Simulation of quantum circuits by ZX-diagram contraction

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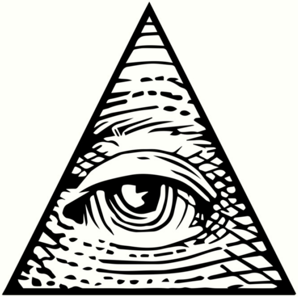
Holy Trinity of quantum circuits

Optimization



Verification

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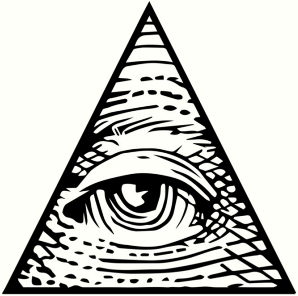

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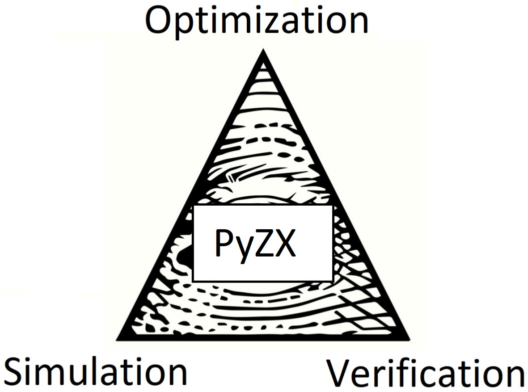

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Why do we care?

- Verification of correctness of circuits.
- Modelling physical systems.
- Understand when *quantum supremacy* has been reached.

Weak versus Strong simulation

Suppose we measure all qubits in computational basis at the end of the circuit. Then we get a probability distribution

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This is **BQP**-complete.

Strong simulation: get any marginal probability of $P(x_1 \cdots x_n)$.
This is **#P**-hard.

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All these methods in a sense rely on *tensor contraction*.
They are all exponential in number of qubits.

Stabilizer decompositions 1

Gottesman-Knill Theorem

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   $|\psi\rangle = \sum_i^n \lambda_i |\phi_i\rangle$.
3. Note: Each $C|\phi_i\rangle$ can be efficiently simulated!

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Given Clifford circuit C and input $|\psi\rangle = \sum_i^n \lambda_i |\phi_i\rangle$ where the $|\phi_i\rangle$ are Clifford. How do we approximate $C|\psi\rangle$?

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We will only use the second approach.

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Can also show that $R(|T\rangle^{\otimes 6}) = 7$, and hence:

$$R(|T\rangle^{\otimes n}) \leqslant 2^{\alpha n} \qquad \text{where } \alpha = \log_2(7)/6 \approx 0.468$$

The goal:
combine tensor contraction
& stabilizer decompositions
using the ZX-calculus.

ZX-diagrams

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Z-spider	X-spider
$\vert 0 \cdots 0\rangle\langle 0 \cdots 0\vert$	$\vert + \cdots +\rangle\langle + \cdots +\vert$
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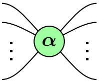	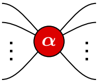

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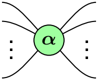
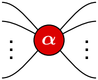
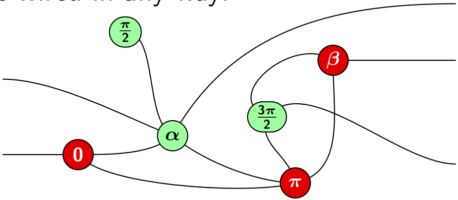
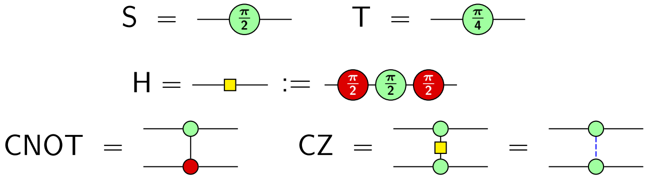
	

Spiders can be wired in any way:



Quantum gates as ZX-diagrams

Every quantum gate can be written as a ZX-diagram:



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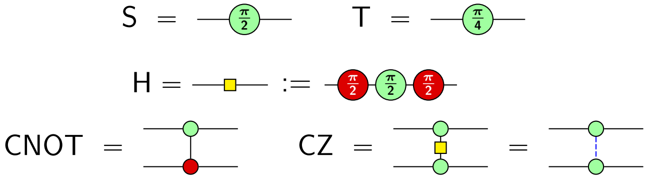
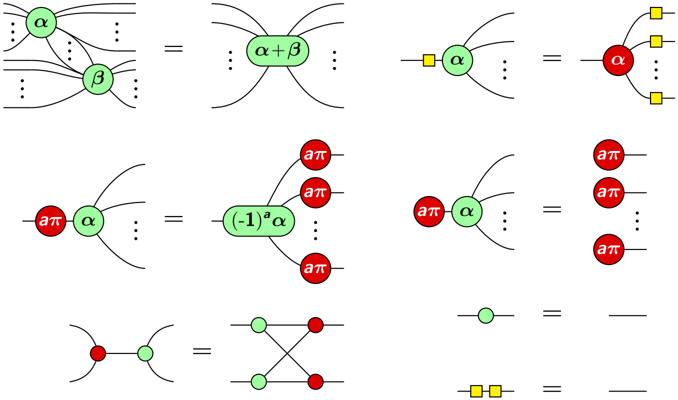


Universality

Any linear map between qubits can be represented as a ZX-diagram.

Rules for ZX-diagrams: The ZX-calculus



$\alpha, \beta \in [0, 2\pi]$, $a \in \{0, 1\}$

Completeness of the ZX-calculus

Theorem

ZX-diagrams representing same linear map,
can be transformed into one another
using previous rules (and some additional ones).

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4. Repeat.
5. ...
6. Profit!

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Same as in previous talk
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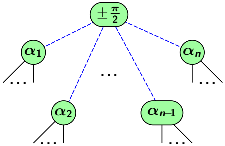
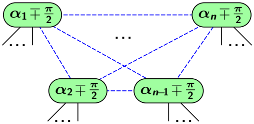
But:

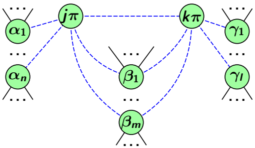
- All rewrites now need to be scalar accurate.
- We no longer care about circuit extraction, so we can do more stuff!

Scalar-accurate local complementation and pivot

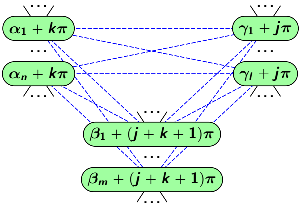


$$= e^{\pm i\pi/4}\sqrt{2}^{\frac{(n-1)(n-2)}{2}}$$





$$= \begin{array}{c} (-1)^{ab}\sqrt{2}^{(n-1)m} \\ \sqrt{2}^{(l-1)m} \\ \sqrt{2}^{(n-1)(l-1)} \end{array}$$



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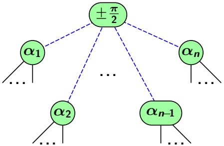

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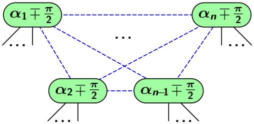

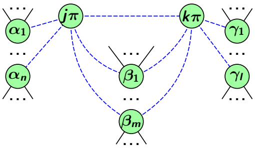

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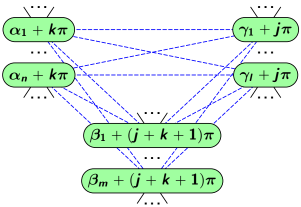
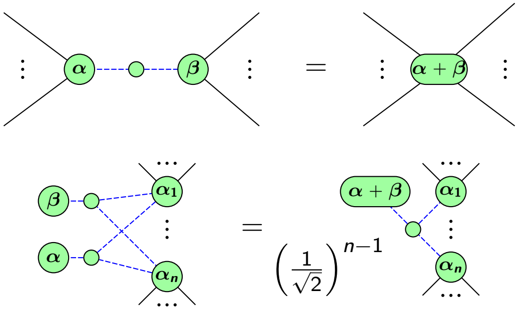


These + variations kill all internal Clifford spiders.

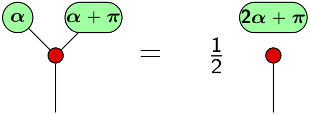
Further optimization

From previous talk:



New rule: Supplementarity

Rule used in ZX for completeness:



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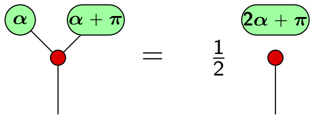
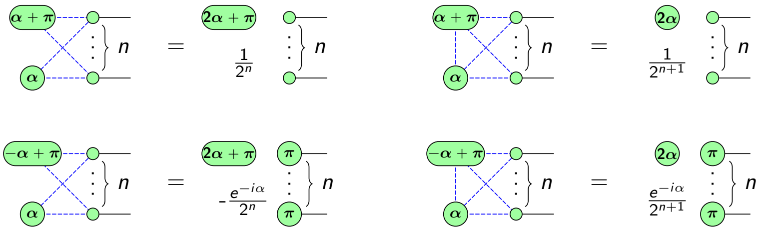
Rule used in ZX for completeness: 

Can be generalised to following four cases:



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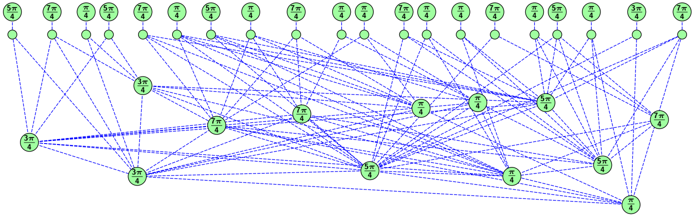
After PyZX simplification: 75 T-gates.
Inputting the state $|++---+-\rangle$ and effect $\langle +011-1-|$,
and further simplifying gives (up to scalar):



This has 33 T-gates.

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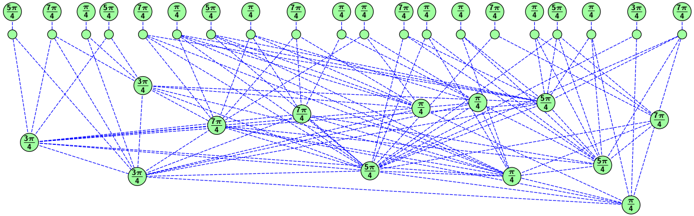

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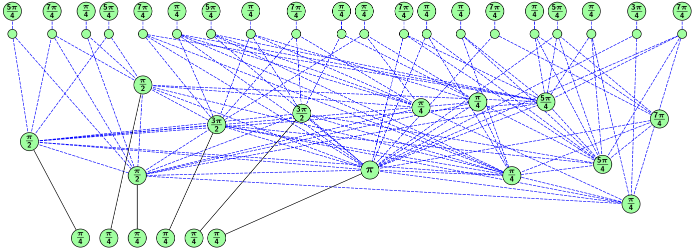
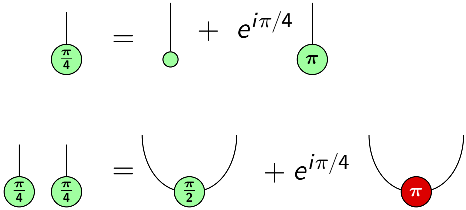


Now we should apply the stabilizer decomposition to these states.

Stabilizer decompositions in ZX



But what about the 6 T-gate rank 7 decomposition?

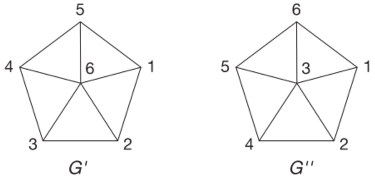


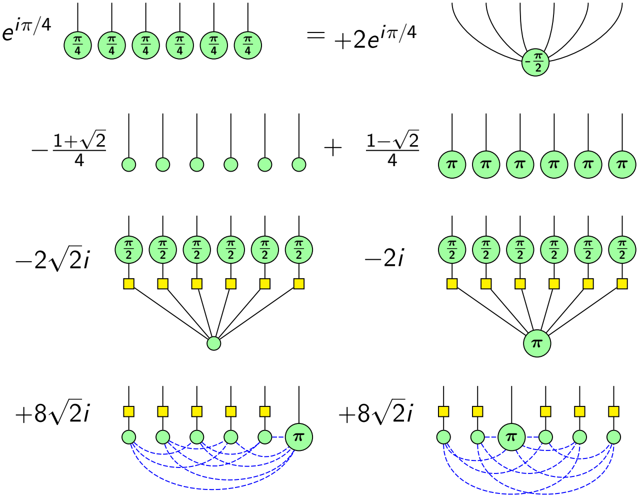
FIG. 3. Graphs G' and G'' used in the definition of stabilizer states ϕ' and ϕ''; see Eq. (11).

$$
\begin{aligned}
|H^{\otimes 6}\rangle = {} & (-16 + 12\sqrt{2})|B_{6,0}\rangle + (96 - 68\sqrt{2})|B_{6,6}\rangle \\
& + (10 - 7\sqrt{2})|E_6\rangle + (-14 + 10\sqrt{2})|O_6\rangle \\
& + (7 - 5\sqrt{2})Z^{\otimes 6}|K_6\rangle + (10 - 7\sqrt{2})|\phi'\rangle \\
& + (10 - 7\sqrt{2})|\phi''\rangle,
\end{aligned} \tag{11}
$$

where

$$
|\phi'\rangle = \prod_{(i,j)\in E'} \Lambda(Z)_{i,j}|O_6\rangle \quad \text{and} \quad |\phi''\rangle = \prod_{(i,j)\in E''} \Lambda(Z)_{i,j}|O_6\rangle.
$$

Source: Sergey Bravyi, Graeme Smith, and John A Smolin.
Trading classical and quantum computational resources (2016).


$e^{i\pi/4}$
$\frac{\pi}{4}$ $\frac{\pi}{4}$ $\frac{\pi}{4}$ $\frac{\pi}{4}$ $\frac{\pi}{4}$ $\frac{\pi}{4}$
$= +2e^{i\pi/4}$
$-\frac{\pi}{2}$
$-\frac{1+\sqrt{2}}{4}$
$+\ \frac{1-\sqrt{2}}{4}$
π π π π π π
$-2\sqrt{2}i$
$\frac{\pi}{2}$ $\frac{\pi}{2}$ $\frac{\pi}{2}$ $\frac{\pi}{2}$ $\frac{\pi}{2}$ $\frac{\pi}{2}$
$-2i$
$\frac{\pi}{2}$ $\frac{\pi}{2}$ $\frac{\pi}{2}$ $\frac{\pi}{2}$ $\frac{\pi}{2}$ $\frac{\pi}{2}$
π
$+8\sqrt{2}i$
π
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π

Demo time

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- With rewriting we can further reduce amount of non-Cliffords in each sub-diagram.
- Even removing just 1 extra spider in every diagram would allow $\approx 15\%$ bigger circuits.

Future work

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- Marginal probabilities possible with CPM construction. Is there a better way?

Thank you for your attention!

github.com/Quantomatic/pyzx   zxcalculus.com/pyzx