Lifting weights

Enriched lenses between transport plans

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SYCO 9
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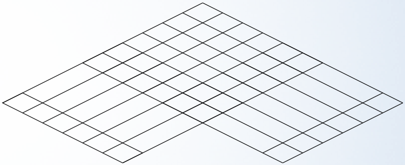
Couplings and optimal transport

Given a (metric) space X, the product space $X \times X$ encodes "transport plans".



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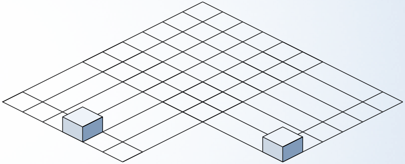

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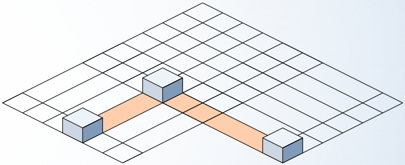
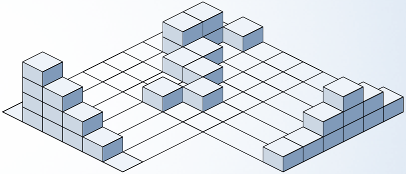


Couplings and optimal transport

The category PX has

- As objects, prob. measures p on X;
- As morphisms, transport plans t.



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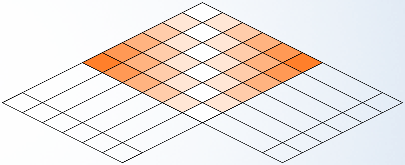

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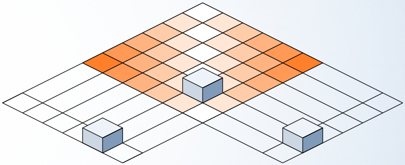

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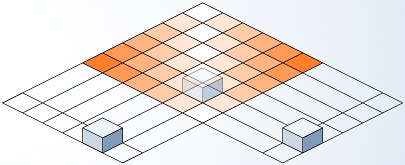
If X has a cost function c,

$$c(x,x) = 0$$



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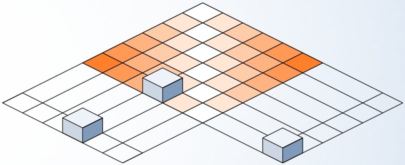

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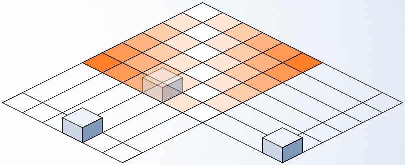
$c(x, y) =$ "cost of transport"



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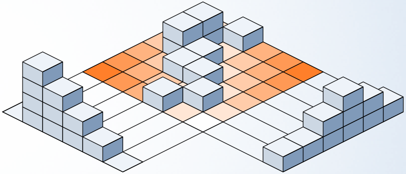
If X has a cost function c,

$$C(t) := \int_{X^2} c(x,y)\, t(dx\, dy).$$



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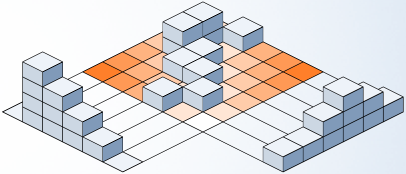
If X has a cost function c,

$$C_k(t) \coloneqq \sqrt[k]{\int_{X^2} c(x,y)^k \, t(dx\,dy)}.$$



Weighted categories and functors

Main definitions

A *weighted category* is a category where each morphism $f : X \to Y$ is equipped with a nonnegative number $w(f)$ called the *weight*, such that

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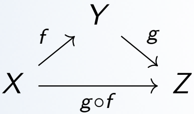

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$$\begin{array}{ccccc} & & Y & & \\ & {\scriptstyle f} \nearrow & & \searrow {\scriptstyle g} & \\ X & & \xrightarrow[g \circ f]{} & & Z \end{array} \qquad\qquad w(g \circ f) \leq w(f) + w(g).$$

A *weighted functor* is a functor $F : \mathsf{C} \to \mathsf{D}$ such that for every morphism f of C,

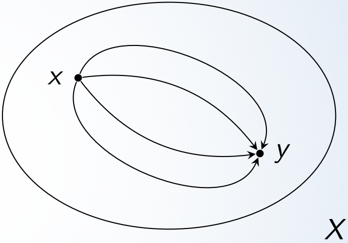
$$w(Ff) \leq w(f).$$

Examples

Categories of paths *of a space*

Let X be a metric space (e.g. $\mathbb{R}^n$). The weighted category $\mathrm{Path}(X)$ has

- As objects, the points of X;
- As morphisms, the curves in X with their length as weight.



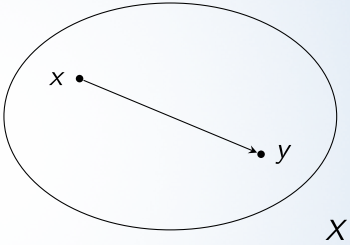
Examples

Generalized metric spaces

A *pseudo-quasi* (or *Lawvere*) *metric space* is a set X with a "cost" function $c : X \times X \to [0, \infty]$ such that

- $d(x, x) = 0$;
- $d(x, z) \leq d(x, y) + d(y, z)$

A pq-metric space is a weighted preorder.



Examples

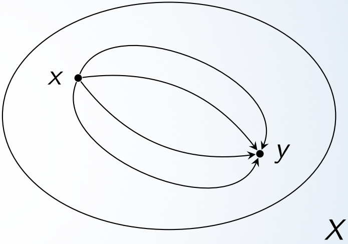
Optimization over paths

Given a weighted category C, for objects X and Y consider the "optimum" weight

$$\inf_{f:X\to Y} w(f)$$

This gives a pq-metric on the objects of C. We call the resulting space Opt(C).

The Wasserstein distances are an example [Villani, 2009].



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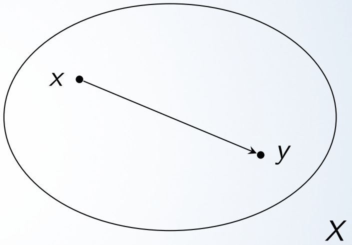
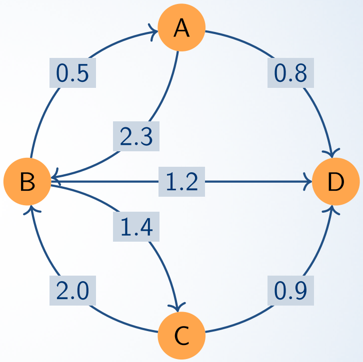


Weighted categories and weighted graphs

Free weighted category

Let G be a weighted graph. The *free weighted category over* G has

- As objects, the vertices of G;
- As morphisms, *walks* in G;
- The weight is the sum of the weights.



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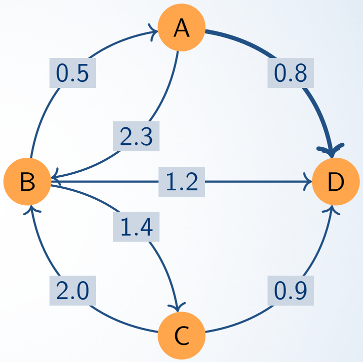

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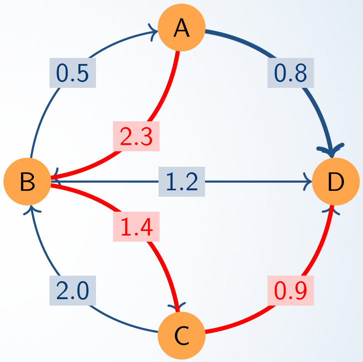

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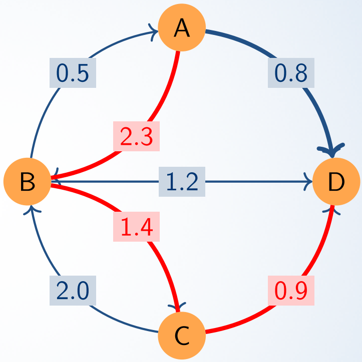
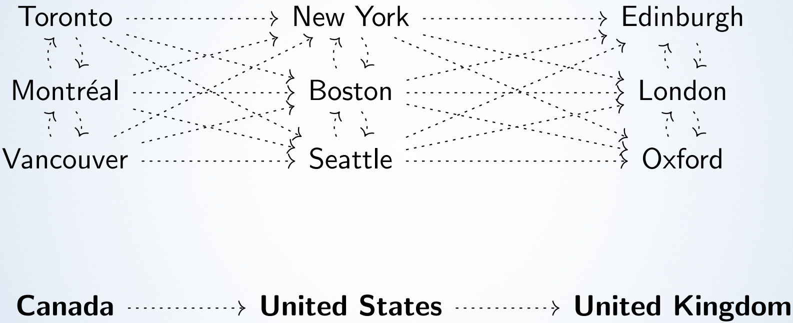
See Jade's talk later today!



Weighted lenses



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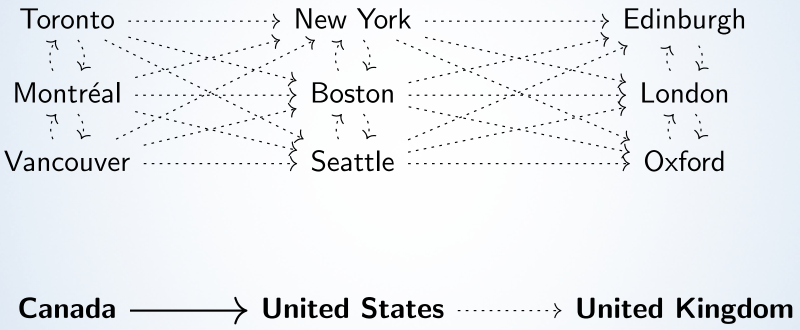

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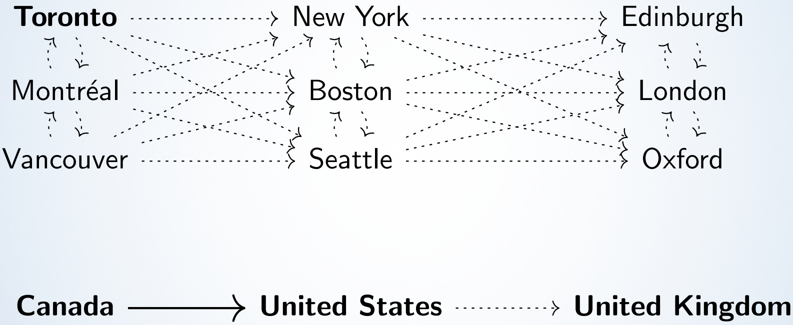

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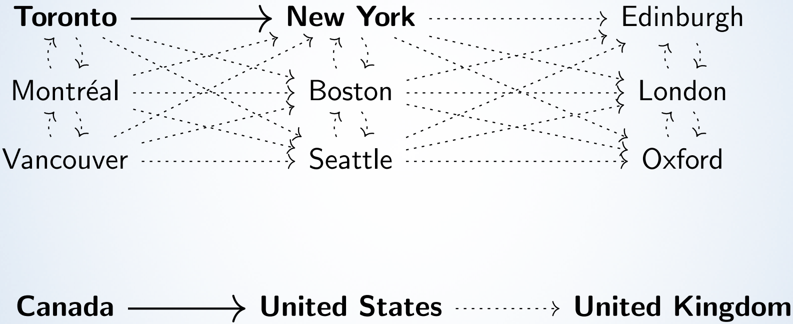

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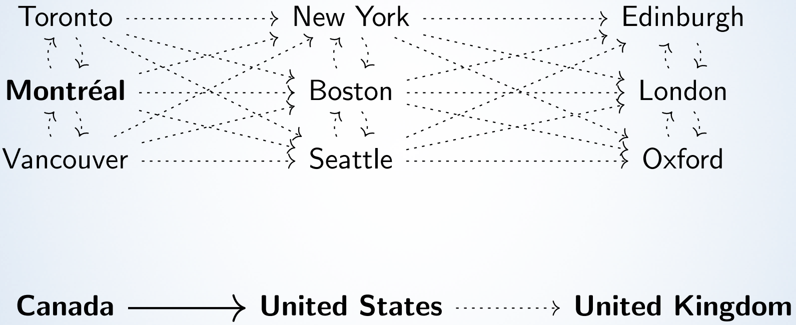

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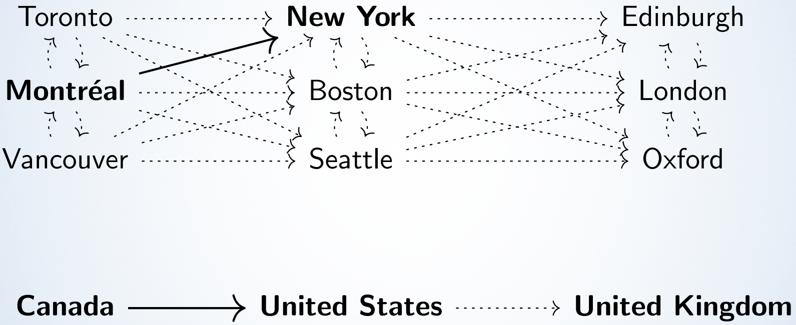

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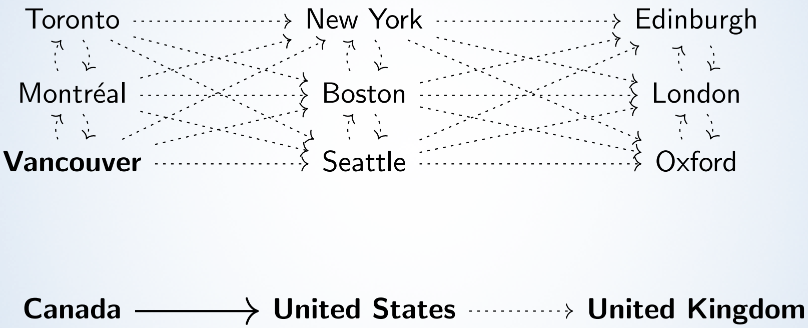

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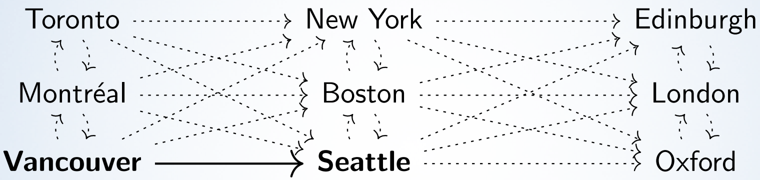

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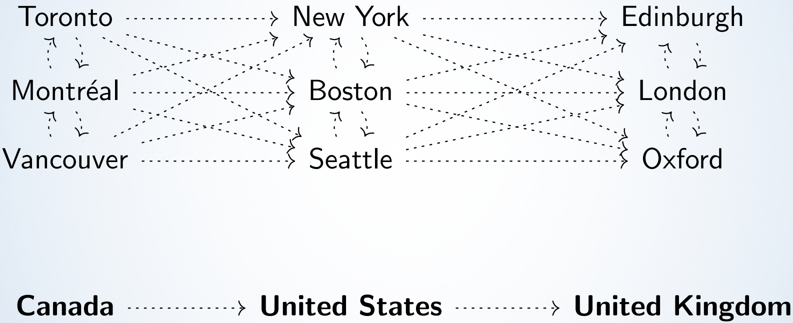

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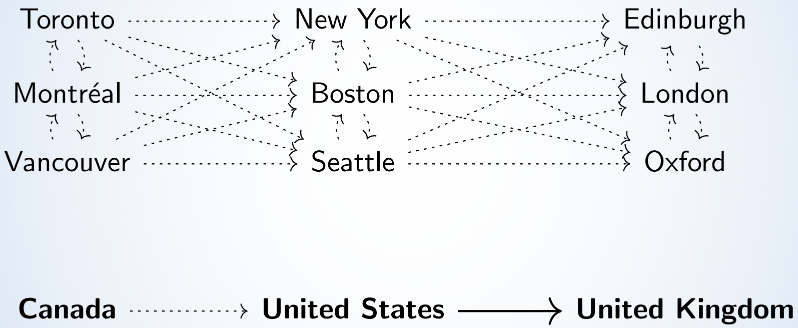

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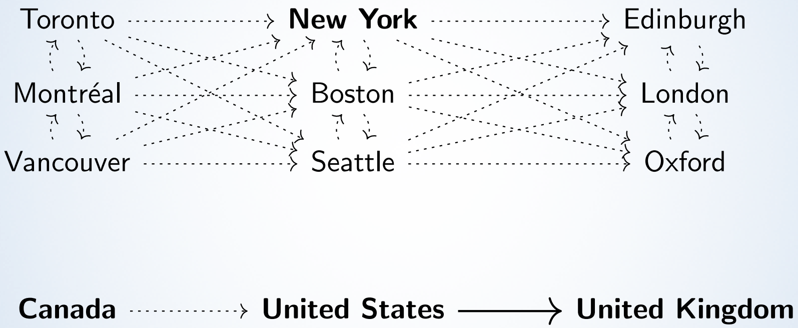

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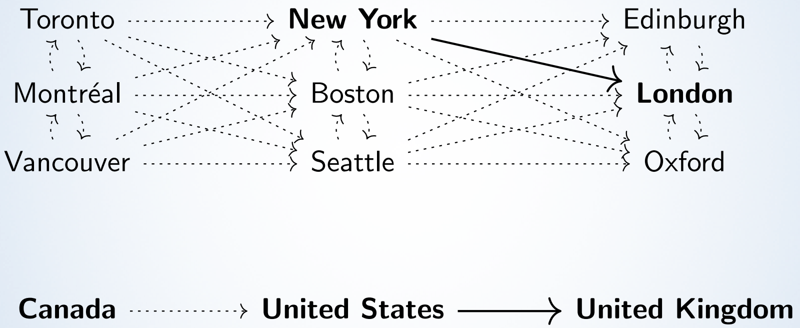

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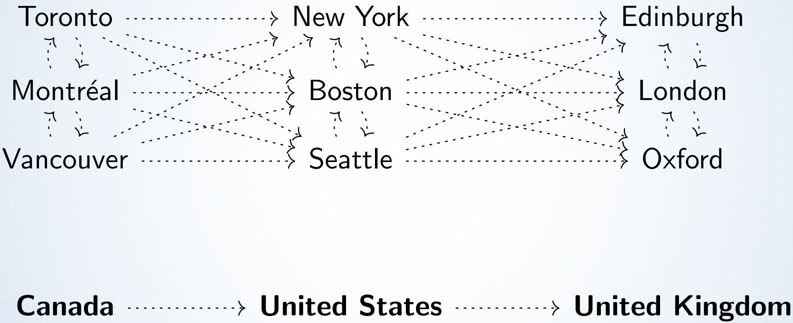

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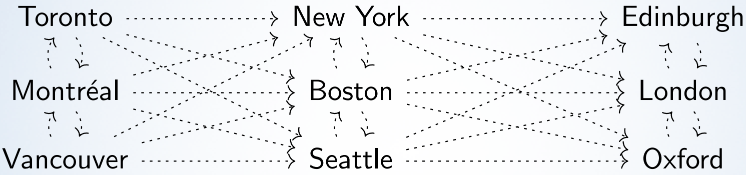
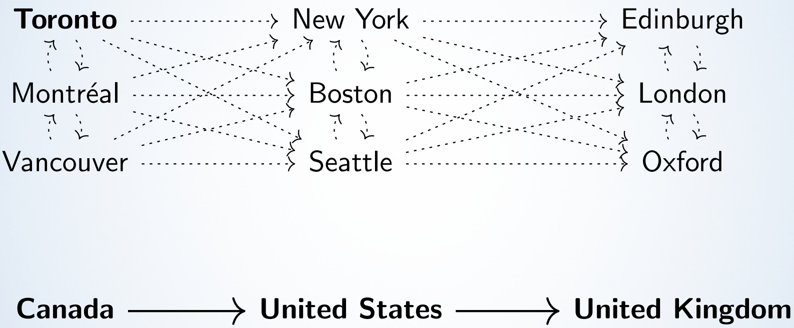


Canada $\longrightarrow$ **United States** $\longrightarrow$ **United Kingdom**

Weighted lenses



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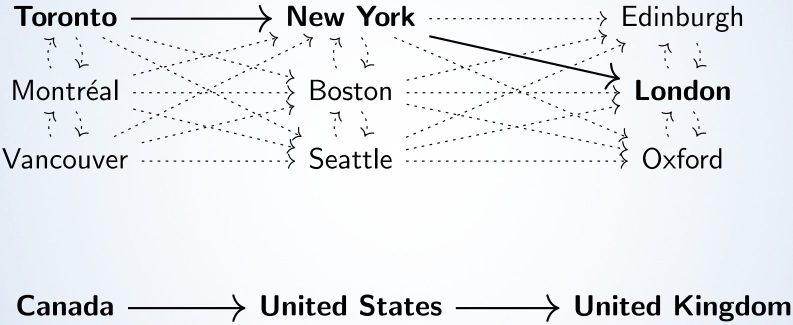
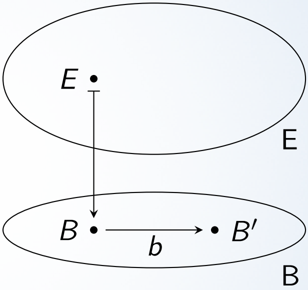


Weighted lenses

Definition: weighted lifting

- Let $F : \mathsf{E} \to \mathsf{B}$ be a functor;
- Let $b : B \to B'$ in B;
- Let $E \in \mathsf{E}$ with $F(E) = B$.

A *lifting of b at E* consists of



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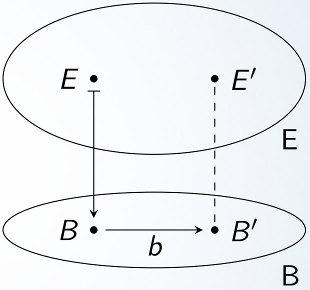

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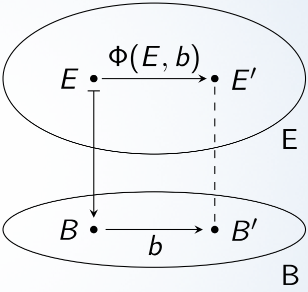
A *lifting of b at E* consists of

- an object $E' \in \mathsf{E}$ with $F(E') = B'$;
- an arrow $\Phi(E, b) : E \to E'$ of E such that $F\big(\Phi(E, b)\big) = b$, with the *same weight* as b.

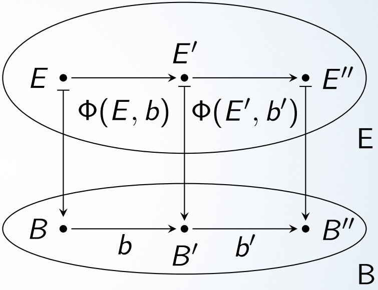


Weighted lenses

Definition: weighted lens

Let E and B be categories. A *weighted lens* from E $\to$ B is

- A functor $F : \mathsf{E} \to \mathsf{B}$;
- For each morphism $b : B \to B'$ of B and each object E of E with $F(E) = B$, a chosen weighted lifting $\Phi(E, b) : E \to E'$, such that
- Identities and compositions are preserved.



Lenses between categories of couplings

Theorem (P 2021)

Let X and Y be pq-metric, standard Borel spaces. Let (f, ϕ) be a weighted lens such that the assignments f and ϕ are measurable.

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Let X and Y be pq-metric, standard Borel spaces. Let (f, ϕ) be a weighted lens such that the assignments f and ϕ are measurable. There is a weighted lens $(f_\sharp, \tilde{\varphi}_\sharp)$ between PX and PY where

- The projection $f_\sharp : PX \to PY$ is the pushforward;

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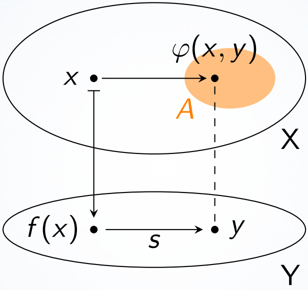
- The projection $f_\sharp : PX \to PY$ is the pushforward;
- The lifting $\tilde{\varphi}_\sharp : PX \times_{PY} P(Y \times Y) \to P(X \times X)$ takes $p \in PX$ and a coupling $s \in P(Y \times Y)$ with first marginal $f_* p$, and returns the coupling $\tilde{\varphi}_\sharp(p, s) \in P(X \times X)$ given for all measurable $A, A' \subseteq X$ by

$$\tilde{\varphi}_\sharp(p, s)(A \times A') := \int_A \int_Y 1_{A'}\big(\varphi(x, y)\big)\, s\big(dy|f(x)\big)\, p(dx).$$

Lenses between categories of couplings

Intuitively, we are “lifting random transitions”.



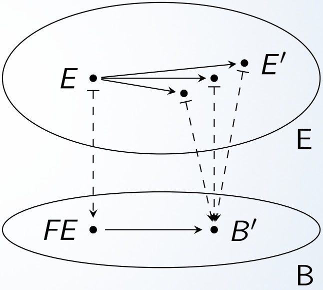
$$\tilde{\varphi}_\sharp(s)(A|x) = \int_Y 1_A\big(\varphi(x,y)\big)\, s\big(dy|f(x)\big)$$

Co-history!

Enriched lenses [Clarke and Di Meglio, 2022]

Let E and B be V-enriched categories. A V-*lens* $(F, \Phi) : \mathsf{E} \to \mathsf{B}$ is

- A V-functor $F : \mathsf{E} \to \mathsf{B}$;



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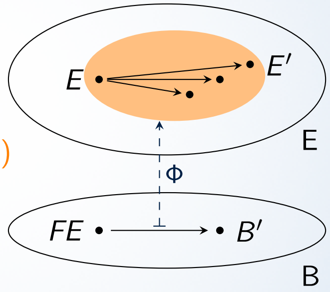
- A V-functor $F : \mathsf{E} \to \mathsf{B}$;
- For each object $E \in \mathsf{E}$ and $B' \in \mathsf{B}$, a V-arrow lifting

$$\mathsf{B}(FE, B') \xrightarrow{\Phi_{E,B'}} \coprod_{E' \in F^{-1}(B')} \mathsf{E}(E, E')$$

and satisfying identity and composition requirements.



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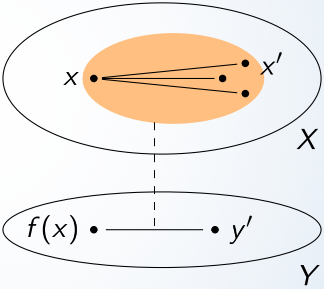
Weak submetries are enriched lenses (Callum Reader)

For $V = [0, \infty]$ ordered downward (Lawvere metric spaces):

$$d_Y\big(f(x), y'\big) = \inf_{x' \in f^{-1}(y)} d(x, x').$$



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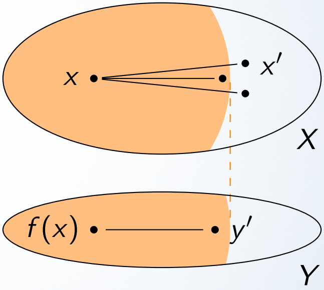
In terms of open balls,

$$f\big(B_X(x, r)\big) = B_Y\big(f(x), r\big)$$

for all $x \in X$ $r \in \mathbb{R}$. In geometry, this is called a *weak submetry*.



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