Control Structures I

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Edinburgh, June 1994

D R A F T . Comments are welcome.

Abstract

Action calculi have been defined as a broad class of action structures with added structure; they are syntactic in nature. Each action calculus $\mathsf{AC}(\mathcal{K})$ is determined essentially by a set $\mathcal{K}$ of *controls*. Different concrete models of concurrent computation such as Petri nets, the typed λ-calculus and the π-calculus, are obtained by different choices of control set $\mathcal{K}$. The aim of this work is to define a class (in fact a category) of action structures with added structure which can be regarded as possible semantic interpretations of the calculus $\mathsf{AC}(\mathcal{K})$; thus $\mathsf{AC}(\mathcal{K})$ is characterized as the initial object in this category. The objects are called *control structures* over $\mathcal{K}$.

The paper contains the technical formulation of the category of control structures and their homomorphisms, and the proof that $\mathsf{AC}(\mathcal{K})$ is initial. Moreover, the category is closed under factorisation by arbitrary congruences on control structures.

The paper concludes with an outline of work in progress to identify particular control structures of interest, such as the quotient of an action calculus for the π-calculus under a bisimulation congruence.

1 Introduction

Action calculi are a syntactic class of action structures which allows a variety of computational calculi to be represented within a single framework. Two presentations of action calculi exist [4]: a direct construction of the syntactic objects called *molecular forms*, and the quotient of a term algebra whose constructors include the operations of action structures. These two presentations have been shown to be isomorphic [3].

Each action calculus $\mathsf{AC}(\mathcal{K})$ is determined essentially by a set $\mathcal{K}$ of *controls*; for example, there is an action calculus for the π-calculus whose controls are $\boldsymbol{\nu}$, **out**, **box** and **rep** (restriction, output, guarding and replication). $\mathsf{AC}(\mathcal{K})$ is also equipped with a set of reaction rules which determine its reaction relation; these rules provide the meaning of the control set $\mathcal{K}$.

As a step towards the treatment of the semantics of $\mathsf{AC}(\mathcal{K})$, for any $\mathcal{K}$, we shall characterize a certain class – in fact a category – of its models algebraically, thereby presenting each action calculus as the initial object of the appropriate category. We call these objects *control structures* over $\mathcal{K}$, and we call the category $\mathbf{CS}(\mathcal{K})$. An important property of the category $\mathbf{CS}(\mathcal{K})$ is closure under quotient by an arbitrary congruence. In particular, it will contain any model derived by factoring the action calculus $\mathsf{AC}(\mathcal{K})$ by a congruence; when the congruence has operational significance, as in the case of bisimulation congruence, this accords with the established practice of giving operational semantics to such calculi.

A significant difficulty to be overcome in defining control structures has been the treatment of names. If a is an action (i.e. an arrow) in $\mathsf{AC}(\mathcal{K})$, then it can be presented in syntactic form, as we shall see; this form may contain free occurrences of names in X, and the set of free names of a – which will be called the *surface* of a – typically represents the means by which a may interact with other actions. Since interaction is semantically significant, it is important that an action a in an arbitrary control structure which is a model of $\mathsf{AC}(\mathcal{K})$ should also possess a surface. If a is no longer a syntactic entity, then its surface cannot be *syntactically* determined; but in control structures as we define them we are able to determine surface *algebraically*, in a way which coincides with the syntactic notion in the special case of action calculi.

The paper is organised as follows. Section 2 contains a review of action structures and action calculi; most of what we need is presented here, but some familiarity with these notions will be an

[1]Robin Milner acknowledges support by a Senior Research Fellowship from the UK Engineering and Physical Sciences Research Council; John Power acknowledges support from ESPRIT Basic Research Action 6453 (Types for Proofs and Programs).

advantage [2, 4, 3]. Section 3 introduces *symmetric action structures*; these are little more than strict symmetric monoidal categories in which the symmetry also respects the additional structure. It is shown that each action calculus $\mathsf{AC}(\mathcal{K})$ is symmetric. Section 3 also introduces the algebraic characterization of surface.

Section 4 contains the main results. First, control structures are introduced as a class of symmetric action structures with additional structure, satisfying further axioms; using these axioms, several alternative characterizations of surface are derived. Next, closure of $\mathbf{CS}(\mathcal{K})$ under congruences is demonstrated. Finally, the initiality of $\mathsf{AC}(\mathcal{K})$ in $\mathbf{CS}(\mathcal{K})$ is proven.

2 Action structures, molecular forms and action calculi

An action structure is a strict monoidal category with additional structure. Its objects form a monoid $(M, \otimes, \epsilon)$, and we call them *arities*; we let k, ℓ, m, n range over arities. The actions of an action structure A are its arrows; $A(m, n)$ is the set of all actions $a : m \to n$ in A. We let $a, b, \ldots$ range over actions, and we write $a \cdot b : k \to m$ (rather than $b \circ a$) for the composite of $a : k \to \ell$ and $b : \ell \to m$. There are also identities $\mathbf{id}_m : m \to m$ and tensor product $a \otimes b : k \otimes m \to \ell \otimes n$, for $a : k \to \ell$ and $b : m \to n$. Each action structure A presupposes a denumerable set X of names, to each of which is assigned an arity. We let u, v, w, x, y, z range over names, and write $x : k$ to mean that x has arity k.

The first element of additional structure in A is the abstraction operation $\mathbf{ab}_x$ for each name x; if $x : k$ and $a : m \to n$ then $\mathbf{ab}_x a : k \otimes m \to k \otimes n$. The abstractors are endofunctors. Thus action structures obey the following eight axioms, in which arities may be assigned in any way which makes sense:

$$
\begin{array}{ll}
a \cdot \mathbf{id} = a = \mathbf{id} \cdot a & a \cdot (b \cdot c) = (a \cdot b) \cdot c \\
a \otimes \mathbf{id}_\epsilon = a = \mathbf{id}_\epsilon \otimes a & a \otimes (b \otimes c) = (a \otimes b) \otimes c \\
\mathbf{id} \otimes \mathbf{id} = \mathbf{id} & (a \cdot b) \otimes (c \cdot d) = (a \otimes c) \cdot (b \otimes d) \\
\mathbf{ab}_x \mathbf{id} = \mathbf{id} & \mathbf{ab}_x (a \cdot b) = (\mathbf{ab}_x a) \cdot (\mathbf{ab}_x b)\ .
\end{array}
$$

The second element of extra structure in A is a preorder $\searrow$ on each $A(m, n)$, called *reaction*; this relation is preserved by composition, product and abstraction. Also each $\mathbf{id}$ is minimal for $\searrow$, i.e. if $\mathbf{id} \searrow a$ then $\mathbf{id} = a$. This preorder represents the dynamics or performance of actions.

We recall that a *homomorphism* $\Phi : A \to B$ between action structures A and B is a map which respects the structure; in particular, it includes a monoid homomorphism from the arities of A to those of B, and a map from the name-set of A to the name-set of B which respects the arity assignment.

In this paper, we restrict consideration to action structures over a fixed name-set X. We shall also assume that the monoid of arities $(M, \otimes, \epsilon)$ is fixed, and we constrain the homomorphisms Φ to act as the identity both upon the names X and upon the arity monoid M. Further we assume that M is freely generated by a set of *prime* arities, that the arity of every name is prime, and that infinitely many names are associated with every prime arity.We shall allow p, q to range over primes. For the present paper, we work entirely in the subcategory of action structures in which these assumptions are obeyed. The assumptions will be relaxed in future work.

An action calculus is determined by a set $\mathcal{K}$ of operators, called *controls*. We let K range over controls.

Definition 2.1 (Controls: statics) *A* control K *is an operator which allows the construction of an action* $K(\vec{a})$ *from a sequence* $\vec{a}$ *of actions, subject to a* rule of arity *having the following form:*

$$
\frac{a_1 : m_1 \to n_1 \quad \cdots \quad a_r : m_r \to n_r}{K(a_1, \ldots, a_r) : m \to n} \quad (\chi)
$$

where the side-condition χ *may constrain the value of the integer* r *and the arities* m_i, n_i, m, n.

Notation Throughout this paper we shall adopt the convention that all names appearing in a vector $\vec{x}$ within round brackets are distinct. We also assume that all terms used are well formed, and that in any equation (including a defining equation) those occurring on each side have the same arity. With each vector of names $\vec{x}$ we associate an arity; if $\vec{x} = x_1 \cdots x_r$ and $x_i : p_i$ $(1 \le i \le r)$ then $\vec{x} : p_1 \otimes \cdots \otimes p_r$.

Definition 2.2 (Molecular Forms) *Let $\mathcal{K}$ be a set of controls. The* molecular forms over $\mathcal{K}$, *denoted* $\mathcal{M}(\mathcal{K})$, *are syntactic objects; they consist of the actions a defined as follows, together with the* molecules μ:

$$a \quad ::= \quad (\vec{x})\ \mu_1 \cdots \mu_r\ \langle \vec{u} \rangle \qquad (\vec{x} : m,\ \vec{u} : n,\ a : m \to n)$$

$$\mu \quad ::= \quad \langle \vec{v} \rangle\, K\vec{b}\,(\vec{y}) \qquad (\vec{v} : k,\ \vec{y} : \ell,\ K\vec{b} : k \to \ell)\ .$$

We let λ, μ range over molecules. In both actions and molecules, whenever a vector $\vec{x}$ occurs in round brackets, its names (which by our convention must be distinct) are binding occurrences with scope extending to the right to the end of the smallest enclosing action, capturing occurrences of the names $\vec{x}$ even within molecule constructions. Names which are not thus bound are free and alpha conversion of bound names is allowed. We assume that no name has more than one binding occurrence in any molecule or action.

In the action a of the above definition, $\vec{x}$ are called the *imported names* and $\vec{u}$ the *exported* ones. The construct $\mu_1 \cdots \mu_r$, called the *body* of a, is a possibly empty *partial* sequence of molecules, where the commutation of any two molecules is allowed provided neither binds a name occurring free in the other.

We shall now define over molecular forms the action structure operations, the control operations and two additional operations: the *datum* $\langle x \rangle$ (for each $x \in X$) and the *discard* $\boldsymbol{\omega}$, which represent provision of (exported) and discarding of (imported) names respectively.

Definition 2.3 *Assume $a = (\vec{u})\, \vec{\lambda}\, \langle \vec{v} \rangle$ and $b = (\vec{x})\, \vec{\mu}\, \langle \vec{y} \rangle$, where no name which is bound in one occurs in the other.*

$$\begin{array}{rcll}
\mathbf{id}_m & \stackrel{\text{def}}{=} & (\vec{x})\langle \vec{x} \rangle & (\vec{x} : m) \\
a \cdot b & \stackrel{\text{def}}{=} & (\vec{u})\, \vec{\lambda}\, \sigma\vec{\mu}\, \langle \sigma\vec{y} \rangle & (\sigma = \{\vec{v}/\vec{x}\}) \\
a \otimes b & \stackrel{\text{def}}{=} & (\vec{u}\vec{x})\vec{\lambda}\, \vec{\mu}\, \langle \vec{v}\vec{y} \rangle & \\
\mathbf{ab}_x a & \stackrel{\text{def}}{=} & (x\vec{u})\, \vec{\lambda}\, \langle x\vec{v} \rangle & \\
\langle x \rangle & \stackrel{\text{def}}{=} & (\ )\langle x \rangle & \\
\boldsymbol{\omega} & \stackrel{\text{def}}{=} & (x)\langle\ \rangle & \\
K(\vec{a}) & \stackrel{\text{def}}{=} & (x)\, \langle \vec{x} \rangle K\vec{a}(\vec{y})\, \langle \vec{y} \rangle & (\vec{x}, \vec{y} \text{ not free in } \vec{a})
\end{array}$$

where $\{\vec{v}/\vec{x}\}$ is simultaneous syntactic substitution for the free occurrences of names in $\vec{x}$ by the corresponding ones in $\vec{v}$.

The following fact will be important later:

Fact 2.4 *The name x is free in a if and only if* $\mathbf{ab}_x a \neq \mathbf{id} \otimes a$.

Proof Let $a = (\vec{u})\vec{\lambda}\langle \vec{v} \rangle$, with $x \notin \vec{u}$. Then $\mathbf{id} \otimes a = (w\vec{u})\vec{\lambda}\langle w\vec{v} \rangle$, where $w \notin \vec{u}$ and w does not occur in a, while $\mathbf{ab}_x a = (x\vec{u})\vec{\lambda}\langle x\vec{v} \rangle$. These two molecular forms are equal (by α-conversion) if and only if x is not free in a. ∎

Definition 2.5 (Controls: dynamics) *Associated with each set of controls $\mathcal{K}$ is a set of* reaction rules, *each of the form*[2]

$$t[\vec{\alpha}] \searrow^1 t'[\vec{\alpha}]$$

where $\vec{\alpha}$ are metavariables for actions and t, t' are terms formed from these metavariables and the operations defined above. An instance *of the rule is gained by replacing the metavariables $\vec{\alpha}$ by actions $\vec{a}$.*

[2] Certain constraints may be placed upon these reaction rules, but they are unimportant for this paper. For example we may require that each name x occurring free in t' also occurs free in t; that each metavariable α in t' also occurs in t; that t contains at least one control (to avoid generating reactions like $\langle xy \rangle \searrow^1 \langle xx \rangle$). In general, we expect a useful classification of action calculi in terms of their reaction rules.

The "single-step" relation $\searrow^1$ over molecular forms is the smallest relation containing every instance of the control rules and closed under product, composition and abstraction. The reaction relation $\searrow$ is the transitive reflexive closure of $\searrow^1$.

An example of reaction rules is provided by the set of controls $\mathcal{K} = \{\boldsymbol{\nu}, \mathbf{out}, \mathbf{box}, \mathbf{rep}\}$ for the π-calculus, which was mentioned in the introduction. Assuming that the arity monoid is the natural numbers under addition, their arity rules are

$$\boldsymbol{\nu} : 0 \to 1 \qquad \mathbf{out} : 1 \otimes m \to 0$$
$$\frac{a : m \to n}{\mathbf{box}\, a : 1 \to n} \qquad \frac{a : m \to 0}{\mathbf{rep}\, a : 1 \to 0}$$

and there are just two reaction rules, as follows:

$$\begin{array}{lcl} (\langle x\rangle \otimes \mathbf{id}_m)\cdot\mathbf{out} \otimes \langle x\rangle\cdot\mathbf{box}\, a & \searrow^1 & a \\ (\langle x\rangle \otimes \mathbf{id}_m)\cdot\mathbf{out} \otimes \langle x\rangle\cdot\mathbf{rep}\, a & \searrow^1 & a \otimes \langle x\rangle\cdot\mathbf{rep}\, a\ . \end{array}$$

It is important to note that the reaction relation need not be preserved by controls; thus from $a \searrow a'$ it does not follow that $\mathbf{box}\, a \searrow \mathbf{box}\, a'$. In fact, the very purpose of $\mathbf{box}$ in the π-calculus is to achieve some sequential control over reactions.

The following is easy to establish [3]:

Proposition 2.6 $(\mathcal{M}(\mathcal{K}), \mathbf{id}, \cdot, \otimes, \mathbf{ab}, \searrow)$ *is an action structure.*

We shall now prepare for Definition 2.14 of the action calculus $\mathsf{AC}(\mathcal{K})$ as a quotient of a term algebra. We assume given a set $\mathcal{K}$ of controls, equipped with arity rules and reaction rules as previously described.

Definition 2.7 (Terms) *The* terms *over $\mathcal{K}$, denoted by $\mathcal{T}(\mathcal{K})$, are generated as follows (we let s, t range over terms):*

$$t ::= \mathbf{id} \mid \langle x\rangle \mid \boldsymbol{\omega} \mid K\vec{t} \mid s\cdot t \mid s \otimes t \mid \mathbf{ab}_x t$$

where $\langle x\rangle : \epsilon \to p$ $(x : p)$ and $\boldsymbol{\omega} : p \to \epsilon$ (for each p), and the other constructions have arities dictated by the arity rules of the constructors. The notions of free name *and* bound name *are standard; $\mathbf{ab}_x$ binds x and $\langle x\rangle$ represents a free occurrence of x. The set of names free in t is denoted by $\mathbf{fn}(t)$.*

Definition 2.8 (Derived operations) *We define an alternative form $(x)t$ of* abstraction, *and the* permutations $\mathbf{p}_{m,n}$, *as follows (together with some abbreviations):*

$$\begin{array}{rcll} (x)t & \stackrel{\text{def}}{=} & \mathbf{ab}_x t \cdot (\boldsymbol{\omega} \otimes \mathbf{id}) & \\ (\vec{x})t & \stackrel{\text{def}}{=} & (x_1)\cdots(x_r)t & (\vec{x} = x_1\cdots x_r,\ \textit{all distinct},\ r \geq 0) \\ \langle\vec{x}\rangle & \stackrel{\text{def}}{=} & \langle x_1\rangle \otimes \cdots \otimes \langle x_r\rangle & (\vec{x} = x_1\cdots x_r,\ r \geq 0) \\ \mathbf{p}_{m,n} & \stackrel{\text{def}}{=} & (\vec{x}\vec{y})\langle\vec{y}\vec{x}\rangle & (\vec{x} : m,\ \vec{y} : n)\ . \end{array}$$

Note that $\mathbf{p}_{m,n}$ is defined using a *particular* vector $\vec{x}\vec{y}$ of distinct variables; when α-conversion is proven, we shall be justified in choosing these variables at will.

We now define substitution upon terms in detail, since we shall need a careful analysis of it later:

Definition 2.9 (Substitution) Substitution $\{y/x\}$ *upon terms is defined as follows:*

$$\begin{array}{ll} \{y/x\}\mathbf{id} \stackrel{\text{def}}{=} \mathbf{id} & \{y/x\}(s \otimes t) \stackrel{\text{def}}{=} \{y/x\}s \otimes \{y/x\}t \\ \{y/x\}\boldsymbol{\omega} \stackrel{\text{def}}{=} \boldsymbol{\omega} & \{y/x\}(s \cdot t) \stackrel{\text{def}}{=} \{y/x\}s \cdot \{y/x\}t \\ \{y/x\}\langle z\rangle \stackrel{\text{def}}{=} \langle z\rangle \quad (z \neq x) & \{y/x\}K(t, \ldots) \stackrel{\text{def}}{=} K(\{y/x\}t, \ldots) \\ \{y/x\}\langle x\rangle \stackrel{\text{def}}{=} \langle y\rangle & \end{array}$$

$$\begin{array}{ll} \{y/x\}\mathbf{ab}_z t \stackrel{\text{def}}{=} \mathbf{ab}_z \{y/x\}t & (z \notin \{x, y\}) \\ \{z/x\}\mathbf{ab}_z t \stackrel{\text{def}}{=} \mathbf{ab}_w \{z/x\}\{w/z\}t & (z \neq x,\ w \notin \mathbf{fn}(t) \cup \{x, z\}) \\ \{y/z\}\mathbf{ab}_z t \stackrel{\text{def}}{=} \mathbf{ab}_z t\ . & \end{array}$$

Note that, in the penultimate equation, some particular w is chosen. We are not assuming α-convertibility at this stage, but it is a consequence of the axioms of action calculi given below.

Lemma 2.10 $\{x/x\}t = t$.

Proof Induction on the structure of terms. ∎

Definition 2.11 (Theory AC) *The* equational theory AC *is the set of equations upon terms generated by the action structure axioms together with the following:*

$$
\begin{array}{lll}
\boldsymbol{\gamma}: & (x)t = \boldsymbol{\omega} \otimes t & (x \notin \mathbf{fn}(t)) \\
\boldsymbol{\delta}: & (x)(\langle x \rangle \otimes \mathbf{id}_m) = \mathbf{id}_{p \otimes m} & (x : p) \\
\boldsymbol{\zeta}: & \mathbf{p}_{k,m} \cdot (t \otimes s) = (s \otimes t) \cdot \mathbf{p}_{\ell,n} & (s : k \to \ell,\ t : m \to n) \\
\boldsymbol{\sigma}: & (\langle y \rangle \otimes \mathbf{id}_m) \cdot (x)t = \{y/x\}t & (t : m \to n)\ .
\end{array}
$$

With some abuse of terminology we shall consider AC to be either the above set of four axioms, or the set of equations inferred from them (a congruence relation). It will be clear from the context which we mean.

Lemma 2.12 *The following are provable in* AC *whenever* $x \notin \mathbf{fn}(t)$*:*

(1) $(x)(s \cdot t) = (x)s \cdot t$

(2) $(x)(s \otimes t) = (x)s \otimes t$

(3) $(x)(t \otimes s) = t \otimes (x)s$, *if* $t : \epsilon \to \ell$

$\boldsymbol{\alpha}$: $(y)t = (x)\{x/y\}t$

(4) $\mathbf{ab}_x s = (x)(\langle x \rangle \otimes s)$

(5) $(\vec{x})(\langle \vec{x} \rangle \otimes \mathbf{id}) = \mathbf{id}$

(6) $\mathbf{ab}_y t = \mathbf{ab}_x \{x/y\}t$.

Proof Reasons for each step appear in the right-hand column; an asterisk indicates repeated use of an equation.

$$
\begin{array}{llcll}
(1) & (x)(s \cdot t) & = & \mathbf{ab}_x s \cdot (x)t & \\
 & & = & \mathbf{ab}_x s \cdot (\boldsymbol{\omega} \otimes t) & \boldsymbol{\gamma} \\
 & & = & \mathbf{ab}_x s \cdot (\boldsymbol{\omega} \otimes \mathbf{id}) \cdot t & \\
 & & = & (x)s \cdot t\ . & \\
 & & & & \\
\dagger & (x)(s \otimes \mathbf{id}) & = & (x)((\langle x \rangle \otimes \mathbf{id}) \cdot (x)s \otimes \mathbf{id}) & 2.10,\ \boldsymbol{\sigma} \\
 & & = & (x)(\langle x \rangle \otimes \mathbf{id} \otimes \mathbf{id}) \cdot ((x)s \otimes \mathbf{id}) & (1) \\
 & & = & (x)s \otimes \mathbf{id}\ . & \boldsymbol{\delta} \\
 & & & & \\
(2) & (x)(s \otimes t) & = & (x)((s \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes t)) & \\
 & & = & (x)(s \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes t) & (1) \\
 & & = & ((x)s \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes t) & \dagger \\
 & & = & (x)s \otimes t\ . & \\
 & & & & \\
(3) & (x)(t \otimes s) & = & (x)(s \cdot (t \otimes \mathbf{id})) & \\
 & & = & (x)s \cdot (t \otimes \mathbf{id}) & (1) \\
 & & = & t \otimes (x)s\ . &
\end{array}
$$

$$\begin{array}{llll}
\boldsymbol{\alpha}: & (x)\{^x\!/\!_y\}t & = (x)(((\langle x\rangle \otimes \mathbf{id}) \cdot (y)t) & \boldsymbol{\sigma} \\
 & & = (x)(\langle x\rangle \otimes \mathbf{id}) \cdot (y)t & (1) \\
 & & = (y)t\ . & \boldsymbol{\delta}
\end{array}$$

$$\begin{array}{llll}
(4) & \mathbf{ab}_x s & = \mathbf{ab}_x s \cdot (x)(\langle x\rangle \otimes \mathbf{id}) & \boldsymbol{\delta} \\
 & & = (x)(s \cdot (\langle x\rangle \otimes \mathbf{id})) & \\
 & & = (x)(\langle x\rangle \otimes s)\ . &
\end{array}$$

(5) Induction on length of $\vec{x}$. Basis true by definition. For the inductive step:

$$\begin{array}{lll}
(x\vec{y})(\langle x\vec{y}\rangle \otimes \mathbf{id}) & = (x)(\vec{y})(((\langle\vec{y}\rangle \otimes \mathbf{id}) \cdot (\langle x\rangle \otimes \mathbf{id})) & \\
 & = (x)((\vec{y})(\langle\vec{y}\rangle \otimes \mathbf{id}) \cdot (\langle x\rangle \otimes \mathbf{id})) & (1)^* \\
 & = (x)(\langle x\rangle \otimes \mathbf{id}) & \text{induction} \\
 & = \mathbf{id}\ . & \boldsymbol{\delta}
\end{array}$$

(6) If $x = y$ then result follows by Lemma 2.10. Assume $x \neq y$.

$$\begin{array}{lll}
\mathbf{ab}_x\{^x\!/\!_y\}t & = (x)(\langle x\rangle \otimes \{^x\!/\!_y\}t) & (4) \\
 & = (x)\{^x\!/\!_y\}(\langle y\rangle \otimes t) & \\
 & = (y)(\langle y\rangle \otimes t) & \boldsymbol{\alpha} \\
 & = \mathbf{ab}_y t\ . & (4)
\end{array}$$

■

It is natural to ask why the axioms AC have been expressed in terms of the derived form $(x)a$ of abstraction, rather than directly in terms of $\mathbf{ab}_x a$. This is mostly for convenience; note especially that the permutations are more directly definable using the derived form. However, there is an equivalent formulation of $\boldsymbol{\gamma}$ using $\mathbf{ab}_x$:

Proposition 2.13 *The theory* AC *is unchanged when the axiom $\boldsymbol{\gamma}$ is replaced by the following axiom:*

$$\boldsymbol{\gamma}': \quad \mathbf{ab}_x t = \mathbf{id} \otimes t \quad (x \notin \mathbf{fn}(t))\ .$$

Proof Assuming $\boldsymbol{\gamma}'$ and $x \notin \mathbf{fn}(t)$, we prove $\boldsymbol{\gamma}$ as follows:

$$(x)t \stackrel{\text{def}}{=} (\mathbf{ab}_x t) \cdot (\boldsymbol{\omega} \otimes \mathbf{id}) = (\mathbf{id} \otimes t) \cdot (\boldsymbol{\omega} \otimes \mathbf{id}) = \boldsymbol{\omega} \otimes t\ .$$

In the other direction, assume $\boldsymbol{\gamma}$ and $x \notin \mathbf{fn}(t)$. Then

$$\begin{array}{lll}
\mathbf{ab}_x t & = (x)(\langle x\rangle \otimes t) & 2.12(4) \\
 & (x)(\langle x\rangle \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes t) & 2.12(1) \\
 & \mathbf{id} \otimes t & \boldsymbol{\delta}\ .
\end{array}$$

■

We are now ready to define action calculus.

Definition 2.14 (Action calculus) *The* action calculus $\mathsf{AC}(\mathcal{K})$ *is defined to be the quotient* $\mathcal{T}(\mathcal{K})/\mathsf{AC}$, *together with a reaction relation which is the smallest generated by the reaction rules* $t[\vec{\alpha}] \searrow^1 t'[\vec{\alpha}]$ *associated with* $\mathcal{K}$ *(with terms* $\vec{s}$ *replacing the metavariables* $\vec{\alpha}$*) and closed under product, composition, abstraction and the equations* AC.

Fact 2.15 $\mathsf{AC}(\mathcal{K})$ *is an action structure.*

The following theorem [3] shows that the molecular forms $\mathcal{M}(\mathcal{K})$ provide an explicit representation of $\mathsf{AC}(\mathcal{K})$.

Theorem 2.16 (Molecular normal form) *For any set of controls* $\mathcal{K}$, *the action structure* $\mathcal{M}(\mathcal{K})$ *of molecular forms is isomorphic to the action calculus* $\mathsf{AC}(\mathcal{K})$.

Discussion The axiomatization of AC, though succinct, is impure in two ways; the axiom $\boldsymbol{\gamma}$ has a syntactic condition upon terms, and the axiom $\boldsymbol{\sigma}$ is expressed using substitution of names into terms. Thus each of these axioms is more exactly an axiom schema – a finite presentation of an infinite set of axioms. We could define a control structure to be an enriched action structure which satisfies this infinite set of axioms, and then by an entirely standard argument we would find $\mathsf{AC}(\mathcal{K})$ to be initial in the category of control structures.

But there are weaknesses in this approach. First, we wish to have a semantic understanding of what it could mean for a name to belong to the "surface" of an action, corresponding to the syntactic notion of free occurrence of a name in a term. This matter was discussed in the introduction. Second, the instances of $\boldsymbol{\gamma}$ (or of $\boldsymbol{\sigma}$), when interpreted in an arbitrary action structure A, constrain only those actions which lie in the image of $\mathsf{AC}(\mathcal{K})$ under any homomorphism; they impose no constraint upon actions of A in general, and therefore they contribute no understanding of A as an algebraic structure. Third, and most simply, a finite set of axioms is more satisfactory than an infinite set.

Bearing these arguments in mind, in the spirit of universal algebra we seek to characterize control structures by a finite set of (pure) axioms, such that $\mathsf{AC}(\mathcal{K})$ is the initial control structure over $\mathcal{K}$. Apart from the elegance of this approach and the greater mathematical insight it provides, it has the advantage that properties such as initiality then follow by standard arguments.

Initiality will be ensured if the new axioms generate exactly the theory AC, i.e. they are equipotent with $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, $\boldsymbol{\zeta}$ and $\boldsymbol{\sigma}$ over the term algebra. This condition does not fully determine the notion of control structure; therefore we must further justify our choice. Our axiomatization has other qualities; it is simple, it is a natural extension of a known categorical structure (symmetric monoidal categories) and it gives a convincing account of the notion of surface.

Before presenting the axioms, let us further analyse the central problem. The greatest difficulty is to replace the axiom schema

$$\boldsymbol{\gamma}' : \mathbf{ab}_x t = \mathbf{id} \otimes t \quad (x \notin \mathbf{fn}(t))$$

(which by Proposition 2.13 is equipotent with $\boldsymbol{\gamma}$) by a finite set of purely algebraic axioms. A less satisfactory solution is to give up the purely algebraic approach, and to postulate that every control structure is equipped with a map **surf** which assigns to each action a a set $\mathbf{surf}(a) \subseteq X$; then one adopts the *single axiom*

$$\boldsymbol{\gamma}'' : \mathbf{ab}_x a = \mathbf{id} \otimes a \quad (x \notin \mathbf{surf}(a)) \ .$$

One also imposes upon **surf** the reasonable condition that, roughly, the surface of each algebraic construction is no greater than the union of the surfaces of its arguments. More precisely, one imposes the following *surface* axioms:

$$\begin{array}{rcl@{\qquad}rcl}
\mathbf{surf}(\mathbf{id}) & = & \emptyset & \mathbf{surf}(\langle x\rangle) & \subseteq & \{x\} \\
\mathbf{surf}(a \otimes b) & \subseteq & \mathbf{surf}(a) \cup \mathbf{surf}(b) & \mathbf{surf}(\boldsymbol{\omega}) & = & \emptyset \\
\mathbf{surf}(a \cdot b) & \subseteq & \mathbf{surf}(a) \cup \mathbf{surf}(b) & \mathbf{surf}(K\vec{a}) & \subseteq & \bigcup_i \mathbf{surf}(a_i) \;\; (a = a_1 \cdots a_r) \\
\mathbf{surf}(\mathbf{ab}_x a) & \subseteq & \mathbf{surf}(a) - \{x\} \ . & & &
\end{array}$$

One then obtains a finite (but not purely algebraic) set of axioms for control structures which ensures that $\mathsf{AC}(\mathcal{K})$ is initial.

This was indeed our first approach. We were then surprised to find that from these axioms one can derive the double implication

$$\mathbf{ab}_x a = \mathbf{id} \otimes a \;\Leftrightarrow\; x \notin \mathbf{surf}(a) \ .$$

To see this, note that one direction ($\Leftarrow$) is already given by $\boldsymbol{\gamma}''$. For the other ($\Rightarrow$), suppose that $\mathbf{ab}_x a = \mathbf{id} \otimes a$. For any y we have

$$a = (\langle y\rangle \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes a) \cdot (\boldsymbol{\omega} \otimes \mathbf{id}) \ ,$$

using $\langle y\rangle \cdot \boldsymbol{\omega} = \mathbf{id}_\epsilon$, which will be ensured by other control structure axioms. It follows that

$$\begin{array}{lcll} \mathbf{surf}(a) & \subseteq & \{y\} \cup \mathbf{surf}(\mathbf{id} \otimes a) & \text{by the surface axioms} \\ & = & \{y\} \cup \mathbf{surf}(\mathbf{ab}_x a) & \text{by assumption} \\ & \subseteq & \{y\} \cup (\mathbf{surf}(a) - \{x\}) & \text{by a surface axiom ,} \end{array}$$

and by choosing $y \neq x$ we deduce $x \notin \mathbf{surf}(a)$. In other words, the surface axioms have constrained $\mathbf{surf}(a)$ to be exactly the set $\{x \mid \mathbf{ab}_x a \neq \mathbf{id} \otimes a\}$. We therefore achieve the same effect if we remove $\boldsymbol{\gamma}''$ and *define* surface by

$$\mathbf{surf}(a) \stackrel{\text{def}}{=} \{x \mid \mathbf{ab}_x a \neq \mathbf{id} \otimes a\} \ .$$

The axioms are still not purely algebraic, since the surface axioms remain; each of these has now become an implication between equations. Our second discovery was that these implications can be replaced (with equivalent power) by a small number of purely equational axioms.

It is convenient to introduce the axioms in two steps. The first step is to define symmetric action structures, an enrichment of symmetric monoidal categories.

3 Symmetric action structures

We begin by recalling the standard notion of symmetric monoidal category [1].

Definition 3.1 (Symmetry) *A* symmetry *on a strict monoidal category is a family of arrows* $\mathbf{c}$ *with components* $\mathbf{c}_{m,n} : m \otimes n \to n \otimes m$ *such that*

S1. $\mathbf{c}$ *is a natural transformation, i.e. the following diagram commutes:*

$$\begin{tikzcd} m \otimes n \arrow[r, "a \otimes b"] \arrow[d, "\mathbf{c}_{m,n}"'] & m' \otimes n' \arrow[d, "\mathbf{c}_{m',n'}"] \\ n \otimes m \arrow[r, "b \otimes a"'] & n' \otimes m' \end{tikzcd}$$

S2. $\mathbf{c}_{m,n} \cdot \mathbf{c}_{n,m} = \mathbf{id}$

S3. $(\mathbf{c}_{m,n} \otimes \mathbf{id}_k) \cdot (\mathbf{id}_n \otimes \mathbf{c}_{m,k}) = \mathbf{c}_{m,n \otimes k}$.

Definition 3.2 (Symmetric action structure) *A* symmetric action structure *is an action structure with a symmetry* $\mathbf{c}$ *on it for which*

S4. $\mathbf{ab}_x \mathbf{c} = \mathbf{id} \otimes \mathbf{c}$

S5. $\mathbf{ab}_x(\mathbf{ab}_x a) = \mathbf{id} \otimes \mathbf{ab}_x a$

S6. $\mathbf{ab}_x(a \otimes \mathbf{id}) = \mathbf{ab}_x a \otimes \mathbf{id}$

S7. *Whenever* $x : k$ *and* $y : \ell$ *are distinct names, the following diagram commutes:*

$$\begin{tikzcd} k \otimes \ell \otimes m \arrow[r, "\mathbf{ab}_x \mathbf{ab}_y a"] \arrow[d, "\mathbf{c}_{k,\ell} \otimes \mathbf{id}_m"'] & k \otimes \ell \otimes m' \arrow[d, "\mathbf{c}_{k,\ell} \otimes \mathbf{id}_{m'}"] \\ \ell \otimes k \otimes m \arrow[r, "\mathbf{ab}_y \mathbf{ab}_x a"'] & \ell \otimes k \otimes m' \end{tikzcd}$$

Lemma 3.3 *Let $a : m \to n$, $b : k \to \ell$, $x : k$ and $y : l$, where x, y are distinct names. The following equations are valid in symmetric action structures:*

(1) $a \otimes b = \mathbf{c}_{m,k} \cdot (b \otimes a) \cdot \mathbf{c}_{\ell,n}$

(2) $\mathbf{ab}_x \mathbf{ab}_y a = (\mathbf{c}_{k,l} \otimes \mathbf{id}) \cdot \mathbf{ab}_y \mathbf{ab}_x a \cdot (\mathbf{c}_{l,k} \otimes \mathbf{id})$

(3) $(\mathbf{id}_k \otimes \mathbf{c}_{m,n}) \cdot (\mathbf{c}_{k,n} \otimes \mathbf{id}_m) = \mathbf{c}_{k \otimes m,n}$

(4) $\mathbf{c}_{m,\epsilon} = \mathbf{id}_m = \mathbf{c}_{\epsilon,m}$.

Proof

(1)
$$\begin{array}{rcll} a \otimes b & = & (a \otimes b) \cdot \mathbf{c}_{n,\ell} \cdot \mathbf{c}_{\ell,n} & \text{S2} \\ & = & \mathbf{c}_{m,k} \cdot (b \otimes a) \cdot \mathbf{c}_{\ell,n} \ . & \text{S1} \end{array}$$

(2)
$$\begin{array}{rcll} \mathbf{ab}_x \mathbf{ab}_y a & = & \mathbf{ab}_x \mathbf{ab}_y a \cdot (\mathbf{c}_{k,l} \otimes \mathbf{id}) \cdot (\mathbf{c}_{l,k} \otimes \mathbf{id}) & \text{S2} \\ & = & (\mathbf{c}_{k,l} \otimes \mathbf{id}) \cdot \mathbf{ab}_y \mathbf{ab}_x a \cdot (\mathbf{c}_{l,k} \otimes \mathbf{id}) \ . & \text{S7} \end{array}$$

(3)
$$\begin{array}{rcll} \mathbf{c}_{k \otimes m,n} & = & \mathbf{c}_{k \otimes m,n} \cdot (\mathbf{c}_{n,k} \otimes \mathbf{id}_m) \cdot (\mathbf{c}_{k,n} \otimes \mathbf{id}_m) & \text{S2} \\ & = & \mathbf{c}_{k \otimes m,n} \cdot (\mathbf{c}_{n,k} \otimes \mathbf{id}_m) \cdot (\mathbf{id}_k \otimes \mathbf{c}_{n,m}) & \\ & & \quad \cdot (\mathbf{id}_k \otimes \mathbf{c}_{m,n}) \cdot (\mathbf{c}_{k,n} \otimes \mathbf{id}_m) & \text{S2} \\ & = & \mathbf{c}_{k \otimes m,n} \cdot \mathbf{c}_{n,k \otimes m} \cdot (\mathbf{id}_k \otimes \mathbf{c}_{m,n}) \cdot (\mathbf{c}_{k,n} \otimes \mathbf{id}_m) & \text{S3} \\ & = & (\mathbf{id}_k \otimes \mathbf{c}_{m,n}) \cdot (\mathbf{c}_{k,n} \otimes \mathbf{id}_m) \ . & \text{S2} \end{array}$$

(4) We show that for every n, $\mathbf{id}_n \otimes \mathbf{c}_{m,\epsilon} = \mathbf{id}_{n \otimes m}$: taking $n = \epsilon$ gives the result.
$$\begin{array}{rcll} \mathbf{id}_{n \otimes m} & = & \mathbf{c}_{n,m} \cdot \mathbf{c}_{m,n} & \text{S2} \\ & = & \mathbf{c}_{n,m} \cdot (\mathbf{c}_{m,n} \otimes \mathbf{id}_\epsilon) \cdot (\mathbf{id}_n \otimes \mathbf{c}_{m,\epsilon}) & \text{S3} \\ & = & \mathbf{id}_n \otimes \mathbf{c}_{m,\epsilon} \ . & \text{S2} \end{array}$$

■

Definition 3.4 (Surface) *Call the the set $\{x \in X \mid \mathbf{ab}_x a \neq \mathbf{id} \otimes a\}$ the* surface *of a, written $\mathbf{surf}(a)$.*

Intuitively, the notion of surface of an action is a generalisation to arbitrary actions of the set of free names of an action in molecular form. Indeed, Fact 2.4 shows that when a is a molecular form then $\mathbf{surf}(a)$ is exactly its set of free names.

Proposition 3.5 (Surface)

(1) $\mathbf{surf}(\mathbf{id}) = \emptyset$

(2) $\mathbf{surf}(\mathbf{c}) = \emptyset$

(3) $\mathbf{surf}(a \cdot b) \subseteq \mathbf{surf}(a) \cup \mathbf{surf}(b)$

(4) $\mathbf{surf}(a \otimes b) \subseteq \mathbf{surf}(a) \cup \mathbf{surf}(b)$

(5) $\mathbf{surf}(\mathbf{ab}_x a) \subseteq \mathbf{surf}(a) - \{x\}$.

Proof (1) and (2) follow trivially from the axioms $\mathbf{ab}_x \mathbf{id} = \mathbf{id}$ and S4 respectively. For (3) and (4) it suffices to show that if $\mathbf{ab}_x a = \mathbf{id} \otimes a$ and $\mathbf{ab}_x b = \mathbf{id} \otimes b$ then $\mathbf{ab}_x (a \cdot b) = \mathbf{id} \otimes (a \cdot b)$ and $\mathbf{ab}_x (a \otimes b) = \mathbf{id} \otimes a \otimes b$. For (5), by S5 we have $x \notin \mathbf{surf}(\mathbf{ab}_x a)$. So, let $y \notin \mathbf{surf}(a)$ with $y \neq x$. Assume $y : \ell$, $x : k$ and $a : m \to n$.

(3)
$$\begin{array}{rcl} \mathbf{ab}_x (a \cdot b) & = & \mathbf{ab}_x a \cdot \mathbf{ab}_x b \\ & = & (\mathbf{id} \otimes a) \cdot (\mathbf{id} \otimes b) \\ & = & \mathbf{id} \otimes (a \cdot b) \ . \end{array}$$

$$
\begin{array}{llll}
(4) & \mathbf{ab}_x(a\otimes b) & = \mathbf{ab}_x((a\otimes\mathbf{id})\cdot(\mathbf{id}\otimes b)) & \\
& & = \mathbf{ab}_x(a\otimes\mathbf{id})\cdot\mathbf{ab}_x(\mathbf{id}\otimes b) & \\
& & = (\mathbf{ab}_x a\otimes\mathbf{id})\cdot\mathbf{ab}_x(\mathbf{id}\otimes b) & \text{S6} \\
& & = (\mathbf{ab}_x a\otimes\mathbf{id})\cdot\mathbf{ab}_x(\mathbf{c}\cdot(b\otimes\mathbf{id})\cdot\mathbf{c}) & 3.3(1) \\
& & = (\mathbf{ab}_x a\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{c})\cdot(\mathbf{ab}_x b\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{c}) & \text{S4, S6} \\
& & = (\mathbf{id}\otimes a\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{c})\cdot(\mathbf{id}\otimes b\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{c}) & \\
& & = (\mathbf{id}\otimes a\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{id}\otimes b) & 3.3(1) \\
& & = \mathbf{id}\otimes a\otimes b\ . &
\end{array}
$$

$$
\begin{array}{llll}
(5) & \mathbf{ab}_y\mathbf{ab}_x a & = (\mathbf{c}_{\ell,k}\otimes\mathbf{id})\cdot\mathbf{ab}_x\mathbf{ab}_y a\cdot(\mathbf{c}_{k,\ell}\otimes\mathbf{id}) & 3.3(2) \\
& & = (\mathbf{c}_{\ell,k}\otimes\mathbf{id})\cdot\mathbf{ab}_x(\mathbf{id}_\ell\otimes a)\cdot(\mathbf{c}_{k,\ell}\otimes\mathbf{id}) & \\
& & = (\mathbf{c}_{\ell,k}\otimes\mathbf{id})\cdot\mathbf{ab}_x(\mathbf{c}_{\ell,m}\cdot(a\otimes\mathbf{id})\cdot\mathbf{c}_{n,\ell})\cdot(\mathbf{c}_{k,\ell}\otimes\mathbf{id}) & 3.3(1) \\
& & = (\mathbf{c}_{\ell,k}\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{c}_{\ell,m})\cdot\mathbf{ab}_x(a\otimes\mathbf{id})\cdot(\mathbf{id}\otimes\mathbf{c}_{n,\ell})\cdot(\mathbf{c}_{k,\ell}\otimes\mathbf{id}) & \text{S6} \\
& & = \mathbf{c}_{\ell,k\otimes m}\cdot(\mathbf{ab}_x a\otimes\mathbf{id})\cdot\mathbf{c}_{k\otimes n,\ell} & \text{S3, } 3.3(3) \\
& & = \mathbf{id}\otimes\mathbf{ab}_x a\ . & 3.3(1)
\end{array}
$$

■

Remarks In Proposition 3.5, clauses (3)–(5) are inclusions rather than equations. In the more refined class of models for action calculi given by control structures (see the following section) we prove a stronger version of (5) with inclusion replaced by equality.

However, equality does not hold for (3) or (4). A counterexample for (3) is given by $\langle x\rangle\cdot\boldsymbol{\omega}=\mathbf{id}_\epsilon$, which holds in any action calculus. For (4), a counterexample is provided by the action structure whose typical element is of the form $(\vec{x})g\langle\vec{y}\rangle$, where g is an element of the free abelian group generated by the names X; thus g takes the form $x_1^{h_1}\cdots x_r^{h_r}$ where $h_1,\ldots,h_r$ are integers. (This example arises as a quotient of the action structure for Synchronous CCS [2].) The tensor product of $a=(\vec{u})f\langle\vec{v}\rangle$ and $b=(\vec{x})g\langle\vec{y}\rangle$ is $(\vec{u}\vec{x})\,f\times g\,\langle\vec{v}\vec{y}\rangle$, where $f\times g$ is the group product. If $a=xy^{-1}$ and $b=yz^2$ are two actions of arity $\epsilon\to\epsilon$, then y lies in the surface of a and of b but not in the surface of $a\otimes b=xz^2$.

Proposition 3.6 *The action calculus* $\mathsf{AC}(\mathcal{K})$ *is a symmetric action structure.*

Proof We take the permutations $\mathbf{p}_{m,n}$ as the symmetry on $\mathsf{AC}(\mathcal{K})$. S1 (naturality) is immediate by $\boldsymbol{\zeta}$. We shall now show that axioms S2–7 are provable in AC. In the proofs that follow assume that $\vec{x},\vec{v}:m$, $\vec{y},\vec{u}:n$ and $\vec{w}:k$ and that these names are all distinct. Reasons for each step appear in the right-hand column; an asterisk indicates repeated use of an equation.

$$
\begin{array}{llll}
\text{S2} & \mathbf{p}_{m,n}\cdot\mathbf{p}_{n,m} & = (\vec{x}\vec{y})\langle\vec{y}\vec{x}\rangle\cdot(\vec{u}\vec{v})\langle\vec{v}\vec{u}\rangle & \\
& & = (\vec{x}\vec{y})(\langle\vec{y}\vec{x}\rangle\cdot(\vec{u}\vec{v})\langle\vec{v}\vec{u}\rangle) & 2.12(1)^* \\
& & = (\vec{x}\vec{y})\langle\vec{x}\vec{y}\rangle & \boldsymbol{\sigma}^* \\
& & = \mathbf{id}\ . & 2.12(5)
\end{array}
$$

$$
\begin{array}{llll}
\text{S3} & (\mathbf{p}_{m,n}\otimes\mathbf{id}_k)\cdot(\mathbf{id}_n\otimes\mathbf{p}_{m,k}) & = ((\vec{x}\vec{y})\langle\vec{y}\vec{x}\rangle\otimes\mathbf{id}_k)\cdot(\mathbf{id}_n\otimes(\vec{v}\vec{w})\langle\vec{w}\vec{v}\rangle) & \\
& & = (\vec{x}\vec{y})(\langle\vec{y}\vec{x}\rangle\otimes\mathbf{id}_k)\cdot(\mathbf{id}_n\otimes(\vec{v}\vec{w})\langle\vec{w}\vec{v}\rangle) & 2.12(2)^* \\
& & = (\vec{x}\vec{y})((\langle\vec{y}\vec{x}\rangle\otimes\mathbf{id}_k)\cdot(\mathbf{id}_n\otimes(\vec{v}\vec{w})\langle\vec{w}\vec{v}\rangle)) & 2.12(1)^* \\
& & = (\vec{x}\vec{y})((\langle\vec{y}\rangle\cdot\mathbf{id}_n)\otimes((\langle\vec{x}\rangle\otimes\mathbf{id}_k)\cdot(\vec{v}\vec{w})\langle\vec{w}\vec{v}\rangle)) & \\
& & = (\vec{x}\vec{y})(\langle\vec{y}\rangle\otimes(\vec{w})\langle\vec{w}\vec{x}\rangle) & \boldsymbol{\sigma}^* \\
& & = (\vec{x}\vec{y}\vec{w})\langle\vec{y}\vec{w}\vec{x}\rangle) & 2.12(3)^* \\
& & = \mathbf{p}_{m,n\otimes k}\ . &
\end{array}
$$

$$
\begin{array}{llll}
\text{S4} & \mathbf{ab}_x\mathbf{p}_{k,n} & = (x)(\langle x\rangle\otimes\mathbf{p}_{k,n}) & 2.12(4) \\
& & = (x)\langle x\rangle\otimes\mathbf{p}_{k,n} & 2.12(2) \\
& & = \mathbf{id}\otimes\mathbf{p}_{k,n}\ . & \boldsymbol{\delta}
\end{array}
$$

$$
\begin{array}{llcll}
\text{S5} & \mathbf{ab}_x(\mathbf{ab}_x t) & = & (x)(\langle x\rangle \otimes \mathbf{ab}_x t) & 2.12(4) \\
 & & = & (x)\langle x\rangle \otimes \mathbf{ab}_x t & 2.12(2) \\
 & & = & \mathbf{id} \otimes \mathbf{ab}_x t\ . & \boldsymbol{\delta} \\
\\
\text{S6} & \mathbf{ab}_x(t \otimes \mathbf{id}) & = & (x)(\langle x\rangle \otimes t \otimes \mathbf{id}) & 2.12(4) \\
 & & = & (x)(\langle x\rangle \otimes t) \otimes \mathbf{id} & 2.12(2) \\
 & & = & \mathbf{ab}_x t \otimes \mathbf{id}\ . & 2.12(4) \\
\\
\text{S7} & \mathbf{ab}_x\mathbf{ab}_y t \cdot (\mathbf{p}_{p,q} \otimes \mathbf{id}) & = & \mathbf{ab}_x\mathbf{ab}_y t \cdot ((xy)\langle yx\rangle \otimes \mathbf{id}) & \\
 & & = & \mathbf{ab}_x\mathbf{ab}_y t \cdot (xy)(\langle yx\rangle \otimes \mathbf{id}) & 2.12(2) \\
 & & = & \mathbf{ab}_x\mathbf{ab}_y t \cdot \mathbf{ab}_x\mathbf{ab}_y(\langle yx\rangle \otimes \mathbf{id}) \cdot (x)(\boldsymbol{\omega} \otimes \mathbf{id}) & \\
 & & = & \mathbf{ab}_x\mathbf{ab}_y(\langle yx\rangle \otimes t) \cdot (x)(\boldsymbol{\omega} \otimes \mathbf{id}) & \\
 & & = & (xy)(\langle yx\rangle \otimes t) & \\
 & & = & (xy)((\langle x\rangle \otimes \mathbf{id}) \cdot (x)(\langle yx\rangle \otimes t)) & 2.10,\boldsymbol{\sigma} \\
 & & = & (xy)((\langle x\rangle \otimes \mathbf{id}) \cdot (\langle y\rangle \otimes \mathbf{id}) \cdot (yx)(\langle yx\rangle \otimes t)) & 2.10,\boldsymbol{\sigma} \\
 & & = & (xy)((\langle yx\rangle \otimes \mathbf{id}) \cdot (yx)(\langle yx\rangle \otimes t)) & \\
 & & = & (xy)((\langle yx\rangle \otimes \mathbf{id}) \cdot (y)(\langle y\rangle \otimes (x)(\langle x\rangle \otimes t))) & 2.12(3) \\
 & & = & (xy)((\langle yx\rangle \otimes \mathbf{id}) \cdot (y)(\langle y\rangle \otimes \mathbf{ab}_x t)) & 2.12(4) \\
 & & = & (xy)((\langle yx\rangle \otimes \mathbf{id}) \cdot \mathbf{ab}_y\mathbf{ab}_x t) & 2.12(4) \\
 & & = & (xy)(\langle yx\rangle \otimes \mathbf{id}) \cdot \mathbf{ab}_y\mathbf{ab}_x t & 2.12(1)^* \\
 & & = & (\mathbf{p}_{p,q} \otimes \mathbf{id}) \cdot \mathbf{ab}_y\mathbf{ab}_x t\ . & 2.12(2)^*
\end{array}
$$

■

4 Control structures

We have prepared the way for the central definition and result of the paper, namely the definition of control structures over $\mathcal{K}$ and the proof that $\mathsf{AC}(\mathcal{K})$ is the initial control structure.

Definition 4.1 (Control structure) *Let A be a symmetric action structure over X, and let $\mathcal{K}$ be a set of controls, equipped with reaction rules. Let A be equipped with*

- *a* datum $\langle x\rangle : \epsilon \to p$ *for each* $x : p \in X$
- *a* discard *operation* $\boldsymbol{\omega} : p \to \epsilon$ *for each prime arity* p
- *a* control *operation* K *for each* $K \in \mathcal{K}$*, obeying the arity rules for* $\mathcal{K}$ *.*

Then A is a control structure over $\mathcal{K}$ *if the reaction $t[\vec{a}] \searrow t'[\vec{a}]$ holds in A for each reaction rule $t[\vec{\alpha}] \searrow^1 t'[\vec{\alpha}]$ of the controls $\mathcal{K}$, and each vector $\vec{a}$ of actions in A of appropriate arity, and moreover the following axioms hold in A:*

$$
\begin{array}{lll}
\text{Surface} & \boldsymbol{\gamma}_1 : & \mathbf{ab}_x\langle y\rangle = \mathbf{id} \otimes \langle y\rangle \qquad \text{if } y \neq x \\
 & \boldsymbol{\gamma}_2 : & \mathbf{ab}_x\boldsymbol{\omega} = \mathbf{id} \otimes \boldsymbol{\omega} \\
\\
\text{Datum} & \boldsymbol{\epsilon} : & (x)\langle x\rangle = \mathbf{id} \\
\\
\text{Substitution} & \boldsymbol{\sigma}_1 : & [x/x]a = a \\
 & \boldsymbol{\sigma}_2 : & [y/x](\langle x\rangle \otimes \langle x\rangle) = \langle y\rangle \otimes \langle y\rangle \\
 & \boldsymbol{\sigma}_3 : & [y/x]K(a_1, \cdots, a_n) = K([y/x]a_1, \cdots, [y/x]a_n)
\end{array}
$$

where

$$
\begin{array}{rcll}
(x)a & \stackrel{\text{def}}{=} & (\mathbf{ab}_x a) \cdot (\boldsymbol{\omega} \otimes \mathbf{id}) & (x : p) \\
{[y/x]a} & \stackrel{\text{def}}{=} & (\langle y\rangle \otimes \mathbf{id}) \cdot (x)a & (x, y : p)\ .
\end{array}
$$

Remarks The operation $[{}^{y}\!/x]$ is called *semantic substitution*. Notice that the axiom $\boldsymbol{\sigma}$ simply asserts that, in $\mathsf{AC}(\mathcal{K})$, semantic substitution agrees with semantic substitution.

The axioms $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ are simple instances of $\boldsymbol{\gamma}$ in AC, $\boldsymbol{\epsilon}$ is an instance of $\boldsymbol{\delta}$, and in the presence of the other axioms $\boldsymbol{\sigma}_1$–$\boldsymbol{\sigma}_3$ correspond to $\boldsymbol{\sigma}$ together with the substitution equations (Definition 2.9).

Note that the employment of symmetry in our formulation has allowed us to avoid the use of vectors of names and also to isolate the treatment of datum, discard and the controls. We shall discuss alternative axiomatization after Proposition 4.9.

The following proposition expresses the interaction between data and discard.

Proposition 4.2 (Absorption) $\langle x\rangle \cdot \boldsymbol{\omega} = \mathbf{id}_\epsilon$.

Proof

$$\begin{array}{rcll}
\mathbf{id}_\epsilon & = & [{}^{x}\!/x]\mathbf{id}_\epsilon & \boldsymbol{\sigma}_1 \\
& = & \langle x\rangle \cdot \mathbf{ab}_x\mathbf{id}_\epsilon \cdot \boldsymbol{\omega} & \\
& = & \langle x\rangle \cdot \mathbf{id} \cdot \boldsymbol{\omega} & \\
& = & \langle x\rangle \cdot \boldsymbol{\omega}\ . &
\end{array}$$

■

The next proposition gives two further alternative characterizations of the surface of an action. Note that these alternatives exist in control structures, but not in arbitrary symmetric action structures, just because of the addition of the data $\langle x\rangle$ and the discard $\boldsymbol{\omega}$.

Proposition 4.3 *For any a and x, the following are equivalent:*

(1) $\mathbf{ab}_x a = \mathbf{id} \otimes a$

(2) $(x)a = \boldsymbol{\omega} \otimes a$

(3) $[{}^{y}\!/x]a = a$, *for all* y .

Proof By definition (1) implies (2) and, by Proposition 4.2 and the definition of $[{}^{y}\!/x]$, (2) implies (3). To show that (3) implies (1) choose $y \neq x$; then

$$\begin{array}{rcll}
\mathbf{ab}_x a & = & \mathbf{ab}_x([{}^{y}\!/x]a) & \\
& = & \mathbf{ab}_x((\langle y\rangle \otimes \mathbf{id}) \cdot (x)a) & \\
& = & \mathbf{ab}_x(\langle y\rangle \otimes \mathbf{id}) \cdot \mathbf{ab}_x(x)a & \\
& = & (\mathbf{id} \otimes \langle y\rangle \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes (x)a) & \mathrm{S6},\boldsymbol{\gamma}_1,\mathrm{S5},\boldsymbol{\gamma}_2 \\
& = & \mathbf{id} \otimes ((\langle y\rangle \otimes \mathbf{id}) \cdot (x)a) & \\
& = & \mathbf{id} \otimes [{}^{y}\!/x]a & \\
& = & \mathbf{id} \otimes a\ . &
\end{array}$$

■

Remark The above proposition can be regarded as the semantic equivalent of $\boldsymbol{\gamma}$. If $x \notin \mathbf{surf}(a)$, then by the definition of surface, $\mathbf{ab}_x a = \mathbf{id} \otimes a$. By Proposition 4.3, $(x)a = \boldsymbol{\omega} \otimes a$.

Proposition 4.4 (Surface)

(1) $\mathbf{surf}(\langle x\rangle) \subseteq \{x\}$

(2) $\mathbf{surf}(\boldsymbol{\omega}) = \emptyset$

(3) $\mathbf{surf}(\mathbf{ab}_x a) = \mathbf{surf}(a) - \{x\}$

(4) $\mathbf{surf}(K(a_1, \cdots, a_r)) \subseteq \bigcup_{1\leq i\leq r} \mathbf{surf}(a_i)$.

Proof (1) and (2) follow trivially from $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ respectively.

For (3) we need only show that $\mathbf{surf}(\mathbf{ab}_x a) \supseteq \mathbf{surf}(a) - \{x\}$ since Proposition 3.5(5) gives the other inclusion. By $\boldsymbol{\sigma}_1$, $a = (\langle x\rangle \otimes \mathbf{id}) \cdot \mathbf{ab}_x a \cdot (\boldsymbol{\omega} \otimes \mathbf{id})$. Hence, by (1), (2) and Proposition 3.5 (3) and (4) we get $\mathbf{surf}(a) \subseteq \{x\} \cup \mathbf{surf}(\mathbf{ab}_x a)$ and the result follows immediately.

To show (4), assume $x \notin \mathbf{surf}(a_i)$, for all a_i in $\vec{a}$. Now by $\boldsymbol{\sigma}_3$, $[{}^y\!/x]K\vec{a} = K[{}^y\!/x]\vec{a}$. By assumption, for each i, $\mathbf{ab}_x a_i = \mathbf{id} \otimes a_i$, so by Proposition 4.3, $[{}^y\!/x]a_i = a_i$. Hence $[{}^y\!/x]K\vec{a} = K\vec{a}$ and by Proposition 4.3, $x \notin \mathbf{surf}(K\vec{a})$. ∎

Remark We do not have $\mathbf{surf}(\langle x\rangle) = \{x\}$ in general, since in the trivial control structure where all terms of the same arity are identified (the terminal control structure) the surface of each term is necessarily empty. Note also that we have refined Proposition 3.5(5) by equality rather than inclusion.

Proposition 4.5 ($\boldsymbol{\delta}$) $(x)(\langle x\rangle \otimes \mathbf{id}) = \mathbf{id}$.

Proof

$$
\begin{array}{rcll}
(x)(\langle x\rangle \otimes \mathbf{id}) & = & \mathbf{ab}_x(\langle x\rangle \otimes \mathbf{id}) \cdot (\boldsymbol{\omega} \otimes \mathbf{id}) & \\
& = & (\mathbf{ab}_x\langle x\rangle \otimes \mathbf{id}) \cdot (\boldsymbol{\omega} \otimes \mathbf{id}) & \text{S6} \\
& = & (\mathbf{ab}_x\langle x\rangle \cdot (\boldsymbol{\omega} \otimes \mathbf{id})) \otimes \mathbf{id} & \\
& = & (x)\langle x\rangle \otimes \mathbf{id} & \\
& = & \mathbf{id}\ . & \boldsymbol{\epsilon}
\end{array}
$$

∎

Proposition 4.6 *The following equations hold in a control structure whenever $x \notin \mathbf{surf}(b)$:*

(1) $(x)(a \cdot b) = (x)a \cdot b$

(2) $(x)(a \otimes b) = (x)a \otimes b$

(3) $(x)(b \otimes a) = b \otimes (x)a$, *if* $b : \epsilon \to \ell$

$\boldsymbol{\alpha}$: $(y)b = (x)[{}^x\!/y]b$

(4) $\mathbf{ab}_x a = (x)(\langle x\rangle \otimes a)$

(5) $(\vec{x})(\langle \vec{x}\rangle \otimes \mathbf{id}) = \mathbf{id}$.

Proof

$$
\begin{array}{lrcll}
(1) & (x)(a \cdot b) & = & \mathbf{ab}_x a \cdot (x)b & \\
& & = & \mathbf{ab}_x a \cdot (\boldsymbol{\omega} \otimes b) & 4.3 \\
& & = & \mathbf{ab}_x a \cdot (\boldsymbol{\omega} \otimes \mathbf{id}) \cdot b & \\
& & = & (x)a \cdot b\ . &
\end{array}
$$

$$
\begin{array}{lrcll}
\dagger & (x)(a \otimes \mathbf{id}) & = & (x)([{}^x\!/x]a \otimes \mathbf{id}) & \boldsymbol{\sigma}_1 \\
& & = & (x)(((\langle x\rangle \otimes \mathbf{id}) \cdot (x)a \otimes \mathbf{id}) & \\
& & = & (x)(\langle x\rangle \otimes \mathbf{id} \otimes \mathbf{id}) \cdot ((x)a \otimes \mathbf{id}) & (1) \\
& & = & (x)a \otimes \mathbf{id} & 4.5\ .
\end{array}
$$

$$
\begin{array}{lrcll}
(2) & (x)(a \otimes b) & = & (x)((a \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes b)) & \\
& & = & (x)(a \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes b) & (1),3.5(1),3.5(4) \\
& & = & ((x)a \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes b) & \dagger \\
& & = & (x)a \otimes b\ . &
\end{array}
$$

$$
\begin{array}{llcll}
(3) & (x)(b \otimes a) & = & (x)(a \cdot (b \otimes \mathbf{id})) & \\
 & & = & (x)a \cdot (b \otimes \mathbf{id}) & (1) \\
 & & = & b \otimes (x)a\ . & \\
\\
\boldsymbol{\alpha}: & (x)[x/y]b & = & (x)(((\langle x\rangle \otimes \mathbf{id}) \cdot (y)b) & \\
 & & = & (x)(\langle x\rangle \otimes \mathbf{id}) \cdot (y)b & (1) \\
 & & = & (y)b\ . & 4.5 \\
\\
(4) & \mathbf{ab}_x a & = & \mathbf{ab}_x a \cdot (x)(\langle x\rangle \otimes \mathbf{id}) & 4.5 \\
 & & = & (x)(a \cdot (\langle x\rangle \otimes \mathbf{id})) & \\
 & & = & (x)(\langle x\rangle \otimes a)\ . &
\end{array}
$$

(5) Induction on length of $\vec{x}$. Basis true by definition. Step:

$$
\begin{array}{lcll}
(x\vec{y})(\langle x\vec{y}\rangle \otimes \mathbf{id}) & = & (x)(\vec{y})((\langle\vec{y}\rangle \otimes \mathbf{id}) \cdot (\langle x\rangle \otimes \mathbf{id})) & \\
 & = & (x)((\vec{y})(\langle\vec{y}\rangle \otimes \mathbf{id})) \cdot (\langle x\rangle \otimes \mathbf{id}) & (1)^* \\
 & = & (x)(\langle x\rangle \otimes \mathbf{id}) & \text{induction} \\
 & = & \mathbf{id}\ . & 4.5
\end{array}
$$

■

Remark When $x \notin \mathbf{surf}(b)$, $\mathbf{ab}_y b = \mathbf{ab}_x[x/y]b$ follows from $\boldsymbol{\alpha}$, (4) and Proposition 4.9(5).

Proposition 4.7 *Define* $\mathbf{p}_{m,n}$ *as* $(\vec{x}\vec{y})\langle\vec{y}\vec{x}\rangle$, *where* $\vec{x} : m$ *and* $\vec{y} : n$. *Then* $\mathbf{p}_{m,n} = \mathbf{c}_{m,n}$.

Proof

$$
\begin{array}{lcll}
(\vec{x}\vec{y})\langle\vec{y}\vec{x}\rangle & = & (\vec{x})((\vec{y})\langle\vec{y}\rangle \otimes \langle\vec{x}\rangle) & 4.6(2)^* \\
 & = & (\vec{x})(\mathbf{id} \otimes \langle\vec{x}\rangle) & 4.6(5) \\
 & = & (\vec{x})((\langle\vec{x}\rangle \otimes \mathbf{id}) \cdot \mathbf{c}_{m,n}) & \text{S1} \\
 & = & (\vec{x})(\langle\vec{x}\rangle \otimes \mathbf{id}) \cdot \mathbf{c}_{m,n} & 3.5(2), 4.6(1)^* \\
 & = & \mathbf{c}_{m,n}\ . & 4.6(5)
\end{array}
$$

■

Corollary 4.8 ($\boldsymbol{\zeta}$) *Assume* $a : m \to n$ *and* $b : k \to \ell$. *Then* $\mathbf{p}_{k,m} \cdot (a \otimes b) = (b \otimes a) \cdot \mathbf{p}_{\ell,n}$.

Proof Immediate by Proposition 4.7 and naturality of symmetries. ■

The following proposition asserts that the semantic substitution $[y/x]$ behaves equationally like the syntactic substitution $\{y/x\}$ as given in Definition 2.9.

Proposition 4.9 (Semantic substitution) *The following properties of semantic substitution hold in any control structure:*

(1) $[y/x]\mathbf{id} = \mathbf{id}$

(2) $[y/x]\boldsymbol{\omega} = \boldsymbol{\omega}$

(3) $[y/x]\langle z\rangle = \langle z\rangle \quad (z \neq x)$

(4) $[y/x]\langle x\rangle = \langle y\rangle$

(5) $[y/x](a \otimes b) = [y/x]a \otimes [y/x]b$

(6) $[y/x](a \cdot b) = [y/x]a \cdot [y/x]b$

(7) $[y/x]K(a, \ldots) = K([y/x]a, \ldots)$

(8) $[y/x]\mathbf{ab}_z a = \mathbf{ab}_z[y/x]a \qquad (z \notin \{x, y\})$

(9) $[z/x]\mathbf{ab}_z a = \mathbf{ab}_w[z/x][w/z]a \quad (z \neq x,\ w \notin \mathbf{surf}(a) \cup \{x, z\})$

(10) $[y/z]\mathbf{ab}_z a = \mathbf{ab}_z a$.

Proof (1), (2), (3) and (10) follow directly from Proposition 4.3(3), and (4) follows directly from $\boldsymbol{\epsilon}$. (7) is exactly $\boldsymbol{\sigma}_3$. For the remaining cases, assume $x, y, z : p$:

$$
\begin{array}{llll}
(5) & [{}^{y}\!/x](a\otimes b) & = & [{}^{y}\!/x]([{}^{x}\!/x]a\otimes[{}^{x}\!/x]b) & \boldsymbol{\sigma}_1\\
& & = & [{}^{y}\!/x](((\langle x\rangle\otimes\mathbf{id}_k)\cdot(x)a)\otimes((\langle x\rangle\otimes\mathbf{id}_m)\cdot(x)b)) & \\
& & = & [{}^{y}\!/x]((\langle xx\rangle\otimes\mathbf{id}_{k\otimes m})\cdot(\mathbf{id}_p\otimes\mathbf{c}_{p,k}\otimes\mathbf{id}_m)\cdot((x)a\otimes(x)b)) & \mathrm{S1}\\
& & = & ((\langle y\rangle\otimes\mathbf{id}_{k\otimes m})\cdot((x)\langle xx\rangle\otimes\mathbf{id}_{k\otimes m})\cdot & \\
& & & \qquad(\mathbf{id}_p\otimes\mathbf{c}_{p,k}\otimes\mathbf{id}_m)\cdot((x)a\otimes(x)b) & 4.6(1,2)\\
& & = & ((\langle yy\rangle\otimes\mathbf{id}_{k\otimes m})\cdot(\mathbf{id}_p\otimes\mathbf{c}_{p,k}\otimes\mathbf{id}_m)\cdot((x)a\otimes(x)b) & \boldsymbol{\sigma}_2\\
& & = & ((\langle y\rangle\otimes\mathbf{id}_k)\cdot(x)a\otimes((\langle y\rangle\otimes\mathbf{id}_m)\cdot(x)b) & \mathrm{S1}\\
& & = & [{}^{y}\!/x]a\otimes[{}^{y}\!/x]b\ . &
\end{array}
$$

$$
\begin{array}{llll}
(6) & [{}^{y}\!/x](a\cdot b) & = & ((\langle y\rangle\otimes\mathbf{id})\cdot(x)(a\cdot b) & \\
& & = & ((\langle y\rangle\otimes\mathbf{id})\cdot(x)((\langle x\rangle\otimes a)\cdot(x)b) & 4.6(4)\\
& & = & [{}^{y}\!/x]((\langle x\rangle\otimes a)\cdot(x)b & \\
& & = & ((\langle y\rangle\otimes[{}^{y}\!/x]a)\cdot(x)b & (4),(5)\\
& & = & [{}^{y}\!/x]a\cdot((\langle y\rangle\otimes\mathbf{id})\cdot(x)b & \\
& & = & [{}^{y}\!/x]a\cdot[{}^{y}\!/x]b\ . &
\end{array}
$$

$$
\begin{array}{llll}
(8) & [{}^{y}\!/x]\mathbf{ab}_z a & = & ((\langle y\rangle\otimes\mathbf{id})\cdot\mathbf{ab}_x\mathbf{ab}_z a\cdot(\boldsymbol{\omega}\otimes\mathbf{id}) & \\
& & = & ((\langle y\rangle\otimes\mathbf{id})\cdot(\mathbf{c}\otimes\mathbf{id})\cdot\mathbf{ab}_z\mathbf{ab}_x a\cdot(\mathbf{c}\otimes\mathbf{id})\cdot(\boldsymbol{\omega}\otimes\mathbf{id}) & 3.3(2)\\
& & = & (\mathbf{id}\otimes\langle y\rangle\otimes\mathbf{id})\cdot\mathbf{ab}_z\mathbf{ab}_x a\cdot(\mathbf{id}\otimes\boldsymbol{\omega}\otimes\mathbf{id}) & \mathrm{S1}\\
& & = & \mathbf{ab}_z((\langle y\rangle\otimes\mathbf{id})\cdot\mathbf{ab}_z\mathbf{ab}_x a\cdot\mathbf{ab}_z(\boldsymbol{\omega}\otimes\mathbf{id}) & \boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2\\
& & = & \mathbf{ab}_z[{}^{y}\!/x]a\ . &
\end{array}
$$

$$
\begin{array}{llll}
(9) & [{}^{z}\!/x]\mathbf{ab}_z a & = & ((\langle z\rangle\otimes\mathbf{id})\cdot\mathbf{ab}_x\mathbf{ab}_z a\cdot(\boldsymbol{\omega}\otimes\mathbf{id}) & \\
& & = & ((\langle z\rangle\otimes\mathbf{id})\cdot(\mathbf{c}\otimes\mathbf{id})\cdot\mathbf{ab}_z\mathbf{ab}_x a\cdot(\mathbf{c}\otimes\mathbf{id})\cdot(\boldsymbol{\omega}\otimes\mathbf{id}) & 3.3(2)\\
& & = & (\mathbf{id}\otimes\langle z\rangle\otimes\mathbf{id})\cdot\mathbf{ab}_z\mathbf{ab}_x a\cdot(\mathbf{id}\otimes\boldsymbol{\omega}\otimes\mathbf{id}) & \mathrm{S1}\\
& & = & (\mathbf{id}\otimes\langle z\rangle\otimes\mathbf{id})\cdot(z)(\langle z\rangle\otimes\mathbf{ab}_x a)\cdot(\mathbf{id}\otimes\boldsymbol{\omega}\otimes\mathbf{id}) & 4.6(4)\\
& & = & (\mathbf{id}\otimes\langle z\rangle\otimes\mathbf{id})\cdot(w)([{}^{w}\!/z](\langle z\rangle\otimes\mathbf{ab}_x a))\cdot(\mathbf{id}\otimes\boldsymbol{\omega}\otimes\mathbf{id}) & \boldsymbol{\alpha}\\
& & = & (\mathbf{id}\otimes\langle z\rangle\otimes\mathbf{id})\cdot(w)(\langle w\rangle\otimes\mathbf{ab}_x[{}^{w}\!/z]a)\cdot(\mathbf{id}\otimes\boldsymbol{\omega}\otimes\mathbf{id}) & (5),(4),(8)\\
& & = & (\mathbf{id}\otimes\langle z\rangle\otimes\mathbf{id})\cdot\mathbf{ab}_w\mathbf{ab}_x[{}^{w}\!/z]a\cdot(\mathbf{id}\otimes\boldsymbol{\omega}\otimes\mathbf{id}) & 4.6(4)\\
& & = & \mathbf{ab}_w((\langle z\rangle\otimes\mathbf{id})\cdot\mathbf{ab}_w\mathbf{ab}_x[{}^{w}\!/z]a\cdot\mathbf{ab}_w(\boldsymbol{\omega}\otimes\mathbf{id}) & \boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2\\
& & = & \mathbf{ab}_w[{}^{z}\!/x][{}^{w}\!/z]a\ . &
\end{array}
$$

■

Discussion There are alternative sets of axioms to the ones given above; which is the most elegant or natural set is arguable. For instance, we can replace $\boldsymbol{\sigma}_2$ by $\boldsymbol{\sigma}'_2 : \mathbf{ab}_x\langle x\rangle = \mathbf{ab}_y\langle y\rangle$. This equation is provable by Proposition 4.6(4) and α-conversion, while $\boldsymbol{\sigma}_2$ is provable (from the alternative set of axioms) as follows:

$$
\begin{array}{llll}
[{}^{y}\!/x](\langle x\rangle\otimes\langle x\rangle) & = & ((\langle y\rangle\otimes\mathbf{id})\cdot(x)(\langle x\rangle\otimes\langle x\rangle) & \\
& = & ((\langle y\rangle\otimes\mathbf{id})\cdot(y)(\langle y\rangle\otimes\langle y\rangle) & \boldsymbol{\sigma}'_2, 4.6(4)\\
& = & [{}^{y}\!/y](\langle y\rangle\otimes\langle y\rangle) & \\
& = & \langle y\rangle\otimes\langle y\rangle\ . & \boldsymbol{\sigma}_1
\end{array}
$$

■

In this new form $\mathbf{ab}_x\langle x\rangle = \mathbf{ab}_y\langle y\rangle$, it becomes clearer that $\boldsymbol{\sigma}_2$ is the only instance of α-conversion which is required to derive α-conversion for arbitrary terms. Finally, the axiom can also be replaced by $[{}^{y}\!/x](\langle x\rangle\otimes\langle x\rangle) = [{}^{y}\!/x]\langle x\rangle\otimes[{}^{y}\!/x]\langle x\rangle$, in the presence of $\boldsymbol{\epsilon}$.

Definition 4.10 (Category of control structures) *The category* $\mathbf{CS}(\mathcal{K})$ *of control structures over* $\mathcal{K}$ *has as* objects *control structures with the (fixed) name-set* X *and arity monoid* M*, and as* morphisms *action structure homomorphisms which act as identity upon* X *and* M *and also preserve the data, discard and control operations.*

It is immediate from the definition of surface that every morphism of control structures reduces it, i.e.

Proposition 4.11 (Surface reduction) *If $\Phi : A \to B$ is a morphism of control structures, then $\mathbf{surf}(\Phi a) \subseteq \mathbf{surf}(a)$ for all $a \in A$.* ■

Since $\mathbf{CS}(\mathcal{K})$ is characterized equationally, it is easy to see that it is closed under taking a congruence. Moreover, we can state precisely what effect the morphism induced by a congruence has upon surface:

Proposition 4.12 (Congruence) *Let $\equiv$ be a congruence over each action-set $A(m,n)$ in a control structure A, i.e. an equivalence which is preserved by the action structure operations, by the control constructions $\mathcal{K}$ and by reaction. Then, the quotient $A/{\equiv}$ is a control structure, with $\Phi : a \mapsto [a]$ as the induced morphism from A to $A/{\equiv}$, where $[a]$ is the congruence class of a. Moreover*

$$\mathbf{surf}([a]) = \bigcap\{\mathbf{surf}(a') \mid a' \in [a]\} \ .$$

Proof The proof is mostly of a kind which is standard in universal algebra. For the last part, we prove each inclusion as follows.

($\subseteq$) It is enough to show that $\mathbf{surf}([a]) \subseteq \mathbf{surf}(a)$ for each a; but this follows from Proposition 4.11.

($\supseteq$) Assuming $x \notin \mathbf{surf}([a])$, it is enough to find $a' \equiv a$ such that $x \notin \mathbf{surf}(a')$. In other words, given $\mathbf{ab}_x a \equiv \mathbf{id} \otimes a$, we wish to find $a' \equiv a$ such that $\mathbf{ab}_x a' = \mathbf{id} \otimes a'$. Pick $a' = [y/x]a$, where $y \neq x$; then $a' \equiv a$ follows from $\mathbf{ab}_x a \equiv \mathbf{id} \otimes a$, and the rest follows much as in Proposition 4.3:

$$\begin{array}{rcll}
\mathbf{ab}_x([y/x]a) & = & \mathbf{ab}_x((\langle y\rangle \otimes \mathbf{id}) \cdot (x)a) & \\
& = & \mathbf{ab}_x(\langle y\rangle \otimes \mathbf{id}) \cdot \mathbf{ab}_x(x)a & \\
& = & (\mathbf{id} \otimes \langle y\rangle \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes (x)a) & \text{S6}, \boldsymbol{\gamma}_1, \text{S5}, \boldsymbol{\gamma}_2 \\
& = & \mathbf{id} \otimes ((\langle y\rangle \otimes \mathbf{id}) \cdot (x)a) & \\
& = & \mathbf{id} \otimes [y/x]a \ . &
\end{array}$$

■

We now proceed to consider initiality among control structures.

Proposition 4.13 *The category $\mathbf{CS}(\mathcal{K})$ has an initial object.*

Proof The proof is again standard; we take the quotient of the term algebra $\mathcal{T}(\mathcal{K})$ by the equivalence induced by all the control structure axioms. ■

Our next task is to establish the rôle played by $\mathsf{AC}(\mathcal{K})$ in the category of control structures.

Proposition 4.14 $\mathsf{AC}(\mathcal{K})$ *is a control structure over $\mathcal{K}$.*

Proof By Proposition 3.6, we already have that $\mathsf{AC}(\mathcal{K})$ is a symmetric action structure. Moreover $\boldsymbol{\epsilon}$ is a special case of $\boldsymbol{\delta}$. By $\boldsymbol{\sigma}$, $\{y/x\}$ agrees with the derived operation $[y/x]$ and $\boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_3$ follow from the equations for $\{y/x\}$. By Lemma 2.10 we have $\boldsymbol{\sigma}_1$. The following proofs give $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$.

$$\begin{array}{llcll}
\boldsymbol{\gamma}_1 & \mathbf{ab}_x\langle y\rangle & = & (x)(\langle x\rangle \otimes \langle y\rangle) & 2.12(4) \\
& & = & (x)(\langle x\rangle \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \langle y\rangle) & 2.12(1) \\
& & = & \mathbf{id} \otimes \langle y\rangle \ . & \boldsymbol{\delta}
\end{array}$$

$$\begin{array}{llcll}
\boldsymbol{\gamma}_2 & \mathbf{ab}_x\boldsymbol{\omega} & = & (x)(\langle x\rangle \otimes \boldsymbol{\omega}) & 2.12(4) \\
& & = & (x)(\langle x\rangle \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \boldsymbol{\omega}) & 2.12(1) \\
& & = & \mathbf{id} \otimes \boldsymbol{\omega} \ . & \boldsymbol{\delta}
\end{array}$$

■

Theorem 4.15 (Initiality) $\mathsf{AC}(\mathcal{K})$ *is initial in* $\mathbf{CS}(\mathcal{K})$.

Proof Since we have shown that the action calculus is a control structure, there is a unique map to it from the initial control structure. That map is obviously onto, so it remains to show that it is one to one. To do that, we must show that whenever the images of two terms are provably equal in AC, then they are equal in the initial control structure. It suffices to show that in the initial control structure, the axioms of AC are valid. By Propositions 4.5 and 4.9, and Corollary 4.8 we get $\boldsymbol{\delta}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\zeta}$ respectively. It remains to derive $\boldsymbol{\gamma}$.

By Proposition 4.3, it suffices to show that whenever $x \notin \mathbf{fn}(t)$ then $x \notin \mathbf{surf}(\widehat{t})$, where $\widehat{t}$ is the action in the initial control structure determined by the term t. This involves an easy induction on the structure of terms $t \in \mathcal{T}(\mathcal{K})$. For instance, consider the case $t \equiv K\vec{s}$; if $x \notin \mathbf{fn}(t)$ then $x \notin \mathbf{fn}(s)$, for each $s \in \vec{s}$; by induction $x \notin \mathbf{surf}(\widehat{s})$; then by Proposition 4.4(4) it follows immediately that $x \notin \mathbf{surf}(\widehat{t})$. ■

Remark Note that Proposition 4.9(9) holds for any $w \notin \mathbf{surf}(a)$. In general, there may may not be any such w; however, in action calculi, there will always be such a w as the surface of any action (i.e. any molecular form) is finite, by Fact 2.4.

The initiality of $\mathsf{AC}(\mathcal{K})$ is significant largely because it has a direct presentation (up to isomorphism) as the action structure of molecular forms $\mathcal{M}(\mathcal{K})$.

5 Discussion and future work

We have presented a restricted category of control structures in which the names X and arities M remain fixed. This restriction can be relaxed, and will be relaxed in future work, but we wished to present the restricted version here for clarity. Let us first observe that it is very easy to extend names to be the free monoid $(X, \otimes, 1)$ generated by X; we only have to extend data and abstraction by the definitions

$$\begin{array}{rcll} \mathbf{ab}_{x_1 \otimes \cdots \otimes x_r} a & \stackrel{\text{def}}{=} & \mathbf{ab}_{x_1} \cdots \mathbf{ab}_{x_r} a & (r \geq 0) \\ \langle x_1 \otimes \cdots \otimes x_r \rangle & \stackrel{\text{def}}{=} & \langle x_1 \rangle \otimes \cdots \otimes \langle x_r \rangle & (r \geq 0) \ . \end{array}$$

The question naturally arises: Can the name structure $(X, \otimes, 1)$ lie in some general class of monoids? Then a morphism of control structures need no longer act as the identity upon names, but should presumably satisfy some condition. The side-condition $x \neq y$ upon the two axioms S7 and $\boldsymbol{\gamma}_1$ provides a clue; we may take a name structure $(X, \otimes, 1, \#)$ to be a monoid equipped with a symmetric binary relation $\#$ called *separation* (which on the *free* monoid is just inequality), we replace the condition $x \neq y$ by $x \# y$ in the two axioms, and we require morphisms to preserve separation. But further research is needed to determine what properties separation should enjoy, and what practical benefit is gained from this generalisation of naming.

Now that we understand at least one category of models for an action calculus $\mathsf{AC}(\mathcal{K})$, at least to the extent that we have a tractable axiomatization of control structures and a satisfactory notion of homomorphism between them, the most important next step is to identify some significant particular control structures.

Some work along these lines is in progress. First, for the action calculus $\mathsf{AC}(\boldsymbol{\nu},\mathbf{out},\mathbf{box},\mathbf{rep})$ for the π-calculus we have defined bisimulation in a natural way, and it appears to induce a congruence (though some details remain to be checked). Thus, in view of Proposition 4.12, by taking a quotient we have a semantic interpretation of the π-calculus as a control structure. It is hoped that this form of bisimulation can be extended to a reasonably wide class of action calculi $\mathsf{AC}(\mathcal{K})$, and we shall then have at least one non-trivial semantic inhabitant of each corresponding category $\mathbf{CS}(\mathcal{K})$.

At the abstract end of the spectrum we have found in each $\mathbf{CS}(\mathcal{K})$ an interesting control structure which we call $\mathsf{SKEL}(\mathcal{K})$, the *action skeleta*. Skeleta are best explained in terms of the unique morphism $\mathbf{skel} : \mathsf{AC}(\mathcal{K}) \rightarrow \mathsf{SKEL}(\mathcal{K})$. For any action a in $\mathsf{AC}(\mathcal{K})$, its skeleton $\mathbf{skel}(a)$ gives a little more information about the "surface structure" of a than $\mathbf{surf}(a)$ does; just enough to ensure that $\mathbf{skel}$ is indeed a morphism of control structures (which of course $\mathbf{surf}$ is not). An example will give the idea of $\mathsf{SKEL}(\mathcal{K})$, without the need for precise definitions. Consider an action of $\mathsf{AC}(\boldsymbol{\nu},\mathbf{out},\mathbf{box},\mathbf{rep})$ in molecular form, for example

$$a = (x)\,[\,\boldsymbol{\nu}(u),\ \langle uy\rangle\mathbf{out},\ \langle u\rangle\mathbf{box}((z)b)\,]\,\langle xuw\rangle\ ,$$

where b is some other action. This action contains three molecules, each built by a different control operator. (It corresponds loosely to the expression $(x)(\boldsymbol{\nu}u)(\,\overline{u}y \mid u(z).b\,)$ of the π-calculus [4], though the latter has nothing which corresponds to the export vector $\langle xuw\rangle$.) Now the surface of a is given by $\mathbf{surf}(a) = \mathbf{surf}(b) - \{x,u,z\} \cup \{y,w\}$. This tells us nothing about *how* each free name, e.g. y or w, is used by a; moreover it says nothing *at all* about how any bound name is used, e.g. the imported name x. In contrast, $\mathbf{skel}(a)$ will tell us of each free or imported name of a whether it is used in a molecule, or in the export vector, or both; but $\mathbf{skel}(a)$ will still say nothing about how names bound by molecules are used, since it will "forget" the molecular construction of a.

In fact a skeleton can be expressed as a kind of degenerate molecular form $(\vec{x})\,S\,\langle\vec{z}\rangle$, where S is a set of names and $\vec{z}$ a vector of names or "blanks" (written here as a star $\star$). The skeleton of a is

$$\mathbf{skel}(a) = (x)\,S\,\langle x \star w\rangle \qquad \text{where } S = \mathbf{surf}(b) - \{u,z\} \cup \{y\}\ ;$$

the blank in the export vector indicates an export name of a which is bound by one of its molecules. Note that the molecular form of a skeleton is completely independent of $\mathcal{K}$; the control structure depends upon $\mathcal{K}$ only in that it possesses (as it must) a control operation for each $K \in \mathcal{K}$, and this is simply given by

$$K\vec{a} \stackrel{\text{def}}{=} (\vec{x})\langle\vec{\star}\rangle\ .$$

We shall not take the details of skeleta any further; they are mentioned here to indicate that control structures offer a wide range of possibilities for abstraction.

It remains a considerable challenge to find other, radically different, examples of control structures. In fact, if the definition of control structures serves to focus the search for semantic models of concurrency then it will have fulfilled its main purpose.

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