Models of Classical Linear Logic via Bifibrations of Polycategories

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Outline

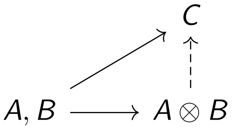
1. Multicategories and Monoidal categories
2. Opfibration of Multicategories
3. Polycategories and Linearly Distributive Categories
4. Bifibration of polycategories

Outline

1. Multicategories and Monoidal categories

Tensor product of vector spaces

- In linear algebra: universal property



- In category theory as a structure: a monoidal product $\otimes$
- Universal property of tensor product needs many-to-one maps
- Category with many-to-one maps $\Rightarrow$ Multicategory

Multicategory[1]

Definition

A *multicategory* $\mathcal{M}$ has:

- A collection of objects
- Γ finite list of objects and A objects
  Set of multimorphisms $\mathcal{M}(\Gamma; A)$
- Identities $id_A : A \to A$
- Composition: $\dfrac{f : \Gamma \to A \qquad g : \Gamma_1, A, \Gamma_2 \to B}{g \circ_i f : \Gamma_1, \Gamma, \Gamma_2 \to B}$
- With usual unitality and associativity and:
  interchange law: $(g \circ f_1) \circ f_2 = (g \circ f_2) \circ f_1$
  where f_1 and f_2 are composed in two different inputs of g

[1] Tom Leinster. *Higher Operads, Higher Categories.* 2004.

Representable multicategory

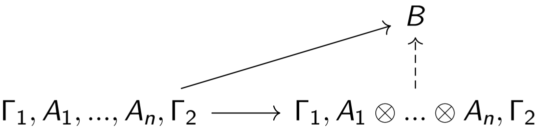
Definition

A multimorphism $u : \Gamma \to A$ is *universal* if any multimap $f : \Gamma_1, \Gamma, \Gamma_2 \to B$ factors uniquely through u.

Definition

A multicategory is *representable* if for any finite list $\Gamma = (A_i)$ there is a universal map $\Gamma \to \bigotimes A_i$.

$$\frac{\Gamma_1, A_1, \ldots, A_n, \Gamma_2 \to B}{\Gamma_1, A_1 \otimes \ldots \otimes A_n, \Gamma_2 \to B}$$



Representable multicategories and Monoidal categories

Let $\mathcal{C}$ be a monoidal category.
There is an underlying representable multicategory $\overrightarrow{\mathcal{C}}$ whose:

- objects are the objects of $\mathcal{C}$
- multimorphisms $f : A_1, ..., A_n \to B$ are morphisms $f : A_1 \otimes ... \otimes A_n \to B$ in $\mathcal{C}$

Conversely any representable multicategory is the underlying multicategory of some monoidal category.

Finite dimensional vector spaces and multilinear maps

Theorem

The multicategory $\overrightarrow{\mathbf{FVect}}$ *of finite dimensional vector spaces and multilinear maps is representable.*

Definition

For normed vect. sp. $(A_i, \| - \|_{A_i}), (B, \| - \|_B)$, $f : A_1, \ldots, A_n \to B$ is *short/contractive* if for any $\vec{x} = x_1, \ldots, x_n$, $\|f(\vec{x})\|_B \leq \prod_i \|x_i\|_{A_i}$

Theorem

The multicategory $\overrightarrow{\mathbf{FBan_1}}$ *of finite dimensional Banach spaces and short multilinear maps is representable.*
Its tensor product is equipped with the projective crossnorm.

Projective crossnorm[2]

Definition

Given two normed vector spaces $(A, \|-\|_A)$ and $(B, \|-\|_B)$ we defined a normed on $A \otimes B$ called the *projective crossnorm* as follows:

$$\|u\|_{A\otimes B} = \inf_{u=\sum_i a_i \otimes b_i} \sum_i \|a_i\|_A \|b_i\|_B$$

Proposition

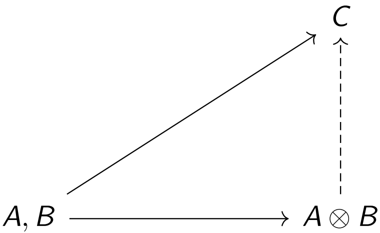
Any (well-behaved) norm $\|-\|$ on $A \otimes B$ is smaller than the projective one:

$$\|u\| \leq \|u\|_{A\otimes B}, \; \forall u \in A \otimes B$$

How does this related to the fact that it is the norm of the tensor product?

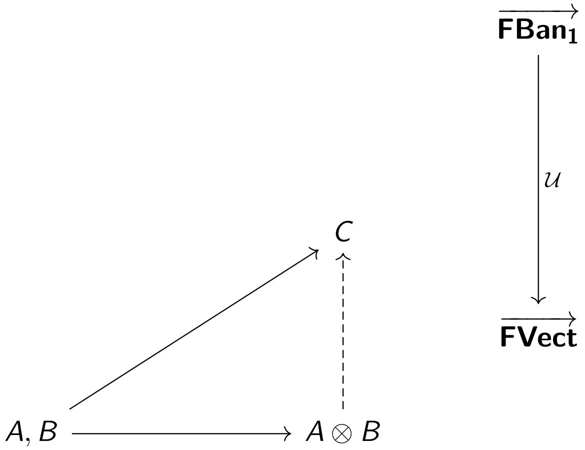
[2] Raymond A. Ryan. *Introduction to Tensor Products of Banach Spaces*. 2002.

Lifting the tensor product of $\overrightarrow{\mathbf{FVect}}$ to $\overrightarrow{\mathbf{FBan_1}}$



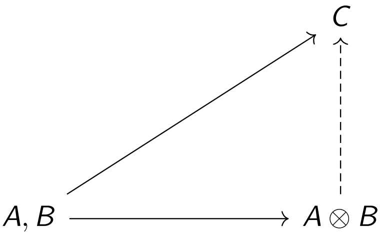
$\overrightarrow{\mathbf{FVect}}$

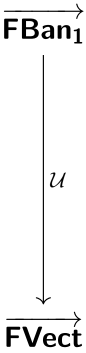
Lifting the tensor product of $\overrightarrow{\mathbf{FVect}}$ to $\overrightarrow{\mathbf{FBan_1}}$



Lifting the tensor product of $\overrightarrow{\mathbf{FVect}}$ to $\overrightarrow{\mathbf{FBan_1}}$

$\| - \|_A, \| - \|_B$





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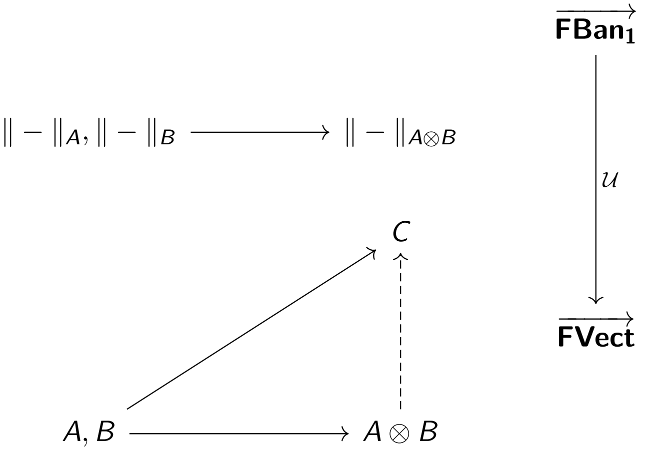

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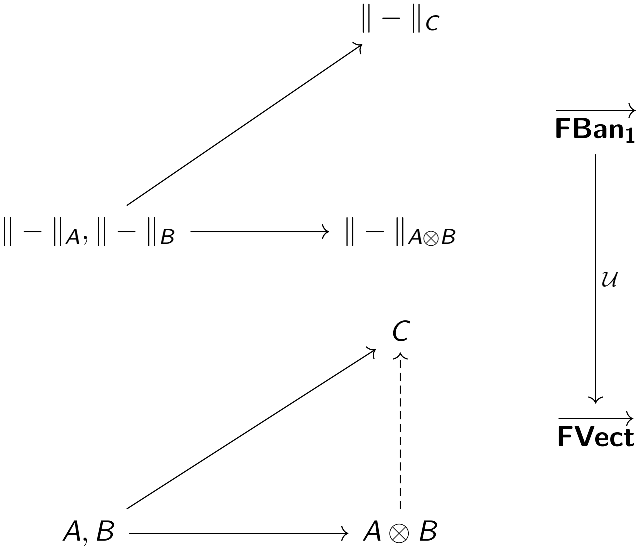

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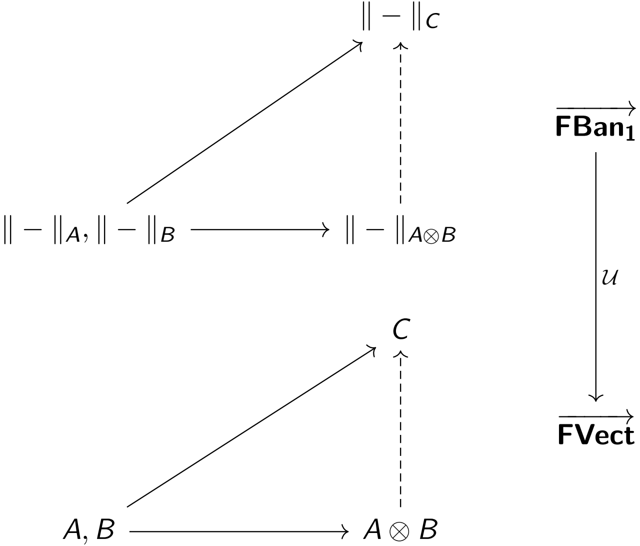

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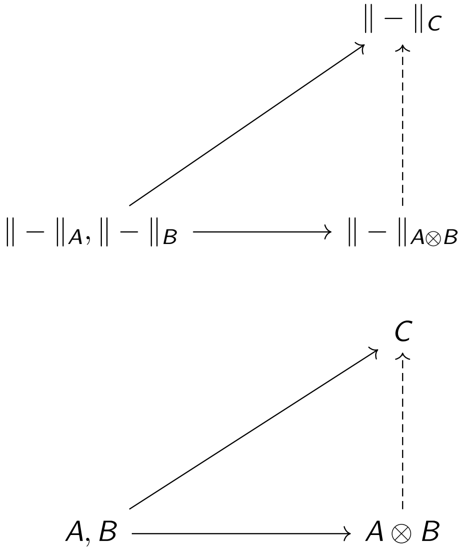


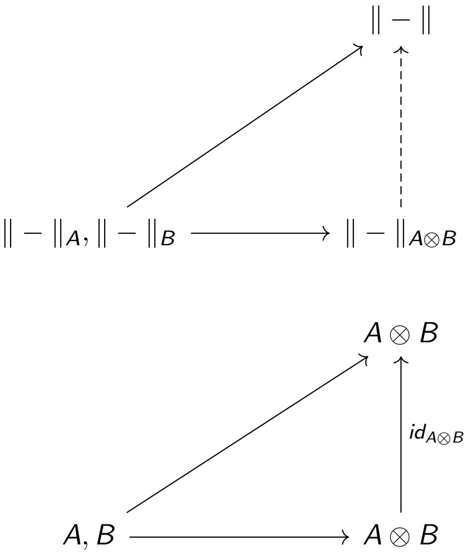
$\overrightarrow{\mathbf{FBan_1}}$

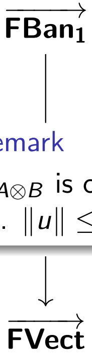
Remark

$\|-\|, \|-\| \to \|-\|_{A\otimes B}$ opcartesian lifting

$\overrightarrow{\mathbf{FVect}}$

Lifting the tensor product of $\overrightarrow{\mathbf{FVect}}$ to $\overrightarrow{\mathbf{FBan_1}}$





Remark

$id_{A \otimes B}$ is contractive, i.e. $\|u\| \leq \|u\|_{A \otimes B}$

Outline

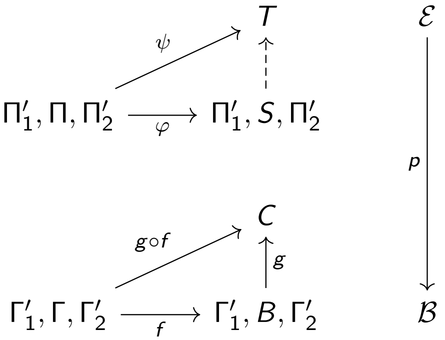
2 Opfibration of Multicategories

Opcartesian multimorphism[3]

$p : \mathcal{E} \to \mathcal{B}$ functor between multicategories

Definition

$\varphi : \Pi \to S$ *opcartesian* if for any multimorphism $\psi : \Pi'_1, \Pi, \Pi'_2 \to T$ lying over $g \circ f$ there is a unique multimorphism $\xi : \Pi'_1, S, \Pi'_2 \to T$ over g such that $\psi = \xi \circ \varphi$.



[3] Claudio Hermida. "Fibrations for abstract multicategories". In: (2004).

Opfibration of multicategories

Definition

A functor $p : \mathcal{E} \to \mathcal{B}$ between multicategories is an *opfibration* if for any multimap $f : \Gamma \to B$ and any Π over Γ there is an object $push_f(\Pi)$ over B and an opcartesian multimorphism $\Pi \to push_f(\Pi)$ lying over f.
$push_f(\Pi)$ is called the pushforward of Π along f.

$$\Pi \qquad -$$

$$\Gamma \xrightarrow[f]{} B$$

Opfibrations lift logical conjunction

Theorem

A multicategory opfibred over a representable one is representable.

Opfibrations lift logical conjunction

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Unfortunately

The forgetful functor $\mathcal{U} : \overrightarrow{\mathbf{FBan_1}} \to \overrightarrow{\mathbf{FVect}}$ is **not** an opfibration.

However it has "enough" opcartesian multimorphism to lift the universal property of $\otimes$.

Opfibrations lift logical conjunction

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A multicategory opfibred over a representable one is representable.

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The forgetful functor $\mathcal{U} : \overrightarrow{\mathbf{FBan_1}} \to \overrightarrow{\mathbf{FVect}}$ is **not** an opfibration.

Proposition

A linear map f (i.e a unary multimorphism in $\overrightarrow{\mathbf{FVect}}$) has opcartesian liftings in $\overrightarrow{\mathbf{FBan_1}}$ if it is surjective.

However it has "enough" opcartesian multimorphism to lift the universal property of $\otimes$.

Fibrational properties of $\otimes$

Proposition

Opcartesian lifting of universal multimorphisms are universal.

Conceptually this comes from the following fact:

Fibrational properties of $\otimes$

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Theorem

A multicategory $\mathcal{P}$ is a representable iff $! : \mathcal{P} \to \mathbb{1}$ is an opfibration.
A multimorphism is universal if it is !-opcartesian.

Definition

The terminal multicategory $\mathbb{1}$ has:

- one object $*$
- one multimorphism $\underline{n} : *^n \to *$ for each arity n

Fibrational properties of $\otimes$

Proposition

Opcartesian lifting of opcartesian multimorphisms are opcartesian.

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Intuitionistic Multiplicative Linear Logic

The multiplicative conjunction $\otimes$ can be seen as a:

- structure on a category: monoidal product $\otimes$
- universal property in a multicategory: universal multimorphism in $\otimes$
- fibrational property in a multicategory: $\otimes$ as a pushforward

We get something similar for $\multimap$:
Multicategories bifibred over $\mathbb{1}$ $\leftrightarrow$ Monoidal closed categories

Classical Multiplicative Linear Logic

- **FVect** and $\mathbf{FBan_1}$ linearly distributive categories
- Two monoidal products $\otimes$ and $\invamp$ interacting well
- $\otimes$ conjunction and $\invamp$ disjunction
- Models of Multiplicative Linear Logic without Negation
- In **FVect**, $\invamp = \otimes$
- In $\mathbf{FBan_1}$, $\invamp$ is the tensor product with the injective crossnorm.
- Sequents for classical MLL are many-to-many.
- We need maps $A \invamp B \to A, B$ for the universal property of $\invamp$

Outline

3 Polycategories and Linearly Distributive Categories

Polycategories[4]

Definition

A *polycategory* $\mathcal{P}$ has:

- A collection of objects
- For Γ, Δ finite list of objects, a set of polymorphisms $\mathcal{P}(\Gamma; \Delta)$
- Identities $id_A : A \to A$
- Composition: $\dfrac{f : \Gamma \to \Delta_1, A, \Delta_2 \qquad g : \Gamma_1, A, \Gamma_2 \to \Delta}{g\,{}_j\!\circ_i f : \Gamma_1, \Gamma, \Gamma_2 \to \Delta_1, \Delta, \Delta_2}$
- Planarity of $\circ$: $(\Gamma_1 = \{\} \vee \Delta_1 = \{\}) \wedge (\Gamma_2 = \{\} \vee \Delta_2 = \{\})$
- With unitality, associativity and two interchange laws

[4] M.E. Szabo. "Polycategories". In: (1975).

Two-tensor polycategory

Definition

A polymorphism $u : \Gamma \to \Delta_1, A, \Delta_2$ is *universal in its i-th variable* if any polymorphism $f : \Gamma_1, \Gamma, \Gamma_2 \to \Delta_1, \Delta, \Delta_2$ factors uniquely through u.

Definition

A *two-tensor polycategory* is a polycategory such that for any finite list $\Gamma = (A_i)$ there are a universal polymap (in its only output) $\Gamma \to \bigotimes A_i$ and a co-universal polymap (in its only input) $\bigparr A_i \to \Gamma$.

$$\frac{\Gamma_1, A_1, ..., A_n, \Gamma_2 \to \Delta}{\Gamma_1, A_1 \otimes ... \otimes A_n, \Gamma_2 \to \Delta}$$

Two-tensor polycategory

Definition

A polymorphism $c : \Gamma_1, A, \Gamma_2 \to \Delta$ is *co-universal in its i-th variable* if for any polymorphism $f : \Gamma_1, \Gamma, \Gamma_2 \to \Delta_1, \Delta, \Delta_2$ we have a unique g with $f = c\,{}_i{\circ}_j\, g$.

Definition

A *two-tensor polycategory* is a polycategory such that for any finite list $\Gamma = (A_i)$ there are a universal polymap (in its only output) $\Gamma \to \bigotimes A_i$ and a co-universal polymap (in its only input) $\bigparr A_i \to \Gamma$.

$$\frac{\Gamma \to \Delta_1, A_1, ..., A_n, \Delta_2}{\Gamma \to \Delta_1, A_1 \parr ... \parr A_n, \Delta_2}$$

Two-tensor polycategories and Linearly distributive categories[5]

Let $\mathcal{C}$ be a linearly distributive category.
There is an underlying two-tensor polycategory $\overleftrightarrow{\mathcal{C}}$ whose:

- objects are the objects of $\mathcal{C}$
- polymorphisms $f : A_1, \ldots, A_m \to B_1, \ldots, B_n$ are morphisms $f : A_1 \otimes \ldots \otimes A_m \to B_1 \parr \ldots \parr B_n$ in $\mathcal{C}$

Conversely any two-tensor polycategory is the underlying polycategory of a linearly distributive category.

[5] J.R.B. Cockett and R.A.G. Seely. "Weakly distributive categories". In: (1997).

Polycategories of f.d. vector spaces

Theorem

There are two-tensor polycategories $\overleftrightarrow{\mathbf{FVect}}$ *and* $\overleftrightarrow{\mathbf{FBan_1}}$ *of finite dimensional vector spaces/Banach spaces.*

This follows from $\mathbf{FVect}$ and $\mathbf{FBan_1}$ being linearly distributive.

Polycategories of f.d. vector spaces

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There are two-tensor polycategories $\overleftrightarrow{\mathbf{FVect}}$ *and* $\overleftrightarrow{\mathbf{FBan_1}}$ *of finite dimensional vector spaces/Banach spaces.*

This follows from **FVect** and $\mathbf{FBan_1}$ being linearly distributive.

Remark

It is possible to define these polycategories without using $⅋$ by taking a polymorphism $f : A_1, ..., A_m \to B_1, ..., B_n$ to be a (short) multilinear morphism $f : A_1 \otimes ... \otimes A_m \otimes B_1^* \otimes ... \otimes B_n^* \to \mathbb{K}$.

Injective crossnorm

Definition

Given two normed vector spaces $(A, \| - \|_A)$ and $(B, \| - \|_B)$ we can define a norm on $A \otimes B$ called the *injective crossnorm* as follows:

$$\|u\|_{A \parr B} := \sup_{\|\varphi\|_{A^*}, \|\psi\|_{B^*} \leq 1} |(\varphi \otimes \psi)(u)|$$

Proposition

For any (well-behaved) norm $\| - \|$ on $A \otimes B$ we have

$$\|x\|_{A \parr B} \leq \|x\| \leq \|x\|_{A \otimes B}$$

Outline

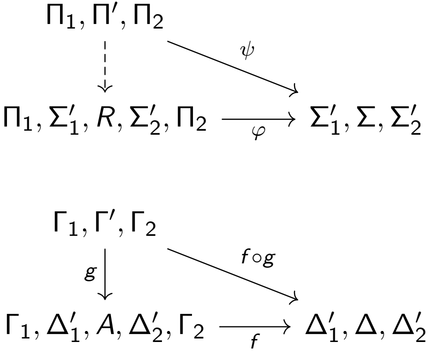
4. Bifibration of polycategories

Cartesian polymorphism

$p : \mathcal{E} \to \mathcal{B}$ between polycategories

Definition

$\varphi : \Pi_1, R, \Pi_2 \to \Sigma$ *cartesian in its i-th variable* if any polymorphism $\psi : \Pi_1, \Pi', \Pi_2 \to \Sigma_1', \Sigma, \Sigma_2'$ lying over $f\,{}_i{\circ}_j\, g$ there is a unique polymorphism $\xi : \Pi' \to \Sigma_1', R, \Sigma_2'$ over g such that $\psi = \varphi\,{}_i{\circ}_j\,\xi$.



Fibration of polycategories

Definition

A functor $p : \mathcal{E} \to \mathcal{B}$ between polycategories is a *fibration* if for any polymap $f : \Gamma_1, A, \Gamma_2 \to \Delta$, any Π_i over Γ_i and any Σ over Δ there is an object $pull_f^k(\Pi_1, \Pi_2; \Sigma)$ over A and a cartesian polymorphism $\Pi_1, pull_f^k(\Pi_1, \Pi_2; \Sigma), \Pi_2 \to \Sigma$ lying over f.
$pull_f^k(\Pi_1, \Pi_2; \Sigma)$ is called the pullback of Σ along f in context Π_1, Π_2.

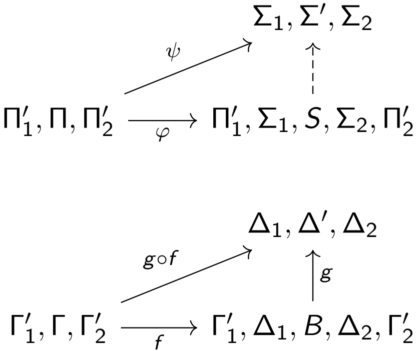
$$\Pi_1, -, \Pi_2 \qquad\qquad \Sigma$$

$$\Gamma_1, A, \Gamma_2 \xrightarrow[f]{} \Delta$$

Opcartesian polymorphism

Definition

$\varphi : \Pi_1 \to \Sigma_1, S, \Sigma_2$ *opcartesian in its i-th variable* if for any polymorphism $\psi : \Pi_1', \Pi, \Pi_2' \to \Sigma_1, \Sigma', \Sigma_2$ lying over $g\,{}_j\!\circ_i f$ there is a unique polymorphism $\xi : \Pi_1', S, \Pi_2' \to \Sigma$ over g such that $\psi = \xi\,{}_j\!\circ_i\,\varphi$.



Opfibration of polycategories

Definition

A functor $p : \mathcal{E} \to \mathcal{B}$ between polycategories is an *opfibration* if for any polymap $f : \Gamma \to \Delta_1, B, \Delta_2$, any Π over Γ and any Σ_i over Δ_i there is an object $push_f^k(\Pi; \Sigma_1, \Sigma_2)$ over B and a cartesian polymorphism $\Pi \to \Sigma_1, push_f^k(\Pi; \Sigma_1, \Sigma_2), \Sigma_2$ lying over f.
$push_f^k(\Pi; \Sigma_1, \Sigma_2)$ is called the pushforward of Π along f in context Σ_1, Σ_2.

$$\begin{array}{ccc} \Pi_1 & & \Sigma_1, -, \Sigma_2 \\ & & \\ \Gamma & \xrightarrow[f]{} & \Delta_1, B, \Delta_2 \end{array}$$

Bifibrations lift logical properties

Theorem

A polycategory bifibred over a two-tensor polycategory is a two-tensor polycategory.

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Unfortunately

The forgetful functor $\mathcal{U} : \overleftrightarrow{\mathbf{FBan_1}} \to \overleftrightarrow{\mathbf{FVect}}$ is **not** a bifibration.

However it has "enough" cartesian and opcartesian polymorphism to lift the logical properties.

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A linear map f (i.e a unary polymorphism in $\overleftrightarrow{\mathbf{FVect}}$) has cartesian (resp. opcartesian) liftings in $\overleftrightarrow{\mathbf{FBan_1}}$ if it is injective (resp. surjective).

However it has "enough" cartesian and opcartesian polymorphism to lift the logical properties.

Fibrational properties of $\otimes$ and $⅋$

Proposition

Opcartesian lifting of universal polymorphisms are universal.

Proposition

Cartesian lifting of co-universal polymorphisms are co-universal.

Conceptually this comes from the following fact:

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Theorem

A polycategory $\mathcal{P}$ is a two-tensor polycategory iff $! : \mathcal{P} \to \mathbb{1}$ is a bifibration. A polymorphism is universal if it is !-opcartesian and co-universal if it is !-cartesian.

Fibrational properties of $\otimes$ and $\parr$

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Opcartesian lifting of opcartesian polymorphisms are opcartesian.

Proposition

Cartesian lifting of cartesian polymorphisms are cartesian.

Conceptually this comes from the following fact:

Theorem

A polycategory $\mathcal{P}$ is a two-tensor polycategory iff $! : \mathcal{P} \to \mathbb{1}$ is a bifibration. A polymorphism is universal if it is !-opcartesian and co-universal if it is !-cartesian.

Conclusion

Three different ways of thinking about logical properties:

- As structures on categories
- As universal properties in polycategories
- As fibrational properties in polycategories

Further work:

- Finding other examples: Higher-order causal processes[6]
- Adding the $*$: some subtilities but possible
- Additive connectors:
  - biproducts $\oplus$ in **FVect**
  - products $\| - \|_\infty$ and coproducts $\| - \|_1$ in $\mathbf{FBan_1}$

[6] Aleks Kissinger and Sander Uijlen. "A categorical semantics for causal structure". In: (2017).