Continuous monoid homomorphisms and geometric morphisms
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1 Categories of actions, and some motivation

2 Semigroup homomorphisms

3 Complete monoids
Section 1

Categories of actions, and some motivation
We may view a monoid $M$ as a one-object category.
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The category of presheaves on a monoid $M$,

$$\text{PSh}(M) := [M^{\text{op}}, \text{Set}] \cong \text{Set}^{-M},$$

coincides with the category of right actions of $M$. 
We might more generally consider a topological monoid \((M, \tau)\).
Continuous monoid actions

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We can then consider the full subcategory of \(\text{PSh}(M)\) on the actions which are continuous with respect to \(\tau\) (when sets are considered as discrete spaces):

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\[
\text{PSh}(M) \leftrightarrow \text{Cont}(M, \tau) : V.
\]

This talk is about categories constructed in this way.

They happen to be examples of Grothendieck toposes.\(^1\)

---

\(^1\)For this talk, it is enough to know that these are a class of (co)complete categories.
Why study these categories?

• One of many ways of constructing toposes.
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- To study monoids, in analogy with studying categories of modules.
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- Stepping stone for understanding actions on more general spaces.
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- One of many ways of constructing toposes.
- To study monoids, in analogy with studying categories of modules.
- Stepping stone for understanding actions on more general spaces.
- As a context for studying specific actions (in computer science, say).
Definition

Let \((M, \tau)\) be a topological monoid, \(X\) an ordinary \(M\)-set. A **necessary clopen** for \(X\) is a set of the form,

\[ I^p_x := \{ m \in M \mid xm = xp \}, \]

where \(x \in X\) and \(p \in M\).
Necessary clopens

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Note that for each \(x \in X\), the subsets \(I^p_x\) partition \(M\), so that if these subsets are all open they are also necessarily closed, hence the name ‘necessary clopen’.
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**Lemma**

Let \((M, \tau)\) be a topological monoid and \(X\) an (ordinary) \(M\)-set. Then \(X\) is continuous with respect to \(\tau\) if and only if all necessary clopens for \(X\) are (cl)open in \(\tau\).
Proposition

Let \((M, \tau)\) be a topological monoid. Then the forgetful functor \(V : \text{Cont}(M, \tau) \to \text{PSh}(M)\) is left exact and comonadic; its right adjoint \(R\) sends an \(M\)-set \(X\) to:

\[
R(X) := \{ x \in X \mid \forall p \in M, \mathcal{I}_x^p \in \tau \}.
\]
Proposition

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$$R(X) := \{x \in X \mid \forall p \in M, \mathcal{I}_x^p \in \tau\}.$$
Lemma

Let \((M, \tau)\) be a topological monoid. Then there is a unique coarsest topology \(\tilde{\tau} \subseteq \tau\) such that the forgetful functor \(\text{Cont}(M, \tilde{\tau}) \rightarrow \text{Cont}(M, \tau)\) is an equivalence. Moreover, \(\tilde{\tau}\) is generated by the Boolean algebra \(\mathcal{V}R(P(M)) \subseteq P(M)\).

Definition

We say a topology \(\tau\) is an action topology if \(\tau = \tilde{\tau}\), and that \((M, \tau)\) is a powder monoid if \(\tau\) is a \(T_0\) action topology.
**Lemma**

Let \((M, \tau)\) be a topological monoid. Then there is a unique coarsest topology \(\tilde{\tau} \subseteq \tau\) such that the forgetful functor \(\text{Cont}(M, \tilde{\tau}) \hookrightarrow \text{Cont}(M, \tau)\) is an equivalence.
**Lemma**

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Moreover, \(\tilde{\tau}\) is generated by the Boolean algebra \(VR(\mathcal{P}(M)) \subseteq \mathcal{P}(M)\).
**Lemma**

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We say a topology \(\tau\) is an **action topology** if \(\tau = \tilde{\tau}\), and that \((M, \tau)\) is a **powder monoid** if \(\tau\) is a \(T_0\) action topology.
The situation

\[ \text{Set} \leftarrow \text{PSh}(M) \leftarrow \text{Cont}(M, \tau) \]

Definitions

Let \( F, E \) be Grothendieck toposes. A geometric morphism \( f: F \to E \) is an adjunction \( (f^* \dashv f_*) \), where \( f^*: F \to E \) and \( f_* \) preserves finite limits.

A point of a Grothendieck topos \( E \) is a geometric morphism \( \text{Set} \to E \).

Thus from the above diagram we recover a canonical point, \((U \circ V \dashv R \circ \text{Hom}_{\text{Set}}(M, -))\), of \( \text{Cont}(M, \tau) \).
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Definitions
The situation

\[
\begin{array}{c}
\text{Set} \\
\downarrow U \\
\text{PSh}(M) \\
\downarrow \text{Hom}_{\text{Set}}(M,-) \\
\end{array}
\begin{array}{c}
\text{Cont}(M,\tau) \\
\downarrow R \\
\end{array}
\quad \xleftarrow{\quad V \quad} \quad \xrightarrow{- \times M}
\]

Definitions

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Thus from the above diagram we recover a canonical point, $(U \circ V \dashv R \circ \text{Hom}_{\textbf{Set}}(M, -))$, of $\text{Cont}(M, \tau)$.
Section 2

Semigroup homomorphisms
Comparing categories of actions

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Comparing categories of actions

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\[
\begin{array}{ccc}
\text{PSh}(M) & \xrightarrow{\nu} & \text{PSh}(M') \\
\downarrow R & & \downarrow R' \\
\text{Cont}(M, \tau) & \xleftarrow{\nu'} & \text{Cont}(M', \tau'),
\end{array}
\]

When is the restricted functor the inverse image of a geometric morphism?
Comparing categories of actions

Suppose that we have two topological monoids \((M, \tau)\) and \((M', \tau')\). Then we have a diagram, and any semigroup homomorphism \(\phi : M \rightarrow M'\) induces an essential geometric morphism between the presheaf categories:

\[
\begin{array}{ccc}
PSh(M) & \xrightarrow{V} & PSh(M') \\
\downarrow R & & \downarrow R' \\
Cont(M, \tau) & & Cont(M', \tau')
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V & \downarrow & V' & \text{Cont}(M, \tau) & \text{Cont}(M', \tau')
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\[
\begin{array}{ccc}
\text{PSh}(M) & \xrightarrow{R} & \text{PSh}(M') \\
\downarrow & & \downarrow \\
\text{Cont}(M, \tau) & \xrightarrow{f^*_R} & \text{Cont}(M', \tau'),
\end{array}
\]

When is the restricted functor the inverse image of a geometric morphism?
Lemma

The following are equivalent:

1. \( f^* : PSh(M') \to PSh(M) \) maps every \((M', \tau')\)-set to an \((M, \tau)\)-set.

2. \( \phi \) is continuous with respect to \( \tau \) and \( \tilde{\tau}' \).

3. The composite functor \( Rf^* V' \) (is left exact and) has a right adjoint \( G \) satisfying \( GR \cong R'f_* \), which is to say that \( f \) restricts along the functors \( V, V' \) to a geometric morphism \( (G \dashv Rf^* V') : \text{Cont}(M, \tau) \to \text{Cont}(M', \tau') \) making the square commute.

Proof sketch:

The precomposition functor \( f^* \) maps \((M', \tau')\)-sets to \((M, \tau)\)-sets if and only if \( \phi^{-1} \) sends necessary clopens to opens, which is equivalent to continuity. It also implies the existence of the right adjoint via the identity \( V(Rf^* V') \cong f^* V' \), which is equivalent to the third statement. \( \square \)
Lemma

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**Proof sketch:** The precomposition functor \( f^* \) maps \((M', \tau')\)-sets to \((M, \tau)\)-sets if and only if \( \phi^{-1} \) sends necessary clopens to opens, which is equivalent to continuity. It also implies the existence of the right adjoint via the identity \( V(Rf^* V') \cong f^* V' \), which is equivalent to the third statement. \( \square \)
In particular, any continuous semigroup homomorphism $\phi : M \to M'$ gives a commuting square of geometric morphisms:
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$$
\begin{array}{ccc}
PSh(M) & \overset{f^*}{\leftarrow} & PSh(M') \\
\downarrow & & \downarrow \\
R & \overset{g^*}{\leftarrow} & R'
\end{array}
$$

$$
\begin{array}{ccc}
Cont(M, \tau) & \overset{f^*}{\leftarrow} & Cont(M', \tau') \\
\downarrow & & \downarrow \\
g^* & \overset{g^*}{\leftarrow} & g^*
\end{array}
$$

Remark

Even if we restrict to powder monoids, it is not the case that all geometric morphisms arise from continuous semigroup homomorphisms; in particular, there are equivalences of such toposes which are not induced by semigroup homomorphisms.
In particular, any continuous semigroup homomorphism $\phi : M \to M'$ gives a commuting square of geometric morphisms:

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**Remark**

Even if we restrict to powder monoids, it is not the case that all geometric morphisms arise from continuous semigroup homomorphisms; in particular, there are equivalences of such toposes which are not induced by semigroup homomorphisms.
The two best-known factorization systems for geometric morphisms are the surjection–inclusion and hyperconnected–localic factorizations, and these are compatible:
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\[
\begin{align*}
\mathcal{F} & \xrightarrow{\text{hyperconnected}} \mathcal{G} \xrightarrow{\text{localic surjection}} \mathcal{H} \xrightarrow{\text{inclusion}} \mathcal{E} \\
\end{align*}
\]

For discrete monoids, the factorization of a geometric morphism induced by a semigroup homomorphism is represented by a factorization of the semigroup homomorphism:

\[
\begin{align*}
\mathcal{M} & \xrightarrow{\phi} \mathcal{L} \xrightarrow{\text{injective monoid hom}} \mathcal{N} \xrightarrow{\text{subsemigroup inclusion}} \mathcal{M}' \\
\end{align*}
\]
Factorizations

The natural next question is what happens for topological monoids?
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\[
\begin{align*}
\text{PSh}(M) & \xrightarrow{s} \text{PSh}(N) \xrightarrow{i} \text{PSh}(M') \\
\text{Cont}(M, \tau) & \xrightarrow{t} \text{Cont}(N, \tau'|_N) \xrightarrow{j} \text{Cont}(M', \tau').
\end{align*}
\]

We deduce that \(t\) is a surjection since \(h''\circ s\) is, but is \(j\) an inclusion?
The natural next question is what happens for topological monoids? Let’s consider the surjection–inclusion factorization. We can endow the intermediate monoid with the subspace topology to get a commuting diagram,

\[
\begin{array}{ccc}
PSh(M) & \xrightarrow{s} & PSh(N) \\
\downarrow{h} & & \downarrow{h''} \\
Cont(M, \tau) & \xrightarrow{t} & Cont(N, \tau'|_N) \\
\downarrow{g} & & \downarrow{h'} \\
& & Cont(M', \tau').
\end{array}
\]

We deduce that \( t \) is a surjection since \( h'' \circ s \) is, but is \( j \) an inclusion?
At this level of generality, $j$ could have a non-trivial hyperconnected factor, and it’s not clear that the intermediate topos will necessarily be a topos of monoid actions.
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Section 3

Complete monoids
Suppose that we are given a topos $\mathcal{E}$, i.e. a category with the basic necessary properties to have a chance of being a category of continuous actions of a monoid, and a point of such a topos:

$$\xymatrix{ \text{Set} \ar[r]^p & \mathcal{E} }$$
Best approximation

Suppose that we are given a topos $\mathcal{E}$, i.e. a category with the basic necessary properties to have a chance of being a category of continuous actions of a monoid, and a point of such a topos:

$$\begin{array}{c}
\text{Set} \\
\downarrow \quad p \\
\mathcal{E}
\end{array}$$

**Theorem**

There is a canonical factorization of $p$ through a topos of topological monoid actions.
Suppose that we are given a topos $\mathcal{E}$, i.e. a category with the basic necessary properties to have a chance of being a category of continuous actions of a monoid, and a point of such a topos:

\[
\begin{array}{ccc}
\text{Set} & \longrightarrow & [L^{\text{op}}, \text{Set}] \\
p & \mapsto & \mathcal{E}
\end{array}
\]

**Theorem**

There is a canonical factorization of $p$ through a topos of topological monoid actions. Namely, consider the monoid $L := \text{End}(p^*)^{\text{op}}$, dual to the monoid of natural endomorphisms of $p^*$. 
Suppose that we are given a topos $\mathcal{E}$, i.e. a category with the basic necessary properties to have a chance of being a category of continuous actions of a monoid, and a point of such a topos:

$$\text{Set} \xrightarrow{\text{p}} [L^{\text{op}}, \text{Set}] \xrightarrow{\text{Cont}(L, \rho)} \mathcal{E}$$

**Theorem**

There is a canonical factorization of $p$ through a topos of topological monoid actions. Namely, consider the monoid $L := \text{End}(p^*)^{\text{op}}$, dual to the monoid of natural endomorphisms of $p^*$. This comes equipped with the coarsest topology making all of the actions of the form $p^*(X)$ continuous.
Complete monoids

In particular, the given point expresses $\mathcal{E}$ as a topos of actions of a topological monoid if and only if the comparison morphism $\text{Cont}(L, \rho) \rightarrow \mathcal{E}$ is an equivalence.

Definition

We call a topological monoid complete if it is isomorphic to the topological monoid of endomorphisms of its canonical point.

Examples

- Any discrete monoid is complete.
- Any prodiscrete monoid is complete. Consider the profinite completion or $p$-adic completion of the integers or the profinite completion of the natural numbers, say.
- The complete monoid corresponding to the real numbers under addition (with their usual topology) is the trivial monoid.
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Complete monoids

Theorem

A topos $\mathcal{E}$ is a topos of topological monoid actions if and only if $\mathcal{E}$ has a point which factors as an essential surjection followed by a hyperconnected morphism.

Proof sketch:

One direction is by the factorization of the canonical geometric morphism. For the other, we perform the factorization we just saw and verify that the comparison morphism is an equivalence. □
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**Theorem**

A topos $\mathcal{E}$ is a topos of topological monoid actions if and only if $\mathcal{E}$ has a point which factors as an essential surjection followed by a hyperconnected morphism.

**Proof sketch:** One direction is by the factorization of the canonical geometric morphism. For the other, we perform the factorization we just saw and verify that the comparison morphism is an equivalence. □
Theorem

Given a topos $\mathcal{E}$, a hyper-connected geometric morphism $h : \text{PSh}(M) \to \mathcal{E}$ realises $\mathcal{E}$ as the topos of continuous actions for a topology $\tau$ on $M$ making $(M, \tau)$ a complete monoid if and only if $h$ is representably full and faithful on essential geometric morphisms, meaning that the functor,

$$h \circ - : \text{EssGeom}(\mathcal{H}, \text{PSh}(M)) \to \text{Geom}(\mathcal{H}, \mathcal{E})$$

is full and faithful for each topos $\mathcal{H}$. Similarly, powder monoids are identified by hyper-connected morphisms which are representably faithful on essential geometric morphisms.
Intrinsic characterization

**Theorem**

Given a topos $\mathcal{E}$, a hyperconnected geometric morphism $h : \text{PSh}(M) \to \mathcal{E}$ realises $\mathcal{E}$ as the topos of continuous actions for a topology $\tau$ on $M$ making $(M, \tau)$ a complete monoid if and only if $h$ is representably full and faithful on essential geometric morphisms, meaning that the functor,

$$\text{id} \circ h : \text{EssGeom}(\mathcal{H}, \text{PSh}(M)) \to \text{Geom}(\mathcal{H}, \mathcal{E})$$

is full and faithful for each topos $\mathcal{H}$. Similarly, powder monoids are identified by hyperconnected morphisms which are representably faithful on essential geometric morphisms.

**Proof sketch:** One can reduce to the case that $\mathcal{H} = \text{Set}$ and consider the essential point of $\text{PSh}(M)$ in one direction, and extend from this case in the other. □
Corollary

For complete monoids \((M, \tau)\) and \((M', \tau')\), the surjection–inclusion factorization of a geometric morphism induced by a continuous semigroup homomorphism \(\phi : M \to M'\) is represented by the factorization of \(\phi\) as a monoid homomorphism followed by an inclusion of semigroups, where the intermediate monoid is endowed with the subspace topology.

Proof sketch: Consider the right-hand square from earlier, since inclusions are representably full and faithful, we deduce that the hyperconnected part of the factorization of \(h' \circ i\) coincides with \(h''\). □
Corollary

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Proof sketch: Consider the right-hand square from earlier,

\[
\begin{array}{c}
\text{PSh}(N) \xrightarrow{i} \text{PSh}(M') \\
\downarrow h'' \quad \downarrow h' \\
\text{Cont}(N, \tau'|_N) \xrightarrow{j} \text{Cont}(M', \tau').
\end{array}
\]

Since inclusions are representably full and faithful, we deduce that the hyperconnected part of the factorization of \(h' \circ i\) coincides with \(h''\). \(\square\)
The other case is more interesting.

**Corollary**

For complete monoids $(M, \tau)$ and $(M', \tau')$, the hyperconnected–localic factorization of a geometric morphism induced by a continuous semigroup homomorphism $\phi: M \rightarrow M'$ is represented by the dense–closed factorization of $\phi$. In particular, a closed subsemigroup of a complete monoid is complete.

Proof sketch: To show that dense morphisms induce hyperconnected morphisms, we directly check the necessary clopens. For the other factor, we consider a general continuous injective monoid homomorphism and observe that the complete monoid induced by the hyperconnected factor is the closure of its image. □
The other case is more interesting.

### Corollary

For complete monoids \((M, \tau)\) and \((M', \tau')\), the hyperconnected–localic factorization of a geometric morphism induced by a continuous semigroup homomorphism \(\phi : M \to M'\) is represented by the **dense–closed** factorization of \(\phi\). In particular, a closed subsemigroup of a complete monoid is complete.

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Thanks for listening!

Any questions?