# A Combinatorial Presentation of the Operad of Plane Graphs

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# Diagrams for Monoidal Categories



- string diagrams graphical language for monoidal categories
- represented by graphs (Selinger, 2011)
  - SMC : directed acyclic graphs
  - traced : can contain cycles
  - autonomous : wires can go the other way
- diagram equality $\longrightarrow$ graph isomorphism
- equational reasoning via rewrite rules
  $\longrightarrow$ double pushout graph rewriting

# Diagrams for non-symmetric Monoidal Categories



Non-symmetric case:

- printing circuits: crossings not possible
- quantum circuits: crossing not for free
- most general case: can add structure on top

Representation:

- no crossing wires
- represented by plane graphs

Implementation:

This project is being implemented in Agda

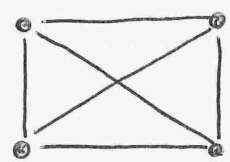
# Plane Graphs



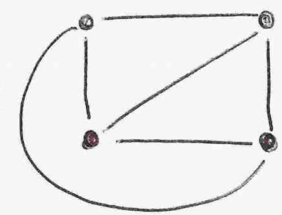
**Definition:**

- graph $G = (V, E)$ consisting of vertices and edges
- embedding of $G$: drawing of $G$ on a surface $S$
- $G$ planar if there exists an embedding into the plane without crossing edges. The embedding is called plane.

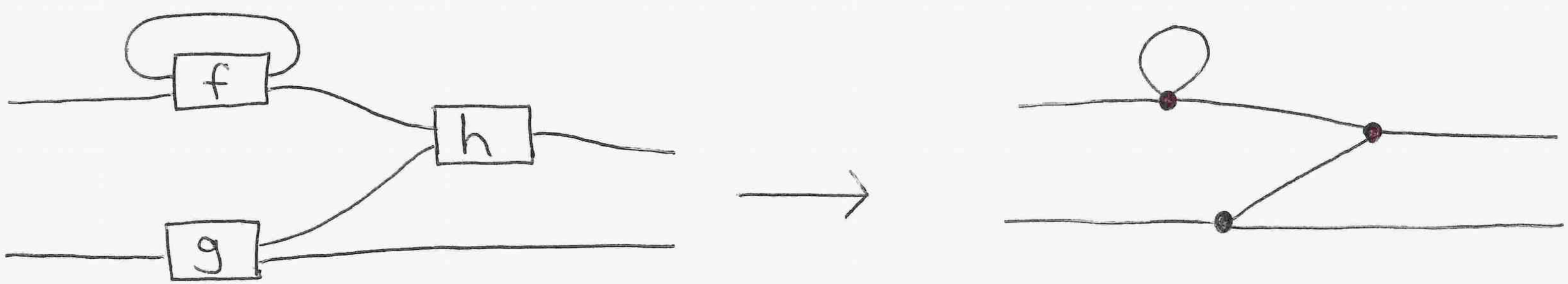
planar graph $G$:



plane embedding of $G$:

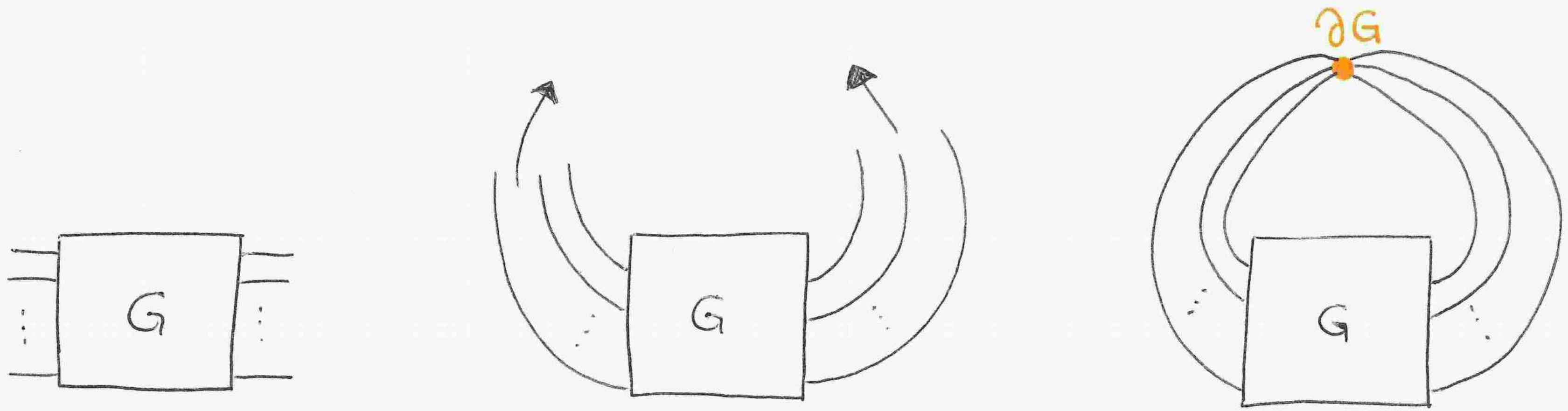


A PRO can be represented as open plane graph:



# Plane Graphs with a Boundary Vertex (1)



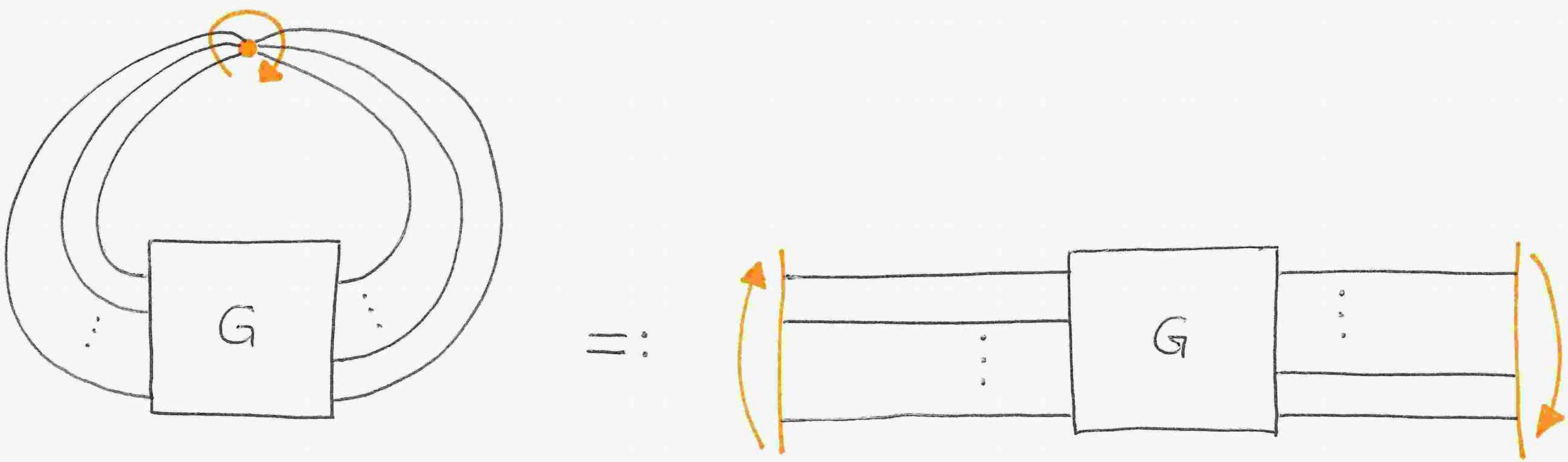



Encoding the dangling input and output wires:

- introduce a new vertex $\partial G$, the boundary vertex
- making the boundary part of the graph

# Plane Graphs with a Boundary Vertex (2)






- boundary vertex part of the plane graph
- connecting parallel graphs
- nice way to represent plane graphs combinatorically

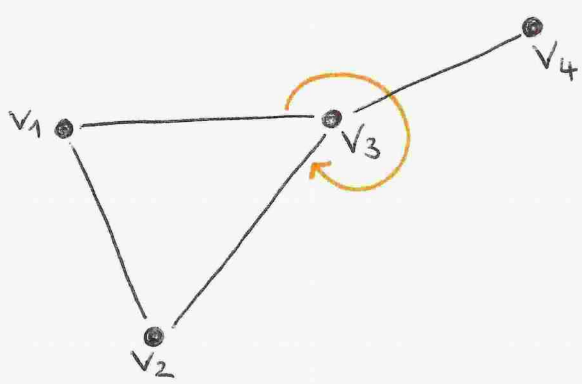
# Rotation Systems (1)



**Definition:**

- rotation of a vertex $v \in V$:
  cyclic ordered list of adjacent vertices
- rotation system: rotation for all vertices in the graph

Here: rotation in clockwise direction




$v_1$ : $v_2, v_3$

$v_2$ : $v_1, v_3$

$v_3$ : $v_1, v_4, v_2$

$v_4$ : $v_3$

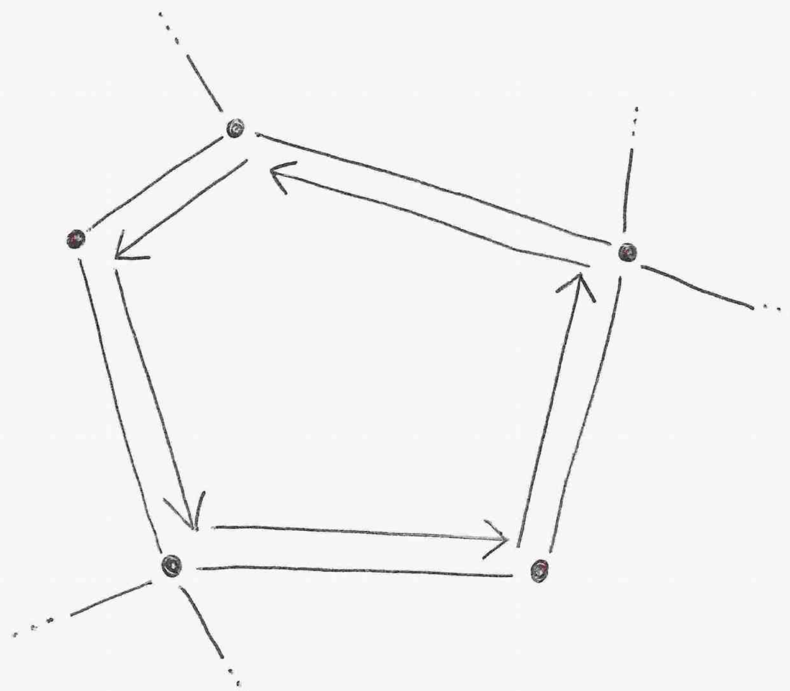
# Rotation Systems (2)



**Lemma:**

A rotation system uniquely defines a (cellular) embedding of a graph (Youngs, 1963)

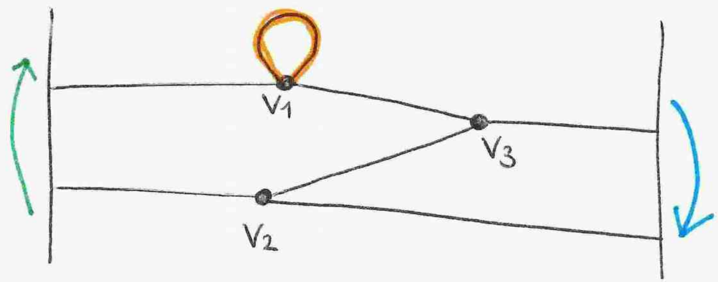
**Proof:**



# Rotation Systems (3)



Example:




$v_1$ : in, $v_1$, $v_1$, $v_3$

$v_2$ : in, $v_3$, out

$v_3$ : $v_1$, out, $v_2$

in : $v_2$, $v_1$

out : $v_3$, $v_2$

- categorically: inputs and outputs non-cyclic ordered lists
- combinatorically: boundary vertex as cyclic ordered list
- special case: multiple self loops (later, maybe)

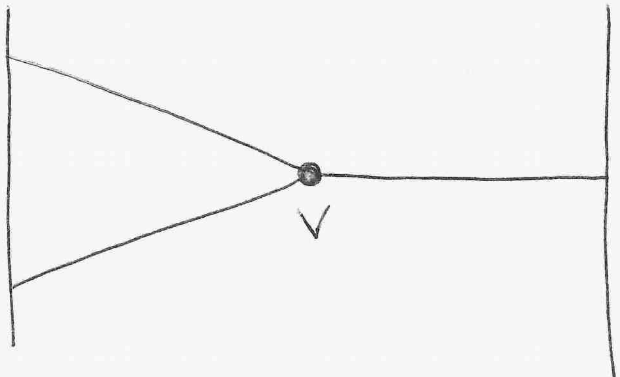
↑

boundary and inner vertices

# Building Graphs – Base Cases (1)



Single vertex:




v : in, in, out

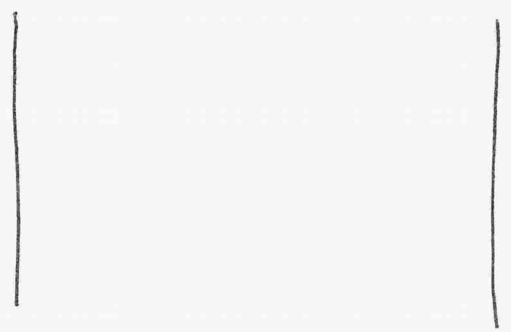
in : v, v

out: v

# Building Graphs – Base Cases (2)



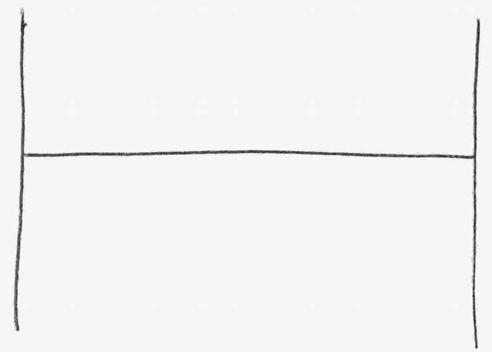
empty graph:



in: []

out: []

identity:



in: out

out: in

# Building Graphs – Base Cases (3)



cap:

in: in, in
out: []

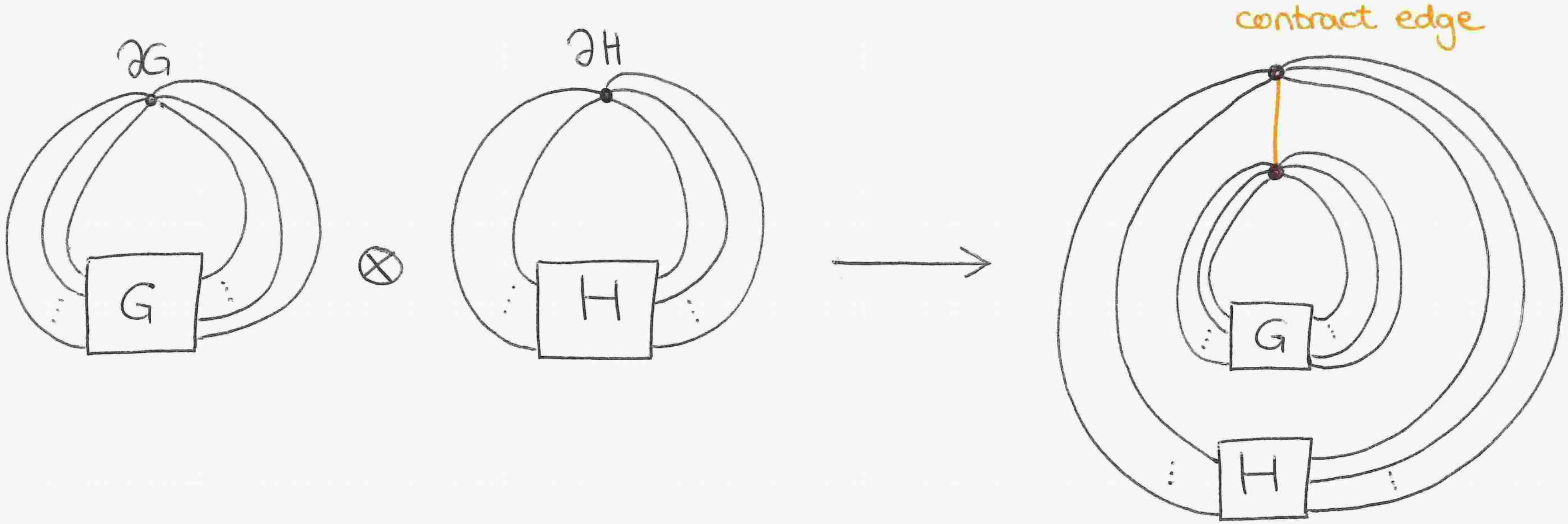
cup:

in: []
out: out, out

cap and cup are self loops at the boundary vertex
(so is the identity!)

# Building Graphs – Parallel Composition (1)



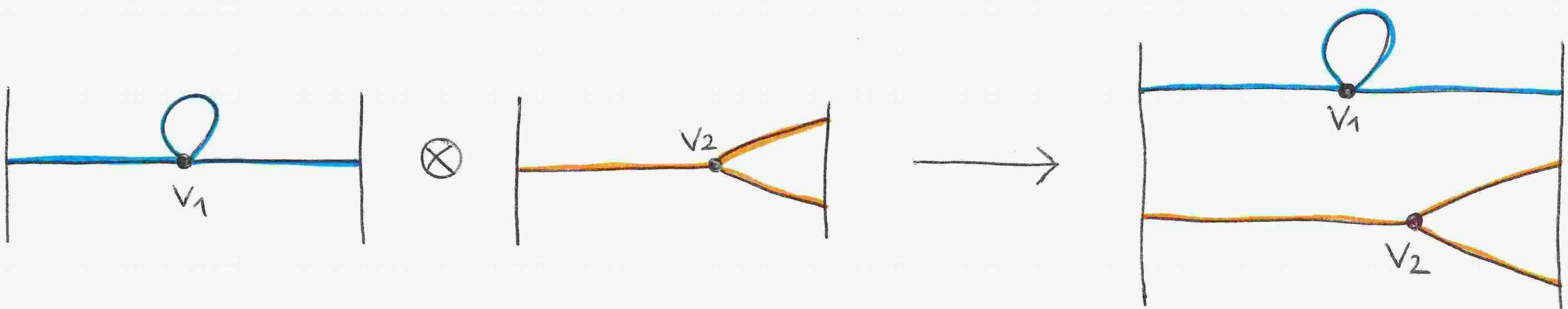



- make names of vertices disjoint
- new rotation system: union of both rotation systems
- new boundary vertex: draw extra edge and contract it

# Building Graphs – Parallel Composition (2)



Example:




$v_1$ : in, $v_1$, $v_1$, out

in : $v_1$

out : $v_1$

$v_2$ : in, out, out

in : $v_2$

out : $v_2$, $v_2$

$v_1$ : in, $v_1$, $v_1$, out

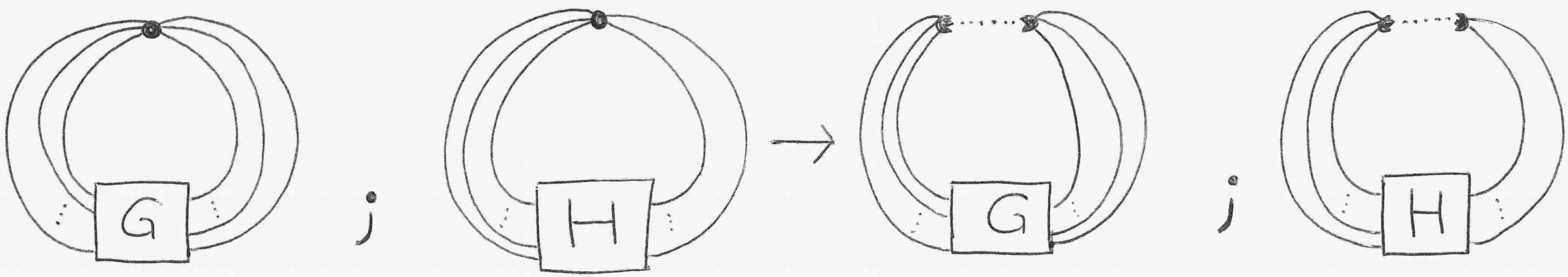
$v_2$ : in, out, out

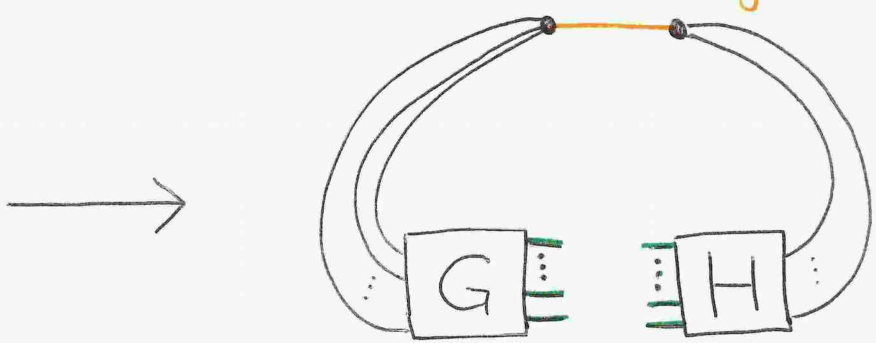
in : $v_2$, $v_1$

out : $v_1$, $v_2$, $v_2$

# Building Graphs – Sequential Composition (1)





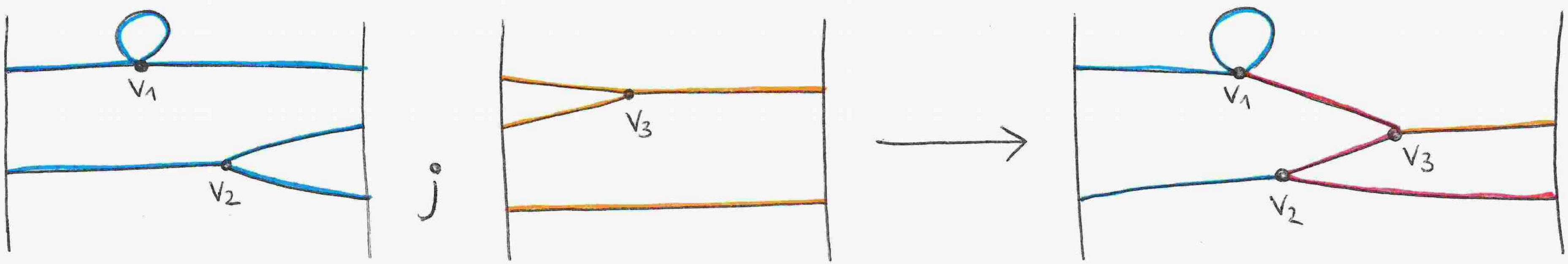




- identify edges at the composition boundary
- update rotation systems on both sides
- new boundary vertex: inputs from the left
  outputs from the right

# Building Graphs – Sequential Composition (2)



Example:




$v_1$: in, $v_1$, $v_1$, out
$v_2$: in, out, out

in: $v_2$, $v_1$
out: $v_1$, $v_2$, $v_2$

$v_3$: in, in, out
in: out, $v_3$, $v_3$
out: $v_3$, in

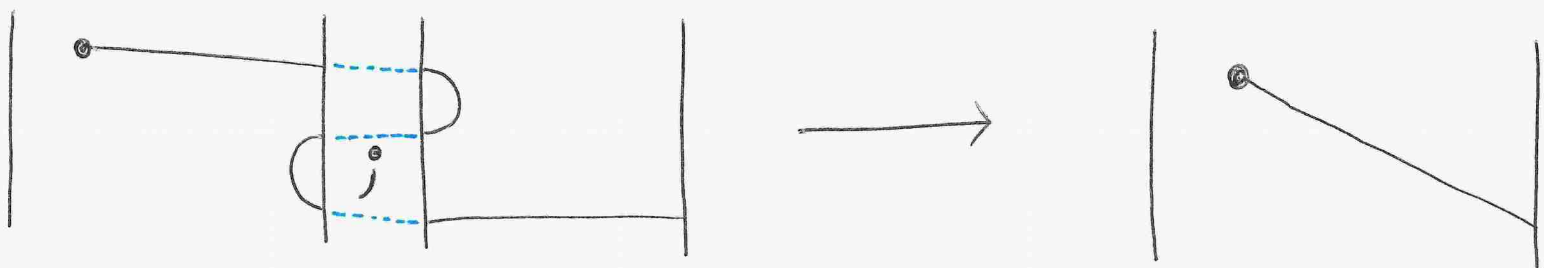
$v_1$: in, $v_1$, $v_1$
$v_2$: in, $v_3$, out
$v_3$: $v_2$, $v_1$, out
in: $v_2$, $v_1$
out: $v_3$, $v_2$

# Building Graphs – Sequential Composition (3)

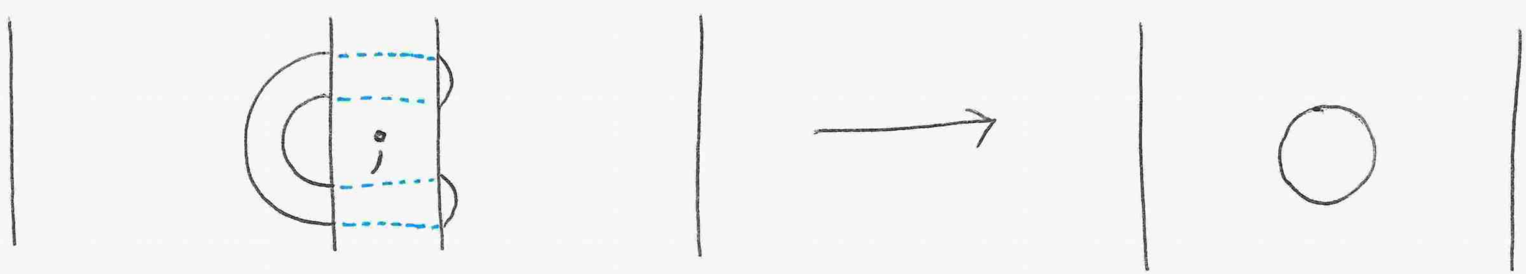


Special cases for sequential composition:

- longer paths:



- cycles:



# Plane Graphs with a Boundary Vertex



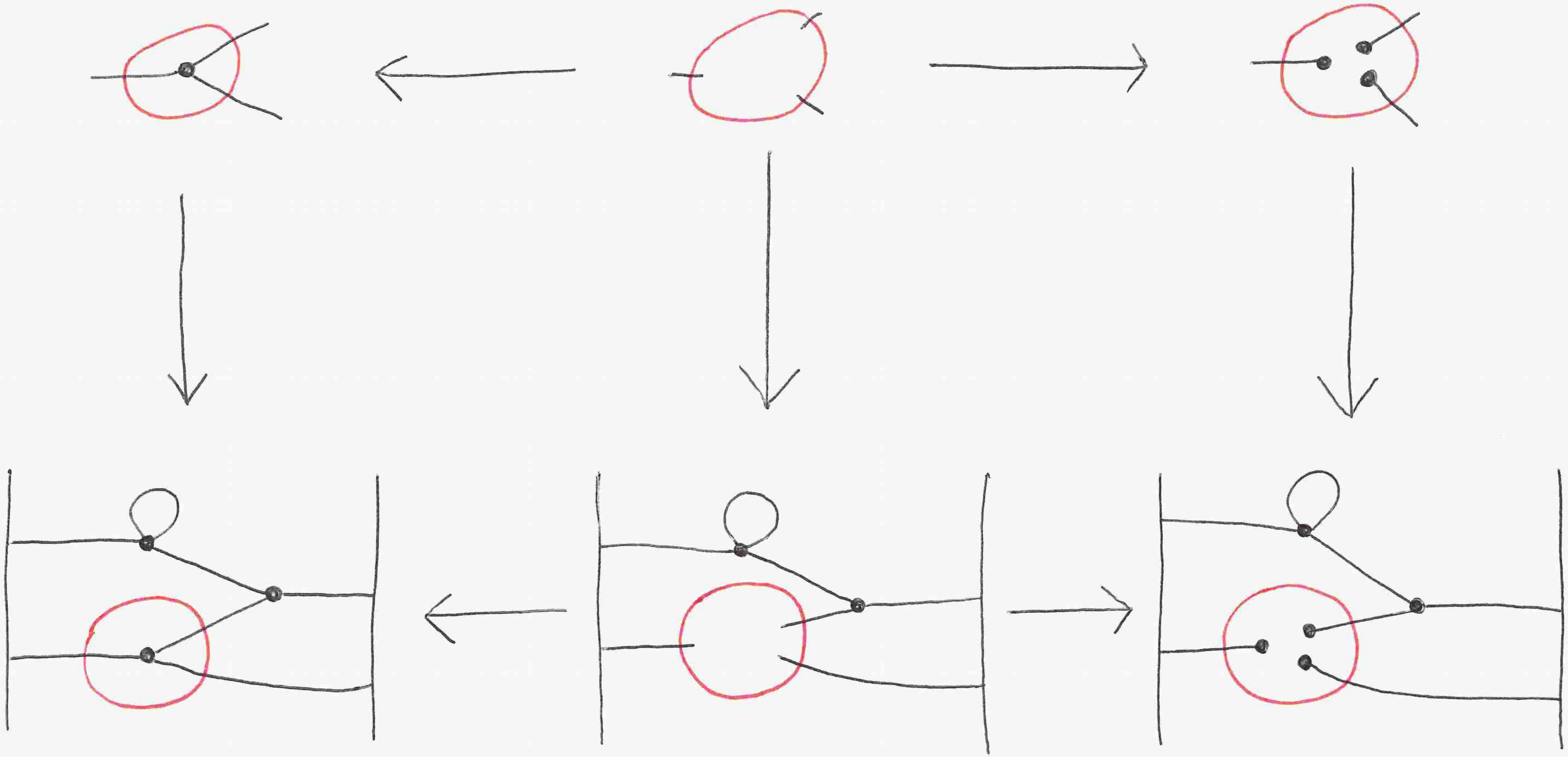
This representation of plane graphs with a boundary vertex defines a strict monoidal category, where

- the objects are lists of types of wires
- the morphisms are graphs
- parallel and sequential composition as defined above

Now: How does rewriting work?

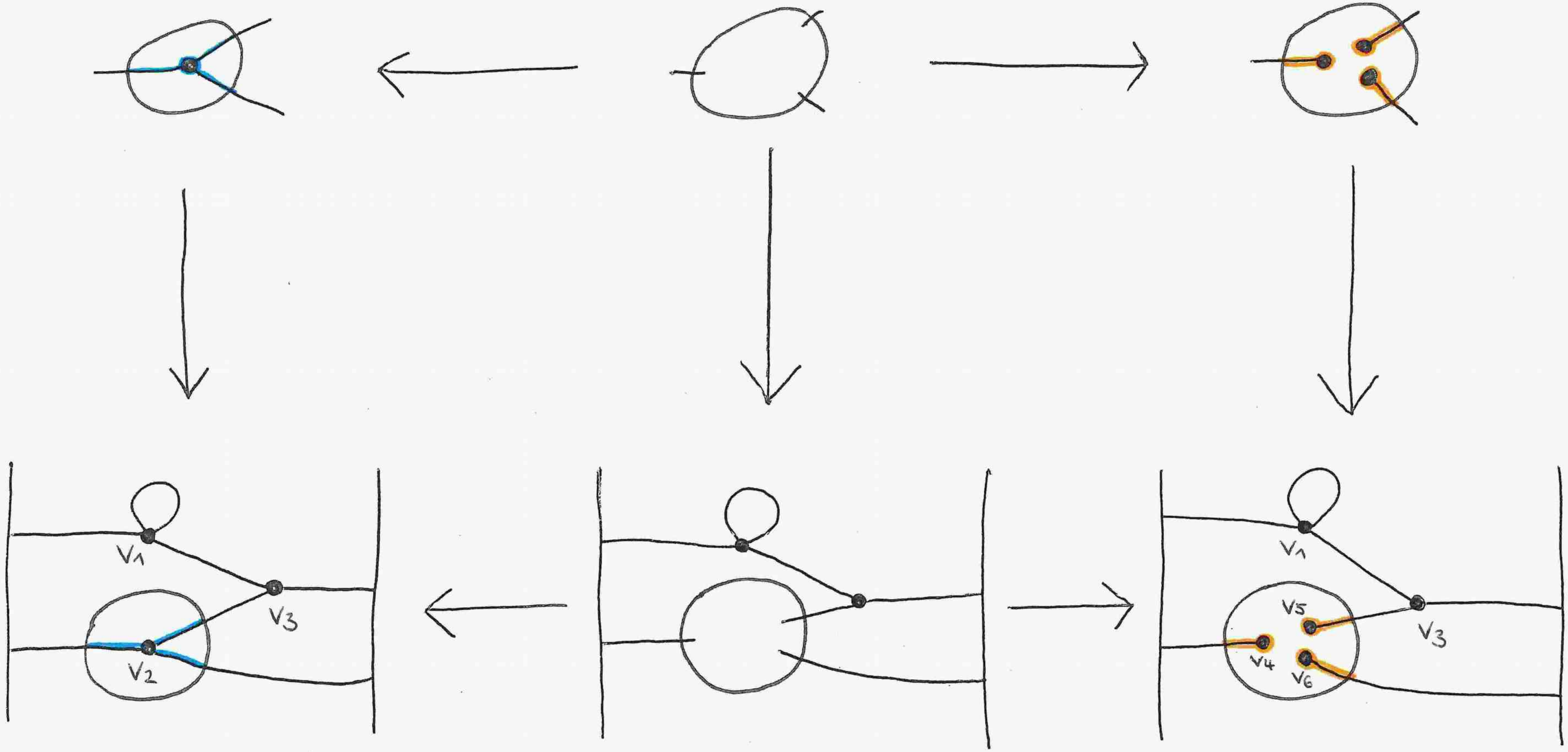
# Graph Rewriting – Double Pushout Approach





# Graph Rewriting – Double Pushout Approach






$v_1$ : in, $v_1$, $v_1$, $v_3$

$v_2$ : in, $v_3$, out

$v_3$ : $v_1$, out, $v_2$

in : $v_2$, $v_1$

out : $v_3$, $v_2$

$v_1$ : in, $v_1$, $v_1$, $v_3$

$v_4$ : in ; $v_5$ : $v_3$ ; $v_6$ : out

$v_3$ : $v_1$, out, $v_5$

in : $v_4$, $v_1$

out : $v_3$, $v_6$

**Lemma:**

Graph rewriting (as defined above) preserves planarity.

**Proof:**

- LHS of rewrite rule is a connected graph
  $\Rightarrow$ can be contracted to a single vertex
  (edge contraction preserves planarity)
- substitution of a plane graph for a vertex preserves planarity

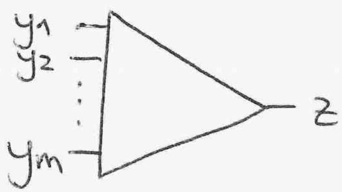
# Operads



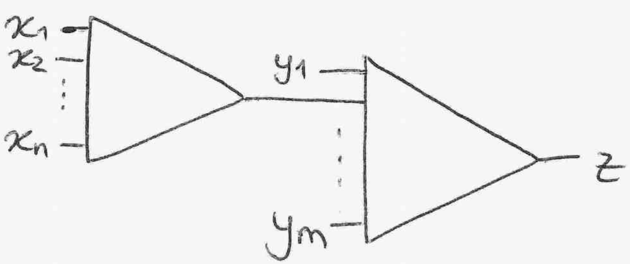
here: coloured operad (= multicategory)

An operad consists of:

- a collection of objects
- a collection of morphisms which take multiple inputs




- composition operation:




- identity $x$ ——— $x$

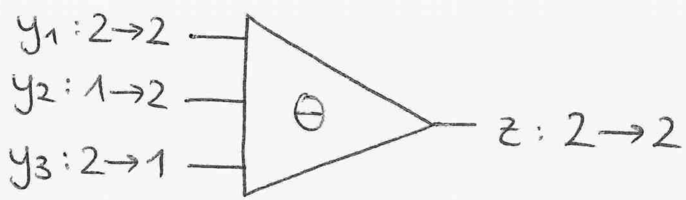
... satisfying the usual identity and associativity laws.

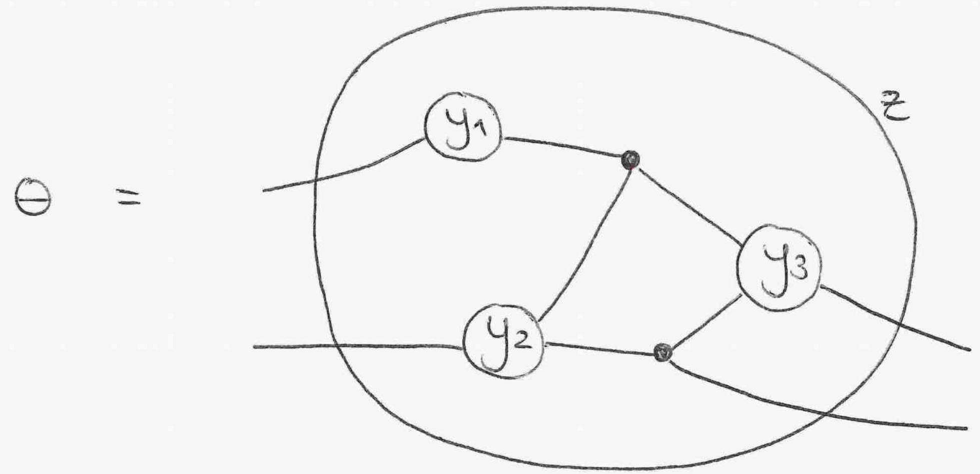
# The Operad of Plane Graphs (1)



- objects : connectivity of graph variables
- morphisms : graphs

**This is a symmetric operad**



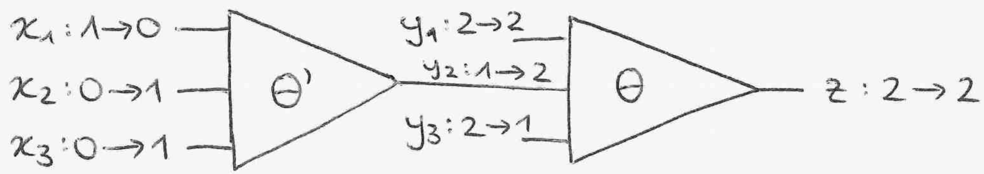




Similar idea to the operad of wiring diagrams (Spivak, 2013)

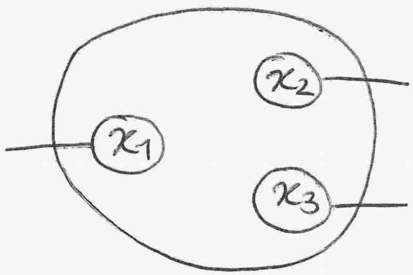
# The Operad of Plane Graphs (2)



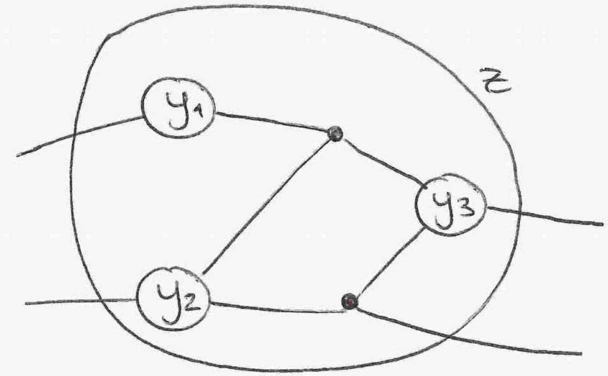
- composition is substitution for a graph variable



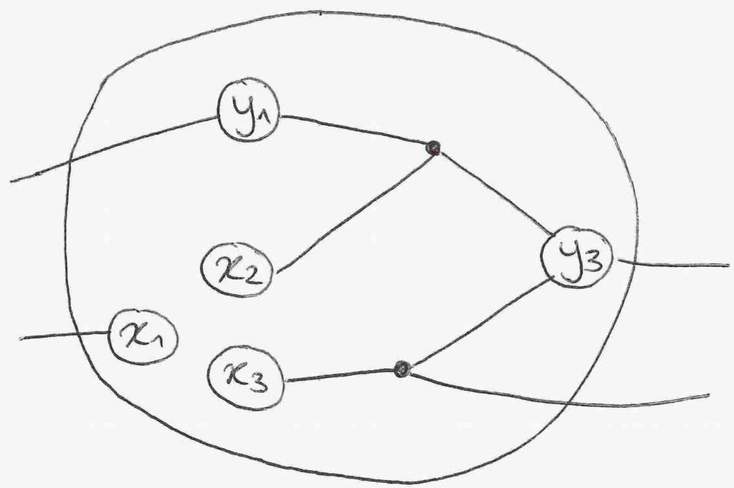

$\theta' =$




$\theta =$




$\theta' ; \theta =$



# Summary



Plane graphs with a boundary vertex form an operad, where the composition operation is substitution.

- representing non-symmetric monoidal categories
- combinatorial presentation via rotation systems

Future work:

- more complex types of wires
- adding geometry information
- cooperads: substitution becomes patternmatching

Thank you for your attention!

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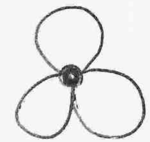
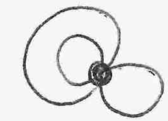
# References


- Selinger, P. (2011). A survey of Graphical Languages for Monoidal Categories, pages 289–355. Springer Berlin Heidelberg.

- Spivak, D. (2013). The Operad of Wiring Diagrams: Formalizing a Graphical Language for Databases, Recursion and Plug-and-Play Circuits. CoRR, abs/1305.0297.

- Youngs, J.W.T. (1963). Minimal Imbeddings and the Genus of a Graph. Journal of Mathematics and Mechanics, 12(2): 303–315.

# Extra: Self loops

- need to distinguish  and 

rotation systems [v,v,v,v,v,v] [v,v,v,v,v,v]

- introduce pointers to other end of edge [v,v,v,v,v,v] [v,v,v,v,v,v]

(validity check: well formed bracketing of pointers.
[v,v,v,v,v,v] is <u>not plane</u>! )

- works for both inner vertices and the boundary