdots, p_m)$ to $(q_1, \cdots, q_n)$ is a function $\phi : \{1, \cdots, n\} \longrightarrow \{1, \cdots, m\}$ such that for all i, $q_i = p_{\phi(i)}$
- composition is given by composition of functions.

Proof Routine calculation. ∎

2.2 Notation Given any category $\mathcal{M}$ with finite products, we can define another category $s(\mathcal{M})$ as follows:

- an object is a pair (k, m) of objects of $\mathcal{M}$
- an arrow from (k, m) to (k', m') is a pair (f, g) of arrows in $\mathcal{M}$ with $f : k \longrightarrow k'$ and $g : k \times m \longrightarrow m'$
- composition is evident.

2.3 Proposition The category $s(\mathcal{M})$ has finite products. Moreover, if $\mathcal{M}$ has strictly associative finite products, so does $s(\mathcal{M})$.

2.4 Proposition In $s(\mathcal{M})$, every map $(f, g) : (k, m) \longrightarrow (k', m')$ factors uniquely as a map of the form (id, h) followed by one of the form (j, π_2).

Proof For the existence, put $h = g$ and $j = f$. It is evident that these are the only possibilities, so we have unicity. ∎

3 Fibrational control structures

We assume the set P is fixed, M is the free monoid on it, and $\mathcal{M}$ is the free category with strictly associative finite products on the set P. So M is the set of objects of $\mathcal{M}$.

A *control signature* $\mathcal{K}$ is a family of tuples, each of the form $K : ((m_1, n_1), \cdots, (m_r, n_r)) \mapsto (m, n)$. The idea is that each K is to be modelled by an operator that allows the construction of an action $K(\vec{a})$ from a sequence $\vec{a}$ of actions, subject to a rule of arity having the form that given the vector $(a_i : m_i \longrightarrow n_i)$ of length r, we have $K(\vec{a}) : m \longrightarrow n$ subject to a side-condition that may constrain the values of r, m_i, n_i, m and n. We assume $\mathcal{K}$ is a fixed control signature.

3.1 Definition A *fibrational control structure* consists of

1. a functor $h : s(\mathcal{M})^{op} \times |\mathcal{M}| \longrightarrow Preord$. The functor h determines a locally preordered graph G_h with vertex set M and with edge preorder from m to n given by the disjoint union over k of $h(k, m, n)$. We denote the underlying ordinary graph by G_{h0}.

2. a strict monoidal locally preordered category structure on G_h such that

   - composition restricts to a family of preorder maps $\cdot : h(k,m,n) \times h(k',n,p) \longrightarrow h(k \times k', m, p)$ natural in the first variable with respect to $s(\mathcal{M})$, meaning that for all $(f,g) : (k,m) \longrightarrow (k'',m'')$ in $s(\mathcal{M})$, the evident diagram commutes, and natural in the second variable with respect to maps of the form $(f',\pi_2) : (k',n) \longrightarrow (k'',n)$
   - the tensor product restricts to a family of maps $\otimes : h(k,m,n) \times h(k',m',n') \longrightarrow h(k \times k', m \times m', n \times n')$, natural in $s(\mathcal{M})$ in both variables, meaning that for all $(f,g) : (k,m) \longrightarrow (k'',m'')$ in $s(\mathcal{M})$, the evident diagram commutes, and the dual
   - id_m (which necessarily lies in $h(1,m,m)$) is maximal in the preorder, as are its weakenings

3. the family $c_{m,n} \in h(1, m \times n, n \times m)$ given by $h(id,c)id_{n\times m}$ provides a symmetry for this strict monoidal structure, modulo the evident composition with $h(c_{k,k'},\pi_2)$

4. for each control K with arity information $((m_1,n_1),\cdots,(m_r,n_r)) \mapsto (m,n)$ and each $(k_1,\cdots,k_r)$, a function $G_{h0}(k_1,m_1,n_1)\times\cdots\times G_{h0}(k_r,m_r,n_r) \longrightarrow G_{h0}(k_1\cdots k_r,m,n)$, natural in each variable with respect to maps of the form $(f_i,\pi_2) : (k_i,m_i) \longrightarrow (k'_i,m_i)$

such that $h(\pi_1,\pi_{k\times m}) : h(j,k \times m,n) \longrightarrow h(j \times k,m,n)$ is invertible for all j,k,m,n. We denote the inverse by $(\ )-$. ■

A fibrational control structure that sends "weakening squares" to pullbacks was called an "abstract" control structure in [HP]. It was proved there that the category of abstract control structures is equivalent to the category of "locally finite" control structures, which amounts to a mild restriction on the category of control structures of [MMP]. The only use made of the weakening squares condition was to make the construction of a control structure from a fibrational control structure: an action with surface $\vec{x}$ has surface contained in any vector that contains $\vec{x}$ as a subsequence, so the axiom was used to identify the surface of an action from the data of a fibrational control structure.

4 Elementary control structures

In this section, we introduce the definition of an elementary control structure. Then we prove that the category of fibrational control structures, for given P and $\mathcal{K}$, is equivalent to that of elementary control structures. Let C_0 denote the underlying ordinary category of a locally preordered category C.

4.1 Definition An *elementary control structure* consists of a strict symmetric monoidal locally preordered category C and an identity on objects strict symmetric monoidal functor $J : \mathcal{M} \longrightarrow C_0$ such that each projection $\pi_2 : k \times m \longrightarrow m$ is maximal in $C(k \times m, m)$, together with, for each control K with arity information $((m_1,n_1),\cdots,(m_r,n_r)) \mapsto (m,n)$ and each $(k_1,\cdots,k_r)$, a function $C_0(k_1 \times m_1, n_1) \times \cdots \times C_0(k_r \times m_r, n_r) \longrightarrow C_0(k_1 \times \cdots \times k_r \times m, n)$, natural in each variable with respect to maps $f_i : k_i \longrightarrow k_i'$ in $\mathcal{M}$.

It is routine to verify that the naturality condition on controls may be expressed in the form that the given family of functions forms a natural transformation between two functors from an r-fold product of copies of $\mathcal{M}$ to Set.

This definition of elementary control structure places it immediately as an instance of the structures described for denotational semantics in [PR].

4.2 Theorem Given a fibrational control structure $(h, \cdot, \otimes, \mathcal{K})$,

1. the locally preordered subcategory of G_h consisting of object set M and homs given by $h(1, m, n)$ for all m and n, is a strict symmetric monoidal subcategory of G_h, which we denote C_h
2. the function sending $g : m \longrightarrow n$ to $h(id_1, g)id_n$ forms an identity on objects strict symmetric monoidal functor $J_h : \mathcal{M} \longrightarrow C_{h0}$
3. for each control K with arity information $((m_1, n_1), \cdots, (m_r, n_r)) \mapsto (m, n)$, the family of functions $C_{h0}(k_1 \times m_1, n_1) \times \cdots \times C_{h0}(k_r \times m_r, n_r) \longrightarrow C_{h0}(k_1 \times \cdots \times k_r \times m, n)$ given by applying the final condition in the definition of fibrational control structure, and using the control data supplied by K and $(k_1, \cdots, k_r)$, is natural in each variable with respect to maps $f_i : k_i \longrightarrow k_i'$ in $\mathcal{M}$.

Proof This all follows by routine checking, largely using the naturality axioms of fibrational control structures. ■

4.3 Corollary Given a fibrational control structure $(h, \cdot, \otimes, \mathcal{K})$, the functor $J_h : \mathcal{M} \longrightarrow C_h$ and the control data of Theorem 4.2 form an elementary control structure.

One can also make the construction of Corollary 4.3 directly in terms of the original definition of control structures: C_h is given by the actions with empty surface.

Moreover, it is routine to see that a fibrational control structure forms an abstract control structure if and only if, in the constructed elementary control structure, for every pair of maps $a : k_1 \times m \times k_2 \times k_3 \longrightarrow n$ and $b : k_1 \times k_2 \times n \times k_3 \longrightarrow n$ in C_h making the evident square commute, there is a unique map $c : k_1 \times k_2 \times k_3 \longrightarrow n$ making the evident two triangles commute.

Now for the inverse construction,

4.4 Theorem Given an elementary control structure $(C, \otimes, c, J : \mathcal{M} \longrightarrow C_0, \mathcal{K})$, the following data form a fibrational control structure:

1. $h_C : s(\mathcal{M})^{op} \times |\mathcal{M}| \longrightarrow Preord$, defined by $h_C(k, m, n) = C(k \times m, n)$, with $h_C(f, g)$ given by precomposition with $J(< \pi_1 \cdot f, g >) : (k, m) \longrightarrow (k', m')$

2. composition $\cdot_C : h_C(k,m,n) \times h_C(k',n,p) \longrightarrow h_C(k \times k', m, p)$, defined by applying $id_{k'} \otimes -$ to $C(k \times m, n)$, then using the composition and symmetry of C

3. tensor $\otimes_C : h_C(k,m,n) \times h_C(k',m',n') \longrightarrow h_C(k \times k', m \times m', n \times n')$, defined by the tensor and symmetry of C

4. for each control K with arity information $((m_1,n_1),\cdots,(m_r,n_r)) \mapsto (m,n)$ and each $(k_1,\cdots,k_r)$, the function $h_{C0}(k_1,m_1,n_1)\times\cdots\times h_{C0}(k_r,m_r,n_r) \longrightarrow h_{C0}(k_1\cdots k_r, m, n)$ defined by the control function $C_0(k_1 \times m_1, n_1) \times \cdots \times C_0(k_r \times m_r, n_r) \longrightarrow C_0(k_1 \times \cdots \times k_r \times m, n)$.

Proof This is routine calculation, with no surprises. ■

It is a triviality that, starting with an elementary control structure, applying the construction of Theorem 4.4, then applying the construction of Theorem 4.2, we regain an elementary control structure that is isomorphic to the original one, because $h_C(1,m,n)$ is defined to be $C(m,n)$, etcetera. It is not quite as obvious, but still follows by routine checking, that starting with a fibrational control structure and applying the two constructions also returns an isomorphic copy of the original fibrational control structure: the key point is that $h(k,m,n)$ is naturally isomorphic to $h(1,k\times m,n)$, so h is fully determined by its naturality and its behaviour on the full subcategory of $s(\mathcal{M})^{op} \times |M|$ determined by triples of the form $(1,m,n)$; similarly for the rest of the data. So we conclude

4.5 Corollary The categories of fibrational control structures and elementary control structures determined by a set P and control signature $\mathcal{K}$ are equivalent.

The equivalence of course is given by the constructions of Theorems 4.2 and 4.4.

5 Semantics

As mentioned after the definition, the notion of elementary control structures fits immediately as an instance of the structures considered by Power and Robinson in their general approach to semantics. Although not explicitly stated therein, the idea was to have a particular base category B, typically with finite products, possibly with more structure, and a "notion of computation" would amount to an identity on objects strict symmetric "premonoidal" functor that may preserve additional structure such as finite sums. One would then seek to add one notion of computation to another by extending the first functor in some uniform way along the second. In order to do that, one would typically need some of the structure (but specifically not higher order structure) to be preserved by the second of the original functors, as we have here. This is supposed to give a modular approach to computation as one would have a uniform methodology for adding a notion of computation to any that have already been incorporated into the picture.

A typical base category B would be the category of small sets Set or the category of small ω-cpo's, $\omega - Cpo$. Of course, in the example we have been studying here, we have as base category the free category with strictly associative finite products on a set. However, that still fits simply even if one insists upon having a fixed base category such as Set, as what we have here is a *theory*, in fact a mild generalisation of a *Lawvere theory*, which is a well known and widely used concept in category theory. This theory may be modelled in Set together with an extension; such an extension can be given immediately by taking any function from our set P into Set, extending it by the universal property, and taking a pushout in the category of symmetric monoidal categories and strict symmetric monoidal functors. Then one can extend the controls freely. In the leading examples, the actions are freely generated by the controls anyway, then they are factored by an appropriate equivalence such as weak bisimulation. So here, it is natural to extend the controls freely, and then the factoring amounts to the category theoretic construction of factoring out a congruence on a category (with added structure), *cf* [AP], where that is precisely how one gives an account of nondeterminism in Power and Robinson's setting.

6 Bibliography

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