String Diagrams for Cartesian Restriction Categories

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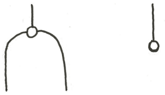
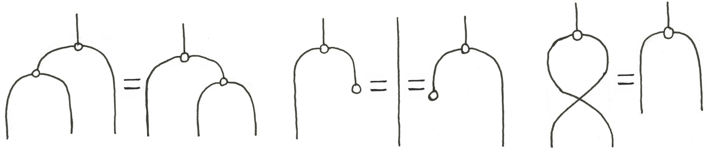
Strings for Products

A well-known happy coincidence of structure (Fox 1976) is that a category with products is the same thing as a symmetric monoidal category in which for each A there are maps $\delta_A : A \to A \otimes A$ and $\varepsilon_A : A \to I$, which we draw:

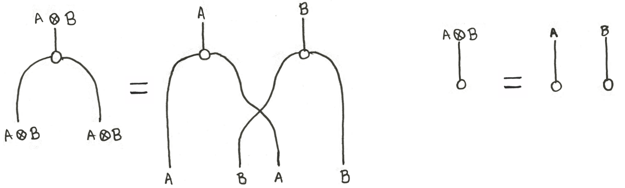


such that **(i)** each $(A, \delta_A, \varepsilon_A)$ is a cocommutative comonoid:

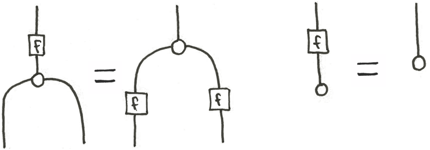


Strings for Products

(ii) the δ and ε maps are *uniform*:

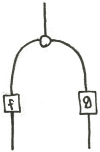


(iii) the δ and ε maps are *natural*:



Strings for Products

Then the product of A and B is $A \otimes B$, the pairing map $\langle f, g \rangle$ is:

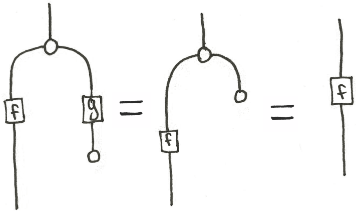


and the projection maps π_0, π_1 are:

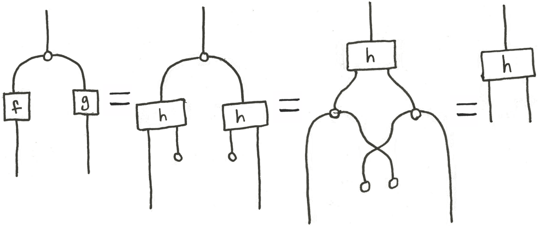
The terminal object is I, with $!_A = \varepsilon_A : A \to I$.

Strings for Products

We have $\langle f, g \rangle \pi_0 = f$ (and similarly $\langle f, g \rangle \pi_1 = g$) by:



For uniqueness, if $h\pi_0 = f$ and $h\pi_1 = g$, we have $\langle f, g \rangle = h$ by:



Restriction Categories

A *restriction category* is a category in which every map $f : X \to Y$ has a *domain of definition* $\overline{f} : X \to X$ satisfying:

- **[R.1]** $\overline{f}\,f = f$
- **[R.2]** $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$
- **[R.3]** $\overline{\overline{f}\,g} = \overline{f}\,\overline{g}$
- **[R.4]** $f\overline{g} = \overline{fg}\,f$

Restriction categories are *categories of partial maps*, where $\overline{f}$ tells us which part of its domain f is defined on (Cockett and Lack 2002).

For example, sets and partial functions form a restriction category, with $\overline{f}(x) = x$ if $f(x)\downarrow$, and $\overline{f}(x)\uparrow$ otherwise.

Restriction Categories

Each homset in a restriction category is a partial order. For $f, g : X \to Y$ say $f \leq g \Leftrightarrow \overline{f} g = f$. (In fact, poset enriched).

A map $f : X \to Y$ in a restriction category $\mathbb{X}$ is called *total* in case $\overline{f} = 1_X$. The total maps of a restriction category form a subcategory, total$(\mathbb{X})$.

Notice that if g is total, then $\overline{f} = \overline{f}\,1 = \overline{f}\,\overline{g} = \overline{f\overline{g}} = \overline{fg}$. If a restriction category $\mathbb{X}$ has products, the projections are total, so $\overline{f} = \overline{\langle f, 1 \rangle} = \overline{\langle f, 1 \rangle \pi_1} = \overline{1} = 1$, and the restriction structure is necessarily trivial (every map is total).

We want limits *and* restriction structure, so we usually work with "restriction limits".

Cartesian Restriction Categories

A restriction category has *restriction products* in case for every pair A, B of objects there is an object $A \times B$ together with total maps $\pi_0 : A \times B \to A$, $\pi_1 : A \times B \to B$ such that whenever we have maps $f : C \to A$ and $g : C \to B$, there is a unique map $\langle f, g\rangle : C \to A \times B$ with $\langle f, g\rangle \pi_0 = \overline{g} f$ and $\langle f, g\rangle \pi_1 = \overline{f} g$.

$$
\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[d, dotted, "\langle f,g\rangle"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] \arrow[ul, phantom, "\geq"] \arrow[ur, phantom, "\leq"] & B
\end{tikzcd}
$$

A restriction category has a *restriction terminal object*, 1, in case for each object A there is a unique total map $!_A : A \to 1$ such that for all $f : A \to B$, $f!_B \leq !_A$.

A restriction category with both of these is called a *cartesian restriction category*.

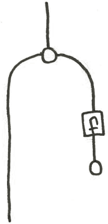
In another happy coincidence of structure (Curien and Obtulowicz 1989), a cartesian restriction category is the same thing as a symmetric monoidal category in which for each A there are maps $\delta_A : A \to A \otimes A$ and $\varepsilon_A : A \to I$ such that

(i) each $(A, \delta_A, \varepsilon_A)$ is a cocommutative comonoid,

(ii) the δ and ε maps are uniform,

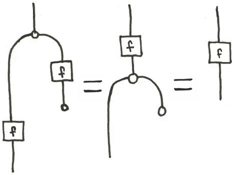
(iii) the δ maps (*but not necessarily the ε maps*) are natural.

For $f : A \to B$ the domain of definition $\overline{f} : A \to A$ is given by:

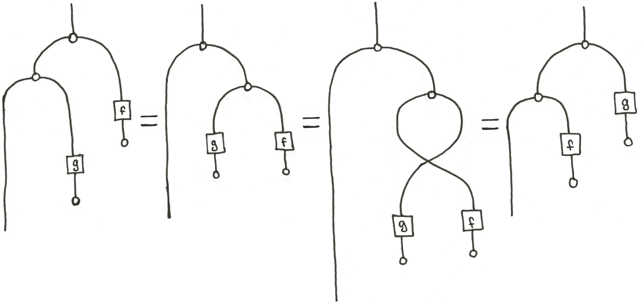


Strings for Cartesian Restriction Categories

We show the restriction axioms hold, beginning with $\overline{f}f = f$:

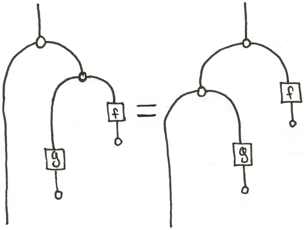


$\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$:

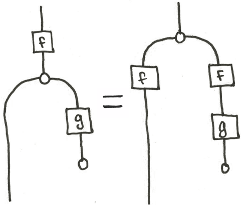


Strings for Cartesian Restriction Categories

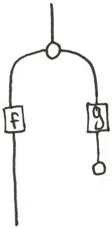
$\overline{\overline{f}\, g} = \overline{f}\,\overline{g}$:



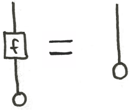
and finally $f\overline{g} = \overline{fg}\,f$:



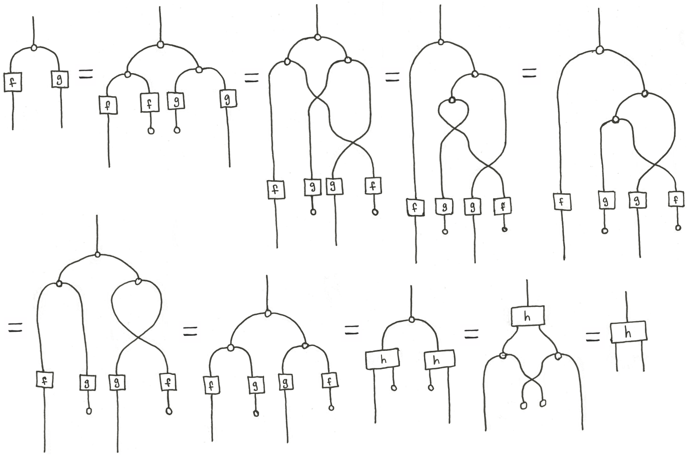
So we have a restriction category. The restriction product of A, B is $A \otimes B$, with the pairing an projection maps the same as they were for products. Notice that $\langle f, g \rangle \pi_0$ is exactly $\overline{g} f$:



Further, a map f is total if and only if



Uniqueness is slightly more involved. If $h\pi_0 = \overline{g}f$ and $h\pi_1 = \overline{f}g$ then $\langle f, g \rangle = h$ by:



Discrete Cartesian Restriction Categories

A *partial inverse* of $f : A \to B$ in a restriction category is a map $f^{(-1)} : B \to A$ such that $ff^{(-1)} = \overline{f}$ and $f^{(-1)}f = \overline{f^{(-1)}}$.

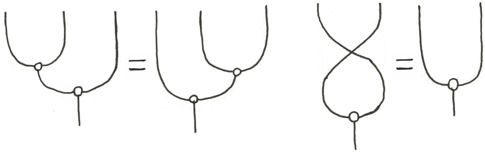
A cartesian restriction category is said to be *discrete* in case for each object A, $\delta_A : A \to A \otimes A$ has a partial inverse.

Discrete cartesian restriction categories are the partial analogue of categories with finite limits. For example, sets and partial functions is a discrete cartesian restriction category with $\delta_A^{(-1)} : A \otimes A \to A$ defined by:

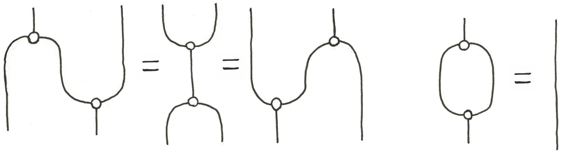
$$\delta_A^{(-1)}(x, y) = \begin{cases} x & \text{if } x = y \\ \uparrow & \text{otherwise} \end{cases}$$

Our next happy coincidence of structure is that a discrete cartesian restriction category is the same thing as a symmetric monoidal category in which for each A there are maps $\delta_A : A \to A \otimes A$, $\varepsilon_A : A \to I$, and $\mu_A : A \otimes A \to A$, which we draw:

such that **(i)** each $(A, \delta_A, \varepsilon_A)$ is a cocommutative comonoid.
(ii) each (A, μ_A) is a commutative semigroup:



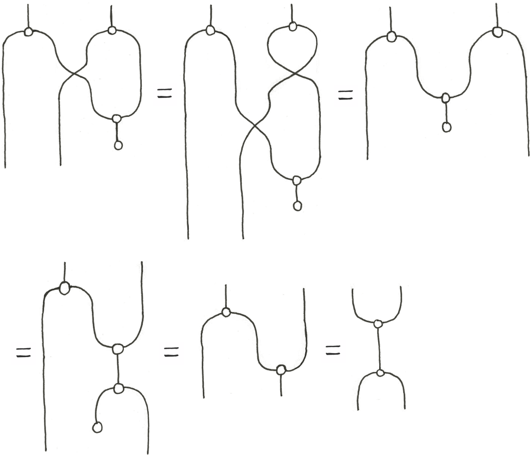
(iii) the δ, ε, and μ maps are uniform.
(iv) the δ maps are natural.
(v) each (A, δ_A, μ_A) is a special semi-frobenius algebra:



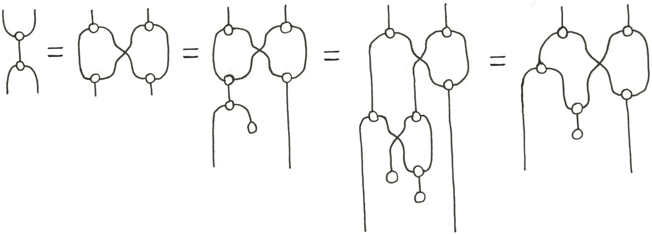
That every discrete cartesian restriction category has this structure with $\delta_A = \Delta_A = \langle 1_A, 1_A \rangle$, $\varepsilon_A =!_A : A \to I$, and $\mu_A = \Delta_A^{(-1)}$ was shown in (Giles 2014). We show both directions ...

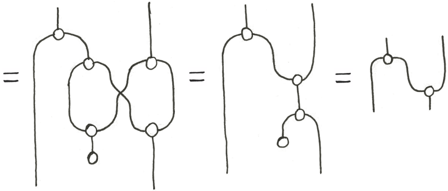
Strings for Discrete Cartesian Restriction Categories

We already know that such a symmetric monoidal category is a cartesian restriction category. The specialness condition says exactly that $\Delta\Delta^{(-1)} = \overline{\Delta} = 1$, so to show that it is discrete we only need that $\overline{\Delta^{(-1)}} = \Delta^{(-1)}\Delta$, which we have by:



Conversely, in a discrete cartesian restriction category we have $\Delta^{(-1)}\Delta = (\Delta \times 1)(1 \times \Delta^{(-1)})$ (and it's mirror) by:



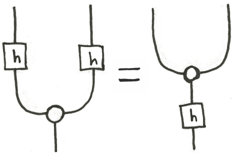


Strings for Discrete Cartesian Restriction Categories

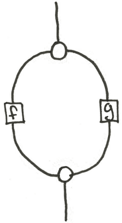
A map $h : A \to B$ in a restriction category is *partial monic* in case for any maps $f, g : C \to A$, if $fh = gh$, then $f\overline{h} = g\overline{h}$.

These maps are important. For example, a *partial topos* is a discrete cartesian closed restriction category in which every partial monic has a partial inverse (Curien and Obtulowicz 1989).

In a discrete cartesian restriction category, h is partial monic if and only if:



Every discrete cartesian restriction category has meets. For every $f, g : A \to B$ there is a map $f \wedge g : A \to B$ satisfying the meet axioms with respect to $\leq$. Define $f \wedge g$ by:

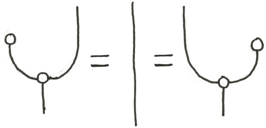


In fact, a cartesian restriction category is discrete *if and only if* it has meets. Further, the meet determines the ordering:

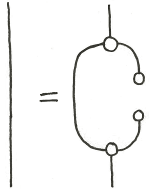
$$f \leq g \Leftrightarrow f \wedge g = f$$

Frobenius Algebras Force Compatibility

A natural question to ask is what happens to a discrete cartesian restriction category when we have a uniform family of maps $\eta_A : I \to A$ such that each (A, μ_A, η_A) is a monoid:

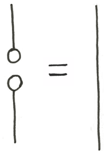


This is equivalent to asking that each $(A, \delta_A, \varepsilon_A, \mu_A, \eta_A)$ is a commutative special frobenius algebra, in which case we have:

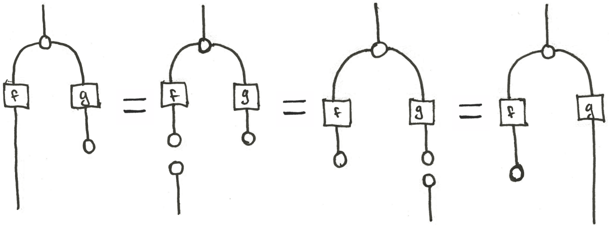
$1_A =$  $= 1_A \cap \varepsilon_A \eta_A \leq \varepsilon_A \eta_A$

Frobenius Algebras Force Compatibility

In a restriction category $1_A \leq f \Rightarrow 1_A = f$, so in fact we have:

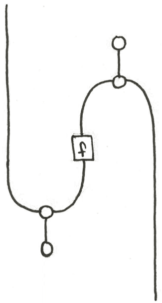


which gives $\overline{f}g = \overline{g}f$ for any parallel maps f, g:

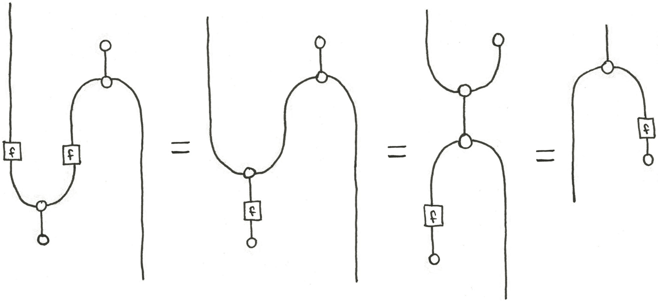


that is, we have a *restriction preorder*.

The η maps also allow the construction of a partial inverse for any partial monic f:

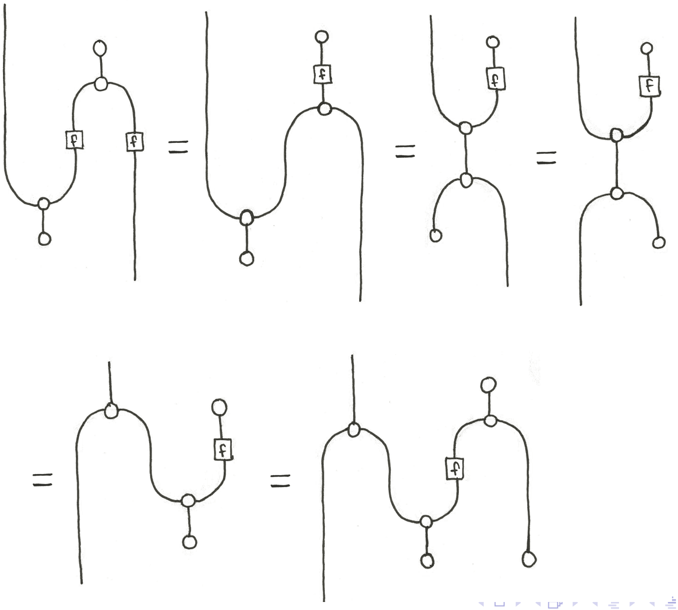


we have $f f^{(-1)} = \overline{f}$ by:

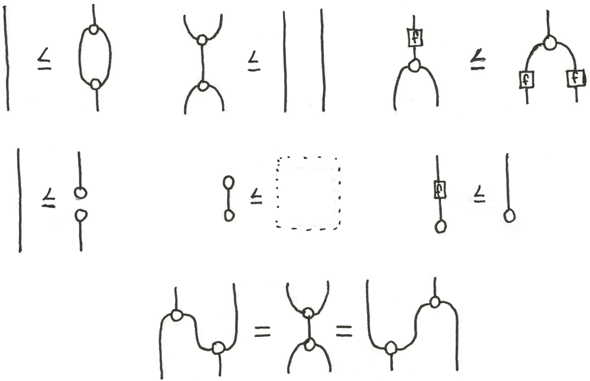


and $f^{(-1)} f = \overline{f^{(-1)}}$ by: ...

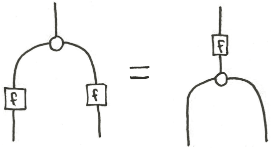
Frobenius Algebras Invert Partial Monics



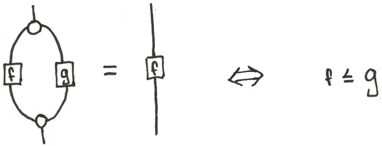
A *cartesian bicategory of relations* (Carboni and Walters 1987) is a poset-enriched symmetric monoidal category in which every object has commutative monoid and comonoid structure satisfying:



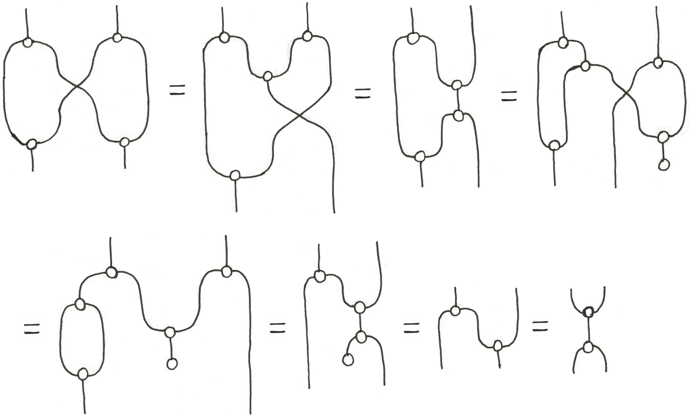
A morphism f in a cartesian bicategory of relations is *deterministic* in case



Cartesian bicategories of relations have meets, defined as in discrete cartesian restriction categories, and the meet determines the ordering on each hom-poset:



We show that the mulitplication of each monoid is deterministic:



It follows that the deterministic maps of a cartesian bicategory of relations form a discrete cartesian restriction category, and the two poset-enrichments coincide.

Range Restriction Categories

A *range restriction category* is a restriction category in which every map $f : X \to Y$ has a *range* $\widehat{f} : Y \to Y$ satisfying:

[RR.1] $\overline{\widehat{f}} = \widehat{f}$

[RR.2] $f\widehat{f} = f$

[RR.3] $\widehat{f\overline{g}} = \widehat{f}\,\overline{g}$

[RR.4] $\widehat{\widehat{f}g} = \widehat{fg}$

The range tells us which part of its codomain f maps something to. (Cockett and Manes 2009).

For example, in sets and partial functions we can define the range of $f : X \to Y$ by

$$\widehat{f}(y) = \begin{cases} y & \text{if } \exists x \in X. f(x) = y \\ \uparrow & \text{otherwise} \end{cases}$$

Range Restriction Categories

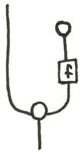
A *(discrete) cartesian range restriction category* is a (discrete) cartesian restriction category with ranges satisfying

$$\widehat{f} \times \widehat{g} = \widehat{f \times g}$$

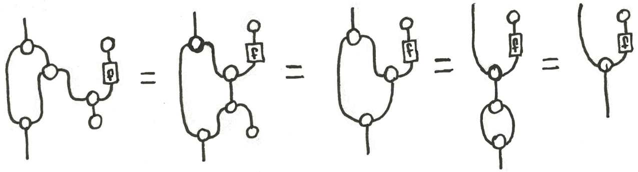
A *regular* restriction category is a discrete cartesian range restriction category in which every partial monic has a partial inverse.

Regular restriction categories are the partial analogue of regular categories. (Cockett, Guo, and Hofstra 2012).

The category of determinsitic maps in a cartesian bicategory of relations has ranges. $\widehat{f}$ is defined by:

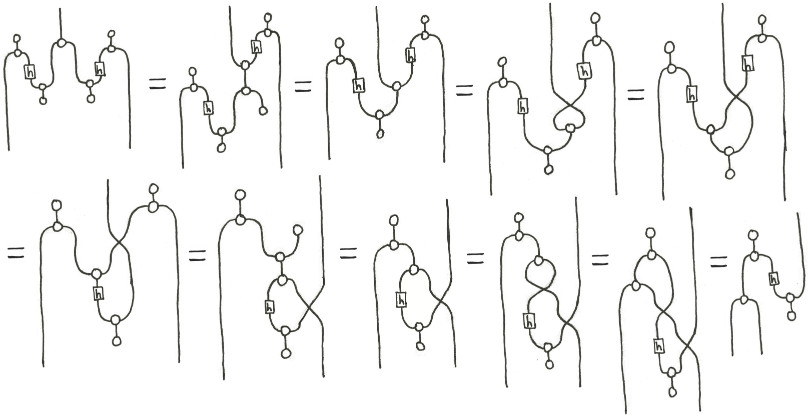


This is deterministic because every map f with $f \leq 1$ is necessarily deterministic, and we have $\widehat{f} \wedge 1 = \widehat{f} \Rightarrow \widehat{f} \leq 1$ by:



Partial Monics Have Deterministic Inverses

In a cartesian bicategory of relations we may construct a partial inverse to any partial monic h as before. We show that $h^{(-1)}$ is deterministic:



Cartesian Bicategories of Relations

Thus, the deterministic maps of a cartesian bicategory of relations form a regular restriction category, which we can reason about with string diagrams

This is particularily exciting as it pertains to the interaction between the domain of definition and range. The combination of $\overline{(\_)}$ and $\widehat{(\_)}$ syntax is unpleasant to work with.

It is almost certainly the case that every regular restriction category arises this way (ongoing work).

Future Work: Joins

A pair of parallel maps $f, g : A \to B$ in a restriction category is said to be *compatible*, written $f \smile g$, in case $\overline{f}g = \overline{g}f$.

A restriction category has *finite joins* in case

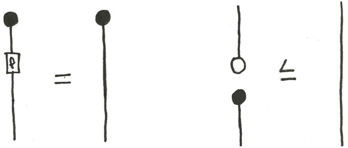
(i) It has *restriction zero maps*: for each A, B there is a map $0_{A,B} : A \to B$ such that $\overline{0_{A,B}} = 0_{A,A}$, and for any $f : A \to B$, $g : C \to D$, $0_{A,B} \leq f$ and $f 0_{B,C} g = 0_{A,D}$.

(ii) Every pair $f, g : A \to B$ of compatible maps has a *join*, $f \vee g$, which is a join with respect to the canonical ordering, and satisfies $h(f \vee g) = (hf \vee hg)$.

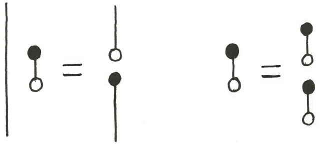
Restriction categories with joins seem to be of central importance in abstract computability.

Future Work: Joins

With JS Lemay: A discrete cartesian restriction category has zero maps if any only if there is a family of maps $z : I \to X$ such that:



The following consequences give a bit more intuition:



Future Work: Joins

Is it possible to derive string diagrams for join restriction categories?

If so, probably only in the presence of additional structure.

In (Bonchi, Pavlovic and Sobociński 2017), the "frobenius theory of commutative monoids" is considered. Models have joins, and joins have a nice diagrammatic representation.

Can we construct similar bicategories of relations whose deterministic maps are regular restriction categories *with joins*?

Other Future Work

Finish story about cartesian bicategories of relations and regular restriction categories. (e.g. tabular corresponds to split, do we get anything when partial monics don't always invert?).

Keep eyes peeled for commutative nonunital special frobenius algebras in nature. One example in infnite dimensional quantum computing thing (Heunen and Abramsky 2011). Do restriction categories say anything interesting here? Are there any other cases like this?

I'd also like to investigate versions of exact and regular completions for categories of partial maps from the viewpoint of cartesian bicategories of relations.

Thanks for listening!