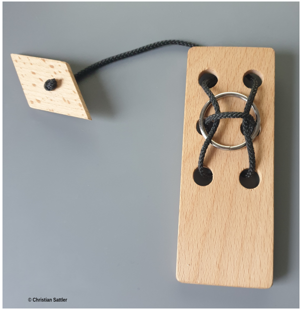
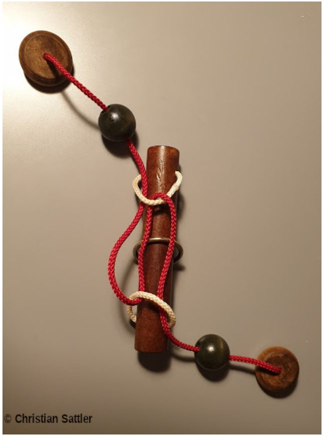
Karlsruher Institut für Technologie

String Diagrams for Strings and Rings

Jakob von Raumer | 8. September 2022

Ring-and-rope puzzles





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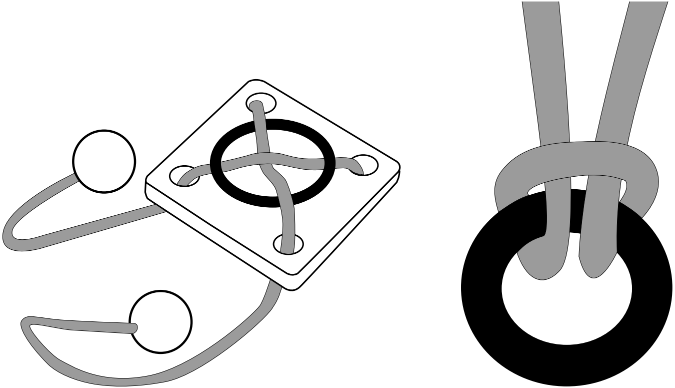
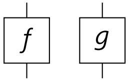


String Diagrams for Monoidal Categories

- Tensor product $A \otimes B$ of objects A and B,
- and tensor product $f \otimes g : A \otimes C \to B \otimes D$ of morphisms f and g,
- obeying associativity, unit laws w. r. t. an object I, naturality (strictly or non-strictly).
- Interchange law represented strictly in String Diagrams.
- Sliding of morphisms



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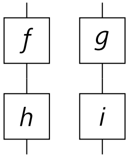

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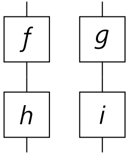

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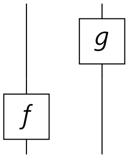

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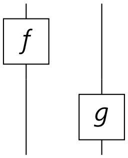
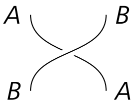


… for Braided Monoidal Categories

- For each objects A and B have a *braiding* $c_{A,B} : A \otimes B \cong B \otimes A$.
- Natural in A and B, and
- Obeying "hexagon laws":

$$\begin{tikzcd}
A \otimes B \otimes C \arrow[rr, "c_{A,B\otimes C}"] \arrow[dr, "c_{A,B}\otimes 1"'] & & B \otimes C \otimes A \\
& B \otimes A \otimes C \arrow[ur, "1\otimes c_{A,C}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \otimes C \arrow[rr, "c_{A\otimes B,C}"] \arrow[dr, "1\otimes c_{B,C}"'] & & C \otimes A \otimes B \\
& A \otimes C \otimes B \arrow[ur, "c_{A,C}\otimes 1"'] &
\end{tikzcd}$$



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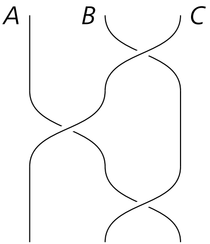

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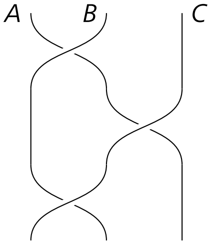

… for Right Rigid Monoidal Categories

- Equip the category with the notion of (right) duals A^* for objects A
- Evaluation morphisms $\epsilon_A : A \otimes A^* \to I$ and coevaluation morphisms $\eta_A : I \to A^* \otimes A$.
- Fulfilling the triangle identities:

$$
\begin{tikzcd}
A \arrow[r, "1 \otimes \eta_A"] \arrow[dr, "1"'] & A \otimes A^* \otimes A \arrow[d, "\epsilon_A \otimes 1"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
A^* \arrow[r, "\eta_A \otimes 1"] \arrow[dr, "1"'] & A^* \otimes A \otimes A^* \arrow[d, "1 \otimes \epsilon_A"] \\
& A^*
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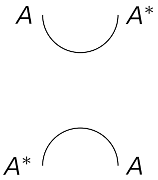

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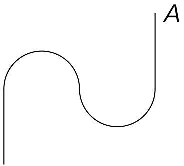
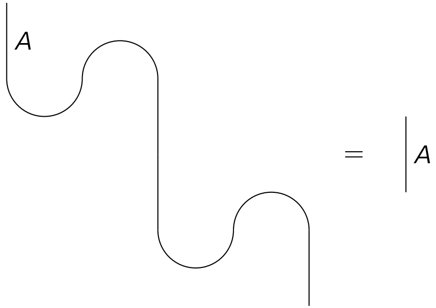


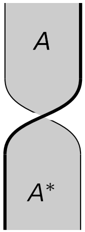
… for Rigid Categories

- A right rigid monoidal category is strictly *pivotal* if we have $i_A : A = A^{**}$.
- Any pivotal right rigid monoidal category is also *left rigid*, with ${}^*A = A^*$.



… for Self-Dual Monoidal Categories

- A *self-duality structure* on s strictly pivotal category is a choice of *half-twists* $h_A : A \cong A^*$ obeying the following equations:
  1. $h_I = 1_I$,
  2. $(f^*)_\sharp = (f_\sharp)^*$ for $f : A \to B$, where the *vertical twist* $f_\sharp$ of f is defined by conjugation with the appropriate half-twists:
  $$f_\sharp := h_B \circ f \circ h_A^{-1} : A^* \to B^*.$$
  3. $(f \otimes g)_\sharp = g_\sharp \otimes f_\sharp$,
  4. $(h_A \otimes h_B \otimes h_C) \circ h_{A \otimes B \otimes C} = (1_A \otimes h_{B \otimes C}) \circ (h_{A \otimes C} \otimes 1_{B^*}) \circ (1_C \otimes h_{A \otimes B})$,
  5. $(1_A \otimes \epsilon_{A^*}) \circ (h_{A \otimes A^*} \otimes 1_A) \circ (1 \otimes \eta_A) = h_{A^*} \circ h_A \circ h_{A^*} \circ h_A$.
- Self-duality induces a braiding via $c_{A,B} := h_{A \otimes B} \circ (h_B^{-1} \otimes h_A^{-1}) : A \otimes B \to B \otimes A$.



The Bicategory of Puzzle Models

Definition

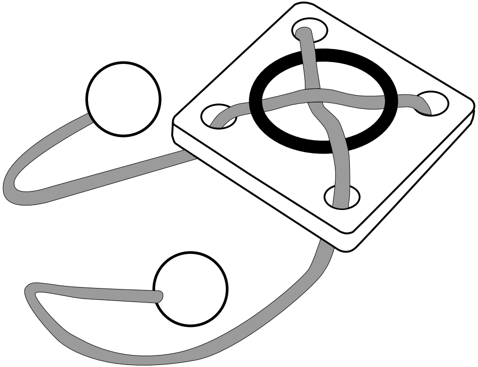
The bicategory $\mathcal{M}$ of models consists of all pointed, strictly monoidal categories which

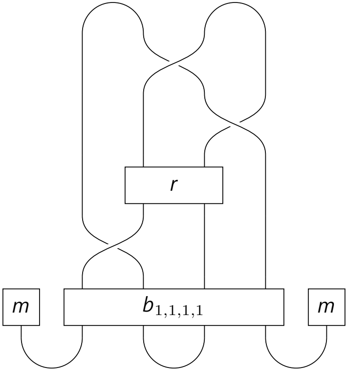
- Are strictly rigid,
- have a self-duality structure (this induces a braiding), and
- model rigid components via (families of) morphisms
  - *marbles* $m_A : I \to A$,
  - *rings* $r_A : A \to A$,
  - for each $k \in \mathbb{N}$ and each list $(A_1, \ldots, A_k)$ of objects of a *board*

$$b_{A_1,\ldots,A_k} : \bigotimes_{i=1}^{k} A_i \to \bigotimes_{i=1}^{k} A_i,$$

satisfying a row of naturality and commutativity laws (see later slide).

String Diagrams for Models of Ring-and-Rope Puzzles





The Bicategory of Puzzle Models, ctd.

Definition

Rigid components are required to satisfy the following equations:

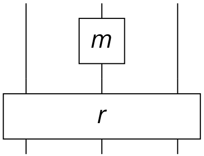
- For all objects A, B and C we have

$$r_{A\otimes B\otimes C} \circ (1_A \otimes m_B \otimes 1_C) = (1_A \otimes m_B \otimes 1_C) \circ r_{A\otimes C}$$

- For f being a evaluation, coevaluation or half-twist we have

$$r_{A\otimes(\mathrm{cod}\, f)\otimes B} \circ (1_A \otimes f \otimes 1_B) = (1_A \otimes f \otimes 1_B) \circ r_{A\otimes(\mathrm{dom}\, f)\otimes B}, \text{ and}$$
$$b_{A_1,\ldots,\mathrm{cod}\, f,\ldots A_k} \circ (1 \otimes f \otimes 1) = (1 \otimes f \otimes 1) \circ b_{A_1,\ldots,\mathrm{dom}\, f,\ldots,A_k}.$$

- Compatibility with half-twists: $h_A \circ m_A = m_{A^*}$, $(b_{A_1,\ldots,A_k})_\sharp = b_{A_k^*,\ldots A_1^*}$.



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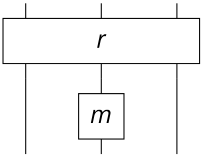
- Compatibility with half-twists: $h_A \circ m_A = m_{A^*}$, $(b_{A_1,\ldots,A_k})_\sharp = b_{A_k^*,\ldots A_1^*}$.



The Initial Model $\mathcal{C}_0$

Definition

Let $\mathcal{C}_0$ the category presented by the following generators

- one generating object S (write $n := S^{\otimes n}$),
- for the pivotal and braided structure morphisms $\eta : 0 \to 2$, $\epsilon : 2 \to 0$, and $c : 2 \to 2$,
- to model the rigid components morphisms $m : 0 \to 1$, and $r_n : n \to n$ for every $n \in \mathbb{N}$, and

$$b_{n_1,\ldots n_k} : (n_1 + \ldots + n_k) \to (n_1 + \ldots + n_k).$$

- for morphisms $f : m \to n$ and $g : k \to l$ a morphism $f \otimes g : m + k \to n + l$,

and the following relations for all morphisms f, g, h and i as well as $m, n \in \mathbb{N}$:

The Initial Model $\mathcal{C}_0$

Definition

and the following relations for all morphisms f, g, h and i as well as $m, n \in \mathbb{N}$:

- $f \otimes 0 = 0 \otimes f = f$,
- $f \otimes (g \otimes h) = (f \otimes g) \otimes h$,
- $(h \otimes i) \circ (f \otimes g) = (h \circ f) \otimes (i \circ g)$,
- $(1_1 \otimes \epsilon) \circ (\eta \otimes 1_1) = 1_1$,
- $c \circ \eta = \eta$,
- $(1_1 \otimes \epsilon) \circ (c \otimes 1_1) \circ (1_1 \otimes c) = \epsilon \otimes 1_1$,
- $(\epsilon \otimes 1_1) \circ (1_1 \otimes c) \circ (c \otimes 1_1) = 1_1 \otimes \epsilon$,
- $(c \otimes 1_1) \circ (1_1 \otimes c) \circ (\eta \otimes 1_1) = 1_1 \otimes \eta$,
- $(1_1 \otimes c) \circ (c \otimes 1_1) \circ (1_1 \otimes \eta) = \eta \otimes 1_1$,
- $(c \otimes 1_1) \circ (1_1 \otimes c) \circ (c \otimes 1_1) = (1_1 \otimes c) \circ (c \otimes 1_1) \circ (1_1 \otimes c)$,
- $m \otimes 1_1 = c \circ (1_1 \otimes m)$,
- $1_1 \otimes m = c \circ (m \otimes 1_1)$,
- $p \otimes 1_1 = c \circ (1_1 \otimes p)$,
- $1_1 \otimes p = c \circ (p \otimes 1_1)$,
- $r_{m+n+2} \circ (1_m \otimes \eta \otimes 1_n) = (1_m \otimes \eta \otimes 1_n) \circ r_{m+n}$,
- $r_{m+n} \circ (1_m \otimes \epsilon \otimes 1_n) = (1_m \otimes \epsilon \otimes 1_n) \circ r_{m+n+2}$,
- $r_{m+n+2} \circ (1_m \otimes c \otimes 1_n) = (1_m \otimes c \otimes 1_n) \circ r_{m+n+2}$,
- $b_{n_1,\dots,n_{p-1},m+l+2,n_{p+1},\dots,n_k} \circ (1 \otimes \eta \otimes 1) = (1 \otimes \eta \otimes 1) \circ b_{n_1,\dots,n_{p-1},m+l,n_{p+1},\dots,n_k}$,
- $b_{n_1,\dots,n_{p-1},m+l,n_{p+1},\dots,n_k} \circ (1 \otimes \epsilon \otimes 1) = (1 \otimes \epsilon \otimes 1) \circ b_{n_1,\dots,n_{p-1},m+l+2,n_{p+1},\dots,n_k}$,
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- $h_{n_1+\dots+n_k} \circ b_{n_1,\dots,n_k} \circ h_{n_1+\dots+n_k} = b_{n_k,\dots n_1}$,
- $(1_1 \otimes b_{n_1,\dots,n_k}) \circ c_{n_1+\dots+n_k,1} = c_{n_1+\dots+n_k,1} \circ (b_{n_1,\dots,n_k} \otimes 1_1)$,
- $(b_{n_1,\dots,n_k} \otimes 1_1) \circ c_{1,n_1+\dots+n_k} = c_{1,n_1+\dots+n_k} \circ (1_1 \otimes b_{n_1,\dots,n_k})$, and
- $r_{m+n+1} \circ (1_m \otimes m \otimes 1_n) = (1_m \otimes m \otimes 1_n) \circ r_{m+n}$,

Initial models

- Conjecture: $\mathcal{C}_0$ is initial in $\mathcal{M}$,
- It is isomorphic to the model of unlabelled string diagrams

Definition

A *ring-and-rope puzzle* is a morphism $f : m \to n$ in $\mathcal{C}_0$. A *solution* for a puzzle f consists of another morphism $f' : m \to n$ together with a proof of the equality

$$f = f' \otimes r_0.$$

Agda Formalisation

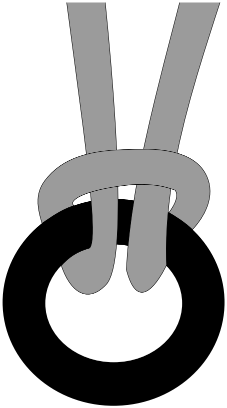
- Initiality suggests formalisation of morphisms as a setoid on an inductive type.
- Code: `https://github.com/javra/strings`

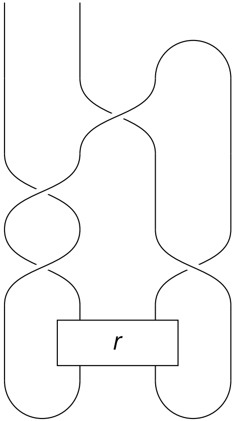
```
data D : ℕ → ℕ → Set where
  ε   : D 0 0                                               -- empty diagram
  |   : D 1 1                                               -- object generator (string)
  _·_ : ∀{m n k} → D m n → D n k → D m k                    -- composition
  _⊗_ : ∀{m n k l} → D m n → D k l → D (m + k) (n + l)      -- tensor product
  ∩   : D 0 2                                               -- coevaluation
  ∪   : D 2 0                                               -- evaluation
  /   : D 2 2                                               -- braiding
  R   : ∀{n} → D n n                                        -- ring
  M   : D 1 0                                               -- marble
  B   : (ns : List ℕ) → D (sum ns) (sum ns)                 -- board
```

Agda Formalisation

```
data _~_ : ∀ {m n} → D m n → D m n → Prop where
...
  _■_   : ∀{m n}{d e f : D m n} → d ~ e → e ~ f → d ~ f
  ··     : ∀{m n k l}{d : D m n}{e : D n k}{f : D k l} → d · (e · f) ~ d · e · f
...
  ⊗⊗     : ∀{m m' m'' n n' n''}{d : D m n}{e : D m' n'}{f : D m'' n''}
             → d ⊗ (e ⊗ f) ~ d ⊗ e ⊗ f
...
  ∩∪     : ∩ ⊗ | · | ⊗ ∪ ~ |
...
  ///    : / ⊗ | · | ⊗ / · / ⊗ | ~ | ⊗ / · / ⊗ | · | ⊗ / -- Reidemeister Type III
  ∩·R    : ∀{l r} → |n⊗|m l ∩ r · R ~ R · |n⊗|m l ∩ r -- string moves through ring
...
  XBX⁻¹  : ∀{ns} → let B' = coeD (sumRev ns) (sumRev ns) (B (reverse ns)) in
                   X · B ns · X ~ B' -- coherence of board with half-twist
  /nB    : ∀{ns} → /n · | ⊗ B ns ~ B ns ⊗ | · /n  -- naturality of braiding wrt board
  /-nB   : ∀{ns} → /-n · B ns ⊗ | ~ | ⊗ B ns · /-n  -- naturality of braiding wrt board
```

Ring on a String



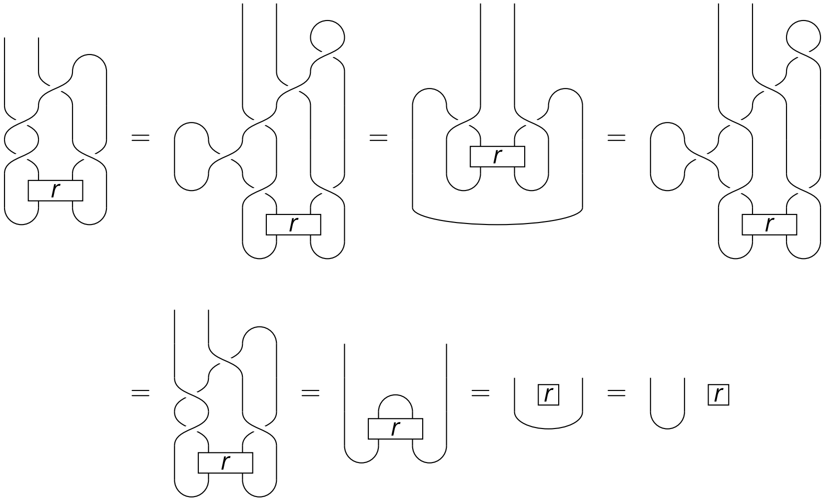



```
puzzle : D 2 0
puzzle =          ⋂n
         ·   /    ⊗   /⁻¹
         · | ⊗    R   ⊗ |
         ·   U    ⊗   U
```

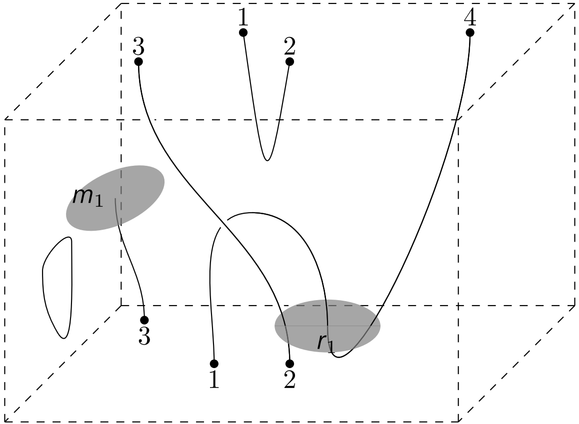
Ring on a String

```
solution : puzzle ~ ⋃ ⊗ R
solution = − ··· ■ ⋂n~⋂-n' ■ ~· ⋂-2 ■ ··
           ■ ~· (−·· ■ ·~ (− ·⊗· ■ ~⊗ /⁻¹/ ■ ⊗~ //⁻¹ ■ ⊗⊗) ■ ·||||)
           ■ ·· ■ ~· (⊗|·⊗| ■ ~⊗ (|⊗·|⊗ ■ ⊗~ (~· (− (⊗ε ■ ε⊗)) ■ ⋂·R)))
           ■ ~· (~⊗ (− |⊗·|⊗) ■ − ⊗|·⊗| ■ ~· (−⊗⊗ ■ ⊗~ R0⊗| ■ ⊗⊗))
           ■ −·· ■ ·~ (~· (~⊗ (⊗~ (⊗ε ■ ε⊗))) ■ ·~ (− ⊗ε) ■ ⊗⋂⊗·⊗⋃⊗ ■ ||·)
           ■ − ·⊗· ■ ~⊗ ||· ■ ⊗~ ·ε
```

Ring on a String



Future Work: A Geometric Model



An illustration of a morphism $(S_1, \ldots, S_5, m_1, r_1)$ in the geometrical model.

Future Work: Strictification à la Vicary's ANCs

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Can make the formalisation more strict by

- Formalising Diagrams as a list of *coupons*,
- which are other components padded by strings.
- Sliding instead of the interchange law as an axiom.

```
data A : ℕ → ℕ → Set where  -- coupons
  / : A 2 2
  ∩ : A 0 2
  ∪ : A 2 0
  R : ∀{n} → A n n
  M : A 0 1
  B : (ns : List ℕ) → A (sum ns) (sum ns)

data C : ℕ → ℕ → Set where  -- padded coupons
  _>A_<_ : ∀{m n} → (l : ℕ) → (a : A m n) → (r : ℕ)
              → C (l + m + r) (l + n + r)

data D : ℕ → ℕ → Set where  -- diagrams
  |n  : ∀ {n} → D n n  -- unit
  _·_ : ∀{m n k} → (d : D m n) → (c : C n k)
            → D m k  -- composition
```

Questions

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- Future work:
  - Prove conjectures regarding the geometric model
  - Generalise to include both of Christian's puzzles
  - Implement a solver for puzzles
- Does it make sense to define models as bicategories themselves?
- Is this all worthwhile?