Cylinder Diagrams: Visualising Multi-parameter Natural Transformations

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SYCO 9
Things I like:

- Functional programs and composition.
- Effects like global store and nondeterminism.
- Parallelism and concurrency.

Dimensions of a page:

- Height.
- Width.
- ...
Effectful composition

\[ f \circ g = \]

How to add parallelism?

h

f

g

f

g

f

g

k
Motivation

Possible solutions:

- Surface diagrams, sheet diagrams.
- Ribbon diagrams, tape diagrams.
- String diagrams with special operations.

Solution of this talk: Cylinder diagrams
Inflate the strings and allow them to contain diagrams.

- Allows for arbitrary nesting.
- Can be neatly flattened to 2 dimensions.

Methods for drawing:

- This presentation will use drawing assembled using tiles with premade illustrations.
- Can also be illustrated by 2-dimensional string diagrams using bracket-strings.
Illustration: Parallel and sequential composition
Core Categorical structure

Given a monoidal category \( \mathcal{C} \), we describe the category consisting of:

- As objects multi-endofunctors on \( \mathcal{C} \).
- As morphisms natural transformations between them.

The resulting category is also monoidal.

It contains \( \mathcal{C} \) as a full subcategory of constant functors.

Main example is taking the category of sets \( \mathcal{C} = \text{Set} \), with the Cartesian product as monoidal product.
Objects and morphisms of \( C \)

Objects are represented (solid) cylinders, and functions by blocks between them:

Object: \( A = \)  

Morphism: \( \left( f : \frac{A}{B} \right) = \)

Special morphisms which exist in **Set** (\( \times \) is the product):

\[
\left( S : \frac{X \times Y}{Y \times X} \right) = \quad C : \frac{X}{X \times X} = \quad D : \frac{X}{1} =
\]
Objects and morphisms of $\mathcal{C}$

Special equations:

Naturality:
Endofunctors of $\mathcal{C}$

Functors are represented by hollow cylinders, within which lies the object or morphism to which it is applied.

\[
A = \hspace{1cm} \iff \hspace{1cm} TA =
\]

\[
\left( f : \frac{A}{B} \right) = \iff \hspace{1cm} \left( Tf : \frac{TA}{TB} \right) =
\]

Functor law for morphism composition:

\[
\left( f : \frac{X}{Y}, g : \frac{Y}{Z} \right) \iff \left( Tg \circ Tf = T(g \circ f) : \frac{TX}{TZ} \right)
\]
Natural Transformations of $\mathcal{C}$

Natural transformations, a parametrized morphisms:

$$\left( \forall X. \ a_X : \frac{TX}{RX} \right)$$

The object $X$ goes through the natural transformation block, illustrating naturality:

$$\left( \forall f : \frac{X}{Y}. \ a_Y \circ Tf = Rf \circ a_X \right)$$
Let there be monads

Definition
A monad is a functor $T$ with two natural transformations

$(\eta_X : \frac{X}{TX}) = \text{ }$ and $(\mu_X : \frac{TTX}{TX}) = \text{ }$, satisfying the following two equations:

(Left and right-unit)

(Associativity)
Strength

Strength is illustrated by pulling in the parameter:

\[ \sigma_{X,Y} : \frac{X \times TY}{T(X \times Y)} = \]

Monoidal strength equations:

Strong monad equations:
Costrength

Using swap, we define costrength as

\[
(\tau_{X,Y} : \frac{TX \times Y}{T(X \times Y)}) = \quad = \quad .
\]

Lemma

A strong monad is a costrong monad.
Costrength

Proof.
Commutativity

How do we *merge* monads?
One method is to use strength and costrength.
Commutativity says the order doesn’t matter.

**Definition**
A monad is commutative if the following equation holds:
Monoidal Monad

Definition

$T$ is a monoidal monad if there is a natural transformation

$$m : \frac{T^2 X \times T^2 Y}{T(X \times Y)}$$

satisfying the following two equations:

\[(\eta)\]

\[(\mu)\]

Lemma

Any strong commutative monad is a monoidal monad.
Monoidal Monad

Proof. We define the following natural transformation:

\[ m : \frac{TX \times TY}{T(X \times Y)} = \]

Using commutativity, we can prove both monoidal monad properties:
Monoidal Monad
Product of Monads

$T$ and $R$ both monads. Assume we have copy and delete.

$$(T \times R)(X) = (TX \times RX) =$$

- The monad unit $\left( \eta^{T \times R}_{X} : \frac{X}{TX \times RX} \right) =$

- The monad multiplication

$$\left( \mu^{T \times R}_{X} : \frac{T(TX \times RX) \times R(TX \times RX)}{TX \times RX} \right) =$$
Product of Monads

This is a monad. Proof:

\[
\begin{align*}
\text{Diagram 1} & \quad = \quad \text{Diagram 2} & \quad = \quad \text{Diagram 3} \\
\text{Diagram 4} & \quad = \quad \text{Diagram 5} & \quad = \quad \text{Diagram 6}
\end{align*}
\]
Product of Monads
Personal motivation: Comomomonad interactions

- Models with both leading/active monadic structure, and following/passive comonad structure.
- Interaction between programs as lead and follow, using interaction laws.
Personal motivation: Comomonad interactions
Multi-functors generically

Functors like $X \times TY$ have multiple arguments. Generically, this can be illustrated by glueing cylinders. Natural transformations can be illustrated by connecting the appropriate arguments.

For example, coproduct in $\textbf{Set}$ can be represented by glued cylinders. Below is an illustration of distributivity.

$$\left( \delta_{X,Y,Z} : \frac{X \times (Y + Z)}{(X \times Y) + (X \times Z)} \right) =$$
Conclusions: Open questions

How to do cylinder diagrams monoidally?

Geometric interpretation of cylinder diagrams.

Considering other generic features.
Conclusions: Pictures

Most images were composed by combining tiles in layers, using the following tilesets:

Working on more automatized methods too.

String drawing method on grid: Link