Continuous monoid homomorphisms and geometric morphisms

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Outline

1. Categories of actions, and some motivation

2. Semigroup homomorphisms

3. Complete monoids

Section 1

Categories of actions, and some motivation

Discrete monoid actions

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The category of presheaves on a monoid M,

$$\mathsf{PSh}(M) := [M^{\mathrm{op}}, \mathbf{Set}] \simeq \mathbf{Set}\text{-}M,$$

coincides with the category of *right actions* of M.

Continuous monoid actions

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This talk is about categories constructed in this way.

They happen to be examples of Grothendieck toposes.[1]

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[1] For this talk, it is enough to know that these are a class of (co)complete categories.

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- One of many ways of constructing toposes.
- To study monoids, in analogy with studying categories of modules.
- Stepping stone for understanding actions on more general spaces.
- As a context for studying specific actions (in computer science, say).

Necessary clopens

Definition

Let (M, τ) be a topological monoid, X an ordinary M-set. A **necessary clopen** for X is a set of the form,

$$\mathcal{I}_x^p := \{m \in M \mid xm = xp\},$$

where $x \in X$ and $p \in M$.

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Note that for each $x \in X$, the subsets $\mathcal{I}_x^p$ partition M, so that if these subsets are all open they are also necessarily closed, hence the name 'necessary clopen'.

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Lemma

Let (M, τ) be a topological monoid and X an (ordinary) M-set. Then X is continuous with respect to τ if and only if all necessary clopens for X are (cl)open in τ.

Right adjoint to V

Proposition

Let (M, τ) be a topological monoid. Then the forgetful functor $V : \mathrm{Cont}(M, \tau) \to \mathrm{PSh}(M)$ is left exact and comonadic; its right adjoint R sends an M-set X to:

$$R(X) := \{x \in X \mid \forall p \in M, \, \mathcal{I}_x^p \in \tau\}.$$

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$$\begin{tikzcd} \mathrm{PSh}(M) \arrow[r, bend right, "R"'] \arrow[r, phantom, "\perp"] & \mathrm{Cont}(M, \tau) \arrow[l, bend right, "V"'] \end{tikzcd}$$

Action topologies

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Let (M, τ) be a topological monoid. Then there is a unique coarsest topology $\tilde{\tau} \subseteq \tau$ such that the forgetful functor $\mathrm{Cont}(M, \tilde{\tau}) \hookrightarrow \mathrm{Cont}(M, \tau)$ is an equivalence.

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Moreover, $\tilde{\tau}$ is generated by the Boolean algebra $VR(\mathcal{P}(M)) \subseteq \mathcal{P}(M)$.

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Moreover, $\tilde{\tau}$ is generated by the Boolean algebra $VR(\mathcal{P}(M)) \subseteq \mathcal{P}(M)$.

Definition

We say a topology τ is an action topology if $\tau = \tilde{\tau}$, and that (M, τ) is a powder monoid if τ is a T_0 action topology.

The situation

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left=50, "-\times M"] \arrow[r, bend right=50, "\mathrm{Hom}_{\mathbf{Set}}(M,-)"'] \arrow[r, phantom, shift left=2ex, "\perp"] \arrow[r, phantom, shift right=2ex, "\perp"] & \mathrm{PSh}(M) \arrow[l, "U"'] \arrow[r, shift right, "R"'] & \mathrm{Cont}(M,\tau) \arrow[l, shift right, "V"'] \arrow[l, phantom, "\perp"]
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Definitions

Let $\mathcal{F}, \mathcal{E}$ be Grothendieck toposes. A geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is an adjunction $(f^* \dashv f_*)$, where $f_* : \mathcal{F} \to \mathcal{E}$ and f^* preserves finite limits.

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\mathbf{Set} \arrow[r, bend left=50, "- \times M"] \arrow[r, bend right=50, "\mathrm{Hom}_{\mathbf{Set}}(M,-)"'] \arrow[r, phantom, bend left=25, "\perp"] \arrow[r, phantom, bend right=25, "\perp"] & \mathrm{PSh}(M) \arrow[l, "U"'] \arrow[r, shift right=1.5, "R"'] \arrow[r, phantom, "\perp"] & \mathrm{Cont}(M, \tau) \arrow[l, shift right=1.5, "V"']
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$$\begin{tikzcd}
\mathbf{Set} \arrow[r, bend left=60, "- \times M"] \arrow[r, bend right=60, "\mathrm{Hom}_{\mathbf{Set}}(M{,}-)"'] \arrow[r, phantom, bend left=30, "\perp"] \arrow[r, phantom, bend right=30, "\perp"] & \mathrm{PSh}(M) \arrow[l, "U"'] \arrow[r, shift right, "R"'] \arrow[r, phantom, "\perp"] & \mathrm{Cont}(M, \tau) \arrow[l, shift right, "V"']
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Thus from the above diagram we recover a canonical point, $(U \circ V \dashv R \circ \mathrm{Hom}_{\mathbf{Set}}(M, -))$, of $\mathrm{Cont}(M, \tau)$.

Section 2

Semigroup homomorphisms

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Suppose that we have two topological monoids (M, τ) and (M', τ'). Then we have a diagram, and any semigroup homomorphism $\phi : M \to M'$ induces an essential geometric morphism between the presheaf categories:

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\mathrm{PSh}(M) \arrow[r, shift left=4, "f_!"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, shift right=4, "f_*"'] \arrow[d, shift left, "R"] & \mathrm{PSh}(M') \arrow[l, "f^*"'] \arrow[d, shift left, "R'"] \\
\mathrm{Cont}(M,\tau) \arrow[u, hook, shift left, "V"] \arrow[u, phantom, "\dashv"] & \mathrm{Cont}(M',\tau'), \arrow[u, hook, shift left, "V'"] \arrow[u, phantom, "\dashv"]
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\end{tikzcd}$$

When is the restricted functor the inverse image of a geometric morphism?

Continuous homomorphisms

Lemma

The following are equivalent:

1. $f^* : \mathsf{PSh}(M') \to \mathsf{PSh}(M)$ maps every (M', τ')-set to an (M, τ)-set.
2. **ϕ is continuous with respect to τ and $\tilde{\tau}'$.**
3. The composite functor Rf^*V' (is left exact and) has a right adjoint G satisfying $GR \cong R'f_*$, which is to say that f restricts along the functors V, V' to a geometric morphism $(G \dashv Rf^*V') : \mathsf{Cont}(M, \tau) \to \mathsf{Cont}(M', \tau')$ making the square commute.

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Proof sketch: The precomposition functor f^* maps (M', τ')-sets to (M, τ)-sets if and only if ϕ^{-1} sends necessary clopens to opens, which is equivalent to continuity. It also implies the existence of the right adjoint via the identity $V(Rf^*V') \cong f^*V'$, which is equivalent to the third statement. $\square$

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\end{tikzcd}$$

Remark

Even if we restrict to powder monoids, it is not the case that all geometric morphisms arise from continuous semigroup homomorphisms; in particular, there are equivalences of such toposes which are not induced by semigroup homomorphisms.

Factorizations

The two best-known factorization systems for geometric morphisms are the surjection–inclusion and hyperconnected–localic factorizations, and these are compatible:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\text{hyperconnected}"'] \arrow[rrr, bend left, "f"] & \mathcal{G} \arrow[r, "\substack{\text{localic}\\ \text{surjection}}"'] & \mathcal{H} \arrow[r, "\text{inclusion}"'] & \mathcal{E}
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\end{tikzcd}$$

For discrete monoids, the factorization of a geometric morphism induced by a semigroup homomorphism is represented by a factorization of the semigroup homomorphism:

$$\begin{tikzcd}
M \arrow[r, "\substack{\text{surjective}\\ \text{monoid hom}}"'] \arrow[rrr, bend left, "\phi"] & L \arrow[r, "\substack{\text{injective}\\ \text{monoid hom}}"'] & N \arrow[r, "\substack{\text{subsemigroup}\\ \text{inclusion}}"'] & M'
\end{tikzcd}$$

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\mathrm{Cont}(M, \tau) \arrow[r, "t"] \arrow[rr, bend right, "g"'] & \mathrm{Cont}(N, \tau'|_N) \arrow[r, "j"] & \mathrm{Cont}(M', \tau').
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\end{tikzcd}$$

We deduce that t is a surjection since $h'' \circ s$ is, but is j an inclusion?

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At this level of generality, j could have a non-trivial hyperconnected factor, and it's not clear that the intermediate topos will necessarily be a topos of monoid actions.
We need to characterize toposes of topological monoid actions, which brings us to a new class of topological monoids.

Section 3

Complete monoids

Best approximation

Suppose that we are given a topos $\mathcal{E}$, i.e. a category with the basic necessary properties to have a chance of being a category of continuous actions of a monoid, and a point of such a topos:

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Theorem

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Namely, consider the monoid $L := \mathrm{End}(p^*)^{\mathrm{op}}$, dual to the monoid of natural endomorphisms of p^*.
This comes equipped with the coarsest topology making all of the actions of the form $p^*(X)$ continuous.

Complete monoids

In particular, the given point expresses $\mathcal{E}$ as a topos of actions of a topological monoid if and only if the comparison morphism $\mathrm{Cont}(L, \rho) \to \mathcal{E}$ is an equivalence.

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Examples

- Any discrete monoid is complete.
- Any prodiscrete monoid is complete. Consider the profinite completion or p-adic completion of the integers or the profinite completion of the natural numbers, say.
- The complete monoid corresponding to the real numbers under addition (with their usual topology) is the trivial monoid.

Complete monoids

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A topos $\mathcal{E}$ is a topos of topological monoid actions if and only if $\mathcal{E}$ has a point which factors as an essential surjection followed by a hyperconnected morphism.

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Proof sketch: One direction is by the factorization of the canonical geometric morphism. For the other, we perform the factorization we just saw and verify that the comparison morphism is an equivalence. ☐

Intrinsic characterization

Theorem

Given a topos $\mathcal{E}$, a hyperconnected geometric morphism $h : \mathrm{PSh}(M) \to \mathcal{E}$ realises $\mathcal{E}$ as the topos of continuous actions for a topology τ on M making (M, τ) a complete monoid if and only if *h is representably full and faithful on essential geometric morphisms*, meaning that the functor,

$$h \circ - : \mathrm{EssGeom}(\mathcal{H}, \mathrm{PSh}(M)) \to \mathrm{Geom}(\mathcal{H}, \mathcal{E})$$

is full and faithful for each topos $\mathcal{H}$. Similarly, powder monoids are identified by hyperconnected morphisms which are *representably faithful* on essential geometric morphisms.

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is full and faithful for each topos $\mathcal{H}$. Similarly, powder monoids are identified by hyperconnected morphisms which are *representably faithful* on essential geometric morphisms.

Proof sketch: One can reduce to the case that $\mathcal{H} = \mathbf{Set}$ and consider the essential point of $\mathrm{PSh}(M)$ in one direction, and extend from this case in the other. □

Surjection–Inclusion

Corollary

For complete monoids (M, τ) and (M', τ'), the surjection–inclusion factorization of a geometric morphism induced by a continuous semigroup homomorphism $\phi : M \to M'$ is represented by the factorization of ϕ as a monoid homomorphism followed by an inclusion of semigroups, where the intermediate monoid is endowed with the subspace topology.

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Proof sketch: Consider the right-hand square from earlier,

$$\begin{tikzcd}
\mathrm{PSh}(N) \arrow[r, "i"'] \arrow[d, "h''"] & \mathrm{PSh}(M') \arrow[d, "h'"] \\
\mathrm{Cont}(N, \tau'|_N) \arrow[r, "j"] & \mathrm{Cont}(M', \tau').
\end{tikzcd}$$

Since inclusions are representably full and faithful, we deduce that the hyperconnected part of the factorization of $h' \circ i$ coincides with h''. □

Hyperconnected–Localic

The other case is more interesting.

Corollary

For complete monoids (M, τ) and (M', τ'), the hyperconnected–localic factorization of a geometric morphism induced by a continuous semigroup homomorphism $\phi : M \to M'$ is represented by the **dense–closed** factorization of ϕ. In particular, a closed subsemigroup of a complete monoid is complete.

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Proof sketch: To show that dense morphisms induce hyperconnected morphisms, we directly check the necessary clopens. For the other factor, we consider a general continuous injective monoid homomorphism and observe that the complete monoid induced by the hyperconnected factor is the closure of its image. $\square$

Fin

Thanks for listening!

Any questions?