Lifting weights
Enriched lenses between transport plans

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Couplings and optimal transport

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If $X$ has a cost function $c$,

$$C(t) := \int_{X^2} c(x, y) \, t(dx \, dy).$$
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If $X$ has a cost function $c$,

$$C_k(t) := \sqrt[k]{\int_{X^2} c(x, y)^k \, t(dx \, dy)}.$$
Weighted categories and functors

Main definitions
A *weighted category* is a category where each morphism $f : X \to Y$ is equipped with a nonnegative number $w(f)$ called the *weight*, such that

\[
w(id) = 0; \\
Y \\
X \\
Z \\
\text{w}(g \circ f) \leq \text{w}(f) + \text{w}(g).
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A **weighted functor** is a functor $F : C \to D$ such that for every morphism $f$ of $C$,

$$w(Ff) \leq w(f).$$
Examples

Categories of paths of a space

Let $X$ be a metric space (e.g. $\mathbb{R}^n$). The weighted category $\text{Path}(X)$ has

- As objects, the points of $X$;
- As morphisms, the curves in $X$ with their length as weight.
Examples

Generalized metric spaces

A *pseudo-quasi* (or *Lawvere*) *metric space* is a set $X$ with a “cost” function $c : X \times X \to [0, \infty]$ such that

- $d(x, x) = 0$;
- $d(x, z) \leq d(x, y) + d(y, z)$

A pq-metric space is a weighted preorder.
Examples

Optimization over paths
Given a weighted category $C$, for objects $X$ and $Y$ consider the “optimum” weight

$$\inf_{f: X \to Y} w(f)$$

This gives a pq-metric on the objects of $C$. We call the resulting space $\text{Opt}(C)$.

The Wasserstein distances are an example [Villani, 2009].
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Free weighted category

Let $G$ be a weighted graph. The *free weighted category over $G$* has

- As objects, the vertices of $G$;
- As morphisms, *walks* in $G$;
- The weight is the sum of the weights.
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See Jade’s talk later today!
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Weighted lenses

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- Montréal
- Vancouver
- New York
- Boston
- Seattle
- Edinburgh
- London
- Oxford

Canada ➔ United States ➔ United Kingdom
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Weighted lenses

**Definition: weighted lifting**
- Let \( F : E \to B \) be a functor;
- Let \( b : B \to B' \) in \( B \);
- Let \( E \in E \) with \( F(E) = B \).

A *lifting of \( b \) at \( E \) consists of

\[
\begin{align*}
E & \quad \Rightarrow \quad B' \\
\downarrow & \quad \downarrow b \\
B & \quad \Rightarrow \quad B'
\end{align*}
\]
Weighted lenses

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A *lifting of $b$ at $E$ consists of*

- an object $E' \in E$ with $F(E') = B'$;
Weighted lenses

Definition: weighted lifting

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- Let $b : B \to B'$ in $B$;
- Let $E \in E$ with $F(E) = B$.

A lifting of $b$ at $E$ consists of

- an object $E' \in E$ with $F(E') = B'$;
- an arrow $\Phi(E, b) : E \to E'$ of $E$ such that $F(\Phi(E, b)) = b$, with the same weight as $b$. 
Weighted lenses

Definition: weighted lens

Let $E$ and $B$ be categories. A \textit{weighted lens} from $E \to B$ is

- A functor $F : E \to B$;
- For each morphism $b : B \to B'$ of $B$ and each object $E$ of $E$ with $F(E) = B$, a chosen weighted lifting $\Phi(E, b) : E \to E'$, such that
- Identities and compositions are preserved.
Lenses between categories of couplings

Theorem (P 2021)
Let $X$ and $Y$ be pq-metric, standard Borel spaces. Let $(f, \phi)$ be a weighted lens such that the assignments $f$ and $\phi$ are measurable.
Lenses between categories of couplings

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Let $X$ and $Y$ be pq-metric, standard Borel spaces. Let $(f, \phi)$ be a weighted lens such that the assignments $f$ and $\phi$ are measurable. There is a weighted lens $(f_\#, \tilde{\phi}_\#)$ between $PX$ and $PY$ where

- The projection $f_\# : PX \to PY$ is the pushforward;
Lenses between categories of couplings

**Theorem (P 2021)**

Let $X$ and $Y$ be pq-metric, standard Borel spaces. Let $(f, \phi)$ be a weighted lens such that the assignments $f$ and $\phi$ are measurable. There is a weighted lens $(f_\#, \tilde{\phi}_\#)$ between $PX$ and $PY$ where

- The projection $f_\# : PX \to PY$ is the pushforward;
- The lifting $\tilde{\phi}_\# : PX \times PY \to P(X \times X)$ takes $p \in PX$ and a coupling $s \in P(Y \times Y)$ with first marginal $f_*p$, and returns the coupling $\tilde{\phi}_\#(p, s) \in P(X \times X)$ given for all measurable $A, A' \subseteq X$ by

$$\tilde{\phi}_\#(p, s)(A \times A') := \int_A \int_Y 1_{A'}(\phi(x, y)) \ s(dy | f(x)) \ p(dx).$$
Lenses between categories of couplings

Intuitively, we are “lifting random transitions”.

\[
\tilde{\varphi}(s)(A|x) = \int_Y 1_A(\varphi(x, y)) s(dy|f(x))
\]
Co-history!

**Enriched lenses [Clarke and Di Meglio, 2022]**

Let $E$ and $B$ be $V$-enriched categories. A $V$-lens $(F, \Phi) : E \to B$ is

- A $V$-functor $F : E \to B$;
- For each object $E \in E$ and $B' \in B$, a $V$-arrow lifting $B(FE, B') \xrightarrow{\Phi_E, B'} \sqcup_{E' \in F^{-1}(B')} E(E, E')$ and satisfying identity and composition requirements.
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Weak submetries are enriched lenses (Callum Reader)

For $V = [0, \infty]$ ordered downward (Lawvere metric spaces):

$$d_Y(f(x), y') = \inf_{x' \in f^{-1}(y)} d(x, x').$$
Weak submetries are enriched lenses (Callum Reader)

For $V = [0, \infty]$ ordered downward (Lawvere metric spaces):

$$d_Y(f(x), y') = \inf_{x' \in f^{-1}(y)} d(x, x').$$

In terms of open balls,

$$f(B_X(x, r)) = B_Y(f(x), r)$$

for all $x \in X$ $r \in \mathbb{R}$. In geometry, this is called a *weak submetry*. 
Some references


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