

Continuous monoid homomorphisms and geometric morphisms

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- ① Categories of actions, and some motivation
- ② Semigroup homomorphisms
- ③ Complete monoids

Section 1

Categories of actions, and some motivation

We may view a monoid M as a one-object category.

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The category of presheaves on a monoid M ,

$$\mathbf{PSh}(M) := [M^{\text{op}}, \mathbf{Set}] \simeq \mathbf{Set}\text{-}M,$$

coincides with the category of *right actions* of M .

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This talk is about categories constructed in this way.

They happen to be examples of Grothendieck toposes.¹

¹For this talk, it is enough to know that these are a class of (co)complete categories.

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- To study monoids, in analogy with studying categories of modules.
- Stepping stone for understanding actions on more general spaces.
- As a context for studying specific actions (in computer science, say).

Definition

Let (M, τ) be a topological monoid, X an ordinary M -set. A **necessary clopen** for X is a set of the form,

$$\mathcal{I}_x^p := \{m \in M \mid xm = xp\},$$

where $x \in X$ and $p \in M$.

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Note that for each $x \in X$, the subsets \mathcal{I}_x^p partition M , so that if these subsets are all open they are also necessarily closed, hence the name 'necessary clopen'.

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Note that for each $x \in X$, the subsets \mathcal{I}_x^p partition M , so that if these subsets are all open they are also necessarily closed, hence the name 'necessary clopen'.

Lemma

Let (M, τ) be a topological monoid and X an (ordinary) M -set. Then X is continuous with respect to τ if and only if all necessary clopens for X are (cl)open in τ .

Proposition

Let (M, τ) be a topological monoid. Then the forgetful functor $V : \text{Cont}(M, \tau) \rightarrow \text{PSh}(M)$ is left exact and comonadic; its right adjoint R sends an M -set X to:

$$R(X) := \{x \in X \mid \forall p \in M, \mathcal{I}_x^p \in \tau\}.$$

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Moreover, $\tilde{\tau}$ is generated by the Boolean algebra $VR(\mathcal{P}(M)) \subseteq \mathcal{P}(M)$.

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Definition

We say a topology τ is an **action topology** if $\tau = \tilde{\tau}$, and that (M, τ) is a **powder monoid** if τ is a T_0 action topology.

The situation

$$\begin{array}{ccc} & \begin{array}{c} \xrightarrow{-\times M} \\ \perp \\ \xrightarrow{U} \\ \perp \\ \xrightarrow{\quad} \end{array} & \\ \mathbf{Set} & \xleftarrow{U} \mathbf{PSh}(M) & \xleftarrow{V} \mathbf{Cont}(M, \tau) \\ & \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xrightarrow{\quad} \end{array} & \\ \text{Hom}_{\mathbf{Set}}(M, -) & & \xrightarrow{R} \end{array}$$

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Definitions

Let \mathcal{F}, \mathcal{E} be Grothendieck toposes. A **geometric morphism** $f : \mathcal{F} \rightarrow \mathcal{E}$ is an adjunction $(f^* \dashv f_*)$, where $f_* : \mathcal{F} \rightarrow \mathcal{E}$ and f^* preserves finite limits.

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Thus from the above diagram we recover a **canonical point**, $(U \circ V \dashv R \circ \text{Hom}_{\mathbf{Set}}(M, -))$, of $\text{Cont}(M, \tau)$.

Section 2

Semigroup homomorphisms

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Suppose that we have two topological monoids (M, τ) and (M', τ') . Then we have a diagram, and any semigroup homomorphism $\phi : M \rightarrow M'$ induces an essential geometric morphism between the presheaf categories:

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$$\begin{array}{ccc} & \xrightarrow{f_!} & \\ \text{PSh}(M) & \xleftarrow{\perp} f^* & \text{PSh}(M') \\ & \xrightarrow{\perp} f_* & \\ V \uparrow \dashv \downarrow R & & V' \uparrow \dashv \downarrow R' \\ \text{Cont}(M, \tau) & \xleftarrow{Rf_* V'} & \text{Cont}(M', \tau'), \end{array}$$

When is the restricted functor the inverse image of a geometric morphism?

Lemma

The following are equivalent:

1. $f^* : \text{PSh}(M') \rightarrow \text{PSh}(M)$ maps every (M', τ') -set to an (M, τ) -set.
2. ϕ is continuous with respect to τ and $\tilde{\tau}'$.
3. The composite functor Rf^*V' (is left exact and) has a right adjoint G satisfying $GR \cong R'f_*$, which is to say that f restricts along the functors V, V' to a geometric morphism $(G \dashv Rf^*V') : \text{Cont}(M, \tau) \rightarrow \text{Cont}(M', \tau')$ making the square commute.

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Proof sketch: The precomposition functor f^* maps (M', τ') -sets to (M, τ) -sets if and only if ϕ^{-1} sends necessary clopens to opens, which is equivalent to continuity. It also implies the existence of the right adjoint via the identity $V(Rf^*V') \cong f^*V'$, which is equivalent to the third statement. \square

Continuous homomorphisms

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 & \xrightarrow{\perp} & \\
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 \end{array}$$

Remark

Even if we restrict to powder monoids, it is not the case that all geometric morphisms arise from continuous semigroup homomorphisms; in particular, there are equivalences of such toposes which are not induced by semigroup homomorphisms.

Factorizations

The two best-known factorization systems for geometric morphisms are the surjection–inclusion and hyperconnected–localic factorizations, and these are compatible:

$$\begin{array}{ccccc} & & f & & \\ & \frown & & \smile & \\ \mathcal{F} & \xrightarrow{\text{hyperconnected}} & \mathcal{G} & \xrightarrow{\text{localic surjection}} & \mathcal{H} & \xrightarrow{\text{inclusion}} & \mathcal{E} \\ & & & & & & \end{array}$$

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For discrete monoids, the factorization of a geometric morphism induced by a semigroup homomorphism is represented by a factorization of the semigroup homomorphism:

$$\begin{array}{ccccc} & & \phi & & \\ & \frown & & \smile & \\ M & \xrightarrow{\text{surjective monoid hom}} & L & \xrightarrow{\text{injective monoid hom}} & N & \xrightarrow{\text{subsemigroup inclusion}} & M' \\ & & & & & & \end{array}$$

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We deduce that t is a surjection since $h'' \circ s$ is, but is j an inclusion?

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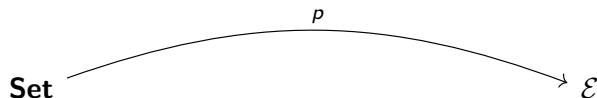
We need to characterize toposes of topological monoid actions, which brings us to a new class of topological monoids.

Section 3

Complete monoids

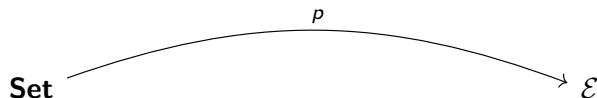
Best approximation

Suppose that we are given a topos \mathcal{E} , i.e. a category with the basic necessary properties to have a chance of being a category of continuous actions of a monoid, and a point of such a topos:



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Suppose that we are given a topos \mathcal{E} , i.e. a category with the basic necessary properties to have a chance of being a category of continuous actions of a monoid, and a point of such a topos:

$$\mathbf{Set} \longrightarrow [L^{\text{op}}, \mathbf{Set}] \xrightarrow{p} \mathcal{E}$$

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$$\mathbf{Set} \longrightarrow [L^{\text{op}}, \mathbf{Set}] \longrightarrow \text{Cont}(L, \rho) \longrightarrow \mathcal{E}$$

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Namely, consider the monoid $L := \text{End}(p^*)^{\text{op}}$, dual to the monoid of natural endomorphisms of p^* .

This comes equipped with the coarsest topology making all of the actions of the form $p^*(X)$ continuous.

Complete monoids

In particular, the given point expresses \mathcal{E} as a topos of actions of a topological monoid if and only if the comparison morphism $\text{Cont}(L, \rho) \rightarrow \mathcal{E}$ is an equivalence.

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Examples

- Any discrete monoid is complete.
- Any prodiscrete monoid is complete. Consider the profinite completion or p -adic completion of the integers or the profinite completion of the natural numbers, say.
- The complete monoid corresponding to the real numbers under addition (with their usual topology) is the trivial monoid.

Theorem

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Proof sketch: One direction is by the factorization of the canonical geometric morphism. For the other, we perform the factorization we just saw and verify that the comparison morphism is an equivalence. \square

Theorem

Given a topos \mathcal{E} , a hyperconnected geometric morphism $h : \text{PSh}(M) \rightarrow \mathcal{E}$ realises \mathcal{E} as the topos of continuous actions for a topology τ on M making (M, τ) a complete monoid if and only if h is *representably full and faithful on essential geometric morphisms*, meaning that the functor,

$$h \circ - : \text{EssGeom}(\mathcal{H}, \text{PSh}(M)) \rightarrow \text{Geom}(\mathcal{H}, \mathcal{E})$$

is full and faithful for each topos \mathcal{H} . Similarly, powder monoids are identified by hyperconnected morphisms which are *representably faithful* on essential geometric morphisms.

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Proof sketch: One can reduce to the case that $\mathcal{H} = \mathbf{Set}$ and consider the essential point of $\text{PSh}(M)$ in one direction, and extend from this case in the other. \square

Corollary

For complete monoids (M, τ) and (M', τ') , the surjection–inclusion factorization of a geometric morphism induced by a continuous semigroup homomorphism $\phi : M \rightarrow M'$ is represented by the factorization of ϕ as a monoid homomorphism followed by an inclusion of semigroups, where the intermediate monoid is endowed with the subspace topology.

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Proof sketch: Consider the right-hand square from earlier,

$$\begin{array}{ccc} \text{PSh}(N) & \xrightarrow{i} & \text{PSh}(M') \\ \downarrow h'' & & \downarrow h' \\ \text{Cont}(N, \tau'|_N) & \xrightarrow{j} & \text{Cont}(M', \tau'). \end{array}$$

Since inclusions are representably full and faithful, we deduce that the hyperconnected part of the factorization of $h' \circ i$ coincides with h'' . \square

The other case is more interesting.

Corollary

For complete monoids (M, τ) and (M', τ') , the hyperconnected–localic factorization of a geometric morphism induced by a continuous semigroup homomorphism $\phi : M \rightarrow M'$ is represented by the **dense–closed** factorization of ϕ . In particular, a closed subsemigroup of a complete monoid is complete.

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For complete monoids (M, τ) and (M', τ') , the hyperconnected–localic factorization of a geometric morphism induced by a continuous semigroup homomorphism $\phi : M \rightarrow M'$ is represented by the **dense–closed** factorization of ϕ . In particular, a closed subsemigroup of a complete monoid is complete.

Proof sketch: To show that dense morphisms induce hyperconnected morphisms, we directly check the necessary clopens. For the other factor, we consider a general continuous injective monoid homomorphism and observe that the complete monoid induced by the hyperconnected factor is the closure of its image. \square

Thanks for listening!

Any questions?